# JOHN NASH'S NONLINEAR ITERATION

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## 1. Introduction

In this note we examine the analytical part of the famous 1954 paper of John F. Nash on the isometric embedding problem [48]. Our aim is to emphasize how Nash's discovery reaches far beyond differential geometry and that, rather than a result in differential geometry, the construction of Nash should be seen as a fully nonlinear iteration scheme that has potential applications for constructing solutions to partial differential equations in

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general. In classical perturbation methods for nonlinear equations, such as the Newton scheme, the linearization of the equation plays the key role. In contrast, in the scheme of Nash the leading order term is quadratic in the perturbation and the linearization becomes negligible. In some sense this means that the Nash scheme is genuinely infinite dimensional, there is no finite dimensional analogue! It is therefore not entirely surprising that a scheme of this type leads to highly irregular solutions.

After giving some of the details of Nash's argument we discuss how we have taken advantage of this point of view in [30] to produce continuous solutions of the incompressible Euler equations which behave in a surprising way. Our paper has given the first approach to a well known conjecture of Lars Onsager in the theory of turbulence, but rather than focusing on the state of the art for the latter and similar problems in the PDE literature (for which we refer to the survey article [27]), here we focus instead on the main underlying ideas and their similarities to Nash's astonishing iteration technique.

1.1. **The Nash-Kuiper theorem.** The existence of isometric immersions (resp. embeddings) of Riemannian manifolds into some Euclidean space is a classical problem, explicitly formulated for the first time by Schläfli, see [57]. Given a Riemannian manifold $(\Sigma, g)$, an immersion (resp. embedding) $u : \Sigma \to \mathbb{R}^n$ is called an *isometry* if it preserves the length of curves, namely if

$$\ell_g(\gamma) = \ell_e(u \circ \gamma) \qquad \text{for any } C^1 \text{ curve } \gamma : I \to \Sigma. \tag{1}$$

Here $\ell_e(\eta)$ denotes the usual euclidean length of a curve $\eta$, namely

$$\ell_e(\eta) = \int |\dot{\eta}(t)|\, dt\,,$$

whereas $\ell_g(\gamma)$ denotes the length of $\gamma$ in the Riemannian manifold $(\Sigma, g)$:

$$\ell_g(\gamma) = \int \sqrt{g(\gamma(t))[\dot{\gamma}(t), \dot{\gamma}(t)]}\, dt\,. \tag{2}$$

If $U \subset \Sigma$ is a coordinate patch, we can express $g$ as customary in local coordinates:

$$g = g_{ij} dx_i \otimes dx_j\,,$$

where we follow the Einstein's summation convention. The square of the integrand in (2) is then

$$g(\gamma(t))[\dot{\gamma}(t), \dot{\gamma}(t)] = g_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\,.$$

Nash started working at this question shortly after his PhD, apparently because of a bet with a colleague at the MIT department, where he had just moved as a young faculty, cf. [47]. The problem was considered a formidable one and at that time comparatively little was known. Janet [39], Cartan [15] and Burstin [14] had proved the existence of local isometric embeddings in the case of analytic metrics. For the very particular case of 2-dimensional spheres endowed with metrics of positive Gauss curvature, Weyl in [62] had raised the question of the existence of isometric embeddings in $\mathbb{R}^3$. The Weyl's problem was solved by Lewy in [44] for analytic metrics and only shortly before Nash's work another brilliant young mathematician, Louis Nirenberg, had settled the case of smooth metrics (in fact $C^4$, see [51] and [52]); the same problem had been solved independently by Pogorolev [54], building upon the work of Alexandrov [1] (see also [55]).

In his two celebrated works on the topic which appeared in 1954 and 1956 (cf. [48, 49]; he wrote a third contribution in the sixties, cf. [50]) Nash completely revolutionized the subject. He first proved a very counterintuitive fact which shocked the geometers of his time, namely the existence of $C^1$ isometric embeddings in codimension 2 in the absence of topological obstructions. He then showed the existence of smooth embeddings in sufficiently high codimension, introducing his celebrated approach to "hard implicit function theorems".

When $u : \Sigma \to \Gamma$ is an immersion of a differentiable manifold $\Sigma$ into a Riemannian manifold $(\Gamma, h)$, it is customary to denote by $u^\sharp h$ the induced pull-back metric on $\Sigma$, which is given by the relation

$$u^\sharp h(p)[X, Y] = h(u(p))[du(X), du(Y)] \qquad \text{for } X, Y \text{ tangent to } \Sigma.$$

If we denote by $e$ the standard euclidean metric on $\mathbb{R}^N$, an isometric immersion of $(\Sigma, g)$ into $\mathbb{R}^N$ is characterized by the identity $u^\sharp e = g$ (and note that the relation itself guarantees that $u$ must be an immersion even if we do not assume it a priori). In a local coordinate patch the latter identity gives a system of partial differential equations:

$$\partial_i u \cdot \partial_j u = g_{ij}\,, \tag{3}$$

consisting of $n(n+1)/2$ equations in $N$ unknowns.

A reasonable guess would therefore be that the system is solvable (at least locally) when $N = n(n+1)/2$: this was in fact what Schläfli conjectured in the nineteenth century and what Janet, Cartan and Burstin proved locally for analytic metrics. However, to quote John Milnor, "Nash was never a reasonable person" (cf. [26]): in his first (very short) paper in 1954 he astonished the geometry world and proved that the only true obstructions to the existence of isometric immersions are topological. As soon as $N \geq n+1$ and there are no such obstructions, then there are in fact plenty of such immersions. Indeed Nash gave a proof for $N \geq n+2$ and just remarked that a similar statement could be proved for $N \geq n+1$: the details were then given in two subsequent notes by Kuiper, [42].

The resulting theorem, which is nowadays called the Nash-Kuiper Theorem on $C^1$ isometric embeddings, is usually stated after introducing the concept of "short map".

**Definition 1.1.** *Let $(\Sigma, g)$ be a Riemannian manifold. An immersion $v : \Sigma \to \mathbb{R}^N$ is short if we have the inequality $v^\sharp e \leq g$ in the sense of quadratic forms. If the inequality $<$ holds we then say that $v$ is* strictly *short.*

As usual, the inequality $h \leq g$ (in the sense of quadratic forms) is the requirement $h_{ij}v^iv^j \leq g_{ij}v^iv^j$ for any tangent vector $v$, whereas $h < g$ means that the strict inequality holds whenever $v$ is nonzero.

The Nash-Kuiper Theorem is then the following

**Theorem 1.2.** *Let $(\Sigma, g)$ be a smooth closed $n$-dimensional Riemannian manifold and $v : \Sigma \to \mathbb{R}^N$ a $C^\infty$ short immersion with $N \geq n+1$. Then, for any $\varepsilon > 0$ there exists a $C^1$ isometric immersion $u : \Sigma \to \mathbb{R}^N$ such that $\|u - v\|_{C^0} \leq \varepsilon$. If $v$ is, in addition, an embedding, then $u$ can be assumed to be an embedding as well.*

A suitable version of this theorem can be proved even for open manifolds and the smoothness of $v$ and $g$ can be considerably relaxed: the metric needs only to be continuous, whereas it is sufficient that the short map is $C^1$.

If $\Sigma$ is a smooth closed manifold, as soon as there is an immersion $v : \Sigma \to \mathbb{R}^N$ we can make it short by simply multiplying it by a small positive constant. Thus, Theorem 1.2 is not merely an existence theorem, but it shows that there exists a huge (essentially $C^0$-dense) set of solutions in rather low codimension, even for the most general manifolds, because the classical Theorem of Whitney guarantees the existence of an embedding already in $\mathbb{R}^{2n-1}$.

This type of abundance of solutions is a central aspect of Gromov's $h$-principle, for which the isometric embedding problem is a primary example (see [35, 32]). For $C^1$ isometric embeddings of surfaces to $\mathbb{R}^3$ the $h$-principle is particularly striking: for classical (e.g. $C^2$) maps preservation of the metric (i.e. being isometric) leads to higher order constraints, most notably the Theorema Egregium of Gauss. In the Weyl problem, i.e. when $(\mathbb{S}^2, g)$ has positive Gauss curvature, this additional constraint is a crucial element in the proof of rigidity: $C^2$ isometric immersions into $\mathbb{R}^3$ are uniquely determined up to a rigid motion ([18, 37], see also [59] for a thorough discussion).

In particular any $C^2$ isometric immersion of the standard sphere in $\mathbb{R}^3$ must map it to the boundary of some ball of radius 1, whereas the Nash-Kuiper theorem implies the existence of $C^1$ isometric embeddings which crumple it in an arbitrarily small region of the 3-dimensional space. It is thus clear that solutions to (3) have a completely different qualitative behavior at low and high regularity (i.e. below and above $C^2$).

1.2. **The Euler equations.** The original $h$-principle of Gromov pertains to various problems in differential geometry, where one expects high flexibility of the moduli space of solutions due to the underdetermined nature of the problem. It was not expected that the same principle and similar methods could be applied to problems in mathematical physics (we quote Gromov's speech at the Balzan Prize [36]: *The class of infinitesimal laws subjugated by the homotopy principle is wide, but it does not include most partial differential equations (expressing infinitesimal laws) of physics with a few exceptions* *in favor of this principle leading to unexpected solutions. In fact, the presence of the h-principle would invalidate the very idea of a physical law as it yields very limited global information effected by the infinitesimal data.* See also the introduction in the book [32]).

In a first paper [28] (see also the survey article [29]) we however found that some well known (and hard to construct) examples of non-uniqueness of solutions due to Scheffer [56] and later to Shnirelman [58] for the incompressible Euler equations

$$\begin{cases} \partial_t v + \mathrm{div}_x (v \otimes v) + \nabla p = 0, \\ \\ \mathrm{div}_x\, v = 0, \end{cases} \tag{4}$$

could be interpreted as some kind of $h$-principle in a very natural way: after introducing a suitable notion of "subsolution" for (4) (in analogy to short maps) we could prove that it can be approximated arbitrarily well by bounded weak solutions, in an appropriate weak topology, following a well-known path in the literature for differential inclusions, see [16, 10, 21, 46].

Dealing with merely bounded (i.e. $v \in L^\infty$) weak solutions of the Euler equations (4) is somewhat reminiscent of dealing with Lipschitz solutions for the isometric embedding problem. Note that in this case one could give two different notions of "weak solution":

- one sticking to requirement (1);
- the other postulating (3) to be valid almost everywhere.

The first one is indeed stronger than the second. Of course if we consider Lipschitz solutions it means that we are allowed to "fold" our Riemannian manifold and thus an $h$-principle statement is much less surprising than the Nash-Kuiper theorem. Moreover in the Lipschitz category one can even impose the target to have the same dimension as the Riemannian manifold, cf. [41].

Coming back to the Euler equations, classical ($C^1$) solutions of (4) are "rigid" in the following sense:

(a) they are uniquely determined by their initial data at time $t = 0$;
(b) the total kinetic energy $\frac{1}{2}\int |v(x,t)|^2\,dx$ is constant in time.

Such additional constraints should be compared to the ones discussed for $C^2$ isometric embeddings of surfaces in $\mathbb{R}^3$: by the Gauss theorem the Gauss curvature is preserved (which is true in general and can be considered as the analog of (b)), and in the particular case of the Weyl problem $C^2$ isometric embeddings are unique up to affine transformations (which can be considered as an analog of (a)). Both of these "rigidity statements" for the Euler equations fail for the $L^\infty$ weak solutions of (4) constructed in [56, 58, 28].

The relation between the constructions in the theory of differential inclusions and those typical of the $h$-principle was first pointed out in an important paper by Müller and Šverak [46] (see also [40]). Inspired by this connection and building upon some of the intuitions of Nash's remarkable work, we were able to prove a very counterintuitive fact, namely the existence of "badly behaved" continuous solutions, cf. [30]. More precisely, if we denote by $\mathbb{T}^3$ the 3-dimensional torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, we then have the following theorem.

**Theorem 1.3.** *Let $e : [0,1] \to \mathbb{R}$ be a positive smooth function. For any $\varepsilon > 0$ there exists a continuous weak solution $(v,p) : \mathbb{T}^3 \times [0,1] \to \mathbb{R}^3 \times \mathbb{R}$ of the Euler equations* (4) *such that $\sup_t \|v\|_{H^{-1}(\mathbb{T}^3)} \leq \varepsilon$ and*

$$\int_{\mathbb{T}^3} |v(x,t)|^2 \, dx = e(t) \qquad \forall t \in [0,1]\,. \tag{5}$$

This theorem shows that the second "rigidity statement" (b) above fails for continuous solutions. Here we concentrate on the issue of energy conservation because of its relevance to 3D turbulence (cf. Section 1.3 below), but similar techniques can be used to show non-uniqueness for continuous solutions as well [38, 22, 23]. Observe however a crucial difference between Theorem 1.2 and Theorem 1.3: Nash's theorem shows that any short map can be approximated with isometries, whereas there is no reference to a similar "density" result in Theorem 1.3. It is however possible to prove an appropriate statement of that kind by improving upon the methods: we refer the reader to the recent note [23] for the precise statement.

In the next sections we will show the similarities and the differences between the proofs of Theorem 1.2 and Theorem 1.3. Since the argument of Nash is short and rather elementary, we give it in full details. We then proceed to the main points in the proof of Theorem 1.3, highlighting how Nash's ideas turned out to be decisive for our proof, but also pointing out some important differences.

1.3. **Rigidity and flexibility: a Hölder threshold?** In both problems seen above a natural question is whether there is a sharp threshold regularity which distinguishes between the two different behaviors of solutions, i.e. between rigidity and $h$-principle statements. In the case of periodic solutions of the Euler equations, this threshold is expected to be $C^{0,\frac{1}{3}}$, due to the following longstanding conjecture of Lars Onsager in the theory of turbulence [53]:

**Conjecture 1.4.** *Consider solutions $(v,p)$ on $\mathbb{T}^3 \times [0,1]$ of* (4) *satisfying the Hölder condition*

$$|v(x,t) - v(x',t)| \leq C|x - x'|^\theta, \tag{6}$$

*where the constant $C$ is independent of $x, x' \in \mathbb{T}^3$ and $t$. Then*

(a) *If $\theta > \frac{1}{3}$, any weak solution $(v,p)$ of* (4) *satisfying* (6) *conserves the energy;*
(b) *For any $\theta < \frac{1}{3}$ there exist weak solutions $(v,p)$ of* (4) *satisfying* (6) *which do not conserve the energy.*

It was proved in the nineties that above this threshold weak solutions satisfy the law of conservation of the energy (see [19, 33]). Concerning the lower regularity range, a first adaptation of the arguments in [30] showed that energy conservation is violated by some solutions in $C^{0,\frac{1}{10}-\varepsilon}$, see [31]. The threshold was later improved in [38] to $\frac{1}{5}$, see also [12], whereas if one gives up the uniform control of the Hölder exponent in time, it is possible to reach $\frac{1}{3}-\varepsilon$: more precisely there are non-conservative solutions in $L^1((0,1), C^{\frac{1}{3}-\varepsilon}(\mathbb{T}^3))$, see [13] for the exact statement (and [11] for an important first step).

For the isometric embedding problem the natural scale of spaces would be $C^{1,\alpha}$ and remarkably the question of rigidity and flexibility for solutions in such spaces was already studied in the fifties and the sixties by Borisov. In a series of papers in the fifties (see [4, 5, 6, 7]) Borisov proved that the rigidity for the Weyl's problem holds in fact when the immersion is $C^{1,\frac{2}{3}+\varepsilon}$ (for a much shorter proof see [20]). Later in [8] he announced a general $h$-principle statement for sufficiently small exponents ($\frac{1}{7}$ in the case of isometric embeddings of 2-dimensional disks in $\mathbb{R}^3$) and published in [9] a proof of a weaker conclusion. A complete proof of Borisov's announced results has been given in [20]. More recently in [25] the thresold for 2-dimensional disks in $\mathbb{R}^3$ has been improved to $\frac{1}{5}$. Differently from the case of the Euler equations, there is no physical motivation to guess the existence of a critical exponent: however Gromov conjectures that the threshold is in fact $\frac{1}{2}$ (see [35]).

In this note we will not discuss how to pass from continuous to Hölder solutions: the basic methods and ideas remain the same, the main points are in carefully estimating the error terms which arise in the iterations leading to the proofs of Theorem 1.2 and Theorem 1.3. This is by no means easy and indeed there are several subtle points and new ideas involved, especially if one is interested in getting the sharp exponents. However in this note we have decided to focus on the most important ideas and in particular on the relations and differences between Theorem 1.2 and Theorem 1.3.

## 2. Nash's $C^1$ iteration "stage"

In this section we introduce the main proposition of Nash's scheme in the proof of Theorem 1.2. We restrict on purpose to the codimension 2 case (the one of Nash's paper) to avoid as many technical points as possible, although the codimension 1 case does not need much more conceptual work. From now on we fix therefore a smooth closed Riemannian manifold $\Sigma$ as in Theorem 1.2 and a corresponding smooth atlas $\mathcal{A} = \{U_\ell\}$ made of finitely many coordinate patches $U_\ell$, whose closures we assume to be topological Euclidean balls.

Given any symmetric $(0,2)$ tensor $h$ on $\Sigma$ we write $h = h_{ij} dx_i \otimes dx_j$ and denote by $\|h\|_{0,U_\ell}$ the supremum of the Hilbert-Schmidt norm of the matrices $h_{ij}(p)$ for $p \in U_\ell$. Similarly, if $v : \Sigma \to \mathbb{R}^N$ is a $C^1$ map, we write

$\|Dv\|_{0,U_\ell}$ for the supremum of the Hilbert-Schmidt norms of the matrices $Dv(p) = (\partial_1 v(p), \ldots, \partial_n v(p))$, where $p \in U_\ell$. Finally we set

$$\|h\|_0 := \sup_\ell \|h\|_{0,U_\ell}$$
$$\|Dv\|_0 := \sup_\ell \|Dv\|_{0,U_\ell} .$$

We are now ready to formulate the main inductive statement, which Nash calls "a stage", cf. [48, Page 391].

**Proposition 2.1.** *Let $(\Sigma, g)$ be as in Theorem 1.2 and $z : \Sigma \to \mathbb{R}^N$ a smooth strictly short immersion. For any $\eta, \delta > 0$ there exists a smooth short $z_1 : \Sigma \to \mathbb{R}^N$ such that*

$$\|z - z_1\|_0 \le \eta \,, \tag{7}$$
$$\|g - z_1^\sharp e\|_0 \le \delta \,, \tag{8}$$
$$\|Dz_1 - Dz\|_0 \le C\sqrt{\|g - z^\sharp e\|_0} \,, \tag{9}$$

*for a constant $C$ which depends only upon $\Sigma$. If $z$ is injective, then we can choose $z_1$ injective.*

If we are only concerned with the case of immersions, Theorem 1.2 can easily be reduced to Proposition 2.1: we start with the short map $v =: v_0$ and assume without loss of generality that it is strictly short. Apply Proposition 2.1 with $z = v_0$, $\eta = \frac{\varepsilon}{4}$ and $\delta = \frac{1}{4}$ to produce a second short immersion $z_1 =: v_1$. We then apply inductively the Proposition to $z = v_i$ with $\eta = \frac{\varepsilon}{2^{i+2}}$ and $\delta = 4^{-i-1}$ to generate $z_1 =: v_{i+1}$. Thus the sequence $v_i$ converge in $C^0$ to a map $u$ with $\|v - u\|_0 \le \frac{\varepsilon}{2}$. On the other hand by (9) we have

$$\|Dv_{i+1} - Dv_i\|_0 \le C\sqrt{\|g - v_i^\sharp e\|_0} \le \frac{C}{2^i} \,,$$

which implies that the convergence of $v_i$ to $u$ is in fact in $C^1$. This fact clearly implies that $v_i^\sharp e$ converges to $u^\sharp e$ uniformly and thus that $u^\sharp e = g$. As already observed, the latter identity guarantees that $u$ is an immersion.

Note next that at each step of the iteration we can ensure that each $v_i$ is indeed injective if the starting map $v_0 = v$ is injective (and hence an embedding). This however does not guarantee the injectivity of the limit. Assume to have performed the $q-1$ step and generated the maps $v_0, v_1, \ldots v_q$, guaranteeing that all of them are injective. Define then the positive numbers

$$2\gamma_i := \min\{|v_i(x) - v_i(y)| : d(x,y) \ge 2^{-i}\} \qquad \text{for } i < q,$$

where $d$ is the geodesic distance induced by the Riemannian metric $g$. We then set $\eta := \min\{2^{-q-2}\varepsilon, 2^{-q-1}\gamma_1, 2^{-q-1}\gamma_2, \ldots, 2^{-q-1}\gamma_{q-1}\}$ and apply the Proposition to $w = v_q$ with $\eta$ and $\delta = 4^{-q-1}$ to generate $v_{q+1}$. Clearly all the conclusions above still apply: we claim however that now the limit $u$ is injective. Fix indeed two points $x \neq y$ and let $q$ be some natural number such that $d(x,y) \geq 2^{-q}$. We can then estimate

$$\begin{aligned}|u(x)-u(y)| \geq& |v_q(x)-v_q(y)| - \sum_{k\geq q} \|v_{k+1}-v_k\|_0\\ \geq& 2\gamma_q - \sum_{k\geq q} 2^{-k-1}\gamma_q \geq \gamma_q > 0\,.\end{aligned}$$

As already said, in this note we wish to focus on the analytic aspects of Nash's paper and for this reason we give the following "local" version of Proposition 2.1. Here "local" is understood in the following sense: (1) we work in a single chart and (2) the metric error $g - z^\sharp e$ is assumed to be in a suitable neighborhood of the flat metric. Such requirements might seem rather restrictive, but in fact even this weaker Proposition is enough to prove Nash's original statement (for the details we refer to [20]).

We define the following cone of positive-definite matrices for any $r < 1$:

$$\mathcal{C}_r := \left\{ A \in \mathrm{Sym}_{n\times n} : \left| \frac{A}{\frac{1}{n}\mathrm{tr}\, A} - \mathrm{Id} \right| < r \right\}.$$

Geometrically $\mathcal{C}_r$ is a convex cone of opening "angle" $r$ centered around the half-line $\{\lambda \mathrm{Id} : \lambda > 0\}$.

**Proposition 2.2.** *There exists a dimensional constant $r_0(n) > 0$ with the following property. Let $U \subset \mathbb{R}^n$ be a bounded simply connected open domain, $g \in C^\infty(\overline{U})$ a smooth metric and $z : \overline{U} \to \mathbb{R}^N$ a smooth short map such that*

$$g - z^\sharp e \in \mathcal{C}_{r_0} \qquad \text{for all } x \in \overline{U}. \tag{10}$$

*For any choice of positive numbers $\delta, \eta > 0$ there exists a smooth short map $z_1 : \overline{U} \to \mathbb{R}^N$ such that*

$$g - z_1^\sharp e \in \mathcal{C}_{r_0} \qquad \text{for all } x \in \overline{U},$$

*and the following estimates hold:*

$$\|z_1 - z\|_0 \leq \eta\,, \tag{11}$$

$$\|g - z_1^\sharp e\|_0 \leq \delta\,, \tag{12}$$

$$\|Dz_1 - Dz\|_0 \leq M\sqrt{\|g - z^\sharp e\|_0}\,, \tag{13}$$

*for a dimensional constant $M$. If $z$ is injective, then we can choose $z_1$ injective.*

2.1. **Nash's spirals.** We now examine how Propositions 2.1 and 2.2 are proved. Nash splits each "stage" into a certain number of steps, where each step aims at decreasing the metric error in a single coordinate direction. To make this precise, we will call a "primitive metric" – following [32] – any $(0,2)$ tensor having the structure $a^2 d\psi \otimes d\psi$ for some pair of smooth functions $a$ and $\psi$. Note that such two-tensor is only positive semidefinite and not a Riemannian metric.

In order to define the steps in a stage, we need to decompose a Riemannian metric on $\Sigma$ into a locally finite sum of primitive metrics. This is the content of the next lemma.

**Lemma 2.3.** *Let $\Sigma$ be a smooth $n$-dimensional manifold, $h$ a smooth positive definite $(0,2)$ tensor on it and $\mathcal{A} = \{U_\ell\}$ a finite atlas of charts on $\Sigma$. Then there is a finite collection $h_j$ of primitive metrics such that each $h_j$ is supported in some $U_\ell$, $\sum h_j = h$ and for each point $p \in \Sigma$ there are at most $K(n) = \frac{n(n+1)^2}{2}$ primitive metrics $h_j$ which do not vanish at $p$.*

The local version, suitable for Proposition 2.2, is the following simple geometric lemma on matrices (we note in passing that this lemma is contained in the paper of Nash [48], but has also proved useful in other contexts, see [34, Lemma 17.13] and [45]):

**Lemma 2.4.** *Consider the space $\mathcal{S}^{n\times n}_+$ of positive definite symmetric matrices. There exists a neighborhood $W$ of the identity matrix $\mathrm{Id}$ and $N(n) = \frac{n(n+1)}{2}$ unit vectors $v_k \in \mathbb{R}^n$ such that any symmetric matrix $A \in W$ can be written in a unique way as a linear combination*

$$A = \sum_k \lambda_k(A) v_k \otimes v_k \tag{14}$$

*with coefficients $\lambda_k(A) \geq \rho_0(n)$, where $\rho_0(n)$ is a positive geometric constant.*

Note that $v_k \otimes v_k$ can be thought as $d\psi_k \otimes d\psi_k$ for the linear function $\psi_k(x) = v_k \cdot x$. The latter lemma is therefore a very natural counterpart of Lemma 2.3: the factor $n+1 = K(n)/N(n)$ distinguishing the number of primitive metrics appearing in the respective decompositions is due to "global geometric aspects", whereas the main (local) idea of both proofs remains the same.

Leaving aside for a moment the (elementary!) proofs of Lemma 2.3 and Lemma 2.4, we turn to the proofs of the Proposition 2.1 and Proposition 2.2.

Let us first focus on Proposition 2.2. In the following we work in local coordinates in the set $U$. For a map $z : \overline{U} \to \mathbb{R}^N$ we denote by $Dz = (\partial_j z^i)_{ij}$ the Jacobian matrix consisting of all first order partial derivatives of $z$. Note that in local coordinates $(z^\sharp e)_{ij} = \langle \partial_i z, \partial_j z\rangle_{\mathbb{R}^N}$, so that in matrix notation the tensor $z^\sharp e$ can be identified with the symmetric $n \times n$ matrix $Dz^T Dz$.

By assumption the matrix $(g - Dz^T Dz)(x)$ is positive definite on $\overline{U}$. Therefore, there exists $\gamma > 0$ so that $g - Dz^T Dz \geq 2\gamma \mathrm{Id}$ on $\overline{U}$. We may assume without loss of generality that $\gamma \leq \delta$. Set

$$h(x) = g(x) - Dz^T(x) Dz(x) - \gamma \mathrm{Id}.$$

Then $h(x) \in \mathcal{C}_{r_0}$ for all $x$. Observe that by Lemma 2.4 and the "pinching" condition, we can write

$$h(x) = \sum_k a_k^2(x) v_k \otimes v_k =: \sum_k h_k(x)\,. \tag{15}$$

Indeed, observe that for any $A \in \mathcal{C}_{r_0}$ with $r_0$ sufficiently small (depending only on $W$ from Lemma 2.3) we have $\frac{n}{\operatorname{tr} A} A \in W$, so that

$$a_k(x) = \sqrt{\tfrac{1}{n} \operatorname{tr} h(x)}\, \lambda_k \left( \frac{n}{\operatorname{tr} h(x)} h(x) \right).$$

Observe moreover that, since clearly the coefficients $\lambda_k(A)$ in (14) depend linearly on $A$, the positive coefficients $a_k(x)$ vary smoothly with $x$.

In the first step we wish to perturb the map $z$ slightly to a new map $\tilde{z}$ for which $D\tilde{z}^T D\tilde{z}$ is approximately $Dz^T Dz + a_1 v_1 \otimes v_1$, with a small error. This is achieved in the proposition below. We then repeat this step a finite number of times to "add" all primitive metrics in the decomposition and estimate the resulting total error.

**Proposition 2.5.** *Let $U \subset \mathbb{R}^n$ be a bounded simply connected open domain, $a_1 \in C^\infty(\overline{U})$, $v_1 \in \mathbb{R}^n$ a unit vector, and $z : \overline{U} \to \mathbb{R}^N$ a smooth short map. For any choice of positive numbers $\tilde{\delta}, \tilde{\eta} > 0$ there exists a smooth short map $\tilde{z} : \overline{U} \to \mathbb{R}^N$ such that the following estimates hold:*

$$\|\tilde{z} - z\|_0 \leq \tilde{\eta}\,, \tag{16}$$

$$\|D\tilde{z}^T D\tilde{z} - Dz^T Dz - a_1^2 v_1 \otimes v_1\|_0 \leq \tilde{\delta}\,, \tag{17}$$

$$\|D\tilde{z} - Dz\|_0 \leq \tilde{M} \|a_1\|_0\,, \tag{18}$$

*for a dimensional constant $\tilde{M}$.*

The map $\tilde{z}$ is completely explicit: Fix two smooth unit length normal fields $\nu, b : U \to \mathbb{R}^N$, i.e. with the properties

- $|\nu(x)| = |b(x)| = 1$ and $\nu(x) \perp b(x)$ for all $x \in U$;
- $\nu(x)$ and $b(x)$ are both orthogonal to $T_{\omega(x)}(z(U))$ for every $x \in U$.

The existence of such vector fields is the consequence of the trivial topology of $U$ and the fact that $N \geq n + 2$. We then set

$$\begin{aligned} \tilde{z}(x) :=& z(x) + z_p(x) \\ =& z(x) + \frac{a_1(x)}{\lambda} \big( \nu(x) \cos(\lambda v_1 \cdot x) + b(x) \sin(\lambda v_1 \cdot x) \big)\,, \end{aligned} \tag{19}$$

where $\lambda \gg 1$ is a large positive parameter, which will be chosen later. The perturbation above makes then very fast spirals around the map $z$. It is clear that $\tilde{z}$ satisfies the estimates (16) and (18) for sufficiently large $\lambda \gg 1$. The heart of the matter is to verify estimate (17) – this will be done in Section 2.2 below.

In the case of Proposition 2.1 we follow a very similar scheme to define $\bar{z}$. Let us fix $\Sigma, g, w, \eta$ and $\delta$ as in the statement. Given the atlas $\mathcal{A}$ on $\Sigma$ we apply first Lemma 2.3 to the metric $h = (1 - \gamma)(g - w^\sharp e)$ with some small $\gamma > 0$, and let $h = \sum_j a_j^2 d\psi_j \otimes d\psi_j$ be the corresponding decomposition in primitive metrics. Recalling that $h_j = a_j^2 d\psi_j \otimes d\psi_j$ we set

$$\begin{aligned}\bar{z}(x) =& z(x) + z_p(x)\\ =& z(x) + \frac{a_1(x)}{\lambda}\big(\nu(x)\cos(\lambda\psi_1(x)) + b(x)\sin(\lambda\psi_1(x))\big)\,. \end{aligned} \tag{20}$$

We see that (19) is a particular case of (20), corresponding to a function $\psi_1(x)$ which is simply linear in the given coordinate patch.

2.2. **An unusual perturbation: the quadratic term wins.** Returning to the local perturbation (19) and Proposition 2.5, we compute the matrix $D\tilde{z}^T D\tilde{z} - Dz^T Dz$. First we calculate

$$\begin{aligned} Dz_p(x) = & \underbrace{-a_1(x)\sin\lambda v_1\cdot x\,\nu(x)\otimes v_1}_{A(x)}\\ & + \underbrace{a_1(x)\cos\lambda v_1\cdot x\, b(x)\otimes v_1}_{B(x)} + E(x)\,, \end{aligned}$$

where $|E(x)| \leq C\lambda^{-1}$, for a constant $C$ which depends on the smooth functions $a_1$, $b$ and $\nu$, but not on $\lambda$ (note that in the line above we understand all summands as $N\times n$ matrices). Next write the tensor $\tilde{h} := \tilde{z}^\sharp e - z^\sharp e$ in coordinates as a symmetric matrix-valued function and observe that then we simply have

$$\tilde{h} = D\tilde{z}^T D\tilde{z} - Dz^T Dz = \underbrace{(Dz_p^T Dz + Dz^T Dz_p)}_{=:L} + \underbrace{Dz_p^T Dz_p}_{=:Q}\,. \tag{21}$$

The decomposition above gives simply the perturbation induced in the metric tensor by the perturbing map $z_p$ as a sum of the parts which are, respectively, linear and quadratic in $z_p$. Recall that $Dz_p = A + B + E$ and, since $\nu$ and $b$ are orthogonal to $z(U)$ we have

$$0 = A^T Dz_p = Dz_p^T A = Dz_p^T B = B^T Dz_p\,.$$

Therefore

$$\|L\|_0 \leq C\lambda^{-1}\,. \tag{22}$$

On the other hand

$$(A+B)^T(A+B) = a_1^2(\cos^2\lambda v_1\cdot x + \sin^2\lambda v_1\cdot x)\, v_1\otimes v_1 = a_1^2 v_1\otimes v_1\,.$$

Hence we have

$$Q = a_1^2 v_1\otimes v_1 + O(\lambda^{-1})\,. \tag{23}$$

Thus a choice of a very large $\lambda$ makes the quadratic part much more important than the linear one: this seems a rather “odd” approach from a classical “PDE” point of view. Nonetheless it achieves the desired goal, namely the addition of a primitive metric with an arbitrary small error. This concludes the proof of Proposition 2.5.

The proof of Proposition 2.2 (and of Proposition 2.1) follow now in a straightforward manner. Guided by the decomposition (15) we define $z_1$ to be the resulting map after adding a finite number of spiraling perturbations as above. In this way we achieve (11) and for any fixed $\varepsilon > 0$

$$\|Dz_1^T Dz_1 - (Dz^T Dz + h)\|_0 \le \varepsilon.$$

Therefore

$$Dz_1^T Dz_1 - g = \gamma \mathrm{Id} + O(\varepsilon),$$

from which we easily deduce the "pinching condition" and (12) for $\varepsilon$ sufficiently small. Taking the trace of relation (15) leads to the estimate $\|a_k\|_0 \le C\sqrt{\|h\|_0}$, whereas from (18) we obtain

$$|Dz_1 - Dz\|_0 \le \tilde{M} \sum_k \|a_k\|_0 \le M\sqrt{\|h\|_0}\,. \tag{24}$$

Finally, when $z$ is injective the injectivity of the map $z_1$ will again follow from taking $\lambda$ very small, since the corresponding perturbation will then be a small normal displacement.

The argument for Proposition 2.1 is entirely analogous. Only slightly more care is needed: although the number of perturbations added might be very large, we do know that at any point $p \in \Sigma$ at most $K(n)$ perturbations give a nonzero contribution.

## 3. Technical points

For the sake of completeness we present the arguments of Lemmas 2.3 and 2.4 as well as the proof of existence of the normal fields $\nu$ and $b$.

*Proof of Lemma 2.4.* We start with Lemma 2.4. Since the set of all matrices of the form $v \otimes v$ is a linear generator of $\mathrm{Sym}_{n\times n}$, there are $N$ such matrices $A_i' = w_i \otimes w_i$ which are linearly independent. Consider $M' := \sum_i A_i'$. Since any pair of positive definite symmetric matrices are similar, we can find a linear isomorphism $L$ of $\mathbb{R}^n$ such that $L^T M' L = \mathrm{Id}$, where Id denotes the identity matrix and thus, if we set $A_i = L^T A_i' L = (Lw_i) \otimes (Lw_i) = v_i \otimes v_i$, we conclude that $\mathrm{Id} = \sum_i A_i$. Next, since $\{v_i \otimes v_i\}$ forms a basis for $\mathrm{Sym}_{n\times n}$, there exist unique linear maps $\mathcal{L}_i : \mathrm{Sym}_{n\times n} \to \mathbb{R}$ such that $A = \sum_i \mathcal{L}_i(A) v_i \otimes v_i$ for every $A \in \mathrm{Sym}_{n\times n}$. The continuity of such maps obviously gives the claim that the coefficients will be positive in a neighborhood of the identity matrix. Note only that the $v_i$'s are not unique vectors: on the other hand they are all nonzero vectors and thus the condition $|v_i| = 1$ after the obvious normalization. □

*Proof of Lemma 2.3.* As for Lemma 2.3, first of all, for each point $p \in \Sigma$ we find a neighborhood $V_p \subset U_\ell$ (for some $\ell$) and $N(n) = \frac{n(n+1)}{2}$ primitive metrics $h_{p1}, \ldots, h_{pN}$ on $V_p$ such that $h = h_{p1} + \ldots + h_{pN}$. This follows easily from Lemma 2.4. Observe that, arguing as above, since we can write any symmetric matrix $M$ as $L^T \mathrm{Id} L$ for some linear isomorphism $L$, Lemma 2.4 holds not only in a neighborhood of the identity matrix, but in fact it is valid in a suitable neighborhood of any symmetric positive definite matrix.

Hence for a suitable choice of $V_p$ we find functions $\psi_i(x) = v_i \cdot x$ (in local coordinates) and smooth $\alpha_i : V_p \to \mathbb{R}$ such that

$$h = \sum_{i=1}^{N} \alpha_i d\psi_i \otimes d\psi_i \,.$$

The "global aspect" of Lemma 2.4 can then be handled using the following elementary combinatorial fact (the reader can consult [20] for instance).

**Lemma 3.1.** *Let $\Sigma$ be a closed differentiable $n$-dimensional manifold and $\{V_\lambda\}$ an open cover of $\Sigma$. Then there is a finite open cover $\{W_\ell\}$ with the properties that:*

(a) *each $W_\ell$ is contained in some $V_\lambda$;*
(b) *the closure of each $W_\ell$ is diffeomorphic to an $n$-dimensional ball;*
(c) *for point $p \in \Sigma$ has a neighborhood contained in at most $n+1$ elements of the cover.*

Apply then Lemma 3.1 and refine the covering $V_p$ to a new covering $W_\ell$ with the properties listed in the Lemma. For each $W_\ell$ we consider a $V_p \supset W_\ell$ and define the corresponding primitive metrics $h_{(\ell 1)} = h_{p1}, \ldots, h_{(\ell N)} = h_{pN}$ (we use the subscript $(\ell j)$ in order to avoid confusions with the explicity expression of the initial tensor $h$ in a given coordinate system!). We then consider compactly supported functions $\beta_\ell \in C^\infty_c(W_\ell)$ with the property that for any point $p$ there is at least a $\beta_\ell$ which does not vanish at $p$ and we set

$$\varphi_\ell := \frac{\beta_\ell}{\sqrt{\sum_j \beta_j^2}} \,.$$

The tensors $\varphi_\ell^2 h_{(\ell j)}$ satisfy all the requirements of the lemma. □

*Construction of normal fields.* The pair of vector fields $\nu$, $b$ exists locally in a sufficiently small neighborhood of any point $p$ following a standard procedure:

- select two orthonormal vectors $\nu(p)$ and $b(p)$ which are normal to $T_{\omega(p)}(\omega(B))$;
- set the functions $\tilde{\nu}$ and $\tilde{b}$ constantly equal to these vectors in a neighborhood of $p$;
- project $\tilde{\nu}(q)$ and $\tilde{b}(q)$ to the vector space normal to $T_{\omega(q)}(\omega(B))$ and then use a Gram-Schmidt orthogonalization procedure to produce $\nu$ and $b$.

Furthermore the problem of passing from the local statement to the global one can be translated into the existence of a suitable section of a fiber bundle: since $B$ is topologically trivial, the existence of a "global" pair $\nu$, $b$ is a classical conclusion.

However one can also use an elementary argument. If we set $U = B_1(0)$ and we consider the set $R$ of radii $r$ for which a pair as required exists in the closed ball $\overline{B}_r(0)$, it is not difficult to use the ideas above to see that such set is both open and closed and thus it must be the entire interval $[0, 1]$. For the full details of this elementary argument we refer to [24].

## 4. The Euler-Reynolds system and the iteration stage for Euler

### 4.1. **The Euler-Reynolds system.**

In order to start an iteration procedure for Theorem 1.3 following the approach of Nash, we need to measure the "distance" of a smooth pair $(v, p)$ from being a solution of (4) and (5). For this reason we introduce a function $\mathring{R}$ which takes values in the space $\mathcal{S}_0^{3\times 3}$ of trace-free symmetric $3\times 3$ matrices. Following [30] a (smooth) triple $(v, p, \mathring{R})$ is a solution of the Euler-Reynolds system if it satisfies the following system of partial differential equations

$$\left\{\begin{array}{l} \partial_t v + \operatorname{div} v \otimes v + \nabla p = \operatorname{div} \mathring{R} \\ \\ \operatorname{div} v = 0\,. \end{array}\right. \tag{25}$$

Note that for $\mathring{R} = 0$ the pair $(v, p)$ solves the incompressible Euler equations.

The tensor $\mathring{R}$ is closely related to the *Reynolds stress tensor*, a classical concept in fluid dynamics. It is generally accepted that the appearance of high-frequency oscillations in the velocity field is the main reason responsible for turbulent phenomena in incompressible flows. One related major problem is therefore to understand the dynamics of the coarse-grained, in other words macroscopically averaged, velocity field. If $\overline{v}$ denotes such macroscopic average, then the Reynolds stress is usually defined as the difference between the average of $v \otimes v$ and $\overline{v} \otimes \overline{v}$. At this formal level the precise definition of averaging plays no role, be it long-time averages, ensemble-averages or local space-time averages. Indeed the latter can be interpreted as taking weak limits: weak limits of Leray solutions of the Navier-Stokes equations with vanishing viscosity have been proposed in the literature as a deterministic approach to turbulence (see [2, 3, 17, 43]).

There is therefore a close analogy with the metric error $v^\sharp e - g$ in Nash's iteration and the Reynolds stress $R$:

- If a sequence of continuous solutions $(v_k, p_k)$ of the Euler equations converge weakly in $L^2$ to some continuous pair $(v, p)$, then the Reynolds stress $R$ is the limit of $v_k \otimes v_k$ minus $v \otimes v$, namely

$$R = \lim_k v_k \otimes v_k - v \otimes v \tag{26}$$

  and we have the identity

$$\left\{\begin{array}{l} \partial_t v + \operatorname{div} v \otimes v + \nabla p = -\operatorname{div} R \\ \\ \operatorname{div} v = 0\,. \end{array}\right. \tag{27}$$

- If a sequence of $C^1$ isometric embeddings $u_k$ of $(\Sigma, g)$ converges uniformly to a $C^1$ map $u$, then the metric error $h$ is the limit of $u_k^\sharp e = g$ minus $u^\sharp e$, namely $h = g - u^\sharp e$.

Note moreover that both $h$ and $R$ are positive semidefinite in the context above: both short maps and the Euler-Reynolds system can be therefore understood as a (convex) relaxation of the corresponding systems of partial differential equations.

Observe that the field $\mathring{R}$ differs from the usual Reynolds stress because of the trace-free condition. However, keeping in mind that in Theorem 1.3 we aim at satisfying in addition (5), a natural analogy of the metric error and the Reynolds stress can be obtained by setting

$$\rho(t) := \frac{1}{3(2\pi)}\left(e(t) - \int_{\mathbb{T}^3} |v(x,t)|^2\,dx\right) \qquad \text{and} \qquad R(x,t) := \rho(t)\mathrm{Id} - \mathring{R}(x,t).$$

Thus, our approximations will consist of smooth solutions $(v, p, R)$ of (27) such that

$$\operatorname{tr} R(t) = \frac{1}{(2\pi)^3}\left(e(t) - \int_{\mathbb{T}^3} |v(x,t)|^2\,dx\right) \tag{28}$$

and we will use $\|R\|_0$ to measure the distance of the pair $(v, p)$ from being a solution of (4)-(5).

Notice that (28) is consistent with (26) if one assumes that

$$e(t) = \int |v_k|^2(x,t)\,dx \qquad \text{for all } k.$$

Furthermore, we will see below that our iteration, analogous to the Nash scheme, will require in addition that $R$ be positive definite, in analogy with short maps.

4.2. **Selection of a good $\mathring{R}$.** Note however an important difference between the Reynolds stress and the metric error: the latter is uniquely determined from the metric $g$ and the short map $u$, whereas the tensor $\mathring{R}$ is not at all uniquely defined from the system (25). However it is possible to select a good "elliptic operator" which solves the equations $\operatorname{div} \mathring{R} = f$ for a trace free symmetric $\mathring{R}$ given a smooth vector field $f$. The relevant technical lemma is the following one.

**Lemma 4.1** (The operator $\mathrm{div}^{-1}$). *There exists a homogeneous Fourier-multiplier operator of order $-1$, denoted*

$$\operatorname{div}^{-1} : C^\infty(\mathbb{T}^3;\mathbb{R}^3) \to C^\infty(\mathbb{T}^3;\mathcal{S}^{3\times 3}_0)$$

*such that, for any $f \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ with average $\fint_{\mathbb{T}^3} f = 0$ we have*

(a) $\operatorname{div}^{-1} f(x)$ *is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$;*
(b) $\operatorname{div}\operatorname{div}^{-1} f = f$.

*Proof.* The proof follows from direct calculation by defining $\operatorname{div}^{-1}$ as

$$\operatorname{div}^{-1} f := \frac{1}{4}\left(D\mathcal{P}g + (D\mathcal{P}g)^T\right) + \frac{3}{4}\left(Dg + (Dg)^T\right) - \frac{1}{2}(\operatorname{div} g)\mathrm{Id},$$

where $g \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ is the solution of $\Delta g = f - \fint_{\mathbb{T}^3} f$ in $\mathbb{T}^3$ and $\mathcal{P}$ is the Leray projector onto divergence-free fields with zero average. □

We recall that the Leray projection operator acts on a general smooth vector field $z$ by returning the divergence-free vector field $\bar z$ with zero average which is closest in $L^2$ to $z$. As it is well known this is just one piece of the Helmholtz decomposition of $z$. Moreover $\bar z = \mathcal{P}(z)$ is achieved as well by solving an elliptic PDE: more precisely if

$$\Delta \alpha = \operatorname{div} z\,,$$

then $\bar z = z - \nabla\alpha - \fint z$. Observe moreover that the operator $\mathcal{Q}(z) = z - \fint z - \mathcal{P}(z)$ is a Fourier-multiplier operator of order $-1$.

4.3. **The iteration stage in Euler.** As in Nash's iteration to prove Theorem 1.2, the proof of Theorem 1.3 is based on an iterative procedure which constructs a sequence of triples $(v_q, p_q, R_q)$ solving (27)-(28) and with $\|R_q\|_0 \to 0$. The exact statement is the following, which we state in a slightly different way from [30, Proposition 2.2] in order to emphasize the similarities to Proposition 2.1 and Proposition 2.2. In particular, we again refer to the cone $\mathcal{C}_r$ from Proposition 2.2.

**Proposition 4.2.** *Let $e$ be as in Theorem 1.3. Then there are positive constants $r < 1$ and $M$ with the following property.*

*Let $\delta, \eta > 0$ be any positive numbers and let $(v, p, R)$ be a smooth solution of the Euler-Reynolds system* (27)-(28) *such that*

$$R(x,t) \in \mathcal{C}_r \quad \text{for all } (x,t). \tag{29}$$

*Then there is a second smooth triple $(v_1, p_1, R_1)$ which solves as well* (27)-(28)*, which satisfies*

$$R_1(x,t) \in \mathcal{C}_r \quad \text{for all } (x,t), \tag{30}$$

*and such that the following estimates hold:*

$$\|v_1 - v\|_{H^{-1}} \le \eta\,, \tag{31}$$

$$\|R_1\|_0 \le \delta\,, \tag{32}$$

*and*

$$\begin{aligned} \|v_1 - v\|_0 &\le M\sqrt{\|R\|_0}\,,\\ \|p_1 - p\|_0 &\le M\|R\|_0\,. \end{aligned} \tag{33}$$

Observe that the tensor $R$ is not only required to be positive definite, but its values are restricted to a small cone around positive multiples of the identity matrix. Hence Proposition 4.2 is really the analog of Proposition 2.2, namely the "local version" of Proposition 2.1. In particular (29) can be thought as a "pinching condition" for the Reynolds stress, in analogy with the "pinching condition" (10).

Concluding Theorem 1.3 from Proposition 4.2 is now a trivial task: one chooses $\delta = \frac{1}{2}$ and applies first the proposition to

- the triple $(v, p, R) = (v_0, p_0, R_0)$ with $(v_0, p_0) = (0, 0)$ and $R_0 = e(t)\mathrm{Id}$,
- $\eta = \frac{\varepsilon}{4}$ and $\delta = \frac{1}{4}$.

We thus produce a second triple $(v_1, p_1, R_1)$.

The proposition is then applied iteratively to

- the triple $(v, p, R) = (v_q, p_q, R_q)$,
- $\eta = \varepsilon 2^{-q-2}$ and $\delta = 4^{-q-1}$,

to produce the next triple $(v_{q+1}, p_{q+1}, R_{q+1})$.

It is therefore obvious that

(a) $\|R_q\|_0$ converges exponentially fast to zero;

(b) The kinetic energy

$$\frac{1}{2}\int |v_q|^2(x,t)\,dx$$

converges exponentially fast to $e(t)$;

(c) $(v_q, p_q)$ converge exponentially fast to a continuous pair $(v, p)$ (in the $\|\cdot\|_0$ norm).

We conclude that $(v, p)$ is a continuous pair which solves the incompressible Euler equations and that the corresponding kinetic energy of $v$ is $e(t)$ at any time $t$.

## 5. The oscillatory ansatz

In analogy with Nash's approach to Proposition 2.1 our strategy for the proof of Proposition 4.2 is to perturb $v$ to $v_1$ by adding a highly oscillatory vector field, which we will call $w_o$. In analogy with the Nash spirals in (19), we make the following *ansatz* on $w_o$:

$$w_o(x,t) = W\Big(v(x,t), \tilde{R}(x,t), \lambda x, \lambda t\Big), \tag{34}$$

where

$$\tilde{R}(x,t) = (1-\gamma)\rho(t)\mathrm{Id} - \mathring{R}(x,t) \tag{35}$$

and $\gamma < \frac{1}{2}$ is a small positive parameter which will be determined later.

A lot of work will be put in determining the correct function $W(v, R, \xi, \tau)$, which turns out to be much more complicated then the corresponding map for Nash's scheme. Since $v_1$ must be $2\pi$-periodic we will impose that $W$ is $2\pi$-periodic in the variable $\xi$.

Notice next that $v_1$ must satisfy the divergence-free condition $\operatorname{div} v_1 = 0$ and $v + w_o$ is not likely to fulfill it. Indeed a stronger analogy with the isometric embedding problem would be to consider first a vector potential for $v$, namely to write $v$ as $\nabla \times z$ for some smooth $z$. Subsequently we would like to perturb $z$ to a new

$$z_1(x,t) = z(x,t) + \frac{1}{\lambda} Z(v(x,t), R(x,t), \lambda x, \lambda t)\,.$$

The resulting map $v_1$ would then be given by

$$v_1(x,t) = v(x,t) + \underbrace{(\nabla_\xi \times Z)(v(x,t), \tilde{R}(x,t), \lambda x, \lambda t)}_{(P)} + O\left(\frac{1}{\lambda}\right)\,,$$

and thus our perturbation $w_o$ in (34) corresponds actually to the term (P). It is therefore natural to expect that one needs to add a further corrector term $w_c(x,t)$ in order to ensure that

$$v_1(x,t) = v(x,t) + \underbrace{w_o(x,t) + w_c(x,t)}_{=:w(x,t)} \tag{36}$$

is divergence free. We can naturally expect that this perturbation will be negligible, namely of order $O(\lambda^{-1})$, for $\lambda$ large.

This is indeed the case and a canonical way of defining $w_c$ is through the Leray projector introduced above, namely

$$w_c = \mathcal{P}\left(w_o - \fint w_o\right) - w_o\,. \tag{37}$$

A rule of thumb here is the following:

(RT) If we apply an operator of order $-1$ to a function $f$ which oscillates at frequency $\lambda$ and has a certain "size" $S$, we then expect an outcome of order $S\lambda^{-1}$.

We will see later that indeed $S$ can be taken to be $\|f\|_0$, at least under appropriate assumptions: this fact will play a crucial role, but for the moment we can ignore it. We also warn the reader that (RT) gives, in many ways, only a very simplistic point of view on a much more complicated picture. For instance, due to the failure of elliptic estimates at the endpoints of the Schauder scale, we can only claim an estimate of type $S\lambda^{-1+\varepsilon}$ for arbitrarily small $\varepsilon > 0$.

5.1. **First condition on the fluctuation profile $W$: stationary phase argument.** We now wish to determine the conditions upon the function $W$ in order to achieve a new triple $(v_1, p_1, R_1)$ which satisfies the conclusions of Proposition 4.2. Recalling (35), we observe that

$$R(x,t) - \tilde{R}(x,t) = \gamma\rho(t)\mathrm{Id}$$

is a function of $t$ only.

Thus, the triple $(v, p, \tilde{R})$ is also a solution of (27) and we can compute:

$$\begin{aligned}
\mathring{R}_1 &= \operatorname{div}^{-1}\Big[\partial_t v_1 + \operatorname{div}(v_1 \otimes v_1) + \nabla p_1\Big] \\
&= \operatorname{div}^{-1}\Big[\partial_t w + v\cdot\nabla w\Big] \\
&\quad + \operatorname{div}^{-1}\Big[\operatorname{div}(w\otimes w - \tilde{R}) + \nabla(p_1 - p)\Big] \\
&\quad + \operatorname{div}^{-1}\Big[w\cdot\nabla v\Big] \\
&= \operatorname{div}^{-1}\Big[\partial_t w_o + v\cdot\nabla w_o\Big] \\
&\quad + \operatorname{div}^{-1}\Big[\operatorname{div}(w_o\otimes w_o - \tilde{R}) + \nabla(p_1 - p)\Big] \\
&\quad + \operatorname{div}^{-1}\Big[w_o\cdot\nabla v\Big] + \mathring{R}_1^{(4)} \qquad (38)
\end{aligned}$$

where $\operatorname{div}^{-1}$ is the operator of order $-1$ from Lemma 4.1 and in $\mathring{R}_1^{(4)}$ we have included all terms involving $w_c$. We expect that $\mathring{R}_1^{(4)}$ is thus negligible, given that $\|w_c\|_0$ should be of size $\lambda^{-1}$. Note also that, since $\operatorname{div}^{-1}$ is an operator of order $-1$, by (RT) we can expect that the term

$$\mathring{R}_1^{(3)} := \operatorname{div}^{-1}\Big[w_o\cdot\nabla v\Big] \qquad (39)$$

has also size $O(\lambda^{-1})$. In fact the term $\mathring{R}_1^{(3)}$ is the analog of the term $L$ in (21) in the isometric embedding problem.

A way to get an intuition for this is by expanding $W(v, \tilde{R}, \xi, \tau)$ as a Fourier series in $\xi$. We then could compute

$$\mathring{R}_1^{(3)} = \operatorname{div}^{-1}\Big[w_o\cdot\nabla v\Big] = \operatorname{div}^{-1}\sum_{k\in\mathbb{Z}^3} c_k(x,t)e^{i\lambda k\cdot x}\,, \qquad (40)$$

where the coefficients $c_k(x,t)$ vary much slower than the rapidly oscillating exponentials. When we apply the operator $\operatorname{div}^{-1}$ we can therefore treat the $c_k$ as constants and gain a factor $\frac{1}{\lambda}$ in the outcome : a typically "stationary phase argument". Note however that this stationary phase argument cannot be applied to the $c_0$ term, there is no reason for $\operatorname{div}^{-1}c_0$ to be of order $\lambda^{-1}$.

Thus, to gain the factor $\frac{1}{\lambda}$ we impose that the trivial mode $c_0$ in the Fourier expansion (40) vanishes. Indicating with $\langle\cdot,\cdot\rangle$ the average in the $2\pi$-periodic $\xi$ variable, this condition is equivalent to

$$\langle W\rangle := \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} W(v, \tilde{R}, \xi, \tau)\,d\xi = 0; \qquad (\text{H1})$$

It also seems quite clear that the error term

$$\mathring{R}_1^{(2)} := \operatorname{div}^{-1}\Big[\operatorname{div}(w_o\otimes w_o - \tilde{R}) + \nabla(p_1 - p)\Big] \qquad (41)$$

corresponds to the quadratic term $Q$ (minus the metric change $\tilde{h}$!) in (21). Therefore we hope to make this term small, namely to exploit the quadratic expression $w_o \otimes w_o$ to cancel the error $R$.

5.2. **Second condition on the fluctuation profile: fast variables.** Observe however that there is a third error term, which is in fact not present in the isometric embedding problem:

$$\mathring{R}_1^{(1)} := \operatorname{div}^{-1}\Big[\partial_t w_o + v \cdot \nabla w_o\Big] . \tag{42}$$

The latter "linear transport" term cannot be handled as $\mathring{R}_1^{(3)}$: since the derivatives inside the square bracket "fall onto" the perturbation $w_o$, we should expect a factor $\lambda$. Even though we can expect to balance such factor $\lambda$ with the factor $\lambda^{-1}$ gained after applying the operator $\operatorname{div}^{-1}$, we still do not have have any reason to expect that $\mathring{R}_1^{(1)}$ will be small.

In order to gain some more insight, we plug in our *ansatz* for $w_o$ and compute explicitly the sum of the remaining error terms $\mathring{R}_1^{(1)} + \mathring{R}_2^{(1)}$:

$$\underbrace{\operatorname{div}^{-1}\Big[\partial_t w_o + v \cdot \nabla w_o\Big] + \operatorname{div}^{-1}\Big[\operatorname{div}(w_o \otimes w_o - \tilde{R}) + \nabla(p_1 - p)\Big]}_{=:E}$$

Assume also that $p_1 - p$ is chosen of the form $P(v(x,t), \tilde{R}(x,t), \lambda x, \lambda t)$. In differentiating the composite functions $w_o$ or $p_1 - p$, there will be terms with a prefactor $\lambda$ (the ones where the outer derivative falls on $\xi$ or $\tau$), and other terms without the factor $\lambda$. We will denote these latter derivatives in the "slow variables" $(v, R)$ as $\partial_t^{slow}$ and $\nabla^{slow}$. Thus:

$$E = \underbrace{\lambda[\partial_\tau W + v \cdot \nabla_\xi W + \operatorname{div}_\xi (W \otimes W) + \nabla_\xi P](v, \mathring{R}, \lambda x, \lambda t)]}_{=:(F)} \tag{43}$$

$$+ \underbrace{[(\partial_t + v \cdot \nabla)^{slow} W]}_{=:(S)} \tag{44}$$

$$+ \underbrace{\operatorname{div}^{slow}[W \otimes W - \tilde{R}] + \nabla^{slow} P}_{=:(Q)} \tag{45}$$

Clearly we would like the "fast term" $(F)$ to disappear and this could be achieved by imposing the condition

$$\begin{cases} \partial_\tau W + v \cdot \nabla_\xi W + \operatorname{div}_\xi (W \otimes W) + \nabla_\xi P = 0 \\ \operatorname{div}_\xi W = 0 . \end{cases} \tag{H3}$$

Furtheremore we could hope to treat the slow term (S) as we have done with $\mathring{R}_1^{(3)}$: assuming that this picture is correct, we only need to understand the quadratic term (Q).

Again, we can use the same idea which helped us estimating $\mathring{R}_1^{(3)}$: we can expand $W \otimes W(v, R, \xi, \tau)$ in Fourier series of $\xi$. In order to gain a factor

$\lambda^{-1}$ from the operator $\operatorname{div}^{-1}$ we need the following condition, analogous to (H1):

$$\langle W \otimes W \rangle = \tilde{R} \tag{H2}$$

Summarizing the preceding discussion, if we could find a fluctuation profile satisfying the conditions (H1), (H2) and (H3), we would be at a good point in our proof of Proposition 4.2.

## 6. Beltrami flows

In order to construct a "fluctuation profile" $W$ satisfying (H1)-(H3), our building blocks (which in a sense play the same role as Nash's spirals in the proof of Proposition 2.1) belong to a special class of stationary periodic solutions to the incompressible Euler equations, called Beltrami flows.

The starting point is the identity

$$\operatorname{div}(U \otimes U) = U \times \operatorname{curl} U - \tfrac{1}{2}\nabla |U|^2 ,$$

for smooth 3-dimensional vector fields $U$. In particular any eigenspace of the curl operator, i.e. the solution space of the system

$$\begin{cases} \operatorname{curl} U &= \lambda_0 U \\ \operatorname{div} U &= 0 \end{cases}$$

for any $\lambda_0$ constant, leads to a *linear* space of stationary flows of the incompressible Euler equations. These can be written as

$$\sum_{|k|=\lambda_0} a_k B_k e^{ik\cdot\xi} \tag{46}$$

for normalized complex vectors $B_k \in \mathbb{C}^3$ satisfying

$$|B_k| = 1, \quad k \cdot B_k = 0 \quad \text{and} \quad ik \times B_k = \lambda_0 B_k,$$

and arbitrary coefficients $a_k \in \mathbb{C}$. Choosing $B_{-k} = -\overline{B_k}$ and $a_{-k} = \overline{a_k}$ ensures that $U$ is real-valued. A calculation then shows

$$\langle U \otimes U \rangle = \frac{1}{2} \sum_{|k|=\lambda_0} |a_k|^2 \Big( \mathrm{Id} - \frac{k \otimes k}{|k|^2} \Big). \tag{47}$$

Moreover, recalling the condition that $W$ must be $2\pi$-periodic in the $\xi$ variable, we impose that

$$k \in \mathbb{Z}^3 . \tag{48}$$

6.1. **Decomposition of the Reynolds stress.** The identity (47) leads to the following decomposition Lemma which is the analogue of Lemma 2.4.

**Lemma 6.1.** *For every $N \in \mathbb{N}$ we can choose $0 < r_0 < 1$ and $\bar{\lambda} > 1$ with the following property. There exist pairwise disjoint subsets*

$$\Lambda_j \subset \{k \in \mathbb{Z}^3 : |k| = \bar{\lambda}\} \qquad j \in \{1, \dots, N\}$$

*and smooth positive functions*

$$\gamma_k^{(j)} \in C^\infty\left(B_{r_0}(\mathrm{Id})\right) \qquad j \in \{1,\dots,N\}, k \in \Lambda_j$$

*such that*

(a) $k \in \Lambda_j$ *implies* $-k \in \Lambda_j$ *and* $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$*;*
(b) *For each* $R \in B_{r_0}(\mathrm{Id})$ *we have the identity*

$$R = \frac{1}{2}\sum_{k\in\Lambda_j}\left(\gamma_k^{(j)}(R)\right)^2\left(\mathrm{Id} - \frac{k}{|k|}\otimes\frac{k}{|k|}\right) \qquad \forall R \in B_{r_0}(\mathrm{Id})\,. \tag{49}$$

Note that the decomposition in (49) is valid only in the $r_0$-neighborhood of the identity matrix. Hence the latter statement is more similar to Lemma 2.4, namely the "single coordinate patch version" of Lemma 2.3. Note also that Lemma 6.1 provides several "independent" decompositions of $R$, corresponding to the $N$ families $(\gamma_k^{(j)})_k$, $j = 1\dots N$. We will return to the significance of this in the next section.

This lemma, taken from [30] (see also [38] for a geometric proof) allows us to choose the amplitudes as

$$a_k = \sqrt{\rho}\,\gamma_k^{(j)}\left(\frac{\tilde{R}}{\rho}\right), \tag{50}$$

where we recall that $\tilde{R} = (1-\gamma)\rho\mathrm{Id} - \mathring{R}$ and $\rho = \frac{1}{3}\mathrm{tr}\,R$. Observe that the restriction for the argument of $\gamma_k$ to lie in $B_{r_0}(\mathrm{Id})$ then translates into the condition

$$|\mathring{R}| \leq (1-\gamma)r_0\rho.$$

In light of the assumption (29) this requirement is satisfied by choosing $r = \frac{r_0}{2}$ (since $\gamma \leq \frac{1}{2}$).

With this choice of $a_k = a_k(\tilde{R})$,

$$W_s(\tilde{R},\xi) := \sum_j \sum_{k\in\Lambda^{(j)}} a_k(\tilde{R})B_k e^{ik\cdot\xi}$$

(defined through the Beltrami-flow relation (46)) satisfies (H1) and (H2).

6.2. **Galilean transformations.** As for (H3), notice that $W_s$ would satisfy it only for $v = 0$. For $v \neq 0$ a quick fix would be to use the Galilean invariance of the Euler equations. Indeed, since (H3) is an equation in the "fast" variables $(\xi,\tau)$ in which $v$ is constant,

$$W(v,\tilde{R},\xi,\tau) := W(\tilde{R},\xi - v\tau) \tag{51}$$

does yield a solution of (H3). Unfortunately this is not a valid solution, but to see why, we need to return once more to the "rule of thumb" (RT) and look more carefully how the $O(\lambda^{-1})$ estimates for $\mathring{R}_1$ are obtained in the next section.

Returning to (40) we see that the main point is to obtain a bound for

$$\mathrm{div}^{-1}\left[c_k(x,t)e^{i\lambda k\cdot x}\right].$$

We have already discussed this point: if $c_k$ were constant, we would immediately obtain the bound $c_k\lambda^{-1}$ (assuming $k\neq 0$ of course). Then, for smooth $c_k$ and large $\lambda$ one may expect that, if at the scale $\lambda^{-1}$ $c_k$ does not vary too much, then we get an estimate of type

$$\left\|\mathrm{div}^{-1}\left[c(x,t)e^{i\lambda k\cdot x}\right]\right\|_0 \le C\frac{\|c_k\|_0}{\lambda}. \tag{52}$$

One way to prove (52) is to use the Schauder theory in an apprioriate way. Unfortunately the latter does not work in $C^0$ for the elliptic operators which define $\mathrm{div}^{-1}$ and we can only reach an the estimate $C_\alpha\|c_k\|_0\lambda^{-1+\alpha}$ for any $\alpha\in(0,1)$, with a constant $C_\alpha$ which blows up when $\alpha\downarrow 0$, (see [30]). Since this small loss in the exponent has little effect on the overall proof, we are going to ignore it in the sequel and assume the validity of (52).

Now, in the case of (40) the coefficients $c_k$ can be bounded by

$$\|c_k\|_0\le\|a_k\|_0\|Dv\|_0,$$

whereas, using the identity (49) we obtain

$$\|a_k\|_0\le C\|R\|_0^{1/2}.$$

Observe the strong similarity with (24) in case of the Nash spirals! Since (46) consists of a fixed finite sum, we conclude

$$\|\mathring{R}_1^{(3)}\|_0\le C\frac{\|R\|_0^{1/2}\|Dv\|_0}{\lambda}.$$

Observe that the same estimate on $a_k$ also yields the square-root estimate (33) – in complete correspondence to (9).

Next, let us again look at $\mathring{R}_1^{(1)}$ and in particular at the "slow derivative term" (S) in (44). We calculate

$$(\partial_t+v\cdot\nabla)^{slow}W=D_vW\,(\partial_t+v\cdot\nabla)v+D_RW\,(\partial_t+v\cdot\nabla)R.$$

The above expression is *linear in* $W$, hence, using (H1) we may (as in (40)) write it as

$$(\partial_t+v\cdot\nabla)^{slow}W=\sum_{k\in\mathbb{Z}^3,k\neq 0}\tilde{c}_k(x,t)e^{i\lambda k\cdot x},$$

for some coefficients $\tilde{c}_k$. Recall that the Beltrami amplitudes $a_k$ are 1/2-homogeneous in $R$, so that $\|D_RW\,(\partial_t+v\cdot\nabla)R\|_0$ can be bounded in terms of $\|R\|_1$. On the other hand, using the Galilean transformation (51) we obtain

$$|D_vW|\sim\tau=\lambda t.$$

This means that, although using (52) we can remove the factor $\lambda$, again we do not arrive at the required smallness. We therefore see that using the Galilean transformation will not work to obtain sufficiently good bounds.

Indeed, there are at least two other problems:

- The Galilean transformation is incompatible with the $2\pi$-periodicity of $W$ in the variable $\xi$, except for very particular values of $v$ (in fact, only when $v \in \mathbb{Z}^3$);
- in our iteration $v$ is not constant, only "slowly varying", compared to the scale $\lambda^{-1}$.

However, the considerations of this section at least suggest which bounds the fluctuation $W$ should satisfy. These should be independent of $\tau$, hence considerations on the homogeneity with respect to $R$ lead to

$$|W| \leq C|R|^{1/2},\ |D_v W| \leq C|R|^{1/2},\ |D_R W| \leq C|R|^{-1/2}\,, \tag{H4}$$

where $C$ is just a geometric constant. Indeed, if we are able to find a fluctuation profile $W = W(v, R, \xi, \tau)$ which satisfies (H1)-(H4), then the oscillatory estimate (52) and arguments as presented in this section lead to

$$\|\mathring{R}_1\|_0 \leq \frac{C}{\lambda},$$

as well as

$$\|v_1 - v\|_0 \leq M\|R\|_0^{1/2}, \quad \|p_1 - p\|_0 \leq M\|R\|_0,$$

where $C$ depends on $v$ and $R$ and $M$ is a geometric constant.

More careful estimates along the same lines would even lead to an iteration which yields a $C^\theta$ Hölder-continuous vector field for any $\theta < 1/3$, namely to a solution of the Onsager's conjecture. For a more detailed exposition we refer to [60] and to the lecture notes [61].

Unfortunately, we are not able to fulfill all criteria (H1)-(H4) as stated, and it will be necessary to introduce additional error terms. This is the main point of departure from the Nash's scheme of Proposition 2.1. The troublesome "fast transport term" corresponds to the "fast derivative term" in the linear part $L$ of (21). Nash can set this term to zero with a suitable choice of the perturbation, namely because the vector fields $\nu$ and $b$ in (19) are perpendicular to the image of the starting map $z$. In Euler this seems impossible.

## 7. A two scale correction to the oscillatory ansatz

As shown in the previous sections, the Beltrami flow (46) with amplitudes given by (50) work well to satisfy (H1)-(H2), but we yet have to deal with the (fast) transport part of (H3). We demonstrated above that a simple Galilean transformation will not work, but we can still use the Galilean invariance of the Euler equations in subtler ways.

### 7.1. **The transport term and the second scale.**

Let us consider the following modification of (46):

$$\sum_{|k|=\lambda_0} a_k(\tilde{R})\phi_k(v,\tau)B_k e^{ik\cdot\xi}\,, \tag{53}$$

where $\phi_k(v,\tau)$ is a "phase function", i.e. a complex valued function with $|\phi_k(v,\tau)| = 1$. Plugging into (H3) leads then to the condition

$$\partial_\tau \phi_k + i(v\cdot k)\phi_k = 0 \tag{54}$$

with exact solution

$$\phi_k(v,\tau) = e^{-i(v\cdot k)\tau}. \tag{55}$$

As we explained above, this choice of $\phi_k$, which corresponds to the Galilean transformation, is incompatible with (H4) because $|\partial_v \phi_k| \sim |\tau|$ is unbounded.

Thus, rather than solving (54) exactly we introduce a suitable "error", namely we aim for

$$\partial_\tau \phi_k + i(v\cdot k)\phi_k = O\left(\mu^{-1}\right), \qquad |\partial_v \phi_k| \leq C\mu\,, \tag{56}$$

where $\mu$ is a second large parameter. There are indeed several ways to solve (56). In [30] we use a suitable partition of unity over the space of velocities (with 8 families). However this approach turns out to have some drawbacks when constructing Hölder continuous solutions: more efficient methods have been found later in [38] and [12] (cf. Remark 7.1).

The introduction of this second scale leads to the following corrections to (H3) and (H4). (H3) is only satisfied approximately:

$$\partial_\tau W + v\cdot\nabla_\xi W = O(\mu^{-1}), \qquad \mathrm{div}_\xi (W\otimes W) + \nabla_\xi P = 0\,. \tag{57}$$

In (H4) the second inequality is replaced by

$$|\partial_v W| \leq C\mu |R|^{1/2}. \tag{58}$$

These changes affect $\mathring{R}_1^{(1)}$, in particular the "fast derivative term" (F) in (43) and the "slow derivative term" (S) in (44). Now (F) is not identically zero, but we can still use the oscillatory estimate (52) on $\partial_\tau W + v\cdot\nabla_\xi W$ so that we obtain

$$\|\mathrm{div}^{-1}\big((\partial_\tau + v\cdot\nabla_\xi)W\big)\|_0 \leq C\|R\|_0^{1/2}\frac{1}{\mu}\,, \tag{59}$$

whereas for (S) we obtain

$$\|\mathrm{div}^{-1}\big((\partial_t + v\cdot\nabla)^{slow} W\big)\|_0 \leq C\|R\|_0^{1/2}\frac{\mu}{\lambda}\,. \tag{60}$$

The optimal relation between the two parameters is then $\mu = \lambda^{1/2}$ (in fact we will only be able to ensure this up to constants, because we require that $\lambda, \mu, \frac{\lambda}{\mu} \in \mathbb{N}$, cf. [30]). Hence the final estimate is

$$\|\mathring{R}_1^{(1)}\|_0 \leq C\frac{\|R\|_0^{1/2}}{\lambda^{1/2}}.$$

7.2. **The energy profile and conclusions.** Overall, with the modified *ansatz* (53) we can ensure that (33) holds and

$$\|\mathring{R}_1\|_0 \leq C\frac{1}{\lambda^{1/2}}$$

with some constant depending on $v$ and $R$. In addition, using again the oscillatory estimate (52), it is not difficult to show that

$$\|v - v_1\|_{H^{-1}} \leq C\frac{\|R\|_0^{1/2}}{\lambda}.$$

Hence, in order to complete the proof of Proposition 4.2 we only to estimate how far is the "kinetic energy" of $v_1$ from the targeted profile $e$ and we need to ensure (30).

Recall first that

$$R_1(x,t) = \rho_1(t)\mathrm{Id} - \mathring{R}_1(x,t) \tag{61}$$

and

$$\rho_1(t) = \frac{1}{3(2\pi)^3}\left(e(t) - \int_{\mathbb{T}^3}|v_1(x,t)|^2\,dx\right). \tag{62}$$

Thus it remains to show that $\rho_1$ is small and that $\mathring{R}_1$ is comparatively smaller (because we need to show (30) as well.

Recalling (H2) we can write

$$w_o \otimes w_o - \tilde{R} = \sum_{k\in\mathbb{Z}^3, k\neq 0} A_k(x,t)e^{i\lambda x\cdot k},$$

where the $A_k$'s are some matrix-valued smooth coefficients determined by $v$ and $R$. In particular, after taking the trace we obtain

$$|w_o|^2 - 3(1-\gamma)\rho(t) = \sum_{k\in\mathbb{Z}^3, k\neq 0} \operatorname{tr} A_k(x,t)e^{i\lambda x\cdot k}.$$

By first integrating over $x \in \mathbb{T}^3$ and then integrating by parts we deduce

$$\left|\fint_{\mathbb{T}^3}|w_o(x,t)|^2\,dx - 3(1-\gamma)\rho(t)\right| \leq \frac{C}{\lambda}.$$

Recalling that $v_1 = v + w_o + w_c$ where $w_c = O(\lambda^{-1})$, we deduce

$$\int_{\mathbb{T}^3}|v_1(x,t)|^2\,dx = \int_{\mathbb{T}^3}|v(x,t)|^2\,dx + 3(2\pi)^3(1-\gamma)\rho(t) + O\left(\frac{1}{\lambda}\right),$$

so that

$$e(t) - \int_{\mathbb{T}^3}|v_1(x,t)|^2\,dx = \gamma\left(e(t) - \int_{\mathbb{T}^3}|v(x,t)|^2\,dx\right) + O\left(\frac{1}{\lambda}\right).$$

Now, since $v$ and $e$ are continuous on the compact interval $\mathbb{T}^3\times[0,1]$ and

$$e(t) > \int_{\mathbb{T}^3}|v(x,t)|^2\,dx \qquad \text{for all } t\in[0,1],$$

there exists $\varepsilon > 0$ so that

$$e(t) - \int_{\mathbb{T}^3} |v(x,t)|^2\,dx \geq \varepsilon \qquad \text{for all } t \in [0,1].$$

Fix $\gamma$ so that $4\gamma \leq \delta$. Then, choosing $\lambda$ sufficiently large we can ensure that (33) and (31) hold, and furthermore $\|\mathring{R}_1\|_0 \leq \frac{r}{2}\gamma\varepsilon$ and

$$\frac{1}{2}\gamma\varepsilon \leq \frac{1}{3(2\pi)^3}\left(e(t) - \int_{\mathbb{T}^3} |v_1(x,t)|^2\,dx\right) \leq 2\gamma.$$

Then

$$\|\mathring{R}_1\|_0 \leq \frac{r}{2}\gamma\varepsilon \leq \frac{r}{3(2\pi)^3}\left(e(t) - \int_{\mathbb{T}^3} |v_1(x,t)|^2\right) = \frac{r}{3}\operatorname{tr} R_1,$$

so that the tensor $R_1$ of (61) satisfies (30) and (32). This concludes the proof of Proposition 4.2.

**Remark 7.1.** *The above is essentially following the ideas from* [30] *and* [31]*. A better modification of the original ansatz was introduced subsequently by P. Isett in* [38]*: there the approximate Galilean phase function* $\phi_k(v,\tau)$ *was replaced by a nonlinear phase function, which is defined using the inverse flow generated by the vector field* $v$*. This correction is key to obtaining improved Hölder estimates, and lead to further progress in Onsager's conjecture (see also* [12]*).*

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