REGULARITY OF AREA MINIMIZING CURRENTS III: BLOW-UP

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Abstract. This is the last of a series of three papers in which we give a new, shorter proof of a slightly improved version of Almgren's partial regularity of area minimizing currents in Riemannian manifolds. Here we perform a blow-up analysis deducing the regularity of area minimizing currents from that of Dir-minimizing multiple valued functions.

0. Introduction

In this paper we complete the proof of a slightly improved version of the celebrated Almgren's partial regularity result for area minimizing currents in a Riemannian manifold (see [1]), namely Theorem 0.3 below.

Assumption 0.1. Let $\varepsilon_0 \in]0,1[$, $m, \bar{n} \in \mathbb{N} \setminus \{0\}$ and $l \in \mathbb{N}$. We denote by

(M) $\Sigma \subset \mathbb{R}^{m+n} = \mathbb{R}^{m+\bar{n}+l}$ an embedded $(m+\bar{n})$-dimensional submanifold of class C^{3,ε_0};

(C) T an integral current of dimension m with compact support $\mathrm{spt}(T) \subset \Sigma$, area minimizing in Σ.

In this paper we follow the notation of [6] concerning balls, cylinders and disks. In particular $\mathbf{B}_r(x) \subset \mathbb{R}^{m+n}$ will denote the Euclidean ball of radius r and center x.

Definition 0.2. For T and Σ as in Assumption 0.1 we define

$$\mathrm{Reg}(T) := \big\{x \in \mathrm{spt}(T) : \ \mathrm{spt}(T) \cap \mathbf{B}_r(x) \text{ is a } C^{3,\varepsilon_0} \text{ submanifold for some } r > 0\big\}, \quad (0.1)$$

$$\mathrm{Sing}(T) := \mathrm{spt}(T) \setminus \big(\mathrm{spt}(\partial T) \cup \mathrm{Reg}(T)\big). \quad (0.2)$$

The partial regularity result proven first by Almgren [1] under the more restrictive hypothesis $\Sigma \in C^5$ gives an estimate on the Hausdorff dimension $\dim_H(\mathrm{Sing}(T))$ of $\mathrm{Sing}(T)$.

Theorem 0.3. *$\dim_H(\mathrm{Sing}(T)) \leq m-2$ for any $m, \bar{n}, l, T$ and Σ as in Assumption 0.1.*

In this note we complete the proof of Theorem 0.3, based on our previous works [3, 5, 4, 6], thus providing a new, and much shorter, account of one of the most fundamental regularity result in geometric measure theory; we refer to [4] for an extended general introduction to all these works. The proof is carried by contradiction: in the sequel we will always assume the following.

Assumption 0.4 (Contradiction)**.** There exist $m \geq 2, \bar{n}, l$, Σ and T as in Assumption 0.1 such that $\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}(T)) > 0$ for some $\alpha > 0$.

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The hypothesis $m \geq 2$ in Assumption 0.4 is justified by the well-known fact that $\mathrm{Sing}(T) = \emptyset$ when $m = 1$ (in this case $\mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ is locally the union of finitely many non-intersecting geodesic segments). Starting from Assumption 0.4, we make a careful blow-up analysis, split in the following steps.

0.1. **Flat tangent planes.** We first reduce to flat blow-ups around a given point, which in the sequel is assumed to be the origin. These blow-ups will also be chosen so that the size of the singular set satisfies a uniform estimate from below (cf. Section 1).

0.2. **Intervals of flattening.** For appropriate rescalings of the current around the origin we take advantage of the center manifold constructed in [6], which gives a good approximation of the average of the sheets of the current at some given scale. However, since it might fail to do so at different scales, in Section 2 we introduce a *stopping condition* for the center manifolds and define appropriate *intervals of flattening* $I_j = [s_j, t_j]$. For each j we construct a different center manifold $\mathcal{M}_j$ and approximate the (rescaled) current with a suitable multi-valued map on the normal bundle of $\mathcal{M}_j$.

0.3. **Finite order of contact.** A major difficulty in the analysis is to prove that the minimizing current has finite order of contact with the center manifold. To this aim, in analogy with the case of harmonic multiple valued functions (cf. [3, Section 3.4]), we introduce a variant of the *frequency function* and prove its almost monotonicity and boundedness. This analysis, carried in Sections 3, 4 and 5, relies on the variational formulas for images of multiple valued maps as computed in [5] and on the careful estimates of [6]. Our frequency function differs from that of Almgren and allows for simpler estimates.

0.4. **Convergence to Dir-minimizer and contradiction.** Based on the previous steps, we can blow-up the Lipschitz approximations from the center manifold $\mathcal{M}_j$ in order to get a limiting Dir-minimizing function on a flat m-dimensional domain. We then show that the singularities of the rescaled currents converge to singularities of that limiting Dir-minimizer, contradicting the partial regularity of [3, Section 3.6] and, hence, proving Theorem 0.3.

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1. Flat tangent cones

Definition 1.1 (Q-points)**.** For $Q \in \mathbb{N}$, we denote by $\mathrm{D}_Q(T)$ the points of density Q of the current T, and set

$$\mathrm{Reg}_Q(T) := \mathrm{Reg}(T) \cap \mathrm{D}_Q(T) \quad \text{and} \quad \mathrm{Sing}_Q(T) := \mathrm{Sing}(T) \cap \mathrm{D}_Q(T).$$

Definition 1.2 (Tangent cones)**.** For any $r > 0$ and $x \in \mathbb{R}^{m+n}$, $\iota_{x,r} : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ is the map $y \mapsto \frac{y-x}{r}$ and $T_{x,r} := (\iota_{x,r})_\sharp T$. The classical monotonicity formula (see [10] and [4, Lemma A.1]) implies that, for every $r_k \downarrow 0$ and $x \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$, there is a subsequence (not relabeled) for which T_{x,r_k} converges to an integral cycle S which is a cone (i.e., $S_{0,r} = S$ for all $r > 0$ and $\partial S = 0$) and is (locally) area-minimizing in $\mathbb{R}^{m+n}$. Such a cone will be called, as usual, *a tangent cone to T at x*.

Fix $\alpha > 0$. By Almgren's stratification theorem (see [10, Theorem 35.3]), for $\mathcal{H}^{m-2+\alpha}$-a.e. $x \in \mathrm{spt}(T)\setminus\mathrm{spt}(\partial T)$, there exists a subsequence of radii $r_k \downarrow 0$ such that T_{x,r_k} converge to an integer multiplicity flat plane. Similarly, for measure-theoretic reasons, if T is as in Assumption 0.4, then for $\mathcal{H}^{m-2+\alpha}$-a.e. $x \in \mathrm{spt}(T)\setminus\mathrm{spt}(\partial T)$ there is a subsequence $s_k \downarrow 0$ such that $\liminf_k \mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(T_{x,s_k})\cap\mathbf{B}_1) > 0$ (see again [10]). Obviously there would then be $Q \in \mathbb{N}$ and $x \in \mathrm{Sing}_Q(T)$ where both subsequences exist. The two subsequences might, however, differ: in the next proposition we show the existence of one point and a single subsequence along which *both* conclusions hold. For the relevant notation (concerning, for instance, excess and height of currents) we refer to [4, 6].

Proposition 1.3 (Contradiction sequence)**.** *Under Assumption 0.4, there are $m, n, Q \geq 2$, Σ and T as in Assumption 0.1, reals $\alpha, \eta > 0$, and a sequence $r_k \downarrow 0$ such that $0 \in \mathrm{D}_Q(T)$ and the following holds:*

$$\lim_{k\to+\infty} \mathbf{E}(T_{0,r_k}, \mathbf{B}_{6\sqrt{m}}) = 0, \tag{1.1}$$

$$\lim_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(T_{0,r_k})\cap\mathbf{B}_1) > \eta, \tag{1.2}$$

$$\mathcal{H}^m\big((\mathbf{B}_1\cap\mathrm{spt}(T_{0,r_k}))\setminus\mathrm{D}_Q(T_{0,r_k})\big) > 0 \quad \forall\, k\in\mathbb{N}. \tag{1.3}$$

The proof is based on the following lemma.

Lemma 1.4. *Let S be an m-dimensional area minimizing integral cone in $\mathbb{R}^{m+n}$ such that $\partial S = 0$, $Q = \Theta(S,0) \in \mathbb{N}$, $\mathcal{H}^m\big(D_Q(S)\big) > 0$ and $\mathcal{H}^{m-1}(\mathrm{Sing}_Q(S)) = 0$. Then, S is an m-dimensional plane with multiplicity Q.*

Proof. For each $x \in \mathrm{Reg}_Q(S)$, let r_x be such that $S\llcorner\mathbf{B}_{2r_x}(x) = Q\,[\![\Gamma]\!]$ for some regular submanifold Γ and set

$$U := \bigcup_{x\in\mathrm{Reg}_Q(S)} \mathbf{B}_{r_x}(x).$$

Obviously, $\mathrm{Reg}_Q(S) \subset U$; hence, by assumption, it is not empty. Fix $x \in \mathrm{spt}(S)\cap\partial U$. Let next $(x_k)_{k\in\mathbb{N}} \subset \mathrm{Reg}_Q(S)$ be such that $\mathrm{dist}\,(x, \mathbf{B}_{r_{x_k}}(x_k)) \to 0$. We necessarily have that $r_{x_k} \to 0$: otherwise we would have $x \in \mathbf{B}_{2r_{x_k}}(x_k)$ for some k, which would imply $x \in \mathrm{Reg}_Q(S) \subset U$, i.e. a contradiction. Therefore, $x_k \to x$ and, by [10, Theorem 35.1],

$$Q = \limsup_{k\to+\infty}\Theta(S,x_k) \leq \Theta(S,x) = \lim_{\lambda\downarrow 0}\Theta(S,\lambda x) \leq \Theta(S,0) = Q.$$

This implies $x \in \mathrm{D}_Q(S)$. Since $x \in \partial U$, we must then have $x \in \mathrm{Sing}_Q(S)$. Thus, we conclude that $\mathcal{H}^{m-1}(\mathrm{spt}(S)\cap\partial U) = 0$. It follows from the standard theory of rectifiable currents (cf. Lemma A.2) that $S' := S\llcorner U$ has 0 boundary in $\mathbb{R}^{m+n}$. Moreover, since S is an area minimizing cone, S' is also an area-minimizing cone. By definition of U we have $\Theta(S',x) = Q$ for $\|S'\|$-a.e. x and, by semicontinuity,

$$Q \leq \Theta(S',0) \leq \Theta(S,0) = Q.$$

We apply Allard's theorem and deduce that S' is regular, i.e. S' is an m-plane with multiplicity Q. Finally, from $\Theta(S',0) = \Theta(S,0)$, we infer $\mathbf{M}(S \llcorner \mathbf{B}_1) = \mathbf{M}(S' \llcorner \mathbf{B}_1)$ and then $S' = S$. □

Proof of Proposition 1.3. Let $m > 1$ be the smallest integer for which Theorem 0.3 fails. By Theorem A.3 there must be an integer rectifiable area minimizing current R of dimension m and a positive integer Q such that the Hausdorff dimension of $\mathrm{Sing}_Q(R)$ is larger than $m-2$ (note that Theorem A.3 is just a corollary of a well known stratification theorem by Almgren, cf. [1, 10, 11]). We fix the smallest Q for which such a current R exists. Recall that, by the upper semicontinuity of the density and a straightforward application of Allard's regularity theorem (see Theorem A.1), $\mathrm{Sing}_1(R) = \emptyset$, i.e. $Q > 1$.

Let $\alpha \in]0,1]$ be such that $\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}_Q(R)) > 0$. By [10, Theorem 3.6] there exists a point $x \in \mathrm{Sing}_Q(R)$ such that $\mathrm{Sing}_Q(R)$ has positive $\mathcal{H}^{m-2+\alpha}_\infty$-upper density: i.e., assuming without loss of generality $x = 0$ and $\partial R \llcorner \mathbf{B}_1 = 0$, there exists $r_k \downarrow 0$ such that

$$\lim_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{Sing}_Q(R_{0,r_k}) \cap \mathbf{B}_1\big) = \lim_{k\to+\infty} \frac{\mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{Sing}_Q(R) \cap \mathbf{B}_{r_k}\big)}{r_k^{m-2+\alpha}} > 0\,.$$

Up to a subsequence (not relabeled) we can assume that $R_{0,r_k} \to S$, with S a tangent cone. If S is a multiplicity Q flat plane, then we set $T := R$ and we are done: indeed, (1.3) is satisfied by Theorem A.1, because $0 \in \mathrm{Sing}(R)$ and $\|R\| \geq \mathcal{H}^m \llcorner \mathrm{spt}(R)$.

Assume therefore that S is *not* an m-dimensional plane with multiplicity Q. Taking into account the convergence of the total variations for minimizing currents [10, Theorem 34.5] and the upper semicontinuity of $\mathcal{H}^{m-2+\alpha}_\infty$ under the Hausdorff convergence of compact sets, we get

$$\mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{D}_Q(S) \cap \bar{\mathbf{B}}_1\big) \geq \liminf_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{D}_Q(R_{0,r_k}) \cap \bar{\mathbf{B}}_1\big) > 0. \tag{1.4}$$

We claim that (1.4) implies

$$\mathcal{H}^{m-2+\alpha}_\infty(\mathrm{Sing}_Q(S)) > 0. \tag{1.5}$$

Indeed, if all points of $\mathrm{D}_Q(S)$ are singular, then this follows from (1.4) directly. Otherwise, $\mathrm{Reg}_Q(S)$ is not empty and, hence, $\mathcal{H}^m(D_Q(S) \cap \mathbf{B}_1) > 0$. In this case we can apply Lemma 1.4 and infer that, since S is not regular, then $\mathcal{H}^{m-1}(\mathrm{Sing}_Q(S)) > 0$ and (1.5) holds.

We can, hence, find $x \in \mathrm{Sing}_Q(S) \setminus \{0\}$ and $r_k \downarrow 0$ such that

$$\lim_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{Sing}_Q(S_{x,r_k}) \cap \mathbf{B}_1\big) = \lim_{k\to+\infty} \frac{\mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{Sing}_Q(S) \cap \mathbf{B}_{r_k}(x)\big)}{r_k^{m-2+\alpha}} > 0.$$

Up to a subsequence (not relabeled), we can assume that S_{x,r_k} converges to S_1. Since S_1 is a tangent cone to the cone S at $x \neq 0$, S_1 splits off a line, i.e. $S_1 = S_2 \times\!\!\!\times [\![\mathbb{R}v]\!]$, for some area minimizing cone S_2 in $\mathbb{R}^{m-1+n}$ and some $v \in \mathbb{R}^{m+n}$ (cf. the arguments in [10, Lemma 35.5]). Since m is, by assumption, the smallest integer for which Theorem 0.3 fails, $\mathcal{H}^{m-3+\alpha}(\mathrm{Sing}(S_2)) = 0$ and, hence, $\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}_Q(S_1)) = 0$. On the other hand, arguing as we did for (1.4), we have

$$\mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(S_1)\cap\bar{\mathbf{B}}_1) \geq \limsup_{k\to+\infty}\mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(S_{x,r_k})\cap\bar{\mathbf{B}}_1) > 0.$$

Thus $\mathrm{Reg}_Q(S_1)\neq\emptyset$ and, hence, $\mathcal{H}^m(D_Q(S_1))>0$. We can apply Lemma 1.4 again and conclude that S_1 is an m-dimensional plane with multiplicity Q. Therefore, the proposition follows taking $T := \tau_\sharp S$, with τ the translation map $y\mapsto y-x$, and Σ the tangent plane at 0 to the original Riemannian manifold. □

2. Intervals of flattening

For the sequel we fix the constant $c_s := \frac{1}{64\sqrt{m}}$ and notice that $2^{-N_0} < c_s$, where N_0 is the parameter introduced in [6, Assumption 1.8]. It is always understood that the parameters $\beta_2, \delta_2, \gamma_2, \kappa, C_e, C_h, M_0, N_0$ in [6] are fixed in such a way that all the theorems and propositions therein are applicable, cf. [6, Section 1.2]. In particular, all constants which will depend upon these parameters will be called *geometric* and denoted by C_0. On the contrary, we will highlight the dependence of the constants upon the parameters introduced in this paper $p_1, p_2, \ldots$ by writing $C = C(p_1, p_2, \ldots)$.

We recall also the notation introduced in [6, Assumption 1.3]. If $\Sigma\cap\mathbf{B}_{7\sqrt{m}}$ has no boundary in $\mathbf{B}_{7\sqrt{m}}$ and for any $p\in\Sigma\cap\mathbf{B}_{7\sqrt{m}}$ there is a map $\Psi_p : T_p\Sigma\supset\Omega\to(T_p\Sigma)^\perp$ parametrizing it, then $\mathbf{c}(\Sigma\cap\mathbf{B}_{7\sqrt{m}}) := \sup_{p\in\Sigma\cap\mathbf{B}_{7\sqrt{m}}}\|D\Psi_p\|_{C^{2,\varepsilon_0}}$. Obviously these assumptions might fail for a general Σ (in fact $\mathbf{c}(\Sigma\cap\mathbf{B}_{7\sqrt{m}})$ need not be well-defined). However, having fixed a point $q\in\Sigma$, given its C^{3,ε_0} regularity, $\mathbf{c}(\iota_{q,r}(\Sigma)\cap\mathbf{B}_{7\sqrt{m}})$ is well-defined whenever r is sufficiently small and converges to 0 as $r\downarrow 0$. In particular, by Proposition 1.3 and simple rescaling arguments, we assume in the sequel the following.

Assumption 2.1. Let $\varepsilon_3\in]0,\varepsilon_2[$. Under Assumption 0.4, there exist $m, n, Q\geq 2$, $\alpha,\eta>0$, T and Σ for which:

(a) there is a sequence of radii $r_k\downarrow 0$ as in Proposition 1.3;
(b) the following holds:

$$T_0\Sigma = \mathbb{R}^{m+\bar{n}}\times\{0\},\quad \mathrm{spt}(\partial T)\cap\mathbf{B}_{6\sqrt{m}}=\emptyset,\quad 0\in D_Q(T), \tag{2.1}$$

$$\|T\|(\mathbf{B}_{6\sqrt{m}r}) \leq r^m\left(Q\,\omega_m(6\sqrt{m})^m+\varepsilon_3^2\right)\quad \text{for all } r\in(0,1), \tag{2.2}$$

$$\mathbf{c}(\Sigma\cap\mathbf{B}_{7\sqrt{m}})\leq\varepsilon_3\,. \tag{2.3}$$

2.1. **Defining procedure.** We set

$$\mathcal{R} := \left\{r\in]0,1] : \mathbf{E}(T,\mathbf{B}_{6\sqrt{m}r})\leq\varepsilon_3^2\right\}. \tag{2.4}$$

Observe that, if $\{s_k\}\subset\mathcal{R}$ and $s_k\uparrow s$, then $s\in\mathcal{R}$. We cover $\mathcal{R}$ with a collection $\mathcal{F}=\{I_j\}_j$ of intervals $I_j=]s_j,t_j]$ defined as follows. $t_0 := \max\{t : t\in\mathcal{R}\}$. Next assume, by induction, to have defined t_j (and hence also $t_0 > s_0\geq t_1 > s_1\geq\ldots > s_{j-1}\geq t_j$) and consider the following objects:

- $T_j := ((\iota_{0,t_j})_\sharp T)\llcorner\mathbf{B}_{6\sqrt{m}}$, $\Sigma_j := \iota_{0,t_j}(\Sigma)\cap\mathbf{B}_{7\sqrt{m}}$; moreover, consider for each j an orthonormal system of coordinates so that, if we denote by π_0 the m-plane $\mathbb{R}^m\times$

$\{0\}$, then $\mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}}, \pi_0) = \mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}})$ (alternatively we can keep the system of coordinates fixed and rotate the currents T_j).

- Let $\mathcal{M}_j$ be the corresponding center manifold constructed in [6, Theorem 1.17] applied to T_j and Σ_j with respect to the m-plane π_0; the manifold $\mathcal{M}_j$ is then the graph of a map $\boldsymbol{\varphi}_j : \pi_0 \supset [-4,4]^m \to \pi_0^\perp$, and we set $\mathbf{\Phi}_j(x) := (x, \boldsymbol{\varphi}_j(x)) \in \pi_0 \times \pi_0^\perp$.

Then, we consider the Whitney decomposition $\mathscr{W}^{(j)}$ of $[-4,4]^m \subset \pi_0$ as in [6, Definition 1.10 & Proposition 1.11] (applied to T_j) and we define

$$s_j := t_j \max \left(\{ c_s^{-1} \ell(L) : L \in \mathscr{W}^{(j)} \text{ and } c_s^{-1}\ell(L) \geq \operatorname{dist}(0, L)\} \cup \{0\} \right) . \tag{2.5}$$

We will prove below that $s_j/t_j < 2^{-5}$. In particular this ensures that $[s_j, t_j]$ is a (nontrivial) interval. Next, if $s_j = 0$ we stop the induction. Otherwise we let t_{j+1} be the largest element in $\mathcal{R} \cap]0, s_j]$ and proceed as above. Note moreover the following simple consequence of (2.5):

(Stop) If $s_j > 0$ and $\bar r := s_j/t_j$, then there is $L \in \mathscr{W}^{(j)}$ with

$$\ell(L) = c_s \, \bar r \qquad \text{and} \qquad L \cap \bar B_{\bar r}(0, \pi_0) \neq \emptyset \tag{2.6}$$

(in what follows $B_r(p,\pi)$ and $\bar B_r(p,\pi)$ will denote the open and closed disks $\mathbf{B}_r(p) \cap (p+\pi)$, $\bar{\mathbf{B}}_r(p) \cap (p+\pi)$);

(Go) If $\rho > \bar r := s_j/t_j$, then

$$\ell(L) < c_s \rho \qquad \text{for all } L \in \mathscr{W}^{j(k)} \text{ with } L \cap B_\rho(0, \pi_0) \neq \emptyset. \tag{2.7}$$

In particular the latter inequality is true for every $\rho \in]0,3]$ if $s_j = 0$.

2.2. **First consequences.** The following is a list of easy consequences of the definition. Given two sets A and B, we define their *separation* as the number $\operatorname{sep}(A,B) := \inf\{|x-y| : x \in A, y \in B\}$.

Proposition 2.2. *Assuming ε_3 sufficiently small, then the following holds:*

(i) $s_j < \frac{t_j}{2^5}$ *and the family $\mathcal{F}$ is either countable and $t_j \downarrow 0$, or finite and $I_j =]0, t_j]$ for the largest j;*

(ii) *the union of the intervals of $\mathcal{F}$ cover $\mathcal{R}$, and for k large enough the radii r_k in Assumption 2.1 belong to $\mathcal{R}$;*

(iii) *if $r \in]\frac{s_j}{t_j}, 3[$ and $J \in \mathscr{W}_n^{(j)}$ intersects $B := \mathbf{p}_{\pi_0}(\mathcal{B}_r(p_j))$, with $p_j := \mathbf{\Phi}_j(0)$, then J is in the domain of influence $\mathscr{W}_n^{(j)}(H)$ (see* [6, Definition 3.3]*) of a cube $H \in \mathscr{W}_e^{(j)}$ with*

$$\ell(H) \leq 3\, c_s\, r \quad \textit{and} \quad \max\{\operatorname{sep}(H,B), \operatorname{sep}(H,J)\} \leq 3\sqrt{m}\, \ell(H) \leq \frac{3r}{16};$$

(iv) $\mathbf{E}(T_j, \mathbf{B}_r) \leq C_0 \varepsilon_3^2 \, r^{2-2\delta_2}$ *for every $r \in]\frac{s_j}{t_j}, 3[$.*

(v) $\sup\{\operatorname{dist}(x, \mathcal{M}_j) : x \in \operatorname{spt}(T_j) \cap \mathbf{p}_j^{-1}(\mathcal{B}_r(p_j))\} \leq C_0\, (\boldsymbol{m}_0^j)^{\frac{1}{2m}} r^{1+\beta_2}$ *for every $r \in]\frac{s_j}{t_j}, 3[$, where $\boldsymbol{m}_0^j := \max\{\mathbf{c}(\Sigma_j)^2, \mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}})\}$.*

Proof. We start by noticing that $s_j \leq \frac{t_j}{2^5}$ follows from the inequality $2^{-N_0} < c_s$ (cf. [6, Assumption 1.8]) because all cubes in the Whitney decomposition have side-length at most 2^{-N_0-6} (cf. [6, Proposition 1.11]). In particular, this implies that the inductive procedure either never stops, leading to $t_j \downarrow 0$, or it stops because $s_j = 0$ and $]0, t_j] \subset \mathcal{R}$, thus proving (i). The first part of (ii) follows straightforwardly from the choice of t_{j+1}, and the last assertion holds from $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r_k}) \to 0$.

Regarding (iii), let $H \in \mathscr{W}_e^{(j)}$ be as in [6, Definition 3.3] and choose $k \in \mathbb{N} \setminus \{0\}$ such that $\ell(H) = 2^k\,\ell(J)$. Observe that $\|D\boldsymbol{\varphi}_j\|_{C^{2,\kappa}} \leq C_0\varepsilon_3$ by [6, Theorem 1.17]. If ε_3 is sufficiently small, we can assume

$$B_{r/2}(0, \pi_0) \subset B \subset B_r(0, \pi_0)\,. \tag{2.8}$$

Now, by [6, Corollary 3.2], $\mathrm{sep}(H, J) \leq 2\sqrt{m}\ell(H)$ and

$$\mathrm{sep}\,(B, H) \leq \mathrm{sep}\,(H, J) + 2\sqrt{m}\ell(J) \leq 3\sqrt{m}\,\ell(H).$$

Both the inequalities claimed in (iii) are then trivial when $r > \frac{1}{4}$, because $\ell(H) \leq 2^{-N_0-6} \leq 2^{-5}c_s \leq 2^{-9}/\sqrt{m}$. Assume therefore $r \leq \frac{1}{4}$ and note that H intersects $B_{2r+3\sqrt{m}\,\ell(H)}$. Let $\rho := 2r + 3\sqrt{m}\ell(H)$. Observe that $2r < \rho < 1$. By the definition of s_j, we have that

$$\ell(H) < c_s\,\left(2r + 3\sqrt{m}\,\ell(H)\right) = 2\,c_s\,r + \frac{3\ell(H)}{16}.$$

Therefore, we conclude that $\ell(H) \leq 3\,c_s r$ and $\mathrm{sep}(H, B) \leq 9\sqrt{m}\,c_s r < 3r/16$.

We now turn to (iv). If $r \geq 2^{-N_0}$, then obviously

$$\mathbf{E}(T_j, \mathbf{B}_r) \leq (4\sqrt{m}\,2^{N_0})^{m+2-2\delta_2} r^{2-2\delta_2}\mathbf{E}(T_j, \mathbf{B}_{4\sqrt{m}}) \leq (4\sqrt{m}\,2^{N_0})^{m+2-2\delta_2} r^{2-2\delta_2}\varepsilon_3^2\,.$$

Otherwise, let $k \geq N_0$ be the smallest natural number such that $2^{-k+1} > r$ and let $L \in \mathscr{W}^{(j),k} \cup \mathscr{S}^{(j),k}$ be a cube so that $0 \in L$ (cf. [6, Definition 1.10], $\ell(H) = 2^{-k}$). By [6, Proposition 4.2(v)], $|p_L| \leq (\sqrt{m} + C_0(\boldsymbol{m}_0^j)^{1/2m}) \leq 2\sqrt{m}\ell(H)$ and so it follows easily that $\mathbf{B}_r \subset \mathbf{B}_L$. From condition (Go) we have $L \notin \mathscr{W}^{(j)}$. Thus, by [6, Proposition 1.11], we get

$$\mathbf{E}(T_j, \mathbf{B}_r) \leq C_0\mathbf{E}(T_j, \mathbf{B}_L) \leq C_0\varepsilon_3^2\,r^{2-2\delta_2}.$$

Finally, (v) follows from [6, Corollary 2.2 (ii)], because by (Go), for every $r \in]\frac{s_j}{t_j}, 3[$, every cube $L \in \mathscr{W}^{(j)}$ which intersects $B_r(0, \pi_0)$ satisfies $\ell(L) < c_s r$. □

3. Frequency function and first variations

Consider the following Lipschitz (piecewise linear) function $\phi : [0 + \infty[\to [0, 1]$ given by

$$\phi(r) := \begin{cases} 1 & \text{for } r \in [0, \frac{1}{2}], \\ 2 - 2r & \text{for } r \in]\frac{1}{2}, 1], \\ 0 & \text{for } r \in]1, +\infty[. \end{cases}$$

For every interval of flattening $I_j =]s_j, t_j]$, let N_j be the normal approximation of T_j on $\mathcal{M}_j$ in [6, Theorem 2.4].

Definition 3.1 (Frequency functions)**.** For every $r \in]0,3]$ we define:

$$\mathbf{D}_j(r) := \int_{\mathcal{M}^j} \phi\left(\frac{d_j(p)}{r}\right) |DN_j|^2(p)\, dp \quad \text{and} \quad \mathbf{H}_j(r) := -\int_{\mathcal{M}^j} \phi'\left(\frac{d_j(p)}{r}\right) \frac{|N_j|^2(p)}{d(p)}\, dp\,,$$

where $d_j(p)$ is the geodesic distance on $\mathcal{M}_j$ between p and $\mathbf{\Phi}_j(0)$. If $\mathbf{H}_j(r) > 0$, we define the *frequency function* $\mathbf{I}_j(r) := \frac{r\,\mathbf{D}_j(r)}{\mathbf{H}_j(r)}$.

The following is the main analytical estimate of the paper, which allows us to exclude infinite order of contact among the different sheets of a minimizing current.

Theorem 3.2 (Main frequency estimate)**.** *If ε_3 is sufficiently small, then there exists a geometric constant C_0 such that, for every $[a,b] \subset [\frac{s_j}{t_j}, 3]$ with $\mathbf{H}_j|_{[a,b]} > 0$, we have*

$$\mathbf{I}_j(a) \leq C_0(1 + \mathbf{I}_j(b)). \tag{3.1}$$

To simplify the notation, in this section we drop the index j and omit the measure $\mathcal{H}^m$ in the integrals over regions of $\mathcal{M}$. The proof exploits four identities collected in Proposition 3.5, which will be proved in the next sections.

Definition 3.3. We let $\partial_{\hat{r}}$ denote the derivative with respect to arclength along geodesics starting at $\mathbf{\Phi}(0)$. We set

$$\mathbf{E}(r) := -\int_{\mathcal{M}} \phi'\left(\tfrac{d(p)}{r}\right) \sum_{i=1}^{Q} \langle N_i(p), \partial_{\hat{r}} N_i(p)\rangle\, dp\,, \tag{3.2}$$

$$\mathbf{G}(r) := -\int_{\mathcal{M}} \phi'\left(\tfrac{d(p)}{r}\right) d(p)\, |\partial_{\hat{r}} N(p)|^2\, dp \quad \text{and} \quad \mathbf{\Sigma}(r) := \int_{\mathcal{M}} \phi\left(\tfrac{d(p)}{r}\right) |N|^2(p)\, dp\,. \tag{3.3}$$

Remark 3.4. Observe that all these functions of r are absolutely continuous and, therefore, classically differentiable at almost every r. Moreover, the following rough estimate easily follows from [6, Theorem 2.4] and the condition (Go):

$$\mathbf{D}(r) \leq \int_{\mathcal{B}_r(\mathbf{\Phi}(0))} |DN|^2 \leq C_0\, \boldsymbol{m}_0\, r^{m+2-2\delta_2} \quad \text{for every} \quad r \in \left]\tfrac{s}{t}, 3\right[\,. \tag{3.4}$$

Indeed, since N vanishes identically on the set $\mathcal{K}$ of [6, Theorem 2.4], it suffices to sum the estimate of [6, Theorem 2.4, (2.3)] over all the different cubes L (of the corresponding Whitney decomposition) for which $\mathbf{\Phi}(L)$ intersects the geodesic ball $\mathcal{B}_r$.

Proposition 3.5 (First variation estimates)**.** *For every γ_3 sufficiently small there is a constant $C = C(\gamma_3) > 0$ such that, if ε_3 is sufficiently small, $[a,b] \subset [\frac{s}{t}, 3]$ and $\mathbf{I} \geq 1$ on $[a,b]$, then the following inequalities hold for a.e. $r \in [a,b]$:*

$$\left|\mathbf{H}'(r) - \tfrac{m-1}{r}\,\mathbf{H}(r) - \tfrac{2}{r}\,\mathbf{E}(r)\right| \leq C\mathbf{H}(r), \tag{3.5}$$

$$\left|\mathbf{D}(r) - r^{-1}\mathbf{E}(r)\right| \leq C\mathbf{D}(r)^{1+\gamma_3} + C\varepsilon_3^2\,\mathbf{\Sigma}(r), \tag{3.6}$$

$$\left|\mathbf{D}'(r) - \tfrac{m-2}{r}\,\mathbf{D}(r) - \tfrac{2}{r^2}\,\mathbf{G}(r)\right| \leq C\mathbf{D}(r) + C\mathbf{D}(r)^{\gamma_3}\mathbf{D}'(r) + Cr^{-1}\mathbf{D}(r)^{1+\gamma_3}, \tag{3.7}$$

$$\mathbf{\Sigma}(r) + r\,\mathbf{\Sigma}'(r) \leq C\, r^2\,\mathbf{D}(r) \leq Cr^{2+m}\varepsilon_3^2. \tag{3.8}$$

We assume for the moment the proposition and prove the theorem.

Proof of Theorem 3.2. Set $\mathbf{\Omega}(r) := \log\big(\max\{\mathbf{I}(r), 1\}\big)$. Fix a $\gamma_3 > 0$ and an ε_3 sufficiently small so that the conclusion of Proposition 3.5. We can thus treat the corresponding constants in the inequalities as geometric ones, but to simplify the notation we keep denoting them by C.

To prove (3.1) it is enough to show $\mathbf{\Omega}(a) \leq C + \mathbf{\Omega}(b)$. If $\mathbf{\Omega}(a) = 0$, then there is nothing to prove. If $\mathbf{\Omega}(a) > 0$, let $b' \in]a, b]$ be the supremum of t such that $\mathbf{\Omega} > 0$ on $]a, t[$. If $b' < b$, then $\mathbf{\Omega}(b') = 0 \leq \mathbf{\Omega}(b)$. Therefore, by possibly substituting $]a, b[$ with $]a, b'[$, we can assume that $\mathbf{\Omega} > 0$, i.e. $\mathbf{I} > 1$, on $]a, b[$. By Proposition 3.5, if ε_3 is sufficiently small, then

$$\frac{\mathbf{D}(r)}{2} \overset{(3.6)\,\&\,(3.8)}{\leq} \frac{\mathbf{E}(r)}{r} \overset{(3.6)\,\&\,(3.8)}{\leq} 2\,\mathbf{D}(r), \tag{3.9}$$

from which we conclude that $\mathbf{E} > 0$ over the interval $]a, b'[$. Set for simplicity $\mathbf{F}(r) := \mathbf{D}(r)^{-1} - r\mathbf{E}(r)^{-1}$, and compute

$$-\mathbf{\Omega}'(r) = \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{\mathbf{D}'(r)}{\mathbf{D}(r)} - \frac{1}{r} \overset{(3.6)}{=} \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{r\mathbf{D}'(r)}{\mathbf{E}(r)} - \mathbf{D}'(r)\mathbf{F}(r) - \frac{1}{r}.$$

Again by Proposition 3.5:

$$\frac{\mathbf{H}'(r)}{\mathbf{H}(r)} \overset{(3.5)}{\leq} \frac{m-1}{r} + C + \frac{2}{r}\,\frac{\mathbf{E}(r)}{\mathbf{H}(r)}, \tag{3.10}$$

$$|\mathbf{F}(r)| \overset{(3.6)}{\leq} C\,\frac{r(\mathbf{D}(r)^{1+\gamma_3} + \mathbf{\Sigma}(r))}{\mathbf{D}(r)\,\mathbf{E}(r)} \overset{(3.9)}{\leq} C\,\mathbf{D}(r)^{\gamma_3-1} + C\,\frac{\mathbf{\Sigma}(r)}{\mathbf{D}(r)^2}, \tag{3.11}$$

$$\begin{aligned}
-\frac{r\mathbf{D}'(r)}{\mathbf{E}(r)} &\overset{(3.7)}{\leq} \left(C - \frac{m-2}{r}\right)\frac{r\mathbf{D}(r)}{\mathbf{E}(r)} - \frac{2}{r}\,\frac{\mathbf{G}(r)}{\mathbf{E}(r)} + C\frac{r\mathbf{D}(r)^{\gamma_3}\mathbf{D}'(r) + \mathbf{D}(r)^{1+\gamma_3}}{\mathbf{E}(r)}\\
&\leq C - \frac{m-2}{r} + \frac{C}{r}\mathbf{D}(r)|\mathbf{F}(r)| - \frac{2}{r}\,\frac{\mathbf{G}(r)}{\mathbf{E}(r)} + C\mathbf{D}(r)^{\gamma_3-1}\mathbf{D}'(r) + C\frac{\mathbf{D}(r)^{\gamma_3}}{r}\\
&\overset{(3.8),\,(3.11)\,\&\,(3.4)}{\leq} C - \frac{m-2}{r} - \frac{2}{r}\,\frac{\mathbf{G}(r)}{\mathbf{E}(r)} + C\mathbf{D}(r)^{\gamma_3-1}\mathbf{D}'(r) + C\,r^{\gamma_3\,m-1}.
\end{aligned} \tag{3.12}$$

By Cauchy-Schwartz, we have

$$\frac{\mathbf{E}(r)}{r\mathbf{H}(r)} \leq \frac{\mathbf{G}(r)}{r\mathbf{E}(r)}. \tag{3.13}$$

Thus, by (3.4), (3.10), (3.12) and (3.13), we conclude

$$\begin{aligned}
-\mathbf{\Omega}'(r) &\leq C + C\,r^{\gamma_3\,m-1} + Cr\mathbf{D}(r)^{\gamma_3-1}\mathbf{D}'(r) - \mathbf{D}'(r)\mathbf{F}(r)\\
&\overset{(3.11)}{\leq} C\,r^{\gamma_3\,m-1} + C\mathbf{D}(r)^{\gamma_3-1}\mathbf{D}'(r) + C\frac{\mathbf{\Sigma}(r)\mathbf{D}'(r)}{\mathbf{D}(r)^2}.
\end{aligned} \tag{3.14}$$

Integrating (3.14) we conclude:

$$\mathbf{\Omega}(a) - \mathbf{\Omega}(b) \leq C + C\left(\mathbf{D}(b)^{\gamma_3} - \mathbf{D}(a)^{\gamma_3}\right) + C\left[\frac{\mathbf{\Sigma}(a)}{\mathbf{D}(a)} - \frac{\mathbf{\Sigma}(b)}{\mathbf{D}(b)} + \int_a^b \frac{\mathbf{\Sigma}'(r)}{\mathbf{D}(r)}\,dr\right] \overset{(3.8)}{\leq} C. \qquad \square$$

The rest of the section is devoted to the proof of Proposition 3.5.

3.1. **Estimates on $\mathbf{H}'$: proof of** (3.5)**.** Set $q := \mathbf{\Phi}(0)$. Let $\exp : B_3 \subset T_q\mathcal{M} \to \mathcal{M}$ be the exponential map and $\mathbf{J}\exp$ its Jacobian. Note that $d(\exp(y), q) = |y|$ for every $y \in B_3$. By the area formula, setting $y = rz$, we can write $\mathbf{H}$ in the following way:

$$\mathbf{H}(r) = -r^{m-1} \int_{T_q\mathcal{M}} \frac{\phi'(|z|)}{|z|} |N|^2(\exp(rz))\, \mathbf{J}\exp(rz)\, dx\,.$$

Therefore, differentiating under the integral sign, we easily get (3.5):

$$\begin{aligned}
\mathbf{H}'(r) = & -(m-1)\, r^{m-2} \int_{T_q\mathcal{M}} \frac{\phi'(|z|)}{|z|} |N|^2(\exp(rz))\, \mathbf{J}\exp(rz)\, dz \\
& - 2\, r^{m-1} \int_{T_q\mathcal{M}} \phi'(|z|) \sum_i \langle N_i(\exp(rz)), \partial_{\hat r} N_i(\exp(rz))\rangle\, \mathbf{J}\exp(rz)\, dz \\
& - r^{m-1} \int_{T_q\mathcal{M}} \frac{\phi'(|z|)}{|z|} |N|^2(\exp(rz))\, \frac{d}{dr}\mathbf{J}\exp(rz)\, dz \\
= & \frac{m-1}{r}\, \mathbf{H}(r) + \frac{2}{r}\, \mathbf{E}(r) + O(1)\, \mathbf{H}(r),
\end{aligned}$$

where we used that $\frac{d}{dr}\mathbf{J}\exp(r\,z) = O(1)$, because $\mathcal{M}$ is a $C^{3,\kappa}$ submanifold and hence exp is a $C^{2,\kappa}$ map (see Proposition A.4). □

3.2. **$\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$: proof of** (3.8)**.** We show the following more precise estimates.

Lemma 3.6. *There exists a dimensional constant $C_0 > 0$ such that*

$$\mathbf{\Sigma}(r) \leq C_0\, r^2\, \mathbf{D}(r) + C_0 r \mathbf{H}(r) \quad \textit{and} \quad \mathbf{\Sigma}'(r) \leq C_0 \mathbf{H}(r), \tag{3.15}$$

$$\int_{\mathcal{B}_r(q)} |N|^2 \leq C_0\, \mathbf{\Sigma}(r) + C_0\, r\, \mathbf{H}(r)\,, \tag{3.16}$$

$$\int_{\mathcal{B}_r(q)} |DN|^2 \leq C_0\, \mathbf{D}(r) + C_0\, r \mathbf{D}'(r). \tag{3.17}$$

In particular, if $\mathbf{I} \geq 1$, then (3.8) *holds and*

$$\int_{\mathcal{B}_r(q)} |N|^2 \leq C_0\, r^2 \mathbf{D}(r). \tag{3.18}$$

Proof. To simplify the notation we drop the subscript $_0$ from the geometric constants. Observe that $\psi(p) := \phi\big(\frac{d(p)}{r}\big)|N|^2(p)$ is a Lipschitz function with compact support in $\mathbf{B}_r(q)$. We therefore use the Poincaré inequality: $\mathbf{\Sigma}(r) = \int_{\mathcal{M}} \psi \leq Cr \int_{\mathcal{M}} |D\psi|$ (the constant C depends on the smoothness of $\mathcal{M}$). We compute

$$\begin{aligned}
\mathbf{\Sigma}(r) &\leq -C \int_{\mathcal{M}} \phi'(r^{-1}d(p))|N|^2(p) + C\, r \int_{\mathcal{M}} \phi(r^{-1}d(p))|N||DN| \\
&\leq Cr\mathbf{H}(r) + C\mathbf{\Sigma}(r)^{1/2} \left(r^2\mathbf{D}(r)\right)^{1/2} \leq Cr\mathbf{H}(r) + \tfrac{1}{2}\mathbf{\Sigma}(r) + C\, r^2\mathbf{D}(r),
\end{aligned}$$

which gives the first part of (3.15). The remaining inequality is straightforward:

$$\mathbf{\Sigma}'(r) = -\int_{\mathcal{M}} \frac{d(p)}{r^2}\,\phi'\left(\frac{d(p)}{r}\right)\,|N|^2(p) \le C\mathbf{H}(r)\,.$$

Since $\phi' = 0$ on $]0, \frac{1}{2}[$ and $\phi' = -2$ on $]\frac{1}{2}, 1[$, we easily deduce

$$\int_{\mathcal{B}_r(q)\setminus\mathcal{B}_{r/2}(q)} |N|^2 \le r\,\mathbf{H}(r),$$

$$r\mathbf{D}'(r) = -\int \frac{d(p)}{r}\phi'\left(\frac{d(p,q)}{r}\right)|DN|^2 \ge \int_{\mathcal{B}_r(q)\setminus\mathcal{B}_{r/2}(q)} |DN|^2\,.$$

On the other hand, since $\phi = 1$ on $[0, \frac{1}{2}]$, (3.16) and (3.17) readily follow. Therefore, in the hypothesis $\mathbf{I} \ge 1$, i.e. $\mathbf{H} \le r\mathbf{D}$, we conclude (3.8) from (3.15). □

3.3. **First variations.** To prove the remaining estimates in Proposition 3.5 we exploit the first variation of T along some vector fields X. The variations are denoted by $\delta T(X)$. We fix a neighborhood $\mathbf{U}$ of $\mathcal{M}$ and the normal projection $\mathbf{p} : \mathbf{U} \to \mathcal{M}$ as in [6, Assumption 2.1]. Observe that $\mathbf{p} \in C^{2,\kappa}$ and [5, Assumption 3.1] holds. We will consider:

- the *outer variations*, where $X(p) = X_o(p) := \phi\left(\frac{d(\mathbf{p}(p))}{r}\right)(p - \mathbf{p}(p))$.
- the *inner variations*, where $X(p) = X_i(p) := Y(\mathbf{p}(p))$ with

$$Y(p) := \frac{d(p)}{r}\,\phi\left(\frac{d(p)}{r}\right)\frac{\partial}{\partial\hat{r}} \quad \forall\, p \in \mathcal{M}$$

($\frac{\partial}{\partial\hat{r}}$ is the unit vector field tangent to the geodesics emanating from $\mathbf{\Phi}(0)$ and pointing outwards).

Note that X_i is the infinitesimal generator of a one parameter family of bilipschitz homeomorphisms Φ_ε defined as $\Phi_\varepsilon(p) := \Psi_\varepsilon(\mathbf{p}(p)) + p - \mathbf{p}(p)$, where Ψ_ε is the one-parameter family of bilipschitz homeomorphisms of $\mathcal{M}$ generated by Y.

Consider now the map $F(p) := \sum_i [\![p + N_i(p)]\!]$ and the current $\mathbf{T}_F$ associated to its image (cf. [5] for the notation). Observe that X_i and X_o are supported in $\mathbf{p}^{-1}(\mathcal{B}_r(q))$ but none of them is *compactly* supported. However, recalling Proposition 2.2 (v) and the minimizing property of T in Σ, we deduce that $\delta T(X) = \delta T(X^T) + \delta T(X^\perp) = \delta T(X^\perp)$, where $X = X^T + X^\perp$ is the decomposition of X in the tangent and normal components to $T\Sigma$. Then, we have

$$\begin{aligned}|\delta\mathbf{T}_F(X)| &\le |\delta\mathbf{T}_F(X) - \delta T(X)| + |\delta T(X^\perp)|\\ &\le \underbrace{\int_{\mathrm{spt}(T)\setminus\mathrm{Im}(F)} |\mathrm{div}_{\vec{T}} X|\;d\|T\| + \int_{\mathrm{Im}(F)\setminus\mathrm{spt}(T)} \left|\mathrm{div}_{\vec{\mathbf{T}}_F} X\right|\;d\|\mathbf{T}_F\|}_{\mathrm{Err}_4} + \underbrace{\left|\int \mathrm{div}_{\vec{T}} X^\perp\,d\|T\|\right|}_{\mathrm{Err}_5}.\end{aligned} \tag{3.19}$$

Set now for simplicity $\varphi_r(p) := \phi\big(\frac{d(p)}{r}\big)$. We wish to apply [5, Theorem 4.2] to conclude

$$\delta \mathbf{T}_F(X_o) = \int_{\mathcal{M}} \Big(\varphi_r\, |DN|^2 + \sum_{i=1}^Q N_i \otimes \nabla\varphi_r : DN_i\Big) + \sum_{j=1}^3 \mathrm{Err}_j^o, \tag{3.20}$$

where the errors Err_j^o correspond to the terms $\mathrm{Err_j}$ of [5, Theorem 4.2]. This would imply

$$\mathrm{Err}_1^o = -Q\int_{\mathcal{M}} \varphi_r \langle H_{\mathcal{M}}, \boldsymbol{\eta}\circ N\rangle, \tag{3.21}$$

$$|\mathrm{Err}_2^o| \leq C_0 \int_{\mathcal{M}} |\varphi_r||A|^2|N|^2, \tag{3.22}$$

$$|\mathrm{Err}_3^o| \leq C_0 \int_{\mathcal{M}} \big(|N||A| + |DN|^2\big)\big(|\varphi_r||DN|^2 + |D\varphi_r||DN||N|\big)\,, \tag{3.23}$$

where $H_{\mathcal{M}}$ is the mean curvature vector of $\mathcal{M}$. Note that [5, Theorem 4.2] requires the C^1 regularity of φ_r. We overcome this technical obstruction applying [5, Theorem 4.2] to a standard smoothing of ϕ and then passing into the limit (the obvious details are left to the reader). Plugging (3.20) into (3.19), we then conclude

$$\big|\mathbf{D}(r) - r^{-1}\mathbf{E}(r)\big| \leq \sum_{j=1}^5 \big|\mathrm{Err}_j^o\big|\,, \tag{3.24}$$

where Err_4^o and Err_5^o correspond respectively to Err_4 and Err_5 of (3.19) when $X = X_o$. With the same argument, but applying this time [5, Theorem 4.3] to $X = X_i$, we get

$$\delta \mathbf{T}_F(X_i) = \frac{1}{2}\int_{\mathcal{M}} \Big(|DN|^2 \mathrm{div}_{\mathcal{M}} Y - 2\sum_{i=1}^Q \langle DN_i : (DN_i \cdot D_{\mathcal{M}} Y)\rangle\Big) + \sum_{j=1}^3 \mathrm{Err}_j^i\,, \tag{3.25}$$

where this time the errors Err_j^i correspond to the error terms Err_j of [5, Theorem 4.3], i.e.

$$\mathrm{Err}_1^i = -Q\int_{\mathcal{M}} \big(\langle H_{\mathcal{M}}, \boldsymbol{\eta}\circ N\rangle\, \mathrm{div}_{\mathcal{M}} Y + \langle D_Y H_{\mathcal{M}}, \boldsymbol{\eta}\circ N\rangle\big)\,, \tag{3.26}$$

$$|\mathrm{Err}_2^i| \leq C_0 \int_{\mathcal{M}} |A|^2 \big(|DY||N|^2 + |Y||N|\,|DN|\big)\,, \tag{3.27}$$

$$|\mathrm{Err}_3^i| \leq C_0 \int_{\mathcal{M}} \Big(|Y||A||DN|^2\big(|N| + |DN|\big) + |DY|\big(|A|\,|N|^2|DN| + |DN|^4\big)\Big)\,. \tag{3.28}$$

Straightforward computations (again appealing to Proposition A.4) lead to

$$D_{\mathcal{M}} Y(p) = \phi'\left(\frac{d(p)}{r}\right)\frac{d(p)}{r^2}\frac{\partial}{\partial\hat r}\otimes\frac{\partial}{\partial \hat r} + \phi\left(\frac{d(p)}{r}\right)\left(\frac{\mathrm{Id}}{r} + O(1)\right)\,, \tag{3.29}$$

$$\mathrm{div}_{\mathcal{M}}\, Y(p) = \phi'\left(\frac{d(p)}{r}\right)\frac{d(p)}{r^2} + \phi\left(\frac{d(p)}{r}\right)\left(\frac{m}{r} + O(1)\right)\,. \tag{3.30}$$

Plugging (3.29) and (3.30) into (3.25) and using (3.19) we then conclude

$$\left|\mathbf{D}'(r) - (m-2)r^{-1}\mathbf{D}(r) - 2r^{-2}\mathbf{G}(r)\right| \leq C_0\mathbf{D}(r) + \sum_{j=1}^{5}\left|\mathrm{Err}_j^i\right| . \tag{3.31}$$

Proposition 3.5 is then proved by the estimates of the errors terms done in the next section.

4. Error estimates

We start with some preliminary considerations, keeping the notation and convention of the previous section (and dropping the subscript when dealing with the maps of Theorem 3.2 and Proposition 3.5).

4.1. **Families of subregions.** Set $q := \mathbf{\Phi}(0)$. We select a family of subregions of $\mathcal{B}_r(p) \subset \mathcal{M}$. Denote by B and ∂B respectively $\mathbf{p}_\pi(\mathcal{B}_r(q))$ and $\mathbf{p}_\pi(\partial\mathcal{B}_r(q))$, where π is the reference m-dimensional plane of the construction of the center manifold $\mathcal{M}$. Since $\|\boldsymbol{\varphi}\|_{C^{3,\kappa}} \leq C\varepsilon_3^{1/m}$ (cf. [6, Theorem 1.17]), by Proposition A.4 we can assume that B is a C^2 convex set which at any boundary point p contains an interior sphere of radius $r/2$ passing through p. Thus:

$$\forall z \in \partial B \quad \text{there is a ball } B_{r/2}(y,\pi) \subset B \text{ whose closure touches } \partial B \text{ at } z. \tag{4.1}$$

Definition 4.1 (Family of cubes)**.** We first define a family $\mathcal{T}$ of cubes in the Whitney decomposition $\mathscr{W}$ as follows:

(i) $\mathcal{T}$ includes all $L \in \mathscr{W}_h \cup \mathscr{W}_e$ which intersect B;
(ii) if $L' \in \mathscr{W}_n$ intersects B and belongs to the domain of influence $\mathscr{W}_n(L)$ of the cube $L \in \mathscr{W}_e$ as in [6, Corollary 3.2], then $L \in \mathcal{T}$.

Definition 4.2 (Associated balls B^L)**.** By Proposition 2.2 (iii), $\ell(L) \leq 3c_s r \leq r$ and $\mathrm{sep}(L,B) \leq 3\sqrt{m}\,\ell(L)$ for each $L \in \mathcal{T}$. Let x_L be the center of L and:

(a) if $x_L \in \overline{B}$, we then set $s(L) := \ell(L)$ and $B^L := B_{s(L)}(x_L,\pi)$;
(b) otherwise we consider the ball $B_{r(L)}(x_L,\pi) \subset \pi$ whose closure touches $\overline{B}$ at exactly one point $p(L)$, we set $s(L) := r(L) + \ell(L)$ and define $B^L := B_{s(L)}(x_L,\pi)$.

Observe that, when $L \in \mathcal{T} \cap \mathscr{W}_h$, then $s(L)$ is at most $(\sqrt{m}+1)\ell(L)$. We proceed to select a countable family $\mathscr{T}$ of pairwise disjoint balls $\{B^L\}$. We let $S := \sup_{L\in\mathcal{T}} s(L)$ and start selecting a maximal subcollection $\mathscr{T}_1$ of pairwise disjoint balls with radii larger than $S/2$. Clearly, $\mathscr{T}_1$ is finite. In general, at the stage k, we select a maximal subcollection $\mathscr{T}_k$ of pairwise disjoint balls which do not intersect any of the previously selected balls in $\mathscr{T}_1 \cup \ldots \cup \mathscr{T}_{k-1}$ and which have radii $r \in]2^{-k}S, 2^{1-k}S]$. Finally, we set $\mathscr{T} := \bigcup_k \mathscr{T}_k$.

Definition 4.3 (Family of cube-ball pairs $(L, B(L)) \in \mathscr{Z}$)**.** Recalling (4.1) and $\ell(L) \leq r$, it easy to see that there exist balls $B_{\ell(L)/4}(q_L,\pi) \subset B^L \cap B$ which lie at distance at least $\ell(L)/4$ from ∂B. We denote by $B(L)$ one such ball and by $\mathscr{Z}$ the collection of pairs $(L, B(L))$ with $B^L \in \mathscr{T}$.

Next, we partition the cubes of $\mathscr{W}$ which intersect B into disjoint families $\mathscr{W}(L)$ labeled by $(L, B(L)) \in \mathscr{Z}$ in the following way (observe that $\mathscr{W}(L)$ and $\mathscr{W}_n(L)$ are different families and should not be confused!). Let $H \in \mathscr{W}$ have nonempty intersection with B. If H is itself in $\mathcal{T}$, we then select $L \in \mathscr{T}$ with $B^L \cap B^H \neq \emptyset$ and assign $H \in \mathscr{W}(L)$. Otherwise H is in the domain of influence of some $J \in \mathscr{W}_e$. By Proposition 2.2, the separation between J and H is at most $3\sqrt{m}\ell(J)$ and, hence, $H \subset B_{4\sqrt{m}\ell(J)}(x_J)$. By construction there is a $B^L \in \mathscr{T}$ with $B^J \cap B^L \neq \emptyset$ and radius $s(L) \geq \frac{s(J)}{2}$. We then prescribe $H \in \mathscr{W}(L)$. Observe that $s(L) \leq 4\sqrt{m}\ell(L)$ and $s(J) \geq \ell(J)$. Therefore, $\ell(J) \leq 8\sqrt{m}\ell(L)$ and $|x_J - x_L| \leq 5s(L) \leq 20\sqrt{m}\ell(L)$. This implies that

$$H \subset B_{4\sqrt{m}\ell(J)}(x_J) \subset B_{4\sqrt{m}\ell(J)+20\sqrt{m}\ell(L)}(x_L) \subset B_{30\sqrt{m}\ell(L)}(x_L)\,.$$

The inclusion $H \subset B_{30\sqrt{m}\ell(L)}(x_L)$ holds also in case $H \in \mathcal{T}$, as can be easily seen simply setting $J = H$ and using the same computations. For later reference, we collect the main properties of the above construction.

Lemma 4.4. *The following holds.*

(i) *If $(L, B(L)) \in \mathscr{Z}$, then $L \in \mathscr{W}_e \cup \mathscr{W}_h$, the radius of $B(L)$ is $\frac{\ell(L)}{4}$, $B(L) \subset B^L \cap B$ and* $\mathrm{sep}\,(B(L), \partial B) \geq \frac{\ell(L)}{4}$.

(ii) *If the pairs $(L, B(L)), (L', B(L')) \in \mathscr{Z}$ are distinct, then L and L' are distinct and $B(L) \cap B(L') = \emptyset$.*

(iii) *The cubes $\mathscr{W}$ which intersect B are partitioned into disjoint families $\mathscr{W}(L)$ labeled by $(L, B(L)) \in \mathscr{Z}$ such that, if $H \in \mathscr{W}(L)$, then $H \subset B_{30\sqrt{m}\ell(L)}(x_L)$.*

4.2. **Basic estimates in the subregions.** For notational convenience, we order the family $\mathscr{Z} = \{(L_i, B(L_i))\}_{i\in\mathbb{N}}$, and set

$$\mathcal{B}^i := \mathbf{\Phi}(B(L_i)) \qquad \mathcal{U}_i = \cup_{H\in\mathscr{W}(L_i)}\mathbf{\Phi}(H) \cap \mathcal{B}_r(q)$$

(recall that $q = \mathbf{\Phi}(0)$). Observe that the separation between $\mathcal{B}^i$ and $\partial\mathcal{B}_r(q)$ is larger than that between $B(L_i)$ and $\partial B = \mathbf{p}_\pi(\partial\mathcal{B}_r(q))$. Thus, by Lemma 4.4 (i), $\varphi_r(p) = \phi\big(\frac{d(p)}{r}\big)$ satisfies

$$\inf_{p\in\mathcal{B}^i} \varphi_r(p) \geq (4r)^{-1}\ell_i\,, \tag{4.2}$$

where $\ell_i := \ell(L_i)$. From this and Lemma 4.4 (iii), we also obtain

$$\sup_{p\in\mathcal{U}_i} \varphi_r(p) - \inf_{p\in\mathcal{U}_i} \varphi_r(p) \leq C\mathrm{Lip}(\varphi_r)\ell_i \leq \frac{C}{r}\,\ell_i \overset{(4.2)}{\leq} C \inf_{p\in\mathcal{B}_i} \varphi_r(p)\,,$$

which translates into

$$\sup_{p\in\mathcal{U}_i} \varphi_r(p) \leq C \inf_{p\in\mathcal{B}^i} \varphi_r(p)\,. \tag{4.3}$$

Moreover, set $\mathcal{V}_i := \mathcal{U}_i \cap (((\mathrm{spt}(\mathbf{T}_F) \setminus \mathrm{spt}(T)) \cup (\mathrm{spt}(T) \setminus \mathrm{spt}(\mathbf{T}_F)))$ and observe that $\mathcal{V}_i \subset \mathcal{U}_i \setminus \mathcal{K}$, where $\mathcal{K}$ is the coincidence set of [6, Theorem 2.4]. From [6, Theorem 2.4], we derive the following estimates:

$$\int_{\mathcal{U}_i} |\boldsymbol{\eta}\circ N| \leq C_0 \boldsymbol{m}_0\, \ell_i^{2+m+\gamma_2/2} + C_0 \int_{\mathcal{U}^i} |N|^{2+\gamma_2}, \tag{4.4}$$

$$\int_{\mathcal{U}_i} |DN|^2 \leq C_0 \boldsymbol{m}_0\, \ell_i^{m+2-2\delta_2}, \tag{4.5}$$

$$\|N\|_{C^0(\mathcal{U}_i)} + \sup_{p\in \mathrm{spt}(T)\cap \mathbf{p}^{-1}(\mathcal{U}_i)} |p-\mathbf{p}(p)| \leq C_0 \boldsymbol{m}_0^{1/2m} \ell_i^{1+\beta_2}, \tag{4.6}$$

$$\mathrm{Lip}(N|_{\mathcal{U}_i}) \leq C_0 \boldsymbol{m}_0^{\gamma_2} \ell_i^{\gamma_2}, \tag{4.7}$$

$$\mathbf{M}(T\llcorner \mathbf{p}^{-1}(\mathcal{V}_i)) + \mathbf{M}(\mathbf{T}_F\llcorner \mathbf{p}^{-1}(\mathcal{V}_i)) \leq C_0 \boldsymbol{m}_0^{1+\gamma_2} \ell_i^{m+2+\gamma_2}. \tag{4.8}$$

To prove these estimates, observe first that $\sum_{H\in\mathscr{W}(L_i)} \ell(H)^m \leq C_0 \ell_i^m$, because all $H \in \mathscr{W}(L_i)$ are disjoint and contained in a ball of radius comparable to ℓ_i. This in turn implies that $\sum_{H\in\mathscr{W}(L_i)} \ell(H)^{m+\varepsilon} \leq C_0 \ell_i^{m+\varepsilon}$, because $\ell(H) \leq \ell_i$ for any $H \in \mathscr{W}(L)$. Thus:

- (4.4) follows summing the estimate of [6, Theorem 2.4 (2.4)] applied with $a = 1$ to $\mathbf{\Phi}(H)$ with $H \in \mathscr{W}(L_i)$;
- (4.5) follows from summing the estimate of [6, Theorem 2.4 (2.3)] applied to $\mathbf{\Phi}(H)$ with $H \in \mathscr{W}(L_i)$;
- (4.6) follows from [6, Theorem 2.4 (2.1)] and [6, Corollary 2.2 (ii)];
- (4.7) follows from [6, Theorem 2.4 (2.1)];
- (4.8) follows summing [6, Theorem 2.4 (2.2)] applied to $\mathbf{\Phi}(H)$ with $H \in \mathscr{W}(L_i)$.

The last ingredient for the completion of the proof of Proposition 3.5 are the following three key estimates which are derived from the analysis of the construction of the center manifold in [6].

Lemma 4.5. *Under the assumptions of Proposition 3.5, it holds*

$$\sum_i \Big(\inf_{\mathcal{B}^i} \varphi_r\Big)\, \boldsymbol{m}_0\, \ell_i^{m+2+\gamma_2/4} \leq C_0 \mathbf{D}(r), \tag{4.9}$$

$$\sum_i \boldsymbol{m}_0\, \ell_i^{m+2+\gamma_2/4} \leq C_0 \left(\mathbf{D}(r) + r\mathbf{D}'(r)\right), \tag{4.10}$$

for some geometric constant C_0. Moreover, for every $t > 0$ there exists $C_0 > 0$ and $a > 0$ such that, for $C(t) = C^t$ and $\gamma(t) = at$ we have

$$\sup_i \boldsymbol{m}_0^t \Big[{\ell_i}^t + \Big(\inf_{\mathcal{B}^i} \varphi_r\Big)^{t/2} \ell_i^{t/2}\Big] \leq C(t)\mathbf{D}(r)^{\gamma(t)}. \tag{4.11}$$

Proof. Recall that, from [6, Propositions 3.1 and 3.4] and (4.2) we have, for some geometric positive constant c_0

$$\int_{\mathcal{B}^i} \varphi_r |N|^2 \geq c_0\, \boldsymbol{m}_0^{1/m} \inf_{\mathcal{B}^i} \varphi_r \ell_i^{m+2+2\beta_2} \geq c_0 \boldsymbol{m}_0^{1/a} \Big[\ell_i^2 + \Big(\inf_{\mathcal{B}^i} \varphi_r\Big)\ell_i\Big]^{1/(2a)} \quad \text{if } L_i \in \mathscr{W}_h, \tag{4.12}$$

$$\int_{\mathcal{B}^i} \varphi_r |DN|^2 \geq c_0\, \boldsymbol{m}_0 \inf_{\mathcal{B}^i} \varphi_r \ell_i^{m+2-2\delta_2} \geq c_0 \boldsymbol{m}_0^{1/a} \Big[\ell_i^2 + \Big(\inf_{\mathcal{B}^i} \varphi_r\Big)\ell_i\Big]^{1/(2a)} \quad \text{if } L_i \in \mathscr{W}_e \tag{4.13}$$

where we just need $a \leq \min\{1/(2(m+2+2\beta_2)), 1/(2(m+2-2\delta_2))\}$ (note that (4.12) follows from [6, Proposition 3.1 (S3)] because $s(L_i) \leq (\sqrt{m}+1)\ell(L_i)$ for $L_i \in \mathscr{W}_h$). Therefore, by Lemma 3.6, (4.2), (4.12) and (4.13), it follows easily that,

$$2^{-t} c_0^{at} \boldsymbol{m}_0^t \Big[\ell_i^t + \Big(\inf_{\mathcal{B}^i} \varphi_r\Big)^{t/2} \ell_i^{t/2}\Big] \leq \left(\int_{\mathcal{B}^i} \varphi_r |DN|^2\right)^{at} + \left(\int_{\mathcal{B}^i} \varphi_r |N|^2\right)^{at}$$
$$\leq 2^t \left(\int_{\mathcal{B}^i} \varphi_r (|DN|^2 + |N|^2)\right)^{at} \stackrel{(3.18)\,\&\,\mathbf{I}\geq 1}{\leq} C_0^t \mathbf{D}(r)^{at}\,.$$

Taking the supremum over i we achieve (4.11). Next, (4.9) follows similarly because the $\mathcal{B}^i$ are disjoint and $8\,\beta_2 < \gamma_2$:

$$\sum_i \big(\inf_{\mathcal{B}^i} \varphi_r\big) \boldsymbol{m}_0\, \ell_i^{m+2+\gamma_2/4} \leq C \sum_i \int_{\mathcal{B}^i} \varphi_r (|DN|^2 + |N|^2) \stackrel{(3.18)\,\&\,\mathbf{I}\geq 1}{\leq} C\mathbf{D}(r)\,.$$

Finally, arguing as above we conclude that

$$\sum_i \boldsymbol{m}_0\, \ell_i^{m+2+\gamma_2/4} \leq C \int_{\mathcal{B}_r(q)} \big(|DN|^2 + |N|^2\big) \stackrel{(3.17)\,\&\,(3.18)}{\leq} C\big(\mathbf{D}(r) + r\mathbf{D}'(r)\big). \tag{4.14}$$

and, hence, (4.10) follows from Lemma 3.6. □

4.3. **Proof of Proposition 3.5:** (3.6) **and** (3.7)**.** We can now pass to estimate the errors terms in (3.6) and (3.7) in order to conclude the proof of Proposition 3.5. Unless otherwise specified, the constants denoted by C will be assumed to be geometric (i.e. to depend only upon the parameters introduced in [6]).

Errors of type 1. By [6, Theorem 1.12], the map $\boldsymbol{\varphi}$ defining the center manifold satisfies $\|D\boldsymbol{\varphi}\|_{C^{2,\kappa}} \leq C\,\boldsymbol{m}_0^{1/2}$, which in turn implies $\|H_{\mathcal{M}}\|_{L^\infty} + \|DH_{\mathcal{M}}\|_{L^\infty} \leq C\,\boldsymbol{m}_0^{1/2}$ (recall that $H_{\mathcal{M}}$ denotes the mean curvature of $\mathcal{M}$). Therefore, by (4.3), (4.4), (4.9) and (4.11), we get

$$|\mathrm{Err}_1^o| \leq C_0 \int_{\mathcal{M}} \varphi_r\, |H_{\mathcal{M}}|\, |\boldsymbol{\eta} \circ N|$$
$$\leq C_0\, \boldsymbol{m}_0^{1/2} \sum_j \Big(\big(\sup_{\mathcal{U}_i} \varphi_r\big)\, \boldsymbol{m}_0\, \ell_j^{2+m+\gamma_2} + C_0 \int_{\mathcal{U}_j} \varphi_r |N|^{2+\gamma_2}\Big)$$
$$\leq C\mathbf{D}(r)^{1+\gamma_3} + C \sum_j \boldsymbol{m}_0^{1/2}\, \ell_j^{\gamma_2(1+\beta_2)} \int_{\mathcal{U}_j} \varphi_r |N|^2 \leq C(\gamma_3)\mathbf{D}(r)^{1+\gamma_3}\,,$$

provided $\gamma_3 > 0$ is sufficiently small depending only upon m, β_2, δ_2 and γ_2. Analogously

$$\big|\mathrm{Err}_1^i\big| \leq C\, r^{-1} \int_{\mathcal{M}} \big(|H_{\mathcal{M}}| + |D_Y H_{\mathcal{M}}|\big) |\boldsymbol{\eta} \circ N|$$
$$\leq C\, r^{-1} \boldsymbol{m}_0^{1/2} \sum_j \Big(\boldsymbol{m}_0\, \ell_j^{2+m+\gamma_2} + C \int_{\mathcal{U}_j} |N|^{2+\gamma_2}\Big) \leq C(\gamma_3)\, r^{-1} \mathbf{D}(r)^{\gamma_3} \big(\mathbf{D}(r) + r\,\mathbf{D}'(r)\big).$$

Errors of type 2. From $\|A\|_{C^0} \leq C\|D\boldsymbol{\varphi}\|_{C^2} \leq C\boldsymbol{m}_0^{1/2} \leq C\varepsilon_3$, it follows that $\mathrm{Err}_2^o \leq C\varepsilon_3^2\boldsymbol{\Sigma}(r)$. Moreover, since $|DX_i| \leq Cr^{-1}$, Lemma 3.6 gives

$$\left|\mathrm{Err}_2^i\right| \leq Cr^{-1}\int_{\mathcal{B}_r(p_0)} |N|^2 + C\int \varphi_r|N||DN| \leq C\mathbf{D}(r)\,.$$

Errors of type 3. Clearly, we have

$$|\mathrm{Err}_3^o| \leq \underbrace{\int \varphi_r\left(|DN|^2|N| + |DN|^4\right)}_{I_1} + \underbrace{C\,r^{-1}\int_{\mathcal{B}_r(q)} |DN|^3|N|}_{I_2} + \underbrace{C\,r^{-1}\int_{\mathcal{B}_r(q)} |DN||N|^2}_{I_3}\,.$$

We estimate separately the three terms (recall that $\gamma_2 > 4\delta_2$):

$$\begin{aligned}
I_1 &\leq \int_{\mathcal{B}_r(p_0)} \varphi_r(|N|^2|DN| + |DN|^3) \leq I_3 + C\sum_j \sup_{\mathcal{U}_j}\varphi_r\boldsymbol{m}_0^{1+\gamma_2}\ell_j^{m+2+\gamma_2/2}\\
&\stackrel{(4.9)\,\&\,(4.11)}{\leq} I_3 + C(\gamma_3)\mathbf{D}(r)^{1+\gamma_3},\\
I_2 &\leq Cr^{-1}\sum_j \boldsymbol{m}_0^{1+1/2m+\gamma_2}\ell_j^{m+3+\beta_2+\gamma_2/2} \stackrel{(4.3)}{\leq} C\sum_j \boldsymbol{m}_0^{1+1/2m+\gamma_2}\ell_j^{m+2+\beta_2+\gamma_2/2}\inf_{\mathcal{B}^j}\varphi_r\\
&\stackrel{(4.9)\,\&\,(4.11)}{\leq} C(\gamma_3)\mathbf{D}(r)^{1+\gamma_3},\\
I_3 &\leq Cr^{-1}\sum_j \boldsymbol{m}_0^{\gamma_2}\ell_j^{\gamma_2}\int_{\mathcal{U}_j}|N|^2 \stackrel{(4.11)}{\leq} C\gamma_3)r^{-1}\mathbf{D}(r)^{\gamma_3}\int_{\mathcal{B}_r(q)}|N|^2 \stackrel{(3.18)}{\leq} C(\gamma_3)\mathbf{D}(r)^{1+\gamma_3}\,,
\end{aligned}$$

provided $\gamma_3 > 0$ is sufficiently small. For what concerns the inner variations, we have

$$|\mathrm{Err}_3^i| \leq C\int_{\mathcal{B}_r(q)}\left(r^{-1}|DN|^3 + r^{-1}|DN|^2|N| + r^{-1}|DN||N|^2\right).$$

The last integrand corresponds to I_3, while the remaining part can be estimated as follows:

$$\begin{aligned}
\int_{\mathcal{B}_r(q)} r^{-1}(|DN|^3 + |DN|^2|N|) &\leq C\sum_j r^{-1}(\boldsymbol{m}_0^{\gamma_2}\ell_j^{\gamma_2} + \boldsymbol{m}_0^{1/2m}\ell_j^{1+\beta_2})\int_{\mathcal{U}_j}|DN|^2\\
&\stackrel{(4.11)}{\leq} C(\gamma_3)\,r^{-1}\mathbf{D}(r)^{\gamma_3}\int_{\mathcal{B}_r(q)}|DN|^2\\
&\leq C(\gamma_3)\mathbf{D}(r)^{\gamma_3}\left(\mathbf{D}'(r) + r^{-1}\mathbf{D}(r)\right)\,.
\end{aligned}$$

Errors of type 4. We compute explicitly

$$|DX_o(p)| \leq 2\,|p - \mathbf{p}(p)|\,\frac{|Dd(\mathbf{p}(p),q)|}{r} + \varphi_r(p)\,|D(p-\mathbf{p}(p))| \leq C\,\left(\frac{|p-\mathbf{p}(p)|}{r} + \varphi_r(p)\right)\,.$$

It follows readily from (3.19), (4.6) and (4.8) that

$$|\mathrm{Err}_4^o| \leq \sum_i C\Big(r^{-1}\boldsymbol{m}_0^{1/2m}\ell_i^{1+\beta_2} + \sup_{\mathcal{U}_i}\varphi_r\Big)\boldsymbol{m}_0^{1+\gamma_2}\ell_i^{m+2+\gamma_2}$$

$$\stackrel{(4.2)\,\&\,(4.3)}{\leq} C\sum_i \left[\boldsymbol{m}_0^{\gamma_2}\ell_i^{\gamma_2/4}\right]\inf_{\mathcal{B}_i}\varphi_r\,\boldsymbol{m}_0\,\ell_i^{m+2+\gamma_2/4} \stackrel{(4.9)\,\&(4.11)}{\leq} C(\gamma_3)\mathbf{D}(r)^{1+\gamma_3}. \tag{4.15}$$

Similarly, since $|DX_i| \leq Cr^{-1}$, we get

$$\mathrm{Err}_4^i \leq Cr^{-1}\sum_j \left(\boldsymbol{m}_0^{\gamma_2}\ell_j^{\gamma_2/2}\right)\boldsymbol{m}_0\,\ell_j^{m+2+\gamma_2/2} \stackrel{(4.10)\,\&\,(4.11)}{\leq} C(\gamma_3)\mathbf{D}(r)^{\gamma_3}\left(\mathbf{D}'(r)+r^{-1}\mathbf{D}(r)\right)\,.$$

Errors of type 5. Integrating by part Err_5, we get

$$\mathrm{Err}_5 = \left|\int\langle X^\perp, h(\vec T(p))\rangle d\|T\|\right| \leq \underbrace{\left|\int\langle X^\perp, h(\vec{\mathbf{T}}_F(p))\rangle d\|\mathbf{T}_F\|\right|}_{I_2}$$
$$+\underbrace{\int_{\mathrm{spt}(T)\setminus\mathrm{Im}(F)}|X^\perp||h(\vec T(p))|d\|T\| + \int_{\mathrm{Im}(F)\setminus\mathrm{spt}(T)}|X^\perp||h(\vec{\mathbf{T}}_F(p)|d\|\mathbf{T}_F\|}_{I_1},$$

where $h(\vec\lambda)$ is the trace of A_Σ on the m-vector $\vec\lambda$, i.e. $h(\vec\lambda) := \sum_{k=1}^m A_\Sigma(v_k,v_k)$ with $v_1,\ldots,v_m$ orthonormal vectors such that $v_1\wedge\ldots\wedge v_m = \vec\lambda$.

Since $|X|\leq C$, I_1 can be easily estimated as Err_4:

$$I_2 \leq C\sum_j (\sup_{\mathcal{U}_i}\varphi_r)\boldsymbol{m}_0^{1+\gamma_2}\,\ell_j^{m+2+\gamma_2} \leq C(\gamma_3)\mathbf{D}^{1+\gamma_3}(r).$$

For what concerns I_2, we argue differently for the outer and the inner variations. For Err_5^o, observe that $|X^{o\perp}(p)| = \varphi_r(\mathbf{p}(p))|\mathbf{p}_{T_p\Sigma^\perp}(p-\mathbf{p}(p))|$. On the other hand, we also have

$$|\mathbf{p}_{T_p\Sigma^\perp}(p-\mathbf{p}(p))| \leq C\mathbf{c}(\Sigma)|p-\mathbf{p}(p)|^2 \leq C\boldsymbol{m}_0^{1/2}|p-\mathbf{p}(p)|^2 \quad \forall\, p\in\Sigma.$$

Therefore, we can estimate

$$I_2^o \leq C\,\boldsymbol{m}_0\int\varphi_r|N|^2 \leq C\,\varepsilon_3^2\mathbf{\Sigma}(r).$$

For the inner variations, denote by $\nu_1,\ldots,\nu_l$ an orthonormal frame for $T_p\Sigma^\perp$ of class C^{2,ε_0} (cf. [5, Appendix A]) and set $h_p^j(\vec\lambda) := -\sum_{k=1}^m\langle D_{v_k}\nu_j(p), v_k\rangle$ whenever $v_1\wedge\ldots\wedge v_m = \vec\lambda$ is an m-vector of $T_p\Sigma$ (with $v_1,\ldots,v_m$ orthonormal). For the sake of simplicity, we write

$$h_p^j := h_p^j(\vec{\mathbf{T}}_F(p)) \quad\text{and}\quad h_p = \sum_{j=1}^l h_p^j\nu_j(p),$$

$$h_{\mathbf{p}(p)}^j := h_{\mathbf{p}(p)}^j(\vec{\mathcal{M}}(\mathbf{p}(p))) \quad\text{and}\quad h_{\mathbf{p}(p)} = \sum_{j=1}^l h_{\mathbf{p}(p)}^j\nu_j(\mathbf{p}(p)).$$

Consider the exponential map $\mathbf{ex}_{\mathbf{p}(p)} : T_{\mathbf{p}(p)}\Sigma\to\Sigma$ and its inverse $\mathbf{ex}_{\mathbf{p}(p)}^{-1}$. Recall that:

- the geodesic distance $d_\Sigma(p,q)$ is comparable to $|p-q|$ up to a constant factor;
- ν_j is C^{2,ε_0} and $\|D\nu_j\|_{C^{1,\varepsilon_0}} \leq C\boldsymbol{m}_0^{1/2}$;

- $\mathbf{ex}_{\mathbf{p}(p)}$ and $\mathbf{ex}^{-1}_{\mathbf{p}(p)}$ are both C^{2,ε_0} and $\|\mathrm{d}\,\mathbf{ex}_{\mathbf{p}(p)}\|_{C^{1,\varepsilon_0}} + \|\mathrm{d}\,\mathbf{ex}^{-1}_{\mathbf{p}(p)}\|_{C^{1,\varepsilon_0}} \leq \boldsymbol{m}_0^{1/2}$;
- $|h^j_p| \leq C\|A_\Sigma\|_{C^0} \leq C\boldsymbol{m}_0^{1/2}$;

where all the constants involved are just geometric. We then conclude that

$$\begin{aligned} h_p - h_{\mathbf{p}(p)} &= \sum_j \nu_j(p)(h^j_p - h^j_{\mathbf{p}(p)}) + \sum_j \big(\nu_j(p) - \nu_j(\mathbf{p}(p))\big) h^j_{\mathbf{p}(p)} \\ &= \sum_j \nu_j(p)(h^j_p - h^j_{\mathbf{p}(p)}) + \sum_j D\nu_j(\mathbf{p}(p)) \cdot \mathbf{ex}^{-1}_{\mathbf{p}(p)}(p)\, h^j_{\mathbf{p}(p)} + O(|p-\mathbf{p}(p)|^2). \end{aligned} \tag{4.16}$$

On the other hand, $X_i(p) = Y(\mathbf{p}(p))$ is tangent to $\mathcal{M}$ in $\mathbf{p}(p)$ and hence orthogonal to $h_{\mathbf{p}(p)}$. Thus

$$\begin{aligned} \langle X_i(p), h_p\rangle = \langle X^i(p), (h_p - h_{\mathbf{p}(p)})\rangle &= \sum_j \langle X_i(\mathbf{p}(p)), D\nu_j(\mathbf{p}(p)) \cdot \mathbf{ex}^{-1}_{\mathbf{p}(p)}(p)\rangle h^j_{\mathbf{p}(p)} \\ &\quad + \sum_j \langle \nu_j(p), X_i(p)\rangle \big(h^j_p - h^j_{\mathbf{p}(p)}\big) + O\left(|p-\mathbf{p}(p)|^2\right) \\ = \sum_j \langle X_i(\mathbf{p}(p)), D\nu_j(\mathbf{p}(p)) &\cdot \mathbf{ex}^{-1}_{\mathbf{p}(p)}(p)\rangle h^j_{\mathbf{p}(p)} \\ &+ O\left(|\vec{\mathbf{T}}_F(p) - \vec{\mathcal{M}}(\mathbf{p}(p))||p-\mathbf{p}(p)| + |p-\mathbf{p}(p)|^2\right), \end{aligned} \tag{4.17}$$

where we used elementary calculus to infer that $|\langle X^i(p), \nu_j(p)\rangle| \leq C|p-\mathbf{p}(p)|$ and

$$|h^j_p - h^j_{\mathbf{p}(p)}| \leq C\left(|\vec{\mathbf{T}}_F(p) - \vec{\mathcal{M}}(\mathbf{p}(p)| + |p-\mathbf{p}(p)|\right).$$

We only need that the constants C appearing in the above inequalities are bounded by a geometric factor: in fact they enjoy explicit bounds in terms of $\boldsymbol{m}_0^{1/2}$ which are at least linear, but such degree of precision is not needed. Finally recalling that $p \in \mathrm{spt}(\mathbf{T}_F)$, we can bound $|p - \mathbf{p}(p)| \leq |N(p)|$ and $|\vec{\mathbf{T}}_F(p) - \vec{\mathcal{M}}(\mathbf{p}(p))| \leq C|DN(\mathbf{p}(p))|$. We therefore conclude the estimate

$$\langle X_i(p), h_p\rangle = \sum_j \langle X_i(\mathbf{p}(p)), D\nu_j(\mathbf{p}(p)) \cdot \mathbf{ex}^{-1}_{\mathbf{p}(p)}(p)\rangle h^j_{\mathbf{p}(p)} + O\big(|N|^2(\mathbf{p}(p)) + |DN|^2(\mathbf{p}(p))\big).$$

We combine it with the expansion of the area functional in [5, Theorem 3.2] to conclude the estimate on I^i_2. Recalling that $\mathbf{p}(F_i(x)) = x$ we get

$$\begin{aligned} I^i_2 &= \left|\int \langle X_i, h_p\rangle d\|\mathbf{T}_F\|\right| = \left|\sum_{i=1}^Q \int_{\mathcal{M}} \langle Y, h_{F_i}\rangle \mathbf{J}F_i\right| \\ &\overset{(4.17)}{\leq} \left|\int_{\mathcal{M}} \sum_{j=1}^l \sum_{i=1}^Q \langle Y(x), D\nu_j(x) \cdot \mathbf{ex}^{-1}_x(F_i(x))\rangle h^j_x d\mathcal{H}^m(x)\right| + C\int_{\mathcal{M}} \varphi_r(|N|^2 + |DN|^2) \end{aligned}$$

Using the Taylor expansion for $\mathbf{ex}_x^{-1}$ at x (and recalling that $F_i(x)-x = N_i(x)$) we conclude

$$\Big|\sum_{i=1}^Q \mathbf{ex}_x^{-1}(F_i(x))\Big| \leq \big|\mathrm{d}\,\mathbf{ex}_x^{-1}(\boldsymbol{\eta}\circ N(x))\big| + O(|N|^2) \leq C|\boldsymbol{\eta}\circ N(x)| + C|N|^2\,.$$

Next consider that $|\langle Y, D\nu_j\cdot v\rangle| \leq C\varphi_r \|A_\Sigma\|_{C^0}|v| \leq C\varphi_r\, \boldsymbol{m}_0^{1/2}|v|$ for every tangent vector v and $|h^j_x| \leq C\|A_\Sigma\|_{C^0} \leq \boldsymbol{m}_0^{1/2}$. We thus conclude with the estimate

$$I_2^i \leq C\,\boldsymbol{m}_0 \int_{\mathcal{M}} \varphi_r\,|\boldsymbol{\eta}\circ N| + C\int_{\mathcal{M}} \varphi_r(|N|^2 + |DN|^2) =: J_1 + J_2\,.$$

Clearly J_1 can be estimated as Err_1^i and J_2 as Err_2^i, thus concluding the proof.

5. Boundedness of the frequency

In this section we prove that the frequency function $\mathbf{I}_j$ remains bounded along the different center manifolds corresponding to the intervals of flattening. To simplify the notation, we set $p_j := \boldsymbol{\Phi}_j(0)$ and write simply $\mathcal{B}_\rho$ in place of $\mathcal{B}_\rho(p_j)$.

Theorem 5.1 (Boundedness of the frequency functions)**.** *Let T be as in Assumption 2.1. If the intervals of flattening are $j_0 < \infty$, then there is $\rho > 0$ such that*

$$\mathbf{H}_{j_0} > 0 \ \textit{on}\ ]0,\rho[\quad \textit{and} \quad \limsup_{r\to 0} \mathbf{I}_{j_0}(r) < \infty\,. \tag{5.1}$$

If the intervals of flattening are infinitely many, then there is a number $j_0 \in \mathbb{N}$ and a geometric constant $j_1 \in \mathbb{N}$ such that

$$\mathbf{H}_j > 0 \ \textit{on}\ ]\tfrac{s_j}{t_j}, 2^{-j_1}3[\ \textit{for all}\ j \geq j_0\,, \qquad \sup_{j\geq j_0}\ \sup_{r\in]\frac{s_j}{t_j},2^{-j_1}3[} \mathbf{I}_j(r) < \infty\,, \tag{5.2}$$

$$\sup\left\{\min\left\{\mathbf{I}_j(r), \frac{r^2\int_{\mathcal{B}_r}|DN_j|^2}{\int_{\mathcal{B}_r}|N_j|^2}\right\} : j\geq j_0 \ \textit{and}\ \max\left\{\frac{s_j}{t_j}, \frac{3}{2^{j_1}}\right\} \leq r < 3\right\} < \infty \tag{5.3}$$

(in the latter inequality we understand $\mathbf{I}_j(r) = \infty$ when $\mathbf{H}_j(r) = 0$)

Proof. Consider the first alternative. We claim that for every $r > 0$ there is a radius $0 < \rho < r$ such that $\mathbf{H}(\rho) = \mathbf{H}_{j_0}(\rho) > 0$. Otherwise N_{j_0} vanishes identically on some $\mathcal{B}_r$. By [6, Propositions 3.1 and 3.4] and Proposition 2.2(iii) this is possible only if no cube of the Whitney decomposition $\mathscr{W}^{(j_0)}$ intersects the projection of $\mathcal{B}_r$ onto the plane π (the reference plane for the construction of the center manifold). But then T_{j_0} would coincide with $Q\,[\![\mathcal{M}]\!]$ in $\mathbf{B}_{3r/4}$ and 0 would be a regular point of T_{j_0} and, therefore, of T.

Next we claim that $\mathbf{H}(r) > 0$ for every $r \leq \rho$. If not, let r_0 be the largest zero of $\mathbf{H}$ which is smaller than ρ. By Theorem 3.2, there is a constant C such that $\mathbf{I}(r) \leq C(1+\mathbf{I}(\rho))$ for every $r \in]r_0,\rho[$. By letting $r \downarrow r_0$, we then conclude

$$r_0\mathbf{D}(r_0) \leq C(1+\mathbf{I}(\rho))\mathbf{H}(r_0) = 0,$$

that is, $N_j|_{\mathcal{B}_{r_0}} \equiv 0$ which we have already excluded. Therefore, since $\mathbf{H} > 0$ on $]0,\rho[$, we can now apply Theorem 3.2 to conclude (5.1).

In the second case, we partition the extrema t_j of the intervals of flattening into two different classes: the class (A) when $t_j = s_{j-1}$ and the class (B) when $t_j < s_{j-1}$. If t_j belongs to (A), set $r := \frac{s_{j-1}}{t_{j-1}}$. Let $L \in \mathscr{W}^{(j-1)}$ be a cube of the Whitney decomposition such that $c_s\, r \leq \ell(L)$ and $L \cap \bar{B}_r(0,\pi) \neq \emptyset$. We are in the position to apply [6, Proposition 3.7] for the comparison of two center manifolds: there exists a constant $\bar{c}_s > 0$ such that

$$\int_{\mathbf{B}_2\cap\mathcal{M}_j} |N_j|^2 \geq \bar{c}_s \boldsymbol{m}_0^j := \bar{c}_s \max\left\{\mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}}), \mathbf{c}(\Sigma_j)^2\right\},$$

which obviously gives $\int_{\mathcal{B}_3} |N_j|^2 \geq c\boldsymbol{m}_0^j$. By [6, (2.7)] (or alternatively by (3.4)), we then conclude

$$\int_{\mathcal{B}_3} |N_j|^2 \geq \bar{c} \int_{\mathcal{B}_3} |DN_j|^2\,, \tag{5.4}$$

where $\bar{c}$ is a positive geometric constant By the Hölder inequality and Sobolev embedding (cf. [3, Proposition 2.11]), there are geometric constants C_0 and $\bar{\alpha} = m(1-\frac{2}{q}) > 0$ such that

$$\begin{aligned}\int_{\mathcal{B}_{\frac{3}{2^J}}} |N_j|^2 \leq \left(\mathcal{H}^m\left(\mathcal{B}_{\frac{3}{2^J}}\right)\right)^{1-2/q} \left(\int_{\frac{3}{2^J}} |N_j|^q\right)^{2/q} &\leq C_0 2^{-J\bar{\alpha}} \int_{\mathcal{B}_3} |N_j|^2 + C_0 2^{-J\bar{\alpha}} \int_{\mathcal{B}_3} |DN_j|^2\\ \leq C_0 2^{-J\bar{\alpha}} \bar{c}^{-1} \int_{\mathcal{B}_3} |N_j|^2 \qquad &\text{for any } J \in \mathbb{N}\end{aligned} \tag{5.5}$$

(in the above we can set $q = 2^\star$ when $m \geq 3$ and choose any $q < \infty$ larger than 2 for $m = 2$; note also that since the curvature of the manifold $\mathcal{M}_j$ is bounded by $\boldsymbol{m}_0^j$, we can assume that $\mathcal{H}^m(\mathcal{B}_\rho)$ is comparable to the m-dimensional volume of the corresponding euclidean ball for every $\rho < 3$). If we choose $J = j_1$ for a large enough j_1 (depending only upon $\bar{c}$, α and C_0) we achieve

$$\int_{\mathcal{B}_3\setminus\mathcal{B}_{\frac{3}{2^{j_1}}}} |N_j|^2 \geq \frac{1}{2}\int_{\mathcal{B}_3} |N_j|^2 \geq \frac{\bar{c}}{2}\int_{\mathcal{B}_3} |DN_j|^2\,. \tag{5.6}$$

In turn we conclude the existence one annulus $\mathcal{A}(k(j)) := \mathcal{B}_{3/(2^{k(j)})} \setminus \mathcal{B}_{3/(2^{k(j)+1})}$ with

$$\int_{\mathcal{A}(k(j))} |N_j|^2 \geq \frac{\bar{c}}{2j_1}\int_{\mathcal{B}_3} |DN_j|^2 \qquad \text{and} \qquad k(j) \leq j_1\,. \tag{5.7}$$

$\mathbf{H}_{N_j}(k(j))$ is bounded from below by the integral on the left hand side of (5.7), whereas the right hand side bounds $\mathbf{D}_{N_j}(2^{-k(j)}3)$ from above. Thus $\mathbf{I}_{N_j}(2^{-k(j)}3)$ is smaller than a constant which depends upon $\bar{c}$ and j_1. Arguing as in the first alternative, we can apply Theorem 3.2 to conclude the positivity of $\mathbf{H}_{N_j}$ and to gain a uniform upper bound for $\mathbf{I}_{N_j}$ on the interval $]\frac{s_j}{t_j}, 2^{-k(j)}3[$: since the latter contains $]\frac{s_j}{t_j}, 2^{j_1}3[$, we conclude the validity of (5.2) (if one or both the intervals are trivial, namely $\frac{s_j}{t_j}$ is larger than the right endpoint, then there is nothing to prove). On the other hand for every $r \in [2^{-k(j)}3, 3[$, by (5.7) we certainly have

$$\int_{\mathcal{B}_r} |N_j|^2 \geq \frac{\bar{c}}{2j_1} \int_{\mathcal{B}_3} |DN_j|^2$$

from which (5.3) readily follows.

In the case t_j belongs to the class (B), then, by construction there is $\eta_j \in]0,1[$ such that $\mathbf{E}((\iota_{0,t_j})_\sharp T, \mathbf{B}_{6\sqrt{m}(1+\eta_j)}) > \varepsilon_3^2$. Up to extraction of a subsequence, we can assume that $(\iota_{0,t_j})_\sharp T$ converges to a cone S: the convergence is strong enough to conclude that the excess of the cone is the limit of the excesses of the sequence. Moreover (since S is a cone), the excess $\mathbf{E}(S, \mathbf{B}_r)$ is independent of r. We then conclude

$$\varepsilon_3^2 \leq \liminf_{j\to\infty, j\in(B)} \mathbf{E}(T_j, \mathbf{B}_3)\,.$$

Thus, by Lemma 5.2 below, we conclude $\liminf_{j\to\infty, j\in(B)} \mathbf{H}_{N_j}(3) > 0$. Since $\mathbf{D}_{N_j}(3) \leq C\boldsymbol{m}_0^j \leq C\varepsilon_3^2$, we achieve that $\limsup_{j\to\infty, j\in(B)} \mathbf{I}_{N_j}(3) < +\infty$, and conclude as before. □

Lemma 5.2. *Assume the intervals of flattening are infinitely many and* $r_j \in]\frac{s_j}{t_j}, 3[$ *is a subsequence (not relabeled) with* $\lim_j \|N_j\|_{L^2(\mathcal{B}_{r_j}\setminus\mathcal{B}_{r_j/2})} = 0$. *If* ε_3 *is sufficiently small, then,* $\mathbf{E}(T_j, \mathbf{B}_{r_j}) \to 0$.

Proof. Note that, if $r_j \to 0$, then necessarily $\mathbf{E}(T_j, \mathbf{B}_{r_j}) \to 0$ by Proposition 2.2(iv). Therefore, up to a subsequence, we can assume the existence of $c > 0$ such that

$$r_j \geq c \quad \text{and} \quad \mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}}) \geq c. \tag{5.8}$$

After the extraction of a further subsequence, we can assume the existence of r such that

$$\int_{\mathcal{B}_r\setminus\mathcal{B}_{\frac{3r}{4}}} |N_j|^2 \to 0, \tag{5.9}$$

and the existence of an area-minimizing cone S such that $(\iota_{0,t_j})_\sharp T \to S$. Note that, by (5.8), S is not a multiplicity Q flat m-plane. Consider the orthogonal projection $\mathbf{q}_j : \mathbb{R}^{m+n} \to \pi_j$, where π_j is the m-dimensional plane of the construction of the center manifold $\mathcal{M}_j$. Assuming ε_3 is sufficiently small, we have $U_j := B_{\frac{15}{16}r}(\pi) \setminus B_{\frac{13}{16}r}(\pi) \subset \mathbf{q}_j(\mathcal{B}_r \setminus \mathcal{B}_{\frac{3}{4}r})$. Consider the Whitney decomposition $\mathscr{W}^{(j)}$ leading to the construction of $\mathcal{M}_j$: if no cube of the decomposition iintersects U^j, then N_j vanishes identically on it. Otherwise, set

$$d_j := \max\left\{\ell(J) : J \in \mathscr{W}^{(j)} \quad \text{and} \quad J \cap U_j \neq \emptyset\right\}.$$

Let $J_j \in \mathscr{W}^{(j)}$ be such that $U_j \cap J_j \neq \emptyset$ and $d_j = \ell(J_j)$. If the stopping condition for J_j is either (HT) or (EX), recalling that $\ell(J_j) \leq c_s r$, we choose a ball $B^j \subset U_j$ of radius $\frac{d_j}{2}$ and at distance at most $\sqrt{m}d_j$ from J_j. If the stopping condition for J_j is (NN), J_j is in the domain of influence of $K_j \in \mathscr{W}_e^{(j)}$. By Proposition 2.2 we can then choose a ball $B^j \subset U_j$ of radius $\frac{\ell(K_j)}{8}$ at distance at most $3\sqrt{m}\ell(K_j)$ from K_j. If the stopping condition is (HT), we then have by [6, Proposition 3.1]

$$\int_{\mathcal{B}_r\setminus\mathcal{B}_{\frac{3r}{4}}} |N_j|^2 \geq \int_{\mathbf{\Phi}_j(B^j)} |N_j|^2 \geq c\left(\boldsymbol{m}_0^j\right)^{\frac{1}{m}} d_j^{m+2+2\beta_2}\,.$$

If the stopping condition is either (NN) or (EX), by [6, Proposition 3.1] and [6, Proposition 3.4] we have

$$\int_{\mathcal{B}_r\setminus\mathcal{B}_{\frac{3r}{4}}} |N_j|^2 \geq \int_{\mathbf{\Phi}_j(B^j)} |N_j|^2 \geq c d_j^2 \int_{\mathbf{\Phi}_j(B^j)} |DN_j|^2 \geq c\boldsymbol{m}_0^j d_j^{m+4-2\delta_2}. \tag{5.10}$$

In both cases we conclude that $d_j \to 0$.

By [6, Corollary 2.2], $\mathrm{spt}(T_j)\cap\mathbf{\Phi}_j(U_j)$ is contained in a d_j-tubular neighborhood of $\mathcal{M}_j$, which we denote by $\hat{\mathbf{U}}_j$. Moreover, again assuming that ε_3 is sufficiently small, we can assume $\mathbf{B}_t\setminus\mathbf{B}_s\cap\mathcal{M}_j\subset\mathbf{\Phi}_j(U_j)$ for some appropriate choice of $s<t$, independent of j. Finally, by [6, Theorem 1.17] we can assume that (up to subsequences) $\mathcal{M}_j$ converges to $\mathcal{M}$ in C^3. We thus conclude that $S\llcorner(\mathbf{B}_t\setminus\bar{\mathbf{B}}_s)$ is supported in $\mathcal{M}\cap(\mathbf{B}_t\setminus\bar{\mathbf{B}}_s)$ and, hence, by the constancy theorem, $S\llcorner(\mathbf{B}_t\setminus\bar{\mathbf{B}}_s) = Q_0\,[\![\mathcal{M}\cap(\mathbf{B}_t\setminus\bar{\mathbf{B}}_s)]\!]$ for some integer Q_0. Observe also that, if $\mathbf{p}_j:\hat{\mathbf{U}}_j\to\mathcal{M}_j$ is the least distance projection onto $\mathcal{M}_j$, by [6, Theorem 2.4] we also have $(\mathbf{p}_j)_\sharp(T_j\llcorner(\mathbf{B}_t\setminus\bar{\mathbf{B}}_s)) = Q\,[\![\mathcal{M}_j\cap(\mathbf{B}_t\setminus\bar{\mathbf{B}}_s)]\!]$. We therefore conclude that $Q_0=Q$. Since S is a cone without boundary, $\partial(S\llcorner\mathbf{B}_t) = Q\,[\![\mathcal{M}\cap\partial\mathbf{B}_t]\!]$, i.e. $S\llcorner\mathbf{B}_t = Q\,[\![0]\!] \times\!\!\!\!\times\, [\![\mathcal{M}\cap\partial\mathbf{B}_t]\!]$. By Allard's regularity theorem (which can be applied because $\Theta(S,0)=\lim_j\Theta(T_j,0)=Q$), S is regular in a neighborhood of 0 and, therefore, it is an m-plane with multiplicity Q, which gives the desired contradiction. □

A corollary of Theorem 5.1 is the following.

Corollary 5.3 (Reverse Sobolev)**.** *Let T be as in Assumption 2.1. Then, there exists a constant $C>0$ which* depends on T but not on j *such that, for every j and for every $r\in]\frac{s_j}{t_j},1]$, there is $s\in]\frac{3}{2}r,3r]$ such that*

$$\int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))} |DN_j|^2 \leq \frac{C}{r^2}\int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))} |N_j|^2\,. \tag{5.11}$$

Proof. If the second alternative in Theorem 5.1 holds, if $r\geq 2^{-j_1}3$ and if $\mathbf{I}_j(3r)$ is larger than the ratio

$$\frac{(3r)^2\int_{\mathcal{B}_{3r}(\mathbf{\Phi}_j(0))}|DN_j|^2}{\int_{\mathcal{B}_{3r}(\mathbf{\Phi}_j(0))}|N_j|^2}\,,$$

then the claim follows from (5.3). Therefore, without loss of generality, we can assume that $\mathbf{I}_j(3r)$ is bounded by a constant $C^\star$, which depends on T but not on j.

We start observing that, by the Coarea Formula,

$$\mathbf{H}_j(3r) = \int_{\mathcal{B}_{3r}(\mathbf{\Phi}_j(0))\setminus\mathcal{B}_{3r/2}(\mathbf{\Phi}_j(0))} 2\frac{|N_j|^2}{d(p)} = 2\int_{3r/2}^{3r}\frac{1}{t}\int_{\partial\mathcal{B}_t(\mathbf{\Phi}_j(0))}|N_j|^2\,dt\,,$$

whereas, using Fubini,

$$\int_{\frac{3}{2}r}^{3r}\int_{\mathcal{B}_t(\mathbf{\Phi}_j(0))}|DN_j|^2\,dt=\int_{\mathcal{M}_j}|DN_j|^2(x)\int_{3r/2}^{3r}\mathbf{1}_{]|x|,\infty[}(t)\,dt\,d\mathcal{H}^m(x)=\frac{3}{2}r\mathbf{D}_j(3r)\,.$$

Since we are assuming that $\mathbf{I}_j(3r)\leq C^\star$

$$\int_{\frac{3}{2}r}^{3r}dt\int_{\mathcal{B}_t(\mathbf{\Phi}_j(0))}|DN_j|^2=\frac{3}{2}r\,\mathbf{D}_j(3r)\leq C^\star\,\mathbf{H}_j(3r)=C^\star\int_{\frac{3}{2}r}^{3r}dt\frac{1}{t}\int_{\partial\mathcal{B}_t(\mathbf{\Phi}_j(0))}|N_j|^2\,.$$

Therefore, there must be $s\in[\frac{3}{2}r,3r]$ such that

$$\int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))}|DN_j|^2\leq\frac{C^\star}{s}\int_{\partial\mathcal{B}_s(\mathbf{\Phi}_j(0))}|N_j|^2\,. \tag{5.12}$$

Fix now any $\sigma\in]s/2,s[$ and any point $x\in\partial\mathcal{B}_s(\Phi_j(0))$. Consider the geodesic line γ passing through x and $\Phi_j(0)$ and let $\hat\gamma$ be the arc on γ having one endpoint $\bar x$ in $\partial\mathcal{B}_\sigma(\Phi_j(0))$ and one endpoint equal to x. Using [3, Proposition 2.1(b)] and the fundamental theorem of calculus, we easily conclude

$$|N_j(x)|^2\leq|N_j(\bar x)|^2+2\int_{\hat\gamma}|DN_j||N_j|\,.$$

Integrating this inequality in x and recalling that $\sigma>s/2$ we then easily conclude

$$\int_{\partial\mathcal{B}_s(\mathbf{\Phi}_j(0))}|N_j|^2\leq C\int_{\partial\mathcal{B}_\sigma(\mathbf{\Phi}_j(0))}|N_j|^2+C\int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))\setminus\mathcal{B}_{s/2}(\mathbf{\Phi}_j(0))}|N_j||DN_j|\,,$$

where the constant C depends only on the curvature of $\mathcal{M}_j$, which is bounded independently of j. We further integrate in σ between $s/2$ and s to achieve

$$\begin{aligned}\frac{s}{2}\int_{\partial\mathcal{B}_s(\mathbf{\Phi}_j(0))}|N_j|^2&\leq C\int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))\setminus\mathcal{B}_{s/2}(\mathbf{\Phi}_j(0))}\big(|N_j|^2+s\,|N_j||DN_j|\big)\\&\leq\frac{s^2}{4C^\star}\int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))}|DN_j|^2+\bar C\int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))}|N_j|^2\,,\end{aligned} \tag{5.13}$$

where $C^\star$ is the constant in (5.12) and the constant $\bar C$ depends on the curvature of $\mathcal{M}_j$ and on $C^\star$. Combining (5.13) with (5.12) we easily conclude (5.11). □

6. Final blow-up sequence and capacitary argument

6.1. **Blow-up maps.** Let T be a current as in the Assumption 2.1. By Proposition 2.2 we can assume that for each radius r_k there is an interval of flattening $I_{j(k)}=]s_{j(k)},t_{j(k)}]$ containing r_k. We define next the sequence of "blow-up maps" which will lead to the proof of Almgren's partial regularity result Theorem 0.3. To this aim, for k large enough, we define $\bar s_k$ so that the radius $\frac{\bar s_k}{t_{j(k)}}\in\,\big]\frac{3}{2}\frac{r_k}{t_{j(k)}},3\frac{r_k}{t_{j(k)}}\big[$ is the radius provided in Corollary 5.3 applied to $r=\frac{r_k}{t_{j(k)}}$. We then set $\bar r_k:=\frac{2\bar s_k}{3t_{j(k)}}$ and rescale and translate currents and maps accordingly:

(BU1) $\bar{T}_k = (\iota_{0,\bar{r}_k})_\sharp T_{j(k)} = ((\iota_{0,\bar{r}_k t_{j(k)}})_\sharp T)\llcorner \mathbf{B}_{6\sqrt{m}/\bar{r}_k}$, $\bar{\Sigma}_k = \iota_{0,\bar{r}_k}(\Sigma_{j(k)})$ and $\bar{\mathcal{M}}_k := \iota_{0,\bar{r}_k}(\mathcal{M}_{j(k)})$;

(BU2) $\bar{N}_k : \bar{\mathcal{M}}_k \to \mathbb{R}^{m+n}$ are the rescaled $\bar{\mathcal{M}}_k$-normal approximations given by

$$\bar{N}_k(p) = \frac{1}{\bar{r}_k} N_{j(k)}(\bar{r}_k p). \tag{6.1}$$

Since by assumption $T_0\Sigma = \mathbb{R}^{m+\bar{n}} \times \{0\}$, the ambient manifolds $\bar{\Sigma}_k$ converge to $\mathbb{R}^{m+\bar{n}} \times \{0\}$ locally in C^{3,ε_0} (more precisely to a "large portion" of $\mathbb{R}^{m+\bar{n}} \times \{0\}$, because $\mathbf{B}_{6\sqrt{m}} \subset \mathbf{B}_{6\sqrt{m}/\bar{r}_k}$). Moreover, since $\frac{1}{2} < \frac{r_k}{\bar{r}_k t_{j(k)}} < 1$, it follows from Proposition 1.3 that

$$\mathbf{E}(\bar{T}_k, \mathbf{B}_{\frac{1}{2}}) \leq C\mathbf{E}(T, \mathbf{B}_{r_k}) \to 0.$$

By the standard regularity theory of area minimizing currents and Assumption 2.1, this implies that $\bar{T}_k$ locally converge (and supports converge locally in the Hausdorff sense) to (a large portion of) a minimizing tangent cone which is an m-plane with multiplicity Q contained in $\mathbb{R}^{m+\bar{n}} \times \{0\}$. Without loss of generality, we can assume that $\bar{T}_k$ locally converge to $Q\,[\![\pi_0]\!]$. Moreover, from Proposition 1.3 it follows that

$$\mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(\bar{T}_k) \cap \mathbf{B}_1) \geq C_0 r_k^{-(m-2+\alpha)} \mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(T) \cap \mathbf{B}_{r_k}) \geq \eta > 0\,, \tag{6.2}$$

where C_0 is a geometric constant.

In the next lemma, we show that the rescaled center manifolds $\bar{\mathcal{M}}_k$ converge locally to the flat m-plane π_0, thus leading to the following natural definition for the blow-up maps $N^b_k : B_3 \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^{m+n})$:

$$N^b_k(x) := \mathbf{h}_k^{-1} \bar{N}_k(\mathbf{e}_k(x))\,, \tag{6.3}$$

where $\mathbf{h}_k := \|\bar{N}_k\|_{L^2(\mathcal{B}_{\frac{3}{2}})}$ and $\mathbf{e}_k : B_3 \subset \mathbb{R}^m \simeq T_{\bar{p}_k}\bar{\mathcal{M}}_k \to \bar{\mathcal{M}}_k$ denotes the exponential map at $\bar{p}_k = \mathbf{\Phi}_{j(k)}(0)/\bar{r}_k$ (here and in what follows we assume, w.l.o.g., to have applied a suitable rotation to each $\bar{T}_k$ so that the tangent plane $T_{\bar{p}_k}\bar{\mathcal{M}}_k$ coincides with $\mathbb{R}^m \times \{0\}$).

Lemma 6.1 (Vanishing lemma)**.** *Under the Assumption 2.1, the following hold:*

(i) *we can assume, without loss of generality,* $\bar{r}_k \boldsymbol{m}_0^{j(k)} \to 0$*;*

(ii) *the rescaled center manifolds* $\bar{\mathcal{M}}_k$ *converge (up to subsequences) to* $\mathbb{R}^m \times \{0\}$ *in* $C^{3,\kappa/2}(\mathbf{B}_4)$ *and the maps* $\mathbf{e}_k$ *converge in* $C^{2,\kappa/2}$ *to the identity map* $\mathrm{id} : B_3 \to B_3$*;*

(iii) *there exists a constant* $C > 0$*, depending only* T*, such that, for every* k*,*

$$\int_{B_{\frac{3}{2}}} |DN^b_k|^2 \leq C. \tag{6.4}$$

Proof. To show (i), note that, if $\liminf_k \bar{r}_k > 0$, we can extract a further subsequence and assume that $\lim_k \bar{r}_k > 0$. Observe that then $\bar{r} := \limsup_k \frac{t_{j(k)}}{r_k} < \infty$. Since $r_k \downarrow 0$, we necessarily conclude that $t_{j(k)} \downarrow 0$ and hence $\mathbf{c}(\Sigma_{j(k)}) \to 0$. Moreover $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_{j(k)}}) \leq C(\bar{r})\mathbf{E}(\bar{T}_k, \mathbf{B}_{6\sqrt{m}\bar{r}_k^{-1}}) \to 0$ because $\bar{T}_k$ converges to $Q\,[\![\pi_0]\!]$. We conclude $\bar{r}_k \boldsymbol{m}_0^{j(k)} \to 0$. On the other hand if $\lim_k \bar{r}_k = 0$ then (i) follows trivially from the fact that $\boldsymbol{m}_0^j$ is a bounded sequence.

Next, using $\bar r_k \boldsymbol{m}_0^{j(k)} \to 0$ and the estimate of [6, Theorem 1.17], it follows easily that $\bar{\mathcal{M}}_k - \bar p_k$ converge (up to subsequences) to a plane in $C^{3,\kappa/2}(\mathbf{B}_4)$. By Proposition 2.2 (v) we deduce easily that such plane is in fact π_0. Since 0 belongs to the support of $T_{j(k)}$ we conclude for the same reason that $\bar{\mathcal{M}}_k$ is converging to π_0 as well. Therefore, by Proposition A.4 the maps $\mathbf{e}_k$ converge to the identity in $C^{2,\kappa/2}$ (indeed, by standard arguments they must converge to the exponential map on the – totally geodesic! – submanifold $\mathbb{R}^m \times \{0\}$). Finally, (iii) is a simple consequence of Corollary 5.3. □

The main result about the blow-up maps N_k^b is the following.

Theorem 6.2 (Final blow-up)**.** *Up to subsequences, the maps N_k^b converge strongly in $L^2(B_{\frac{3}{2}})$ to a function $N_\infty^b : B_{\frac{3}{2}} \to \mathcal{A}_Q(\{0\} \times \mathbb{R}^{\bar n} \times \{0\})$ which is Dir-minimizing in B_t for every $t \in]\frac{5}{3}, \frac{3}{2}[$ and satisfies $\|N_\infty^b\|_{L^2(B_{\frac{3}{2}})} = 1$ and $\boldsymbol{\eta} \circ N_\infty^b \equiv 0$.*

We postpone the proof of Theorem 6.2 to the next section and show next Theorem 0.3.

6.2. **Proof of Theorem 0.3: capacitary argument.** Let N_∞^b be as in Theorem 6.2 and

$$\Upsilon := \left\{x \in \bar B_1 : N_\infty^b(x) = Q\,[\![0]\!]\right\}.$$

Since $\boldsymbol{\eta} \circ N_\infty^b \equiv 0$ and $\|N_\infty^b\|_{L^2(B_{3/2})} = 1$, from the regularity of Dir-minimizing Q-valued functions (cf. [3, Proposition 3.22]), we know that $\mathcal{H}_\infty^{m-2+\alpha}(\Upsilon) = 0$. We show in the next three steps that this contradicts Assumption 0.4.

Step 1. We cover Υ by balls $\{\mathbf{B}_{\sigma_i}(x_i)\}$ in such a way that $\sum_i \omega_{m-2+\alpha}(4\sigma_i)^{m-2+\alpha} \le \frac{\eta}{2}$, where η is the constant in (6.2). By the compactness of Υ, such a covering can be chosen finite. We can therefore choose a $\bar\sigma > 0$ so that the $5\bar\sigma$-neighborhood of Υ is covered by $\{\mathbf{B}_{\sigma_i}(x_i)\}$. Denote by Λ_k the set of multiplicity Q points of $\bar T_k$ far away from the singular set Υ:

$$\Lambda_k := \{p \in D_Q(\bar T_k) \cap \mathbf{B}_1 : \mathrm{dist}(p, \Upsilon) > 4\bar\sigma\}.$$

Clearly, $\mathcal{H}_\infty^{m-2+\alpha}(\Lambda_k) \ge \frac{\eta}{2}$. Let $\mathbf{V}$ denote the neighborhood of Υ of size $2\bar\sigma$. By the Hölder continuity of Dir-minimizing functions (cf. [3, Theorem 2.9]) there is a positive constant $1 > \vartheta > 0$ such that $|N_\infty^b(x)|^2 \ge 2\vartheta$ for every $x \notin \mathbf{V}$. We next introduce a parameter $\sigma > 0$ whose choice will be specified only at the very end: throughout the rest of the proof it will only required to be sufficiently small. In particular, $\sigma < \bar\sigma$ will surely imply that

$$\fint_{B_{2\sigma}(x)} |N_\infty^b|^2 \ge 2\,\vartheta \qquad \forall\, x \in B_{\frac{5}{4}} \text{ with } \mathrm{dist}(x, \Upsilon) \ge 4\bar\sigma.$$

Therefore, from Theorem 6.2 we infer that, for sufficiently large k's,

$$\fint_{\mathcal{B}_{2\sigma}(x)} \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta} \circ \bar N_k]\!])^2 \ge \vartheta \mathbf{h}_k^2 \qquad \forall\, x \in \Gamma_k := \mathbf{p}_{\bar{\mathcal{M}}_k}(\Lambda_k). \tag{6.5}$$

Step 2. For every $p \in \Lambda_k$, consider $\bar z_k(p) = \mathbf{p}_{\pi_k}(p)$ (where π_k is the reference plane for the center manifold related to $T_{j(k)}$) and $\bar x_k(p) := (\bar z_k(p), \bar r_k^{-1} \boldsymbol{\varphi}_{j(k)}(\bar r_k z_k(p)))$. Observe that $\bar x_k(p) \in \bar{\mathcal{M}}_k$. We next claim the existence of a suitably chosen geometric constant

$1 > c_0 > 0$ (in particular, independent of σ) such that, when k is large enough, for each $p \in \Lambda_k$ there is a radius $\varrho_p \leq 2\sigma$ with the following properties:

$$\frac{c_0\,\vartheta}{\sigma^\alpha}\mathbf{h}_k^2 \leq \frac{1}{\varrho_p^{m-2+\alpha}}\int_{\mathcal{B}_{\varrho_p}(\bar{x}_k(p))} |D\bar{N}_k|^2, \tag{6.6}$$

$$\mathcal{B}_{\varrho_p}(\bar{x}_k(p)) \subset \mathbf{B}_{4\varrho_p}(p)\,. \tag{6.7}$$

In order to show this claim, fix such a point p, consider the point $q_k := \bar{r}_k p$, $z_k := \bar{r}_k \bar{z}_k(p)$ and $x_k = \bar{r}_k\bar{x}_k(p) = (z_k, \boldsymbol{\varphi}_{j(k)}(z_k))$. Observe that $q_k \in D_Q(T_{j(k)})$. By [6, Proposition 3.1], z_k cannot belong to some $L \in \mathscr{W}_h^{(j(k))}$ (otherwise $\mathbf{B}_{16r_L}(p_L)$ would contain a multiplicity Q point of $T_{j(k)}$, contradicting statement (S1) in [6, Proposition 3.1]). We thus distinguish two possibilities:

(Exc) either z_k belongs to some $L_k \in \mathscr{W}_e^{(j(k))} \cup \mathscr{W}_n^{(j(k))}$;
(Con) or it belongs to the set $\Gamma_{j(k)}$.

Case (Exc). Observe that if $L_k \in \mathscr{W}_n^{(j(k))}$, by Proposition 2.2 (iii), there exists a cube $H_k \in \mathscr{W}_e^{(j(k))}$ such that L_k belongs to the domain of influence of H_k and $\mathrm{sep}(B_{\bar{r}_k}, H_k) \leq 3\bar{r}_k/16$. Thus H_k intersects $B_{19\bar{r}_k/16}(0,\pi)$.

We wish now to apply [6, Proposition 3.5] with s in there equal to $\bar{r}_k$ and T in there equal to $T_{j(k)}$: the aim is to infer

$$\bar{\ell}_k := \bar{r}_k^{-1}\sup\left\{\ell(L) : L \in \mathscr{W}_e^{(j(k))} \text{ and } L \cap B_{19\bar{r}_k/16}(0,\pi) \neq \emptyset\right\} = o(1). \tag{6.8}$$

Observe first that, taking into account the inequality $1 \leq \bar{r}_k t_{j(k)}/r_k \leq 2$, a simple scaling argument gives

$$\mathcal{H}^{m-2+\alpha}(D_Q(T_{j(k)}), \mathbf{B}_{\bar{r}_k}) \geq \left(\tfrac{r_k}{t_{j(k)}}\right)^{m-2+\alpha}\mathcal{H}^{m-2+\alpha}(D_Q(T_{0,r_k}) \cap \mathbf{B}_{t_{j(k)}\bar{r}_k/r_k})$$
$$\geq \left(\tfrac{r_k}{t_{j(k)}}\right)^{m-2+\alpha}\mathcal{H}^{m-2+\alpha}(D_Q(T_{0,r_k}) \cap \mathbf{B}_1) \overset{(1.2)}{\geq} \eta\left(\tfrac{\bar{r}_k}{2}\right)^{m-2+\alpha},$$

which verifies [6, (3.4)]. We next need to verify [6, (3.3)] and consider therefore $L \in \mathscr{W}^{(j)}$ which intersects $B_{3\bar{r}_k}(0,\pi)$. Since $\bar{r}_k > s_{j(k)}/t_{j(k)}$, by (Go) we have $\ell(L) < 3c_s\bar{r}_k \leq \bar{r}_k$. Now, for any fixed $\hat{\alpha} > 0$ we can apply [6, Proposition 3.5] provided $\min\{\bar{r}_k, \boldsymbol{m}_0^{j(k)}\}$ is small enough, which is the case for k large enough by Lemma 6.1(i). Thus [6, Proposition 3.5] implies $\limsup_k \bar{\ell}_k \leq \hat{\alpha}$ and the arbitrariness of the latter parameter implies (6.8).

For k large enough, we can then apply [6, Proposition 3.6] with $\eta_2 = \frac{\vartheta}{4}$ (in particular this condition on how large k must be is *independent* of the point p) . The Proposition will be applied to L_k, if $L_k \in \mathscr{W}_e^{(j(k))}$, or to H_k above, if $L_k \in \mathscr{W}_n^{(j(k))}$. We thus set

$$J_k = \begin{cases} H_k & \text{if } L_k \in \mathscr{W}_n^{(j(k))},\\ L_k & \text{if } L_k \in \mathscr{W}_e^{(j(k))}\end{cases}$$

and conclude the existence of a constant $\bar{s} < 1$ such that

$$\fint_{\mathcal{B}_{\bar{s}\ell(J_k)}(x_k)} \mathcal{G}(N_{j(k)}, Q\,[\![\boldsymbol{\eta}\circ N_{j(k)}]\!])^2 \leq \frac{\vartheta}{4\omega_m \ell(J_k)^{m-2}}\int_{\mathcal{B}_{\ell(J_k)}(x_k)} |DN_{j(k)}|^2\,.$$

By (6.8) have, provided k is large enough, $t(p) := \frac{\ell(L)}{\bar r_k} \leq \bar\ell_k \leq \sigma$. Therefore, rescaling to $\bar{\mathcal{M}}_k$, there exists $t(p) \leq \bar\ell_k$ such that

$$\fint_{\mathcal{B}_{\bar s t(p)}(\bar x_k(p))} \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta}\circ \bar N_k]\!])^2 \leq \frac{\vartheta}{4\omega_m t(p)^{m-2}} \int_{\mathcal{B}_{t(p)}(\bar x_k(p))} |D\bar N_k|^2\,. \tag{6.9}$$

Moreover, from Proposition 2.2 (v) and Lemma 6.1, for k large enough, we get

$$|p - \bar x_k(p)| \leq C(\boldsymbol{m}_0^{j(k)})^{\frac{1}{2m}} \bar r_k^{\beta_2} t(p) < \bar s\, t(p). \tag{6.10}$$

Case (Con). In case q_k belongs to the contact set $\mathbf{\Phi}_{j(k)}(\mathbf{\Gamma}_{j(k)})$, then $p = x_k(p)$ and $N_{j(k)}(x_k(p)) = Q\,[\![0]\!]$. Therefore

$$\lim_{t\downarrow 0} \fint_{\mathcal{B}_t(\bar x_k(p))} \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta}\circ \bar N_k]\!])^2 = 0$$

and we choose $t(p) < \sigma$ such that

$$\fint_{\mathcal{B}_{\bar s t(p)}(\bar x_k(p))} \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta}\circ \bar N_k]\!])^2 \leq \frac{\vartheta}{4}\mathbf{h}_k^2\,. \tag{6.11}$$

Observe also that (6.10) holds trivially.

Having chosen $t(p)$ in both cases, we next show the existence of $\varrho_p \in]\bar s\, t(p), 2\sigma[$ such that (6.6) holds. Observe that (6.7) will be an obvious consequence of (6.10). Notice that if

$$\frac{1}{\omega_m t(p)^{m-2}} \int_{\mathcal{B}_{t(p)}(\bar x_k(p))} |D\bar N_k|^2 \geq \mathbf{h}_k^2\,, \tag{6.12}$$

then (6.6) follows with $\varrho_p = t(p)$. If (6.12) does not hold, then

$$\fint_{\mathcal{B}_{\bar s t(p)}(\bar x_k(p))} \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta}\circ \bar N_k]\!])^2 \leq \frac{\vartheta}{4}\mathbf{h}_k^2\,. \tag{6.13}$$

Indeed we can use (6.9) in the case (Exc) (in the case (Con) we have already shown it: see (6.11)).

We now argue by contradiction to infer the existence of $\varrho_p \in]\bar s t(p), 2\sigma]$ such that (6.6) holds. Indeed, if this were not the case, we set for simplicity $f := \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta}\circ \bar N_k]\!])$,

$$\bar f_r := \fint_{\mathcal{B}_r(\bar x_k(p))} f$$

and, letting j be the smallest integer such that $2^{-j}\sigma \le \bar{s}t(p)$, we can estimate as follows:

$$\left(\fint_{\mathcal{B}_{2\sigma}(\bar{x}_k(p))} f^2\right)^{1/2} \le \left(\fint_{\mathcal{B}_{2\sigma}(\bar{x}_k(p))} (f-\bar{f}_{2\sigma})^2\right)^{1/2} + \sum_{i=0}^{j-1} |\bar{f}_{2^{1-i}\sigma} - \bar{f}_{2^{-i}\sigma}| + |\bar{f}_{2^{1-j}\sigma} - \bar{f}_{\bar{s}t(p)}|$$
$$+ \left(\fint_{\mathcal{B}_{\bar{s}t(p)}(\bar{x}_k(p))} |f-\bar{f}_{\bar{s}t(p)}|^2\right)^{1/2} + \left(\fint_{\mathcal{B}_{\bar{s}t(p)}(\bar{x}_k(p))} f^2\right)^{1/2}$$
$$\stackrel{(6.13)}{\le} C\sum_{i=0}^{j-1}\left(\frac{1}{(2^{1-i}\sigma)^{m-2}}\int_{\mathcal{B}_{2^{1-i}\sigma}(\bar{x}_k(p))} |D\bar{N}_k|^2\right)^{1/2} + \sqrt{\frac{\vartheta}{2}}\mathbf{h}_k\,. \tag{6.14}$$

In the previous lines we have used repeatedly $|Df| \le |D\bar{N}_k|$, the classical Poincaré inequality and the following simple Morrey-type estimate (which is also a consequence of the Poincaré inequality)

$$(\bar{f}_{2t} - \bar{f}_t)^2 \le \frac{C_0}{t^{m-2}}\int_{\mathcal{B}_{2t}(\bar{x}_k(p))} |Df|^2\,.$$

Note that such constant C_0 (and the constant for the Poincaré inequality) depends only upon the regularity of the underlying manifold $\bar{\mathcal{M}}_k$, and, hence, can be assumed independent of k. Summarizing, if (6.6) were to fail for every radius in the interval $]\bar{s}t(p), 2\sigma]$, from (6.14) we would conclude

$$\fint_{\mathcal{B}_{2\sigma}(\bar{x}_k(p))} f^2 \le \mathbf{h}_k^2\vartheta\left(\frac{1}{\sqrt{2}} + Cc_0\frac{1}{\sigma^{\alpha/2}}\sum_{i=0}^{j-1}(2^{1-j}\sigma)^{\alpha/2}\right)^2 \le \mathbf{h}_k^2\vartheta\left(\frac{1}{\sqrt{2}} + c_0C(\alpha)\right)^2$$

Since $C(\alpha)$ depends on α, m and Q, but does not depend on k, for c_0 chosen sufficiently small the latter inequality would contradict (6.5). Note that (6.7) follows by a simple triangular inequality.

Step 3. Finally, we show that (6.6) and (6.7) lead to a contradiction. Consider a covering of Λ_k with balls $\mathbf{B}^i := \mathbf{B}_{20\varrho_{p_i}}(p_i)$ with the property that the corresponding balls $\mathbf{B}_{4\varrho_{p_i}}(p_i)$ are disjoint. We then can estimate

$$\frac{\eta}{2} \le C_0\sum_i \varrho_{p_i}^{m-2+\alpha} \stackrel{(6.6)}{\le} \frac{C_0}{c_0}\frac{\sigma^\alpha}{\vartheta\mathbf{h}_k^2}\sum_i\int_{\mathcal{B}_{\varrho_{p_i}}(\bar{x}_k(p_i))} |D\bar{N}_k|^2$$
$$\le \frac{C_0}{c_0}\frac{\sigma^\alpha}{\vartheta\mathbf{h}_k^2}\int_{\mathcal{B}_{\frac{3}{2}}} |D\bar{N}_k|^2 \stackrel{(6.4)}{\le} C\frac{\sigma^\alpha}{\vartheta},$$

where $C_0 > 0$ is a dimensional constant. In the last line we have used that, thanks to (6.7), the balls $\mathcal{B}_{\varrho_{p_i}}(\mathbf{p}_{\bar{\mathcal{M}}_k}(p_i))$ are pairwise disjoint and that, provided σ is smaller than $\frac{1}{32}$ and k large enough, they are all contained in $\mathcal{B}_{\frac{3}{2}}$. Since ϑ and c_0 are independent of σ, the above inequality reaches the desired contradiction as soon as σ is fixed sufficiently small. This will only require a sufficiently small $\bar{\ell}_k$, which by (6.8) is ensured for k sufficiently large.

7. Harmonicity of the limit

In this section we prove Theorem 6.2 and conclude our argument. We continue to follow the notation of the previous section, in particular recall the maps defined in (BU1) and (BU2) of Section 6.1

7.1. **First estimates.** Without loss of generality we might translate the manifolds $\bar{\mathcal{M}}_k$ so that the rescaled points $\bar p_k = \bar r_k^{-1}\boldsymbol{\Phi}_{j(k)}(0)$ coincide all with the origin. Let $\bar F_k : \mathcal{B}_{\frac{3}{2}} \subset \bar{\mathcal{M}}_k \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ be the multiple valued map given by $\bar F_k(x) := \sum_i [\![x + (\bar N_k)_i(x)]\!]$ and, to simplify the notation, set $\mathbf{p}_k := \mathbf{p}_{\bar{\mathcal{M}}_k}$. We start by showing the existence of a suitable exponent $\gamma > 0$ such that

$$\mathrm{Lip}(\bar N_k|_{\mathcal{B}_{3/2}}) \leq C\mathbf{h}_k^\gamma \quad \text{and} \quad \|\bar N_k\|_{C^0(\mathcal{B}_{3/2})} \leq C(\boldsymbol{m}_0^{j(k)}\bar r_k)^\gamma, \tag{7.1}$$

$$\mathbf{M}((\mathbf{T}_{\bar F_k} - \bar T_k)\llcorner(\mathbf{p}_k^{-1}(\mathcal{B}_{\frac{3}{2}})) \leq C\mathbf{h}_k^{2+2\gamma}, \tag{7.2}$$

$$\int_{\mathcal{B}_{\frac{3}{2}}} |\boldsymbol{\eta}\circ \bar N_k| \leq C\mathbf{h}_k^2\,. \tag{7.3}$$

Indeed, set $p_{j(k)} = \boldsymbol{\Phi}_{j(k)}(0)$. Using the domain decomposition of Section 4.1 (note that $\frac{3}{2}\bar r_k \in]\frac{s_{j(k)}}{t_{j(k)}}, 3[$) and arguing in an analogous way we infer that

$$\|N_{j(k)}\|_{C^0(\mathcal{B}_{\frac{3}{2}\bar r_k}(p_{j(k)}))} \leq C(\boldsymbol{m}_0^{j(k)})^{\frac{1}{2m}}\bar r_k^{1+\beta_2} \quad \text{and} \quad \mathrm{Lip}(N_{j(k)}|_{\mathcal{B}_{\frac{3}{2}\bar r_k}(p_{j(k)})}) \leq C(\boldsymbol{m}_0^{j(k)})^{\gamma_2}\max_i \ell_i^{\gamma_2}$$

$$\mathbf{M}\big((\mathbf{T}_{F_{j(k)}} - T_{j(k)})\llcorner\mathbf{p}_k^{-1}(\mathcal{B}_{\frac{3}{2}\bar r_k}(p_{j(k)}))\big) \leq \sum_i (\boldsymbol{m}_0^{j(k)})^{1+\gamma_2}\ell_i^{m+2+\gamma_2},$$

$$\int_{\mathcal{B}_{\frac{3}{2}\bar r_k}(p_{j(k)})} |\boldsymbol{\eta}\circ N_{j(k)}| \leq C\boldsymbol{m}_0^{j(k)}\bar r_k \sum_i \ell_i^{2+m+\gamma_2/2} + \frac{C}{\bar r_k}\int_{\mathcal{B}_{\frac{3}{2}\bar r_k}(p_{j(k)})} |N_{j(k)}|^2\,,$$

where this time, for the latter inequality we have used [6, Theorem 2.4 (2.4)] with $a = \bar r_k$. On the other hand, again by the arguments of Section 4.1 (see for instance (4.12), (4.13) and (4.14)) and Corollary 5.3, we see that

$$\sum_i \boldsymbol{m}_0^{j(k)}\ell_i^{m+2+\frac{\gamma_2}{4}} \leq C_0\int_{\mathcal{B}_{\frac{3}{2}\bar r_k}(p_{j(k)})} (|DN_{j(k)}|^2 + |N_{j(k)}|^2) \leq C\bar r_k^{-2}\int_{\mathcal{B}_{\bar s_k}(p_{j(k)})} |N_{j(k)}|^2, \tag{7.4}$$

from which (7.1)-(7.3) follow by a simple rescaling (the constant C on the right hand side of (7.4) depends on T but not on k).

It is then clear that the strong L^2 convergence of N_k^b is a consequence of these bounds and of the Sobolev embedding (cf. [3, Proposition 2.11]); whereas, by (7.3),

$$\int_{\mathcal{B}_{\frac{3}{2}}} |\boldsymbol{\eta}\circ N_\infty^b| = \lim_{k\to+\infty}\int_{\mathcal{B}_{\frac{3}{2}}} |\boldsymbol{\eta}\circ N_k^b| \leq C\lim_{k\to+\infty}\mathbf{h}_k = 0\,.$$

Finally, note that N_∞^b must take its values in $\{0\}\times\mathbb{R}^{\bar n}\times\{0\}$. Indeed, considering the tangential part of $\bar N_k$ given by $\bar N_k^T(x) := \sum_i [\![\mathbf{p}_{T_x\bar\Sigma_k}(\bar N_k(x))_i]\!]$, it is simple to verify that

$\mathcal{G}(\bar N_k, \bar N_k^T) \leq C_0 |\bar N_k|^2$, which leads to

$$\int_{\mathcal{B}_{3/2}} \mathcal{G}(N_k^b, \mathbf{h}_k^{-1} \bar N_k^T \circ \mathbf{e}_k)^2 \leq C_0 \mathbf{h}_k^{-2} \int_{\mathcal{B}_{3/2}} |\bar N_k|^4 \stackrel{(7.1)}{\leq} C(\boldsymbol{m}_0^{j(k)} \bar r_k)^{2\gamma} \to 0 \quad \text{as } k \to +\infty,$$

and, by the convergence of $\bar\Sigma_k$ to $\mathbb{R}^{m+\bar n} \times \{0\}$, gives the claim.

7.2. **A suitable trivialization of the normal bundle.** By Lemma 6.1, we can consider for every $\bar{\mathcal{M}}_k$ an orthonormal frame of $(T\bar{\mathcal{M}}_k)^\perp$, $\nu_1^k, \ldots, \nu_{\bar n}^k, \varpi_1^k, \ldots \varpi_l^k$ with the property that $\nu_j^k(x) \in T_x \bar\Sigma_k$, $\varpi_j^k(x) \perp T_x \bar\Sigma_k$ and (cf. [5, Lemma A.1])

$$\nu_j^k \to e_{m+j} \quad \text{and} \quad \varpi_j^k \to e_{m+\bar n+j} \quad \text{in } C^{2,\kappa/2}(\bar{\mathcal{M}}_k) \text{ as } k \uparrow \infty$$

(for every j: here $e_1, \ldots, e_{m+\bar n+l}$ is the standard basis of $\mathbb{R}^{m+\bar n+l} = \mathbb{R}^{m+n}$). We next claim the existence of maps $\psi_k : \bar{\mathcal{M}}_k \times \mathbb{R}^{\bar n} \to \mathbb{R}^l$ converging to 0 in $C^{2,\kappa/2}$ (uniformly bounded in $C^{2,\kappa}$) and of $\delta > 0$ (independent of k) such that, for every $v \in T_p \bar{\mathcal{M}}_k$ with $|v| \leq \delta$,

$$p + v \in \bar\Sigma_k \iff v^\perp = \psi_k(p, v^T),$$

with $v^T = (\langle v, \nu_1^k \rangle, \ldots, \langle v, \nu_{\bar n}^k \rangle) \in \mathbb{R}^{\bar n}$ and $v^\perp = (\langle v, \varpi_1^k \rangle, \ldots, \langle v, \varpi_l^k \rangle) \in \mathbb{R}^l$. To see this, consider the map

$$\Phi_k : \bar{\mathcal{M}}_k \times \mathbb{R}^{\bar n} \times \mathbb{R}^l \ni (p, z, w) \mapsto p + z^j \nu_j^k + w^j \varpi_j^k \in \mathbb{R}^{m+n},$$

where we use the Einstein convention of summation over repeated indices. It is simple to show that the frame can be chosen so that $D\Phi_k(0,0) = \mathrm{Id}$ and, hence, $\Phi_k^{-1}(\bar\Sigma_k)$ can be written locally as a graph of a function ψ_k satisfying the claim above.

Note that, by construction we also have that $\psi_k(p, 0) = |D_w \psi_k(p, 0)| = 0$ for every $p \in \bar{\mathcal{M}}_k$, which in turn implies

$$|D_x \psi_k(x, w)| \leq C|w|^{1+\kappa}, \quad |D_w \psi_k(x, w)| \leq C|w| \quad \text{and} \quad |\psi_k(x, w)| \leq C|w|^2. \tag{7.5}$$

Given now any Q-valued map $u = \sum_i [\![u_i]\!] : \bar{\mathcal{M}}_k \to \mathcal{A}_Q(\{0\} \times \mathbb{R}^{\bar n} \times \{0\})$ with $\|u\|_{L^\infty} \leq \delta$, we can consider the map $\mathbf{u}_k := \psi_k(x, u)$ defined by

$$x \mapsto \sum_i \left[\!\!\left[(u_i)^j \nu_j^k(x) + \psi_k^j(x, u_i(x)) \varpi_j^k(x) \right]\!\!\right],$$

where we set $(u_i)^j := \langle u_i(x), e_{m+j} \rangle$, $\psi_k^j(x, u_i(x)) := \langle \psi_k(x, u_i(x)), e_{m+\bar n+j} \rangle$ (again we use Einstein's summation convention). Then, the differential map $D\mathbf{u}_k := \sum_i [\![D(\mathbf{u}_k)_i]\!]$ is given by

$$\begin{aligned} D(\mathbf{u}_k)_i = {} & D(u_i)^j \nu_j^k + \left[D_x \psi_k^j(x, u_i) + D_w \psi_k^j(x, u_i) D u_i \right] \varpi_j^k \\ & + (u_i)^j D\nu_j^k + \psi_k^j(x, u_i) D\varpi_j^k . \end{aligned}$$

Taking into account that $\|D\nu_i^k\|_{C^0} + \|D\varpi_j^k\|_{C^0} \to 0$ as $k \to +\infty$, by (7.5) we deduce that

$$\left| \int \left(|D\mathbf{u}_k|^2 - |Du|^2 \right) \right| \leq C \int \left(|Du|^2 |u| + |Du| |u|^{1+\kappa} + |u|^{2+2\kappa} \right) + o(1) \int \left(|u|^2 + |Du|^2 \right) . \tag{7.6}$$

Now we clearly have $\bar N_k(x) = \psi_k(x, \bar u_k)$ for some Lipschitz $\bar u_k : \bar{\mathcal{M}}_k \to \mathcal{A}_Q(\mathbb{R}^{\bar n})$ with $\|\bar u_k\|_{L^\infty} = o(1)$ by (7.1). Setting $u_k^b := \bar u_k \circ \mathbf{e}_k$, we conclude from (5.11), (7.1) and (7.6) that

$$\lim_{k\to+\infty} \int_{B_{\frac32}} \left(|DN_k^b|^2 - \mathbf{h}_k^{-2}|Du_k^b|^2\right) = 0, \tag{7.7}$$

and N_∞^b is the limit of $\mathbf{h}_k^{-1} u_k^b$.

7.3. **Competitor function.** We now show the Dir-minimizing property of N^b_∞. Clearly, there is nothing to prove if its Dirichlet energy vanishes. We can therefore assume that there exists $c_0 > 0$ such that

$$c_0 \mathbf{h}_k^2 \le \int_{\mathcal{B}_{\frac32}} |D\bar N_k|^2 . \tag{7.8}$$

Assume there is a radius $t \in \left]\frac54, \frac32\right[$ and a function $f : B_{\frac32} \to \mathcal{A}_Q(\mathbb{R}^{\bar n})$ such that

$$f|_{B_{\frac32}\setminus B_t} = N^b_\infty|_{B_{\frac32}\setminus B_t} \quad \text{and} \quad \mathrm{Dir}(f, B_t) \le \mathrm{Dir}(N^b_\infty, B_t) - 2\,\delta,$$

for some $\delta > 0$. We can apply [4, Proposition 3.5] to the functions $\mathbf{h}_k^{-1}u_k^b$ and find $r \in]t, 2[$ and competitors v_k^b such that, for k large enough,

$$v_k^b|_{\partial B_r} = u_k^b|_{\partial B_r}, \quad \mathrm{Lip}(v_k^b) \le C\mathbf{h}_k^\gamma, \quad |v_k^b| \le C(\boldsymbol{m}_0^k \bar r_k)^\gamma,$$
$$\int_{B_{\frac32}} |\boldsymbol{\eta}\circ v_k^b| \le C\mathbf{h}_k^2 \quad \text{and} \quad \int_{B_{\frac32}} |Dv_k^b|^2 \le \int |Du_k^b|^2 - \delta\,\mathbf{h}_k^2\,,$$

where $C > 0$ is a constant independent of k and γ the exponent of (7.1)-(7.3). Clearly, by Lemma 6.1 and (7.5), the maps $\tilde N_k = \psi_k(x, v_k^b \circ \mathbf{e}_k^{-1})$ satisfy

$$\tilde N_k \equiv \bar N_k \quad \text{in } \mathcal{B}_{\frac32}\setminus \mathcal{B}_t, \quad \mathrm{Lip}(\tilde N_k) \le C\mathbf{h}_k^\gamma, \quad |\tilde N_k| \le C(\boldsymbol{m}_0^k \bar r_k)^\gamma,$$
$$\int_{\mathcal{B}_{\frac32}} |\boldsymbol{\eta}\circ \tilde N_k| \le C\mathbf{h}_k^2 \quad \text{and} \quad \int_{\mathcal{B}_{\frac32}} |D\tilde N_k|^2 \le \int_{\mathcal{B}_{\frac32}} |D\bar N_k|^2 - \delta \mathbf{h}_k^2.$$

7.4. **Competitor current.** Consider finally the map $\tilde F_k(x) = \sum_i [\![x + \tilde N_i(x)]\!]$. The current $\mathbf{T}_{\tilde F_k}$ coincides with $\mathbf{T}_{\bar F_k}$ on $\mathbf{p}_k^{-1}(\mathcal{B}_{\frac32}\setminus\mathcal{B}_t)$. Define the function $\varphi_k(p) = \mathrm{dist}_{\bar{\mathcal{M}}_k}(0, \mathbf{p}_k(p))$ and consider for each $s \in \left]t, \frac32\right[$ the slices $\langle \mathbf{T}_{\tilde F_k} - \bar T_k, \varphi_k, s\rangle$. By (7.2) we have

$$\int_t^{\frac32} \mathbf{M}(\langle \mathbf{T}_{\tilde F_k} - \bar T_k, \varphi_k, s\rangle) \le C\mathbf{h}_k^{2+\gamma}\,.$$

Thus we can find for each k a radius $\sigma_k \in \left]t, \frac32\right[$ on which $\mathbf{M}(\langle \mathbf{T}_{\tilde F_k} - \bar T_k, \varphi_k, \sigma_k\rangle) \le C\mathbf{h}_k^{2+\gamma}$. By the isoperimetric inequality (see [4, Remark 4.3]) there is a current S_k such that

$$\partial S_k = \langle \mathbf{T}_{\tilde F_k} - \bar T_k, \varphi_k, \sigma_k\rangle, \quad \mathbf{M}(S_k) \le C\mathbf{h}_k^{(2+\gamma)m/(m-1)} \quad \text{and} \quad \mathrm{spt}(S_k) \subset \bar\Sigma_k.$$

Our competitor current is, then, given by

$$Z_k := \bar T_k \llcorner (\mathbf{p}_k^{-1}(\bar{\mathcal{M}}_k \setminus \mathcal{B}_{\sigma_k})) + S_k + \mathbf{T}_{\tilde F_k}\llcorner(\mathbf{p}_k^{-1}(\mathcal{B}_{\sigma_k})).$$

Note that Z_k is supported in $\bar\Sigma_k$ and has the same boundary as $\bar T_k$. On the other hand, by (7.2) and the bound on $\mathbf{M}(S_k)$, we have

$$\mathbf{M}(\tilde T_k) - \mathbf{M}(\bar T_k) \leq \mathbf{M}(\mathbf{T}_{\bar F_k}) - \mathbf{M}(\mathbf{T}_{\tilde F_k}) + C\mathbf{h}_k^{2+2\gamma} \,. \tag{7.9}$$

Denote by A_k and by H_k respectively the second fundamental forms and mean curvatures of the manifolds $\bar{\mathcal{M}}_k$. Using the Taylor expansion of [5, Theorem 3.2], we achieve

$$\mathbf{M}(\tilde T_k) - \mathbf{M}(\bar T_k) \leq \frac{1}{2}\int_{\mathcal{B}_\rho} \Big(|D\tilde N_k|^2 - |D\bar N_k|^2\Big) + C\|H_k\|_{C^0}\int \Big(|\boldsymbol{\eta}\circ \bar N_k| + |\boldsymbol{\eta}\circ\tilde N_k|\Big)$$
$$+ \|A_k\|^2_{C^0}\int \Big(|\bar N_k|^2 + |\tilde N_k|^2\Big) + o(\mathbf{h}_k^2) \leq -\frac{\delta}{2}\mathbf{h}_k^2 + o(\mathbf{h}_k^2)\,, \tag{7.10}$$

where in the last inequality we have taken into account Lemma 6.1. Clearly, (7.10) and (7.9) contradict the minimizing property of $\bar T_k$ for k large enough and concludes the proof.

Appendix A. Some technical Lemmas

The following is a special case of Allard's ε-regularity theory (see [10, Chapter 5]).

Theorem A.1. *Assume T is area minimizing, $x \in \mathrm{D}_Q(T)$ and $\|T\|((\mathrm{spt}(T)\cap U)\setminus D_Q) = 0$ in some neighborhood U of x. Then, $x \in \mathrm{Reg}(T)$. In particular, $\mathrm{D}_1(T) \subset \mathrm{Reg}(T)$.*

Proof. By simple considerations on the density, the tangent cones at x must necessarily be all m-dimensional planes with multiplicity Q. This allows to apply Allard's theorem and conclude that, in a neighborhood of x, $\mathrm{spt}(T)$ is necessarily the graph of a C^{1,κ_0} function for some $\kappa_0 > 0$. Let $u : \mathbb{R}^m \to \mathbb{R}^{\bar n + l}$ be the corresponding function and $\Psi : \mathbb{R}^{m+\bar n} \to \mathbb{R}^l$ a C^{3,ε_0} function whose graph describes Σ. Let $\bar u$ consist of the first $\bar n$ coordinates functions of u. We then have that $\bar u$ minimizes an elliptic functional of the form $\int \Phi(x, \bar u(x), D\bar u(x))\, dx$ where $(x, v, p) \mapsto \Phi(x, v, p)$ and $(x, v, p) \to D_p\Phi(x, v, p)$ are of class C^{2,ε_0}. We can then apply the classical regularity theory to conclude that $\bar u \in C^{3,\varepsilon_0}$ (see, for instance, [9, Theorem 9.2]), thereby concluding that x belongs to $\mathrm{Reg}(T)$ according to Definition 0.2. Fix next *any* $x \in D_1(T)$. By the upper semicontinuity of the density Θ (cf. [10]), $\Theta \leq \frac{3}{2}$ in a neighborhood U of x, which implies $\|T\|((\mathrm{spt}(T) \cap U) \setminus D_1) = 0$. □

Next, we prove the following technical lemma.

Lemma A.2. *Let T be an integer rectifiable current of dimension m in $\mathbb{R}^{m+n}$ with locally finite mass and U an open set such that $\mathcal{H}^{m-1}(\partial U \cap \mathrm{spt}(T)) = 0$ and $(\partial T)\llcorner U = 0$. Then $\partial(T\llcorner U) = 0$.*

Proof. Consider $V \subset\subset \mathbb{R}^{m+n}$. By the slicing Theorem [7, 4.2.1] applied to $\mathrm{dist}(\cdot, \partial U)$ we conclude that $S_r := T\llcorner(V \cap U \cap \{\mathrm{dist}\,(x, \partial U) > r\})$ is a normal current in $\mathbf{N}_m(V)$ for a.e. r. Since $\mathbf{M}(T\llcorner(V \cap U) - S_r) \to 0$ as $r \downarrow 0$, we conclude that $T\llcorner(U \cap V)$ is in the $\mathbf{M}$ closure of $\mathbf{N}_m(V)$. Thus, by [7, 4.1.17], $T\llcorner U$ is a flat chain in $\mathbb{R}^{m+n}$. By [7, 4.1.12], $\partial(T\llcorner U)$ is also a flat chain. It is easy to check that $\mathrm{spt}(\partial(T\llcorner U)) \subset \partial U \cap \mathrm{spt}(T)$. Thus we can apply [7, Theorem 4.1.20] to conclude that $\partial(T\llcorner U) = 0$. □

Recall the following theorem (for the proof see [10, Theorem 35.3]).

Theorem A.3. *If T is an integer rectifiable area minimizing current in Σ, then*

$$\mathcal{H}^{m-3+\alpha}_\infty\Big(\mathrm{spt}(T)\setminus\Big(\mathrm{spt}(\partial T)\cup\bigcup_{Q\in\mathbb{N}}\mathrm{D}_Q(T)\Big)\Big)=0\quad\forall\,\alpha>0.$$

We finally prove the following result (first proved by Allard in an unpublished note and hence reported in [1]).

Proposition A.4. *Set $\pi:=\mathbb{R}^m\times\{0\}\subset\mathbb{R}^{m+n}$ and let $\mathcal{M}$ be the graph of a $C^{3,\kappa}$ function $\boldsymbol{\varphi}:\pi\supset B_3(0)\to\mathbb{R}^m$, with $\boldsymbol{\varphi}(0)=0$. Then the exponential map $\exp:B_3(0)\to\mathcal{M}$ belongs to the class $C^{2,\kappa}$. Moreover, if $\|\boldsymbol{\varphi}\|_{C^{3,\kappa}}$ is sufficiently small, then the set $\mathbf{p}_\pi(\exp(B_r(0)))\subset\pi$ is (for all $r<3$) a convex set and the maximal curvature of its boundary is less than $\frac{2}{r}$.*

Proof. Consider any $C^{3,\kappa}$ chart $x:\mathcal{M}\to\Omega$, for instance the one induced by the graphical structure. It is then obvious that the components g_{ij} of the Riemannian metric (induced by the Euclidean ambient space on the submanifold $\mathcal{M}$) are $C^{2,\kappa}$. We let ∇ be the Levi-Civita connection on $\mathcal{M}$ for which g is parallel and consider the corresponding Christoffel symbols Γ^i_{jk} (in the fixed coordinate patch). Using the standard formula which expresses the Christoffel symbols Γ^i_{jk} in terms of the metric g_{sr} (see for instance [8, Proposition 2.54]), it is easy to see that the former are $C^{1,\kappa}$. The careful reader will notice that these objects are usually defined in standard textbooks assuming that the metric is C^∞, but in order to have a unique Levi-Civita connection it is enough that the metric is C^1, see the proof of [8, Theorem 2.51] (in fact the Levi-Civita connection on $\mathcal{M}$ can also be recovered by differentiating in the Euclidean ambient space and projecting the result onto the tangent space to $\mathcal{M}$). Similarly, for C^2 metrics we can use the intrinsic definition of the Riemann curvature tensor as in [8, Definition 3.3]. From the standard formula in [8, 3.16] we easily conclude that the components of this tensor are $C^{0,\kappa}$. However, by [3, Lemma A.1] we can choose a $C^{2,\kappa}$ orthonormal frame $\nu_1,\ldots,\nu_n:\Omega\to\mathbb{R}^{m+n}$ for the normal bundle of $\mathcal{M}$ and the curvature tensor can be computed via the Gauss' equations as in [8, 5.8b)]: we thus conclude that the components of the Riemann tensor are in fact $C^{1,\kappa}$. Again, although the references above carry on all computations in the C^∞ setting, it can be easily checked that these work in a straightforward way under our regularity assumptions.

Let next $\Phi(t,v):=\exp(vt)$ (the fact that the exponential map is well defined will be justified in few lines). Fix a $C^{3,\kappa}$ coordinate patch on $\mathcal{M}$ where 0 is the origin, using the graphical structure of $\mathcal{M}$ over $T_0\mathcal{M}$. Set $t\mapsto\gamma(t)=\Phi(t,v)$ and use the notation γ'_j for the components of γ' in the fixed chart $x:\Omega\to\mathcal{M}$ (so, $\gamma'(t)=\sum_j\gamma'_j(t)\frac{\partial}{\partial x_j}$). γ satisfies the system of differential equations

$$\gamma''_j(t)+\sum_{ik}\Gamma^j_{ik}(\gamma(t))\gamma'_i(t)\gamma'_k(t)=0,$$

with the initial conditions $\gamma(0)=0$ and $\gamma'(0)=v$, cf. [8, Definition 2.77]. It follows thus that the maps Φ and $\partial_t\Phi$ are $C^{1,\kappa}$; incidentally, this shows that the exponential map is well-defined (in fact, standard textbooks on ODEs only provide C^1 regularity; however the usual proof of C^1 regularity via Gronwall Lemma on the linearized ODEs for the derivative $\partial_v\Phi$ can be easily modified to prove $\partial_w\Phi\in C^{0,\alpha}$; cf. [2, Section 9]).

Fix now a tangent vector e at 0, a point $p = \exp(v) \in \mathcal{M}$ and perform a parallel transport of e along the ("radial") geodesic segment $[0,1] \ni t \mapsto \exp(tv)$ to define $e(p)$. We claim that the corresponding vector field is $C^{1,\kappa}$. Indeed, fix any orthonormal tangent frame $f_1, \ldots f_m$ which is $C^{2,\kappa}$. Let

$$e(\exp(tv)) = \sum_i \alpha_{v,i}(t) f_i(\Phi(t,v)) = \sum_{i,k} \alpha_{v,i}(t) \sum_k \varphi_{ik}(\Phi(t,v)) \frac{\partial}{\partial x_k}\,,$$

where the functions φ_{ik} are $C^{2,\kappa}$. Recall that the a vector field $X(t) = \sum_j X_j(t) \frac{\partial}{\partial x_j}$ along a C^1 curve c with tangent $c'(t) = \sum_i c_i'(t) \frac{\partial}{\partial x_i}$ is parallel if and only if

$$X_i'(t) = -\sum_{j,k} \Gamma^i_{jk}(c(t)) c_j'(t) X_k(t)\,,$$

cf. [8, Theorem 2.68 and equation (2.69)]. We therefore conclude that the coefficients $\alpha_{v,i}(t)$ must satisfy a system of ODEs of the form

$$\alpha_{v,i}'(t) = -\sum_j \alpha_{v,j}(t) F_{ij}(\Phi(t,v), \partial_t \Phi(t,v))$$

where $(t,v) \mapsto F_{ij}(\Phi(t,v), \partial_t\Phi(t,v))$ are $C^{1,\kappa}$ maps. Thus the existence of e and the claimed regularity of $(t,v) \mapsto \alpha_{v,i}(t)$ follow from the standard theory of ODEs.

Recall also that the parallel transport keeps the angle between vectors constant, cf. [8, Proposition 2.74]. We conclude that there exists an orthonormal frame $e_1, \ldots, e_m$ of class $C^{1,\kappa}$ which is parallel along geodesic rays emanating from the origin. Next, consider the map $(w,v,t) \mapsto \partial_w \Phi(t,v)$ where w varies in $\mathbb{R}^m$. Fix w and v and consider again the curve $\gamma(t)$ above and the vector $\eta_{v,w}(t) = \partial_w\Phi(t,v)$. We claim that η satisfies the Jacobi equation along the geodesic γ, with initial data $\eta_{v,w}(0) = 0$ and $\eta'_{v,w}(0) = w$. More precisely, if we write the vector field in the frame e_i as $\eta(t) = \sum_i \eta_i(t) e_i(\gamma(t))$, the Jacobi equation is

$$\eta''_{v,w,i}(t) = -\sum_j R_{\gamma(t)}(e_j(\gamma(t)), \gamma'(t), \gamma'(t), e_i(\gamma(t)))\eta_{v,w,j}(t)\,, \tag{A.1}$$

where R depends on the Riemann tensor (cf. [8, Theorem 3.43]). Note that we do not have the usual smoothness assumptions under which (A.1) is derived in standard textbooks. We can however proceed by regularizing our manifold $\mathcal{M}$ via convolution of the function of which the manifold is a graph. We fix the corresponding graphical charts for the regularized manifolds and observe that the exponential maps in these coordinates have uniform $C^{1,\kappa}$ bounds from the corresponding ODEs and thus will converge to Φ in C^1. Similarly one concludes the obvious convergence statements for the Riemann tensor and thus the right hand side of (A.1) for the corresponding objects converge uniformly. This justifies, in the limit, that $\eta_{v,w,i}$ is twice differentiable (in time) and that (A.1) holds.

Taking into account that $\gamma(t) = \Phi(t,v)$ and $\gamma'(t) = \partial_t\Phi(t,v)$ we conclude that $\eta_{v,w,i}$ satisfies an ODE of the type $\eta''_{v,w,i}(t) = \Lambda(t,v,\eta_{v,w,i}(t))$ where the function Λ is $C^{1,\kappa}$ in all its entries. We thus conclude that the map $(v,w,t) \mapsto \eta_{v,w}(t) = \partial_w\Phi(t,v)$ is a $C^{1,\kappa}$ map. Since $d\exp(v)(w) = \partial_w\Phi(v,1)$, this implies that the exponential map is $C^{2,\kappa}$.

As for the last assertion, for $\|\varphi\|_{C^{3,\kappa}}$ sufficiently small we conclude from the discussion above that $\mathbf{p}_\pi \circ \exp$ is C^2 close to the identity, which implies the desired statement. □

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