Some remarks on the distributional Jacobian

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Abstract

In this paper we prove a strong coarea-type formula and a chain rule for distributional Jacobians of some classes of maps. In particular we give a partial answer to a question arisen by Jerrard and Soner in Jerrard and Soner (Indiana Univ. Math. J. 51 (2002) 645).
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1. Introduction

The starting point of this paper is a "weak" coarea formula involving distributional Jacobians proved by Jerrard and Soner in [9]. The notion of distributional Jacobian is based on the fact that maps $u \in C^1(\mathbf{R}^m, \mathbf{R}^n)$, with $m \geqslant n$, satisfy $\mathrm{d}(u_1 \, \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n) = \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n$. This identity holds in the sense of distributions for maps $u \in W_{\mathrm{loc}}^{1,p}$ with $p \geqslant n$ but fails when $p < n$. However if $p \geqslant n-1$ and $u \in L^\infty$ then $v = u_1 \, \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n$ is an L^1 function and we can define the distributional Jacobian $[Ju]$ as the exterior derivative of v in the sense of distributions (we refer to [9] for an account of its applications and of the main papers on the argument). We notice that v has a natural action (via the classical Hodge operator) as $m-n+1$-dimensional current and we can think of $[Ju]$ as the boundary of v (see Definitions 3, 4 and Remark 5; for a more general discussion of the properties of $[Ju]$ as current we refer to [5]).

If we define $u^y = (u-y)/|u-y|$ for every $y \in \mathbf{R}^n$ such that $\mathcal{L}^m(u^{-1}(y)) = 0$, then we have the following:

Theorem 1 (see [9, Section 4]). *If $u \in W^{1,n-1} \cap L^\infty$ then $u^y \in W^{1,n-1} \cap L^\infty$ for a.e. $y \in \mathbf{R}^n$ and*

$$\langle [Ju], \omega \rangle = \frac{1}{\omega_n} \int_{\mathbf{R}^n} \langle [Ju^y], \omega \rangle \, \mathrm{d}y \tag{1}$$

for every smooth $m-n$ differential form ω (with ω_n we denote the Lebesgue measure of the unit ball in $\mathbf{R}^n$). If $u \in W^{1,n-1} \cap L^\infty$ and $F \in C^1(\mathbf{R}^n, \mathbf{R}^n)$ then

$$\langle [J(F(u))], \omega \rangle = \frac{1}{\omega_n} \int_{\mathbf{R}^n} \det \nabla F(y) \langle [Ju^y], \omega \rangle \, \mathrm{d}y. \tag{2}$$

It is natural to think of (2) as an extension of the chain rule for Jacobians of smooth maps, hence we will refer to it as *weak chain rule*. Of course when u is smooth we have the much stronger relation $[J(F(u))] = \det \nabla F(u)[Ju]$. For general u we cannot even give a meaning to the right-hand side of this equation, because $[Ju]$ is only a distribution. However when $[Ju]$ is a measure we might hope that there exists a pointwise representative $\bar{u}$ of u which is Borel measurable and satisfies $[J(F(u))] = \det \nabla F(\bar{u})[Ju]$ (where the right-hand side is now well defined because $\det \nabla F(\bar{u})$ is a Borel function). In general this identity does not hold; if it holds we say that u satisfies a *strong chain rule*.

If for every open set $A \subset \Omega$ we define

$$\|Ju\|(A) := \sup\{\langle [Ju], \omega \rangle \,|\, \omega \in C_c^\infty(A), \|\omega\|_\infty \leqslant 1\}$$

then a consequence of the previous Theorem is

$$\|Ju\|(A) \leqslant \frac{1}{\omega_n} \int_{\mathbf{R}^n} \|Ju^y\|(A) \, \mathrm{d}y. \tag{3}$$

A natural question is to ask under what conditions the equality holds. When $n = 1$ the problem reduces to the coarea formula on the level sets of a scalar function (see for example Theorem 3.40 of [1]). When $n > 1$ Jerrard and Soner proved that the equality holds for $u \in W^{1,p}$ with $p > n$. In this case $[Ju]$ is equal to the natural action as current of $\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n$ and a comparison with the classical coarea formula for approximately differentiable functions (see for example Theorem 3.2.3 of [6]) implies that $[Ju^y]$ is the integer rectifiable current supported by $u^{-1}(y)$ with the usual orientation. Motivated by these facts we will say that equation (1) represents a *weak coarea formula* and that u satisfies the *strong coarea formula* if the equality holds in (3).

In general, for the validity of (3), it is not sufficient to assume merely that $\|Ju\|$ is a Radon measure (see [9] for some counterexamples based on earlier ones of Giaquinta et al. [7]). However it is conjectured in [9] that the strong coarea formula holds for $W^{1,n}$ functions and for continuous functions such that $\|Ju\|$ is a Radon measure.

In this paper we prove that if u satisfies the strong chain rule then the strong coarea formula holds (see Theorem 13). This theorem provides a different proof of the strong coarea formula when $u \in W^{1,p}$, $p > n$, which can be extended to $u \in W^{1,n}$; more generally it implies that the strong coarea formula holds whenever u is a Cartesian map in the sense of Giaquinta et al. (see Remarks 16 and 18).

In Section 4 we prove (using arguments involving classical degree of continuous maps and distributional Jacobians) that if u is a continuous BnV function in $W^{1,p}$ with

$p > n - 1$ and $\|Ju^y\|(\Omega)$ is an integrable function of y then u satisfies the strong chain rule (see Theorem 14). This provides a partial answer to the conjecture of Jerrard and Soner. However we notice that in this case the strong coarea formula can be shown with a direct proof without passing through the strong chain rule. Unfortunately we are not able to prove that if $\|Ju^y\|(\Omega) \notin L^1_{loc}$ and u is continuous then $\|Ju\|(\Omega) = \infty$, which would prove the full conjecture in the case $p > n - 1$.

2. Preliminaries

We begin this section by stating some basic definitions and properties of distributional Jacobians. We refer to [9] for a more general treatment and to [5] for definitions and proofs in the framework of Geometric Measure Theory. We notice that in [5] we used the "metric theory" of currents introduced by Ambrosio and Kirchheim in [2] because we think that it simplifies notations and proofs. However to make this paper more readable and self-contained we state definitions and theorems of this section in the language of "classical" Geometric Measure Theory.

We will denote by $\Lambda^n(\mathbf{R}^m)$ the vector space of n-covectors and we will endow it with the usual norm $|\cdot|$ as Hilbert space (see for example Section 1 of [6]). If $\Omega \subset \mathbf{R}^m$ is an open set we will call n- *form on* Ω a function $\omega : \Omega \to \Lambda^n(\mathbf{R}^m)$. We will endow $C^\infty_c(\Omega, \Lambda^n(\mathbf{R}^m))$ with the usual C^k seminorms and we will call n- *dimensional current on* Ω a continuous linear functional on it. For the sake of simplicity when the regularity of a form ω is not specified we suppose that it is C^∞. The exterior derivative on forms and the boundary operator on currents are defined as usual.

From now on Ω will denote an open set and $B_r(x)$ the open ball of radius r centered on x. Moreover if μ is a Radon measure on Ω and $A \subset \Omega$ is a Borel set then $\mu \llcorner A$ will indicate the measure v given by $v(E) = \mu(A \cap E)$.

Definition 2. If T is a k-dimensional current on Ω and $A \subset \Omega$ is an open set then we define

$$\|T\|_h(A) := \sup\{T(\omega)|\mathrm{supp}(\omega) \subset\subset A, \|\omega\|_\infty \leqslant 1\}$$

If $\|T\|_h(\Omega) < \infty$ then $\|T\|_h$ can be extended to a Radon measure.

To simplify notations we will use $\|T\|$ in place of $\|T\|_h$ but we warn the reader that this notation is different from the one used in the literature. Indeed in the literature $\|T\|$ usually denotes the *mass of* T, which can be defined on open sets A as

$$\|T\|_M(A) := \sup\{T(\omega)|\mathrm{supp}(\omega) \subset\subset A, \|\omega\|_\infty \leqslant 1 \text{ and } \omega \text{ is } simple\}.$$

The mass can be extended to a Radon measure when $\|T\|_M(\Omega) < \infty$. Moreover $\|T\|_M \leqslant \|T\|$ and there is a constant k_1 (which depends on the dimensions of Ω and of T) such that $\|T\| \leqslant k_1\|T\|_M$. Of course if T is a 0-dimensional current the two measures coincide.

Definition 3. Let us fix a simple k-form $v = u_0 \, du_1 \wedge \cdots \wedge du_k$ in $\Omega \subset \mathbf{R}^m$ with $u \in L^\infty$, $\nabla u_i \in L^k$ for every i and $m \geqslant k$. We define the $m-k$-dimensional current H_v via the action

$$H_v(f \, dg_1 \wedge \cdots \wedge dg_{m-k}) := \int_{\mathbf{R}^m} u_0 f \det(\nabla g_1, \ldots, \nabla g_{m-k}, \nabla u_1, \ldots, \nabla u_k)$$

extended to general differential forms by linearity.

Definition 4. Let $u \in W^{1,n-1}(\Omega, \mathbf{R}^n) \cap L^\infty$ with $\Omega \subset \mathbf{R}^m$. Then we define $j^*(u)$ as the $(n-1)$-form

$$\sum_{i=1}^n (-1)^{i-1} u_i \, du_1 \wedge \cdots \wedge \widehat{du_i} \wedge \cdots \wedge du_n$$

and $j(u)$ as the current on Ω given by $(-1)^n H_{j^*(u)}$. Moreover we call $[Ju] := \partial j(u)/n$ *weak Jacobian* of u and we say that $u \in \mathrm{BnV}$ (or u has *bounded higher variation*) if $\|Ju\|(\Omega)$ is finite. In this paper to simplify the notation we will write Ju instead of $[Ju]$.

Remark 5. We remark that Ju can be thought as the natural action as current of the distributional exterior derivative of $j^*(u)/n$. Indeed let us fix a system of coordinates in $\mathbf{R}^m$ and suppose that $j^*(u) = \sum_{I \in J_{n-1}} f_I \, dx_I$ where:

(i) J_k is the collection of all subsets of $\{1, \ldots, m\}$ with cardinality k;
(ii) $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ if $I = \{i_1, \ldots, i_k\}$.

Let us denote by $\partial_j f_I$ the partial derivative of f_I with respect to x_j in the sense of distributions and, given a distribution T, let us agree that the formal writing $S = T \, dx_1 \wedge \cdots \wedge dx_k$ acts as a current via

$$\langle S, f \, dg_1 \wedge \cdots \wedge dg_{m-k} \rangle := T(f \det(\nabla x_1, \ldots, \nabla x_k, \nabla g_1, \ldots, \nabla g_{m-k})).$$

Hence it is not difficult to see that

$$Ju = \frac{1}{n} \sum_{I \in J_{n-1}} \sum_{j=1}^m \partial_j f_I \, dx_I \wedge dx_j := \sum_{I \in J_n} T_I \, dx_I.$$

Moreover u is in BnV if and only if every T_I is a measure with bounded variation. In this case Ju can be thought as a measure which takes values in the Hilbert space $\Lambda^n(\mathbf{R}^m)$: in this setting $\|Ju\|$ is exactly the total variation measure of Ju.

When $u \in \mathrm{BnV}$ we can use the last remark to give a meaning to $\langle Ju, \omega \rangle$ for every ω which is Borel measurable.

Theorem 6. *Let us suppose that $u \in W^{1,n-1} \cap L^\infty(\Omega, \mathbf{R}^n)$, where $\Omega \subset \mathbf{R}^n \times \mathbf{R}^{m-n}$. If we put $v_z(w) = u(w,z)$ and $\Omega_z = \Omega \cap (\mathbf{R}^n \times \{z\})$, then*

$$\langle Ju, g \, dz_1 \wedge \cdots \wedge dz_{m-n} \rangle = \int_{\mathbf{R}^n} \langle Jv_z, g(\cdot, z) \rangle \, dz.$$

Moreover if $u \in \mathrm{BnV}$ *then* $v_z \in \mathrm{BnV}(\Omega_z, \mathbf{R}^n)$ *for a.e.* z *and*

$$\|Ju\|(A) \geqslant \int_{\mathbf{R}^n} \|Jv_z\|(A \cap \Omega_z) \, \mathrm{d}z$$

for every measurable set $A \subset \Omega$.

We refer to [9] or to [5] for the proof. The next lemmas will focus on the relations between classical degree theory and weak Jacobians (see [11, Section 8]). If $\Omega \subset \mathbf{R}^n$ and $u \in C(\bar{\Omega}, \mathbf{R}^n)$ then we will denote by $\deg(u, \Omega, y)$ the degree of u at y. If M and N are smooth compact manifolds of the same dimension and $u \in C(M, N)$ then we denote by $\deg(u, M, N)$ the degree of the map u.

Lemma 7. *Assume* $\Omega \subset \mathbf{R}^n$ *has smooth boundary and let* $u \in C^\infty(\Omega, \mathbf{R}^n)$. *Then for every vector field* $\varphi \in C_c^\infty(\mathbf{R}^n, \mathbf{R}^n)$ *we have*

$$\int_{\mathbf{R}^n} \nabla \cdot \varphi(y) \deg(u, \Omega, y) \, \mathrm{d}y = \frac{1}{n} \int_{\partial \Omega} \varphi(u) \cdot M_j^i(\nabla u) \cdot v \, \mathrm{d}\mathcal{H}^{n-1},$$

where v *denotes the unit normal to* $\partial\Omega$.

After a change of variables the proof of this Lemma follows from an integration by parts.

Remark 8. When $\Omega \subset \mathbf{R}^n$, $j(u)$ can be seen as the vector field given by $v := u \cdot M_j^i(\nabla u)$ (where M_j^i is the cofactor matrix of ∇u). Indeed let us take a form $\omega = g \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-1}$: then we have that

$$\langle j(u), \omega \rangle = \int_{\mathbf{R}^n} g(x) v_n(x) \, \mathrm{d}x.$$

Hence the 0-current Ju is the divergence of v/n in the sense of distributions.

Given a set $K \subset \mathbf{R}^n$ and a real number $\varepsilon > 0$ we denote by K_ε the set $\{x \in \mathbf{R}^n \mid \mathrm{dist}(x, K) < \varepsilon\}$.

Lemma 9. *Let* $u \in \mathrm{BnV}(\Omega, \mathbf{R}^n)$ *be a continuous map (with* $\Omega \subset\subset \mathbf{R}^n$). *For every* $\varepsilon > 0$ *there exists an open set* M *with smooth boundary such that*

(a) $M \subset \Omega$ *and* $\Omega \subset M_\varepsilon$;
(b) $u \in W^{1,n-1}(\partial M)$;
(c) *it holds*

$$\frac{1}{n} \int_{\partial M} u \cdot M_j^i(\nabla u) \cdot v \, \mathrm{d}\mathcal{H}^{n-1} = Ju(M).$$

Proof. First of all we choose an open set $M' \subset \Omega$ with smooth boundary such that $\Omega \subset M'_{\varepsilon/2}$. A well known theorem on tubular neighborhoods implies that there exists a $\delta < \varepsilon/2$ such that for every $0 < t < \delta$ the sets

$$S_{-t} = \{x \mid \mathrm{dist}(x, \partial M') = t\} \cap M', \quad S_t = \{x \mid \mathrm{dist}(x, \partial M') = t\} \backslash M'$$

are smooth manifolds. Moreover we can define a signed distance function d such that the sets S_{-t}, S_t are the level sets of d. Hence for every $|s| < \delta$, $d^{-1}(s)$ is the boundary of an open set M_s which satisfies (a). Applying the coarea formula to d we conclude that for a.e. s condition (b) holds for M_s.

Now let us put $v := u \cdot M_j^i(\nabla u)$; we have $v \in L^1(\Omega, \mathbf{R}^n)$ and the measure Ju is the divergence of v (see the previous Remark). Using standard approximation arguments we can find a sequence of C^∞ vector fields v_n such that, for a.e. s, $v_n \to v$ strongly in $L^1(\partial M_s)$ and

$$\frac{1}{n} \int_{M_s} \nabla \cdot v_n \to Ju(M_s).$$

Hence it follows that condition (c) is satisfied by M_s for a.e. s. $\square$

Lemma 10. *If* $M \subset\subset \mathbf{R}^n$ *is an open set with smooth boundary and* $u \in W^{1,n-1}$ $(\partial M, S^{n-1})$ *is continuous then*

$$\omega_n \deg(u, \partial M, S^{n-1}) = \frac{1}{n} \int_{\partial M} u \cdot M_j^i(\nabla u) \cdot v \, \mathrm{d}\mathscr{H}^{n-1}.$$

Proof. A straightforward calculation shows that $(1/n)u \cdot M_j^i(\nabla u) \cdot v$ is the Jacobian determinant of u as map from ∂M to S^{n-1}. Then the statement of the Lemma follows from the integral formula for the degree as can be found for example in Brezis and Nirenberg [3]. $\square$

Lemma 11. *Let us suppose that* $u \in C(\bar{\Omega}, \mathbf{R}^n) \cap W^{1,n-1}$, *where* $\Omega \subset \mathbf{R}^n$. *Then for a.e.* $y \in \mathbf{R}^n$ *we have*

$$\omega_n |\deg(u, \partial\Omega, y)| \leqslant \|Ju^y\|(\Omega). \tag{4}$$

Proof. As it is stated in Theorem 1, $u^y \in W^{1,n-1}(\Omega, S^{n-1})$ for a.e. y (see Lemma 6 in [9, Section 4] for a proof of this fact). If $y \in u(\partial\Omega)$ then $|\deg(u, \partial\Omega, y)| = 0$ and the statement is obviously true; in an analogous way we reason if $\|Ju^y\|(\Omega) = \infty$. Hence let us fix a $y \notin u(\partial\Omega)$ such that $u^y \in W^{1,n-1}(\Omega, S^{n-1})$ and $\|Ju^y\|(\Omega) < \infty$. If we take an open set M with smooth boundary such that $u^y \in W^{1,n-1} \cap C(\partial M)$ then

$$\omega_n \deg(u^y, \partial M, S^{n-1}) = \frac{1}{n} \int_{\partial M} u^y \cdot M_j^i(\nabla u^y) \cdot v \, \mathrm{d}\mathscr{H}^{n-1}.$$

Using Lemma 9 we conclude that for ε sufficiently small there is an open set M such that $M \subset \Omega \subset M_\varepsilon$, $u^y \in W^{1,p} \cap C(\partial M)$ and

$$\omega_n \deg(u^y, \partial M, S^{n-1}) = Ju^y(M).$$

We notice that for ε small enough we have

$$\deg(u^y, \partial M, S^{n-1}) = \deg(u, \partial M, y) = \deg(u, \partial\Omega, y)$$

and this completes the proof. $\square$

We end this section with a corollary of the area formula (see [10] or Corollary 3.2.20 in [6] for the proof).

Lemma 12. *Let Γ be an oriented C^∞ $(n-1)$- dimensional manifold and suppose that $u \in W^{1,p}(\Gamma, \mathbf{R}^n) \cap C$ with $p > n-1$. Then for any $\mathcal{H}^{n-1}$ measurable set $A \subset \Gamma$ we have*

$$\mathcal{H}^{n-1}(u(A)) \leqslant (n-1)^{(1-n)/2} \int_A |Du|^{n-1} \, \mathrm{d}\mathcal{H}^{n-1}$$

3. Strong coarea formula

In this section we prove that the strong chain rule implies the strong coarea formula. In particular we have the following:

Theorem 13. *Let $u \in \mathrm{BnV}(\Omega, \mathbf{R}^n)$, where $\Omega \subset \mathbf{R}^m$, and suppose that there is a point-wise representative $\bar{u}$ of u such that $J(F(u)) = \det \nabla F(\bar{u}) Ju$ for every $F \in C^1(\mathbf{R}^n, \mathbf{R}^n)$. Then*

$$\omega_n \int_{\mathbf{R}^n} \|Ju^y\|(\Omega) = \|Ju\|(\Omega). \tag{5}$$

In the following we deal with family of currents parameterized by variables in $\mathbf{R}^n$: for the sake of clarity if T_y is a family of currents on $\mathbf{R}^m$ parameterized by y and ω is a form sometimes we will indicate with $\langle T_y(x), \omega(x) \rangle$ the real numbers $T_y(\omega)$.

Proof. To simplify notations we will identify u and $\bar{u}$. Moreover we notice that it is sufficient to prove

$$\|Ju\|(\Omega) \geqslant \frac{1}{\omega_n} \int_{\mathbf{R}^n} \|Ju^y\|(\Omega) \, \mathrm{d}y. \tag{6}$$

Let us define the current T on $\Omega_x \times \mathbf{R}_y^n$ in the following way:

(a) if $\omega = \sum f_I(x, y) \, \mathrm{d}x_I$ then

$$T(\omega) = \frac{1}{\omega_n} \int_{\mathbf{R}^n} \sum \langle Ju^y, f_I(x, y) \, \mathrm{d}x_I \rangle \, \mathrm{d}y;$$

(b) if $\omega = f(x, y) v \wedge \mathrm{d}y_j$ then $\langle T, \omega \rangle = 0$;
(c) T is extended by linearity to the whole space of smooth forms.

(In more "technical" words we have

$$T(\omega) = \frac{1}{\omega_n} \int_{\mathbf{R}^n} \langle \overline{Ju}_y, \omega \rangle \, \mathrm{d}y$$

where $\overline{Ju}_y$ is the push-forward of Ju^y via the $i_y : \Omega \to \Omega \times \mathbf{R}^n$ given by $i_y(x) = (x, y)$.) We will prove that

$$\|Ju\|(\Omega) \geqslant \|T\|(\Omega \times \mathbf{R}^n) \geqslant \frac{1}{\omega_n} \int_{\mathbf{R}^n} \|Ju^y\|(\Omega). \tag{7}$$

Step 1: We prove the second inequality of (7). First of all we recall that

$$\int_{\mathbf{R}^n} \|j(u^y)\|_{L^1} \, \mathrm{d}y = \omega_n \|j(u)\|_{L^1} < \infty. \tag{8}$$

Let W' be the set of points y_0 such that

$$\lim_{\varepsilon \to 0} \frac{1}{\omega_n \varepsilon^n} \int_{B_\varepsilon(y_0)} \|j(u^{y_0}) - j(u^y)\|_{L^1} \, \mathrm{d}y = 0.$$

Then Eq. (8) gives that $\mathscr{L}^n(\mathbf{R}^n \backslash W') = 0$. We claim that, fixed an $\eta > 0$, for every $z \in W'$ there is an $\varepsilon(z)$ such that

(A) For every $\delta < \varepsilon(z)$ there is a smooth form v such that $\|v\|_\infty \leqslant 1$, $\mathrm{supp}(v) \subset \Omega \times B_\delta(z)$ and $T(v) \geqslant (1-\eta)^3 \delta^n \|Ju^z\|(\Omega)$.

Using (A) and standard covering arguments it is easy to find a sequence of smooth n-forms $v_k(x, y)$ on $\Omega \times \mathbf{R}^n$ such that $\|v_k\|_\infty \leqslant 1$ and

$$\|T\|(\Omega \times \mathbf{R}^n) \geqslant \limsup_{k \to \infty} T(\omega_k) \geqslant (1-\eta)^3 \frac{1}{\omega_n} \int_{\mathbf{R}^n} \|Ju^y\|(\Omega) \, \mathrm{d}y.$$

If we let η go to 0 we have the desired inequality.

To prove (A) we reason as follows. First we choose $\omega \in C_c^\infty(\Omega, \Lambda^{m-n}(\mathbf{R}^m))$ such that $\|\omega\|_\infty \leqslant 1$ and $\langle Ju^z, \omega \rangle \geqslant (1-\eta)\|Ju^z\|(\Omega)$. Then we notice that there is an ε such that for $\delta \leqslant \varepsilon$

$$\frac{1}{\omega_n \delta^n} \int_{B_\delta(z)} \langle Ju^y(x), \omega(x) \rangle \, \mathrm{d}y = \frac{1}{\omega_n \delta^n} \int_{B_\delta(z)} \langle j(u^y)(x), \mathrm{d}\omega(x) \rangle \, \mathrm{d}y$$

$$\geqslant (1-\eta)\langle j(u^z), \mathrm{d}\omega \rangle = (1-\eta)\langle Ju^z, \omega \rangle.$$

Moreover with the same idea it is not difficult to find for every δ a function $\varphi \in C_c^\infty(B_\delta(z))$ such that

$$\int_{\mathbf{R}^n} \langle Ju^y(x), \varphi(y)\omega(x) \rangle \, \mathrm{d}y \geqslant (1-\eta) \int_{B_\delta(z)} \langle Ju^y, \omega \rangle \, \mathrm{d}y.$$

Then the form $v(x, y) = \varphi(y)\omega(x)$ satisfies condition (A) for δ.

Step 2: We will prove that $\|T\|(\Omega \times \mathbf{R}^n) \leqslant \|Ju\|(\Omega)$. First we notice that the strong chain-rule and Theorem 14 imply

$$\langle \det \nabla F(y)\omega(x), T(x, y) \rangle = \frac{1}{\omega_n} \int_{\mathbf{R}^n} \det \nabla F(y) \langle Ju^y(x), \omega(x) \rangle \, \mathrm{d}y$$

$$= \langle [\det \nabla F(u(x))]\omega(x), Ju(x) \rangle \tag{9}$$

for every ω with $\mathrm{supp}(\omega) \subset\subset \Omega$ and for every $F \in C^1(\mathbf{R}^n, \mathbf{R}^n)$. Now let us take a real function $\psi \in C^1(\mathbf{R}^n)$: it is easy to see that we can find two functions ψ_1, ψ_2 such that $\psi_1 + \psi_2 = \psi$, $\psi_i \in C^1$ and $\psi_1 \geqslant k$, $\psi_2 \leqslant -k$ for some $k > 0$. From a result of Dacorogna and Moser (see [4]) it follows that for every ball $B \subset \mathbf{R}^n$ we can find functions

$F_B^i \in C^1(\mathbf{R}^n, \mathbf{R}^n)$ such that

$$\det \nabla F_B^i(y) = \psi_k(y) \quad \text{for every } y \in B.$$

If we fix a form ω with compact support we can find a ball B such that $u(\mathrm{supp}(\omega)) \subset B$, hence Eq. (9) applied to F_B^i and the multilinearity of T give

$$\langle \psi(y)\omega(x), T(x, y)\rangle = \langle \psi(u(x))\omega(x), Ju(x)\rangle.$$

Now we would like to use density arguments to find

$$\langle \psi(x, y)\omega(x), T(x, y)\rangle = \langle \psi(x, u(x))\omega(x), Ju(x)\rangle \tag{10}$$

for every $\psi \in C_c^1(\Omega \times \mathbf{R}^n)$. We notice that (8) and the relation $\partial j(u^y) = Ju^y$ imply

$$\lim_{k \to \infty} T(\psi_k(x, y)\omega(x)) = T(\psi(x, y)\omega(x))$$

if $\psi_k \to \psi$ in C^1. Moreover we remark that the vector space generated by

$$\{\phi \in C_c^1(\Omega \times \mathbf{R}^n) | \phi \text{ is of the form } g(x)f(y)\}$$

is dense in $C_c^1(\Omega \times \mathbf{R}^n)$ in the strong topology. These facts imply that (10) holds.

By the definition of T we have that if v is a smooth form on $\Omega \times \mathbf{R}^n$ with compact support then there are a scalar function $\psi \in C^\infty(\Omega \times \mathbf{R}^n)$ and an $(m-n)$-form ω on Ω such that

(a) $T(v) = T(\psi(x, y)\omega(x))$;
(b) $\|\psi\omega\|_\infty \leqslant \|v\|_\infty$.

We conclude that (10) gives $\|T\|(\Omega \times \mathbf{R}^n) \leqslant \|Ju\|(\Omega)$. $\quad \square$

4. Strong chain rule

In this section we will prove the following

Theorem 14. *Let $u \in W^{1,p}(\Omega, \mathbf{R}^n)$ be continuous, with $\Omega \subset \mathbf{R}^m$ and $p > n - 1$. Moreover suppose that*

$$\int_{\mathbf{R}^n} \|Ju^y\|(\Omega)\,\mathrm{d}y < \infty. \tag{11}$$

If $F \in C^1(\mathbf{R}^n, \mathbf{R}^n)$ then $F(u) \in \mathrm{BnV}$ and $J(F(u)) = \det \nabla F(u)Ju$.

First of all we will prove the Theorem when $m = n$ and then we will recover the general case with a slicing argument. The key ingredient for the case $m=n$ is the lemma stated below, where we use some ideas of Müller and Spector (see [11, Section 8]). We remark that the hypothesis $p > n - 1$ is needed in the proof of the first step of Lemma 15. Indeed in this proof we use Lemma 12 which in general does not hold when $p = n - 1$.

The continuity of u is used heavily in the second step of Lemma 15 and the technical hypothesis (11) is employed to derive condition (C) below. Condition (C) is

crucial for passing to the limit in formula (15) because it allows to use the Dominated
Convergence Theorem. Indeed it could be true that the relation

$$\int_{\mathbf{R}^n} \deg(u, A, y) \, \mathrm{d}y = Ju(A) \tag{12}$$

holds true for every continuous BnV function u and for every open set A such that
$\mathscr{L}^n(u(A)) = 0$. This would imply that the strong chain rule, and hence the strong
coarea formula, are true for every continuous BnV function in $W^{1,p}$, $p > n - 1$. How-
ever at the present stage we do not know how to derive equality (12) without using
condition (C).

Lemma 15. *Let $u \in \mathrm{BnV}(\Omega, \mathbf{R}^n) \cap W^{1,p}$, where $\Omega \subset \mathbf{R}^n$ and $p > n - 1$. Suppose that
u is continuous and that there exists a function $N \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ such that*

(C) *for all open $\Omega' \subset\subset \Omega$, $|\deg(u, y, \Omega')| \leqslant N(y)$ for a.e. y.*

Then for every open set $A \subset\subset \Omega$ such that $\mathscr{L}^n(u(\partial A)) = 0$ we have

$$\int_{\mathbf{R}^n} \deg(u, y, A) \, \mathrm{d}y = Ju(A), \qquad \int_{\mathbf{R}^n} |\deg(u, y, A)| \, \mathrm{d}y \leqslant \|Ju\|(A). \tag{13}$$

Proof. We split the proof into two steps.

First step: We prove the first equation of (13) when A is an open set with smooth
boundary such that $u \in W^{1,p}(\partial A)$ and

$$\frac{1}{n} \int_{\partial A} u \cdot M^i_j(\nabla u) \cdot v \, \mathrm{d}\mathscr{H}^{n-1} = Ju(A). \tag{14}$$

Let us approximate u strongly in L^∞ and in $W^{1,p}(A) \cap W^{1,p}(\partial A)$ with C^∞ functions
u_k. The formula of integration by parts of Lemma 7 gives

$$\int_{\mathbf{R}^n} \nabla \cdot \varphi(y) \deg(u_k, A, y) \, \mathrm{d}y = \frac{1}{n} \int_{\partial A} \varphi(u) \cdot M^i_j(\nabla u_k) \cdot v \, \mathrm{d}\mathscr{H}^{n-1}$$

for every C^∞_c vector field. This implies that $f_k := \deg(u_k, A, y)$ are functions of
bounded variation and

$$\|Df_k\|(\mathbf{R}^n) \leqslant \frac{1}{n} \int_{\partial A} \|M^i_j(\nabla u_k)\| \, \mathrm{d}\mathscr{H}^{n-1} \leqslant \frac{1}{n} \|\nabla u_k\|^{n-1}_{L^{n-1}(\partial A)}.$$

Moreover the supports of f_k are all contained in a ball big enough. From the Poincaré
inequality for BV functions it follows that their BV norms are equibounded, hence the
sequence f_k is weakly compact in L^1. Moreover $\deg(u_k, A, y)$ converges to $\deg(u, A, y)$
for every $y \notin u(\partial A)$ and Lemma 12 implies that $u(\partial A)$ is negligible, hence we have
$f_k = \deg(u_k, A, y) \to \deg(u, A, y)$ strongly in L^1. Passing to the limit we conclude that

$$\int_{\mathbf{R}^n} \deg(u, y, A) = \frac{1}{n} \int_{\partial A} u \cdot M^i_j(\nabla u) \cdot v \, \mathrm{d}\mathscr{H}^{n-1}$$

and from (14) we obtain the first equation of (13).

Second step: Let us fix an open set A with $\mathscr{L}^n(u(\partial A)) = 0$: using Lemma 9 we
choose open sets $A_k \uparrow A$ such that every A_k has smooth boundary and satisfies (14). By

classical arguments we have $\deg(u, A_k, y) \to \deg(u, A, y)$ for a.e. y and the Dominated convergence Theorem allows us to conclude

$$\int_{\mathbf{R}^n} \deg(u, A, y) \, dy = \lim_{k \to \infty} \int_{\mathbf{R}^n} \deg(u, A_k, y) \, dy$$

$$= \lim_{k \to \infty} Ju(A_k) = Ju(A). \tag{15}$$

Now let us define for every $k \in \mathbf{Z}$

$$L_k := \{ y | \deg(u, y, A) = k \} \quad B_k = u^{-1}(L_k) \cap A.$$

Then we have that every L_k is an open set and its boundary is contained in $u(\partial A)$. It follows that every B_k is an open set and $u(\partial B_k)$ has measure zero. Moreover

$$|\deg(u, y, A)| = \sum_k |\deg(u, y, B_k)|$$

and we conclude

$$\int_{\mathbf{R}^n} |\deg(u, y, A)| \, dy = \sum_k |Ju(B_k)| \leqslant \|Ju\|(A). \qquad \square$$

Proof of Theorem 14. *First step*: $\Omega \subset \mathbf{R}^n$.

Eq. (11), the weak coarea formula and the weak chain rule imply that u and $F(u)$ are BnV function, hence the writing $\det \nabla F(u)Ju$ is formally correct in the distributional sense because it is the product of a measure by a continuous function. Moreover condition (11) implies condition (C) of the previous Lemma (see Lemma 11). Then, thanks to standard measure-theoretic arguments, we only have to check*

$$J(F(u))(B) = \int_B \det \nabla F(u(x)) \, dJu(x).$$

for every ball $B \subset\subset \Omega$.

We notice that the previous lemma implies

$$J(F(u))(B) = \int_{\mathbf{R}^n} \deg(F(u), y, B) \, dy$$

$$= \int_{\mathbf{R}^n} \det \nabla F(y) \deg(u, y, B) \, dy.$$

By standard covering arguments and the continuity of u we can find a countable family of pairwise disjoint balls B_i centered on x_i such that

(a) $\|Ju\|(B) = \sum_i \|Ju\|(B_i)$;
(b) $\|J(F(u))\|(B) = \sum_i \|J(F(u))\|(B_i)$;
(c) $|\det \nabla F(y) - \det \nabla F(u(x_i))| \leqslant \varepsilon$ for every $y \in u(B_i)$.

First of all we notice that for a.e. y

$$\deg(u, y, B) = \sum_i \deg(u, y, B_i). \tag{16}$$

Indeed Lemma 12 implies $\mathscr{L}^n(u(\partial B_i)) = 0$, hence we have

$$\int_{\mathbf{R}^n} \left| \deg(u, y, B) - \sum_{i=1}^N \deg(u, y, B_i) \right| \mathrm{d}y$$

$$\int_{\mathbf{R}^n} \left| \deg\left(u, y, B \setminus \bigcup_{i=1}^N \bar{B}_i\right) \right| \leqslant \|Ju\| \left(B \setminus \bigcup_{i=1}^N \bar{B}_i\right)$$

and if we let N go to infinity condition (b) gives (16). Moreover conditions (b) and (c) imply that

$$\Delta_i := \left| \det \nabla F(u(x_i)) Ju(B_i) - \int_{\mathbf{R}^n} \deg(F(u), y, B_i) \, \mathrm{d}y \right|$$

$$\leqslant \int |\det \nabla F(y) - \det \nabla F(u(x_i))| |\deg(u, y, B_i)| \, \mathrm{d}y \leqslant \varepsilon \|Ju\|(B_i)$$

and that

$$\Lambda_i := \left| \det \nabla F(u(x_i)) Ju(B_i) - \int_{B_i} \det \nabla F(u(x)) \, \mathrm{d}Ju(x) \right| \leqslant \varepsilon \|Ju\|(B_i).$$

Hence we have

$$\left| J(F(u))(B) - \int_B \det \nabla F(u(x)) \, \mathrm{d}Ju(x) \right|$$

$$\leqslant \sum_i \left| J(F(u))(B_i) - \int_{B_i} \det \nabla F(u(x)) \, \mathrm{d}Ju(x) \right|$$

$$\leqslant \sum_i (\Delta_i + \Lambda_i) \leqslant 2\varepsilon \sum_i \|Ju\|(B_i) = 2\varepsilon \|Ju\|(B).$$

If we let $\varepsilon \downarrow 0$ we conclude $J(F(u))(B) = [\det \nabla F(u) Ju](B)$.

Second step: Again as in the previous step we remark that both $F(u)$ and u are functions of bounded higher variation. For sake of simplicity we choose an orthogonal system of coordinates and we denote by $x_1, \ldots, x_{m-n}$ the first $m - n$ and by $z_1, \ldots, z_n$ the remaining ones. Moreover let us fix a form ω of type $f \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{m-n}$ and define $v_x(z) := u(x, z)$, $\Omega_x := \Omega \cap (\{x\} \times \mathbf{R}^n)$. Of course we have $v_x^y(z) = u^y(x, z)$, hence applying Theorem 6 and Fubini Tonelli Theorem we have

$$\int_{\mathbf{R}^{m-n}} \int_{\mathbf{R}^n} \|Jv_x^y\|(\Omega_x) \, \mathrm{d}y \, \mathrm{d}x = \int_{\mathbf{R}^n} \int_{\mathbf{R}^{m-n}} \|Jv_x^y\|(\Omega_x) \, \mathrm{d}x \, \mathrm{d}y$$

$$\leqslant \int_{\mathbf{R}^n} \|Ju^y\|(\Omega) \, \mathrm{d}y < \infty.$$

It follows that for a.e. x

$$\int_{\mathbf{R}^n} \|Jv_x^y\|(\Omega_x) \, \mathrm{d}y < \infty.$$

From the slicing arguments and the previous step we know that

$$\langle J(F(u)), \omega \rangle = \int_{\mathbf{R}^{m-n}} \langle f(\cdot, x), J(F(v_x)) \rangle \, \mathrm{d}x$$

$$= \int_{\mathbf{R}^{m-n}} \langle \det \nabla F(v_x) f(\cdot, x), J v_x \rangle \, \mathrm{d}x$$

$$= \langle J u, \omega \det \nabla F(u) \rangle.$$

We can reason in the same way for every differential form $\omega = f \, d\pi$, where π is a projection on $m - n$ coordinates, and this completes the proof. $\quad\square$

Remark 16. If $u \in W^{1,p}(\Omega, \mathbf{R}^n)$ with $p \geqslant n$ then the strong chain rule holds without any other assumption. For sake of simplicity let us suppose $\Omega \subset \mathbf{R}^n$. If we approximate u strongly in $W^{1,p}$ with C^∞ functions u_k we have that $Ju = \det \nabla u \mathscr{L}^n \llcorner \Omega$ and $\det \nabla u_k \to \det \nabla u$ strongly in $L^{p/n}$. Moreover if $F \in C^1(\mathbf{R}^n, \mathbf{R}^n)$, then

$$J(F(u_k)) = \det \nabla F(u_k) \det \nabla u_k \, \mathscr{L}^n \llcorner \Omega. \tag{17}$$

Since $J(F(u_k)) \to J(F(u))$, letting $k \uparrow \infty$ in (17), we obtain $J(F(u)) = \det \nabla F(u) Ju$. In the same way we can recover the general case $\Omega \subset \mathbf{R}^m$ with $m \geqslant n$.

5. Further remarks

As corollary of Theorem 13 and of Theorem 14 we have

Theorem 17. *Let $u \in W^{1,p}(\Omega, \mathbf{R}^n)$, where $\Omega \subset \mathbf{R}^m$ and $p > n - 1$. If u is continuous and $\|Ju^y\|(\Omega) \in L^1(\mathbf{R}^n)$ then the strong coarea formula holds.*

However a direct proof of Theorem 17 can be given as follows. First of all we can observe that

$$\int_{u^{-1}(B)} \varphi \, \mathrm{d}Ju = \int_B \langle Ju^y, \varphi \rangle \, \mathrm{d}y$$

for every $m - n$ form $\varphi \in C_c^1(u^{-1}(B))$ and for every ball B. Hence using the condition $\|Ju^y\|(\Omega) \in L^1$ and an approximation argument we can argue that the same formula holds for every form $\varphi \in C(u^{-1}(\bar{B}))$. At this point using an argument similar to that of step 1 of the proof of Theorem 13 we get the strong coarea formula.

Remark 18. Another interesting consequence of Theorem 13 is the following. Let us suppose that $u \in W^{1,p} \cap L^\infty$ is a Cartesian map, i.e. the graph of u is a rectifiable current which has no boundary in the interior of $\Omega \times \mathbf{R}^n$ (see [8] for the theory of Cartesian maps and related questions). Then it is easy to prove that $\|Ju\|$ is an absolutely continuous measure and that its density coincides with the pointwise Jacobian. Hence using the invariance of the definition of Cartesian map by composition of smooth maps we point out that

$$J(F(u)) = \det \nabla F(u) Ju \quad \text{for every } F \in C^1(\mathbf{R}^n, \mathbf{R}^n).$$

Applying Theorem 13 we conclude that the strong coarea formula holds for u. Moreover for Sobolev functions we have a classical coarea formula involving the pointwise Jacobian (see [8] or Theorem 3.2.3 of [6]): a comparison gives that if u is a Cartesian map then there exists a set $\Omega' \subset \Omega$ of full measure such that $\|Ju^y\| = \mathscr{H}^{m-n}(u^{-1}(y) \cap \llcorner \Omega')$ for a.e. y.

Finally we notice that Remark 16 and Theorem 13 give a proof of the strong coarea formula in the case of $u \in W^{1,p}$ with $p > n$ which is different from the one given in [9] and can be extended to the case $p = n$.

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References

[1] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Clarendon Press, Oxford, 2000.
[2] L. Ambrosio, B. Kirchheim, Currents on metric spaces, Acta Math. 185 (2000) 1–80.
[3] H. Brezis, L. Nirenberg, Degree theory and BMO: Part 1, compact manifolds without boundaries, Selecta Math. (N.S.) 1 (1995) 197–263.
[4] B. Dacorogna, J. Moser, On a partial differential equation involving the Jacobian determinant, Ann. Inst. H. Poincare Anal. Non Lineare 7 (1990) 1–26.
[5] C. De Lellis, Fine properties of currents and applications to distributional Jacobians, Proc. Roy. Soc. Edinburgh 132A (2002) 815–842.
[6] H. Federer, Geometric Measure Theory, Springer, Berlin, 1969.
[7] M. Giaquinta, G. Modica, J. Souček, Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity, Arch. Rational Mech. Anal 106 (1989) 97–159.
[8] M. Giaquinta, G. Modica, J. Souček, Cartesian Currents in the Calculus of Variations, Springer, Berlin, 1998.
[9] L. Jerrard, M. Soner, Functions of higher bounded variation, Indiana Univ. Math. J. 51 (2002) 645–677.
[10] M. Marcus, V.J. Mizel, Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational integrals, Bull. Amer. Math. Soc. 79 (1973) 790–795.
[11] S. Müller, S. Spector, An existence theory for nonlinear elasticity that allows for cavitation, Arch. Rational Mech. Anal. 131 (1995) 1–66.