HIGHER INTEGRABILITY OF THE GRADIENT FOR MINIMIZERS OF THE $2d$ MUMFORD-SHAH ENERGY

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Abstract. We prove the existence of an exponent $p > 2$ with the property that the approximate gradient of any local minimizer of the 2-dimensional Mumford-Shah energy belongs to L^p_{loc}.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and denote by

$$\mathrm{MS}(v, A) = \int_A |\nabla v|^2 dx + \mathcal{H}^1(J_v \cap A), \tag{1.1}$$

the Mumford-Shah energy of $v \in SBV(\Omega)$ on the open subset $A \subseteq \Omega$. In case $A = \Omega$ we shall drop the dependence on the set of integration. In what follows, the letter u will always denote a *local minimizer* of the energy (1.1), that is any function $u \in SBV(\Omega)$ with $\mathrm{MS}(u) < +\infty$ and such that

$$\mathrm{MS}(u) \leq \mathrm{MS}(w) \quad \text{whenever } \{w \neq u\} \subset\subset \Omega.$$

The class of all local minimizers shall be denoted by $\mathcal{M}(\Omega)$. The aim of this note is to prove the following higher integrability result that was conjectured by De Giorgi in all space dimensions (cp. with [9, Conjecture 1]).

Theorem 1.1. *There is $p > 2$ such that $\nabla u \in L^p_{\mathrm{loc}}(\Omega)$ for all $u \in \mathcal{M}(\Omega)$ and for all open sets $\Omega \subseteq \mathbb{R}^2$.*

Our interest is motivated by the paper [1], where the authors investigated the connection between the higher integrability of ∇u and the Mumford-Shah conjecture, which we recall for the reader's convenience.

Conjecture 1.2 (Mumford-Shah [20])**.** If $u \in \mathcal{M}(\Omega)$, then $\overline{J_u}$ is the union of (at most) countably many injective C^1 arcs $\gamma_i : [a_i, b_i] \to \Omega$ with the following properties:

- Any compact $K \subset \Omega$ intersects at most finitely many arcs;
- Two arcs can have at most an endpoint p in common and if this is the case, then p is in fact the endpoint of three arcs, forming equal angles of $\frac{2\pi}{3}$.

If Conjecture 1.2 does hold, then $\nabla u \in L^p_{loc}$ for all $p < 4$ (cp. with [1, Proposition 6.3] under $C^{1,1}$ regularity assumptions on $\overline{J_u}$, see also Proposition 1.5 below). Viceversa, the higher integrability can be translated into an estimate for the size of the singular set of $\overline{J_u}$ (see [1, Corollary 5.7]): in particular this set has Hausdorff dimension $2 - \frac{p}{2}$ under the apriori assumption that $\nabla u \in L^p_{loc}$ for some $p > 2$. In fact [1] proves also an higher-dimensional analog of this second result.

Following a classical path, the key ingredient to establish Theorem 1.1 is a reverse Hölder inequality for the gradient, which we state independently.

Theorem 1.3. *For all $q \in (1, 2)$ there exist $\rho \in (0, 1)$ and $C > 0$ such that*

$$\|\nabla u\|_{L^2(B_\rho)} \leq C\|\nabla u\|_{L^q(B_1)} \qquad \text{for any } u \in \mathcal{M}(B_1). \tag{1.2}$$

Using the obvious scaling invariance of (1.1), Theorem 1.3 yields a corresponding reverse Hölder inequality for balls of arbitrary radius: Theorem 1.1 is then a consequence of (by now) classical arguments (see for instance [15]). The exponent p could be explicitely estimated in terms of q, C and ρ. However, since our argument for Theorem 1.3 is indirect, we do not have any explicit estimate for C (ρ can instead be computed). Hence, combining Theorem 1.1 with [1] we can only conclude that the dimension of the singular set of $\overline{J_u}$ is strictly smaller than 1. This was already proved in [8] using different arguments and, though not stated there, Guy David pointed out to the first author that the corresponding dimension estimate could be made explicit. In fact, after discussing the present result, he suggested to the first author that also the constant C in Theorem 1.3 might be estimated: a viable strategy would combine the core argument of this paper with some ideas from [8] (see Remark 6.1 below; note that the proof of Theorem 1.3 given here makes already a fundamental use of the paper [8], but depends only on the ε-regularity theorem for "spiders" and "segments", cp. with Theorem 2.1). However, the resulting estimate would give an extremely small number, whereas the proof would very likely become much more complicated. Since we do not see any way to make further progress, we have decided not to pursue this issue here. We remark instead that a basic ingredient of our proof, namely the compactness Theorem 5.1, gives a more elementary approach, valid in *any* dimension, to identify the limits of sequence of minimizers in the regime of small gradients. Similar results appear in [1] using Almgren's minimal sets and stationary varifolds, whereas our strategy is based only on the concept of minimal Caccioppoli partitions: therefore not only is the proof less technical but the limiting objects satisfy a stronger variational property. As shown in [12], Theorem 5.1 allows to derive the results of [1] directly from the regularity theory for minimal Caccioppoli partitions.

Moreover, as a side effect of our considerations, we remark a small improvement of the result in [1] in the 2-dimensional case: a weaker form of the Mumford-Shah conjecture in 2d is equivalent to a sharp L^p estimate of the gradient of the minimizers.

Conjecture 1.4. If $u \in \mathcal{M}(\Omega)$, then $\overline{J_u}$ is the union of (at most) countably many injective C^0 arcs $\gamma_i : [a_i, b_i] \to \Omega$ which are C^1 on $]a_i, b_i[$ and satisfy the two conditions of Conjecture 1.2.

Proposition 1.5. *The Conjecture 1.4 holds true for $u \in \mathcal{M}(\Omega)$ if and only if $\nabla u \in L^{4,\infty}_{loc}(\Omega)$, i.e. if for all $\Omega' \subset\subset \Omega$ there is a constant $K = K(\Omega') > 0$ such that*

$$|\{x \in \Omega' : |\nabla u(x)| > \lambda\}| \le K\lambda^{-4}.$$

The if direction of Proposition 1.5 is achieved by first proving that $\overline{J_u}$ has locally finitely many connected components and then invoking the result of Bonnet [4]. In turn, the proof that the connected components are locally finite is a fairly simple application of David's ε-regularity theorem. The subtle difference between Conjecture 1.2 and Conjecture 1.4 is in the following point: assuming Conjecture 1.4 holds, if $p = \gamma_i(a_i)$ is a "loose end" of the arc γ_i, i.e. does not belong to any other arc, then the techniques in [4] show that any blowup is a cracktip, but do not give the uniqueness. In particular, Bonnet is not able to exclude the possibility that γ_i "spirals" around p infinitely many times (compare with the discussion at the end of [4, Section 1]). As far as we know this point is still open.

Several minor lemmas and propositions reported in this paper, such as Lemma 2.5, Proposition 3.2 (see for instance [18, Section 30.3] or [19]), Lemma A.1 and Lemma A.2 are well known in the literature. On the other hand we have not been able to find a precise reference: we therefore provide a proof just for completeness.

1.1. Sketch of the proof of Theorem 1.3.

We fix an exponent $q \in (1, 2)$ and a suitable radius ρ (whose choice will be specified later). Assuming that (1.2) is false, we consider a sequence $(u_k)_{k\in\mathbb{N}} \in \mathcal{M}(B_1)$ such that

$$\|\nabla u_k\|_{L^2(B_\rho)} \ge k\|\nabla u_k\|_{L^q(B_1)}. \tag{1.3}$$

Since the Mumford-Shah energy of $u \in \mathcal{M}(B_1)$ can be easily bounded apriori, we have $\|\nabla u_k\|_{L^q(B_1)} \to 0$. A suitable competitor argument then shows that:

(a) The L^2 energy of the gradients of u_k converge to 0;
(b) The jump set J_{u_k} of u_k converges to a set J which is a (locally finite) union of minimal connections.

Though this last statement is, intuitively, quite clear, it is technically demanding, because we do not have any apriori control of the norms $\|u_k\|_{L^1}$. Very similar results are contained in [1, Proposition 5.3, Theorem 5.4] under the stronger assumption that $\|\nabla u_k\|_{L^2}$ converges to 0. As already mentioned such results hinge upon the notion of Almgren's area minimizing sets, and thus need a delicate study of the behaviour of the composition of SBV functions with Lipschitz deformations that are not necessarily one-to-one. Instead, in Theorem 5.1 below we shall set the analysis into the framework of Caccioppoli partitions, naturally related to the SBV theory. Because of this, as pointed out in item (a) above, the fact that the Dirichlet energy of u_k is infinitesimal turns out to be a consequence of (1.3) and of the energy upper bound for functions in $\mathcal{M}(B_1)$.

Having established (a) and (b), an elementary argument shows the existence of a universal constant ρ such that the intersection of J with $B_{2\rho}$ is:

(i) either empty;
(ii) or a straight segment;
(iii) or a spider, i.e. three segments meeting at a common point with equal angles.

We use then the regularity theory developed by David (see [8]) to conclude that, if k is large enough, $\overline{J_{u_k}} \cap B_{2\rho}$ is diffeomorphic to (and a small perturbation of) one of these three cases. Finally a variational argument (based on a simple "Fubini and competitor" trick) shows the existence of a constant C (independent of k) with the property that

$$\|\nabla u_k\|_{L^2(B_\rho)} \leq C\|\nabla u_k\|_{L^q(B_1)} \tag{1.4}$$

which contradicts (1.3). This last elementary argument is similar to the one used by the first author and Emanuele Spadaro in the work [13].

1.2. **Outline of the paper.** Section 2 contains a summary of the regularity theory needed in our proof, a simple trace inequality which plays a key role in proving (1.4) and a few important properties of minimal connections. Section 3 relates minimal 2-dimensional partitions to minimal networks: the main proposition is well-known but, since we have not been able to find a reference, we provide a proof. Section 4 contains the first key ingredient: the argument which gives the alternatives (i)-(ii)-(iii) listed above. Section 5 contains a proof of the compactness properties (a) and (b) for sequences $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}(B_1)$ with $\|\nabla u_k\|_{L^q} \to 0$, $q \geq 1$. Section 6 collects all the technical statements of the previous sections to give a rigorous proof of Theorem 1.3 following the argument sketched above. Finally, in Section 7 we prove Proposition 1.5.

2. Preliminaries

2.1. **Regularity results for $\mathcal{M}(\Omega)$.** In case Ω is a ball $B_\rho(x)$, a simple comparison argument gives the following energy upper bound which we shall repeatedly invoke in the sequel,

$$\sup_{\mathcal{M}(B_\rho(x))} \mathrm{MS}(u, B_\rho(x)) \leq 2\pi\rho. \tag{2.1}$$

Throughout the whole paper we shall take advantage of several results available in literature for functions in $\mathcal{M}(\Omega)$. We shall quote precise references (mainly referring to the book [2]) when needed. Here, we limit ourselves to recall two main properties: the density lower bound and David's ε-regularity Theorem.

The density lower bound estimate by De Giorgi, Carriero and Leaci, reported below in the form proved by the last two authors, establishes the existence of a constant $\theta_0 > 0$ such that

$$\mathcal{H}^1(J_u \cap B_\rho(x)) \geq \theta_0 \rho \qquad \text{for any } u \in \mathcal{M}(\Omega),\ x \in \overline{J_u} \text{ and } \rho \in (0, \mathrm{dist}(x, \partial\Omega)) \tag{2.2}$$

(see [10], [5], [7] and [2, Theorem 7.21]). In the two dimensional setting an alternative derivation of the property above and an explicit estimate on the constant θ_0 has been recently obtained by the authors (see [11]).

An obvious corollary of (2.2) and of standard density estimates is that J_u is essentially closed, i.e. $\mathcal{H}^1(\overline{J_u} \setminus J_u) = 0$.

We next summarize the ε-regularity theorem first proved by David (cp. with [8, Proposition 60.1]; see also [2, Theorem 8.2] for a weaker version in any dimension). To this aim we call *minimal cone* any set which is either a line or a spider, i.e., the union of three half-lines meeting with angles $\frac{2}{3}\pi$ in a point called center. Moreover, we denote by $\mathrm{dist}_{\mathcal{H}}$ the Hausdorff distance.

Theorem 2.1. *There exists $\varepsilon > 0$ and an absolute constant $c \in (0,1)$ with the following properties. If $u \in \mathcal{M}(\Omega)$, $x \in \overline{J_u}$, $B_r(x) \subset \Omega$ and $\mathscr{C}$ is a minimal cone such that*

$$\int_{B_r(x)} |\nabla u|^2\, dx + \mathrm{dist}_{\mathcal{H}}(\overline{J_u} \cap B_r(x), \mathscr{C} \cap B_r(x)) \leq \varepsilon\, r, \tag{2.3}$$

then there exists a C^1-diffeomorphism ϕ of $B_r(x)$ onto its image with

$$\overline{J_u} \cap B_{cr}(x) = \phi\,(\mathscr{C}) \cap B_{cr}(x).$$

In addition, for any given $\delta \in (0, 1/2)$, there is $\varepsilon > 0$ such that, if (2.3) holds, then $\overline{J_u} \cap (B_{(1-\delta)r}(x) \setminus B_{\delta r}(x))$ is δ-close, in the C^1 norm, to $\mathscr{C} \cap (B_{(1-\delta)r}(x) \setminus B_{\delta r}(x))$.

Remark 2.2. The last sentence of Theorem 2.1 is not contained in [8, Proposition 60.1]. However it is a simple consequence of the theory developed in there. By scaling, we can assume $r = 1$ and $x = 0$. Fix a cone $\mathscr{C}$, a $\delta > 0$ and a sequence $\{u_k\} \subset \mathcal{M}(B_1)$ for which the left hand side of (2.3) goes to 0. If $\mathscr{C}$ is a segment, then it follows from [8] (or [2]) that there are uniform $C^{1,\alpha}$ bounds on $\overline{J_{u_k}} \cap B_{1-\delta}$. We can then use the Ascoli-Arzelà Theorem to conclude that $\overline{J_{u_k}}$ is converging in C^1 to $\mathscr{C}$.

In case the minimal cone $\mathscr{C}$ is a spider, then observe that $\mathscr{C} \cap (B_1 \setminus B_{\delta/2})$ consists of three distinct segments at distance $\delta/2$ from each other. Covering each of these segments with balls of radius comparable to δ and centered in a point belonging to the segment itself, we can argue as above and conclude that, for k large enough, $\overline{J_{u_k}} \cap (B_{1-\delta} \setminus B_\delta)$ consist of three arcs, with uniform $C^{1,\alpha}$ estimates. Once again the Ascoli-Arzelà Theorem shows that $\overline{J_{u_k}} \cap (B_{1-\delta} \setminus B_\delta)$ is converging in C^1 to $\mathscr{C} \cap (B_{1-\delta} \setminus B_\delta)$.

2.2. **A simple trace lemma.** The following is a simple fact which will play a key role in our proof.

Lemma 2.3. *For any $q \in (1,2)$ there exists $C = C(q) > 0$ such that the following holds. For any arc $\gamma \subseteq \partial B_1$ and any $g \in W^{1,q}(\gamma)$, there exists $w \in W^{1,2}(B_1)$ with trace g on γ and*

$$\|\nabla w\|_{L^2(B_1)} \leq \frac{C}{(2\pi - \mathcal{H}^1(\gamma))^{1-\frac{1}{q}}} \|g'\|_{L^q(\gamma)}. \tag{2.4}$$

Proof. Let α, $\beta \in \partial B_1$ denote the extreme points of γ. By the Hölder inequality

$$|g(\alpha) - g(\beta)| = \left| \int_\gamma g' d\mathcal{H}^1 \right| \leq (\mathcal{H}^1(\gamma))^{1-\frac{1}{q}} \|g'\|_{L^q(\gamma)} .$$

Linearly interpolating g on $\partial B_1 \setminus \gamma$, we get an extension $h \in W^{1,p}(\partial B_1)$ of g satisfying the estimate

$$\|h'\|^q_{L^q(\partial B_1 \setminus \gamma)} = (2\pi - \mathcal{H}^1(\gamma))^{1-q} |g(\alpha) - g(\beta)|^q \leq \left(\frac{\mathcal{H}^1(\gamma)}{2\pi - \mathcal{H}^1(\gamma)} \right)^{q-1} \|g'\|^q_{L^q(\gamma)} . \tag{2.5}$$

In turn, if we set $k := h - \fint_{\partial B_1} h$, the Poincaré inequality and (2.5) yield

$$\|k\|^q_{L^q(\partial B_1)} \leq C \|h'\|^q_{L^q(\partial B_1)} \leq C \left(\frac{2\pi}{2\pi - \mathcal{H}^1(\gamma)} \right)^{q-1} \|g'\|^q_{L^q(\gamma)} . \tag{2.6}$$

The embedding $W^{1,q}(\partial B_1) \to H^{1/2}(\partial B_1)$ provides us with a function $v \in W^{1,2}(B_1)$ with boundary trace k and such that

$$\|\nabla v\|_{L^2(B_1)} \leq C \|k\|_{H^{1/2}(\partial B_1)} \leq C \|k\|_{W^{1,q}(\partial B_1)} \overset{(2.6)}{\leq} \frac{C}{(2\pi - \mathcal{H}^1(\gamma))^{1-\frac{1}{q}}} \|g'\|_{L^q(\gamma)}.$$

By the latter inequality the function $w := v + \fint_{\partial B_1} h$ fulfills the assertions of the Lemma. $\square$

2.3. **Minimal connections.**

Definition 2.4. A minimal connection of $\{q_1, \ldots, q_N\} \subset \mathbb{R}^2$ is any minimizer Γ of the Steiner problem

$$\min\left\{\mathcal{H}^1(\Sigma) : \Sigma \text{ closed and connected and } q_1, \ldots, q_N \in \Sigma\right\}. \tag{2.7}$$

It is well known that minimizers for (2.7) exist (for instance cp. with [21, Theorem 1.1]). In the next lemma we collect some results for minimal connections that we shall use repeatedly in the forthcoming sections.

Lemma 2.5.

(a) If Γ is a minimal connection of $\{q_1, \ldots, q_N\}$, then Γ is the union of finitely many segments $\{\sigma_i = [\alpha_i, \beta_i]\}_{i=1}^M$ such that
 (a_1) either $\sigma_i \cap \sigma_j = \emptyset$ or $\sigma_i \cap \sigma_j = \{p\} \subset \{\alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_M\}$;
 (a_2) if α_i (resp. β_i) $\notin \{q_1, \ldots, q_N\}$, then it is the endpoint of three σ_j's, meeting at angles $\frac{2}{3}\pi$ (and hence forming a spider in a neighborhood of α_i).

(b) If in addition $\{q_1, \ldots, q_N\} \subset \partial B_\rho$, then
 (b_1) $\Gamma \subset \overline{B}_\rho$ and $\Gamma \cap \partial B_\rho = \{q_1, \ldots, q_N\}$;
 (b_2) each q_i is the endpoint of at most *two σ_j, meeting at an angle $\geq 2\pi/3$;*

(c) If $(\{q_1^k, \ldots, q_L^k\})_{k\in\mathbb{N}}$ converges in the sense of Hausdorff to $\{q_1, \ldots q_N\}$ and Γ_k are minimal connections of $\{q_1^k, \ldots, q_L^k\}$, then a subsequence of $(\Gamma_k)_{k\in\mathbb{N}}$ converges in the Hausdorff sense to a minimal connection Γ of $\{q_1, \ldots, q_N\}$ and

$$\lim_k \mathcal{H}^1(\Gamma_k) = \mathcal{H}^1(\Gamma);$$

(d) There exists $\delta > 0$ such that, for all $N \geq 4$ and all N-tuple of distinct points $q_i \in \partial B_\rho$, any minimal connection Γ of the q_i's satisfies

$$\mathcal{H}^1(\Gamma) \leq (N - \delta)\rho. \tag{2.8}$$

Proof of Lemma 2.5. The properties listed in items (a) and (b) are classical and we refer to [21, Theorem 1.2] for a recent account and an elegant elementary approach.

We next address (c). Let U be a bounded neighborhood of $\{q_1, \ldots, q_N\}$. For k large enough $\{q_1^k, \ldots, q_L^k\} \subset U$ and a simple projection argument implies that Γ_k is contained in the closed convex hull C of U. Hence, by compactness we may find a subsequence of $(\Gamma_k)_{k\in\mathbb{N}}$ (not relabeled) converging in the Hausdorff sense to a closed connected set $\Gamma \subseteq C$. Gołab's theorem (see [3, Theorem 4.4.7]) implies then

$$\mathcal{H}^1(\Gamma) \leq \liminf_k \mathcal{H}^1(\Gamma_k).$$

Because of the Hausdorff convergence, given $\varepsilon > 0$, there is n_0 large enough such that, for any $k \geq n_0$ and any q_i^k, there is a $q_{i'}$ at distance at most ε from q_i^k. Therefore, adding to Γ L segments with length at most ε we find a connected closed set Σ_k containing the points $\{q_1^k, \ldots, q_L^k\}$. Σ_k is a competitor for problem (2.7), thus by minimality of Γ_k we have

$$\mathcal{H}^1(\Gamma_k) \leq \mathcal{H}^1(\Sigma_k) \leq \mathcal{H}^1(\Gamma) + L\varepsilon\,.$$

Letting first $k \uparrow \infty$ and then $\varepsilon \downarrow 0^+$ we infer

$$\limsup_k \mathcal{H}^1(\Gamma_k) \leq \mathcal{H}^1(\Gamma).$$

Arguing in the same fashion we conclude that Γ is a minimizer of the Steiner problem.

Finally, we show (d). Without loss of generality we can assume $\rho = 1$. Since $\mathcal{H}^1(\partial B_1) = 2\pi < 7$ the inequality is obvious for $N \geq 7$ and we assume, therefore, $N \in \{4, 5, 6\}$. Assume by contradiction that (2.8) does not hold. For some $N \in \{4, 5, 6\}$, there exists a sequence of N-tuples of distinct points $(\{q_1^k, \ldots, q_N^k\})_{k \in \mathbb{N}}$ of ∂B_1 such that, if Γ_k is a corresponding minimal connection,

$$\mathcal{H}^1(\Gamma_k) \geq N - \frac{1}{k}\,.$$

Upon the extraction of subsequences, we assume that each sequence $(q_i^k)_{k \in \mathbb{N}}$ converges to a point $q_i \in \partial B_1$, $1 \leq i \leq N$. By (c) a subsequence of $(\Gamma_k)_{k \in \mathbb{N}}$ (not relabeled) converges in the Hausdorff sense to a minimal connection Γ of $\{q_1, \ldots, q_N\}$ with

$$\mathcal{H}^1(\Gamma) = \lim_k \mathcal{H}^1(\Gamma_k) \geq N. \tag{2.9}$$

For each q_i let γ_i be the closed segment $[0, q_i]$, which obviously has length one. Consider the closed connected set $\Sigma = \gamma_1 \cup \ldots \cup \gamma_N$. Since $\mathcal{H}^1(\Sigma) \leq N$, the inequality (2.9) and the minimality of Γ imply that all the q_i's must be distinct and that Σ is a minimal connection as well. However, since $N \geq 4$, Σ violates (a$_2$). $\square$

3. Caccioppoli partitions I

Definition 3.1. A Caccioppoli partition of Ω is a countable partition $\mathscr{E} = \{E_i\}_{i=1}^{\infty}$ of Ω in sets of (positive Lebesgue measure and) finite perimeter with $\sum_{i=1}^{\infty} \mathrm{Per}(E_i, \Omega) < \infty$.

For each Caccioppoli partition $\mathscr{E}$ we set

$$J_{\mathscr{E}} := \bigcup_i \partial^* E_i\,.$$

The partition $\mathscr{E}$ is said to be minimal if

$$\mathcal{H}^1(J_{\mathscr{E}}) \leq \mathcal{H}^1(J_{\mathscr{F}})$$

for all Caccioppoli partitions $\mathscr{F}$ for which there exists an open subset $\Omega' \subset\subset \Omega$ with $\sum_{i=1}^{\infty} \mathcal{L}^2\left((F_i \triangle E_i) \cap (\Omega \setminus \Omega')\right) = 0$.

Note that any Caccioppoli partition satisfies

$$\sum_{i=1}^{\infty} \mathrm{Per}(E_i) = 2\mathcal{H}^1(J_{\mathscr{E}}). \tag{3.1}$$

In addition, if $\Omega = B_\rho(x)$ for some $\rho > 0$ and $x \in \mathbb{R}^2$, an elementary comparison argument implies the following energy upper bound

$$\mathcal{H}^1(J_{\mathscr{E}}) \leq 2\pi\rho. \tag{3.2}$$

We quote [2, Section 4.4] and the papers [6], [16] as main references for the theory of Caccioppoli partitions.

Minimal Caccioppoli partitions are linked to minimal connections in a natural way.

Proposition 3.2. *Let $\mathscr{E}$ be a minimal Caccioppoli partition. Then $J_{\mathscr{E}}$ is essentially closed. Moreover, if we denote by J its closure, then any sphere $\partial B_\rho(x) \subset\subset \Omega$ intersects J in finitely many points, each connected component K of $J \cap \overline{B_\rho(x)}$ satisfies $\mathcal{H}^0(K \cap \partial B_\rho(x)) \geq 2$, and it is a minimal connection of $K \cap \partial B_\rho(x)$.*

The statement of this last proposition is a well-known fact, but since we have not been able to find a reference, we include below its proof for the reader's convenience.

Proof. Let us first prove that $J_{\mathscr{E}}$ is essentially closed, i.e. $\mathcal{H}^1(J \setminus J_{\mathscr{E}}) = 0$ (recall that $J = \overline{J_{\mathscr{E}}}$). We shall actually show that

$$\Omega \setminus J = \left\{x \in \Omega : \mathcal{H}^1(B_r(x) \cap J_{\mathscr{E}}) < r, \text{ for some } r \in (0, d(x, \partial\Omega))\right\}, \tag{3.3}$$

the latter equality together with standard density estimates imply the conclusion.

Denote by $\Omega_{\mathscr{E}}$ the set on the right hand of (3.3). Clearly $\Omega \setminus J \subseteq \Omega_{\mathscr{E}}$. To prove the opposite inclusion let $x \in \Omega_{\mathscr{E}}$. The Co-Area formula (see [2, Theorem 2.93]) implies that the set $\{\rho \in (0, r) : \mathcal{H}^0(\partial B_\rho(x) \cap J_{\mathscr{E}}) = 0\}$ has positive length. Therefore, we can find a radius ρ for which $\partial B_\rho(x)$ belongs to a single set of the Caccioppoli partition $\mathscr{E}$, which for convenience we denote by E_0.

We consider the new partition $\mathscr{F} := \{E_0 \cup B_\rho(x)\} \cup \cup_{i>0}\{E_i \setminus B_\rho(x)\}$. $\mathscr{F}$ is an admissible competitor for $\mathscr{E}$ and hence $\mathcal{H}^1(J_{\mathscr{E}}) \leq \mathcal{H}^1(J_{\mathscr{F}})$. This obviously implies that $\mathcal{H}^1(J_{\mathscr{E}} \cap B_\rho(x)) = 0$. We have proved that $\Omega_{\mathscr{E}} \subseteq \Omega \setminus J_{\mathscr{E}}$; since $\Omega_{\mathscr{E}}$ is open we conclude $\Omega_{\mathscr{E}} \subseteq \Omega \setminus J$.

Note that $\mathscr{E}$ can therefore be seen as a classical partition of Ω in a countable collection of open sets $\{E_i\}_{i \in \mathbb{N}}$ and the closed set $J = \overline{J_{\mathscr{E}}}$ of finite length and is the union of $\partial E_i \cap \Omega$. From now on we omit this set from $\mathscr{E}$. Moreover, we consider the new partition given by the connected components of $\Omega \setminus J$. This new partition must be minimal as well and, by abuse of notation, we keep denoting it by $\mathscr{E} = \{E_i\}_{i\in\mathbb{N}}$.

Given $x \in \Omega$, we consider the family of concentric balls $\{B_\rho(x) \subset \Omega : \rho > 0\}$. Without loss of generality we assume $x = 0$. The Co-Area formula implies that $\mathcal{H}^0(J \cap \partial B_\rho) < +\infty$ for a.a. ρ. Let $\rho > 0$ be such that $B_\rho \subset\subset \Omega$ and $J \cap \partial B_\rho$ is finite. We will now show the last statement of the Proposition for this particular ρ, that is:

(Cl) each connected component H of $J \cap \overline{B_\rho}$ is a minimal connection for $H \cap \partial B_\rho$.

This would conclude the proof of the Proposition, because for any $B_r \subset\subset \Omega$, we can choose a $\rho > r$ such that $B_\rho \subset\subset \Omega$ and $J \cap \partial B_\rho$ is finite. By Lemma 2.5 we then would conclude that $\overline{B}_\rho \cap J$ consists of finitely many segments, and hence that $\partial B_r \cap J$ is finite.

We now come to the proof of (Cl), which will be split in several steps. From now on without loss of generality we assume that $\rho = 1$, and introduce the notation A_i to denote the connected components of $B_1 \setminus J$.

Step 1. *Each A_i is simply connected.*

Otherwise, one of them, which for convenience we denote by A_0, contains a simple closed curve γ which is not contractible in $B_1 \setminus J$. By the Jordan-Schoenflies Theorem (see [22, Corollary 2.9]) γ bounds a topological disk U contained in B_1. Since the curve is not contractible in $B_1 \setminus J$, U must contain at least a point of J. By (3.3), $\mathcal{H}^1(U \cap J) > 0$. Denote by E_0 the element of $\mathscr{E}$ containing A_0. Note that $\mathscr{F} = \{E_0 \cup U\} \cup \bigcup_{i>0}\{E_i \setminus U\}$ would then be a competitor with $\mathcal{H}^1(J_{\mathscr{F}}) < \mathcal{H}^1(J_{\mathscr{E}})$, which is a contradiction.

Step 2. *$\partial A_i \setminus J \neq \emptyset$ for all i.*

Indeed, first of all observe that each $x \in J$ must be in the closure of two A_j's. Otherwise there would be a neighborhood U of $x \in J$ such that $U \setminus J$ is contained in one single connected component A_j, which in turn is contained in a single element $E_j \in \mathscr{E}$. But then we could redefine E_j as $E_j \cup U$ decreasing $\mathcal{H}^1(J_{\mathscr{E}})$.

Next assume the existence of A_i such that $\partial A_i \subset J$. By the observation above it follows that $\partial A_i \subset \bigcup_{j\neq i} \partial A_j$. Hence there must be a $j \neq i$ such that $\mathcal{H}^1(\partial A_i \cap \partial A_j) > 0$. Observe that A_i coincides necessarily with an element of the partition, which we denote by E_i, whose closure is contained in B_1. Instead, A_j is contained in one element E_ℓ of the partition. Since we are assuming that the E_k's are the connected component of $\Omega \setminus J$, E_ℓ is necessarily distinct from E_i (otherwise there would be a continuous path γ joining a point $x \in A_i$ and a point $y \in A_j$; this path cannot cross ∂B_1 because

$\overline{A_i} \subset B_1$; but this would be a contradiction because then A_i and A_j would be the same connected component of $B_1 \setminus J$).

We next define the following new partition $\mathscr{F} = \{F_k\}_{k\in\mathbb{N}}$, where $F_k = E_k$ if $k \notin \{\ell, i\}$, $F_\ell = E_\ell \cup E_i \cup (\partial E_i \cap \partial E_\ell)$ and $F_i = \emptyset$. Observe that $\mathscr{F}$ is a competitor for $\mathscr{E}$. Moreover,

$$\mathcal{H}^1(J_{\mathscr{F}}) = \mathcal{H}^1(J_{\mathscr{E}}) - \mathcal{H}^1(\partial E_i \cap \partial E_\ell) = \mathcal{H}^1(J_{\mathscr{E}}) - \mathcal{H}^1(\partial A_i \cap \partial A_\ell) < \mathcal{H}^1(J_{\mathscr{E}}),$$

which contradicts the minimality of $\mathscr{E}$.

Step 3. *The connected components of $J \cap \overline{B_1}$ are finitely many and they all contain at least one point of ∂B_1.*

Recall that J intersects ∂B_1 in finitely many points and hence divides it into finitely many arcs. Since $\partial A_i \setminus J \neq \emptyset$, each ∂A_i must intersect one of these arcs, which we call γ. For any $x \in \gamma$ there is $r > 0$ sufficiently small such that $B_r(x) \cap B_1 \subset B_1 \setminus J$. But then there is an open set U containing γ such that $U \cap B_1 \subset B_1 \setminus J$ and $U \cap B_1$ is connected. This implies that $\gamma \subset \partial A_i$ and $\gamma \cap \partial A_j = \emptyset$ for every $j \neq i$. We conclude therefore that there are finitely many A_i's. Since each A_i is a bounded topological open disk of $\mathbb{R}^2$, its boundary must be connected (see Lemma A.2 for an elementary proof). Moreover, $\partial A_i \subset \partial B_1 \cup J$, which has finite length. By a well-known theorem about continua, ∂A_i must be arcwise connected (see [14, Lemma 3.12]). Let now H be a connected component of $J \cap \overline{B_1}$. H intersects some ∂A_i in a point x. There exists then a continuous curve $\eta : [0,1] \to \partial A_i$ such that $\eta(0) \in \partial A_i \cap H$ and $\eta(1) \in \partial B_1$. Let $s \in [0,1]$ be the least number such that $\eta(s) \in \partial B_1$. Then $\eta([0,s])$ must be contained in J and hence in H (because H is a connected component of $J \cap \overline{B_1}$). Moreover $\eta(s) \in \partial B_1$. Thus H must contain at least one point of $J \cap \partial B_1$, which is the claim of this step.

Step 4. *Each connected component H of $J \cap \overline{B_1}$ contains at least two distinct points of $J \cap \partial B_1$.*

Assume by contradiction that $H \cap \partial B_1$ consists of exactly one point, which we call $\{p\}$. Set $K = (J \cap \overline{B_1}) \setminus H$ and consider the connected component Ω' of $B_1 \setminus K$ such that $\partial\Omega' \ni p$. Ω' is a topological disk. Indeed, if it were not simply connected, it would contain a simple curve γ which is not contractible: if U is the topological disk bounded by γ, we would have $U \subset B_1$ and being γ not contractible in Ω' we would necessarily have $\partial\Omega' \cap U \neq \emptyset$. Since $\partial\Omega' \cap B_1 \subset K$, this would mean that $K \cap U \neq \emptyset$. But since $\partial U \subset \Omega'$, K does not intersect ∂U. This means that at least one connected component of K is contained in U. Since each connected component of K is a connected component of J, this contradicts Step 3.

$\partial\Omega'$ is a compact connected set with finite length. Then there exists a Lipschitz curve $\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma([0,1]) = \partial\Omega'$ (see [14, Exercise 3.5]). Thus $\partial\Omega'$ is the continuous image of a locally connected set and it is therefore locally connected (see the last paragraph of page 19 of [22]). We can then apply the [22, Continuity Theorem, page 18] to conclude that there is a continuous map $z : \overline{B_1} \to \overline{\Omega'}$ such hat $z|_{B_1}$ is a (conformal) homeomorphism onto Ω'. It is obvious that z maps ∂B_1 onto $\partial\Omega'$. It is also true that $z^{-1}(q)$ consists of one single point whenever $q \in (\partial B_1 \cap \partial\Omega') \setminus K$. This follows from the fact that such q's do not disconnect $\partial\Omega'$, see [22, Section 2.3]. However we have not found a simple proof for this quite intuitive fact and we provide a rather subtle one in the appendix (see Lemma A.1).

Consider now the connected component H. $H \setminus \{p\}$ is obviously contained in Ω'. Moreover, by the remark above there is a ball $B_\rho(p)$ such that each point of $B_\rho(p) \cap \overline{\Omega'}$ has one single counterimage through z. This means that z is an homeomorphism between $B_\rho(p) \cap \overline{\Omega'}$ and $U = z^{-1}(B_\rho(p) \cap \overline{\Omega'})$. We conclude therefore that $H' = z^{-1}(H)$ intersects ∂B_1 at one single point which we denote by p'.

Any connected component of $\Omega' \setminus H$ is a connected component of $B_1 \setminus J$. Recall that z is an homeomorphism of B_1 onto Ω'. Thus, if $\{\Xi_i\}_{i\in\mathbb{N}}$ are the connected components of $B_1 \setminus H'$, $\{z(\Xi_i)\}_{i\in\mathbb{N}}$ are all the (distinct) connected components of $\Omega' \setminus H$. Let $q \in \partial B_1 \setminus \{p'\}$. Then $B_r(q) \cap B_1 \subset B_1 \setminus H'$ provided r is sufficiently small. Since $B_r(q) \cap B_1$ is connected, there is one and only one i such that $q \in \partial\Xi_i$. However, since H' intersects ∂B_1 in one single point, for every pair $q, q' \in \partial B_1 \setminus \{p'\}$ we can easily construct a continuous curve $\gamma : [0,1] \to \overline{B_1}$ such that

$$\gamma(0) = q, \quad \gamma(1) = q' \quad \text{and} \quad \gamma(]0,1[) \subset B_1 \setminus H' \qquad \text{(see Figure 1).} \tag{3.4}$$

Thus, $\partial B_1 \setminus \{p'\}$ is contained in the boundary of a single Ξ_i and without loss of generality we assume $i = 1$. If there is a second distinct connected component Ξ_2, then $\partial\Xi_2 \subset H'$. Thus $A_2 = z(\Xi_2)$ is a connected component of $B_1 \setminus J$ with the property that $\partial A_2 \subset z(H') = H \subset J$. But then A_2 would contradict Step 2. We conclude that $B_1 \setminus H'$ is connected and so is $A_1 = \Omega' \setminus H$. This means that H is all contained in the boundary of a connected component A_1 of $B_1 \setminus J$ and does not intersect any other connected component. Once again we could define a new partition by setting $\mathscr{F} = \{A_1 \cup H\} \cup \bigcup_{i\neq 1} A_i$, violating the minimality of $\mathscr{E}$.

Step 5. *Each connected component H of $J \cap \overline{B_1}$ is a minimal connection of $H \cap \partial B_1$.*

Recall that in Step 3 we have shown that $B_1 \setminus J = \cup_{s=1}^{\ell} A_s$. Let γ_1 and γ_2 be two arcs of $\partial B_1 \setminus J$. Each γ_i is contained in a single ∂A_{s_i}. Assume $s_1 \neq s_2$. Let $H_1, \dots H_N$ be the connected component of $J \cap \overline{B_1}$ (they are finitely many by Step 4). Then there is one

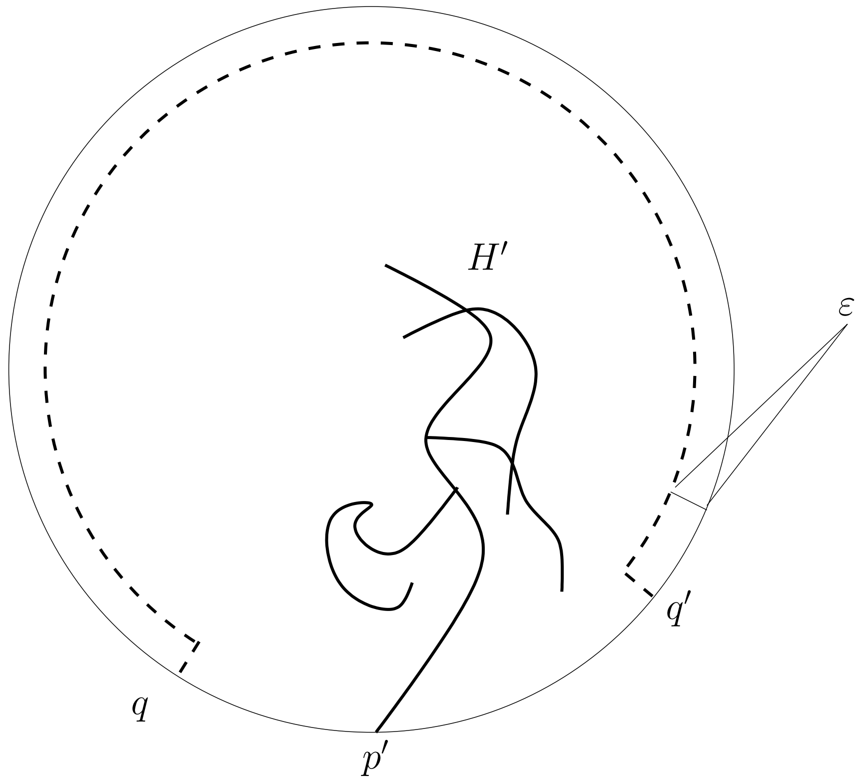

FIGURE 1. If ε is chosen sufficiently small, the curve in the picture satisfies (3.4)

H_j with the property that the γ_i's belong to the boundaries of two distinct connected components of $B_1 \setminus H_j$. However, by the same construction of Figure 1, this implies that the γ_i's must belong to distinct connected components of $\partial B_1 \setminus H_j$. Thus there are two points $p, q \in H_j \cap \partial B_1$ dividing ∂B_1 into two arcs, each containing one of the γ_i's. Let K_j be a minimal connection for $H_j \cap \partial B_1$. K_j then contains a piecewise smooth injective arc joining p and q and it is obvious that the γ_i's belong to the boundaries of distinct connected components of $B_1 \setminus K_j$.

For every i consider therefore a minimal connection K_i of $H_i \cap \partial B_1$ and the corresponding distinct connected components $O_1, \ldots, O_L$ of $B_1 \setminus \cup_{i=1}^N K_i$. The argument above implies that for each i there is an $s(i)$ such that $\partial O_i \cap \partial B_1 \subseteq \partial A_{s(i)}$, which means that there is a $\sigma(i)$ such that $\partial O_i \cap \partial B_1 \subset E_{\sigma(i)}$.

We therefore define a competitor $\mathscr{F}$ in the following way:

$$F_\tau := (E_\tau \setminus B_1) \cup \bigcup_{i:\sigma(i)=\tau} O_i \, .$$

It is easy to check that $\mathscr{F}$ is a competitor for $\mathscr{E}$ and

$$\sum_{i=1}^{N} \mathcal{H}^1(H_i) + \mathcal{H}^1(J \cap (\Omega \setminus \overline{B_1})) = \mathcal{H}^1(J_{\mathscr{E}}) \leq \mathcal{H}^1(J_{\mathscr{F}}) \leq \sum_{i=1}^{N} \mathcal{H}^1(K_i) + \mathcal{H}^1(J \cap (\Omega \setminus \overline{B_1}))\,.$$

On the other hand by the minimality of K_i we have $\mathcal{H}^1(H_i) \geq \mathcal{H}^1(K_i)$. We conclude therefore that each H_i is a minimal connection of $H_i \cap \partial B_1$. □

4. Caccioppoli partitions II

Lemma 4.1. *There exists a radius $\rho_0 \in (0,1)$ with the following property. Assume $\mathscr{E}$ is a minimal Caccioppoli partition of B_1. Then, for all $\rho \in (0, \rho_0]$*

$$\mathcal{H}^0(J_{\mathscr{E}} \cap \partial B_\rho) \leq 3, \quad \textit{and} \;\; \mathcal{H}^1(J_{\mathscr{E}} \cap B_\rho) \leq 3\rho. \tag{4.1}$$

Proof. We divide the proof into two steps. In the first one we take advantage of Lemma 2.5 and a compactness argument to show that minimal Caccioppoli partitions with jump set $J_{\mathscr{E}}$ intersecting ∂B_ρ in $N \in \{4,5,6\}$ points, for some $\rho \in (0,1)$, have length uniformly less than $N\rho$ itself. The second step iterates this estimate to show that one can always reduce to the case of at most three intersections. To simplify the notation, we set $J = J_{\mathscr{E}}$.

Step 1. *There exists $\delta \in (0,1)$ such that, if $\mathscr{E}$ is as in the statement with additionally $\mathcal{H}^0(J \cap \partial B_\rho) \in \{4,5,6\}$, for some ρ, then*

$$\mathcal{H}^1(J \cap B_\rho) \leq \big(\mathcal{H}^0(J \cap \partial B_\rho) - \delta\big)\, \rho.$$

By scaling, we can assume that $\rho = 1$. Arguing by contradiction we assume that there is a sequence $(\mathscr{E}_k)_{k\in\mathbb{N}}$ of minimal Caccioppoli partitions of B_1 such that, if $J_k = J_{\mathscr{E}_k}$, then

(i) $\mathcal{H}^0(J_k \cap \partial B_1) \in \{4,5,6\}$;
(ii) $\mathcal{H}^1(J_k \cap B_1) > \mathcal{H}^0(J_k \cap \partial B_1) - \frac{1}{k}$.

Upon the extraction of subsequences (not relabeled in what follows) we may assume that $\mathcal{H}^0(J_k \cap \partial B_1)$ is a constant value $N \in \{4,5,6\}$. Recall next that, by Proposition 3.2, the connected components of J_k are minimal connections (and hence they are at most three). In what follows L_k denotes a connected component of J_k. Obviously, joining each point of $L_k \cap \partial B_1$ with 0, we conclude the trivial estimate

$$\mathcal{H}^1(L_k \cap B_1) \leq \mathcal{H}^0(L_k \cap \partial B_1)\,. \tag{4.2}$$

Combining (4.2) with (ii) we then conclude

$$\mathcal{H}^1(L_k \cap B_1) > \mathcal{H}^0(L_k \cap \partial B_1) - \frac{1}{k} \tag{4.3}$$

Given any sequence $\{L_k\}_{k\in\mathbb{N}}$ we can, after extracting a subsequence, assume that $\mathcal{H}^0(L_k \cap \partial B_1)$ is a constant $\bar{N} \in \{2,3,4,5,6\}$, that $L_k \cap \partial B_1$ converges to a set E consisting of at most $\bar{N}$ points and that $L_k \cap \overline{B}_1$ converges to a minimal connection L of E (we apply Lemma 2.5). Thus

$$\bar{N} = \mathcal{H}^1(L) \leq \mathcal{H}^0(E) \leq \bar{N}.$$

This implies that $\bar{N}$ is at most 3 by Lemma 2.5 and indeed that L is either a diameter of B_1 or is a spider centered at its origin.

Thus, for k large enough, each connected component of J_k must be either close to a centered spider or to a diameter in the Hausdorff distance. Since $N \geq 4$ there are at least two such connected components and since they have to be disjoint sets, none of them can be a spider. They therefore must all be close to a diameter, which must be the same for all of them. Hence, upon extraction of a subsequence, each $J_k \cap B_1$ consists either of three or of two (nonintersecting) straight segments converging to a diameter of B_1.

If k is large enough, there exists then a single closed connected set H_k contained in $\overline{B}_1$ with $H_k \cap \partial B_1 = J_k \cap \partial B_1$ and $\mathcal{H}^1(H_k) \leq 3$. Without loss of generality, we can assume that the boundary of each connected component A_j of $B_1 \setminus H_k$ intersects $\partial B_1 \setminus J_k$. Recall that $J_k = \cup_i \partial^* E_i$, with $\mathscr{E}_k = \{E_i\}_{i\in\mathbb{N}}$ minimal Caccioppoli partition of $B_{1/\rho}$. Since H_k is connected, each $(\partial A_j \cap \partial B_1) \setminus J_k$ is contained in a single set $E_{i(j)}$. But then we can define a new Caccioppoli partition $\mathscr{F}_k = \{(E_i \setminus B_1) \cup \bigcup_{j:\, i(j)=i} A_j\}_{i\in\mathbb{N}}$. Using this partition as a competitor, we get

$$\mathcal{H}^1(J_k) \leq \mathcal{H}^1(J_{\mathscr{F}_k}) = \mathcal{H}^1(J_k \setminus B_1) + \mathcal{H}^1(H_k) \leq \mathcal{H}^1(J_k \setminus B_1) + 3,$$

which is obviously a contradiction in view of (i) and (ii).

Step 2. *Conclusion.*

Fix $\lambda \in (2\pi/7, 1)$, by the energy upper bound (3.2) and the Co-Area formula we may find $\rho_1 \in (1-\lambda, 1)$ such that $\mathcal{H}^0(J \cap \partial B_{\rho_1}) \leq 6$. By Step 1 we infer $\mathcal{H}^1(J \cap B_{\rho_1}) \leq (6-\delta)\rho_1$, so that a radius $\rho_2 \in \left(\frac{\delta}{7}\rho_1, \rho_1\right)$ can be selected satisfying $\mathcal{H}^0(J \cap \partial B_{\rho_2}) \leq 5$. Iterating twice this argument shows the existence of a radius $\rho_4 \in \left(\frac{\delta^3}{7^3}(1-\lambda), 1\right)$ such that

$$\mathcal{H}^0(J \cap \partial B_{\rho_4}) \leq 3.$$

Proposition 3.2 guarantees that $J \cap B_{\rho_4}$ is a minimal connection for $J \cap \partial B_{\rho_4}$. Hence three different configurations are then possible:

(a) $\mathcal{H}^0(J \cap \partial B_{\rho_4}) = 0$, and then $J \cap B_{\rho_4} = \emptyset$;

(b) $\mathcal{H}^0(J \cap \partial B_{\rho_4}) = 2$, and then $J \cap B_{\rho_4}$ is a segment;

(c) $\mathcal{H}^0(J \cap \partial B_{\rho_4}) = 3$, and then $J \cap B_{\rho_4}$ is a spider.

In any event, the conclusion follows by setting $\rho_0 := \frac{\delta^3}{7^3}(1-\lambda)$. □

5. Sequences in $\mathcal{M}(B_1)$ with $\|\nabla u_k\|_{L^1} \to 0$

In what follows we analyze the compactness properties of sequences of local minimizers with vanishing gradient energy: the conclusions are summarized in Theorem 5.1 below. Observe that we do not assume any uniform L^p bound, since the theorem will be later applied to sequences of minimizers for which any L^p norm might indeed blow up. This lack of control upon the size of the functions makes the argument slightly involved.

We point out that Theorem 5.1 below is stated and proved only in the two dimensional case of interest here. In spite of this, the analogous statement in any dimension can be obtained only with straightforward notational changes in the proof below.

Furthermore, Theorem 5.1 should be compared with [1, Proposition 5.3, Theorem 5.4] where under the stronger assumption that $\|\nabla u_k\|_{L^2}$ is infinitesimal, it is proved that any weak-$*$ limit of $\mathcal{H}^{n-1} \llcorner S_{u_k}$ is a $(n-1)$-rectifiable measure with multiplicity one concentrated on an area minimizing set according to Almgren.

In what follows we agree to identify each measurable set E with its measure theoretic closure given by those points where the density of E is strictly positive.

Theorem 5.1. *Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}(B_1)$ be such that*

$$\lim_k \|\nabla u_k\|_{L^1(B_1)} = 0. \tag{5.1}$$

Then, (up to the extraction of a subsequence not relabeled for convenience) there exists a minimal Caccioppoli partition $\mathscr{E} = \{E_i\}_{i\in\mathbb{N}}$ such that $(\overline{J_{u_k}})_{k\in\mathbb{N}}$ converges locally in the Hausdorff distance to $\overline{J_{\mathscr{E}}}$ and

$$\lim_k \mathrm{MS}(u_k, A) = \lim_k \mathcal{H}^1(J_{u_k} \cap A) = \mathcal{H}^1(J_{\mathscr{E}} \cap A) \qquad \textit{for all open sets } A \subset B_1. \tag{5.2}$$

Proof. The sequence $(u_k)_{k\in\mathbb{N}}$ does not satisfy, apriori, any L^p bound, thus in order to gain some insight on the asymptotic behaviour of the corresponding jump sets we first construct a new sequence $(w_k)_{k\in\mathbb{N}}$ with null gradients introducing an infinitesimal error on the length of the jump set of w_k with respect to that of u_k. Then, we investigate the limit behaviour of the corresponding Caccioppoli partitions.

Step 1. *There exists a sequence $(w_k)_{k\in\mathbb{N}} \subseteq SBV(B_1)$ satisfying*

(i) $\nabla w_k = 0$ $\mathcal{L}^2$ *a.e. on* B_1,

(ii) $\|u_k - w_k\|_{L^\infty(B_1)} \leq 2\|\nabla u_k\|^{1/2}_{L^1(B_1)}$,

(iii) $\mathcal{H}^1(J_{w_k} \setminus (J_{u_k} \cup H_k)) = 0$ *for some Borel measurable set H_k, with $\mathcal{H}^1(H_k) = o(1)$ as $k \uparrow \infty$.*

Note that in turn item (iii) implies that

$$\mathrm{MS}(w_k) = \mathcal{H}^1(J_{w_k}) \le \mathcal{H}^1(J_{u_k}) + o(1) \le \mathrm{MS}(u_k) + o(1). \tag{5.3}$$

In Step 2 below we shall eventually show that $|\mathrm{MS}(w_k) - \mathrm{MS}(u_k)| \le o(1)$.

Recall that the BV Co-Area formula (see [2, Theorem 3.40]) establishes

$$\int_{B_1} |\nabla u_k| dx = |Du_k|(B_1 \setminus J_{u_k}) = \int_{\mathbb{R}} \mathrm{Per}\,(\partial^*\{u_k \ge t\} \setminus J_{u_k})\, dt. \tag{5.4}$$

Denote by I_i^k a partition of $\mathbb{R}$ of intervals of equal length $\|\nabla u_k\|_{L^1(B_1)}^{1/2}$. Equation (5.4) and the Mean value Theorem provide the existence of levels $t_i^k \in I_i^k$ satisfying

$$\sum_{i=1}^{\infty} \mathrm{Per}\left(\partial^*\{u_k \ge t_i^k\} \setminus J_{u_k}\right) \le \|\nabla u_k\|_{L^1(B_1)}^{1/2}. \tag{5.5}$$

Then define the functions w_k to be equal to t_i^k on $\{u_k \ge t_i^k\} \setminus \{u_k \ge t_{i+1}^k\}$. The choice of the I_i^k's, (5.5) and the very definition yield that w_k belongs to $SBV(B_1)$ and that it satisfies properties (i) and (ii). To conclude, note that $\mathcal{H}^1\left(J_{w_k} \setminus (\cup_i \partial^*\{u_k \ge t_i^k\} \cup J_{u_k})\right) = 0$ by construction, thus item (iii) follows at once from (5.5).

Step 2. *Compactness for the jump sets.*

Each function w_k determines a Caccioppoli partition $\mathscr{E}_k = \{E_i^k\}_{i\in\mathbb{N}}$ of B_1 (see [6, Lemma 1.11]). In addition, upon reordering the sets E_i^k's, we may assume that $\mathcal{L}^2(E_i^k) \ge \mathcal{L}^2(E_j^k)$ if $i < j$. Then, the compactness theorem for Caccioppoli partitions (see [16, Theorem 4.1, Proposition 3.7] and [2, Theorem 4.19]) provides us with a subsequence (not relabeled) and a Caccioppoli partition $\mathscr{E} := \{E_i\}_{i\in\mathbb{N}}$ such that

$$\lim_j \sum_{i=1}^{\infty} \mathcal{L}^2(E_i^k \triangle E_i) = 0, \qquad \text{and} \qquad \sum_{i=1}^{\infty} \mathrm{Per}(E_i, A) \le \liminf_k \sum_{i=1}^{\infty} \mathrm{Per}(E_i^k, A) \tag{5.6}$$

for all open subsets A in B_1. We claim that $\mathscr{E}$ determines a minimal Caccioppoli partition and in proving this we will also establish (5.2).

We start off observing that the first identity (5.6) and the Co-Area formula yield the existence of a set $I \subset (0,1)$ of full measure such that

$$\liminf_k \sum_{i=1}^{\infty} \mathcal{H}^1\left((E_i^k \triangle E_i) \cap \partial B_\rho\right) = 0 \qquad \forall \rho \in I\,. \tag{5.7}$$

Define the measures μ_k as $\mu_k(A) := \mathrm{MS}(u_k, A) + \mathrm{MS}(w_k, A)$ (A being an arbitrary Borel subset of B_1). Condition (2.1) and item (iii) in Step 1 ensure that, upon the extraction of a further subsequence, μ_k converges weakly* to a finite measure μ on B_1. W.l.o.g. we may assume that for all $\rho \in I$ we have, in addition, $\mu(\partial B_\rho) = 0$.

Let us now fix a Caccioppoli partition $\mathscr{F} := \{F_i\}_{i\in\mathbb{N}}$ suitable to test the minimality of $\mathscr{E}$, i.e. $\sum_{i=1}^{\infty}\mathcal{L}^2\left((F_i\triangle E_i)\cap(B_1\setminus\overline{B_t})\right)=0$ for some $t\in(0,1)$. Moreover, we may also suppose that $\sum_{i=1}^{\infty}\mathcal{H}^1\left((F_i\triangle E_i)\cap\partial B_\rho\right)=0$ for all $\rho\in I\cap(t,1)$. Let then ρ and r be radii in $I\cap(t,1)$ with $\rho<r$ and assume, after passing to a subsequence (not relabeled) that the lim inf in (5.7) is actually a lim for these two radii. We define

$$\omega_k := \begin{cases} w_k & \text{on } B_1\setminus\overline{B_\rho} \\ t_i^k & \text{on } F_i\cap B_\rho. \end{cases}$$

Note that $\omega_k\in SBV(B_1)$ with $\nabla\omega_k=0$ $\mathcal{L}^2$ a.e. on B_1, and since $t<\rho\in I$ it follows

$$\mathcal{H}^1\left(J_{\omega_k}\triangle\left((J_{\mathscr{F}}\cap B_\rho)\cup(\cup_{i\in\mathbb{N}}(E_i^k\triangle E_i)\cap\partial B_\rho)\cup(J_{w_k}\cap(B_1\setminus\overline{B_\rho}))\right)\right)=0.$$

Consider $\varphi\in\mathrm{Lip}\cap C_c(B_1,[0,1])$ with $\varphi|_{B_r}\equiv 1$, and $|\nabla\varphi|\le(1-r)^{-1}$ on B_1, and set $v_k:=\varphi\,\omega_k+(1-\varphi)\,u_k$. Clearly, v_k is admissible to test the minimality of u_k.

Consider next any open set A containanining $\overline{B_t}$. Simple calculations lead to

$$\begin{aligned}
\mathrm{MS}(u_k,A) &\le \mathrm{MS}(v_k,A)\\
&\le \mathrm{MS}(\omega_k,A)+2\mathrm{MS}(u_k,B_1\setminus\overline{B_r})+\frac{2}{(1-r)^2}\|u_k-\omega_k\|^2_{L^2(B_1\setminus\overline{B_r})}\\
&\le \mathcal{H}^1\left(J_{\mathscr{F}}\cap A\right)+\sum_{i\in\mathbb{N}}\mathcal{H}^1\left((E_i^k\triangle E_i)\cap\partial B_\rho\right)+\mathcal{H}^1\left(J_{w_k}\cap(A\setminus\overline{B_\rho})\right)\\
&\quad+2\mathrm{MS}(u_k,B_1\setminus\overline{B_r})+\frac{2}{(1-r)^2}\|u_k-w_k\|^2_{L^2(B_1\setminus\overline{B_r})}\\
&\le \mathcal{H}^1\left(J_{\mathscr{F}}\cap A\right)+\sum_{i\in\mathbb{N}}\mathcal{H}^1\left((E_i^k\triangle E_i)\cap\partial B_\rho\right)+3\mu_k(B_1\setminus\overline{B_\rho})\\
&\quad+\frac{2}{(1-r)^2}\|u_k-w_k\|^2_{L^\infty(B_1)}.
\end{aligned}\tag{5.8}$$

Note that in the third inequality we have used that ω_k and w_k coincide on $B_1\setminus\overline{B_\rho}$, and that $\rho<r$. By letting $k\uparrow\infty$ in (5.8), we infer

$$\begin{aligned}
\mathcal{H}^1(J_{\mathscr{E}}\cap A) &\le \liminf_k \mathcal{H}^1(J_{u_k}\cap A)\le\liminf_k \mathrm{MS}(u_k,A)\le\limsup_k \mathrm{MS}(u_k,A)\\
&\le \limsup_k \mathrm{MS}(v_k,A)\le\mathcal{H}^1\left(J_{\mathscr{F}}\cap A\right)+3\mu(B_1\setminus\overline{B_\rho}),
\end{aligned}$$

where we have used that r and ρ belong to I, inequality (5.3), the convergence $\mu_k\rightharpoonup^*\mu$, and the limit (5.7). Finally, by letting $\rho\in I$ tend to 1^- we conclude

$$\mathcal{H}^1(J_{\mathscr{E}}\cap A)\le\mathcal{H}^1\left(J_{\mathscr{F}}\cap A\right),\tag{5.9}$$

which proves the minimality of $\mathscr{E}$ in A (and hence, in particular, in B_1). Therefore, $J_{\mathscr{E}}$ satisfies the density lower bound $\mathcal{H}^1(J_{\mathscr{E}}\cap B_r(x))\ge 1$ for all $x\in\overline{J_{\mathscr{E}}}$ (see Step 1 of

Proposition 3.2), hence it is essentially closed. Using the De Giorgi, Carriero, Leaci density lower bound (see formula (2.2)), we conclude that $(\overline{J_{u_k}})_{k\in\mathbb{N}}$ converges to $\overline{J_{\mathscr{E}}}$ in the local Hausdorff topology on $\overline{B_1}$. In addition, choosing $\mathscr{E} = \mathscr{F}$ (which therefore allows us to take A arbitrary), we infer (5.2). □

6. Proof of Theorem 1.3

Fix any exponent $q \in (1, 2)$ and set $\rho = \rho_0/8$, where ρ_0 is the radius provided by Lemma 4.1.

We argue by contradiction and assume that a sequence $(u_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}(B_1)$ exists with

$$\int_{B_\rho} |\nabla u_k|^2 dx \geq k \left(\int_{B_1} |\nabla u_k|^q dx \right)^{2/q} . \tag{6.1}$$

The energy upper bound (2.1) then leads to

$$\lim_k \int_{B_1} |\nabla u_k|^q dx = 0.$$

Thus, Theorem 5.1 gives us a subsequence (not relabeled for convenience) and a Caccioppoli partition $\mathscr{E}$ such that all the conclusions there hold true. By Lemma 4.1, we have $\mathcal{H}^0(\bar{J}_{\mathscr{E}} \cap \partial B_{\rho_0}) \leq 3$.

Since $J_{\mathscr{E}}$ and J_{u_k} are both essentially closed, from now on we use, by a slight abuse of notation, the same names for their closures. We can distinguish three different cases:

- (a$_1$) $\mathcal{H}^0(J_{\mathscr{E}} \cap \partial B_{\rho_0}) = 0$, then set $\varrho := \rho_0$;
- (a$_2$) $\mathcal{H}^0(J_{\mathscr{E}} \cap \partial B_{\rho_0}) = 2$, hence $J_{\mathscr{E}} \cap B_{\rho_0}$ is a segment and $\partial B_{\rho_0} \setminus J_{\mathscr{E}}$ is the union of two arcs. Then, either both arcs have length less than $\frac{4\pi}{3}\rho_0$, or $J_{\mathscr{E}} \cap B_{\rho_0/2} = \emptyset$. In the first alternative we set $\varrho := \rho_0$, in the latter $\varrho := \rho_0/2$;
- (a$_3$) $\mathcal{H}^0(J_{\mathscr{E}} \cap B_{\rho_0}) = 3$, $J_{\mathscr{E}}$ is a (possibly off-centered) spider and $\partial B_{\rho_0} \setminus J_{\mathscr{E}}$ is the union of three arcs. Then, either all of them have length less than $(2\pi - \frac{1}{8})\rho_0$ and in this case we set $\varrho := \rho_0$, or $\mathcal{H}^0(J_{\mathscr{E}} \cap B_{\rho_0/2}) = 2$. In this last event we are back in the setting of item (ii) above with $\frac{\rho_0}{2}$ playing the role of ρ_0. Thus $\partial B_{\rho_0/2} \setminus J_{\mathscr{E}}$ is either the union of two arcs, both with length smaller than $\frac{2}{3}\pi\rho_0$ (and we set $\varrho := \frac{\rho_0}{2}$), or $J_{\mathscr{E}} \cap B_{\rho_0/4} = \emptyset$, and then set $\varrho := \frac{\rho_0}{4}$.

Summarizing: $\varrho \geq \rho_0/4$ and

- (b$_1$) either $J_{\mathscr{E}} \cap B_\varrho = \emptyset$;
- (b$_2$) or $J_{\mathscr{E}} \cap B_\varrho$ is a segment and $\partial B_\varrho \setminus J_{\mathscr{E}}$ is the union of two arcs each with length $< \frac{4\pi}{3}\varrho$;
- (b$_3$) of $J_{\mathscr{E}} \cap B_\varrho$ is a spider and $\partial B_\varrho \setminus J_{\mathscr{E}}$ the union of three arcs each with length $< (2\pi - \frac{1}{8})\varrho$.

By (5.2) in Theorem 5.1 and the local Hausdorff convergence of $(\overline{J_{u_k}})_{k\in\mathbb{N}}$ to $\overline{J_{\mathscr{E}}}$ on $\overline{B_1}$, it is possible to select $L>0$ such that for all $k\geq L$ the following condition holds true

$$\int_{B_\varrho}|\nabla u_k|^2\,dx+\mathrm{dist}_{\mathcal{H}}(\overline{J_{\mathscr{E}}}\cap B_\varrho,\overline{J_{u_k}}\cap B_\varrho)\leq\varepsilon\,\varrho.$$

By Theorem 2.1 (we keep the notation introduced there), we may find a constant $\beta\in(0,1/3)$ such that for all $k\geq L$ one of the following alternatives happens

- (c_1) $J_{u_k}\cap B_\varrho=\emptyset$;
- (c_2) For each $t\in((1-\beta)\varrho,\varrho)$, $\partial B_t\setminus J_{u_k}$ is the union of two arcs γ_1^k and γ_2^k each with length $<(2\pi-\frac{1}{9})t$, whereas $J_{u_k}\cap B_t$ is connected and divides B_t in two components B_1^k, B_2^k with $\partial B_i^k=\gamma_i^k\cup(J_{u_k}\cap\overline{B_t})$;
- (c_3) For each $t\in((1-\beta)\varrho,\varrho)$, $\partial B_t\setminus J_{u_k}$ is the union of three arcs γ_1^k, γ_2^k and γ_3^k each with length $<(2\pi-\frac{1}{9})t$, whereas $B_t\cap J_{u_k}$ is connected and divides B_t in three connected components B_1^k, B_2^k and B_3^k with $\partial B_i^k\subset\gamma_i^k\cup(J_{u_k}\cap\overline{B_t})$.

We finally choose $r\in((1-\beta)\varrho,\varrho)$ and a subsequence, not relabeled, such that

- (A) $g_k:=u_k|_{\partial B_r}$ belongs to $W^{1,q}(\gamma)$ for any connected component γ of $\partial B_r\setminus J_{u_k}$;
- (B) g_k satisfies

$$\int_{\partial B_r\setminus J_{u_k}}|g_k'|^q d\mathcal{H}^1\leq\frac{1}{\beta\varrho}\int_{B_\varrho}|\nabla u_k|^q\,dx\leq\frac{4}{\beta\rho_0}\int_{B_1}|\nabla u_k|^q\,dx.$$

Let us conclude our argument by showing that (6.1) is violated for k sufficiently big. To this aim we note first that the choices of ρ, β and ϱ yield $r>\rho$.

In case (c_1) holds, $\partial B_r\cap J_{u_k}=\emptyset$ and u_k is the harmonic extension of its boundary trace g_k. Hence, for some constant $C>0$ (independent of k)

$$\begin{aligned}\int_{B_\rho}|\nabla u_k|^2 &\leq \int_{B_r}|\nabla u_k|^2\leq C\min_c\|g_k-c\|^2_{H^{1/2}(\partial B_r)}\\ &\leq C\left(\int_{\partial B_r}|g_k'|^q\,d\mathcal{H}^1\right)^{2/q}\overset{(B)}{\leq}C\left(\frac{4}{\beta\rho_0}\int_{B_1}|\nabla u_k|^q\,dx\right)^{2/q},\end{aligned}$$

contradicting (6.1).

In case (c_2) or (c_3) hold the construction is similar. Denote by K_k the minimal connection relative to $J_{u_k}\cap\partial B_r$. Then K_k splits $\overline{B_r}$ into two (case (c_2)) or three (case (c_3)) regions denoted by B_r^i. Let γ^i be the arc of ∂B_r contained in the boundary of B_r^i. By Lemma 2.3 we find a function $w_k^i\in W^{1,2}(B_r)$ with boundary trace g_k and satisfying for some absolute constant $C>0$

$$\int_{B_r}|\nabla w_k^i|^2\,dx\leq C\left(\int_{\gamma^i}|g_k'|^q\,d\mathcal{H}^1\right)^{2/q}. \tag{6.2}$$

Denote by w_k the function equal to w_k^i on B_k^i. It is easy to check that $w_k \in SBV(B_r)$, and that $J_{w_k} \subseteq K_k$. The minimality of u_k implies then

$$\begin{aligned}\int_{B_\rho} |\nabla u_k|^2 \quad &\leq \quad \int_{B_r} |\nabla u_k|^2 \leq \int_{B_r} |\nabla w_k|^2 + \mathcal{H}^1(K_k) - \mathcal{H}^1(J_{u_k} \cap B_r) \leq \int_{B_r} |\nabla w_k|^2 \\ &\overset{(6.2)}{\leq} C \left(\int_{\partial B_r \setminus J_{u_k}} |g_k'|^q \, d\mathcal{H}^1 \right)^{2/q} \overset{(B)}{\leq} C \left(\frac{4}{\beta \rho_0} \int_{B_1} |\nabla u_k|^q \, dx \right)^{2/q}, \end{aligned} \tag{6.3}$$

contradicting (6.1). $\square$

Remark 6.1. After the first technical step in which we reduce to the case where the sets J_{u_k} have a nice structure, the core of the argument is the construction of the competitor w_k. Our knowledge of J_{u_k} is used to make J_{w_k} shorter than J_{u_k}, which is a key point for (6.3).

In order to give an explicit estimate for the constant C in Theorem 1.3 it would then suffice to find a variational argument which avoids the first compactness step of the proof, i.e. an argument which works without any apriori knowledge of the structure of J_{u_k}. To this aim one would like to construct a competitor w_k enjoying the bounds

$$\int_{B_r} |\nabla w_k|^2 \leq C \left(\sum_i \int_{\gamma^i} |g_k'|^q d\mathcal{H}^1 \right)^{\frac{2}{q}} \tag{6.4}$$

and

$$\mathcal{H}^1(J_{u_k} \cap B_r) - \mathcal{H}^1(J_{w_k} \cap B_r) \leq C \left(\int_{B_1} |\nabla u_k|^q \, dx \right)^{2/q} . \tag{6.5}$$

Under the present assumptions we do not know, however, whether J_{u_k} "separates" those pairs of arcs γ^i, γ^j for which

$$\left| \fint_{\gamma^j} g_k - \fint_{\gamma^i} g_k \right|$$

is large compared to $\|g_k'\|_{L^q}$. To overcome this difficulty we could enlarge J_{u_k} so that J_{w_k} does separate those pairs of arcs. In this case the total added length should then be estimated in terms of ∇u_k. As suggested by Guy David to the first author, this might be done by adding portions of level sets of u_k, which in turn can be estimated in terms of ∇u_k using the coarea formula. Some technical lemmas exploiting this idea are already present in [8].

7. A remark on the Mumford-Shah conjecture

In this section we shall prove Proposition 1.5, for which we need the following preliminary observation.

Lemma 7.1. *Let $f \in L^{4,\infty}_{loc}(\Omega)$, then for all $\varepsilon > 0$ the set*

$$D_\varepsilon := \left\{ x \in \Omega : \liminf_r \frac{1}{r} \int_{B_r(x)} f^2(y)\, dy \geq \varepsilon \right\} \tag{7.1}$$

is locally finite.

Proof. We shall show in what follows that if $f \in L^{4,\infty}(\Omega)$ then D_ε is finite, an obvious localization argument then proves the general case.

Let $\varepsilon > 0$ and consider the set D_ε in (7.1) above. First note that, for any $B_r(x) \subset \Omega$ and any $\lambda > 0$ we have the estimate

$$\begin{aligned} \int_{\{y \in B_r(x) :\, |f(y)| \geq \lambda\}} f^2(y)\, dy &\leq \int_{\{y \in \Omega :\, |f(y)| \geq \lambda\}} f^2(y)\, dy \\ &= 2 \int_\lambda^{+\infty} t\, |\{y \in \Omega : \, |f(y)| \geq t\}| dt \\ &\leq \int_\lambda^{+\infty} \frac{2K}{t^3} dt = \frac{K}{\lambda^2}. \end{aligned} \tag{7.2}$$

If $x \in D_\varepsilon$ and $r > 0$ satisfy

$$\int_{B_r(x)} f^2(y)\, dy \geq \frac{\varepsilon}{2} r, \tag{7.3}$$

choosing $\lambda = 2(K/r\varepsilon)^{1/2}$ in (7.2) we conclude

$$\int_{\{y \in B_r(x) :\, |f(y)| < 2(\frac{K}{r\varepsilon})^{1/2}\}} f^2(y)\, dy \geq \frac{\varepsilon}{4} r. \tag{7.4}$$

Furthermore, the trivial estimate

$$\int_{\{y \in B_r(x) :\, |f(y)| < \lambda\}} f^2(y)\, dy < \pi \lambda^2 r^2,$$

implies for $\lambda = (\varepsilon/8\pi r)^{1/2}$

$$\int_{\{y \in B_r(x) :\, |f(y)| < (\frac{\varepsilon}{8\pi r})^{1/2}\}} f^2(y)\, dy < \frac{\varepsilon}{8} r. \tag{7.5}$$

By collecting (7.4) and (7.5) we infer

$$\int_{\{y \in B_r(x) :\, (\frac{\varepsilon}{8\pi r})^{1/2} \leq |f(y)| < 2(\frac{K}{r\varepsilon})^{1/2}\}} f^2(y)\, dy \geq \frac{\varepsilon}{8} r,$$

that in turn implies

$$|\{y \in B_r(x) : \, |f(y)| \geq (\frac{\varepsilon}{8\pi r})^{1/2}\}| \geq \frac{\varepsilon^2 r^2}{32K}. \tag{7.6}$$

Let $\{x_1,\dots,x_N\} \subseteq D_\varepsilon$ and $r>0$ be a radius such that the balls $B_r(x_i) \subseteq \Omega$ are disjoint and (7.3) holds for each x_i. Then, from (7.6) and the fact that $f \in L^{4,\infty}(\Omega)$, we infer

$$N\frac{\varepsilon^2 r^2}{32K} \le |\{y\in\Omega:\,|f(y)| \ge (\frac{\varepsilon}{8\pi r})^{1/2}\}| \le \frac{K(8\pi r)^2}{\varepsilon^2} \Longrightarrow N \le \frac{2^{11}K^2\pi^2}{\varepsilon^4},$$

and the conclusion follows at once. □

We are now ready to give the proof of Proposition 1.5.

Proof of Proposition 1.5. To prove the direct implication we assume without loss of generality that $\Omega = B_R$ for some $R>1$, being the result local. In addition, we may also suppose that $\overline{J_u} \cap \partial B_1 = \{y_1,\dots,y_M\}$. Theorem 2.1 and Theorem 5.1 yield that there exists some $\varepsilon_0>0$ such that for all points $x \in B_R \setminus D_{\varepsilon_0}$ the set $\overline{J_u} \cap B_r(x)$ is either empty or diffeomorphic to a minimal cone, for some $r>0$. In particular, in the latter event $B_r(x) \setminus \overline{J_u}$ is not connected.

Supposing that $D_{\varepsilon_0} \cap B_1 = \{x_1,\dots,x_N\}$, and setting

$$\Omega_k := B_{1-1/k} \setminus \bigcup_{i=1}^{N} B_{1/k}(x_i)\,,$$

a covering argument and the last remark give that for every $x \in \Omega_k \cap \overline{J_u}$ there is a continuous arc $\gamma_k : [0,1] \to \overline{J_u}$ with $\gamma_k(0)=x$ and $\gamma_k(1) = y \in \partial\Omega_k$. Then, the sequence $(\widetilde{\gamma}_k)_{k\in\mathbb{N}}$ of reparametrizations of the γ_k's by arc length converges to some arc $\gamma:[0,1]\to\overline{J_u}$ with $\gamma(0)=x$ and $\gamma(1) \in \{x_1,\dots,x_N,y_1,\dots,y_M\}$.

From this, we deduce that $\overline{B_1} \cap \overline{J_u}$ has a finite number of connected components. Bonnet's regularity results [4, Theorems 1.1 and 1.3] then provide the thesis.

To conclude we prove the opposite implication. To this aim we consider $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and suppose that $\overline{J_u}\cap\Omega''$ is a finite union of C^1 arcs of finite length. Observe that these arcs are locally C^∞ (see for instance [2]). Denote by $\{x_1,\dots,x_N\}$ the end points of the arcs in Ω' and let $r>0$ be such that $B_{4r}(x_i) \subseteq \Omega'$ for all i, and $B_{4r}(x_i) \cap B_{4r}(x_j) = \emptyset$ if $i \ne j$. [2, Theorem 7.49] (or [8, Proposition 17.15]) implies that ∇u has a $C^{0,\alpha}$ extension on both sides of $(\Omega''\cap\overline{J_u}) \setminus \cup_i \overline{B_r(x_i)}$ for all $\alpha<1$. In particular, ∇u is bounded on $\overline{\Omega'} \setminus \cup_i B_{2r}(x_i)$.

Next consider the sequence $r_k = r/2^{k-1}$, $k \ge 0$, and fix $i \in \{1,\dots,N\}$. Then, by [8, Proposition 37.8] (or [4, Theorem 2.2]) we can extract a subsequence $k_j \uparrow \infty$ along which the blow-up functions $u_j(x) := r_{k_j}^{-1/2} u(x_i + r_{k_j}x) - a_j$ converge to some w in $W^{1,2}_{loc}(B_4 \setminus K)$, for some piecewise constant function $a_j : \Omega \setminus \overline{J_{u_j}} \to \mathbb{R}$, and $(\overline{J_{u_j}})_{j\in\mathbb{N}}$ converges to some set K in the Hausdorff metric.

By Bonnet's blow-up theorem [4, Theorem 4.1] only two possibilities occur: either x_i is a spider point, i.e., K is a spider and w is locally constant on $B_4 \setminus K$, or x_i is a spiral point, i.e., up to a rotation $K = \{(x,0) : x \le 0\}$ and $w(\rho,\theta) = C \pm \sqrt{\frac{2}{\pi}\rho} \cdot \sin(\theta/2)$ for $\theta \in (-\pi,\pi)$, $\rho > 0$ and some constant $C \in \mathbb{R}$ (note that in principle the blow-up limit in this case might be non unique, as if $\overline{J_u}$ was a slow-turning spiral ending in x_i (cp. with [8, Theorem 69.29])).

In both cases, we claim that ∇u_j has a $C^{0,\alpha}$ extension on the closure of each connected component of $U_j := (B_3 \setminus \overline{B_1}) \setminus \overline{J_{u_j}}$ with $\sup_j \|\nabla u_j\|_{L^\infty(U_j)} \le C$. This follows as in [2, Theorem 7.49] (or [8, Proposition 17.15], see also Remark 2.2) locally straightening $\overline{J_{u_j}} \cap (B_4 \setminus \overline{B_{1/2}})$ onto $K \cap (B_4 \setminus \overline{B_{1/2}})$ via a $C^{1,\alpha}$ conformal map, a reflection argument and standard Schauder estimates for the laplacian. Scaling back the previous estimate gives

$$|\nabla u(x)| \le C\,|x - x_i|^{-1/2} \qquad \text{for } x \in \cup_{j\in\mathbb{N}}(\overline{B_{3r_{k_j}}(x_i)} \setminus B_{r_{k_j}}(x_i)),$$

in turn from this, the maximum principle and Hopf's lemma we infer

$$|\nabla u(x)| \le C\, r_k^{-1/2} \qquad \text{for } x \in B_{2r}(x_i) \setminus B_{r_k}(x_i).$$

The latter inequality finally implies $\nabla u \in L^{4,\infty}(B_{2r}(x_i))$.

Eventually, we are able to conclude $\nabla u \in L^{4,\infty}(\Omega')$, being on one hand ∇u bounded on $\overline{\Omega'} \setminus \cup_i B_{2r}(x_i)$, and on the other hand belonging to $L^{4,\infty}(\cup_i B_{2r}(x_i))$. □

Appendix A

Lemma A.1. *Let $\Omega \subset B_1$ be a topological disk with $\partial\Omega$ locally connected. Assume that $\partial\Omega = \alpha \cup L$, where α is a closed arc of ∂B_1 with (distinct) extrema a and b and L a compact set with $L \cap \alpha = \{a,b\}$. If $p \in \alpha \setminus \{a,b\}$, then $\partial\Omega \setminus \{p\}$ is connected.*

Proof. We apply [22, Continuity Theorem, page 18] to conclude that there is a continuous map $z : \overline{B_1} \to \overline{\Omega}$ such that $z|_{B_1}$ is a (conformal) homeomorphism onto Ω. By [22, Proposition 2.5], p disconnects $\partial\Omega$ if and only if $z^{-1}(p)$ consists of more than one point. Observe that if q is another point of $\alpha \setminus \{a,b\}$, then $\partial\Omega \setminus \{p\}$ is connected if and only if $\partial\Omega \setminus \{q\}$ is connected. Therefore, either each point $p \in \alpha \setminus \{a,b\}$ has a single counterimage through z or they all have more than one counterimage. Assume by contradiction that each $p \in \alpha \setminus \{a,b\}$ has at least two counterimages. By [22, Corollary 2.19] the set of p's with more than two counterimages is countable.

Consider now any open arc $\beta \subset\subset \alpha$ with endpoints a', b' such that $z^{-1}(a')$ and $z^{-1}(b')$ consist both of two points. $z^{-1}(\beta)$ is an open subset of $\mathbb{S}^1$ and hence consists of (at most) countably many disjoint arcs η_i. The endpoints of each η_i are, by continuity contained in $z^{-1}(\{a',b'\}$. Hence there are exactly two such arcs. Consider a point

$c \in \alpha \setminus \{a, b\}$ having exactly two distinct counterimages c_1 and c_2 and let β_i be a sequence of arcs as above with $\cap_i \beta_i = \{c\}$. Obviously $\cap_i z^{-1}(\beta_i) = \{c_1, c_2\}$. Thus, for i sufficiently large there are at least two connected components η_1 and η_2 of $z^{-1}(\beta_i)$, at positive distance, one containing c_1 and the other containing c_2. η_1 and η_2 are two arcs. Let d_i, e_i be their respective extrema and let a', b' be the extrema of $\beta_i =: \beta$. We can (after relabiling the extrema) distinguish two cases.

Case 1 $z(d_1) = z(e_1) = a'$ and $z(d_2) = z(e_2) = b'$. Consider the open arcs delimited by d_1 and c_1 and by c_1 and e_1. They are both mapped onto the arc delimited by a' and c. Now, since the points with more than 2 preimages are countably many, the restriction of z to each arc must be injective. Passing to a smaller arc, we find then an open arc $\omega \subset \alpha$ and two open arcs ω_1, $\omega_2 \subset \mathbb{S}^1$ such that $z|_{\omega_i}$ is an homeomorphism onto ω and the distance between the ω_i is positive.

Case 2 $z(d_1) = z(d_2) = a'$ and $z(e_1) = z(e_2) = b'$. Then the two arcs ω_1 and ω_2 are precisely given by η_1 and η_2, whereas ω can be chosen equal to β: indeed, again by the countability of the points with more than two preimages, $z|_{\eta_i}$ must be injective, which means that z maps each η_i homeomorphically onto β.

We fix the arcs ω_1, ω_2 and ω found above. Let $q \in \omega$ be such that $z^{-1}(q)$ consists of two points. Observe that if r belongs to a sufficiently small neighborhood of q, then $z^{-1}(r)$ consists also of two points. Otherwise there would be a sequence $(r_k)_{k\in\mathbb{N}}$ converging to q with $z^{-1}(r_k)$ consisting each of at least three points. Since $z^{-1}(q) \cap \omega_i$ consists of exactly one point, this would give a sequence $(r'_k)_{j\in\mathbb{N}} \subset \mathbb{S}^1 \setminus (\omega_1 \cup \omega_2)$ such that $z(r'_k) = r_k$. But then there must be a point $r'_\infty \in \mathbb{S}^1 \setminus (\omega_1 \cup \omega_2)$ with $z(r'_\infty) = q$. Since each ω_i contains a preimage of q, we conclude that q has at least three preimages, which is a contradiction.

Therefore, if we make ω smaller, we can assume that $z^{-1}(\omega) = \omega_1 \cup \omega_2$, as well as that $z|_{\omega_i}$ is an homeomorphism onto ω.

Let d and e be the endpoints of ω and consider a point $P \in \Omega$. Let S be the open sector delimited by the segments $[P, d]$, $[P, e]$ and the arc ω. If ω is sufficiently small and the point P sufficiently close to ω, the sector S is containd in Ω. We then define the map $R : [0, 1] \times (\overline{B_1} \setminus \{P\}) \to \overline{B_1}$ as the usual retraction: if $x \in \overline{B_1}$, we let s be the halfline originating in P and containing x and we define $R(1, x) = s \cap \partial B_1$ and $R(\lambda, x) = (1 - \lambda)x + \lambda R(1, x)$. Consider the map $\zeta = R(1, z)$ (recall that $\Omega \subseteq B_1$). R is an homotopy between $z|_{\partial B_1}$ and ζ. We define $\deg(\zeta, P)$ as the degree in P of any continuous extension of ζ to $\overline{B_1}$ (note that this degree does not depend upon the chosen extension, see [17, Theorem 2.14]). Since P is not in the image through $R(\lambda, z)$

of ∂B_1, by [17, Theorem 2.12] we have $\deg(z,P) = \deg(\zeta,P)$. On the other hand, since $z|_{B_1}$ is a diffeomorphism onto Ω and $P \in \Omega$, $\deg(z,P)$ is either 1 or -1. Without loss of generality, we can assume that $\deg(z,P) = 1$. Thus $\deg(\zeta,P) = 1$ as well. But since ζ maps $\mathbb{S}^1 = \partial B_1$ into itself, $\deg(\zeta,P)$ is the winding number W of ζ (see page 20 of [17]).

Observe next that $R(1,\cdot)$ is the identity on ω and that it maps any point outside the sector S in $\partial B_1 \setminus \omega$. Therefore, $\zeta^{-1}(\omega) = \omega_1 \cup \omega_2$ and $\zeta|_{\omega_i} = z|_{\omega_i}$. It is easy to see that ζ can be realized as the uniform limit of smooth maps $\zeta_k : \mathbb{S}^1 \to \mathbb{S}^1$ retaining the properties that $\zeta_k^{-1}(\omega) = \omega_1 \cup \omega_2$ and that $\zeta_k|_{\omega_i}$ is an homeomorphism onto ω. So, for k large enough the winding number W of ζ_k must be 1. However, if we take a regular point O of ζ_k, we can compute W using the formula

$$W = \sum_{q \in \zeta_k^{-1}(O)} \mathrm{sign}\left(d\zeta_k(q)\right).$$

But for $O \in \omega$, the set $\zeta_k^{-1}(O)$ consists of exactly two points and hence W is 2, 0 or -2. This is a contradiction and completes the proof. $\square$

Lemma A.2. *Let $A \subset \mathbb{R}^2$ be a bounded open set homeomorphic to the disk B_1. Then ∂A is connected.*

Proof. Let $z : B_1 \to A$ be an homeomorphism. For all $k \in \mathbb{N} \setminus \{0\}$ set

$$E_k := B_1 \setminus \overline{B_{1-1/k}} \qquad \text{and} \qquad G_k := \overline{z(E_k)}\,.$$

We claim that

$$\bigcap_k G_k = \partial A\,. \tag{A.1}$$

From (A.1) the claim of the lemma follows easily. Indeed each E_k is connected and so is $z(E_k)$, since z is an homeomorphism. But then G_k is the closure of a connected set, and hence connected. We conclude that the compact sets G_k converge in the sense of Hausdorff to ∂A and the connectedness of ∂A follows easily (see for instance [14, Theorem 3.18]).

In order to show (A.1) we first observe that $z(E_k) \subset A$ and hence $G_k \subset \bar{A}$. On the other hand, if $x \in A$, $y = z^{-1}(x) \in B_1$ and there esxists $\rho > 0$ such that $B_\rho(y) \subset\subset B_1$. Thus, for k large enough, $z(B_\rho(y)) \cap z(E_k) = \emptyset$, and, since $z(B_\rho(y))$ is a neighborhood of x, $x \notin G_k$. We therefore conclude $\cap_k G_k \subset \partial A$. Next, consider $x \in \partial A$. Then there is a sequence $x_k \to x$ with $(x_k)_{k\in\mathbb{N}} \subset A$. A subsequence of $(z^{-1}(x_k))_{k\in\mathbb{N}}$ converges then to an element $y \in \overline{B_1}$ and y must necessarily belong to ∂B_1. Thus, for any fixed k, $z^{-1}(x_k) \in E_k$ provided k is large enough. But this easily implies $x \in G_k = \overline{z(E_k)}$. Hence we have shown the inclusion $\partial A \subset \cap_k G_k$, which concludes the proof. $\square$

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