REGULARITY THEORY FOR 2-DIMENSIONAL ALMOST MINIMAL CURRENTS I: LIPSCHITZ APPROXIMATION

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ABSTRACT. We construct Lipschitz Q-valued functions which approximate carefully integral currents when their cylindrical excess is small and they are almost minimizing in a suitable sense. This result is used in two subsequent works to prove the discreteness of the singular set for the following three classes of 2-dimensional integral currents: area minimizing in Riemannian manifolds, semicalibrated and spherical cross sections of 3-dimensional area minimizing cones.

This paper is the second in a series of works aimed at establishing an optimal regularity theory for 2-dimensional integral currents which are almost minimizing in a suitable sense. Building upon the monumental work of Almgren [2], Chang in [5] established that 2-dimensional area-minimizing currents in Riemannian manifolds are classical minimal surfaces, namely they are regular (in the interior) except for a discrete set of branching singularities. The argument of Chang is however not entirely complete since a key starting point of his analysis, the existence of the so-called "branched center manifold", is only sketched in the appendix of [5] and requires the understanding (and a suitable modification) of the most involved portion of the monograph [2].

An alternative proof of Chang's theorem has been found by Rivière and Tian in [21] for the special case of J-holomorphic curves. Later on the approach of Rivière and Tian has been generalized by Bellettini and Rivière in [4] to handle a case which is not covered by [5], namely that of special Legendrian cycles in $\mathbb{S}^5$ (see also [3] for a further generalization).

Meanwhile the first and second author revisited Almgren's theory giving a much shorter version of his program for proving that area-minimizing currents are regular up to a set of Hausdorff codimension 2, cf. [6, 8, 7, 9, 10]. In this note and its companion papers [11, 12] we build upon the latter works in order to give a complete regularity theory which includes both the theorems of Chang and Bellettini-Rivière as special cases, in particular recovering the fine description of the structure of singular points proven by Chang and extending this picture to the cases of semicalibrated currents and spherical cross-sections (we refer to [11, 12] for more precise statements).

We start by introducing the following terminology (cf. [13, Definition 0.3]).

Definition 0.1. Let $\Sigma \subset R^{m+n}$ be a C^2 submanifold and $U \subset \mathbb{R}^{m+n}$ an open set.

    (a) An m-dimensional integral current T with finite mass and $\mathrm{spt}(T) \subset \Sigma \cap U$ is area-minimizing in $\Sigma \cap U$ if $\mathbf{M}(T + \partial S) \geq \mathbf{M}(T)$ for any $(m+1)$-dimensional integral current S with $\mathrm{spt}(S) \subset\subset \Sigma \cap U$.

(b) A semicalibration (in Σ) is a C^1 m-form ω on Σ such that $\|\omega_x\|_c \leq 1$ at every $x \in \Sigma$, where $\|\cdot\|_c$ denotes the comass norm on $\Lambda^m T_x \Sigma$. An m-dimensional integral current T with $\mathrm{spt}(T) \subset \Sigma$ is *semicalibrated* by ω if $\omega_x(\vec{T}) = 1$ for $\|T\|$-a.e. x.

(c) An m-dimensional integral current T supported in $\partial \mathbf{B}_R(p) \subset \mathbb{R}^{m+n}$ is a *spherical cross-section of an area-minimizing cone* if $p \mathbin{\text{\textzero}} T$ is area-minimizing.

Calibrated submanifolds, namely currents T as in (b) where the calibrated form is closed, have been central objects of study in several areas of differential geometry and mathematical physics since the seminal work of Harvey and Lawson, cf. [17]. Two primary examples are holomorphic subvarieties and special Lagrangians in Calabi-Yau manifolds, which play an important role in string theory (especially regarding mirror symmetry, cf. [18, 27]) but also emerge naturally in gauge theory (see [28]). Semicalibrations are a natural generalization of calibrations: since the condition $d\omega = 0$ on the calibrating form is rather rigid and in particular very unstable under deformations. In fact semicalibrations were considered already in [28] (cf. Section 6 therein) and around the same time they became rather popular in string theory, when several authors directed their attention to non-Calabi-Yau manifolds (the subject is nowadays known as "flux compactification", cf. [14]): in that context the natural notion to consider is indeed a special class of semicalibrating forms (see for instance the works [15, 16], where these are called quasi calibrations).

In what follows, given an integer rectifiable current T, we denote by $\mathrm{Reg}(T)$ the subset of $\mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ consisting of those points x for which there is a neighborhood U such that $T \llcorner U$ is a (constant multiple of) a regular oriented submanifold. Correspondingly, $\mathrm{Sing}(T)$ is the set $\mathrm{spt}(T) \setminus (\mathrm{spt}(\partial T) \cup \mathrm{Reg}(T))$. Observe that $\mathrm{Reg}(T)$ is relatively open in $\mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ and thus $\mathrm{Sing}(T)$ is relatively closed. The main result of this and the works [11, 12] is then the following

Theorem 0.2. *Let Σ and ω be as in Definition 0.1, let T be as in (a), (b) or (c) and assume in addition that $m = 2$, that Σ is of class $C^{3,\alpha}$ and ω of class $C^{2,\alpha}$ for some positive α. Then $\mathrm{Sing}(T)$ is discrete.*

Clearly Chang's result is covered by case (a). As already pointed out, the proof of Theorem 0.2 gives in fact more information, namely an accurate description of the behavior of T around any singular point. This is the exact analog of the singularity description provided by Chang [5] for the area minimizing case, whose validity is therefore extended to both cases (b) and (c) of Definition 0.1. The results of Theorem 0.2 are optimal, because of the well-known examples of area minimizing currents induced by singular complex curves. Note, however, that there are many singular semicalibrated currents which are not calibrated, and we give an example in the appendix.

The program of extending the Almgren-Chang regularity theory to general semicalibrated currents was started by Rivière and Tian in [22] and indeed their alternative proof of Chang's theorem was meant as a first step towards case (b) of Theorem 0.2 (cf. [22, page 743]). The first notable contribution which goes beyond the Almgren-Chang result is due to Pumberger and Rivière in [20], and important groundbreaking results were then achieved by Bellettini and Rivière in [4] and by Bellettini in [3]. In particular [4] proved

the theorem above for Legendrian cycles in $\mathbb{S}^5$, which form a special subclass of both (b) and (c). The result and the methods were then extended in [3] to a class of 2-dimensional semicalibrated currents in 5-dimensional manifolds which, roughly speaking, are based on Legendrian cycles as local models. In this and the notes [11, 12] we give a complete answer in the general 2-dimensional case. In higher dimensions Almgren's famous bound on the Hausdorff dimension of the singular set has been extended to semicalibrated currents by the third author in [25].

Following the Almgren-Chang program, Theorem 0.2 will be established through a suitable "blow-up argument" which requires four essential tools:

(i) The uniqueness of tangent cones for T. This result is a, by now classical, theorem of White for area-minimizing 2-dimensional currents in the Euclidean space, cf. [29]. Chang extended it to case (a) in the appendix of [5], whereas Pumberger and Rivière covered case (b) in [20]. A general derivation of these results for a wide class of almost minimizers has been given in [13]: the theorems in there cover, in particular, all the cases of Definition 0.1.

(ii) The theory of multiple-valued functions, pioneered by Almgren in [2], for which we will use the results and terminology of the papers [6, 8].

(iii) A suitable approximation procedure for integer rectifiable currents with graphs of multiple valued functions. The one needed in case (a) is already contained in [7], but the latter reference does not cover the cases (b) and (c): the purpose of this note is to extend the theorems in [7] to these cases.

(iv) The so-called "center manifold": this will be constructed in [11], whereas the final argument for Theorem 0.2 will then be given in [12].

In fact this note does more than just providing (iii) for the cases of (b) and (c), because we give an approximation theorem for almost minimal currents in any dimension m, see Definition 1.1 for the precise condition. Indeed, relaxing the minimizing condition in the regularity theory is a central theme in geometric measure theory: from the one hand it could be the first step towards the analysis of different elliptic functionals, from the other hand it has many applications in a variety of problems in which the minimizing condition must be weakened (the examples are numerous: we just cite the fundamental work of Almgren on elliptic variational problems with constraints [1] and the celebrated paper of Schoen and Simon on stable hypersurfaces, [23]). However, there are very few results in this direction in higher codimension: compared to the codimension one case the task is much harder, since several delicate arguments of the Almgren-Chang depend sensibly upon the minimizing assumption. This note gives a first contribution by establishing a strong approximation theorem under a very natural condition, see Definition 1.1 for the precise formulation.

1. NOTATION AND STATEMENT OF THE MAIN THEOREM

We introduce the notion of almost minimizers that we are going to use in the paper.

Definition 1.1 ($\mathbf{\Omega}$-minimality)**.** Let $\mathbf{\Omega}$ be a positive constant. An integer rectifiable m-dimensional current with compact support in $\mathbb{R}^{m+n}$ is called $\mathbf{\Omega}$-*minimal* if

$$\mathbf{M}(T) \leq \mathbf{M}(T + \partial S) + \mathbf{\Omega} \, \mathbf{M}(S) \qquad \forall S \in \mathbf{I}_{m+1}(\mathbb{R}^{m+n}) \quad \text{with compact support.} \qquad (1.1)$$

In order to state the main result, we need to introduce some notation. With $\mathbf{B}_r(p)$ and $B_r(x)$ we denote, respectively, the open ball with radius r and center p in $\mathbb{R}^{m+n}$ and the open ball with radius r and center x in $\mathbb{R}^m$. $\mathbf{C}_r(x)$ will always denote the cylinder $B_r(x) \times \mathbb{R}^n$ and the point x will be omitted when it is the origin. In fact, by a slight abuse of notation, we will often treat the center x as a point in $\mathbb{R}^{m+n}$, avoiding the correct, but more cumbersome, $(x, 0)$. e_i will denote the unit vectors in the standard basis, π_0 the (oriented) plane $\mathbb{R}^m \times \{0\}$ and $\vec{\pi}_0$ the m-vector $e_1 \wedge \ldots \wedge e_m$ orienting it. We denote by $\mathbf{p}$ and $\mathbf{p}^\perp$ the orthogonal projections onto, respectively, π_0 and its orthogonal complement $\pi_0^\perp$. In some cases we need orthogonal projections onto other planes π and their orthogonal complements $\pi^\perp$, for which we use the notation $\mathbf{p}_\pi$ and $\mathbf{p}_\pi^\perp$. For what concerns integral currents we use the definitions and the notation of [24]. We isolate the main assumption of our approximation theorem in the following

Assumption 1.2. For some open cylinder $\mathbf{C}_{4r}(x)$ (with $r \leq 1$) and some positive integer Q,

$$\mathbf{p}_\sharp T = Q \, [\![B_{4r}(x)]\!] \quad \text{and} \quad \partial T \llcorner \mathbf{C}_{4r}(x) = 0 \,. \qquad (1.2)$$

The following is the notion of *excess*, which represents the main regularity parameter for integral currents.

Definition 1.3 (Excess)**.** For a current T as in Assumption 1.2 we define the *cylindrical excess* $\mathbf{E}(T, \mathbf{C}_r(x))$, the *excess measure* $\mathbf{e}_T$ and its *density* $\mathbf{d}_T$:

$$\mathbf{E}(T, \mathbf{C}_r(x)) := \frac{\|T\|(\mathbf{C}_r(x))}{\omega_m r^m} - Q,$$

$$\mathbf{e}_T(A) := \|T\|(A \times \mathbb{R}^n) - Q \, |A| \qquad \text{for every Borel } A \subset B_r(x),$$

$$\mathbf{d}_T(y) := \limsup_{s \to 0} \frac{\mathbf{e}_T(B_s(y))}{\omega_m \, s^m} = \limsup_{s \to 0} \mathbf{E}(T, \mathbf{C}_s(y)),$$

where ω_m is the measure of the m-dimensional unit ball (the subscripts $_T$ will be omitted if clear from the context).

The main theorem of the paper is then the following approximation result (for the notation concerning multiple valued functions and their graphs we refer to [6, 8, 7]).

Theorem 1.4. *There exist constants* $M, C_{21}, \beta_0, \varepsilon_{21} > 0$ *(depending on* m, n, Q*) with the following property. Assume that* $T \in \mathbf{I}_m(\mathbb{R}^{m+n})$ *is* $\mathbf{\Omega}$-*minimal, it satisfies* (1.2) *in the cylinder* $\mathbf{C}_{4r}(x)$ *and* $E = \mathbf{E}(T, \mathbf{C}_{4r}(x)) < \varepsilon_{21}$*. Then, there exist a map* $f \colon B_r(x) \to \mathcal{A}_Q(\mathbb{R}^n)$ *and a closed set* $K \subset B_r(x)$ *such that*

$$\mathrm{Lip}(f) \leq C_{21} E^{\beta_0}, \qquad (1.3)$$

$$\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n) \quad and \quad |B_r(x) \setminus K| \leq C_{21} E^{\beta_0} (E + r^2 \mathbf{\Omega}^2) r^m \,, \tag{1.4}$$

$$\left| \|T\| \left(\mathbf{C}_r(x) \right) - Q \omega_m r^m - \frac{1}{2} \int_{B_r(x)} |Df|^2 \right| \leq C_{21} E^{\beta_0} (E + r^2 \mathbf{\Omega}^2) r^m \,, \tag{1.5}$$

$$\operatorname{osc}(f) \leq C_{21} \mathbf{h}(T, \mathbf{C}_{4r}(x)). \tag{1.6}$$

The proof of Theorem 1.4 will be achieved in the next three sections. The first one contains the most significant new ideas compared to the approximation of mass minimizing currents as done in [7]: here, indeed, we show how to improve upon the almost minimal condition under the assumption that the cylindrical excess is small, thus leading to a refined estimate. In the two subsequent sections we modify accordingly the computations of [7] to prove Theorem 1.4. Finally, in the last section we show how Theorem 1.4 applies to the currents considered in Definition 0.1 and state for later reference the approximation result which will be used in [11, 12] to prove Theorem 0.2. From now on constants which depend only upon m, n and Q will be called *dimensional constants*.

2. Homotopy lemma

Before proving the main Lipschitz approximation theorem we need a lemma which estimates carefully the difference in mass between an $\mathbf{\Omega}$-almost minimizer and a competitor in terms of a power of the excess and the constant $\mathbf{\Omega}$. The key idea is to choose the surface S in (1.1) to be a (suitable perturbation of the) homotopy between two accurately chosen preliminary Lipschitz approximations of T and R. To this regard we introduce the notion of E^β-approximation as in [7, Definition 5.1]. According to [7, Proposition 2.2 & Definition 5.1] we then have

Theorem 2.1. *There exist dimensional constants $\varepsilon_0, C_{21} > 0$ such that, if T is as in Theorem 1.4 in a cylinder $\mathbf{C}_{4r}(x)$, $E := \mathbf{E}(T, \mathbf{C}_{4r}(x)) < \varepsilon_0$ and $0 < \beta \leq \frac{1}{2m}$, then the following holds. There is a function $u \in \operatorname{Lip}(B_{7r/2}(x), \mathcal{A}_Q(\mathbb{R}^n))$, called E^β-approximation of T, such that*

$$\operatorname{Lip}(u) \leq C_{21} E^\beta,$$

$$\mathbf{G}_u \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n),$$

$$\mathbf{M}\left((T - \mathbf{G}_u) \llcorner (B_{7r/2}(x) \setminus K) \right) + |B_{7r/2}(x) \setminus K| \leq C_{21} r^m E^{1-2\beta}.$$

By using the E^β-approximations we get the following improvement of the $\mathbf{\Omega}$-minimality in the case of small excesses.

Lemma 2.2 (Homotopy Lemma)**.** *Let T be an $\mathbf{\Omega}$-almost minimizer which satisfies (1.2). There are positive dimensional constants ε_{22} and C_{25} such that, if $E = \mathbf{E}(T, \mathbf{C}_{4r}(x)) \leq \varepsilon_{22}$, then the following holds. For every $R \in \mathbf{I}_m(\mathbf{C}_{3r}(x))$ such that $\partial R = \partial(T \llcorner \mathbf{C}_{3r}(x))$, we have*

$$\|T\|(\mathbf{C}_{3r}(x)) \leq \mathbf{M}(R) + C_{25} r^{m+1} \mathbf{\Omega} E^{1/2} \,. \tag{2.1}$$

Moreover, let $\beta \leq \frac{1}{2m}$, $s \in]r, 2r[$, $R = \mathbf{G}_g \llcorner \mathbf{C}_s(x)$ for some Lipschitz map $g \colon B_s \to \mathcal{A}_Q(\mathbb{R}^n)$ with $\mathrm{Lip}(g) \leq 1$ and f be the E^β-approximation of T in $\mathbf{C}_{3r}$. If $f = g$ on ∂B_s and $P \in \mathbf{I}_m(\mathbb{R}^{m+n})$ is such that $\partial P = \partial((T - \mathbf{G}_f)\llcorner \mathbf{C}_s)$, then

$$\|T\|(\mathbf{C}_s(x)) \leq \mathbf{M}(\mathbf{G}_g) + \mathbf{M}(P) + C_{25}\Omega\left(E^{3/4}r^{m+1} + (\mathbf{M}(P))^{1+1/m} + \int_{B_s(x)} \mathcal{G}(f,g)\right). \tag{2.2}$$

Proof. We will first show (2.1): in fact (2.2) follows easily from a portion of the same argument, as it will be highlighted at the end.

Without loss of generality we assume $x = 0$. If $\|T\|(\mathbf{C}_{3r}) \leq \mathbf{M}(R)$ then there is nothing to prove. Hence we can suppose

$$\mathbf{M}(R) \leq \|T\|(\mathbf{C}_{3r}). \tag{2.3}$$

Define the current $R' \in \mathbf{I}_m(\mathbf{C}_{4r})$ by $R' := R + T \llcorner (\mathbf{C}_{4r} \setminus \mathbf{C}_{3r})$. Observe that $\partial(T - R') = 0$. So $\partial(\mathbf{p}_\sharp(T - R')) = 0$. On the other hand $\mathbf{p}_\sharp(T - R') = k \llbracket B_{4r} \rrbracket$ for some constant k and thus we conclude $\mathbf{p}_\sharp(T - R') = 0$. Therefore R' satisfies (1.2). Moreover we notice that, thanks to (2.3), the cylindrical excess of R' enjoys the following bound:

$$\mathbf{E}(R', C_{4r}) = \frac{\mathbf{M}(R')}{\omega_m(4r)^m} - Q \overset{(2.3)}{\leq} \frac{\mathbf{M}(T)}{\omega_m(4r)^m} - Q = \mathbf{E}(T, C_{4r}) =: E.$$

Let $f, h \colon B_{7r/2} \to \mathcal{A}_Q(\mathbb{R}^n)$ be the E^β-Lipschitz approximations of T and R' respectively, in the cylinders $\mathbf{C}_{7r/2}$ for some $\beta \in (0, 1/2m]$. Then there exist sets $K_T, K_{R'} \subset B_{7r/2}(x)$ such that $T \llcorner (K_T \times \mathbb{R}^n) = \mathbf{G}_f \llcorner (K_T \times \mathbb{R}^n)$ and $R' \llcorner (K_{R'} \times \mathbb{R}^n) = \mathbf{G}_h \llcorner (K_{R'} \times \mathbb{R}^n)$, fulfilling the following estimates:

$$\mathbf{M}((T - \mathbf{G}_f)\llcorner C_{7r/2}) \leq C_{21}r^m E^{1-2\beta} \quad \text{and} \quad \mathbf{M}((R' - \mathbf{G}_h)\llcorner C_{7r/2}) \leq C_{21}r^m E^{1-2\beta}, \tag{2.4}$$

$$|B_{7r/2} \setminus K_T| \leq C_{21}r^m E^{1-2\beta} \quad \text{and} \quad |B_{7r/2} \setminus K_{R'}| \leq C_{21}r^m E^{1-2\beta}, \tag{2.5}$$

$$\mathrm{Lip}(f) \leq C_{21}E^\beta \quad \text{and} \quad \mathrm{Lip}(h) \leq C_{21}E^\beta. \tag{2.6}$$

Next we set $K := K_T \cap K_{R'}$ and we notice that by (2.5)

$$|B_{7r/2} \setminus K| \leq Cr^m E^{1-2\beta}. \tag{2.7}$$

Let $|\cdot|$ be the function $|(x, y)| := |x|_2$ for every $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, where $|x|_2$ is the Euclidean norm of the vector x. By the slicing theory, (2.4), (2.7) and Fubini's Theorem there exists $s \in (3r, 7/2r)$ such that

$$\mathbf{M}(\langle T - \mathbf{G}_f, |\cdot|, s\rangle) + \mathbf{M}(\langle R' - \mathbf{G}_h, |\cdot|, s\rangle) \leq Cr^{m-1}E^{1-2\beta} \tag{2.8}$$

and

$$|\partial B_s \setminus K| \leq Cr^{m-1}E^{1-2\beta}. \tag{2.9}$$

By the Isoperimetric Inequality, there exists $P_T, P_R \in \mathbf{I}_m(\mathbb{R}^{m+n})$ such that

$$\partial P_T = \langle T - \mathbf{G}_f, |\cdot|, s\rangle \qquad \partial P_R = \langle R' - \mathbf{G}_h, |\cdot|, s\rangle$$

and

$$\mathbf{M}(P_T) + \mathbf{M}(P_R) \leq C\big(\mathbf{M}(\langle T - \mathbf{G}_f, |\cdot|, s\rangle)\big)^{m/(m-1)} + C\big(\mathbf{M}(\langle R' - \mathbf{G}_h, |\cdot|, s\rangle)\big)^{m/(m-1)}$$
$$\leq Cr^m E^{m(1-2\beta)/(m-1)}.$$

Therefore, we can conclude that

$$\partial((T - \mathbf{G}_f) \llcorner \mathbf{C}_s) = \partial P_T \qquad \partial((R' - \mathbf{G}_h) \llcorner \mathbf{C}_s) = \partial P_R \tag{2.10}$$

and, since $\beta \leq \frac{1}{2m}$, also

$$\mathbf{M}(P_T) + \mathbf{M}(P_R) \leq Cr^m E \tag{2.11}$$

Next consider the functions

$$f' := \boldsymbol{\xi} \circ f \colon B_{7r/2} \to \mathcal{Q} \subset \mathbb{R}^{N(Q,n)} \quad \text{and} \quad h' := \boldsymbol{\xi} \circ h \colon B_{7r/2} \to \mathcal{Q} \subset \mathbb{R}^{N(Q,n)} \,.$$

where $\boldsymbol{\xi} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^{N(Q,n)}$ is the bilipschitz embedding in [6, Section 2.1], and the homotopy between them, defined by

$$\tilde{H} \colon [0,1] \times B_{7r/2} \ni (t,x) \to (x, tf'(x) + (1-t)h'(x)) \in \mathbb{R}^m \times \mathbb{R}^N \,.$$

Consider the Lipschitz map

$$\phi \colon \mathbb{R}^m \times \mathbb{R}^N \ni (x,y) \to (x, \boldsymbol{\xi}^{-1}(\boldsymbol{\rho}(y))) \in \mathbb{R}^m \times \mathcal{A}_Q(\mathbb{R}^n)$$

where $\boldsymbol{\rho} : \mathbb{R}^{N(Q,n)} \to \mathcal{Q} := \boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$ is the Lipschitz retraction in [6, Section 2.1], and define $H := \phi \circ \tilde{H}$. H can be seen as a Q-valued map $H : [0,1] \times B_{7r/2 \to \mathcal{A}_Q(\mathbb{R}^{m+n})}$. Without changing notation for H we restrict it to $[0,1] \times B_s$ and following the notation of [8, Definition 1.3] we define $S := \mathbf{T}_H$. If we set $G := H|_{[0,1] \times \partial B_s}$ we can use [8, Theorem 2.1] to conclude that

$$\partial S = (\mathbf{G}_f - \mathbf{G}_h) \llcorner \mathbf{C}_s - \mathbf{T}_G = (\mathbf{G}_f - \mathbf{G}_h) \llcorner \mathbf{C}_s - P \,, \tag{2.12}$$

where $P := \mathbf{T}_G$. We now want to estimate $\mathbf{M}(S)$ and $\mathbf{M}(P)$ and we will do it using the Q-valued area formula in [8, Lemma 1.9]. We start with $\mathbf{M}(S)$. We fix a point of differentiability p where $DH = \sum \llbracket DH_i \rrbracket$. On $[0,1] \times B_s$ we use the coordinates (t,x) and on the target space $\mathbb{R}^{m+n}$ the coordinates (x,y). Let $p = (t_0, x_0)$. It is then obvious that the matrix DH_i can be decomposed as

$$DH_i(p) = \begin{pmatrix} 0_{m \times 1} & I_{m \times m} \\ v_{n \times 1} & A_{n \times m} \,. \end{pmatrix}$$

where the matrices A and v can be bound using the following observation. If we consider the map $t \mapsto \Phi(t) := H(t, x_0)$ and $x \mapsto \Lambda(x) := t_0 f'(x) + (1 - t_0)h'(x)$, we then have $|v| \leq C\mathrm{Lip}(\Phi)$ and $|A| \leq C\mathrm{Lip}(\Lambda)$, where the constant C depends only on n and Q. On the other hand, it is easy to see that $\mathrm{Lip}(\Phi) \leq C\mathcal{G}(f(x_0), h(x_0))$ and $\mathrm{Lip}(\Lambda) \leq C(\mathrm{Lip}(h) + \mathrm{Lip}(f)) \leq E^\beta$. Thus we can estimate

$$\mathbf{J}H_i(p) := \sqrt{\det(DH_i^*(p) \cdot DH_i(p))} \leq C\mathcal{G}(f(x_0), h(x_0)) \,.$$

Using [8, Lemma 1.9] we then conclude

$$\mathbf{M}(S) \leq C \int_{B_s} \mathcal{G}(f,h)$$

and, arguing in a similar fashion,

$$\mathbf{M}(P) \leq C \int_{\partial B_s} \mathcal{G}(f,h) \,.$$

Observe that f and h coincide, respectively, with the slices of the currents T and R' on any $x_0 \in K$. On the other hand, $s > 3r$ and $T \llcorner \mathbf{C}_{4r} \setminus \mathbf{C}_{3r} = R' \llcorner \mathbf{C}_{4r} \setminus \mathbf{C}_{3r}$. We thus conclude that $h = f$ on $K \cap \partial B_s$. Let $x \in \partial B_s \setminus K$. By (2.9), there exists $x_0 \in K \cap \partial B_s$ such that $|x - x_0| \leq CrE^{(1-2\beta)/(m-1)} = CrE^{2\beta}$ (recall that $\beta \leq \frac{1}{2m}$). Thus

$$\mathcal{G}(f(x), h(x)) \leq (\mathrm{Lip}(f) + \mathrm{Lip}(h)) |x - x_0| \leq CrE^{3\beta},$$

and so we conclude

$$\mathbf{M}(P) \leq C \int_{\partial B_s} \mathcal{G}(f, h) \leq CrE^{3\beta}|\partial B_s \setminus K| \leq Cr^m E^{1+\beta} \leq Cr^m E. \qquad (2.13)$$

On the other hand, we recall that, by a standard variant of the Poincaré inequality (cf., for example, [30, 4.4.7]),

$$\int_{B_s} \mathcal{G}(f, h) \leq Cr\|\mathcal{G}(f, h)\|_{L^1(\partial B_s)} + Cr\|D(\mathcal{G}(f, h))\|_{L^1(B_s)}$$

$$\overset{(2.13)}{\leq} Cr^{m+1}E + Cr^{1+m/2} \left(\int (|Df|^2 + |Dh|^2) \right)^{1/2} \leq Cr^{m+1}E^{1/2}. \qquad (2.14)$$

Thus,

$$(\mathbf{G}_f - \mathbf{G}_h) \llcorner \mathbf{C}_s = \partial S + P \qquad (2.15)$$

with

$$\mathbf{M}(P) \leq Cr^m E \quad \text{and} \quad \mathbf{M}(S) \leq Cr^{m+1}E^{1/2}. \qquad (2.16)$$

Now observe that

$$0 = \partial(T - R') = \partial((\mathbf{G}_f - \mathbf{G}_h) \llcorner \mathbf{C}_s) + \partial(P_T - P_R) = \partial\partial S + \partial P + \partial(P_T - P_R).$$

Hence, $\partial(P + P_T - P_R) = 0$ and, by the isoperimetric inequality, there is an S' with $\mathbf{M}(S') \leq Cr^{m+1}E^{1+1/m}$ and $\partial S' = P + P_T - P_R$. Additionally, again using the isoperimetric inequality, there are currents S_T and S_R such that

$$\partial S_T = (T - \mathbf{G}_f) \llcorner \mathbf{C}_s - P_T$$
$$\partial S_R = (R' - \mathbf{G}_h) \llcorner \mathbf{C}_s - P_R$$

and

$$\mathbf{M}(S_T) \leq C \left(\|T - \mathbf{G}_f\|(\mathbf{C}_s) + \mathbf{M}(P_T) \right)^{(m+1)/m} \leq CE^{3/4}r^{m+1}$$
$$\mathbf{M}(S_R) \leq C \left(\|R' - \mathbf{G}_h\|(\mathbf{C}_s) + \mathbf{M}(P_R) \right)^{(m+1)/m} \leq CE^{3/4}r^{m+1}.$$

In the latter inequalities we have used $\|R' - \mathbf{G}_h\|(\mathbf{C}_s) + \|T - \mathbf{G}_f\|(\mathbf{C}_s) \leq CE^{1-2\beta}r^m$: in particular $(1-2\beta)(m+1)/m \geq 1 - 1/m^2 \geq 3/4$; observe that this estimate explains the exponent of E in the third summand of the right hand side of (2.2).

Thus, setting $S'' = S + S_T - S_R + S'$ we finally achieve $(T - R') \llcorner \mathbf{C}_s = \partial S''$ and $\mathbf{M}(S'') \leq Cr^{m+1}E^{1/2}$. Recalling that $s > 3r$ and that $R' = R + T \llcorner (\mathbf{C}_{4r} \setminus \mathbf{C}_{3r})$ we conclude $\partial S'' = (T - R) \llcorner \mathbf{C}_{3r}$. Applying now the $\mathbf{\Omega}$-minimality of T we conclude

$$\|T\|(\mathbf{C}_{3r}) \leq \mathbf{M}(R) + C_{25}r^{m+1}\mathbf{\Omega}E^{1/2}.$$

For the proof of (2.2) we conclude with the same computations, except that this time $f = g$ on ∂B_s and the current R is already given by $\mathbf{G}_g \llcorner \mathbf{C}$. The modifications to the argument are then straightforward, given the remark of the previous paragraph. $\square$

3. Harmonic approximation and gradient L^p estimates

In this and in the next section we follow largely [7] with minor modifications: on the one hand we have the additional $\boldsymbol{\Omega}$-error terms, but on the other hand the ambient Riemannian manifold is the Euclidean space. Thus the arguments are somewhat less technical.

3.1. **Harmonic Approximation.** In this subsection we prove that if T is an almost minimizer then its E^β-Lipschitz approximation is close to a Dir-minimizing function w with estimates which are infinitesimal in the excess.

Theorem 3.1 (First harmonic approximation). *For every $\eta_1, \delta > 0$ and every $\beta \in (0, \frac{1}{2m})$, there exists a constant $\varepsilon_{23} > 0$ with the following property. Let T be an $\boldsymbol{\Omega}$-almost minimizer which satisfies Assumption 1.2 in $\mathbf{C}_{4r_0}(x)$. If $E = \mathbf{E}(T, \mathbf{C}_{4r_0}(x)) \leq \varepsilon_{23}$ and $r_0\, \boldsymbol{\Omega} \leq \varepsilon_{23} E^{1/2}$, then the E^β-Lipschitz approximation f in $\mathbf{C}_{3r_0}(x)$ satisfies*

$$\int_{B_{2r_0}(x) \setminus K} |Df|^2 \leq \eta_1 E\, \omega_m \,(4\,r_0)^m = \eta_1\, \mathbf{e}_T(B_{4r_0}(x)). \tag{3.1}$$

Moreover, there exists a Dir-minimizing function w such that

$$r_0^{-2} \int_{B_{2r_0}(x)} \mathcal{G}(f, w)^2 + \int_{B_{2r_0}(x)} \left(|Df| - |Dw| \right)^2 \leq \eta_1 E\, \omega_m\, (4\, r_0)^m = \eta_1\, \mathbf{e}_T(B_{4r_0}(x)), \tag{3.2}$$

$$\int_{B_{2r_0}(x)} |D(\boldsymbol{\eta} \circ f) - D(\boldsymbol{\eta} \circ w)|^2 \leq \eta_1 E\, \omega_m\, (4\, r_0)^m = \eta_1\, \mathbf{e}_T(B_{4r_0}(x)) \,. \tag{3.3}$$

Proof. The proof of the theorem is at all analogous to the one given in [7, Theorem 3.2]: for this reason, we provide here only the principal parts, leaving the details to the readers. By rescaling and translating, it is not restrictive to assume that $x = 0$ and $r_0 = 1$. The proof is by contradiction: assume there exist a constant $c_1 > 0$, a sequence of positive real numbers $(\varepsilon_l)_l$, a sequence of $\boldsymbol{\Omega}_l$-minimal currents $(T_l)_{l \in \mathbb{N}}$ and corresponding E_l^β-Lipschitz approximations $(f_l)_{l \in \mathbb{N}}$ such that

$$E_l := \mathbf{E}(T_l, \mathbf{C}_4) \leq \varepsilon_l \to 0, \quad \boldsymbol{\Omega}_l \leq \varepsilon_l E_l^{1/2} \quad \text{and} \quad \int_{B_2 \setminus K_l} |Df_l|^2 \geq c_1\, E_l, \tag{3.4}$$

where $K_l := \{x \in B_3 : \mathbf{me}_{T_l}(x) < E_l^{2\beta}\}$ with $\mathbf{me}_{T_l}$ denoting the "non-centered" maximal function of $\mathbf{e}_{T_l}$:

$$\mathbf{me}_{T_l}(y) := \sup_{y \in B_s(w) \subset B_4(x)} \frac{\mathbf{e}_{T_l}(B_s(w))}{\omega_m\, s^m} = \sup_{y \in B_s(w) \subset B_4(x)} \mathbf{E}(T_l, \mathbf{C}_s(w)).$$

Set $\Gamma_l := \{x \in B_4 : \mathbf{me}_{T_l}(x) \leq 2^{-m} E_l^{2\beta}\}$ and observe that $\Gamma_l \cap B_3 \subset K_l$. From the Lipschitz approximation in [7, Proposition 3.2], it follows that

$$\mathrm{Lip}(f_l) \leq C_{22} E_l^\beta, \tag{3.5}$$

$$|B_r \setminus K_l| \le C_{22} E_l^{-2\beta} \mathbf{e}_T \big(B_{r+r_0(l)} \setminus \Gamma_l \big) \quad \text{for every } r \le 3 \,, \tag{3.6}$$

where $r_0(l) = 16 \, E_l^{(1-2\beta)/m} < \frac{1}{2}$. Then, (3.4), (3.5) and (3.6) give

$$c_1 \, E_l \le \int_{B_2 \setminus K_l} |Df_l|^2 \le C_{22} \, \mathbf{e}_{T_l}(B_s \setminus \Gamma_l) \quad \forall \, s \in \left[\tfrac{5}{2}, 3 \right] .$$

Setting $c_2 := c_1/(2C_{22})$, we have $2c_2 E_l \le \mathbf{e}_{T_l}(B_s \setminus \Gamma_l) = \mathbf{e}_{T_l}(B_s) - \mathbf{e}_{T_l}(B_s \cap \Gamma_l)$, thus leading to

$$\mathbf{e}_{T_l}(\Gamma_l \cap B_s) \le \mathbf{e}_{T_l}(B_s) - 2 \, c_2 \, E_l \,, \tag{3.7}$$

for l large enough. Next observe that $\omega_m 4^m E_l = \mathbf{e}_{T_l}(B_4) \ge \mathbf{e}_{T_l}(B_s)$, because $\mathbf{e}_{T_l}$ is a positive measure under the Assumption 1.2. Therefore, by the Taylor expansion in [8, Corollary 3.3], (3.7) and $E_l \downarrow 0$, it follows that, for every $s \in [5/2, 3]$,

$$\int_{\Gamma_l \cap B_s} \frac{|Df_l|^2}{2} \le (1 + C \, E_l^{2\beta}) \, \mathbf{e}_{T_l}(\Gamma_l \cap B_s)$$

$$\le (1 + C \, E_l^{2\beta}) \left(\mathbf{e}_{T_l}(B_s) - 2 \, c_2 \, E_l \right) \le \mathbf{e}_{T_l}(B_s) - c_2 \, E_l. \tag{3.8}$$

Our aim is to show that (3.8) contradicts the $\mathbf{\Omega}_l$-almost minimizing property (1.1) of T_l. This is shown by constructing a suitable competitor S_l for T_l, via a careful modification of the E_l^β-approximations f_l. The construction of the competitor S_l is identical to the one done in [7, pages 1854-1857], actually simplified by the fact that our currents T_l are supported in $\mathbb{R}^{m+n}$ and not in a Riemannian manifold. Therefore, we omit here the details of the computations (which can be found in full details in the PhD thesis of the third author, [26]) and recall only the conclusion: there exist integer rectifiable currents S_l such that $\partial S_l = \partial (T_l \llcorner \mathbf{C}_4)$ and

$$\mathbf{M}(S_l) - \mathbf{M}(T_l) \le - \frac{c_2 \, E_l}{4} + C \, E_l^{1+\gamma} \,. \tag{3.9}$$

Now using (2.1) of the Homotopy Lemma 2.2 we have the upper bound

$$\mathbf{M}(S_l) - \mathbf{M}(T_l) \ge -C_{25} \mathbf{\Omega}_l E_l^{1/2} \ge -C_{25} \varepsilon_l E_l.$$

Combining this inequality with (3.9) we obtain

$$\frac{c_2 E_l}{4} \le C E_l^{1+\gamma} + C \varepsilon_l E_l$$

which for E_l, ε_l sufficiently small (and hence for l large enough) provides the desired contradiction.

For what concerns (3.2), we argue similarly. Let $(T_l)_l$ be a sequence with vanishing $E_l := \mathbf{E}(T_l, \mathbf{C}_4)$, contradicting the second part of the statement and perform the same analysis as before. Up to subsequences, one of the following statement must be false:

   (i) $\lim_l \int_{B_2} |Dg_l|^2 = \int_{B_2} |Dh_{l_0}|^2$, for any l_0 (recall that $\int_{B_2} |Dh_l|^2$ is constant);
   (ii) h_l is Dir-minimizing in B_2.

If (i) is false, then there is a positive constant c_2 such that, for every $r \in [5/2, 3]$,

$$\int_{B_r} \frac{|Dh_l|^2}{2} \leq \int_{B_r} \frac{|Dg_l|^2}{2} - c_2 \leq \frac{\mathbf{e}_{T_l}(B_r)}{E_l} - \frac{c_2}{2},$$

for l large enough. Therefore we can argue exactly as in the proof of (3.1) (using h_l instead of H_l to construct the competitors) and reach a contradiction. If (ii) is false, then h_l is not Dir-minimizing in $B_{5/2}$. This implies (cp. [7, pages 1857-1859]) that we can find a competitor F_l satisfying, for any $r \in [5/2, 3]$,

$$\int_{B_r} \frac{|DF_l|^2}{2} \leq \int_{B_r} \frac{|Dh_l|^2}{2} - c_2 \leq \lim_l \int_{B_r} \frac{|Dg_l|^2}{2} - 2\, c_2 \leq \frac{\mathbf{e}_T(B_r)}{E_l} - \frac{c_2}{2},$$

provided l is large enough (where $c_2 > 0$ is a constant independent of r and l). On the other hand, since $F_l = h_l$ on $B_3 \setminus B_{5/2}$, $\|\mathcal{G}(F_l, g_l)\|_{L^2(B_3 \setminus B_{5/2})} \to 0$ and we argue as above with F_l in place of H_l and reach a contradiction in this case as well. (The details of this argument are also reported in the PhD thesis of the third author [26]). □

3.2. **Improved excess estimate.** The higher integrability of the Dir-minimizing functions (cp. [7, Theorem 6.1]) and the harmonic approximation in Theorem 3.1 lead to the following estimate, which we call "weak" since we will improve it in the next section with Theorem 4.1.

Proposition 3.2 (Weak excess estimate). *For every $\eta_2 > 0$, there exist $\varepsilon_{24}, C_{26} > 0$ with the following property. Let T be an $\boldsymbol{\Omega}$-almost minimizer and assume it satisfies (1.2) in $\mathbf{C}_{4s}(x)$. If $E = \mathbf{E}(T, \mathbf{C}_{4s}(x)) \leq \varepsilon_{24}$, then*

$$\mathbf{e}_T(A) \leq \eta_2\, \mathbf{e}_T(B_{4s}(x)) + C_{26}\, \boldsymbol{\Omega}^2\, s^{m+2}, \tag{3.10}$$

for every $A \subset B_s(x)$ Borel with $|A| \leq \varepsilon_{24}|B_s(x)|$ (C_{26} depends only on η_2, m, n and Q).

Proof. The proof is a minor modification of [7, Proposition 6.4]: nevertheless, being very short, we provided here a brief account of all the arguments.

Without loss of generality, we can assume $s = 1$ and $x = 0$. We distinguish the two regimes: $\hat{\varepsilon}^2 E \leq \boldsymbol{\Omega}^2$ and $\boldsymbol{\Omega}^2 \leq \hat{\varepsilon}^2 E$, where $\hat{\varepsilon} \leq \varepsilon_{24}$ is a parameter whose choice will be specified later. In the former, clearly $\mathbf{e}_T(A) \leq C\, E \leq C\, \boldsymbol{\Omega}^2$. In the latter, we let f be the $E^{1/4m}$-Lipschitz approximation of T in $\mathbf{C}_3$. By a Fubini-type argument as the ones already used in the previous sections, we find a radius $r \in (1, 2)$ and a current P with $\mathbf{M}(P) \leq CE^{1+\gamma}$ and $\partial((T - \mathbf{G}_f) \llcorner \mathbf{C}_r) = \partial P$ for some $\gamma(m) > 0$. We can thus apply the Homotopy Lemma 2.2 to $R = \mathbf{G}_f \llcorner \mathbf{C}_r + P + T \llcorner (\mathbf{C}_3 \setminus \mathbf{C}_r)$:

$$\|T\|(\mathbf{C}_r) \leq \mathbf{M}(R \llcorner \mathbf{C}_r) + C\boldsymbol{\Omega}E^{1/2} \leq \|\mathbf{G}_f\|(\mathbf{C}_r) + C\hat{\varepsilon}E + CE^{1+\gamma}$$

$$\leq Q\,|B_r| + \int_{B_r} \frac{|Df|^2}{2} + C\hat{\varepsilon}E + C\, E^{1+\gamma}, \tag{3.11}$$

for some positive γ (possibly smaller than the previous one), where we used the Taylor expansion in [8, Corollary 3.3].

On the other hand, using the Taylor expansion for the part of the current which coincides with the graph of f, we deduce as well that

$$\|T\|(\mathbf{C}_r) = \|T\|((B_r \setminus K) \times \mathbb{R}^n) + \|T\|((B_r \cap K) \times \mathbb{R}^n)$$

$$\geq \|T\|((B_r \setminus K) \times \mathbb{R}^n) + Q \, |B_r \cap K| + \int_{B_r \cap K} \frac{|Df|^2}{2} - C \, E^{1+\gamma}. \qquad (3.12)$$

Subtracting (3.12) from (3.11), we then have

$$\mathbf{e}_T(B_r \setminus K) \leq \int_{B_r \setminus K} \frac{|Df|^2}{2} + C\hat{\varepsilon}E + CE^{1+\gamma}, \qquad (3.13)$$

and we recall that the constant C is independent of $\hat{\varepsilon}$. Therefore, taking into account (3.1) of Theorem 3.1, we conclude that the excess on the exceptional set $B_r \setminus K$ is infinitesimal with respect to the E if ε_{24} is chosen small enough, namely

$$\mathbf{e}_T(B_r \setminus K) \leq \eta \, E, \qquad (3.14)$$

for a suitable $\eta > 0$. Let now $A \subset B_1$ be such that $|A| \leq \varepsilon_{24} \, \omega_m$. Combining (3.14) with the Taylor expansion and with (3.2) of Theorem 3.1, we have

$$\mathbf{e}_T(A) \leq \mathbf{e}_T(A \setminus K) + \int_A \frac{|Df|^2}{2} + C \, E^{1+\gamma} \leq \int_A \frac{|Dw|^2}{2} + 2 \, \eta \, \mathbf{e}_T(B_4), \qquad (3.15)$$

where w is a Dir-minimizing and ε_{24} is assumed small enough. Hence, we infer the conclusion (3.10) from the higher integrability of the gradient of Dir-minimizing functions given in [7, Theorem 6.1] (see [7, page 1861] for the simple argument). $\qquad\square$

3.3. **Gradient L^p estimate.** One of the key points of the proof of Theorem 1.4 is to show an L^p estimate, for some $p > 1$, for the density $\mathbf{d}$ of the excess measure of an $\mathbf{\Omega}$-almost minimizer.

Theorem 3.3 (Gradient L^p estimate). *There exist constants $p_2 > 1$ and $C, \varepsilon_{25} > 0$ (depending on n, Q) with the following property. Assume T satisfies (1.2) in the cylinder $\mathbf{C}_4$. If T is an $\mathbf{\Omega}$-almost minimizer and $E = \mathbf{E}(T, \mathbf{C}_4) < \varepsilon_{25}$, then*

$$\int_{\{\mathbf{d} \leq 1\} \cap B_2} \mathbf{d}^{p_2} \leq C \, E^{p_2 - 1} \left(E + \mathbf{\Omega}^2 \right). \qquad (3.16)$$

Proof. The proof is the same as the proof of [7, Theorem 2.3], where [7, Proposition 6.4] is replaced by our Proposition 3.2. $\qquad\square$

4. Strong excess estimate and proof of Theorem 1.4

4.1. **Almgrem's strong excess estimate.** Thanks to the higher integrability of Theorem 3.3, we can control the excess where $\mathbf{d} \leq 1$. To control it outside this region, we prove the following strengthened version of Proposition 3.2.

Theorem 4.1 (Almgren's strong excess estimate). *There are constants $\varepsilon_{21}, \gamma_2, C_{27} > 0$ (depending on n, Q) with the following property. Assume T satisfies Assumption 1.2 in $\mathbf{C}_4$ and is $\mathbf{\Omega}$ almost minimizing. If $E = \mathbf{E}(T, \mathbf{C}_4) < \varepsilon_{21}$, then*

$$\mathbf{e}_T(A) \le C_{27}\left(E^{\gamma_2} + |A|^{\gamma_2}\right)\left(E + \mathbf{\Omega}^2\right) \quad \textit{for every Borel } A \subset B_1. \tag{4.1}$$

Proof. The proof follows the same scheme in [7]. First of all, by a regularization by convolution technique, we construct a subset of radii $B \subset [1, 2]$ with $|B| > \frac{1}{2}$ with the property that, for every $\sigma \in B$, there exists a Q-valued function $g \in \mathrm{Lip}(B_\sigma, \mathcal{A}_Q(\mathbb{R}^n))$ such that

$$g|_{\partial B_\sigma} = f|_{\partial B_\sigma}, \quad \mathrm{Lip}(g) \le C_{28}\, E^{\beta_1} \tag{4.2}$$

$$\int_{B_\sigma} |Dg|^2 \le \int_{B_\sigma \cap K} |Df|^2 + C_{28}\, E^{\gamma_3}\left(E + \mathbf{\Omega}^2\right), \tag{4.3}$$

where f is the E^{β_1}-Lipschitz approximation of the $\mathbf{\Omega}$-minimal current T and γ_3, C_{28} are dimensional positive constants. The proof of the above estimates is given in [7, Proposition 7.3].

Using now the isoperimetric inequality and a slicing argument, we find a radius $\sigma \in B$ and $P \in \mathbf{I}_m(\mathbb{R}^{m+n})$ with $\partial P = \partial((T - \mathbf{G}_f) \llcorner \mathbf{C}_s)$ and $\mathbf{M}(P) \le C E^{1+\gamma}$. We can therefore apply the Homotopy Lemma 2.2 to conclude that

$$\|T\|(\mathbf{C}_\sigma) \le \|\mathbf{G}_g\|(\mathbf{C}_\sigma) + C\mathbf{\Omega} \int_{B_\sigma} \mathcal{G}(g, f) + C E^{1+\gamma} + C\,\mathbf{\Omega}\, E^{3/4}. \tag{4.4}$$

Then, from (4.4), (4.3), the inequality $2\mathbf{\Omega}\, E^{3/4} \le E^\gamma \mathbf{\Omega}^2 + E^{3/2-\gamma}$ (for any $\gamma < 1/2$) and the Taylor expansion for $\mathbf{M}(\mathbf{G}_g)$ we achieve

$$\|T\|(\mathbf{C}_\sigma) \le Q\,|B_\sigma| + \int_{B_\sigma \cap K} \frac{|Df|^2}{2} + C E^\gamma(E + \mathbf{\Omega}^2) + C\mathbf{\Omega} \int_{B_\sigma} \mathcal{G}(g, f), \tag{4.5}$$

for some $\gamma > 0$. On the other hand, by the Taylor's expansion in [8, Corollary 3.3],

$$\|T\|(\mathbf{C}_s) = \|T\|((B_s \setminus K) \times \mathbb{R}^n) + \|\mathbf{G}_f\|((B_s \cap K) \times \mathbb{R}^n)$$

$$\ge \|T\|((B_s \setminus K) \times \mathbb{R}^n) + Q\,|K \cap B_s| + \int_{K \cap B_s} \frac{|Df|^2}{2} - C\, E^{1+\gamma}, \tag{4.6}$$

possibly changing the value of $\gamma > 0$. Hence, from (4.5) and (4.6), we get

$$\mathbf{e}_T(B_s \setminus K) \le C\, E^\gamma\left(E + \mathbf{\Omega}^2\right) + C\mathbf{\Omega} \int_{B_\sigma} \mathcal{G}(g, f). \tag{4.7}$$

Next note that, by the Taylor expansion of the mass of the graph of f, it follows that $|Df|^2 \le C\,\mathbf{d}_T \le C E^{2\beta} < 1$ a.e. in K: indeed, in all Lebesgue points of K and $|Df|^2$ we have that

$$|Df|^2(x) = \lim_{s \to 0} \frac{\int_{B_s(x) \cap K} |Df|^2}{\omega_m\, s^m} \le C \lim_{s \to 0} \frac{\mathbf{e}_{\mathbf{G}_f}(B_s(x) \cap K)}{\omega_m\, s^m} \le C \limsup_{s \to 0} \frac{\mathbf{e}_T(B_s(x))}{\omega_m\, s^m} = C\mathbf{d}_T(x).$$

Therefore, for every $A \subset B_1$ Borel set, we can use the higher integrability of $|Df|$ in K given by Theorem 3.3 to get

$$
\begin{aligned}
\mathbf{e}_T(A) &\leq \mathbf{e}_T(A \cap K) + \mathbf{e}_T(A \setminus K) \\
&\leq \int_{A \cap K} \frac{|Df|^2}{2} + C\, E^{1+\gamma} + C\, E^\gamma\, (E + \mathbf{\Omega}^2) + C\mathbf{\Omega} \int_{B_\sigma} \mathbf{G}(g, f) \\
&\leq C\, |A \cap K|^{\frac{p_2 - 1}{p_2}} \left(\int_{A \cap K} |Df|^{q_2} \right)^{2/q_2} + C\, E^\gamma\, (E + \mathbf{\Omega}^2) + C\mathbf{\Omega} \int_{B_\sigma} \mathbf{G}(g, f) \\
&\leq C\, |A|^{\frac{p_2 - 1}{p_2}} \left(E + \mathbf{\Omega}^2 \right) + C\, E^\gamma\, (E + \mathbf{\Omega}^2) + C\mathbf{\Omega} \int_{B_\sigma} \mathbf{G}(g, f).
\end{aligned}
$$

In order to conclude the proof we need only to estimate the term $\int_{B_\sigma} \mathbf{G}(g, f)$. For this part of the argument it is important to recall the construction of the map g in [7]. We introduce the following notation. Given two (vector-valued) functions h_1 and h_2 and two radii $0 < s < r$, we denote by $\mathrm{lin}(h_1, h_2)$ the linear interpolation in $B_r \setminus \bar{B}_s$ between $h_1|_{\partial B_r}$ and $h_2|_{\partial B_s}$, i.e., if $(\theta, t) \in \mathbb{S}^{m-1} \times [0, \infty)$ are spherical coordinates, then

$$
\mathrm{lin}(h_1, h_2)(\theta, t) = \frac{r - t}{r - s}\, h_2(\theta, s) + \frac{t - s}{r - s}\, h_1(\theta, r).
$$

Next, we fix two parameters $\delta > 0$ and $\varepsilon > 0$, radii $1 < r_1 < r_2 < r_3 < 2$, given by

$$
r_3 = \sigma, \quad r_2 = r_3 - s \quad \text{and} \quad r_1 = r_2 - s,
$$

with $\sigma \in B$ the radius in the estimates (4.2) and (4.3) (whose existence is established in [7]) and with $\varepsilon = E^a$, $\delta = E^b$ and $s = E^c$, where

$$
a = \frac{1 - 2\beta_1}{2m}, \quad b = \frac{1 - 2\beta_1}{4m\,(n\,Q + 1)} \quad \text{and} \quad c = \frac{1 - 2\beta_1}{8^{nQ}\, 4m\,(n\,Q + 1)}.
$$

Fix also $\varphi \in C_c^\infty(B_1)$ a standard nonnegative mollifier. We set $f' := \boldsymbol{\xi} \circ f$. Recall the Lipschitz maps $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_\delta^\star$ of [6, Theorem 2.1] and [7, Proposition 7.2], respectively, and define:

$$
g' := \begin{cases}
\sqrt{E}\, \boldsymbol{\rho} \circ \mathrm{lin}\left(\frac{f'}{\sqrt{E}}, \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}} \right) \right) & \text{in } B_{r_3} \setminus B_{r_2}, \\[2mm]
\sqrt{E}\, \boldsymbol{\rho} \circ \mathrm{lin}\left(\boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}} \right), \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon \right) \right) & \text{in } B_{r_2} \setminus B_{r_1}, \\[2mm]
\sqrt{E}\, \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon \right) & \text{in } B_{r_1}.
\end{cases} \tag{4.8}
$$

Finally set $g := \boldsymbol{\xi}^{-1} \circ g'$. In particular, recalling that $\boldsymbol{\xi}^{-1}$ is Lipschitz continuous and $f = \boldsymbol{\xi}^{-1} \circ f$, we can estimate as follows

$$\int_{B_\sigma} \mathcal{G}(f, g) \le C \underbrace{\int_{B_\sigma \setminus B_{\sigma-s}} \left| f' - \sqrt{E} \boldsymbol{\rho} \circ \mathrm{lin}\left(\frac{f'}{\sqrt{E}}, \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}}\right)\right)\right|}_{I_1} +$$

$$+ C \underbrace{\int_{B_{\sigma-s} \setminus B_{\sigma-2s}} \left| f' - \sqrt{E} \boldsymbol{\rho} \circ \mathrm{lin}\left(\boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}}\right), \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\right)\right)\right|}_{I_2} +$$

$$+ C \underbrace{\int_{B_{\sigma-2s}} \left| f' - \sqrt{E} \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\right)\right|}_{I_3}.$$

We will estimate I_1, I_2, I_3 separately. For what concerns I_1, we recall that $\boldsymbol{\rho} \circ f' = f'$, $\boldsymbol{\rho}$ is Lipschitz continuous and $\lambda \boldsymbol{\rho}(P) = \boldsymbol{\rho}(\lambda P)$, for every $\lambda > 0, P \in \mathcal{Q}$, since $\mathcal{Q}$ is a cone; therefore,

$$I_1 \le C \int_{\sigma-s}^\sigma \int_{\partial B_t} \sqrt{E} \left| \frac{f'}{\sqrt{E}} - \frac{t+s-\sigma}{s} \frac{f'}{\sqrt{E}} - \frac{\sigma-t}{s} \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}}\right)\right| dt$$

$$= C\sqrt{E} \int_{\sigma-s}^\sigma \frac{\sigma-t}{s} \int_{\partial B_t} \left| \frac{f'}{\sqrt{E}} - \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}}\right)\right| dt \le C\sqrt{E} \delta^{8-nQ} \left| B_\sigma \setminus B_{\sigma-s}\right| \le C E^{1/2+c}$$

where we used $\left|\boldsymbol{\rho}_\delta^\star(P) - P\right| \le C \delta^{8-nQ}$ from [7, Proposition 7.2] and $\left| B_\sigma \setminus B_{\sigma-s}\right| \le Cs \le CE^c$. We next bound I_2: similarly as for I_1

$$I_2 \le C\sqrt{E} \int_{\sigma-2s}^{\sigma-s} \int_{\partial B_t} \left| \frac{f'}{\sqrt{E}} - \frac{t+2s-\sigma}{s} \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}}\right) - \frac{\sigma-s-t}{s} \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\right)\right|$$

$$\le C\sqrt{E} \int_{\sigma-2s}^{\sigma-s} \int_{\partial B_t} \left(\left| \frac{f'}{\sqrt{E}} - \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}}\right)\right| + \frac{\sigma-s-t}{s} \left| \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}}\right) - \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\right)\right|\right) dt$$

$$\le C E^{1/2+c} + C \int_{B_{\sigma-s} \setminus B_{\sigma-2s}} \left| f' - f' * \varphi_\varepsilon\right|$$

where we have used the fact that $\boldsymbol{\rho}_\delta^\star$ is Lipschitz. The estimate for I_3 is similarly given by

$$I_3 \le C\sqrt{E} \int_{B_{\sigma-2s}} \left(\left| \frac{f'}{\sqrt{E}} - \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}}\right)\right| + \left| \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}}\right) - \boldsymbol{\rho}_\delta^\star\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\right)\right|\right)$$

$$\le C E^{1/2+c} + C \int_{B_{\sigma-2s}} \left| f' - f' * \varphi_\varepsilon\right|.$$

We therefore achieve the estimate

$$I_2 + I_3 \le C E^{1/2+c} + \int_{B_{\sigma-s}} \left| f' - f' * \varphi_\varepsilon\right|$$

and to conclude, we compute

$$\int_{B_{\sigma-s}} |f' - f' * \varphi_\varepsilon| \leq \int_{B_{\sigma-s}} \int_{B_\varepsilon} \varphi_\varepsilon(x)|f'(y - x) - f'(y)| \, dy \, dx$$

$$\leq \int_{B_{\sigma-s}} \int_{B_\varepsilon} \int_0^1 \varphi_\varepsilon(x)|Df'(y - tx) \cdot x| \, dt \, dy \, dx$$

$$\leq \int_0^1 \int_{B_\varepsilon} \varphi_\varepsilon(x)\varepsilon \int_{B_{\sigma-s}} |Df(y - tx)| \, dy \, dx \, dt \leq \varepsilon \, \|Df\|_{L^1(B_\sigma)} \leq C E^{1/2+a},$$

(where we have used the fact that $\varepsilon \leq s$). Putting everything together we conclude that

$$\mathbf{\Omega} \int_{B_\sigma} \mathcal{G}(f, g) \leq C \, \mathbf{\Omega} \, E^{1/2+\gamma} \leq C \, E^\gamma (E + \mathbf{\Omega}^2)$$

for a suitable $\gamma > 0$, thus concluding the proof of the Theorem. □

4.2. **Proof of Theorem 1.4.** Without loss of generality, we can assume $r = 1$ and $x = 0$. Choose $\beta_2 < \min\{\frac{1}{2m}, \frac{\gamma_3}{2(1+\gamma_3)}\}$, where γ_3 is the constant in Theorem 4.1. Let f be the E^{β_2}-Lipschitz approximation of T. Clearly (1.3) follows directly from [7, Proposition 3.2] if $\beta_0 < \beta_2$. Set next $A := \{\mathbf{me}_T > 2^{-m}E^{2\beta_2}\} \cap B_{9/8}$. By [7, Proposition 3.2], $|A| \leq C E^{1-2\beta_2}$. Apply estimate (4.1) to A to conclude:

$$|B_1 \setminus K| \leq C \, E^{-2\beta_2} \, \mathbf{e}_T(A) \leq C \, E^{\gamma_3 - 2\beta_2(1+\gamma_3)}(E + \mathbf{\Omega}^2).$$

By our choice of γ_3 and β_2, this gives (1.4) for some positive β_0. Finally, set $S = \mathbf{G}_f$. Recalling the strong Almgren's estimate (4.1) and the Taylor expansion in [8, Corollary 3.3], we conclude:

$$\left| \|T\|(\mathbf{C}_1) - Q\,\omega_m - \int_{B_1} \frac{|Df|^2}{2} \right| \leq \mathbf{e}_T(B_1 \setminus K) + \mathbf{e}_S(B_1 \setminus K) + \left| \mathbf{e}_S(B_1) - \int_{B_1} \frac{|Df|^2}{2} \right|$$

$$\leq C \, E^{\gamma_3}(E + \mathbf{\Omega}^2) + C \, |B_1 \setminus K| + C \, \mathrm{Lip}(f)^2 \int_{B_1} |Df|^2 \leq C \, E^{\gamma_1}(E + \mathbf{\Omega}^2).$$

The L^∞ bound follows straightforwardly from [7, Proposition 3.2].

5. Approximation of 2-dimensional almost minimizing currents

As mentioned in the introduction, we state here the approximation result for two dimensional currents as in (a), (b) and (c) of Theorem 0.2, which will be used in our subsequent notes [11, 12]. The following are the main assumptions.

Assumption 5.1. In case (a) $\Sigma \subset \mathbb{R}^{m+n}$ is a C^2 submanifold of dimension $m + \bar{n} = m + n - l$, which is the graph of an entire function $\Psi : \mathbb{R}^{m+\bar{n}} \to \mathbb{R}^l$ and satisfies the bounds

$$\|D\Psi\|_0 \leq c_0 \quad \text{and} \quad \mathbf{A} := \|A_\Sigma\|_0 \leq c_0, \tag{5.1}$$

where c_0 is a positive (small) dimensional constant. ω is a C^1 m-form. T is an integral current of dimension 2 with bounded support. Moreover it satisfies one of the three conditions (a), (b) or (c) in Definition 0.1. In particular in case (a) we have $\mathrm{spt}(T) \subset \Sigma$ and T

is area-minimizing in Σ. In case (b) we assume $\Sigma = \mathbb{R}^{m+n}$ and T is semicalibrated by ω. In case (c) we have that Σ coincides with a portion of $\partial\mathbf{B}_R(p)$, which is the graph of a map $\Psi : \Omega \to \mathbb{R}$ satisfying (5.1), for some $\Omega \subset \mathbb{R}^{m+n-1}$. Finally, for some open cylinder $\mathbf{C}_{4r}(x)$ (with $r \leq 1$) and some positive integer Q, we assume that Assumptions 1.2 still holds.

Theorem 5.2. *There exist constants $M, C_{21}, \beta_0, \varepsilon_{21} > 0$ (depending on $m, n, \bar{n}, Q$) with the following property. Assume that T satisfies Assumption 1.2 in the cylinder $\mathbf{C}_{4r}(x)$ and $E = \mathbf{E}(T, \mathbf{C}_{4r}(x)) < \varepsilon_{21}$. Then, there exist a map $f : B_r(x) \to \mathcal{A}_Q(\mathbb{R}^n)$, with $\{x\} \times \mathrm{spt}(f(x)) \subset \Sigma$ for every x, and a closed set $K \subset B_r(x)$ such that*

$$\mathrm{Lip}(f) \leq C_{21}E^{\beta_0} + C_{21}\mathbf{\Omega}r \quad \text{in case (a) and (c)}, \tag{5.2}$$

$$\mathrm{Lip}(f) \leq C_{21}E^{\beta_0} \qquad \text{in case (b)} \tag{5.3}$$

$$\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n) \quad and \quad |B_r(x) \setminus K| \leq C_{21}E^{\beta_0}\left(E + r^2\mathbf{\Omega}^2\right)r^m, \tag{5.4}$$

$$\left|\|T\|(\mathbf{C}_r(x)) - Q\omega_m r^m - \frac{1}{2}\int_{B_r(x)}|Df|^2\right| \leq C_{21}E^{\beta_0}\left(E + r^2\mathbf{\Omega}^2\right)r^m, \tag{5.5}$$

where $\mathbf{\Omega} = \mathbf{A}$ in case (a), $\mathbf{\Omega} = \|d\omega\|_0$ in case (b) and $\mathbf{\Omega} = \frac{3}{R}$ in case (c). If in addition $\mathbf{h}(T, \mathbf{C}_{4r}(x)) := \sup\{|\mathbf{p}^\perp(x) - \mathbf{p}^\perp(y)| : x, y \in \mathrm{spt}(T) \cap \mathbf{C}_{4r}(x)\} \leq r$, then

$$\mathrm{osc}(f) \leq C_{21}\mathbf{h}(T, \mathbf{C}_{4r}(x)) + C_{21}(E^{1/2} + r\mathbf{\Omega})r \quad \text{in case (a) and (c)}, \tag{5.6}$$

$$\mathrm{osc}(f) \leq C_{21}\mathbf{h}(T, \mathbf{C}_{4r}(x)) \quad \text{in case (b)}. \tag{5.7}$$

Proof. Case (a) is proved in [7, Theorem 2.4], while case (b) follows directly from Theorem 1.4 after recalling that semicalibrated currents are $\mathbf{\Omega}$-minimal currents for $\mathbf{\Omega} = \|d\omega\|_0$ by [13, Proposition 1.2].

It remains to handle case (c). Again by [13, Proposition 1.2], a current satisfying (c) of Definition 0.1 is an $\mathbf{\Omega}$-minimal current for $\mathbf{\Omega} = \frac{3}{R}$. Therefore, we can apply Theorem 1.4. However, the graph of the map f so obtained is not necessarily contained in Σ. We show here how to modify it in such a way to fulfill the requirements of Theorem 5.2. We assume that Ψ is a function whose graph coincides with Σ (the connected component of $\partial\mathbf{B}_R(p) \cap \mathbf{C}_{4r}(x)$ containing $\mathrm{spt}(T)$) and arguing as in [7, Remark 1.5] we can assume that $\|\Psi_0\| \leq CE^{1/2}r + C\mathbf{\Omega}r^2$, $\|D\Psi\|_0 \leq CE^{1/2} + C\mathbf{\Omega}r$ and $\|D^2\Psi\|_0 \leq C\mathbf{\Omega}$. The domain of Ψ is a subset of $B_{4r}(x) \times \mathbb{R}^{n-1}$. Let now $f = \sum_i [\![f_i]\!]$ be the function given by Theorem 1.4 and let $\bar{f} = \sum_i [\![\bar{f}_i]\!]$, where $\bar{f}_i(y)$ gives the first $n-1$ coordinates of $f_i(y)$. Observe that on the set K we necessarily have

$$f(y) = \sum_i [\![(\bar{f}_i(y), \Psi(y, \bar{f}_i(y)))]\!] .$$

We then can extend $\bar{f}$ to $B_r(x) \setminus K$ with $\mathrm{Lip}(\bar{f}) \leq C\mathrm{Lip}(f)$ and $\mathrm{osc}(\bar{f}) \leq C\mathrm{osc}(f)$ and hence define $\hat{f}(y) = \sum_i [\![(\bar{f}_i(y), \Psi(y, \bar{f}_i(y)))]\!]$ for *every* $y \in B_r(x)$ (it must be shown that $(y, \bar{f}_i(y))$ belongs to the domain of definition of Ψ, but this follows easily from the smallness

of osc $(\bar{f})$). Obviously $f = \hat{f}$ on K. On the other hand it is straightforward to check that

$$\text{Lip}(\hat{f}) \leq C \, \text{Lip}(\bar{f}) + C(\text{Lip}(\bar{f}) + 1)\|D\Psi_0\| \leq CE^{\beta_0} + C\mathbf{\Omega}r$$
$$\text{osc}\,(\hat{f}) \leq C \, \text{osc}\,(f) + \|\Psi\|_0 \leq C\mathbf{h}(T, \mathbf{C}_{4r}(x)) + C(E^{1/2} + \mathbf{\Omega}r)r \,.$$

In addition we conclude

$$\left| \int_{B_r(x)} |Df|^2 - \int_{B_r(x)} |D\hat{f}|^2 \right| \leq (\text{Lip}(f)^2 + \text{Lip}(\hat{f})^2)|B_r(x) \setminus K| \leq C|B_r(x) \setminus K| \,.$$

Thus the estimates in Theorem 1.4 complete the proof.   □

Appendix A. A singular semicalibrated current

Here we give an explicit example of a 2-dimensional current with a singular point that is semicalibrated by a differential form which is not closed.

Consider a function $u : \mathbb{R}^2 \to \mathbb{R}^2$ which is C^∞ but not analytic, and assume that

$$u(0) = |\nabla u(0)| = 0.$$

We need to introduce the following:

(1) $E_1, E_2 \in \mathcal{T}\big(\text{Gr}(u)\big)$ and $E_3, E_4 \in \mathcal{N}\big(\text{Gr}(u)\big)$ smooth sections of the tangent and the normal bundles of $\text{Gr}(u) \subset \mathbb{R}^4$ considered as a smooth oriented submanifold such that

$$E_i(p) \cdot E_j(p) = \delta_{ij} \quad \forall\, p \in \text{Gr}(u), \,\forall\, i, j = 1, \dots, 4;$$

moreover we assume that $E_i(0) = e_i$ for every $i = 1, \dots, 4$, where $\{e_i\}_{i=1,\dots,4}$ is the standard basis of $\mathbb{R}^4$;

(2) $\theta^1, \dots, \theta^4$ the dual fields:

$$\theta^i(p)\big(E_j(p)\big) = \delta^i_j \quad \forall\, p \in \text{Gr}(u), \,\forall\, i, j = 1, \dots, 4;$$

(3) $\mathbf{p}_u$ the nearest point projection on $\text{Gr}(u)$, which exists in a tubular neighborhood of the submanifold $\text{Gr}(u)$ and therefore, in particular, in B_{r_0} for some $r_0 > 0$;

(4) $[\![D]\!] := (e_3 \wedge e_4) \cdot \mathcal{H}^2 \llcorner \{x_1 = x_2 = 0\}$ the current associated to the oriented integration on the vertical plane $D = \{x_1 = x_2 = 0\}$.

It is now elementary to verify the following claims:

(i) the smooth 2-dimensional differential form

$$\omega(x) := \theta^1(\mathbf{p}_u(x)) \wedge \theta^2(\mathbf{p}_u(x)) + \theta^3(\mathbf{p}_u(x)) \wedge \theta^4(\mathbf{p}_u(x))$$

is a semicalibration in B_{r_0};

(ii) the current $T := \mathbf{G}_u + [\![D]\!]$ is semicalibrated by ω in B_{r_0} and $0 \in \text{Sing}(T)$.

Note that ω is not a closed form, for in this case T would be an area minimizing current thus implying that $\text{spt}(T) \setminus \{0\}$ is locally the graph of an analytic map (cf. [19, Theorem 5.5]): this is obviously not the case for a generic smooth u.

Actually, following the same principles, it is simple to construct many more examples. In particular it is possible to construct examples where the semicalibrated current has a branching singularity. However the corresponding computations are slightly more involved.

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