THE ONSAGER THEOREM

CAMILLO DE LELLIS

Abstract. In his famous 1949 paper on hydrodinamic turbulence, Lars Osanger advanced a remarkable conjecture on the energy conservation of weak solutions to the Euler equations: all Hölder continuous solutions with Hölder exponent strictly larger than $\frac{1}{3}$ preserves the kinetic energy, while there are Hölder continuous solutions with any exponent strictly smaller than $\frac{1}{3}$ which do not preserve the kinetic energy.

While the first statement was proved by Constantin, E and Titi in 1994, the second was proved only recently by P. Isett building upon previous works of László Székelyhidi Jr. and the author. This paper is a survey on the proof of the conjecture and on several other related discoveries which have been made in the last few years.

1. Introduction

The incompressible Euler equations describe the motion of a perfect incompressible fluid. Written down by L. Euler over 250 years ago, they are a system of partial differential equations in two unknowns: a vector function v, which gives the velocity of the fluid particle passing through the point x at the time t, and a scalar field p, the hydrodinamic pressure. The system takes the following form

$$\begin{cases} \partial_t v + (v\cdot\nabla)v + \nabla p = 0 \\ \\ \operatorname{div} v = 0, \end{cases} \tag{1}$$

where the components of the advective term $(v\cdot\nabla)v$ are given by

$$[(v\cdot\nabla)v]_i = \sum_j v_j \frac{\partial v_i}{\partial x_j} = \sum_j v_j \partial_j v_i\,.$$

In particular, the divergence free condition on v implies that, for C^1 solutions (v,p), the system can be quivalently written as

$$\begin{cases} \partial_t v + \operatorname{div}(v\otimes v) + \nabla p = 0 \\ \\ \operatorname{div} v = 0\,. \end{cases} \tag{2}$$

The first, vectorial, equation expresses the conservation of momentum for every fluid region and it assumes that there is no dissipation of energy through friction. The second equation expresses instead the conservation of mass: the fluid is therefore assumed to be incompressible and to have constant density, which for convenience we can normalize to 1.

For the moment let us consider the 3-dimensional case with periodic boundary conditions. In other words we take the spatial domain to be the flat 3-dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$. Thus $(v,p) : \mathbb{T}^3 \times I \to \mathbb{R}^3 \times \mathbb{R}$, where I is a time interval, say $I = [0,T]$. In the rest of this note we will call (v,p) a classical solution if $(v,p) \in C^1(\mathbb{T}^3 \times [0,T])$ and satisfies (2) pointwise.

The total kinetic energy of the fluid is given by

$$E(t) := \frac{1}{2} \int_{\mathbb{T}^3} |v(x,t)|^2\, dx$$

and, as expected, it is a conserved quantity for classical solutions, namely $E(t) = E(0)$ for every $t \in I$. The proof is an easy computation. After scalar multiplying the first equation by v we derive

$$\sum_j v_j \partial_t v_j + \sum_j v_j \sum_k v_k \partial_k v_j + \sum_j v_j \partial_j p = 0\,,$$

which we can rewrite as

$$\partial_t \frac{|v|^2}{2} + (v \cdot \nabla) \left(\frac{|v|^2}{2} + p\right) = 0\,.$$

We then use the divergence free condition to infer

$$\partial_t \frac{|v|^2}{2} + \mathrm{div} \left(\left(\frac{|v|^2}{2} + p\right) v\right) = 0\,, \tag{3}$$

which integrated in the space variable gives

$$2\frac{dE}{dt} = \frac{d}{dt} \int_{\mathbb{T}^3} |v|^2(x,t)\, dx = 0\,. \tag{4}$$

The smoothness of the pair (especially that of v) has been crucially used to derive (3) and a natural question is whether its validity can be extended to less regular solutions (of course if we decrease the regularity of the solutions in a substantial way, we need to specify in which sense we understand the equations; this issue will be addressed thouroughly in a few paragraphs). Although this might seem a rather academic question, it was actually posed much before the modern trend of studying weak solutions of partial differential equations took place. In his famous 1949 paper [75] on hydrodinamic turbulence, Lars Onsager, a celebrated theoretical physicist of the 20th century, stated the following remarkable conjecture.

Conjecture 1.1. *Consider periodic* 3*-dimensional weak solutions of* (2)*, where the velocity v satisfies the uniform Hölder condition*

$$|v(x,t) - v(x',t)| \le C|x - x'|^\theta, \tag{5}$$

for constants C and θ independent of x, x' and t.

(a) If $\theta > \frac{1}{3}$, then the total kinetic energy of v is constant;

(b) For any $\theta < \frac{1}{3}$ there are solutions v for which the total kinetic energy is not *constant and, in particular, it is dissipated, namely*

$$E(t) \le E(s) \text{ for all } t \le s \text{ and } E(\tau) < E(0) \text{ for some } \tau > 0. \tag{6}$$

As already mentioned, in order to get a clearly stated mathematical problem one needs to specify what weak solution means in this context, in particular because Hölder functions are not pointwise differentiable. Remarkably Onsager in [75] gives a concise, elegant and mathematically precise definition of what he means by weak solution, which in fact adheres to what a modern mathematician would call a distributional square summable solution. Below we will give an account of Onsager's motivation for raising the conjecture and we will discuss various definitions of weak solutions of the Euler equations.

Part (a) of Conjecture 1.1 has been settled in the nineties by an elegant work of Constantin, E and Titi (see [29]) after a slighly weaker statement was first proved by Eyink in [48]. The remaining part of the Conjecture has been settled in the last couple of years and, therefore, we could now call it the Onsager theorem. The aim of this note is to give an account of the proof and to provide also a short survey of the literature that, although not directly related to the conjecture, has stemmed from the attempts at solving it in the last 12 years. In particular the methods used to prove the conjecture have been applied to a large range of other problems, raising challenging open questions and uncovering surprising connections between different fields in geometry and analysis.

2. Weak solutions

We will propose here several different definitions of weak solutions of (2).

2.1. **Fourier series.** Consider a classical solution (v, p) on the periodic torus. Let

$$v(x,t) = \sum_{k\in\mathbb{Z}^3} \hat{v}_k(t) e^{ik\cdot x}$$

be its Fourier expansion. Observe that the divergence-free condition is thus equivalent to

$$k \cdot \hat{v}_k = 0 \qquad \text{for all } k \in \mathbb{Z}^3. \tag{7}$$

As for the first equation in (2) we can rewrite it as

$$\hat{v}_k'(t) + i \sum_{j+\ell=k} \hat{v}_\ell(t) \cdot j \hat{v}_j(t) + ik\hat{p}_k(t) = 0 \tag{8}$$

where $p(x,t) = \hat{p}_k(t)e^{ik\cdot x}$ is the Fourier expansion of the pressure. After scalar multiplying with k the above relation we find

$$|k|^2 \hat{p}_k = -\sum_{j+\ell} \hat{v}_\ell \cdot j\hat{v}_j \cdot k\,.$$

In particular, if we introduce the symmetric bilinear map

$$\mathcal{B}_k(\hat{v},\hat{v}) = \sum_{j+\ell=k} \hat{v}_\ell \cdot k\, \hat{v}_j - \frac{k}{|k|^2} \sum_{j+\ell=k} \hat{v}_\ell \cdot k\, \hat{v}_j \cdot k\,.$$

we can rewrite (8) as

$$\hat{v}_k' = -i\mathcal{B}_k(\hat{v},\hat{v})\,. \tag{9}$$

It is easy to check that, if (7) holds at some time $\tau \in [0,T]$ and the Fourier coefficients satisfy the infinite system of ODEs (9), then in fact (7) holds at all times $t \in [0,T]$. Thus, considering that we want to solve (2) subject to the initial condition $v(\cdot,0) = v^0$ for some solenoidal initial data v_0, it suffices to solve the infinite system (9) coupled with the initial conditions $\hat{v}_k(0) = \hat{v}^0_k$.

Observe now that the bilinear operator $\mathcal{B}_k(\hat{v},\hat{v})$ is well defined and continuous on $\ell^2(\mathbb{Z}^3,\mathbb{R}^3)$. Namely, if $v(\cdot,t) \in L^2$ for every t, the right hand side of (9) is an absolutely converging series for every k and the system (9) is well-defined. We can thus say that $v \in L^\infty([0,T], L^2(\mathbb{T}^3,\mathbb{R}^3))$ is a weak solution of (2) if:

- each $t \mapsto \hat{v}_k(t)$ is an absolutely continuous function and (9) holds at almost every time $t \in [0,T]$;
- and (7) holds at the initial time.

2.2. **Distributional solutions.** Since the only nonlinearity in (2) is the quadratic term $v \otimes v$ in the first equation, distributional solutions can be defined as pairs $v \in L^2(\mathbb{T}^3 \times [0,T],\mathbb{R}^3)$ and $P \in \mathcal{D}'(\mathbb{R}^3\times]0,T[)$ satisfying

$$\int v(x,t)\cdot \nabla\phi(x,t)\,dxdt = 0 \tag{10}$$

$$\int \left[v(x,t)\cdot\partial_t\psi(x,t) + v\otimes v(x,t) : D\psi(x,t)\right]\,dxdt + P(\mathrm{div}\,\psi) = 0 \tag{11}$$

for every pair of tests $\varphi \in C^\infty_c(\mathbb{T}^3\times]0,T[)$ and $\psi \in C^\infty_c(\mathbb{T}^3\times]0,T[,\mathbb{R}^3)$. As it is the case for the Fourier approach, the pressure can be eliminated from the equations by requiring that (11) holds for divergence-free tests.

2.3. **Continuum mechanics.** If Ω is a smooth open domain and ν denotes the exterior unit normal to Ω, then

$$\int_{\partial\Omega} p(x,t)\nu(x)\,dx$$

is the total force exerted at time t by the fluid outside Ω upon the portion of fluid inside Ω. Note that p is then well-defined up to an arbitrary function of time, since

$$\int_{\partial\Omega} \nu = 0$$

for every smooth bounded open set Ω. This arbitrariness in the definition of p can be seen directly from (1) and it is natural to mod it out by normalizing p so that $\int_{\mathbb{T}^3} p(x,t)\,dx = 0$.

As already mentioned the two equations in (1) express simply the conservation of mass and momentum. Indeed, if (v,p) is a pair of C^1 functions satisfying (1) and Ω an arbitrary domain, the divergence theorem implies

$$\int_{\partial\Omega} v \cdot \nu = 0 \tag{12}$$

$$\frac{d}{dt}\int_{\Omega} v = \int_{\partial\Omega} v(v\cdot\nu) + \int_{\partial\Omega} p\nu\,. \tag{13}$$

The identity (12) has a very intuitive interpretation: the total amount of fluid particles "getting out" of Ω is balanced by the total amount "getting in" and as a result the total amount of fluid occupying the region Ω remains constant. The identity (13) is the counterpart of the conservation of momentum: the rate of change of the momentum of the fluid contained in Ω is given by the sum of the flux of momentum through $\partial\Omega$ and the total force exerted on Ω by the portion of fluid lying outside.

In continuum mechanics it is often the case that balance laws as in (12) and (13) (valid for any "fluid element" Ω) are derived, under suitable assumptions, from first principles, whereas the differential equations (as (1)) are deduced as consequences when the functions are sufficiently smooth. In the case at hand (1) can be easily derived from (12)-(13) if the pair (v, p) is C^1. However we can make sense of (12) and (13) even if (v, p) are much less smooth: the continuity of the pair is, for instance, enough to make sense of all the integrals in (12) and (13) whenever Ω has C^1 (or even Lipschitz) boundary. This provides a third way of defining weak solutions.

Finally let us observe that the three definitions are all equivalent when they all make sense. The careful reader will have noticed that we have given them under slightly different assumptions and that in particular the "distributional approach" requires the least amount of regularity on the pair (v, p): the technicalities can actually be adjusted so to make sense of the other two approaches under the same minimal smoothness.

3. Anomalous dissipation

Let us briefly describe the considerations leading Onsager to his conjecture. We start by introducing the Navier Stokes equations, namely the system

$$\begin{cases} \partial_t v + \operatorname{div}(v\otimes v) + \nabla p = \nu\Delta v \\ \\ \operatorname{div} v = 0\,, \end{cases} \tag{14}$$

where the viscosity ν is a positive (and, for our purposes, small) constant. For solutions of (14) the energy conservation law is given by

$$\frac{d}{dt}\int_{\mathbb{T}^3} |v|^2(x,t)\,dx = -2\nu\int_{\mathbb{T}^3} |Dv|^2(x,t)\,dx\,. \tag{15}$$

For sufficiently smooth solutions it can be derived by scalar multiplying the first equation by u and performing essentially the same computations leading to (4).

If we were considering a family of solutions v_ν with $\nu \to 0$ and if these solutions were to converge to a classical solution of (1), then the right hand side of (15) would behave as $O(\nu)$. However, in the theory of hydrodynamic turbulence it is expected that, in 3-dimensions and for "typical" turbulent solutions of (14), the right hand side of (15) is independent of the viscosity. Thus, one may advance the hypothesis that the dissipation of the energy is not primarily driven by the viscous term $\nu\Delta u$ and that the main responsible for this dissipation is indeed the nonlinear term of the equations, which appear as well in (1).

This hypothesis and a corresponding "energy spectrum" law have been first put forward by Kolmogorov in [66] (nowadays often cited as K41 theory) and, as pointed out by Onsager in [75], rediscovered independently at least twice (in [74] and [94]; see also [52], which refers to [78]). We briefly explain here the motivations given by [75] for the Kolmogorov's law (and refer to [49] for a nice and much more detailed analysis of Onsager's discoveries).

Recall that we denote by E the total kinetic energy $E(t) = \frac{1}{2}\int |u|^2(x,t)dx$. In addition let Q be its rate of dissipation $\frac{dE}{dt}$ and let L be the "macroscale" of the flow (in our case we can suppose this is the side length of the torus, i.e. 2π). If we assume that Q depends only on L and E a simple dimensional analysis suggests the law

$$Q = -\frac{dE}{dt} = cE^{\frac{3}{2}}L^{-1} \tag{16}$$

where c is a dimensionless constant. Indeed, if σ denotes the unit of space and τ the unit of time, then E is measured in σ^2/τ^2, Q in σ^2/τ^3 and L in σ: it can be readily checked that the law (16) is the only possible one of the form $cE^\alpha L^\beta$ for which c is a dimensionless constant. The law (16) has been verified extensively in experiments and it turns out to be valid as long as the viscosity is very small compared to $E(t)$.

In order to get into Onsager's explanation of how this might be possible, we proceed as in the derivation of (9) to conclude that the Fourier coefficients $\hat{v}_k$ of a solution of (14) satisfy the identities

$$\hat{v}_k'(t) = -i\mathcal{B}_k(\hat{v}(t), \hat{v}(t)) - \nu|k|^2\hat{v}_k(t)\,. \tag{17}$$

In particular, multiplying by the complex conjugate of $\hat{v}_k$ (which in fact equals $\hat{v}_{-k}$ by the reality of the velocity v) we derive

$$\frac{d}{dt}|\hat{v}_k|^2 = -2|k|^2\nu|\hat{v}_k|^2 - i\mathcal{B}_k(\hat{v},\hat{v})\cdot\hat{v}_{-k}\,. \tag{18}$$

Using the expression for $\mathcal{B}_k$ we can compute

$$-i\mathcal{B}_k(\hat{v},\hat{v})\cdot\hat{v}_{-k} = \underbrace{\sum_\ell \left(-2\mathrm{Im}\,\left((\hat{v}_{k+\ell}\cdot\ell)(\hat{v}_{-k}\cdot\hat{v}_{-\ell}) + (\hat{v}_{\ell-k}\cdot k)(a_k\cdot a_{-\ell})\right)\right)}_{=:Q(k,\ell)}\,.$$

Note that $Q(k,\ell) = -Q(\ell,k)$: this term accounts for the "energy exchange" between different Fourier modes. As long as $-\nu|k|^2$ is small (i.e. for sufficiently small k), we can assume that the term $Q(k,\ell)$ is the dominating one in (18).

The picture proposed by Onsager for a "typical" chaotic flow is the following: in the infinite sum at the right hand side of (18) only the terms where $\hat{v}_k, \hat{v}_\ell$, $\hat{v}_{k+\ell}$ have a comparable wavelength are relevant. So, the energy gets redistributed from wave lengths of a certain size to wave lengths of, say, double that size. As λ grows the redistribution process happens faster and faster, so that after a short time (i.e. before E becomes too smal for the validity of (16)) the energy is redistributed at all scales. If this transfer is a chaotic process, after few steps the information about the low wave numbers (i.e. the macroscopic features of the flow) is lost. It is therefore plausible that the energy flux of the energy distribution depends only on the total dissipation rate $Q = -\frac{dE}{dt}$ and on the modulus of the wave number $|k|$.

If we set $f(\lambda) := \sum_{|k|\leq\lambda} |\hat{v}_k|^2$ the energy distribution $E(\lambda)$ is "formally" $\frac{df}{d\lambda}$, so that $E = \int E(\lambda)d\lambda$. Since the frequency is measured in σ^{-1}, $E(\lambda)$ is measured in $\sigma^3\tau^{-2}$. The same dimensional analysis leading to (16) gives then

$$E(\lambda) = \beta Q^{\frac{2}{3}}\lambda^{-\frac{5}{3}} \tag{19}$$

where β is a dimensional constant. The last identity is the famous Kolmogorov's law.

In the final paragraph of his note Onsager writes:

It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily in the absence of viscosity. In fact it is possible to show that the velocity field in such "ideal turbulence" cannot obey any Lipschitz condition of the form $|v(x) - v(y)| \leq C|x|^\alpha$ *for any* α *greater than* $\frac{1}{3}$*; otherwise the energy is conserved. ... The detailed conservation of energy* (18) *does not imply conservation of the total energy if the number of steps in the cascade is infinite, as expected, and the double sum of* $Q(\ell,k)$ *converges only conditionally.*

Onsager considers thus the possibility of setting $\nu = 0$ rather than assuming it small. He further claims that a closer inspection of the identity (18) shows that the total conservation of the energy can be inferred from the weak formulation of the equation (1) only when the solution is Hölder continuous with exponent larger than $\frac{1}{3}$, whereas this might fail for smaller exponents.

The exponent $\frac{1}{3}$ has a direct sinificance in isotropic turbulence, since it is related to another famous law of the Kolmogorov's theory, namely the fact that, in isotropic turbulent flows, the spatial variance of velocities is comparable to the distance to the power $\frac{2}{3}$ (see the discussion in the paper [49]). The latter law is also derived in the literature using scaling arguments. In fact to our knowledge this is the only type of "theoretical arguments" present in the literature on fully developed turbulence. The recent proof of the Onsager's conjecture gives a first justification purely based on rigorous mathematical considerations pertaining to the equations of motions.

4. Energy conservation: the proof of Constantin, E and Titi

Following Onsager's suggestion, the claim about the energy conservation has been shown by Eyink in [48] under the assumption that

$$\sum_k |k|^{\frac{1}{3}+\varepsilon}|\hat{v}_k| < \infty$$

(which does imply the $(\frac{1}{3}+\varepsilon)$-Hölder regularity, in space, of the function v, but it is obviously a stronger condition). Onsager's exact claim has then, namely Part (a) of Conjecture 1.1, been shown by Constantin, E and Titi in [29] with an elegant and fairly short argument (we refer also to [20] for more precise results), which we expose here.

Fix a weak solutions (v,p) satisfying the Hölder condition (5) for some positive θ and consider a standard family of (radially symmetric) mollifiers $\varphi_\varepsilon(x) = \varepsilon^{-3}\varphi(\frac{x}{\varepsilon})$ *in space*. As usual we will denote by f_ε the mollification of any summable function f defined on $\mathbb{R}^3 \times [0,T]$ with the kernel φ_ε, namely

$$f_\varepsilon(x) = \int_{\mathbb{T}^3} f(x-y,t)\varphi_\varepsilon(y)\,dy = \frac{1}{\varepsilon^3}\int_{\mathbb{T}^3} f(x-y,t)\varphi\left(\frac{y}{\varepsilon}\right)\,dy\,.$$

In order to simplify some computations, it is useful to consider f as a periodic funcion defined on $\mathbb{R}^3 \times [0,T]$, whereas φ, which is supported in a neighborhood of the origin in $\mathbb{T}^3$, identified in this case with $[-\pi,\pi]^3 \subset \mathbb{R}^3$, will be considered also as a funcion on $\mathbb{R}^3$ by setting it equal to 0 outside its support, rather than extending it periodically. With this convention we can then write

$$f(x) = \int_{\mathbb{R}^3} f(x-\varepsilon z,t)\varphi(z)\,dz\,.$$

Clearly

$$\operatorname{div} v_\varepsilon = 0\,.$$

On the other hand the momentum balance in the Euler equations has a nonlinear term and for this reason v_ε is not a solution. We can however regard it as an "approximate solution":

$$\partial_t v_\varepsilon + \operatorname{div} v_\varepsilon \otimes v_\varepsilon + \nabla p_\varepsilon = \operatorname{div}\big(\underbrace{v_\varepsilon \otimes v_\varepsilon - (v\otimes v)_\varepsilon}_{=:T_\varepsilon}\big)\,. \tag{20}$$

Note that v is, by assumption, bounded. Moreover, since p satisfies

$$\Delta p = -\operatorname{div}\operatorname{div} v\otimes v$$

and we can normalize it to have mean 0, the classical Schauder estimates imply that the analog of (5) holds for p as well. In particular both p_ε and v_ε and $(v\otimes v)_\varepsilon$ have bounded spatial derivatives of every order. The equation

(20) implies in turn that $\partial_t v_\varepsilon$ is also bounded. v_ε is thus a Lipschitz function of time and space and we can scalar multiply (20) by v_ε to derive the identity

$$\partial_t \frac{|v_\varepsilon|^2}{2} + \operatorname{div}\left(\left(\frac{|v_\varepsilon|^2}{2} + p_\varepsilon\right) v_\varepsilon\right) = T_\varepsilon \cdot v_\varepsilon\,, \tag{21}$$

In particular, integrating in time we conclude

$$\frac{d}{dt}\int_{\mathbb{T}^3} |v_\varepsilon(x,t)|^2\, dx = 2\int_{\mathbb{T}^3} T_\varepsilon(x,t)\cdot v_\varepsilon(x,t)\, dx\,.$$

Since v_ε converges to v uniformly, it suffices to show that

$$\lim_{\varepsilon\downarrow 0} \underbrace{\int_{\mathbb{T}^3} T_\varepsilon(x,t)\cdot v_\varepsilon(x,t)\, dx}_{=:S_\varepsilon(t)} = 0 \qquad \forall t \tag{22}$$

to conclude that the conservation of the total kinetic energy is valid for v.

We first integrate by parts and write

$$S_\varepsilon(t) = \int_{\mathbb{T}^3} [(v\otimes v)_\varepsilon - v_\varepsilon \otimes v_\varepsilon](x,t) : Dv_\varepsilon(x,t)\, dx\,. \tag{23}$$

Next observe

$$\begin{aligned} Dv_\varepsilon(x,t) &= \frac{1}{\varepsilon}\int_{\mathbb{R}^3} v(x-\varepsilon y,t)\otimes \nabla\varphi(y)\, dy \\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^3} [(v(x-\varepsilon y,t) - v(x,t)] \otimes \nabla\varphi(y)\, dy\,. \end{aligned}$$

In particular, we conclude

$$|Dv_\varepsilon(x,t)| \le C[v(\cdot,t)]_\theta \varepsilon^{\theta-1}\,, \tag{24}$$

where C is a constant depending only on the mollifier φ and

$$[v(\cdot,t)]_\theta := \sup_{x\neq y} \frac{|v(x,t)-v(y,t)|}{|x-y|^\theta}$$

is the usual Hölder seminorm.

Furthermore we can write

$$\begin{aligned} [(v\otimes v)_\varepsilon - v_\varepsilon\otimes v_\varepsilon](x,t) &= \int (v\otimes v)(x-\varepsilon y,t)\varphi(y)\, dy \\ &\quad - \int\int v(x-\varepsilon y,t)\otimes v(x-\varepsilon z,t)\varphi(y)\varphi(z)\, dy\, dz \\ = \int\int v(x-\varepsilon y,t)\otimes [v(x-\varepsilon y,t) &- v(x-\varepsilon z,t)]\varphi(y)\varphi(z)\, dy\, dz\,. \end{aligned}$$

Symmetrizing the latter expression in y and z we infer

$$\begin{aligned} &[(v\otimes v)_\varepsilon - v_\varepsilon\otimes v_\varepsilon](x,t) \\ &= \frac{1}{2}\int\int [v(x-\varepsilon y,t) - v(x-\varepsilon z,t)]\otimes [v(x-\varepsilon y,t) - v(x-\varepsilon z,t)] \\ &\qquad \varphi(y)\varphi(z)\, dy\, dz\,. \end{aligned}$$

Thus, using (5) we easily conclude

$$|[(v\otimes v)_\varepsilon - v_\varepsilon \otimes v_\varepsilon](x,t)| \leq C[v(\cdot,t)]_\theta^2 \varepsilon^{2\theta}\,. \tag{25}$$

Inserting (24) and (25) in (23) we achieve

$$|S_\varepsilon(t)| \leq C[v(\cdot,t)]_\theta^3 \varepsilon^{3\theta-1}\,.$$

The latter inequality shows (22) when $\theta > \frac{1}{3}$.

5. Energy dissipation: L^2 and L^∞

Concerning Part (b) of Conjecture 1.1, the first construction ever of an L^2 solution that violates the energy conservation is due to Scheffer in [79]. The main theorem of [79] states the existence of a non-trivial weak solution in $L^2(\mathbb{R}^2\times\mathbb{R})$ with compact support in space and time. Later on Shnirelman in [81] gave a different proof of the existence of a non-trivial weak solution in $L^2(\mathbb{T}^2\times\mathbb{R})$ with compact support in time. In these constructions the energy is clearly not conserved, because it is identically 0 at any sufficiently large time and it is positive on a set of measure zero. Moreover, since the authors have no control on the regularity of the total kinetic energy $E(t)$ (which is not even known to be bounded for every t), there could in principle be no interval in which the function E is monotone. The existence of a solution in three space dimension which does not conserve the kinetic energy but respects however the expected monotonicity on some time interval was achieved later by Shnirelman in [82].

In the paper [39] we provided a relatively simple proof of the following stronger statement.

Theorem 5.1. *There exist infinitely many compactly supported bounded weak solutions of the incompressible Euler equations in any space dimension. There exist also infinitely many bounded weak solutions which do not preserve the kinetic energy but for which the latter is monotone nonincreasing.*

In fact our proof shows the existence of solutions of the Euler equations which violate the usual conservation energy and the uniqueness of the Cauchy problem in several different ways (see also [40]). The key was to regard solutions of the system (1) as divergence-free matrix fields satisfying a suitable algebraic constraint: in particular we realized that this point of view allowed to use well established techniques from the theory of differential inclusions, cf. [19, 9, 35, 72, 65].

5.1. **Differential inclusions.** In order to explain some of the ideas in [39], let us recall the concept of Reynolds stress. It is generally accepted that the appearance of high-frequency oscillations in the velocity field is the main reason responsible for turbulent phenomena in incompressible flows. One related major problem is therefore to understand the dynamics of the coarse-grained, in other words macroscopically averaged, velocity field. If $\overline{v}$ denotes the macroscopically averaged velocity field, then it satisfies

$$\begin{cases} \partial_t \overline{v} + \operatorname{div}(\overline{v} \otimes \overline{v} + R) + \nabla \overline{p} = 0 \\ \\ \operatorname{div} \overline{v} = 0, \end{cases} \tag{26}$$

where

$$R = \overline{v \otimes v} - \overline{v} \otimes \overline{v}\,.$$

The latter quantity is called Reynolds stress and arises because the averaging does not commute with the nonlinearity $v \otimes v$. The careful reader will have noticed the analogy with the commutator $-T_\varepsilon$ considered in the previous section. On this formal level the precise definition of averaging plays no role, be it long-time averages, ensemble-averages or local space-time or space averages: the commutators T_ε in (20) is thus the Reynolds stress for the macroscopic average v_ε.

A slightly more general version of this type of averaging follows the framework introduced by Tartar [92, 93] and DiPerna [46] in the context of conservation laws. We start by separating the linear equations from the nonlinear constitutive relations. Accordingly, we write (26) as

$$\begin{cases} \partial_t \overline{v} + \operatorname{div} \overline{u} + \nabla \overline{q} = 0 \\ \\ \operatorname{div} \overline{v} = 0, \end{cases}$$

where $\overline{u}$ is the traceless part of $\overline{v} \otimes \overline{v} + R$. Since one can write

$$R = \overline{(v - \overline{v}) \otimes (v - \overline{v})},$$

it is clear that $R \geq 0$, i.e. R is a symmetric positive semidefinite matrix. In terms of the coarse-grained variables $(\overline{v}, \overline{u})$ this inequality can be written as

$$\overline{v} \otimes \overline{v} - \overline{u} \leq \tfrac{2}{n} \overline{e}\, I,$$

where I is the $n \times n$ identity matrix and

$$\overline{e} = \overline{\tfrac{1}{2}|v|^2}$$

is the macroscopic kinetic energy density. Motivated by these calculations, we define subsolutions as follows (here and in the rest of the note we use the notation $\mathcal{S}^{n\times n}$ for the vector space of symmetric $n \times n$ matrices and we denote by $\mathcal{S}^{n\times n}_0$ the subspace of symmetric matrices with zero trace).

Definition 5.2 (Subsolutions). *Let $\overline{e} \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ with $\overline{e} \geq 0$. A subsolution to the incompressible Euler equations with given kinetic energy density $\overline{e}$ is a triple*

$$(v, u, q) : \mathbb{R}^n \times (0,T) \to \mathbb{R}^n \times \mathcal{S}^{n\times n}_0 \times \mathbb{R}$$

with the following properties:

- $v \in L^2_{loc}$, $\quad u \in L^1_{loc}$, $\quad$ *q is a distribution;*

- $$\left\{\begin{array}{l}\partial_t v + \operatorname{div} u + \nabla q = 0\\[2ex] \operatorname{div} v = 0,\end{array}\right. \qquad \textit{in the sense of distributions;} \tag{27}$$
- $$v \otimes v - u \leq \tfrac{2}{n}\bar{e}\, I \qquad \textit{a.e.}\ . \tag{28}$$

Observe that subsolutions automatically satisfy $\frac{1}{2}|v|^2 \leq \bar{e}$ a.e. (the inequality follows from taking the trace in (28)). If in addition we have the equality sign $\frac{1}{2}|v|^2 = \bar{e}$ a.e., then the v component of the subsolution is in fact a weak solution of the Euler equations. As mentioned above, in passing to weak limits (or when considering any other averaging process), the high-frequency oscillations in the velocity are responsible for the appearance of a non-trivial Reynolds stress. Equivalently stated, this phenomenon is responsible for the inequality sign in (28).

5.2. **Iteration.** The key point in our approach to prove Theorem 5.1 is that, starting from a subsolution, an appropriate iteration process reintroduces the high-frequency oscillations. In the limit of this process one obtains weak solutions. However, since the oscillations are reintroduced in a very non-unique way, in fact this generates *many* solutions from the same subsolution. In the next theorem we give a precise formulation of the previous discussion.

Theorem 5.3 (Subsolution criterion)**.** *Let $\bar{e} \in C(\mathbb{R}^n \times (0,T))$ and $(\bar{v}, \bar{u}, \bar{q})$ be a smooth, strict subsolution, i.e.*

$$(\bar{v}, \bar{u}, \bar{q}) \in C^\infty(\mathbb{R}^n \times (0,T)) \ \textit{satisfies}\ (27) \tag{29}$$

and

$$\bar{v} \otimes \bar{v} - \bar{u} < \tfrac{2}{n}\bar{e} \qquad \textit{on}\ \mathbb{R}^n \times (0,T). \tag{30}$$

Then there exist infinitely many weak solutions $v \in L^\infty_{loc}(\mathbb{R}^n \times (0,T))$ of the Euler equations such that

$$\begin{array}{rcl}\frac{1}{2}|v|^2 & = & \bar{e},\\ p & = & \bar{q} - \frac{2}{n}\bar{e}\end{array}$$

almost everywhere. Infinitely many among these belong to $C((0,T), L^2)$. If in addition

$$\bar{v}(\cdot, t) \rightharpoonup v_0(\cdot) \ \textit{in}\ L^2_{loc}(\mathbb{R}^n) \ \textit{as}\ t \to 0, \tag{31}$$

then all the v's so constructed take the initial data v_0 at time 0.

Following the references [39, 40], the point of view above has been taken by several other authors in a variety of situations, see for instance [31, 83, 86, 44, 87, 88, 2, 21, 23, 62, 61, 47, 10, 22, 33, 69, 24, 34, 63, 70, 50].

6. Differential inclusions and the Nash-Kuiper Theorem

As pointed out in the important paper [72] by Müller and Šverak, the results in the theory of differential inclusions which construct many "unusual solutions" (see, for instance, [19, 9, 35, 64, 65]) have a close relation to Gromov's h-principle in geometry. In particular the method of convex integration, introduced by Gromov and extended by Müller and Šverak to Lipschitz mappings, provides a very powerful tool to construct such examples. Essentially in the paper [39] these tools were suitably modified and used for the first time to explain Scheffer's theorem and go way beyond it.

The origin of Gromov's convex integration lies in the famous Nash-Kuiper theorem on isometric embeddings of Riemannian manifold. Let Σ be a smooth compact manifold of dimension $n \geq 2$, equipped with a Riemannian metric g. A map $u : \Sigma \to \mathbb{R}^N$ is *isometric* if it preserves the length of curves, i.e. if

$$\ell_g(\gamma) = \ell_e(u \circ \gamma) \qquad \text{for any } C^1 \text{ curve } \gamma \subset \Sigma, \tag{32}$$

where $\ell_g(\gamma)$ denotes the length of γ with respect to the metric g:

$$\ell_g(\gamma) = \int \sqrt{g(\gamma(t))[\dot\gamma(t), \dot\gamma(t)]}\, dt\,. \tag{33}$$

If $u \in C^1(M^n; \mathbb{R}^N)$ this means that the pull back of the Euclidean metric $u^\sharp e$ agrees with g. In local coordinates this amounts to the system

$$\partial_i u \cdot \partial_j u = g_{ij} \tag{34}$$

consisting of $s_n = \frac{n}{2}(n+1)$ equations in m unknowns. If in addition u is injective, it is an isometric embedding.

The existence of isometric immersions (resp. embeddings) of Riemannian manifolds into some Euclidean space is a classical problem, explicitly formulated for the first time by Schläfli, see [80]: in the latter Schläfli conjectured that the system is solvable *locally* if the dimension N of the target is at least s_n. In the first half of the twentieth century Janet [60], Cartan [18] and Burstin [17] proved Schläfli's conjecture for analytic metrics.

For the very particular case of 2-dimensional spheres endowed with metrics of positive Gauss curvature, Weyl in [95] raised the question of the existence of (global!) isometric embeddings in $\mathbb{R}^3$. The Weyl's problem was solved by Lewy in [68] for analytic metrics and Nirenberg settled the case of smooth metrics in his PhD thesis in 1949; a different proof was given independently by Pogorelov [76] around the same time, building upon the work of Alexandrov [1] (see also [77]).

An important aspect of the Weyl's problem is the rigidity of the solutions found by Lewy, Nirenberg and Pogorelov. Indeed, already before the work of Lewy, Cohn–Vossen and Herglotz proved independently that C^2 isometric immersions of positively curved spheres are uniquely determined up to rigid motions, cf. [27, 53] and see also [84] for a thorough discussion.

6.1. **Nash's surprising discovery.** Before the appearance of Nash's celebrated works, it was natural to expect that the assumption of C^2 regularity in the works of Cohn–Vossen and Herglotz was just of technical nature. But in his 1954 note [73] Nash astonished the geometry world and proved that the only true obstructions to the existence of isometric immersions are topological and that as soon as $N \geq n+1$ and there are no such obstructions, then there are in fact plenty of such immersions. Nash's Theorem was therefore in stark contrast with the intuition that codimension 1 smooth isometric immersions are rather rigid for $n = 2$ and that for $n > 2$, given that the system (34) is heavily overdetermined, existence of solutions should occur rarely.

In order to state Nash's Theorem we need some terminology.

Definition 6.1. *Let (Σ, g) be a Riemannian manifold. An immersion $v : \Sigma \to \mathbb{R}^N$ is short if it "shrinks" the length of curves. For C^1 immersions and in local coordinates such condition is equivalent to the inequality*

$$(\partial_i v \cdot \partial_j v) w^i w^j \leq g_{ij} w^i w^j \qquad \text{for any tangent vector } w. \tag{35}$$

Theorem 6.2. *Let (Σ, g) be a smooth closed n-dimensional Riemannian manifold and $v : \Sigma \to \mathbb{R}^N$ a C^∞ short immersion with $N \geq n+1$. Then, for any $\varepsilon > 0$ there exists a C^1 isometric immersion $u : \Sigma \to \mathbb{R}^N$ such that $\|u - v\|_{C^0} \leq \varepsilon$. If v is, in addition, an embedding, then u can be assumed to be an embedding as well.*

Indeed Nash gave a proof of Theorem 6.2 for $N \geq n+2$ and just remarked that it could be proved for $N \geq n+1$ with some additional work; the details were then given in two subsequent notes by Kuiper, [67]. For this reason Theorem 6.2 is called nowadays the Nash–Kuiper Theorem on C^1 isometric embeddings.

6.2. $C^{1,\alpha}$ **isometric embeddings.** In the specific case of the Weyl problem, where $n = 2$ and $N = 3$, the classical rigidity results of Herglotz and Cohn-Vossen imply that the concluson of Theorem 6.2 is necessarily false for C^2 isometries v. An interesting question, which shares a striking formal analogy with the Onsager's conjecture, is to understand if and where there is a sharp border on the Hölder scale $C^{1,\theta}$, $\theta \in (0,1)$ between the dramatically different behavior of solutions of the Weyl problem for low versus high θ.

In a series of papers in the 1950s, cf. [3, 4, 5, 6], Yu. Borisov showed that the rigidity of the Weyl problem can in fact be extended to $C^{1,\theta}$ immersions provided θ is sufficiently large.

Theorem 6.3. *Let $(\mathbb{S}^2, g)$ be a surface with C^2 metric and positive Gauss curvature, and let $u \in C^{1,\theta}(\mathbb{S}^2; \mathbb{R}^3)$ be an isometric immersion with $\theta > 2/3$. Then $u(\mathbb{S}^2)$ is the boundary of an open convex set.*

Borisov's Theorem is more general, but the statement above avoids the introduction of Pogorelov's concept of bounded extrinsic curvature, cf. [30]:

Borisov proves such property without any assumption on the topology of the surface and then exploits the work of Pogorelov, [77], to conclude the local convexity of the image. A much shorter proof of Borisov's Theorem has been discovered in [30] expoliting the same key computation of Constantin-E-Titi's proof of part (a) of Onsager's conjecture: another remarkable analogy between these seemingly unrelated areas!

On the other hand for sufficiently small Hölder exponents the Nash-Kuiper construction remains valid:

Theorem 6.4. *Let (Σ, g) be a C^2 Riemannian manifold of dimension n. Any short immersion $u : \Sigma \to \mathbb{R}^{n+1}$ can be uniformly approximated with $C^{1,\theta}$ isometric immersions with*

(a) $\theta < \frac{1}{1+n(n+1)}$ *when Σ is a closed ball;*
(b) $\theta < \frac{1}{1+n(n+1)^2}$ *when Σ is a general compact n-manifold.*

The maps can be chosen to be embeddings if u is an embedding.

Case (a) of this theorem was announced in [7] by Yu. Borisov, based on his habilitation thesis, under the additional assumption that g be analytic. A proof with $n = 2$ appeared more than 40 years later, cf. [8]. The general statement of Theorem 6.4 has been proved in [30].

Observe that in the first interesting case of 2-dimensional disks we have $\frac{1}{7}$: there is thus a significant gap between this and the "rigidity threshold" $\frac{2}{3}$ in Theorem 6.3. It is of course very tempting to ask whether there is a single sharp interface distinguishing between the two behaviors. Gromov [51] mentions $\frac{1}{2}$ (cf. Question 36 therein) as a possible threshold. In the case of 2-dimensional disks the recent paper [38] gave the first improvement of Borisov's local exponent, namely we have the following

Theorem 6.5. *Let $\overline{D} \subset \mathbb{R}^2$ be a closed disk and g a C^2 metric on it. Then any short immersion $u : \overline{D} \to \mathbb{R}^3$ can be uniformly approximated with $C^{1,\theta}$ isometric immersions if $\theta < \frac{1}{5}$. The maps can be chosen to be embeddings if u is an embedding.*

7. C^0 dissipative solutions of Euler

There is a clear formal analogy between (34)-(35) and (1)-(26). First of all, note that the Reynolds stress measures the defect to being a solution of the Euler equations and it is in general a nonnegative symmetric tensor, whereas $g_{ij} - \partial_i u \cdot \partial_j u$ measures the defect to being isometric and, for a short map, is also a nonnegative symmetric tensor. More precisely (34) can be formulated for the deformation gradient $A := Du$ as the coupling of the linear constraint

$$\operatorname{curl} A = 0$$

with the nonlinear relation

$$A^t A = g.$$

In this sense short maps are "subsolutions" to the isometric embedding problem in the spirit of Definition 5.2. Along this line of thought, Theorem 5.3 is then an analogue for the Euler equations of the Nash-Kuiper Theorem 6.2. However note that, strictly speaking, the "best analog" of the Nash-Kuiper theorem would hold if we could replace the L^∞ regularity of the solutions of Theorem 5.3 with the continuity of the solutions $A = Du$ of Theorem 6.2.

The intuition above drove László Székelyhidi and myself to propose a suitable approach à la Nash as a line of attack for part (b) of Conjecture 1.1. The feasibility of such strategy was confirmed at first by the following result, which we proved in [41], using a suitable "convex integration scheme".

Theorem 7.1. *Given any positive smooth function E on $[0,T]$ there is a pair $(v,p) : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3 \times \mathbb{R}$ of continuous functions which solves* (1) *in the distributional sense and satisfies $\frac{1}{2}\int_{\mathbb{T}^3} |v|^2(x,t)\, dx = E(t)$.*

The construction of continuous solutions of (1) follows, loosely speaking, the same philosophy of Nash's proof and of the proof of Theorem 5.3, in the sense that at each step of the iteration we add a highly oscillatory correction to our subsolution and improve its defect to being a solution.

Indeed we construct a sequence of subsolutions (v_q, p_q, R_q), i.e. solutions of

$$\begin{cases} \partial_t v_q + \operatorname{div} v_q \otimes v_q + \nabla p_q = -\operatorname{div} R_q \\ \\ \operatorname{div} v_q = 0 \end{cases} \tag{36}$$

and iteratively remove the error R_q, which is a symmetric 3×3 matrix field. As a first observation note that if one is only interested in measuring the "distance" of a smooth pair (v_q, p_q) from being a solution of (1), then only the traceless part of R_q is relevant: we can write

$$R_q = \rho_q \mathrm{Id} + \mathring{R}_q,$$

where $\mathring{R}_q$ is a traceless 3×3 symmetric matrix, since $\operatorname{div}(\rho_q \mathrm{Id}) = \nabla \rho_q$. Hence if $\mathring{R}_q = 0$ then v_q is a solution of the Euler equations (perhaps with a different pressure).

Recall that we also aim in Theorem 7.1 at satisfying a certain energy profile for the total kinetic energy. We choose therefore a sequence $E_q = E_q(t)$ with $E_q(t) \to E(t)$ and set

$$\rho_q(t) := \frac{1}{3(2\pi)^3}\left(E_{q+1}(t) - \frac{1}{2}\int_{\mathbb{T}^3} |v_q(x,t)|^2\, dx\right),$$

$$R_q(x,t) := \rho_q(t)\mathrm{Id} + \mathring{R}_q(x,t)\,.$$

We now explain the key points of the iteration and also which kind of Hölder regularity one could expect for the final solution.

Our aim is to build a sequence of triples $(v_q, p_q, \mathring{R}_q)$ solving (36) which converge uniformly to a triple $(v, p, 0)$ (actually in what follows we will mostly focus on the velocity v). The sequence will be achieved iteratively by adding a suitable perturbation to v_q and p_q. We thus set

$$w_q = v_q - v_{q-1}.$$

The size of w_q will be controlled with two parameters. The *amplitude* δ_q bounds the C^0 norm:

$$\|w_q\|_0 \lesssim \delta_q^{1/2}. \tag{37}$$

Up to negligible errors the Fourier transform of the perturbation w_q will be localized in a shell centered around a given *frequency* λ_q. Hence

$$\|\nabla w_q\|_0 \lesssim \delta_q^{1/2}\lambda_q. \tag{38}$$

Along the iteration we will have $\delta_q \to 0$ and $\lambda_q \to \infty$ at a rate that is at least exponential. For the sake of definiteness we may think

$$\lambda_q := \lambda^q \quad \text{ and } \delta_q := \lambda_q^{-2\theta_0} \tag{39}$$

for some $\lambda > 1$ (although in the actual proofs a slightly super-exponential growth is required). The positive number θ_0 is the threshold Hölder regularity which we are able to achieve through the iteration, since it can be easily shown by interpolation that $\|v_q - v_{q-1}\|_\alpha = \|w_q\|_\alpha \lesssim \delta_q^{1/2}\lambda_q^\alpha \lesssim \lambda_q^{\alpha-\theta_0}$ and thus $\{v_q\}_q$ is a Cauchy sequence in C^α whenever $\alpha < \theta_0$.

The perturbation w_{q+1} is added to "balance" the error R_q and indeed we will see that $R_q \sim w_{q+1} \otimes w_{q+1}$. For this reason we will have

$$\|\mathring{R}_q\|_0 \leq c_0\delta_{q+1} \tag{40}$$

$$\|\nabla\mathring{R}_q\|_0 \lesssim \delta_{q+1}\lambda_q \tag{41}$$

The main part of the perturbation w_{q+1} satisfies (ideally, as we will see later) an Ansatz of the type

$$w_o(x,t) = W\Big(v_q(x,t), R_q(x,t), \lambda_{q+1}x, \lambda_{q+1}t\Big), \tag{42}$$

where W is a function which we are going to specify next. The pressure p_{q+1} will be defined similarly as $p_{q+1} = p_q + P(v_q, R_q, \lambda_{q+1}x, \lambda_{q+1}t)$, but we will not enter into the details in our discussion, since its role is anyway secondary.

First of all, the oscillatory nature of the perturbation requires us to impose that W is periodic in the variable $\xi \in \mathbb{T}^3$. Next, observe that v_{q+1} must satisfy the divergence-free condition $\operatorname{div} v_{q+1} = 0$ and $v + w_o$ is not likely to fulfill this: we need to add a suitable correction w_c in order to satisfy it. Consider therefore a vector potential for v_q, namely write v_q as $\nabla \times z_q$ for some smooth z_q. Subsequently we would like to perturb z_q to a new

$$z_{q+1}(x,t) = z_q(x,t) + \frac{1}{\lambda_{q+1}}Z(v(x,t), R(x,t), \lambda_{q+1}x, \lambda_{q+1}t).$$

Computing $v_{q+1} := \nabla \times z_{q+1}$ we get

$$v_{q+1}(x,t) = v_q(x,t) + \underbrace{(\nabla_\xi \times Z)(v(x,t), \tilde{R}(x,t), \lambda x, \lambda t)}_{(P)} + O\left(\frac{1}{\lambda}\right).$$

The term (P) would correspond to w_o if we were able to find a vector potential Z for W which is *periodic in* ξ. This requires $\operatorname{div}_\xi W = 0$ and $\langle W \rangle = 0$, where we use the notation $\langle , \rangle$ to denote the average in the ξ variable.

Similar considerations (see for instance [85]) lead to the following set of conditions that we would like to impose on W:

- $\xi \mapsto W(v, R, \xi, \tau)$ is 2π-periodic with vanishing average, i.e.

$$\langle W \rangle := \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} W(v, R, \xi, \tau)\, d\xi = 0; \tag{H1}$$

- The average stress is given by R, i.e.

$$\langle W \otimes W \rangle = R\,; \tag{H2}$$

- The "cell problem" is satisfied:

$$\begin{cases} \partial_\tau W + v \cdot \nabla_\xi W + \operatorname{div}_\xi (W \otimes W) + \nabla_\xi P = 0 \\ \\ \operatorname{div}_\xi W = 0\,, \end{cases} \tag{H3}$$

  where $P = P(v, R, \xi, \tau)$ is a suitable pressure;
- W is smooth in all its variables and satisfies the estimates

$$|W| \lesssim |R|^{1/2},\ |\partial_v W| \lesssim |R|^{1/2},\ |\partial_R W| \lesssim |R|^{-1/2}. \tag{H4}$$

As a consequence of (H1)-(H2) we obtain

$$\int_{\mathbb{T}^3} |v_{q+1}|^2\, dx \sim \int_{\mathbb{T}^3} |v_q|^2\, dx + \int_{\mathbb{T}^3} \langle |W|^2 \rangle\, dx = \int_{\mathbb{T}^3} |v_q|^2\, dx + 3(2\pi)^3 \rho_q(t)$$

and thus the total kinetic energy of the v_{q+1} is (up to small errors) $e_{q+1}(t)$.

Having defined the couple (v_{q+1}, p_{q+1}) we face the problem of finding a suitable stress tensor $\mathring{R}_{q+1}$. An important remark is that it is possible to select a good "elliptic operator" which solves the equations $\operatorname{div} \mathring{R} = f$. The relevant technical lemma is the following one.

Lemma 7.2 (The operator div^{-1})**.** *There exists a homogeneous Fourier-multiplier operator of order -1, denoted*

$$\operatorname{div}^{-1} : C^\infty(\mathbb{T}^3; \mathbb{R}^3) \to C^\infty(\mathbb{T}^3; \mathcal{S}_0^{3\times 3})$$

such that, for any $f \in C^\infty(\mathbb{T}^3; \mathbb{R}^3)$ with average $\fint_{\mathbb{T}^3} f = 0$ we have

(a) $\operatorname{div}^{-1} f(x)$ *is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$;*
(b) $\operatorname{div} \operatorname{div}^{-1} f = f$.

Assuming the existence of an ideal profile W, the next stress tensor $\mathring{R}_{q+1}$ would then be defined through

$$\begin{aligned}\mathring{R}_{q+1} &= - \operatorname{div}^{-1}\Big[\partial_t v_{q+1} + \operatorname{div}(v_{q+1}\otimes v_{q+1}) + \nabla p_{q+1}\Big]\\ &= - \underbrace{\operatorname{div}^{-1}\Big[\partial_t w_{q+1} + v_q\cdot\nabla w_{q+1}\Big]}_{=:\mathring{R}^{(1)}_{q+1}}\\ &\quad - \underbrace{\operatorname{div}^{-1}\Big[\operatorname{div}(w_{q+1}\otimes w_{q+1} - R_q) + \nabla(p_{q+1}-p_q)\Big]}_{=:\mathring{R}^{(2)}_{q+1}}\\ &\quad - \underbrace{\operatorname{div}^{-1}\Big[w_{q+1}\cdot\nabla v_q\Big]}_{=:\mathring{R}^{(3)}_{q+1}} \qquad (43)\end{aligned}$$

where div^{-1} is the operator of order -1 from Lemma 7.2. Since we are assuming that the size of the corrector w_c is negligible compared to w_o, we will discuss the corresponding terms where w_o replaces w_q.

The main issues are therefore

- to show that indeed it is possible to send δ_q to 0 as $q\uparrow\infty$ (so that the scheme converges)
- and to obtain a relation between δ_q and λ_q in the form of (39).

If we were able to find a "profile" W satisfying (H1)-(H2)-(H3)-(H4), then the iteration proposed so far would lead to a proof of the Onsager's conjecture. In order to see this first expand $W(v,R,\xi,\tau)$ as a Fourier series in ξ. We then could compute

$$\mathring{R}^{(3)} = \operatorname{div}^{-1}\Big[w_o\cdot\nabla v_q\Big] = \operatorname{div}^{-1}\sum_{k\in\mathbb{Z}^3, k\neq 0} c_k(x,t)e^{i\lambda_{q+1}k\cdot x}\,, \qquad (44)$$

where the coefficients $c_k(x,t)$ vary much slower than the rapidly oscillating exponentials. When we apply the operator div^{-1} we can therefore treat the c_k as constants and gain a factor $\frac{1}{\lambda_{q+1}}$ in the outcome: a typically "stationary phase argument". Note that it is crucial that c_0 vanishes: this is in fact the content of condition (H1).

Using (H4) we can estimate the size of each term c_k as

$$\|c_k\|_0 \lesssim \|W\|_0\|\nabla v_q\|_0 \lesssim \|R_q\|_0^{1/2}\|\nabla v_q\|_0.$$

Applying (38) and (40) we arrive at

$$\|\mathring{R}^{(3)}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}\,. \qquad (45)$$

In fact in our computations so far we are ignoring a lot of technical issues: the relevant estimates are much more complicated and affected by several other terms which we are neglecting.

Similar arguments for the two other error tensors $\mathring{R}^{(1)}_{q+1}$ and $\mathring{R}^{(2)}_{q+1}$ would lead to an estimate of type

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}. \tag{46}$$

Of course, this is just one of the estimates for $(v_{q+1}, p_{q+1}, R_{q+1})$ and similar ones should be obtained for all the other quantities (and for other norms). However, (46) already implies a relation between δ_q and λ_q. Indeed, comparing it with (40), the inductive step requires

$$\delta_{q+2} \sim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}.$$

Assuming $\lambda_q \sim \lambda^q$ for some fixed $\lambda \gg 1$, this would lead to

$$\delta_q^{1/2} \sim \lambda^{-q/3} \sim \lambda_q^{-1/3}, \tag{47}$$

which gives $\theta_0 = 1/3$ as the critical Hölder regularity.

8. First Höder regularity

It tuns out that almost all conditions on the function $W = W(v, R, \xi, \tau)$ can be fulfilled, as shown in [41]. Let us first examine the simple case in which we set $v = 0$: it is then possible to construct a function $W_s(R, \xi) = W(0, R, \xi, \tau)$ satisfying the constraints (H1)-(H4). The basic building block is given by Beltrami flows. For the details we refer the reader to [41], but one important aspect of Beltrami flows is that one can construct several different W_s with the property that any linear combinations of them still satisfy (H1)-(H3)-(H4) (and a suitable version of (H2)). In fact W_s takes the form

$$W_s(R, \xi) = \sum_{k\in\Lambda} a_k(R) B_k e^{ik\cdot\xi}$$

where Λ is a subset of $\mathbb{Z}^3$ with the property that $|k|$ is a fixed constant for every $k \in \Lambda$ and B_k is related to k by a precise algebraic formula. Note in particular that two distinct profiles W_s^1 and W_s^2 whose linear combination is still a profile can be obtained by choosing two disjoint faimilies Λ's in the same sphere intersected with the lattice $\mathbb{Z}^3$.

Another aspect which is important about Beltrami flows is that the corresponding stationary profiles W_s are only defined for R in a suitably small cone $\mathcal{C}$ of tensors R, whose axis is the half-line $\{\lambda \mathrm{Id} : \lambda \in \mathbb{R}^+\}$.

Having obtained a profile $W(0, R, \xi, \tau) = W_s(R, \xi)$, it seems natural to extend W by imposing that $\partial_\tau W + v \cdot \nabla_\xi W = 0$, leading to the formula

$$W(v, R, \xi, \tau) = W_s(R, \xi - v\tau) = \sum_{k\in\Lambda} a_k(R) B_k e^{i(k-v\tau)\cdot\xi}\,. \tag{48}$$

However the latter fails to satisfy (H4), because $|\partial_v W(v, R, \xi, \tau)| \sim |R|^{1/2}|\tau|$. This is a serious problem: observing that τ is the "fast time" variable, in the construction (42) $\tau = \lambda_{q+1} t$, leading to an additional factor λ_{q+1} in the estimates for $\mathring{R}^{(1)}_{q+1}$ and $\mathring{R}^{(2)}_{q+1}$: this loss destroys any hope that the scheme might converge.

In [41] a "phase function" $\phi_k(v,\tau)$ was introduced to deal with the transport part of the cell problem. By considering W of the form

$$\sum_{|k|=\lambda_0} a_k(R)\phi_k(v,\tau)B_k e^{ik\cdot\xi} \tag{49}$$

the cell problem in (H3) leads to the equation

$$\partial_\tau \phi_k + i(v\cdot k)\phi_k = 0\,.$$

Since the exact solution $\phi_k(v,\tau) = e^{-i(v\cdot k)\tau}$ is incompatible with the requirement (H4), an approximation is used such that

$$\partial_\tau \phi_k + i(v\cdot k)\phi_k = O\left(\mu_q^{-1}\right), \qquad |\partial_v \phi_k| \lesssim \mu_q$$

for some new parameter μ_q. To be precise, the approximation involves a partition of unity over the space of velocities and the use of 8 distinct families $\Lambda^{(j)}$.

This leads to the following corrections to (H3) and (H4): (H3) is only satisfied approximately,

$$\partial_\tau W + v\cdot\nabla_\xi W + \mathrm{div}_\xi (W\otimes W) + \nabla_\xi P = O(\mu_q^{-1})$$

and in (H4) the second inequality is replaced by

$$|\partial_v W| \lesssim \mu_q |R|^{1/2}.$$

In [42] the approach above was subsequently used to show the first example of Hölder flows with prescribed energy profiles, more precisely:

Theorem 8.1. *Given any positive smooth function E on $[0,T]$ and any $\alpha < \frac{1}{10}$ there is a pair $(v,p) : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3 \times \mathbb{R}$ of C^α functions which solves* (1) *in the distributional sense and satisfies $\frac{1}{2}\int_{\mathbb{T}^3} |v|^2(x,t)\,dx = E(t)$.*

After the works [41] and [42] the same point of view has been taken in several different situations, cf. [26, 36, 25, 59, 58, 43, 89, 90, 91, 28, 45, 15, 16, 56, 57, 71]

9. $\frac{1}{5}$-Hölder regularity

A further improvement was obtained in [12], following an idea first introduced by Isett in [54]. We change the Ansatz (49) on W and look for a perturbation w_o which has the form

$$w_o(x,t) = W_s(R_q(x,t), \lambda_{q+1}\Phi_q(x,t)) = \sum_{k\in\Lambda^{(1)}} a_k(R_q(x,t))B_k e^{i\lambda_{q+1}\Phi_q(x,t)}\,, \tag{50}$$

where Φ_q solves the transport equation

$$\partial_t \Phi_q + (v_q\cdot\nabla_x)\Phi_q = 0\,. \tag{51}$$

With (50), we would have

$$\mathring{R}^{(1)}_{q+1} = \sum_{k\in\Lambda^{(1)}} \nabla a_k(R_q)(\partial_t R_q + (v_q\cdot\nabla)R_q)e^{i\lambda_{q+1}\Phi_q}\,. \tag{52}$$

Assuming that $D\Phi_q(x,t)$ is not too far from the identity, the stationary phase argument leads to

$$\|\mathring{R}^{(1)}\|_0 \lesssim \delta_{q+1}^{3/2}\delta_q^{1/2}\lambda_q\lambda_{q+1}^{-1}\,. \tag{53}$$

In fact in the latter estimates we are also assuming that the advective derivative $\partial_t R_q + (v_q\cdot\nabla)R_q$ satisfies a better bound than the usual derivative DR_q. This is indeed correct, as first pointed out by Isett in [54], and intuitively it can be justified by observing that even the advective derivative $(\partial_t v_q + v_q\cdot\nabla)v_q$ satisfies a better bound than Dv_q.

However, since $\|Dv_q\|_0 \to \infty$, we expect the deformation matrix $D\Phi_q$ to be controllable only for short times. More precisely, by a well-known elementary estimate on ODEs, if $\Phi_q(x,t_0)=x$, then

$$\|D\Phi_q(\cdot,t)-\mathrm{Id}\|_0 \lesssim \|\nabla v_q\|_0|t-t_0| \lesssim \delta_q^{1/2}\lambda_q|t-t_0| \tag{54}$$

for $|t-t_0|\lesssim(\delta_q^{1/2}\lambda_q)^{-1}$. The latter is a typical "CFL condition", cf. [32].

To handle this problem we proceed as in [12] and consider a partition of unity $(\chi_j)_j$ on the time interval $[0,T]$ such that the support of each χ_j is an interval I_j of size $\frac{1}{\mu_q}$ for some $\mu_q \gg 1$. In each time interval I_j we set $\Phi_{q,j}$ to be the solution of the transport equation (51) which satisfies

$$\Phi_{q,j}(x,t_j)=x,$$

where t_j is the center of the interval I_j. Recalling that $\|Dv_q\|_0\lesssim\delta_q^{1/2}\lambda_q$, (54) leads to

$$\|D\Phi_{q,j}\|_0 = O(1) \quad\text{ and }\quad \|D\Phi_{q,j}-\mathrm{Id}\|_0 \lesssim \frac{\delta_q^{1/2}\lambda_q}{\mu_q} \tag{55}$$

provided

$$\mu_q \geq \delta_q^{1/2}\lambda_q, \tag{56}$$

an estimate we will henceforth assume. Observe also that $|\partial_t\chi_j|\lesssim\mu_q$.

The new fluctuation will take the form

$$w_o = \sum_j \chi_j(t)\sum_{k\in\Lambda^{(i(j))}} a_k(R_q)B_k e^{i\lambda_{q+1}k\cdot\Phi_{q,j}} \tag{57}$$

where:

- $i(j)$ equals 1 if j is odd and 2 if j is even;
- $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are two disjoint families.

The new Ansatz leads then to the following estimate

$$\|\mathring{R}_{q+1}\|_0 \lesssim \delta_{q+1}^{1/2}\mu_q\lambda_{q+1}^{-1} + \delta_{q+1}\delta_q^{1/2}\lambda_q\mu_q^{-1} \tag{58}$$

Optimizing in μ_q we then reach

$$\|\mathring{R}_{q+1}\|_0 \lesssim \delta_{q+1}^{3/4}\delta_q^{1/4}\lambda_q^{1/2}\lambda_{q+1}^{-1/2}\,, \tag{59}$$

namely

$$\delta_{q+2} \sim \delta_{q+1}^{3/4}\delta_q^{1/4}\lambda_q^{1/2}\lambda_{q+1}^{-1/2}\,.$$

The latter relation leads to a threshold $\theta_0 = \frac{1}{5}$ and hence to the following theorem

Theorem 9.1. *Given any positive smooth function e on $[0,T]$ and any $\alpha < \frac{1}{5}$ there is a pair $(v,p) : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3 \times \mathbb{R}$ of C^α functions which solves (1) in the distributional sense and satisfies $\frac{1}{2}\int_{\mathbb{T}^3} |v|^2(x,t)\,dx = e(t)$.*

10. First Onsager-critical construction

In [11] Buckmaster observed that, by choosing the cut-off functions χ_i appropriately in (57) it is possible to show that the solution produced in the proof of Theorem 9.1 enjoys $C^{1/3-\varepsilon}$ regularity at almost every time-slice. The idea is to make the cut-off flat on large portions of their supports while paying very steep time derivatives on small portions. The price to pay is that the "global" Hölder control gets much weaker: the solutions is just slightly better than continuous (i.e. it has a very small Hölder exponent, depending on ε). In [13], jointly with Buckmaster and Székelyhidi we exploited a quantitative version of the latter idea to reach the first nonconservative solutions up Onsager's threshold 1/3, albeit in a weaker form than as stated in his conjecture.

Theorem 10.1. *For every $\alpha < \frac{1}{3}$ there are a nontrivial continuous compactly supported solution $(v,p) : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}$ of (1) and an L^1 function $C : \mathbb{R} \to \mathbb{R}^+$ such that*

$$|v(x,t) - v(y,t)| \leq C(t)|x-y|^\alpha \qquad \forall t \in \mathbb{R}, \forall x,y \in \mathbb{T}^3\,.$$

11. h-principle

The Beltrami flows together with the transport Ansatz explained in the previous sections settle the issue of convergence (at least for Hölder exponents $\theta < 1/5$), but are not sufficient to conclude an h-principle statement which is a satisfactory counterpart of Theorem 6.2. The reason is that the stationary profiles W_s defined through Beltrami flows are only defined for R's belonging to a suitably small cone of tensors.

Nevertheless, there is a very simple set of stationary flows (which we will call "Mikado flows") based on pipe flow, which can generate all R. These flows were introduced by Daneri and Székelyhidi in [37].

Lemma 11.1. *For any compact subset $\mathcal{N}$ consisting of positive definite 3×3 matrices there exists a smooth vector field*

$$W_s : \mathcal{N} \times \mathbb{T}^3 \to \mathbb{R}^3, \quad i = 1,2$$

such that, for every $R \in \mathcal{N}$

$$\left\{\begin{array}{l}\operatorname{div}_\xi (W_s(R,\xi) \otimes W(R,\xi)) = 0, \\ \\ \operatorname{div}_\xi W_s(R,\xi) = 0,\end{array}\right. \tag{60}$$

and

$$\langle W_s \rangle = 0, \tag{61}$$

$$\langle W_s \otimes W_s \rangle = R. \tag{62}$$

In particular, in [37] the authors could prove the following h-principle result

Theorem 11.2. *Let* $(\bar v, \bar p, \bar R)$ *be a smooth solution of*

$$\left\{\begin{array}{l}\partial_t \bar v + \operatorname{div}(\bar v \otimes \bar v) + \nabla \bar p = -\operatorname{div} \bar R \\ \\ \operatorname{div} \bar v = 0\end{array}\right. \tag{63}$$

on $\mathbb{T}^3 \times [0,T]$ *such that* $\bar R(x,t)$ *is positive definite for all* x, t*. Then for any* $\alpha < 1/5$ *there exists a sequence* $\{(v_k, p_k)\} \subset C^\alpha$ *of weak solutions of* (1) *such that*

$$v_k \overset{*}{\rightharpoonup} \bar v \quad \text{and} \quad v_k \otimes v_k \overset{*}{\rightharpoonup} \bar v \otimes \bar v + \bar R \quad \text{in } L^\infty$$

uniformly in time and furthermore for all $t \in [0,T]$

$$\int_{\mathbb{T}^3} v_k \otimes v_k \, dx = \int_{\mathbb{T}^3} (\bar v \otimes \bar v + \bar R) \, dx.$$

12. Isett's proof of Onsager's conjecture

The stationary profile W_s reached through the Mikado flows in [37] have not only the feature of being defined on any compact subset of positive definite R's, but they also yield improved estimates in several error terms in the iteration. On the other hand they are not compatible with the "patching in time" used in (57) and indeed in [37] they are only used for finitely many steps of the iteration, whereas the "tail" of the series $\sum_q w_q$ still consists of oscillatory perturbations whose building blocks are Beltrami flows.

In [55] Isett has been able to overcome this last obstruction by introducing a different "patching strategy". Isett's key idea can be easily explained as follows. Considered a given triple $(v_q, p_q, \mathring{R}_q)$ reached at a certain step of the iteration, satisfying all the estimates outlined in the previous subsection. The obstruction to using Mikado flows could be overcome if $\mathring{R}_q$ were supported in a union of disjoint time-stripes of the form $\mathbb{T}^3 \times [a_k, b_k]$, where $b_k - a_k \sim (\delta_q^{1/2} \lambda_q)^{-1}$, compatibly with the CFL condition (54). In this case there would be no need of "patching" the oscillatory perturbations, since they would be supported on disjoint time-stripes where the CFL condition holds and the flows $\Phi_{q,j}$ of the previous subsections are close to the identity.

In order to reach this ideal situation, Isett in [55] partitions the whole time interval in smaller intervals $[c_k, c_{k+1}]$ with size $\sim (\delta_q^{1/2}\lambda_q)^{-1}$. In intervals of comparable size it is possible to find exact solutions (z_k, r_k) of the incompressible Euler equations with $z_k(\cdot, c_k) = v_q(\cdot, c_k)$. Patching such solutions with a partition of unity one can obtain new velocity and pressure fields $(\tilde v_q, \tilde p_q)$ together with "separate" time-stripes where they are exact solutions of the Euler equations. The remaining regions consist of time-stripes where we have to find a new stress tensor $\tilde R_q$. Isett shows that such tensor can be found so that its size is not much larger than $\mathring{R}_q$: in fact it essentially satisfies the same estimates with worse constants.

Now there are no obstructions to apply the oscillatory perturbations of [37] to the new triple $(\tilde v_q, \tilde p_q, \tilde R_q)$ and therefore one can reach the following statement

Theorem 12.1. *For every $\alpha < \frac{1}{3}$ there is a nontrivial continuous compactly supported solution $(v,p) \in C^\alpha(\mathbb{T}^3 \times \mathbb{R})$ of (1).*

13. h-principle and Onsager's conjecture with dissipative solutions

In the previous "patching" of exact solutions of the Euler equations a canonical choice of the stress tensor $\tilde R_q$ would be

$$\tilde R_q := \operatorname{div}^{-1}[\partial_t \tilde v_q + (\tilde v_q \cdot \nabla)\tilde v_q + \nabla \tilde p_q]\,. \tag{64}$$

However [55] generates $\tilde R_q$ with a different, more complicated, procedure, since the author is not able to reach the desired estimate through the operator div^{-1}. A suboptimal outcome is that Theorem 12.1 does not produce "dissipative solutions".

This has been instead accomplished in [14], where in a joint work with Buckmaster, Székelyhidi and Vicol we derive appropriate estimates for the "canonical" $\tilde R_q$ as defined in (64). We can therefore derive the existence of dissipative solutions in the whole range of Hölder exponents of the second part of Onsager's conjecture. Indeed such a statement is obtained as a corollary of the exact counterpart of the h-principle result in [37]:

Theorem 13.1. *Let $(\bar v, \bar p, \bar R)$ be a smooth solution of (63) on $\mathbb{T}^3 \times [0,T]$ such that $\bar R(x,t)$ is positive definite for all x,t. Then for any $\alpha < 1/3$ there exists a sequence $\{(v_k, p_k)\} \subset C^\alpha$ of weak solutions of (1) such that*

$$v_k \overset{*}{\rightharpoonup} \bar v \quad \text{and} \quad v_k \otimes v_k \overset{*}{\rightharpoonup} \bar v \otimes \bar v + \bar R \quad \text{in } L^\infty$$

uniformly in time and furthermore for all $t \in [0,T]$

$$\int_{\mathbb{T}^3} v_k \otimes v_k \, dx = \int_{\mathbb{T}^3} (\bar v \otimes \bar v + \bar R)\, dx.$$

Corollary 13.2. *For every $\alpha < \frac{1}{3}$ and every positive smooth $E : [0,T] \to \mathbb{R}$ there exists a solution $(v,p) \in C^{1/3}(\mathbb{T}^3 \times [0,T])$ of (1) such that*

$$\frac{1}{2}\int_{\mathbb{T}^3} |v(x,t)|^2\, dx = E(t)\,.$$

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Institut für Mathematik, Universität Zürich, CH-8057 Zürich
E-mail address: camillo.delellis@math.uzh.ch