Ill-posedness for Bounded Admissible Solutions of the 2-dimensional p-system

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ABSTRACT. Consider the p–system of isentropic gas dynamics in n space dimensions, where $n \geq 2$. In a recent joint work with László Székelyhidi we showed bounded initial data for which this system has infinitely many admissible solutions. Moreover, the solutions and the initial data are bounded away from the void. Our result builds on an earlier work, where we introduced a new tool to generate wild solutions to the Euler equations for incompressible fluids.

1. Introduction

The p–system of isentropic gas dynamics in Eulerian coordinates is perhaps the oldest hyperbolic system of conservation laws. The unknowns of the system, which consists of $n+1$ equations, are the density ρ and the velocity v of the gas:

$$
\left\{\begin{array}{l}
\partial_t \rho + \mathrm{div}_x(\rho v) \;=\; 0\\
\partial_t(\rho v) + \mathrm{div}_x(\rho v\otimes v) + \nabla[p(\rho)] \;=\; 0\\
\rho(0,\cdot) \;=\; \rho^0\\
v(0,\cdot) \;=\; v^0
\end{array}\right. \tag{1.1}
$$

(cf. (3.3.17) in [**2**] and Section 1.1 of [**9**] p7). The pressure p is a function of ρ, which is determined from the constitutive thermodynamic relations of the gas in question and satisfies the assumption $p' > 0$. A typical example is $p(\rho) = k\rho^\gamma$, with constants $k > 0$ and $\gamma > 1$, which gives the constitutive relation for a polytropic gas (cf. (3.3.19) and (3.3.20) of [**2**]). Weak solutions of (1.1) are bounded functions in $\mathbb{R}^n$, which solve it in the sense of distributions. Thus, weak solutions satisfy the following identities for every test function $\psi, \varphi \in C^\infty_c(\mathbb{R}^n \times [0,\infty[)$:

$$
\int_0^\infty \int_{\mathbb{R}^n} \Big[\rho\partial_t\psi + \rho v\cdot\nabla_x\psi\Big]\,dx\,dt \;+\; \int_{\mathbb{R}^n} \rho^0(x)\,\psi(x,0)\,dx = 0, \tag{1.2}
$$

$$
\int_0^\infty \int_{\mathbb{R}^n} \Big[\rho v\cdot\partial_t\varphi + \rho\langle v\otimes v, \nabla\varphi\rangle\Big]\,dx\,dt \;+\; \int_{\mathbb{R}^n} \rho^0(x)v^0(x)\cdot\varphi(x,0)\,dx \;=0. \tag{1.3}
$$

We also recall that weak solutions can be redefined an a set of measure zero so that the map $t\mapsto(\rho(\cdot,t),v(\cdot,t)) \in L^\infty(\mathbb{R}^n)$ is weakly* continuous.

Admissible solutions have to satisfy an additional inequality, coming from the conservation law for the energy of the system. More precisely, consider the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ given through the law $p(r) = r^2 \varepsilon'(r)$.

DEFINITION 1.1. A weak solution $v \in C([0, +\infty[, L^\infty_{w^*}(\mathbb{R}^n))$ of (1.1) is admissible if the following inequality holds for every nonnegative $\psi \in C^\infty_c(\mathbb{R}^n \times \mathbb{R})$:

$$\int_0^\infty \int_{\mathbb{R}^n} \left[\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}\right)\partial_t\psi + \left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right) v\cdot\nabla_x\psi\right]$$

$$+ \int_{\mathbb{R}^n} \left(\rho^0\varepsilon(\rho^0) + \frac{\rho^0|v^0|^2}{2}\right)\psi(\cdot, 0) \;\geq\; 0\,. \tag{1.4}$$

In the paper [**4**] we have given a proof of the following result.

THEOREM 1.2. *Let $n \geq 2$. Then, for any given function p, there exist bounded initial data (ρ^0, v^0) with $\rho^0 \geq c > 0$ for which there are infinitely many bounded admissible solutions (ρ, v) of (1.1) with $\rho \geq c > 0$.*

REMARK 1.3. In fact, all the solutions constructed in our proof of Theorem 1.2 satisfy the energy *equality*, that is, the equality sign holds in (1.4). They are therefore also entropy solutions of the full compressible Euler system (see for instance example (d) of Section 3.3 of [**2**]) and they show nonuniqueness in this case as well.

2. Ill–posedness for incompressible Euler

Theorem 1.2 draws on some ideas which we have recently introduced to understand some celebrated examples of wild solutions to the Euler equations of incompressible fluid dynamics. Consider indeed the Cauchy problem

$$\begin{cases} \partial_t v + \operatorname{div}(v\otimes v) + \nabla p \;=\; 0, \\ \operatorname{div} v \;=\; 0, \\ v(x,0) \;=\; v^0(x), \end{cases} \tag{2.1}$$

where the initial data v^0 satisfies the compatibility condition

$$\operatorname{div} v^0 \;=\; 0\,. \tag{2.2}$$

In his pioneering work [**8**] V. Scheffer showed that weak solutions to (2.1) are not unique. In particular Scheffer constructed a nontrivial weak solution which is compactly supported in space and time, thus disproving uniqueness for (2.1) even when $v^0 = 0$. A simpler construction was later proposed by A. Shnirelman in [**10**].

In a recent paper [**3**], we have shown how the general framework of convex integration [**1, 7, 5**] combined with Tartar's programme on oscillation phenomena in conservation laws [**12**] (see also [**6**] for an overview) can be applied to (2.1). In this way, one can easily recover Scheffer's and Shnirelman's counterexamples in all dimensions and with bounded velocity and pressure. Moreover, the construction yields as a simple corollary the existence of energy–decreasing solutions, thus recovering another groundbreaking result of Shnirelman [**11**], again with the additional features that our examples have bounded velocity and pressures and can be shown to exist in any dimension.

These results left open the question of whether one might achieve the uniqueness of weak solutions by imposing a form of the energy inequality. In the work [**4**] we answered this question in the negative for several known criteria. Though the motivation for (1.4) comes from the theory of shock waves, which are obviously absent in incompressible Euler, these admissibility criteria are formally very similar to that of Definition 1.1, Therefore, the ideas introduced in [**4**] can be successfully exported to admissible solutions of the p–system, yielding Theorem 1.2 as a corollary.

3. Plane wave analysis of Euler's equations

We start by briefly explaining Tartar's framework [**12**]. One considers nonlinear PDEs that can be expressed as a system of linear PDEs (conservation laws)

$$\sum_{i=1}^{m} A_i \partial_i z = 0 \tag{3.1}$$

coupled with a pointwise nonlinear constraint (constitutive relations)

$$z(x) \in K \subset \mathbb{R}^d \text{ a.e.}, \tag{3.2}$$

where $z : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ is the unknown state variable. The idea is then to consider *plane wave* solutions to (3.1), that is, solutions of the form

$$z(x) = a h(x \cdot \xi), \tag{3.3}$$

where $h : \mathbb{R} \to \mathbb{R}$. The *wave cone* Λ is given by the states $a \in \mathbb{R}^d$ such that for any choice of the profile h the function (3.3) solves (3.1), that is,

$$\Lambda := \left\{ a \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^m \setminus \{0\} \quad \text{with} \quad \sum_{i=1}^{m} \xi_i A_i a = 0 \right\}. \tag{3.4}$$

The oscillatory behavior of solutions to the nonlinear problem is then determined by the compatibility of the set K with the cone Λ.

The incompressible Euler equations can be naturally rewritten in this framework. The domain is $\mathbb{R}^m = \mathbb{R}^{n+1}$, and the state variable z is defined as $z = (v, u, q)$, where

$$q = p + \frac{1}{n}|v|^2, \text{ and } u = v \otimes v - \frac{1}{n}|v|^2 I_n,$$

so that u is a symmetric $n \times n$ matrix with vanishing trace and I_n denotes the $n \times n$ identity matrix. From now on the linear space of symmetric $n \times n$ matrices will be denoted by $\mathcal{S}^n$ and the subspace of trace–free symmetric matrices by $\mathcal{S}^n_0$. The following lemma is straightforward.

Lemma 3.1. *Suppose* $v \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}^n)$, $u \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathcal{S}^n_0)$, *and* $q \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ *solve*

$$\begin{aligned} \partial_t v + \mathit{div}\, u + \nabla q &= 0, \\ \mathit{div}\, v &= 0, \end{aligned} \tag{3.5}$$

in the sense of distributions. If in addition

$$u = v \otimes v - \frac{1}{n}|v|^2 I_n \qquad \textit{a.e. in } \mathbb{R}^n_x \times \mathbb{R}_t, \tag{3.6}$$

then v *and* $p := q - \frac{1}{n}|v|^2$ *are a solution to* (2.1) *with* $f \equiv 0$. *Conversely, if* v *and* p *solve* (2.1) *distributionally, then* v, $u := v \otimes v - \frac{1}{n}|v|^2 I_n$ *and* $q := p + \frac{1}{n}|v|^2$ *solve* (3.5) *and* (3.6).

Consider the $(n+1)\times(n+1)$ symmetric matrix in block form

$$U = \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix}, \tag{3.7}$$

where I_n is the $n \times n$ identity matrix. Notice that by introducing new coordinates $y = (x, t) \in \mathbb{R}^{n+1}$ the equation (3.5) becomes simply

$$\mathrm{div}_y U = 0.$$

Here, as usual, a divergence–free matrix field is a matrix of functions with rows that are divergence–free vectors. Therefore the wave cone corresponding to (3.5) is given by

$$\Lambda = \left\{ (v, u, q) \in \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R} : \det \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix} = 0 \right\}.$$

REMARK 3.2. A simple linear algebra computation shows that for every $v \in \mathbb{R}^n$ and $u \in \mathcal{S}^n_0$ there exists $q \in \mathbb{R}$ such that $(v, u, q) \in \Lambda$, revealing that the wave cone is very large. Indeed, let $V^\perp \subset \mathbb{R}^n$ be the linear space orthogonal to $v \neq 0$ and consider on $V^\perp$ the quadratic form $\xi \mapsto \xi \cdot u\xi$. Then, $\det U = 0$ if and only if $-q$ is an eigenvalue of this quadratic form.

In order to exploit this fact for constructing irregular solutions to the nonlinear system, one needs plane wave–like solutions to (3.5) which are localized in space. Clearly an exact plane–wave as in (3.3) has compact support only if it is identically zero. Therefore this can only be done by introducing an error in the range of the wave, deviating from the line spanned by the wave state $a \in \mathbb{R}^d$. A crucial point is, therefore, to control this error.

4. The generalized energy and subsolutions

Next, for every $r \geq 0$, we consider the set of *Euler states of speed r*

$$K_r := \left\{ (v, u) \in \mathbb{R}^n \times \mathcal{S}^n_0 : u = v \otimes v - \frac{r^2}{n} I_n,\ |v| = r \right\} \tag{4.1}$$

(cp. with [**3**] and [**4**]). Lemma 3.1 says simply that solutions to the incompressible Euler equations can be viewed as evolutions on the manifold of Euler states subject to the linear conservation laws (3.5).

Next, we denote by K^{co}_r the convex hull in $\mathbb{R}^n \times \mathcal{S}^n_0$ of K_r. This convex set has been computed in [**4**].

LEMMA 4.1. *For any $w \in \mathcal{S}^n$ let $\lambda_{max}(w)$ denote the largest eigenvalue of w. For $(v, u) \in \mathbb{R}^n \times \mathcal{S}^n_0$ let*

$$e(v, u) := \frac{n}{2} \lambda_{max}(v \otimes v - u). \tag{4.2}$$

Then

(i) $e : \mathbb{R}^n \times \mathcal{S}^n_0 \to \mathbb{R}$ *is convex;*
(ii) $\frac{1}{2}|v|^2 \leq e(v, u)$, *with equality if and only if* $u = v \otimes v - \frac{|v|^2}{n} I_n$*;*
(iii) $|u|_\infty \leq 2\frac{n-1}{n} e(v, u)$, *where* $|u|_\infty$ *denotes the operator norm of the matrix;*
(iv) *The* $\frac{1}{2}r^2$*–sublevel set of* e *is the convex hull of* K_r*, i.e.*

$$K^{co}_r = \left\{ (v, u) \in \mathbb{R}^n \times \mathcal{S}^n_0 : e(v, u) \leq \frac{r^2}{2} \right\}. \tag{4.3}$$

(v) *If* $(u,v) \in \mathbb{R}^n \times \mathcal{S}_0^n$, *then* $\sqrt{2e(v,u)}$ *gives the smallest* ρ *for which* $(u,v) \in K_\rho^{co}$.

In view of (ii), if a triple (v,u,q) solving (3.5) corresponds a solution of the incompressible Euler equations via the correspondence in Lemma 3.1, then $e(v,u)$ is simply the energy density of the solution. In view of this remark, if (v,u,q) is a solution of (3.5), $e(v,u)$ will be called the *generalized energy density*, and $E(t) = \int_{\mathbb{R}^n} e(v(x,t),u(x,t))dx$ will be called the *generalized energy*.

The key proposition of [**4**] states, roughly speaking, that, given an energy profile $\bar{e}$ satisfying certain technical assumptions, the existence of some suitable "subsolution" for the Cauchy problem (2.1)–(2.2) implies the existence of weak solutions having energy density $\bar{e}$.

Proposition 4.2. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set (not necessarily bounded) and let*

$$\bar{e} \in C\big(\overline{\Omega} \times]0,T[\big) \cap C\big([0,T]; L^1(\Omega)\big).$$

Assume there exists (v_0,u_0,q_0) *smooth solution of* (3.5) *on* $\mathbb{R}^n \times]0,T[$ *with the following properties:*

$$v_0 \in C\big([0,T]; L^2_w\big), \tag{4.4}$$

$$\operatorname{supp}(v_0(\cdot,t),u_0(\cdot,t)) \subset\subset \Omega \text{ for all } t \in]0,T[, \tag{4.5}$$

$$e\big(v_0(x,t),u_0(x,t)\big) < \bar{e}(x,t) \text{ for all } (x,t) \in \Omega \times]0,T[\,. \tag{4.6}$$

Then there exist infinitely many weak solutions v *of the Euler equations* (2.1) *with pressure*

$$p = q_0 - \frac{1}{n}|v|^2 \tag{4.7}$$

such that

$$v \in C\big([0,T]; L^2_w\big), \tag{4.8}$$

$$v(\cdot,t) = v_0(\cdot,t) \qquad \text{for } t = 0,T, \tag{4.9}$$

$$\frac{1}{2}|v(\cdot,t)|^2 = \bar{e}(\cdot,t)\,\mathbf{1}_\Omega \qquad \text{for every } t \in]0,T[. \tag{4.10}$$

Proposition 4.2 is proved in [**4**] combining Tartar's plane wave analysis with the so called Baire category argument. A different approach is instead given by the Lipschitz convex integration (see for instance [**3**] and [**13**]).

In order to give an idea of this second mechanism, consider the particular case $\bar{e} \equiv 1$ and $v_0 \equiv 0$. Moreover, let us neglect the technical condition (4.8). Our goal would then be to construct a weak bounded solution of Euler which

- is supported in $\Omega \times [0,T]$;
- takes the values 0 at the times 0 and T;
- has energy identically equal to 1 on $\Omega \times]0,T[$.

Note that such solution can be extended to 0 for times $t \notin [0,T]$, thus achieving the celebrated example of Scheffer. In fact, this is the solution constructed in [**3**].

The idea of the Lipschitz convex integration would be to construct (v,u) as an infinite sum

$$(v,u) = \sum_{i=1}^{\infty} (v_i, u_i)$$

with the properties that

(a) the partial sums $S_k = \sum_{i=0}^{k}(v_i, u_i)$ are smooth solutions of (3.5), compactly supported in Ω;
(b) S_k takes its values in the interior of the set K_1^{co};
(c) (v, u) takes its values in the extremal points K_1 a.e. in Ω.

Now, (a) is achieved because in fact each summand (v_i, u_i) is a smooth solution of (3.5), compactly supported in Ω. As for (c) the key is that:

- S_k converges strongly in L^1_{loc};
- the summand (v_i, u_i) is chosen inductively so to let $\|\mathrm{dist}\,(S_k, K_1)\|_{L^1}$ tend to 0.

Each (v_{k+1}, u_{k+1}) is in fact the sum of finitely many localized waves with disjoint supports, which "move" S_k closer to the set K_1. Note the existence of these waves require (b). The strong convergence is triggered by the choice of the frequencies λ_k of the localized waves, which grow very fast.

The Baire category method, instead, achieves the sequence S_k using a "stability argument". As a byproduct we obtain that, if one looks at the weak* closure of smooth maps S_k's satisfying (a) and (b), a "typical" element of this set takes its values in K_1. One advantage of the Baire category method is therefore that it produces automatically infinitely many solutions.

5. Construction of suitable initial data

Another main discovery of [**4**] is the existence of "interesting" subsolutions. Consider for instance the example discussed in the previous section: $\bar{e} \equiv 1$ and $v_0 \equiv 0$. It is then obvious that, for any v exhibited by Proposition 4.2, we have

- $v(0, \cdot) \equiv 0$;
- $\int |v(x,t)|^2\, dx = |\Omega|$ for $t \in]0, T[$.

Therefore, the initial data is taken in a "weak sense", that is, for $t \downarrow 0$, $v(\cdot, t)$ converges only weakly to 0, but not strongly. Thus, v violates any reasonable generalization of the classical energy identity.

In order to achieve solutions v which fulfill an energy inequality we then need a subsolution v_0 which satisfies $|v_0(0,x)|^2 = 2\bar{e}(0,x)$ for a.e $x \in \Omega$. On the other hand, the existence of such subsolutions is not obvious. For instance, the typical "weak–strong uniqueness" of admissible solutions (valid both for incompressible Euler and for hyperbolic systems of conservation laws; see, for instance, the appendix of [**4**] and [**2**]) implies that such v_0 is necessarily nonsmooth at $t = 0$.

In [**4**] we showed the existence of some "interesting subsolutions". Having fixed a bounded open set $\Omega \subset \mathbb{R}^n$ we indeed have

Proposition 5.1. *There exist triples $(\bar{v}, \bar{u}, \bar{q})$ solving (3.5) in $\mathbb{R}^n \times \mathbb{R}$ and enjoying the following properties:*

$$\bar{q} = 0,\ (\bar{v}, \bar{u}) \text{ is smooth in } \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}) \text{ and } \bar{v} \in C\big(\mathbb{R}; L^2_w\big)\,, \tag{5.1}$$

$$\mathrm{supp}\,(\bar{v}, \bar{u}) \subset \Omega \times]-T, T[\,, \tag{5.2}$$

$$\mathrm{supp}\,(\bar{v}(\cdot,t), \bar{u}(\cdot,t)) \subset\subset \Omega \text{ for all } t \neq 0\,, \tag{5.3}$$

$$e\big(\bar{v}(x,t), \bar{u}(x,t)\big) < 1 \text{ for all } (x,t) \in \mathbb{R}^n \times (\mathbb{R} \setminus \{0\})\,. \tag{5.4}$$

Moreover

$$\frac{1}{2}|\bar{v}(x,0)|^2 = 1 \ \textit{a.e. in } \Omega.$$

It is interesting to note that the proof of this Proposition is just an adaptation of the same ideas of the proof of Proposition 4.2.

6. Proof of Theorem 1.2

In this section we show how the setting described so far, though centered on the incompressible Euler equations, yields in fact a short proof of Theorem 1.2.

We let Ω be the unit ball, $T = 1/2$ and $(\bar{v}, \bar{u})$ be as in Proposition 5.1. Define $\bar{e} := 1$, $q_0 := 0$,

$$v_0(x,t) := \begin{cases} \bar{v}(x,t) & \text{for } t \in [0,1/2] \\ \bar{v}(x,t-1/2) & \text{for } t \in [1/2,1], \end{cases} \tag{6.1}$$

$$u_0(x,t) := \begin{cases} \bar{u}(x,t) & \text{for } t \in [0,1/2] \\ \bar{u}(x,t-1/2) & \text{for } t \in [1/2,1]. \end{cases} \tag{6.2}$$

It is easy to see that the triple (v_0, u_0, q_0) satisfies the assumptions of Proposition 4.2 with $\bar{e} \equiv 1$. Therefore, there exists infinitely many solutions $v \in C([0,1], L^2_w)$ of (2.1) in $\mathbb{R}^n \times [0,1]$ with

$$v(x,0) = \bar{v}(x,0) = v(x,1) \text{ for a.e. } x \in \Omega,$$

and such that

$$\frac{1}{2}|v(\cdot,t)|^2 = \mathbf{1}_\Omega \qquad \text{for every } t \in]0,1[\,. \tag{6.3}$$

Since $\frac{1}{2}|v_0(\cdot,0)|^2 = \mathbf{1}_\Omega$ as well, it turns out that the map $t \mapsto v(\cdot,t)$ is continuous in the strong topology of L^2.

Each such v can be extended to $\mathbb{R}^n \times [0,\infty[$ 1-periodically in time, by setting $v(x,t) = v(x,t-k)$ for $t \in [k,k+1]$. Summarizing, we have found infinitely many solutions (v,p) of (2.1) with the following properties:

- $v \in C([0,\infty[, L^2)$ and $|v|^2 = 2\,\mathbf{1}_{\Omega\times[0,\infty[}$;
- $p = -|v|^2/n = -2n^{-1}\,\mathbf{1}_{\Omega\times[0,\infty[}$.

Consider now the isentropic Euler system (1.1) and let $p(\rho)$ be the pressure as a function of the density. Let $\alpha := p(1)$, $\beta := p(2)$ and $\gamma = \beta - \alpha$. Recalling that $p' > 0$, we conclude $\gamma > 0$. Since now p denotes the function $\rho \mapsto p(\rho)$, the pairs (v,p) constructed in the paragraphs above will be instead denoted by $(v,\tilde{p})$.

Next, note that the *incompressible* Euler equations are invariant under the rescalings $v(x,t) \mapsto \lambda v(\lambda t, x)$, $\tilde{p}(t,x) \mapsto \lambda^2 \tilde{p}(\lambda t, x)$. Thus, we can rescale the solutions considered above so to achieve $\tilde{p} = -\gamma\,\mathbf{1}_{\Omega\times[0,\infty[}$ (and thus $|v|^2 = n\gamma\,\mathbf{1}_{\Omega\times[0,\infty[}$). We are also free of adding an arbitrary constant to $\tilde{p}$. Adding the constant γ we then achieve

$$\tilde{p} \;=\; (\beta-\gamma)\,\mathbf{1}_{\Omega\times[0,\infty[} + \beta\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[} \;=\; \alpha\,\mathbf{1}_{\Omega\times[0,\infty[} + \beta\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[}\,.$$

Therefore, we conclude that

$$\partial_t v + \operatorname{div} v \otimes v + \nabla\big(\alpha\,\mathbf{1}_{\Omega\times[0,\infty[} + \beta\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[}\big) \;=\; 0 \tag{6.4}$$

and

$$\operatorname{div} v \;=\; 0\,. \tag{6.5}$$

Fix any such v and set

$$\rho \;=\; \mathbf{1}_{\Omega\times[0,\infty[} + 2\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[}\,.$$

Since $v = 0$ outside of Ω, obviously

$$\rho v = v \qquad \text{and} \qquad \rho v\otimes v = v\otimes v\,. \tag{6.6}$$

Moreover,

$$p(\rho) \;=\; \alpha\,\mathbf{1}_{\Omega\times[0,\infty[} + \beta\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[}\,. \tag{6.7}$$

So,

$$\begin{aligned}
\partial_t(\rho v) + \mathrm{div}\,[\rho v\otimes v] \;&\overset{(6.6)}{=}\; \partial_t v + \mathrm{div}\, v\otimes v\\
&\overset{(6.4)}{=}\; -\nabla[\alpha\,\mathbf{1}_{\Omega\times[0,\infty[} + \beta\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[}] \;\overset{(6.7)}{=}\; -\nabla[p(\rho)]\,.
\end{aligned} \tag{6.8}$$

Moreover, notice further that

$$\partial_t\rho \;=\; 0\,. \tag{6.9}$$

Hence,

$$\partial_t\rho + \mathrm{div}\,(\rho v) \;\overset{(6.9)}{=}\; \mathrm{div}\,(\rho v) \;\overset{(6.6)}{=}\; \mathrm{div}\, v \;\overset{(6.5)}{=}\; 0\,. \tag{6.10}$$

Thus, the pair (ρ, v) is a weak solution of (1.1) with initial data (ρ^0, v^0), where $\rho_0 = \mathbf{1}_\Omega + 2\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega}$.

Each such solution is admissible. Indeed, since ε depends only on ρ and $|v|^2 = n\gamma$ in Ω and vanishes outside, we obviously have

$$\partial_t\left[\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}\right] \;=\; 0\,. \tag{6.11}$$

For the same reason, we have

$$\mathrm{div}_x\left[\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right)v\right] \;=\; \left(\varepsilon(1) + p(1) + \frac{n\gamma}{2}\right)\mathrm{div}\, v \;=\; 0\,. \tag{6.12}$$

Therefore, (ρ, v) solves

$$\partial_t\left[\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}\right] + \mathrm{div}_x\left[\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right)v\right] \;=\; 0 \tag{6.13}$$

in the sense of distributions inside $\mathbb{R}^n\times]0,\infty[$.

That is, (1.4) holds (with the equality sign!) for every $\psi \in C^\infty_c(\mathbb{R}^n\times]0,\infty[)$. However, since $(\rho(\cdot,t), v(\cdot,t)) \to (\rho^0, v^0)$ strongly in L^2_{loc}, the same equality holds even for test functions ψ which do not vanish at $t = 0$.

7. Final comments

Clearly, the solutions constructed in the previous section are discontinuous along the interface $\partial\Omega\times[0,\infty[$. However this discontinuity is not at all a shock wave.

Let us indeed analyse closer the mathematical concept of shock wave. Even in the most general mathematical framework, a shock wave is an (at least) rectifiable set S of codimension 1 along which the solution undergoes a "generalized jump discontinuity". By this we mean that the solution has, in the strong L^1 sense, one-sided traces on S.

Consider now $S := \partial\Omega\times[0,\infty[$. Along this interace, v *does not* have a generalized jump discontinuity. Indeed v has strong trace approaching $\partial\Omega$ from the exterior. But from the interior v does not have a trace in the strong L^1 sense. This is not at all surprising. Consider the construction outlined in Section 4, where v is build as a series $\sum v_i$ of compactly supported oscillatory solutions. It is quite obvious that these oscillations become extremely fast as we let $k \uparrow \infty$ and approach the boundary of Ω, since we have to use very steep cut-off functions to keep the v_i supported in Ω.

In particular, though the normal trace of v at $\partial\Omega \times [0, \infty[$ is zero in a "weak sense" (for divergence–free fields, left and right normal traces coincide), the normal trace of $v \otimes v$ is not related to the trace of v by the obvious algebraic formula which would be a consequence of a strong trace. In particular we cannot infer that the (weak) normal trace of $v \otimes v$ vanishes. This implies, for instance, that we cannot write the usual Rankine–Hugoniot condition for the solution (v, ρ) on the interface $\partial\Omega \times [0, \infty[$.

Our discussion should bring yet an important point to the attention of the reader. The ill–posedness proved in our works do not seem to bear any relation to the formation of shock waves. One might instead conjecture that it is an effect of accumulation of vorticity. This explains loosely why our arguments fit both the incompressible and the compressible equations.

However, the reader should be extremely cautious in interpreting our solutions in terms of classical concepts of fluid dynamics. As an example, let us come back to the solutions v produced by Proposition 4.2 when $\bar{e} \equiv 1$ and $v_0 \equiv 0$. If any such solution described the motion of an incompressible *physical* fluid, we would have a number of paradoxes:

- The fluid would be totally at rest at time 0, move at positive time and go back at rest at time T, in spite of the total absence of external forces;
- the fluid would in fact remain at rest outside Ω and move *any* particle in Ω at speed 1, but the interface would not resemble at all that of a classical shear flow;
- the pressure would be constant inside Ω, displaying a total absence of interaction for the particles staying in Ω.

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