1. Section 3.2.1

In the proof of Proposition 3.5, the conclusion

$$|D(\vartheta \circ f)| < |Df| \quad \text{a.e. on } E' \tag{1}$$

from point (b) of Lemma 3.7 is incorrect: even in the classical (Euclidean) setting, a Lipschitz function g could satisfy the inequality $|g(x) - g(y)| < |x - y|$ for all distinct x and y and still have $|Dg| = 1$ on a set of positive measure (where with $|Dg|$ we denote the operator norm). Moreover, there are distinct points $S_1, S_2 \in B_{2r}(T) \setminus B_r(T)$ for which $\mathcal{G}(\vartheta(S_1), \vartheta(S_2)) = \mathcal{G}(S_1, S_2)$, so that the claim (b) of Lemma 3.7 is also incorrect. I wish to thank Daniele Semola for having pointed this out to me.

However (1) is indeed true for a slightly different ϑ compared to the one of Lemma 3.7 under the assumption that $r < \frac{s(T)}{8}$ (which is slightly stronger than the one of Proposition 3.5). First of all we fix $\rho > 2r$ such that $\rho < s(T)/4$. Then we define $\vartheta(S)$ to be equal to T outside $B_\rho(T)$, to be the identity in $B_r(T)$ and to be equal to

$$\sum_{j=1}^{J} \sum_{l=1}^{k_j} \left[\!\!\left[\frac{\rho - \mathcal{G}(T,s)}{(\rho/r - 1)\mathcal{G}(T,s)} (S_{l,j} - Q_j) + Q_j \right]\!\!\right] \tag{2}$$

otherwise. For this ϑ we claim the following slightly stronger version of Lemma 3.7(b) , from which the (1) easily follows

(b') for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\mathcal{G}(\vartheta(S_1), \vartheta(S_2)) \le (1 - \delta)\mathcal{G}(S_1, S_2) \tag{3}$$

for every $S_1 \notin B_{r+\varepsilon}(T)$ and every $S_2 \neq S_1$.

In order to prove (b') we use the following elementary fact

Lemma 1.1. *Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as*

$$\Phi(z) = \begin{cases} z & \text{if } |z| \le r \\ \frac{\rho - |z|}{(\rho/r - 1)|z|} z & \text{if } r \le |z| \le \rho \\ 0 & \text{if } |z| \ge \rho\,. \end{cases}$$

Then

(b") *for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$|\Phi(z_1) - \Phi(z_2)| \le (1 - \delta)|z_1 - z_2| \tag{4}$$

for every $|z_1| > r + \varepsilon$ and every $z_2 \neq z_1$.

(b') is then an easy consequence: first of all observe that the inequality (3) is trivial if $S_1 \notin B_\rho(T)$ or $S_2 \notin B_\rho(T)$. Hence assume $S_1, S_2 \in B_\rho(T)$ and observe that we can write

$$S_1 = \sum [\![S_{1,j}]\!] \qquad S_2 = \sum [\![S_{2,j}]\!]$$

where $S_{i,j} \in B_\rho(k_j [\![Q_j]\!]) \subset \mathcal{A}_{k_j}$. Given the separation between distinct Q_j and $Q_{j'}$ we actually have

$$(\mathcal{G}(S_1, S_2))^2 = \sum_j (\mathcal{G}(S_{1,j}, S_{2,j}))^2 .$$

In fact for each j and j' distinct we have $|S_{1,j,\ell} - S_{1,j',\ell'}| \geq |Q_j - Q_{j'}| - 2\rho > 2\rho$ (of course with $S_{i,j} = \sum_\ell [\![S_{i,j,\ell}]\!]$) , whereas $|S_{1,j,\ell} - S_{1,j,\ell'}| < 2\rho$.

Moreover we can order the $S_{i,j,\ell}$ so that

$$(\mathcal{G}(S_{1,j}, S_{2,j}))^2 = \sum_\ell |S_{1,j,\ell} - S_{2,j,\ell}|^2 .$$

Consider the vectors $w_{i,j,\ell} := S_{i,j,\ell} - Q_j$ and the vectors $z_1, z_2 \in \mathbb{R}^{Qn}$ given by

$$z_i = (w_{i,1,1}, w_{i,1,2}, \ldots, w_{i,1,k_1}, w_{i,2,1}, \ldots, w_{i,J,1}, \ldots, w_{i,J,k_j})$$

We then obviously have

$$|z_1 - z_2| = \mathcal{G}(S_1, S_2) \tag{5}$$

whereas

$$|z_i| = \mathcal{G}(S_i, T) .$$

On the other hand

$$\vartheta(S_i) = \sum_{j,\ell} \left[\!\!\left[\frac{\rho - |z_i|}{(\rho/r - 1)|z_i|} w_{i,j,\ell} \right]\!\!\right]$$

and so

$$\mathcal{G}(\vartheta(S_1), \vartheta(S_2)) \leq \left| \frac{\rho - |z_1|}{(\rho/r - 1)|z_1|} z_1 - \frac{\rho - |z_2|}{\rho/r - |z_2|} z_2 \right| = |\Phi(z_1) - \Phi(z_2)| . \tag{6}$$

Hence (3) is an easy consequence of (5), (6) and (4).

In order to show Lemma 1.1 we fix, without loss of generality, $r = 1$ (the general case follows from a simple scaling argument) and $\varepsilon > 0$. Note that Φ is Lipschitz on the whole space and C^1 in each of the three regions $\{|z| < 1\}, \{1 < |z| < \rho\}$ and $\{|z| > \rho\}$. Moreover $\rho > 2r = 2$. Since $D\Phi$ is the identity in the first region and vanishes in the third, it suffices to show that its operator norm is strictly less than 1 in the second region. In such region we compute

$$D\Phi(z) = \frac{\rho - |z|}{(\rho - 1)|z|} Id - \frac{\rho}{(\rho - 1)|z|^3} z \otimes z .$$

Observe that the matrix is symmetric and hence it suffices to show that all eigenvalues are contained in the open interval $(-1, 1)$. Now on the subspace orthogonal to z the eigenvalue is $\frac{\rho - |z|}{(\rho-1)|z|} = \frac{\rho}{(\rho-1)|z|} - \frac{1}{\rho - 1} < \frac{\rho}{\rho - 1} - \frac{1}{\rho-1}$ (because $|z| > 1$). Thus that eigenvalue is positive and strictly smaller than 1 in the region of interest. On the eigenspace generated by z the eigenvalue is $-\frac{1}{\rho-1}$ and hence negative and strictly larger than -1 (because $\rho > 2$).

2. Section 4.3.2

In a previous version of this Errata, I had the following minor corrections, pointed out to me by Ryan Scott:

In equation (4.29) the prefactor in front of the integral should be k^{m-3} and not $\frac{1}{k^{m+1}}$. The same correction applies to the displayed equation right after (4.29).

In the displayed equation before (4.30) the prefactor should be Ck^{m-1} instead of $Ck^{-(m-1)}$.

In the last displayed equation of Section 4.3 a power 2 is missing in the left hand side and the factor $\frac{C}{k^{m+2}}$ on the right hand side should be Ck^{m-3}.

However the proof of Section 4.3.2 contains a major problem, which was pointed out to me by Jonas Hirsch: when, after the cubical sudvisions, we claim that we can argue as in Theorem 1.7 extending the map first to the 1-skeleton, it must be noticed that in this case we are not dealing with cubes, as in Theorem 1.7. Thus the claimed Lipschitz estimates are not correct. The extrinsic proof given in Lemma 2.14 is instead correct and it is indeed possible to give an intrinsic proof, following a strategy which still shares some similarities with the wrong one used in Section 4.3.2. Below I include the details of this proof when interpolating between two flat cubes: the interpolation between two concentric spheres follows from simple adjustments.

Proposition E.1. *There is a constant C depending only on m, n and Q with the following properties. Assume that $0 < \varepsilon \leq 1$ and that $f, g : [-\varepsilon, 1+\varepsilon]^m \to \mathcal{A}_Q(\mathbb{R}^n)$ are two maps in $W^{1,2}$. Then there is a map $h \in W^{1,2}([0,1]^m \times [0,\varepsilon], \mathcal{A}_Q(\mathbb{R}^n))$ such that:*

$$h(x,0) = f(x) \qquad \text{and} \qquad h(x,\varepsilon) = g(x) \qquad \forall x \in [0,1]^m\,, \tag{7}$$

$$\int_{[0,1]^m\times[0,\varepsilon]} |Dh|^2 \leq C\varepsilon \int_{[0,1]^m} (|Df|^2 + |Dg|^2) + \frac{C}{\varepsilon}\int \mathcal{G}(f,g)^2\,. \tag{8}$$

The proof below (which is close in spirit to the proof of Luckhaus in [1] of what is nowadays called the Luckaus Lemma in the literature of harmonic maps) is based an a cubical subdivision and on the following lemma. It is useful to set the following terminology. If $L := x_0 + \delta\mathbb{Z}^N$ is a square lattice in $\mathbb{R}^N$, we call *cubical decomposition induced by L* the collection $\mathscr{C}$ of closed cubes in $\mathbb{R}^N$ with sidelength δ and vertices belonging to L.

Lemma E.2. *There is a constant C depending only on m, n and Q with the following properties. Assume that $0 < \varepsilon \leq 1$, $D = [0,\varepsilon]^m + x$ and $\varphi \in W^{1,2}(\partial D, \mathcal{A}_Q(\mathbb{R}^n))$. Then there is an extension ψ of φ to D such that*

$$\mathrm{Dir}(\psi, D) \leq C\varepsilon \mathrm{Dir}(\varphi, \partial D)\,. \tag{9}$$

Proof. Observe that, by a simple scaling argument, it is sufficient to prove the lemma when $D = [0,1]^m$. On the other hand, since D is biLipschitz equivalent to the uniti ball, it then suffices to show the claim when $D = \overline{B}_1$. In this case the claim is a weaker form of Proposition 3.9, where the constant C is also shown to satisfy a certain precise bound. In fact, in the case $m = 2$ we can simply take the proof of Proposition 3.9 given in Section 3.3.2, which does not make any use of the maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$. In the case $m \geq 3$ the proof of

Proposition 3.9 uses actually Proposition E.1, but only to reach a constant C smaller than $\frac{1}{m-2}$: if we follow the first paragraph of 3.3.3 we conclude that (9) holds with $C=(m-2)^{-1}$ if we simply set $\psi(x) := \varphi(\varepsilon x/|x|)$ for every $x\neq 0$. □

Proof of Proposition E.1. Fix any vector $v\in[0,\varepsilon]^m$ and for each such vector consider the cubical decomposition $\mathcal{C}$ of $\mathbb{R}^m$ induced by the lattice $v+\varepsilon\mathbb{Z}^m$. We then define accordingly the k-dimensional skeleta contained in $[-\varepsilon,1+\varepsilon]^m$, which are the families $\mathcal{S}^k(v)$ of all (closed) k-dimensional faces of the cubes of $\mathcal{C}$ contained in $[-\varepsilon,1+\varepsilon]^m$. By Fubini, for every $k\geq 1$ and for a.e. v we have that, $f|_F, g|_F\in W^{1,2}$ for every $F\in\mathcal{S}^k(v)$. Moreover

$$\begin{aligned}&\int_{[0,\varepsilon]^m}\Big(\sum_{F\in\mathcal{S}^k(v)}\int_F\left(|Df|^2+|Dg|^2+\mathcal{G}(f,g)^2\right)\Big)\,dv\\ \leq &C(k,m)\varepsilon^k\int_{[-\varepsilon,1+\varepsilon]^m}\left(|Df|^2+|Dg|^2+\mathcal{G}(f,g)^2\right).\end{aligned}$$

By standard arguments we can then choose v such that:

(r) For every $k\geq 1$, for each $F\in\mathcal{S}^k(v)$ and for all $G\in\mathcal{S}^{k-1}(v)$ with $G\subset F$, the restrictions $f|_F$, $f|_G$, $g|_F$ and $g|_G$ are all $W^{1,2}$ and moreover the traces of $f|_F$ and $g|_F$ at G are precisely $f|_G$ and $g|_G$;

(e1) For every $k\geq 1$ we have

$$\sum_{F\in\mathcal{S}^k(v)}\int_F(|Df|^2+|Dg|^2)\leq C\varepsilon^{k-m}\int_{[-\varepsilon,1+\varepsilon]^m}(|Df|^2+|Dg|^2)\,; \tag{10}$$

(e2) For $k=0$ we have

$$\sum_{p\in\mathcal{S}^0(v)}\mathcal{G}(f(p),g(p))^2\leq C\varepsilon^{-m}\int_{[-\varepsilon,1+\varepsilon]^m}\mathcal{G}(f,g)^2\,. \tag{11}$$

Having fixed such a v, we consider the cubical decomposition $\mathscr{C}$ of $\mathbb{R}^{m+1}$ induced by the lattice $\varepsilon\mathbb{Z}^{m+1}+(v,0)$. Define then $\mathscr{S}^k$ to be the families of k-dimensional faces belonging to cubes of $\mathscr{C}$ contained in the slab $[-\varepsilon,1+\varepsilon]^m\times[0,\varepsilon]$. For eack $k<m+1$ we let $\mathscr{U}^k$ and $\mathscr{L}^k$ be respectively the "horizontal lower and upper faces" in $\mathscr{S}^k$, namely those faces contained in $[-\varepsilon,1+\varepsilon]^m\times\{\varepsilon\}$ and $[-\varepsilon,1+\varepsilon]^m\times\{0\}$, respectively. The remaining faces $\mathscr{V}^k$ will be called "vertical" faces. Observe that all elements in $\mathscr{S}^0$ are horizontal and that all elements in $\mathscr{S}^{m+1}$ are vertical: $\mathscr{S}^0$ consists of points $(p,0)$ and (p,ε) for $p\in\mathcal{S}^0(v)$ (where the first belong to $\mathscr{L}^0$ and the second to $\mathscr{U}^0$); $\mathscr{S}^{m+1}=\mathscr{V}^{m+1}$ is the collection of those cubes of $\mathscr{C}$ contained in the slab $[-\varepsilon,1+\varepsilon]^m\times[0,\varepsilon]$. Observe that, for $1\leq k\leq m$, $\mathscr{L}^k$ and $\mathscr{U}^k$ consist, respectively, of faces $F\times\{0\}$ and $F\times\{\varepsilon\}$ for $F\in\mathcal{S}^k(v)$.

We are now ready to define the map h. First of all h is defined on the horizontal faces in the obvious way so that (7) holds. Consider next any vertical segment σ. Its two endpoints are gievn by $(p,0)\in\mathscr{L}^0$ and $(p,\varepsilon)\in\mathscr{U}^0$, where $p\in\mathcal{S}^0$ (v). We then can extend "linearly" h to σ achieving the simple bound

$$\int_\sigma|Dh|^2\leq\varepsilon^{-1}\mathcal{G}(f(p),g(p))^2\,. \tag{12}$$

We proceed in this way for all vertical segments: since they are disjoint, h is well defined. In particular, by (11) we reach the bound

$$\sum_{\sigma\in\mathscr{V}^1}\int_\sigma |Dh|^2 \leq C\varepsilon^{-1-m}\int_{[-\varepsilon,1+\varepsilon]^m}\mathcal{G}(f,g)^2\,. \tag{13}$$

Pick next a vertical 2-dimensional face τ. Its boundary consists of two horizontal segments and two vertical segments. Let us denote the horizontal ones as $\alpha\in\mathscr{L}^1$ and $\beta\in\mathscr{U}^1$ and observe that the two vertical ones join two pairs of points $(p,0),(p,\varepsilon)$ and $(q,0),(q,\varepsilon)$, with $q\in\mathcal{S}^0(v)$. It turns out that $h|_{\partial\tau}$ is in $W^{1,2}$, because of (r). Using Lemma E.2 and (14) we can extend h to τ with the estimate

$$\int_\tau |Dh|^2 = \varepsilon\int_\alpha |Df|^2 + \varepsilon\int_\beta |Dg|^2 + \mathcal{G}(f(p),g(p))^2 + \mathcal{G}(f(q),g(q))^2\,.$$

Summing all these bounds and using (10) and (14) we then achieve

$$\sum_{\tau\in\mathscr{V}^2}\int_\tau |Dh|^2 \leq C\varepsilon^{2-m}\int_{[-\varepsilon,1+\varepsilon]^m}(|Df|^2+|Dg|^2) + C\varepsilon^{-m}\int_{[-\varepsilon,1+\varepsilon]^m}\mathcal{G}(f,g)^2\,. \tag{14}$$

We then proceed inductively over $\mathscr{V}^k$. Observe that at the final step, namely that of "vertical" $(m+1)$-dimensional cube, we reach a map h which is $W^{1,2}$ in each element of $\mathscr{V}^{m+1}$. Moreover, given any pair of cubes $H,K\in\mathscr{V}^{m+1}$ with a common face $F\in\mathscr{V}^m$, the traces of $h|_H$ and $h|_K$ on F coincide. So h is a $W^{1,2}$ function on the union of the cubes $C\in\mathscr{V}^{m+1}$: such set contains the domain $[0,1]^m\times[0,\varepsilon]$ and thus we can regard h as an element of $W^{1,2}([0,1]^m\times[0,\varepsilon],\mathcal{A}_Q(\mathbb{R}^n))$. Similarly, the traces on the lower and upper horizontal m-dimensional faces coincide with f and g and so (7) holds. Finally, the inductive proecure gives the estimates

$$\sum_{\tau\in\mathscr{V}^k}\int_\tau |Dh|^2 \leq C\varepsilon^{k-m}\int_{[-\varepsilon,1+\varepsilon]^m}(|Df|^2+|Dg|^2) + C\varepsilon^{k-m-2}\int_{[-\varepsilon,1+\varepsilon]^m}\mathcal{G}(f,g)^2\,. \tag{15}$$

But we also have

$$\int_{[0,1]^m\times[0,\varepsilon]}|Dh|^2 \leq \sum_{K\in\mathscr{V}^{m+1}}\int_K |Dh|^2\,.$$

The latter inequality combined with the case $k=m+1$ of (15) gives (8). □

References

[1] S. Luckhaus, *Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold.*, Indiana Univ. Math. J. **37**, (1988), No. 2, 349–368.