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SURPRISING SOLUTIONS TO THE ISENTROPIC EULER SYSTEM OF GAS DYNAMICS

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Abstract. In a recent paper, jointly with Elisabetta Chiodaroli and Ondřej Kreml we consider the Cauchy problem for the isentropic compressible Euler system in 2 space dimensions, with initial data which assume two different constant values and have a discontinuity across a line. If we consider selfsimilar solutions we then encounter a classical 1-dimensional Riemann problem for the corresponding hyperbolic system of conservation laws. We show that for some suitable choice of the pressure and of the initial data there exist infinitely many bounded admissible solutions which are not selfsimilar and indeed are genuinely 2-dimensional. We also show that some of these Riemann data are generated by a 1-dimensional compression wave. Our theorem leads therefore to Lipschitz initial data for which there are infinitely many global bounded admissible weak solutions. Each of these solutions coincide as long as the classical (Lipschitz) solution exists and they differentiate themselves immediately after the first blow-up time. Our approach is heavily influenced by a work of László Székelyhidi which provides a similar result in the case of the classical vortex-sheet problem for the incompressible Euler equations.

1. **Introduction.** Consider the isentropic compressible Euler equations of gas dynamics in n space dimensions. This system consists of $n+1$ scalar equations, which state the conservation of mass and linear momentum. The unknowns are the density ρ and the velocity v and the system takes the the form:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho v) \; = \; 0 \\ \\ \partial_t(\rho v) + \mathrm{div}_x \left(\rho v \otimes v\right) + \nabla_x [p(\rho)] \; = \; 0 \end{cases} \tag{1}$$

The pressure p is a function of ρ determined from the constitutive thermodynamic relations of the gas under consideration and it is assumed to satisfy $p' > 0$ (this hypothesis guarantees also the hyperbolicity of the system on the regions where

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ρ is positive). A common choice is the polytropic pressure law $p(\rho) = \kappa\rho^\gamma$ with constants $\kappa > 0$ and $\gamma > 1$. The classical kinetic theory of gases predicts exponents $\gamma = 1 + \frac{2}{d}$, where d is the degree of freedom of the molecule of the gas.

A lot of attention has been devoted in the literature to the *Cauchy problem* which consists of solving (1) on a domain of the form $\mathbb{R}^2 \times [0, T[$ (where T might also be infinite), subject to an initial condition of type

$$\begin{cases} \rho(\cdot, 0) \;=\; \rho^0 \\ \\ v(\cdot, 0) \;=\; v^0 \,. \end{cases} \tag{2}$$

It is well known that, even starting from extremely regular initial data, the solutions of the Cauchy problem for the system (1) develop singularities in finite time. It is also well-known that after the appearance of the first singularity weak solutions (i.e. solutions in the usual distributional sense, see Definition 2.1 for the precise formulation) are not unique: the standard example is provided by "non-physical" shocks, which can however be ruled out imposing that the weak solutions satisfy some further admissibility condition. Much effort has been put in understanding how this approach can give well-posedness results after the appearance of the first singularity, leading to a quite mature and successful theory in one space dimension (we refer the reader to the monographs [1],[8] and [19]).

Here we consider the case of two space dimensions and restrict our attention to bounded weak solutions of (1) which satisfy the following additional inequality in the sense of distributions (called usually *entropy inequality*, although for the specific system (1) this is rather a weak form of the energy balance):

$$\partial_t \left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2}\right) + \mathrm{div}_x \left[\left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} + p(\rho)\right) v\right] \;\le\; 0 \tag{3}$$

where the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is given through the law $p(r) = r^2\varepsilon'(r)$. Indeed, admissible solutions are required to satisfy a slightly stronger condition, i.e. a form of (3) which involves also the initial data, see Definition 2.2.

Starting from the work [12] it was observed that (3) is in this case not enough to restore uniqueness of admissible *bounded* solutions. The methods used in [12], inspired by techniques developed in the theory of differential inclusions, show a rather surprising abundance of admissible solutions to the Cauchy problem with certain particular initial data. However those specific initial data were rather irregular, leaving open the question whether this fact alone was responsible for such behavior.

The investigations of [12] have been pushed further in [5] and in [6]: in the latter paper, we have shown that the same nonuniqueness result holds even for *Lipschitz* initial data, therefore leading to the following theorem.

Theorem 1.1. *Let $p(\rho) = \rho^2$. Then there are* Lipschitz *initial data ρ^0 and v^0, with $\rho^0 \ge c_0 > 0$ for which there are* infinitely many *admissible bounded weak solutions (ρ, v) of the Cauchy problem (1)-(2), with* $\inf \rho > 0$. *All these solutions coincide with the classical one as long as it exists and differ immediately after the formation of the first singularity.*

2. **Main results.** We recall here the usual definitions of weak and admissible solutions to (1).

Definition 2.1. By a *weak solution* of (1)-(2) on $\mathbb{R}^2\times[0,\infty[$ we mean a pair $(\rho, v)\in L^\infty(\mathbb{R}^2\times[0,\infty[)$ such that the following identities hold for every test functions $\psi\in C_c^\infty(\mathbb{R}^2\times[0,\infty[,\mathbb{R})$, $\phi\in C_c^\infty(\mathbb{R}^2\times[0,\infty[,\mathbb{R}^2)$:

$$\int_0^\infty\int_{\mathbb{R}^2}\left[\rho\partial_t\psi+\rho v\cdot\nabla_x\psi\right]dxdt+\int_{\mathbb{R}^2}\rho^0(x)\psi(x,0)dx \;=\; 0 \tag{4}$$

$$\int_0^\infty\int_{\mathbb{R}^2}\left[\rho v\cdot\partial_t\phi+\rho v\otimes v:D_x\phi+p(\rho)\operatorname{div}_x\phi\right]+\int_{\mathbb{R}^2}\rho^0(x)v^0(x)\cdot\phi(x,0)dx \;=\; 0. \tag{5}$$

Definition 2.2. A bounded weak solution (ρ, v) of (1)-(2) is admissible if it satisfies the following inequality for every nonnegative test function $\varphi\in C_c^\infty(\mathbb{R}^2\times[0,\infty[)$:

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^2}\left[\left(\rho\varepsilon(\rho)+\rho\frac{|v|^2}{2}\right)\partial_t\varphi+\left(\rho\varepsilon(\rho)+\rho\frac{|v|^2}{2}+p(\rho)\right)v\cdot\nabla_x\varphi\right]\\&+\int_{\mathbb{R}^2}\left(\rho^0(x)\varepsilon(\rho^0(x))+\rho^0(x)\frac{\left|v^0(x)\right|^2}{2}\right)\varphi(x,0)\,dx\;\geq\;0\,.\end{aligned} \tag{6}$$

The following is then the theorem proved in [12]:

Theorem 2.3 (De Lellis - Székelyhidi)**.** *For any $p\in C^1$ with $p'>0$ there are pairs $\rho^0, v^0\in L^\infty$ such that there are infinitely many bounded admissible solutions (ρ, v) of (1)-(2) with $\inf\rho>0$.*

As already mentioned, the initial data constructed in [12] were however very irregular. It was then proved by Chiodaroli that indeed the ill-posedness of Theorem 2.3 still holds even if ρ^0 is regular. More precisely

Theorem 2.4 (Chiodaroli)**.** *For any $p\in C^1$ with $p'>0$ and any $\rho^0\in C^1$ with $\inf\rho^0>0$ there is $v^0\in L^\infty$ such that there are infinitely many bounded admissible solutions (ρ, v) of (1)-(2) with $\inf\rho>0$.*

In [6] we consider first initial data of a very particular form. We denote the space variable as $x=(x_1,x_2)\in\mathbb{R}^2$ and set

$$(\rho^0(x), v^0(x)) := \begin{cases}(\rho_-, v_-) & \text{if } x_2<0\\[2ex] (\rho_+, v_+) & \text{if } x_2>0,\end{cases} \tag{7}$$

where $\rho_\pm, v_\pm$ are constants.

It is well-known that for some special choices of these constants there are solutions of (1) which are *rarefaction waves*, i.e. self-similar solutions depending only on t and x_2 which are locally Lipschitz for positive t and constant on lines emanating from the origin (see [8, Section 7.6] for the precise definition). Reversing their order (i.e. exchanging $+$ and $-$) the very same constants allow for a *compression wave* solution, i.e. a solution on $\mathbb{R}^2\times]-\infty,0[$ which is locally Lipschitz and converges, for $t\uparrow 0$, to the jump discontinuity of (7). When this is the case we will then say that the data (7) are *generated by a classical compression wave.*

It follows from the usual treatment of the 1-dimensional Riemann problem that for data as in (7) uniqueness holds if the admissible solutions are also required to be self-similar, i.e. of the form $(\rho, v)(x,t)=\left(r\left(\frac{x_2}{t}\right), w\left(\frac{x_2}{t}\right)\right)$, and to have locally bounded variation. In fact in this case the solutions are obtained "gluing" together rarefaction waves and jump discontinuities across interfaces of type $\{(x,t) : x_2 = \nu t\}$.

In the paper [6] we show the existence of bounded admissible solutions which are *not* selfsimilar. Although we expect such solutions to exist for very general pressure laws, we show them only for some particular choice of the pressure p.

Theorem 2.5 (Chiodaroli-De Lellis-Kreml)**.** *There are smooth pressures p with $p' > 0$ and constants $\rho_\pm, v_\pm$ for which, if (ρ^0, v^0) are as in (7), then there are infinitely many bounded admissible solutions (ρ, v) of (1)-(2) with* $\inf \rho > 0$.

Among the pressure laws of Theorem 2.5 there is also the quadratic law $p(\rho) = \rho^2$. The strongest results of [6] are indeed proved fur such law. More precisely we have the following strengthened version of Theorem 2.5.

Theorem 2.6 (Chiodaroli-De Lellis-Kreml)**.** *Assume $p(\rho) = \rho^2$. Then there are constants $\rho_\pm, v_\pm$ for which the conclusion of Theorem 2.5 holds and such that (ρ^0, v^0) are generated by a classical compression wave.*

Theorem 1.1 is then a simple corollary of Theorem 2.6: the solutions of Theorem 1.1 are simply obtained "patching" a classical compression wave with the nonstandard solutions of Theorem 2.6.

3. ***h*-principle and differential inclusions.** The proof of Theorem 1.1 relies heavily on the works of the first author and László Székelyhidi, who in the paper [11] introduced methods from the theory of differential inclusions to explain the existence of compactly supported nontrivial weak solutions of the *incompressible* Euler equations (discovered in the pioneering work of Scheffer [20]; see also [21]). Indeed the paper [12] is based on the observation that these methods could be applied to the compressible Euler equations and lead to the ill-posedness of bounded admissible solutions, see [12].

The link with the incompressible Euler equations is provided by the following elementary remark.

Remark 1. Assume $\Omega \subset \mathbb{R}^2 \times \mathbb{R}$ and let (ρ, v) be a distributional solution of (1) with constant density ρ. Then the pair $(v, 0)$ is a weak solution of the incompressible Euler equations

$$\begin{cases} \partial_t v + \operatorname{div} v \otimes v + \nabla q = 0 \\ \\ \operatorname{div} v = 0 \,. \end{cases} \tag{8}$$

Or in other words a "pressureless" solution, where $q = 0$: note however that q could be set to be any given constant.

Although classical solutions of the incompressible Euler equations with constant pressure are rather rare, the methods of [11] show that there are many such weak solutions. In fact the constraints posed by the equations for weak solutions are so much weaker than those posed for classical solutions, that all these irregular ones can be constructed to satisfy the additional constraint $|v| = \text{const.}$. In particular these methods yield the following crucial lemma (cf. with [6, Lemma 3.7]; here $\mathcal{S}_0^{2\times 2}$ denotes the set of symmetric traceless 2×2 matrices and Id is the identity matrix).

Lemma 3.1. *Let $(\tilde{v}, \tilde{u}) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ and $C > 0$ be such that*

$$\tilde{v} \otimes \tilde{v} - \tilde{u} < \frac{C}{2} \mathrm{Id} \,. \tag{9}$$

For any open set $\Omega \subset \mathbb{R}^2 \times \mathbb{R}$ *there are infinitely many maps* $(\underline{v}, \underline{u}) \in L^\infty(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ *with the following property*

(i) $\underline{v}$ *and* $\underline{u}$ *vanish identically outside* Ω*;*
(ii) $\operatorname{div}_x \underline{v} = 0$ *and* $\partial_t \underline{v} + \operatorname{div}_x \underline{u} = 0$*;*
(iii) $(\tilde{v} + \underline{v}) \otimes (\tilde{v} + \underline{v}) - (\tilde{u} + \underline{u}) = \frac{C}{2}\mathrm{Id}$ *a.e. on* Ω*.*

For the relevance of the condition (9) and the techniques used to prove these type of theorems we refer the reader to the survey article [13]: we give here just a brief comment. Observe that, inside Ω, the pair $(v, u) = (\tilde{v} + \underline{v}, \tilde{u} + \underline{u})$ solves the linear identities

$$\begin{cases} \partial_t v + \operatorname{div} u = 0 \\ \\ \operatorname{div} v = 0 \end{cases} \tag{10}$$

and the algebraic constraint

$$u = v \otimes v - \frac{C}{2}\mathrm{Id}\,. \tag{11}$$

Since C is a constant, plugging (11) into (10) we actually conclude that v is a solution (in Ω) of (8) with constant pressure (such constant being free for us to decide). Moreover, since u is trace-free, we conclude that $|v|^2$ equals the constant C. So, (9) can be interpreted as a relaxation of (11), i.e. an inequality which would be automatically satisfied by any weak limit of sequences of solutions as above. The methods of [11]-[12] essentially show that for any such "candidate weak limit" there are indeed many sequences of *exact solutions* converging to it (although such solutions are rather irregular).

4. **The main geometric idea.** Coming back to *compressible Euler* consider now any constant $\rho_0 > 0$. The pair $(\rho, v) = (\rho_0, \tilde{v} + \underline{v})$ is then a weak solution of (1) in Ω, as it can be easily verified by the identities

$$\partial_t \rho_0 + \operatorname{div}(\rho_0 v) = \rho_0 \operatorname{div} v$$

and

$$\partial_t(\rho_0 v) + \operatorname{div}(\rho_0 v \otimes v) + \nabla[p(\rho_0)] = \rho_0(\partial_t v + \operatorname{div} v \otimes v)\,.$$

In fact it also an admissible solution: since

$$\rho_0 \varepsilon(\rho_0) + \rho_0 \frac{|v|^2}{2}$$

and

$$\rho_0 \varepsilon(\rho_0) + \rho_0 \frac{|v|^2}{2} + p(\rho_0)$$

are both constants, (3) amounts to $\operatorname{div} v = 0$.

Observe however that the pair (ρ, v) ceases to give a solution of compressible Euler on the *whole* space-time. Assume now to chop $\mathbb{R}^2 \times \mathbb{R}$ into finitely many open subsets Ω_i and repeat on each Ω_i the construction of the previous section, starting from arbitrary constants for v, u and ρ. Define a resulting pair (ρ, v) by setting it equal, in each separate Ω_i, to the various functions given by Lemma 3.1 (in particular we are free to set the constants ρ_i for the value of the pressure). Although (ρ, v) is an admissible solution of (1) in each separate open set, it might fail to do so on the entire space-time. However, a careful computation shows that in order to be an admissible solution on the entire space-time, we just need to satisfy some compatibility conditions at the interfaces, which are reminiscent of the Rankine-Hugoniot conditions. We observe that some care is needed: due to the oscillatory nature of the solutions, the traces of $|v|^2$ do not coincide with the moduli squared of the traces of v!

The relevant computations show that these compatibility conditions depend only on the chosen "starting" constants. We are therefore ready to give an outline of the main idea behind the construction in [6], which indeed stems out of several conversations of the authors with László Székelyhidi and it is inspired by his work [23].

Consider first some data as in (7). We then partition the upper half space $\{t > 0\}$ in regions contained between half-planes meeting all at the line $\{t = x_2 = 0\}$, see Definition 5.1 and cf. Figure 1. We then define the density function $\rho = \overline{\rho}$ to be constant in each region: this density function will indeed give the final ρ for all the solutions we construct and it is therefore required to take the constant values $\rho_\pm$ in the outermost regions $P_\pm$.

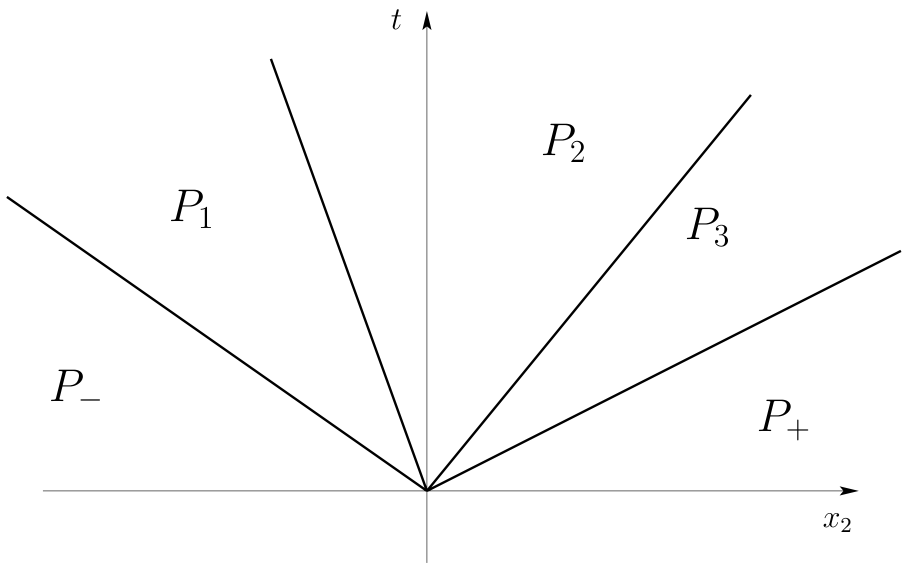

FIGURE 1. A "fan partition" in five regions.

We then solve the compressible Euler equations (1) in each region $P_1, \ldots, P_N$ using Lemma 3.1, so imposing that the modulus of the velocity is constant (in each region): its square will be denoted by C_i.

The corresponding constant values (ρ_i, v_i, u_i) will then give a globally defined (piecewise constant) function $(\overline{\rho}, \overline{v}, \overline{u})$, which will be called a *fan subsolution* of the compressible Euler equations. We then wish to choose our subsolution so that, after solving (1) in each region P_i with the methods of [12], the resulting globally defined (ρ, v) are admissible *global* solutions of (1). This leads to a suitable system of PDEs for the piecewise constant functions $(\overline{\rho}, \overline{v}, \overline{u})$ which are summarized in the Definitions 5.2 and 5.3.

5. **Subsolutions.** The approach sketched in the previous section leads to the following rigorous definitions (cf. Definitions 3.3, 3.4 and 3.5 in [6]).

Definition 5.1 (Fan partition)**.** A *fan partition* of $\mathbb{R}^2\times]0,\infty[$ consists of finitely many open sets $P_-, P_1, \ldots, P_N, P_+$ of the following form

$$P_- = \{(x,t) : t > 0 \quad \text{and} \quad x_2 < \nu_- t\} \tag{12}$$

$$P_+ = \{(x,t) : t > 0 \quad \text{and} \quad x_2 > \nu_+ t\} \tag{13}$$

$$P_i = \{(x,t) : t > 0 \quad \text{and} \quad \nu_{i-1} t < x_2 < \nu_i t\} \tag{14}$$

where $\nu_- = \nu_0 < \nu_1 < \ldots < \nu_N = \nu_+$ is an arbitrary collection of real numbers.

Definition 5.2 (*Fan Compressible* subsolutions)**.** A *fan subsolution* to the compressible Euler equations (1) with initial data (7) is a triple $(\overline{\rho}, \overline{v}, \overline{u}) : \mathbb{R}^2\times]0,\infty[\to (\mathbb{R}^+, \mathbb{R}^2, \mathcal{S}_0^{2\times 2})$ of piecewise constant functions satisfying the following requirements.

(i) There is a fan partition $P_-, P_1, \ldots, P_N, P_+$ of $\mathbb{R}^2\times]0,\infty[$ such that

$$(\overline{\rho}, \overline{v}, \overline{u}) = \sum_{i=1}^{N} (\rho_i, v_i, u_i)\mathbf{1}_{P_i} + (\rho_-, v_-, u_-)\mathbf{1}_{P_-} + (\rho_+, v_+, u_+)\mathbf{1}_{P_+}$$

where ρ_i, v_i, u_i are constants with $\rho_i > 0$ and $u_\pm = v_\pm \otimes v_\pm - \frac{1}{2}|v_\pm|^2 \mathrm{Id}$;

(ii) For every $i \in \{1, \ldots, N\}$ there exists a positive constant C_i such that

$$v_i \otimes v_i - u_i < \frac{C_i}{2} \mathrm{Id}\,. \tag{15}$$

(iii) The triple $(\overline{\rho}, \overline{v}, \overline{u})$ solves the following system in the sense of distributions:

$$\partial_t \overline{\rho} + \mathrm{div}_x(\overline{\rho}\,\overline{v}) \;=\; 0 \tag{16}$$

$$\partial_t(\overline{\rho}\,\overline{v}) + \mathrm{div}_x\,(\overline{\rho}\,\overline{u}) + \nabla_x \left(p(\overline{\rho}) + \frac{1}{2}\left(\sum_i C_i \rho_i \mathbf{1}_{P_i} + \overline{\rho}|\overline{v}|^2 \mathbf{1}_{P_+ \cup P_-} \right)\right) = 0 \tag{17}$$

Definition 5.3 (Admissible fan subsolutions)**.** A fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ is said to be *admissible* if it satisfies the following inequality in the sense of distributions

$$\partial_t\left(\overline{\rho}\varepsilon(\overline{\rho})\right) + \mathrm{div}_x\left[(\overline{\rho}\varepsilon(\overline{\rho}) + p(\overline{\rho}))\,\overline{v}\right] + \partial_t\left(\overline{\rho}\frac{|\overline{v}|^2}{2}\mathbf{1}_{P_+\cup P_-}\right) + \mathrm{div}_x\left(\overline{\rho}\frac{|\overline{v}|^2}{2}\overline{v}\mathbf{1}_{P_+\cup P_-}\right)$$
$$+ \sum_{i=1}^{N}\left[\partial_t\left(\rho_i\,\frac{C_i}{2}\,\mathbf{1}_{P_i}\right) + \mathrm{div}_x\left(\rho_i\,\overline{v}\,\frac{C_i}{2}\,\mathbf{1}_{P_i}\right)\right] \;\leq\; 0\,. \tag{18}$$

The discussion of the previous section can then be summarized in the following proposition.

Proposition 5.4. *Let p be any C^1 function and $(\rho_\pm, v_\pm)$ be such that there exists at least one admissible fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ of (1) with initial data (7). Then there are infinitely many bounded admissible solutions (ρ, v) to (1)-(7) such that $\rho = \overline{\rho}$.*

6. **The algebra.** As already mentioned, the various conditions given in the above definitions can be easily reduced to Rankine-Hugoniot conditions on the (flat) interfaces dividing the various regions. As shown in [6] it suffices consider fan subsolutions with a fan partition consisting of only three sets, namely P_-, P_1 and P_+: this rather restrictive assumption is already enough to show that subsolutions exist.

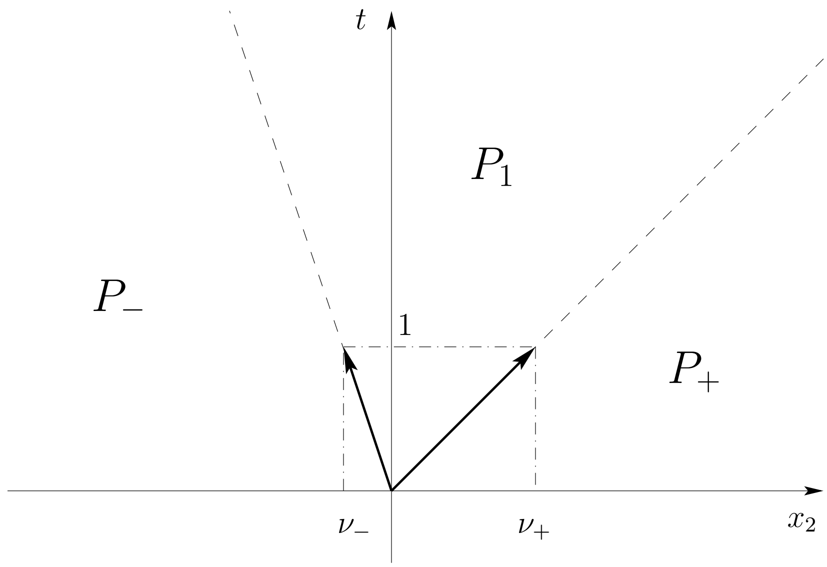

FIGURE 2. The fan partition in three regions.

We introduce therefore the real numbers $\alpha, \beta, \gamma, \delta, v_{-1}, v_{-2}, v_{+1}, v_{+2}$ such that

$$v_1 = (\alpha, \beta), \tag{19}$$
$$v_- = (v_{-1}, v_{-2}) \tag{20}$$
$$v_+ = (v_{+1}, v_{+2}) \tag{21}$$
$$u_1 = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix}. \tag{22}$$

We are now ready to report the algebraic conditions that such numbers must satisfy and which correspond to Proposition 5.1 in [6].

Proposition 6.1. *Let $N = 1$ and P_-, P_1, P_+ be a fan partition as in Definition 5.1. The constants $v_1, v_-, v_+, u_1, \rho_-, \rho_+, \rho_1$ as in (19)-(22) define an admissible fan subsolution as in Definitions 5.2-5.3 if and only if the following identities and inequalities hold:*

- *Rankine-Hugoniot conditions on the left interface:*

$$\nu_-(\rho_- - \rho_1) = \rho_- v_{-2} - \rho_1 \beta \tag{23}$$
$$\nu_-(\rho_- v_{-1} - \rho_1 \alpha) = \rho_- v_{-1} v_{-2} - \rho_1 \delta \tag{24}$$
$$\nu_-(\rho_- v_{-2} - \rho_1 \beta) = \rho_- v_{-2}^2 + \rho_1 \gamma + p(\rho_-) - p(\rho_1) - \rho_1 \frac{C_1}{2}\,; \tag{25}$$

- *Rankine-Hugoniot conditions on the right interface:*

$$\nu_+(\rho_1 - \rho_+) = \rho_1 \beta - \rho_+ v_{+2} \tag{26}$$
$$\nu_+(\rho_1 \alpha - \rho_+ v_{+1}) = \rho_1 \delta - \rho_+ v_{+1} v_{+2} \tag{27}$$
$$\nu_+(\rho_1 \beta - \rho_+ v_{+2}) = -\rho_1 \gamma - \rho_+ v_{+2}^2 + p(\rho_1) - p(\rho_+) + \rho_1 \frac{C_1}{2}\,; \tag{28}$$

- *Subsolution condition:*

$$\alpha^2 + \beta^2 < C_1 \tag{29}$$

$$\left(\frac{C_1}{2} - \alpha^2 + \gamma\right)\left(\frac{C_1}{2} - \beta^2 - \gamma\right) - (\delta - \alpha\beta)^2 > 0\,; \tag{30}$$

- *Admissibility condition on the left interface:*

$$\nu_-(\rho_-\varepsilon(\rho_-) - \rho_1\varepsilon(\rho_1)) + \nu_-\left(\rho_-\frac{|v_-|^2}{2} - \rho_1\frac{C_1}{2}\right)$$

$$\leq [(\rho_-\varepsilon(\rho_-) + p(\rho_-))v_{-2} - (\rho_1\varepsilon(\rho_1) + p(\rho_1))\beta] + \left(\rho_- v_{-2}\frac{|v_-|^2}{2} - \rho_1\beta\frac{C_1}{2}\right)\ ; \tag{31}$$

- *Admissibility condition on the right interface:*

$$\nu_+(\rho_1\varepsilon(\rho_1) - \rho_+\varepsilon(\rho_+)) + \nu_+\left(\rho_1\frac{C_1}{2} - \rho_+\frac{|v_+|^2}{2}\right)$$

$$\leq [(\rho_1\varepsilon(\rho_1) + p(\rho_1))\beta - (\rho_+\varepsilon(\rho_+) + p(\rho_+))v_{+2}] + \left(\rho_1\beta\frac{C_1}{2} - \rho_+ v_{+2}\frac{|v_+|^2}{2}\right)\,. \tag{32}$$

Although there seems to be an abundance of constants satisfying the requirements of the proposition above, it has proved rather difficult to find an efficient way of finding them. A large portion of the paper [6] is spent to give two different methods to generate *some* constants fulfilling the inequalities and identities (23)-(32).

7. **Specific solutions.** The first of these methods makes the specific choice $p(\rho) = \rho^2$. It is with this specific pressure law that we reach Theorem 2.6 and hence our main result Theorem 1.1. More precisely we show that there are constants satisfying the requirements of Proposition 6.1 for which, in addition, the initial data (7) is generated by a compression wave. Such data are in fact easy to characterize, following classical computations.

Lemma 7.1. *Let* $0 < \rho_- < \rho_+$, $v_+ = (-\frac{1}{\rho_+}, 0)$ *and* $v_- = (-\frac{1}{\rho_+}, 2\sqrt{2}(\sqrt{\rho_+} - \sqrt{\rho_-}))$. *Then there is a pair* $(\rho, v) \in W^{1,\infty}_{loc} \cap L^\infty(\mathbb{R}^2 \times]-\infty, 0[, \mathbb{R}^+ \times \mathbb{R}^2)$ *such that*

(i) $\rho_+ \geq \rho \geq \rho_- > 0$;

(ii) The pair solves the hyperbolic system

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho v) \;=\; 0 \\ \\ \partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] \;=\; 0 \end{cases} \tag{33}$$

with $p(\rho) = \rho^2$ *in the classical sense (pointwise a.e. and distributionally);*

(iii) for $t \uparrow 0$ *the pair* $(\rho(\cdot, t), v(\cdot, t))$ *converges pointwise a.e. to* (ρ^0, v^0) *as in* (7);

(iv) $(\rho(\cdot, t), v(\cdot, t)) \in W^{1,\infty}$ *for every* $t < 0$.

A clever choice of some of the constants combined with some careful algebraic computations show then the following

Lemma 7.2. *Let* $p(\rho) = \rho^2$. *There exist* $\rho_\pm, v_\pm$ *satisfying the assumptions of Lemma 7.1 and* $\rho_1, C_1, v_1, u_1, \nu_\pm$ *satisfying the algebraic identities and inequalities* (23)-(32).

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