The h-principle and Onsager's conjecture

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1 Introduction

The h-principle is a concept introduced by Gromov which pertains to various problems in differential geometry, where one expects high flexibility of the moduli spaces of solutions due to the high-dimensionality (or underdetermined nature) of the problem. Interestingly, in some cases a form of the h-principle holds even for systems of partial differential relations which are, formally, not underdetermined.

Perhaps the most famous instance is the Nash-Kuiper theorem on C^1 isometric Euclidean embeddings of n-dimensional Riemannian manifolds. In the classical situation of embedding (two-dimensional) surfaces in three-space, the resulting maps comprise three unknown functions that must satisfy a system of three independent partial differential equations. This is a determined system and, indeed, sufficiently regular solutions satisfy additional constraints (C^2, i.e. continuous second order derivatives, suffices). The oldest example of such a constraint is the Theorema Egregium of Gauss: the determinant of the differential of the Gauss map (a-priori an "extrinsic" quantity) equals a function which can be computed directly from the metric, i.e. the intrinsic Gauss curvature of the original surface.

At a global level there are much more restrictive consequences: for instance any (C^2) isometric embedding u of the standard 2-sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ can be extend in a unique way to an isometry of $\mathbb{R}^3$ and must therefore map $\mathbb{S}^2$ affinely onto the boundary of a unitary ball. In other words C^2 isometric embeddings of the standard 2-sphere in $\mathbb{R}^3$ are rigid; in fact the same holds for any metric on the 2-sphere which has positive Gauss curvature.

Nevertheless, the outcome of the Nash-Kuiper theorem is that C^1 solutions are very flexible and all forms of the aforementioned rigidity are lost. In a sense, in this situation low regularity serves as a replacement for high-dimensionality.

A similar phenomenon has been found recently for solutions of a very classical system of partial differential equations in mathematical physics: the Euler equations for ideal incompressible fluids. Regular (C^1) solutions of this system are determined by the boundary and initial data, whereas continuous solutions are not unique and might even violate the law of conservation of kinetic energy. Although at a rigorous mathematical level this was proved only recently, the latter phenomenon was predicted in 1949 by Lars Onsager in his famous note [41] about statistical hydrodynamics. Onsager conjectured a threshold regularity for the conservation of the kinetic energy. The conjecture is still open and the threshold has deep connections with the Kolmogorov's theory of fully developed turbulence.

In this brief note we will first review the isometric embedding problem, emphasizing the h-principle aspects. We will then turn to some "h-principle-type statements" in the theory of differential inclusions, proved in the last three decades by several authors. These results were developed independently of Gromov's work, but a fruitful relation was pointed out in a groundbreaking paper by Müller and Šverak fifteen years ago, see [38]. There is however a fundamental difference: in differential geometry the h-principle results are in the "C^0 category", whereas the corresponding statements in the theory of differential inclusions hold in the "L^∞ category". Indeed "L^∞ h-principle statements" in differential geometry are usually trivial, whereas "C^0 h-principle statements" in the theory of differential inclusions are usually false. Surprisingly both aspects are present and nontrivial when dealing with solutions of the incompressible Euler equations: the last two sections of this note will be devoted to them.

2 Nash and the isometric embedding problem

Let M^n be a smooth compact manifold of dimension $n \geq 2$, equipped with a Riemannian metric g. An isometric embedding of (M^n, g) into $\mathbb{R}^m$ is a continuous map u which preserves the length of curves. Obviously this implies that u is a Lipschitz homeomorphism of M and $u(M)$. For $u \in C^1$ the length-preserving condition amounts, in local coordinates, to the system

$$\partial_i u \cdot \partial_j u = g_{ij} . \tag{1}$$

consisting of $n(n+1)/2$ equations in m unknowns. The system obviously guarantees that any C^1 solution is an immersion: the property of being an embedding is than simply equivalent to the injectivity of the map u. We will therefore use the term "isometric immersion" for C^1 solutions of (1) which are not necessarily injective.

We note in passing that one may also study "weak" solutions of the system (1). Recall that a Lipschitz mapping $u : M \to \mathbb{R}^m$ is, in virtue of the classical Rademacher theorem, differentiable almost everywhere. Then, we say that $u \in Lip$ is a weak isometry if (1) holds almost everywhere on M. However, being a weak isometry does not imply that the length of curves is preserved. As pointed out by Gromov in [29], such a map may – and in fact, generically will (see [36]) – contract whole submanifolds of M into single points.

Before the fundamental works of Nash in the fifties, only the existence of local analytic embeddings for analytic metrics was known (for $m = \frac{n(n+1)}{2}$), see [32] and [11]. Assuming for the moment that $g \in C^\infty$, the pioneering ideas introduced by Nash culminated into two "classical" theorems concerning the (global!) solvability of (1):

Theorem 1 (Nash [40], Gromov [29]). *Let $m \geq (n+2)(n+3)/2$ and $v : M \to \mathbb{R}^m$ be a* short *embedding (resp. immersion) of M, i.e. a C^1 embedding (resp. immersion) satisfying the inequality $\partial_i v \cdot \partial_j v \leq g_{ij}$ in the sense of quadratic forms. Then v can be uniformly approximated by isometric embeddings (resp. immersions) of class C^∞.*

Theorem 2 (Nash [39], Kuiper [37]). *If $m \geq n+1$, then any short embedding (resp. immersion) can be uniformly approximated by isometric embeddings (resp. immersions) of class C^1.*

Theorems 1 and 2 are not merely existence theorems, they show that there exists a huge (essentially C^0-dense) set of solutions. This type of abundance of solutions is a central aspect of Gromov's h-principle. Naively, such "flexibility" could be expected for high codimension as in Theorem 1, since then there are many more unknowns than equations in (1). The h-principle for C^1 isometric embeddings is on the other hand rather striking, especially when compared to the classical rigidity result concerning the Weyl problem (see [43] for a thorough discussion):

Theorem 3 (Cohn Vossen [14], Herglotz [30]). *If $(\mathbb{S}^2, g)$ is a compact Riemannian surface with positive Gauss curvature and $u \in C^2$ is an isometric immersion into $\mathbb{R}^3$, then $u(M)$ is uniquely determined up to a rigid motion.*

It is intuitively clear that weak (i.e. Lipschitz) isometries cannot enjoy any rigidity property of this type. One can think for instance of folding a piece of paper. The folding preserves length, is therefore isometric, but the resulting map is clearly not C^1: the tangent vector is not continuous across folds. The difficulty of the Nash-Kuiper theorem is precisely to obtain a continuous tangent vector, and this requires a complicated "high dimensional" construction.

Thus it is clear that isometric immersions have a completely different qualitative behaviour at low and high regularity (i.e. below and above C^2).

Both Theorems 1 and 2 make use of a certain extra freedom or "extra dimensions" in the problem. The proof of Theorem 1 relies on the Nash-Moser implicit function theorem and yields solutions which are not only isometric but also *free* - the $n + n(n + 1)/2$ vectors of first and second partial derivatives of the map u are linearly independent in $\mathbb{R}^m$ at each point x. The presence of "extra dimensions" in the proof of Theorem 2 is more subtle and manifests itself as low regularity. Naively, one might think of low regularity in this context as having a large number of active Fourier modes.

The iteration technique in the proof of Theorem 2, called *convex integration*, was subsequently developed by Gromov [28, 29] into a very powerful and very general tool to prove the h-principle in a wide variety of geometric-topological problems (see also [25, 45]). In such situations typically the sought-after solution must satisfy a pointwise *inequality* rather than an equality. An example is to find n divergence free vectorfields on a parallelizable n-dimensional manifold which are linearly independent at any point - the inequality here arises from the pointwise linear independence. Convex integration in this context is essentially a homotopic-theoretic method. In contrast, for *equalities* there is no general method except in certain cases (so-called ample relations), which do not include Theorem 2 or the applications to fluid mechanics below.

In general the regularity of solutions obtained using convex integration (for ample relations) agrees with the highest derivatives appearing in the equations (see [44]). An interesting question raised in [29, p. 219] is how one could extend convex integration to produce more regular solutions. Essentially the same question, in the case of isometric embeddings, is also mentioned as Problem 27 in [46]. In the latter context, for high codimension this was resolved by Källen in [33]. In codimension 1 the problem was first considered by Borisov who in [6] announced that, if g is analytic, then the h-principle holds for local isometric embeddings $u \in C^{1,\alpha}$ with $\alpha < \frac{1}{1+n+n^2}$ ($C^{k,\alpha}$ is the usual notation for spaces of C^k maps u such that each partial derivative w of order k is Hölder continuous with exponent α, namely satisfies the bound $|w(x) - w(y)| \leq C(d(x, y))^\alpha$, where d is the Riemannian distance). A proof for the case $n = 2$ appeared in [7] and for a proof in any dimension valid also for C^2 metrics we refer the reader to [16]. Borisov also pointed out that the optimal regularity for rigidity statements is *not* C^2: in particular in a series of papers [1, 2, 3, 4, 5] he showed that Theorem 3 is valid for $C^{1,\alpha}$ isometric immersions u when $\alpha > \frac{2}{3}$ (see [16] for a short proof).

3 The h-principle as a relaxation statement

The h-principle amounts to the vague statement that local constraints do not influence global behaviour. In differential geometry this leads to the fact that certain problems can be solved by purely topological or homotopic-theoretic methods, once the "softness" of the local (differential) constraints has been shown. In turn, this softness of the local constraints can be seen as a kind of relaxation property.

In order to gain some intuition let us again look at the system of partial differential equations (1) with some fixed smooth g. Obviously any sequence of solutions

$$\{u^k\}_k, \qquad u^k : \Omega \to \mathbb{R}^m,$$

of (1) enjoys a uniform bound upon the maximum of $|\partial_i u^k|$ and thus the Arzelà-Ascoli theorem guarantees the uniform convergence, up to subsequences, to some limit map u. The limit u must be Lipschitz and an interesting question is whether from the equations we can recover some better convergence, for instance in the C^1 category. As we have learned from the previous section, this depends on the codimension and the apriori assumptions on the smoothness of the sequence. For instance, for surfaces in 3-space if the metric g has positive curvature and the maps u^k are sufficiently smooth, their images will be (portions of) convex surfaces: this, loosely speaking, amounts to some useful information about second derivatives which will improve the convergence of u^k and result in a limit u with convex image.

If instead we only assume that the sequence u^k consists of approximate solutions, for instance in the sense that

$$\partial_i u^k \cdot \partial_i u^k - g_{ij} \to 0 \text{ uniformly},$$

then even if g has positive curvature and the u^k are smooth, their images will not necessarily be convex. Let us nonetheless see what we can infer about the limit u. Consider a smooth curve $\gamma \subset \Omega$. Then $u^k \circ \gamma$ is a C^1 Euclidean curve. As already noticed, if we denote by $L_e(u^k \circ \gamma)$ the "Euclidean length" and by $L_g(\gamma)$ the length of γ in the Riemannian manifold (Ω, g), then

$$L_e(u^k \circ \gamma) - L_g(\gamma) \to 0. \tag{2}$$

On the other hand the curves $u^k \circ \gamma$ converge uniformly to the (Lipschitz) curve $u \circ \gamma$ and it is well-known that under such type of convergence the length might shrink but cannot increase. In other words we conclude that

$$L_e(u \circ \gamma) \leq L_g(\gamma). \tag{3}$$

Recall that, by Rademacher's theorem, u is differentiable almost everywhere: it is a simple exercise to see that, when (2) holds for every curve γ in Ω, then

$$\partial_i u \cdot \partial_j u \leq g_{ij} \qquad \text{a.e. in } \Omega, \tag{4}$$

(as above, the latter inequality should be understood in the sense of quadratic forms).

Thus, loosely speaking, one possible interpretation of the Theorems 1-2 is that the system of partial differential inequalities (4) is the "relaxation" of (1) (resp. in the C^∞ and C^1 category) with respect to the C^0 topology. In order to explain this better, let us simplify the situation further, and consider the case $\Omega \subset \mathbb{R}^n$ with the flat metric $g_{ij} = \delta_{ij}$, to be embedded isometrically into $\mathbb{R}^m$. Then the system (1) is equivalent to the condition that the full matrix derivative $Du(x)$ is a linear isometry at every point x, i.e. that

$$Du(x) \in O(n, m) \tag{5}$$

for every x. Note also that the inequality (4) is similarly equivalent to

$$Du(x) \in \operatorname{co} O(n, m), \tag{6}$$

where, for a compact set K we denote by co K its convex hull.

More generally, given a compact set of matrices $K \subset \mathbb{R}^{m\times n}$ one considers the differential inclusion

$$Du(x) \in K \tag{7}$$

and its relaxation - the latter may be given by the convex hull co K, but it might also be a strictly smaller set. The local aspect of the h-principle amounts to the statement that solutions of the original inclusion (7) are dense in C^0 in the potentially much larger relaxation.

This might seem very surprising, but consider the following one-dimensional problem, i.e. the case $n = m = 1$. Thus, setting $\Omega = [0, 1]$, we are looking at the inclusion problem $u'(x) \in \{-1, 1\}$. Of course C^1 solutions need to have constant derivative ± 1, but Lipschitz solutions may be rather wild. In fact, it is not difficult to show that the closure in C^0 of the set

$$\mathcal{S} := \{u \in \mathrm{Lip}[0, 1] : |u'| = 1 \text{ a.e. }\}$$

coincides with the convex hull

$$\mathcal{R} := \{u \in \mathrm{Lip}[0, 1] : |u'| \leq 1 \text{ a.e. }\}.$$

Since the topology of uniform convergence in this setting (uniform Lipschitz bound) is equivalent to weak* convergence of the derivative in L^∞, the latter statement can be interpreted as a form of the Krein-Milman theorem. Moreover, it was observed in [12] that $\mathcal{R} \setminus \mathcal{S}$ is a meager set in the Baire Category sense, cf. also [8].

For general differential inclusions with $m, n \geq 2$, the situation is more complicated, but there is - as a rule of thumb - a kind of dichotomy, depending on the set K: either

- (a) one has a large relaxation and a Krein-Milman type result as above, or
- (b) one has rigidity (and essentially "no relaxation").

These two situations have been studied in detail in the context of nonlinear elasticity (see [17, 38, 34, 35]. Case (a) can be interpreted as a form of the h-principle, albeit a weak form, as in general the solutions to the corresponding problem (7) will be Lipschitz but not necessarily C^1.

As discussed above, in the weak isometric map problem (i.e. the case where $K = O(n, m)$) solutions can be intuitively constructed by folding (see also [18, 19]): such maps have an altogether different structure from the Nash-Kuiper C^1 solutions. In this example the existence of many Lipschitz solutions is not as surprising as Theorem 2. Next, we discuss the Euler equations, where already a weak form of the h-principle is rather striking.

4 The Euler equations as a differential inclusion

The incompressible Euler equations is perhaps the oldest system of partial differential equations in fluid dynamics and it was derived by Euler more than 250 years ago. The unknowns are the velocity v of the fluid and the (mechanical) pressure field p. Both of them depend upon a space variable x (ranging in some domain Ω of $\mathbb{R}^2$ and $\mathbb{R}^3$, or in the periodic tori $\mathbb{T}^2$, $\mathbb{T}^3$) and a time variable t. The system can be written as follows:

$$\begin{cases} \partial_t v + \operatorname{div}_x (v \otimes v) + \nabla p = 0, \\ \operatorname{div}_x v = 0\,. \end{cases} \tag{8}$$

For C^1 functions v the nonlinearity $\operatorname{div}_x(v \otimes v)$ equals the advective derivative $(v \cdot \nabla)v$, which in components is expressed as

$$[(v \cdot \nabla)\, v]_i = \sum_j v_j \partial_j v_i\,.$$

The reader familiar with the theory of distributions will recognize that, after writing the Euler equations as in (8), we can naturally introduce a concept of weak solutions as soon as v is a square summable function. We refrain however from defining such "distributional solutions" formally. Rather, we describe a possible route to such concept.

Assume for the moment that the pair (v, p) is smooth and satisfies (8). Consider a "fluid element", namely a region $U \subset\subset \Omega$ with smooth boundary ∂U. If we integrate the second equation of (8) in the space variable and use the divergence theorem we achieve

$$\int_{\partial U} v(x, t) \cdot n(x)\, dS(x) = 0\,, \tag{9}$$

where n denotes the outward unit normal to ∂U (and we use the notation $\int_\Sigma f(x)\, dS(x)$ for surface integrals). If we instead integrate the first equation on $U \times [a, b]$ we then achieve

$$\int_a^b \left[\int_{\partial U} \big((v(x,t) \cdot n(x)) v(x,t) + p(x,t) n(x)\big)\, dS(x) \right] dt$$
$$= \int_U (v(x, a) - v(x, b))\, dx. \tag{10}$$

Both identities make perfect sense when (v, p) are merely continuous functions and express the balance of mass and momentum for the portion of the fluid which occupies the region U. (9) simply expresses the conservation of mass, since it requires that the amount of fluid particles leaving U balances that of particles entering U. (10) expresses the variation of the momentum, which can change only for two reasons:

- particle fluids leave the region U, carrying different momenta compared to those entering;
- the fluid occupying the "external regions", namely the complement of the fluid element U, exerts a force on the portion occupying U; such force is directed along the unit normal to the boundary ∂U at it is proportional to the mechanical pressure p.

It is not difficult to see that *continuous* (v, p) are distributional solutions of (8) if and only if the identities (9) and (10) hold for every smooth $U \subset\subset \Omega$. However the weak formulation through (9)-(10) is very natural and interesting per se: it is indeed common to derive the equations governing a continuous system by first considering the laws of conservation of motion in different regions. The corresponding partial differential equations are then derived following a process which is the reverse of the one outlined above.

Finally, if we wish to abandon the requirement that (v, p) is continuous, general distributional solutions can be suitably characterized as maps satisfying (9) and (10) for "almost all" fluid elements U.

The system (8) is, for classical C^1 solutions, deterministic: when supplied with appropriate boundary conditions such solutions are unique. The most common condition (when the space domain $\Omega = \mathbb{R}^2, \mathbb{R}^3, \mathbb{T}^2, \mathbb{T}^3$ and the space-time domain is $\Omega \times [0, T]$)), is the initial value

$$v(\cdot, 0) = v_0 .$$

Classical solutions are then uniquely determined by the initial data v_0 (but in the cases $\Omega = \mathbb{R}^2, \mathbb{R}^3$ some additional assumption upon the decay at spatial infinity is needed: $v(\cdot, t) \in L^2$ is the most natural one and it is sufficient).

Surprisingly, Scheffer proved in [42] that the situation is completely different for irregular weak solutions.

Theorem 4 (Scheffer 1993)**.** *There is a nontrivial, compactly supported $v \in L^2(\mathbb{R}^2 \times \mathbb{R})$ which solves* (8) *in the sense of distributions.*

In [22] we have shown that the latter theorem can be derived very naturally as a corollary of a suitable h-principle statement, or relaxation result, in the spirit of the previous section. There are several powerful general versions of such statement, which restrict severely the natural attempts to give a definition of "admissible weak solutions" enjoying uniqueness, see for instance [23]. To keep our discussion as simple as possible, we restrict here to a rather easy version. But we first need to introduce the system of "partial differential inequalities" which is the appropriate relaxation of (8).

Definition 5 (Subsolutions)**.** Let $\bar{e} \in C^\infty \cap L^1(\mathbb{R}^n \times \mathbb{R})$ with $\bar{e} \geq 0$. A triple of smooth compactly supported functions

$$(v, u, q) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R}$$

is a subsolution of (8) with energy density $\bar{e}$ if the following properties hold:

(i) u takes values in the subspace of symmetric trace-free matrices and $\operatorname{spt}(v, u, q) \subset \operatorname{spt}(\bar{e})$;

(ii) (v, u, q) solves

$$\left\{ \begin{array}{l} \partial_t v + \operatorname{div} u + \nabla q = 0 \\ \operatorname{div} v = 0, \end{array} \right. \tag{11}$$

(iii) the following inequality holds, in the sense of quadratic forms, on the set $\{\bar{e} > 0\}$

$$v_i v_j - u_{ij} < \tfrac{2}{n} \bar{e} \delta_{ij} . \tag{12}$$

Theorem 6 ([22])**.** *Let $\bar{e} \in C^\infty \cap L^1(\mathbb{R}^n \times \mathbb{R})$ and $(\bar{v}, \bar{u}, \bar{q})$ be a subsolution with kinetic energy $\bar{e}$. Then there exists a sequence of bounded weak solutions (v^k, p^k) of* (8) *on $\mathbb{R}^n \times \mathbb{R}$ such that*

$$\tfrac{1}{2}|v^k|^2 = \bar{e} \qquad \textit{almost everywhere} \tag{13}$$

and $v^k \to v$ weakly in L^2.

The analogy with the aforementioned results in the theory of differential inclusions is rather striking. But, perhaps more surprisingly, in the case of the Euler equations a similar statement can be proved in the C^0 category.

Theorem 7 ([24])**.** *Let $E \in C^\infty([0, T])$ with $E > 0$. Then there exists a sequence of continuous weak solutions (v^k, p^k) of* (8) *on $\mathbb{T}^3 \times [0, T]$ such that*

$$\frac{1}{2} \int |v^k|^2(x, t)\, dx = \overline{E}(t) \qquad \textit{fotr every } t \tag{14}$$

and $v^k \to 0$ weakly in L^2.

In fact, it is possible to extend the preceeding theorem and produce sequences which converge not to zero but to a vectorfield v from a certain class of "subsolutions", as in Theorem 6. However, presently there is no full characterization of the corresponding "relaxed problem" (for some results in this direction see however [20]) . Note also that Theorem 7 remains valid in two space dimensions as well (see [13]).

From Theorem 7 we conclude that continuous solutions of the Euler equations do not necessarily preserve the kinetic energy. This phenomenon was in fact predicted long ago by Lars Onsager and we will discuss it in the next section.

5 Onsager's conjecture and continuous dissipative solutions

One of the fundamental problems in the theory of turbulence is to find a satisfactory mathematical framework linking the basic continuum equations of fluid motion to the highly chaotic, apparently random behaviour of fully developed turbulent flows. Consider the incompressible Navier-Stokes equations

$$\left\{ \begin{array}{l} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \mu \Delta v \\ \\ \operatorname{div} v = 0 , \end{array} \right. \tag{15}$$

describing the motion of an incompressible viscous fluid. The coefficient $\mu > 0$ is the viscosity, which, after appropriate non-dimensionalizing, equals the reciprocal of the Reynolds number Re. As μ becomes smaller (more precisely, the Reynolds number becomes larger), the observed motion becomes more and more complex, at some stage becoming chaotic. The statistical theory of turbulence, whose foundations were laid by Kolmogorov in 1941, aims to describe universal patterns in this chaotic, turbulent flow sufficiently far away from the domain boundaries, by postulating that generic flows can be seen as realizations of random fields, and by using the symmetry and scaling properties of the Navier-Stokes equations; we refer the reader to [27].

One of the cornerstones of the theory is the famous Kolmogorov-Obukhov 5/3 law. It states that the energy spectrum $E(k)$, defined to be the kinetic energy per unit mass and unit wavenumber, behaves like a power law

$$E(k) \sim k^{-5/3}. \tag{16}$$

This power law, which is supposed to be valid in a certain intermediate range of wave numbers k - called the inertial range -, away from the large scales (affected by the boundaries of the domain and external forces) and away from the very small scales (affected by dissipation), agrees remarkably well with experiments and numerical simulations. Closely related to the 5/3 law is the idea of an energy cascade, originally due to Richardson. The energy is introduced at large scales and through nonlinear interaction it cascades to smaller and smaller scales until it is dissipated by the viscosity in the very small scales, cf. [27]. Indeed, a key hypothesis of the K41 theory is that the mean rate of energy dissipation ϵ is strictly positive and independent of μ in the infinite Reynolds number limit ($\mu \to 0$). This effect in turbulent flows is known as *anomalous dissipation*.

Extending the inertial range to infinitely small scales (i.e. $k \to \infty$) corresponds in a certain sense to the limit $\mu \to 0$, when (15) becomes the incompressible Euler equations

$$\begin{cases} \partial_t v + \operatorname{div}_x(v \otimes v) + \nabla p = 0, \\ \operatorname{div}_x v = 0\,. \end{cases} \tag{17}$$

A classical calculation shows that for smooth solutions (v, p) of (17) the kinetic energy is conserved

$$\int |v(x,t)|^2\,dx = \int |v(x,0)|^2\,dx. \tag{18}$$

Lars Onsager suggested in his famous note [41] the possibility of anomalous dissipation for *weak solutions* of the Euler equations as a consequence of the energy cascade. It is worth emphasizing that, although the K41 theory and the theory of turbulence in general is a *statistical theory*, concerned with ensemble averages of solutions of the Navier-Stokes equations, the suggestion of Onsager turns this into the following "pure PDE" question

Conjecture 8. *For weak solutions (v, p) of (17) with*

$$|v(x,t) - v(y,t)| \le C|x-y|^\theta \qquad \forall x, y, t \tag{19}$$

(where the constant C is independent of x, y, t), we have:
(a) If $\theta > 1/3$, the energy is conserved by any solution, i.e. (18) holds;
(b) For $\theta < 1/3$ there are solutions which do not conserve the energy.

The space of functions satisfying (19) is usually denoted by $L^\infty(0,T;C^\theta(\mathbb{T}^3))$ and belongs naturally to the hierarchy of spaces $L^p(0,T;C^\theta(\mathbb{T}^3))$: a function in the latter space is assumed to satisfy (19) at a.e. t with a time-dependent constant $C(t)$ such that $\int C(t)^p dt < \infty$.

The first part of the conjecture, i.e. assertion (a), has been shown by Eyink in [26], following some original computations of Onsager, and by Constantin, E and Titi in [15]. The proof amounts to giving a rigorous justification of the formal computation leading to (18) and in [15] this is done via a suitable regularization of the equation and a commutator estimate (whereas Onsager's original calculations are based on convergence of Fourier series).

Concerning the second part of the conjecture, clearly the first mathematical statement in that direction is Theorem 4. Theorem 7 showed for the first time rigorously that C^0 solutions can *dissipate* the kinetic energy. The two statements are prototypical of a series of recent results concerning point (b) of Conjecture 8, which take the techniques of [24] as starting point. Therefore, having fixed a certain specific space of functions X, these results can be classified in the following two categories:
(A) There exists a nontrivial weak solution $v \in X$ of (17) with compact support in time.
(B) Given any smooth positive function $E = E(t) > 0$, there exists a weak solution $v \in X$ of (17) with

$$\int_{\mathbb{T}^3} |v(x,t)|^2\,dx = E(t) \quad \forall\, t.$$

Obviously both types lead to non-conservation of energy and would therefore conclude part (b) of Onsager's conjecture if proved for the space $X = L^\infty(0,T;C^{1/3-\epsilon}(\mathbb{T}^3))$ for every $\epsilon > 0$. So far the best results are as follows.

Theorem 9. *Let ϵ be any positive number smaller than $\frac{1}{5}$. Then*
- *Statement (A) is true for $X = L^1(0,T;C^{1/3-\epsilon}(\mathbb{T}^3))$.*
- *Statement (B) is true for $X = L^\infty(0,T;C^{1/5-\epsilon}(\mathbb{T}^3))$.*

Statement (B) has been shown for $X = L^\infty(0,T;C^{1/10-\epsilon})$ in [21], whereas P. Isett in [31] was the first to prove Statement (A) for $X = L^\infty(0,T;C^{1/5-\epsilon})$, thereby reaching the current best "uniform" Hölder exponent for Part (b) of Onsager's conjecture. Subsequently, T. Buckmaster, the two authors and P. Isett proved Statement (B) for $X = L^\infty(0,T;C^{1/5-\epsilon})$ in [9]. Finally, Statement (A) for $X = L^1(0,T;C^{1/3-\epsilon}(\mathbb{T}^3))$ has been proved very recently in [10].

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