# SUPPORTING DEGREES OF MULTI-GRADED LOCAL COHOMOLOGY MODULES

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Abstract. For a finitely generated graded module $M$ over a positively-graded commutative Noetherian ring $R$, the second author established in 1999 some restrictions, which can be formulated in terms of the Castelnuovo regularity of $M$ or the so-called $a^*$-invariant of $M$, on the supporting degrees of a graded-indecomposable graded-injective direct summand, with associated prime ideal containing the irrelevant ideal of $R$, of any term in the minimal graded-injective resolution of $M$. Earlier, in 1995, T. Marley had established connections between finitely graded local cohomology modules of $M$ and local behaviour of $M$ across $\mathrm{Proj}(R)$.

The purpose of this paper is to present some multi-graded analogues of the above-mentioned work.

## 0. Introduction

Very briefly, the purpose of this paper is to explore multi-graded analogues of some results in the algebra of modules, and particularly local cohomology modules, over a commutative Noetherian ring that is graded by the additive semigroup $\mathbb{N}_0$ of non-negative integers.

To describe the results that we plan to generalize, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be such a 'positively-graded' commutative Noetherian ring. Any unexplained notation in this Introduction will be as in Chapters 12 and 13 of our book [4]. In particular, the *injective envelope of a graded $R$-module $M$ will be denoted by ${}^*E(M)$ (see [4, §13.2]), and, for $t \in \mathbb{Z}$, the $t$th shift functor (on the category ${}^*\mathcal{C}(R)$ of all graded $R$-modules and homogeneous $R$-homomorphisms) will be denoted by $(\,\bullet\,)(t)$ (see [4, §12.1]).

Let $\mathbb{N}$ denote the set of positive integers; set $R_+ := \bigoplus_{n \in \mathbb{N}} R_n$, the irrelevant ideal of $R$. For a graded $R$-module $M$ and $\mathfrak{p} \in {}^*\mathrm{Spec}(R)$ (the set of homogeneous prime ideals of $R$), we use $M_{(\mathfrak{p})}$ to denote the homogeneous localization of $M$ at $\mathfrak{p}$. For $i \in \mathbb{N}_0$, the ordinary Bass number $\mu^i(\mathfrak{p}, M)$ is equal to the rank of the homogeneous localization $\left({}^*\mathrm{Ext}^i_R(R/\mathfrak{p}, M)\right)_{(\mathfrak{p})}$ as a (free) module over $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ (see R. Fossum and H.-B. Foxby [6, Corollary 4.9]).

Let $i \in \mathbb{N}_0$, and consider a direct decomposition given by a homogeneous isomorphism

$$ {}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-n_\alpha), $$

for an appropriate family $(\mathfrak{p}_\alpha)_{\alpha \in \Lambda_i}$ of graded prime ideals of $R$ and an appropriate family $(n_\alpha)_{\alpha \in \Lambda_i}$ of integers. (See [4, §13.2].)

Suppose that the graded prime ideal $\mathfrak{p}$ contains the irrelevant ideal $R_+$. In this case, the graded ring $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ is concentrated in degree 0, and its 0th component is a field isomorphic to $k_{R_0}(\mathfrak{p}_0)$, the residue field of the local ring $(R_0)_{\mathfrak{p}_0}$. Thus,

$$ \mu^i(\mathfrak{p}, M) = \dim_{k_{R_0}(\mathfrak{p}_0)} \left({}^*\mathrm{Ext}^i_R(R/\mathfrak{p}, M)\right)_{(\mathfrak{p})} = \sum_{t \in \mathbb{Z}} \dim_{k_{R_0}(\mathfrak{p}_0)} \left(\left({}^*\mathrm{Ext}^i_R(R/\mathfrak{p}, M)\right)_{(\mathfrak{p})}\right)_t. $$

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In [15], it was shown that the graded $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$-module $\left({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)\right)_{(\mathfrak{p})}$ carries information about the shifts '$-n_\alpha$' for those $\alpha \in \Lambda_i$ for which $\mathfrak{p}_\alpha = \mathfrak{p}$. One has

$$ {}^*E(R/\mathfrak{p})(n) \ncong {}^*E(R/\mathfrak{p})(m) \text{ in } {}^*\mathcal{C}(R) \quad \text{for } m, n \in \mathbb{Z} \text{ with } m \neq n, $$

and, for a given $t \in \mathbb{Z}$, the cardinality of the set $\{\alpha \in \Lambda_i : \mathfrak{p}_\alpha = \mathfrak{p} \text{ and } n_\alpha = t\}$ is equal to

$$ \dim_{k_{R_0}(\mathfrak{p}_0)} \left(\left({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)\right)_{(\mathfrak{p})}\right)_t, $$

the dimension of the $t$th component of $\left({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)\right)_{(\mathfrak{p})}$.

Let ${}^*\operatorname{Var}(R_+) := \{\mathfrak{q} \in {}^*\operatorname{Spec}(R) : \mathfrak{q} \supseteq R_+\}$. Let $\mathfrak{p} \in {}^*\operatorname{Var}(R_+)$, let $i \in \mathbb{N}_0$ and let $t \in \mathbb{Z}$. We say that $t$ is an *$i$th level anchor point of* $\mathfrak{p}$ *for* $M$ if

$$ \left(\left({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)\right)_{(\mathfrak{p})}\right)_t \neq 0; $$

the set of all $i$th level anchor points of $\mathfrak{p}$ for $M$ is denoted by $\operatorname{anch}^i(\mathfrak{p}, M)$; also, we write

$$ \operatorname{anch}(\mathfrak{p}, M) = \bigcup_{j \in \mathbb{N}_0} \operatorname{anch}^j(\mathfrak{p}, M), $$

and refer to this as the set of *anchor points of* $\mathfrak{p}$ *for* $M$. Thus $\operatorname{anch}^i(\mathfrak{p}, M)$ is the set of integers $h$ for which, when we decompose

$$ {}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-n_\alpha) $$

by means of a homogeneous isomorphism, there exists $\alpha \in \Lambda_i$ with $\mathfrak{p}_\alpha = \mathfrak{p}$ and $n_\alpha = h$. Note that $\operatorname{anch}^i(\mathfrak{p}, M) = \emptyset$ if $\mu^i(\mathfrak{p}, M) = 0$, and that $\operatorname{anch}^i(\mathfrak{p}, M)$ is a finite set when $M$ is finitely generated.

It was also shown in [15] that, when the graded $R$-module $M$ is non-zero and finitely generated, the Castelnuovo regularity $\operatorname{reg}(M)$ of $M$ is an upper bound for the set

$$ \bigcup_{\mathfrak{p} \in {}^*\operatorname{Var}(R_+)} \operatorname{anch}(\mathfrak{p}, M) $$

of all anchor points of $M$. Consequently, for each $i \geq 0$, every *indecomposable *injective direct summand $F$ of ${}^*E^i(M)$ with associated prime containing $R_+$ must have $F_j = 0$ for all $j > \operatorname{reg}(M)$.

In §§2,3 we shall present an analogue of this theory for a standard multi-graded commutative Noetherian ring $S = \bigoplus_{\mathbf{n} \in \mathbb{N}_0{}^r} S_\mathbf{n}$ (where $r \in \mathbb{N}$ with $r \geq 2$). There is a satisfactory generalization of anchor point theory to the multi-graded case, but we must stress now that we have not uncovered any links between our multi-graded anchor point theory and the fast-developing theory of multi-graded Castelnuovo regularity (see, for example, Huy Tài Hà [9] and D. Maclagan and G. G. Smith [12]). This may be because our multi-graded anchor point theory only yields information about multi-graded local cohomology modules with respect to ${\mathbb{N}_0}^r$-graded ideals of $S$ that contain one of the components $S_{(0,\ldots,0,1,0,\ldots,0)}$, whereas the ideal $S_+ := \bigoplus_{\mathbf{n} \in \mathbb{N}^r} S_\mathbf{n}$, which is relevant to multi-graded Castelnuovo regularity, normally does not have that property.

The short §4 provides some motivation for our work in §5, where we provide multi-graded analogues of work of T. Marley [14] about finitely graded local cohomology modules. We say that a graded $R$-module $L = \bigoplus_{n \in \mathbb{Z}} L_n$ is *finitely graded* precisely when $L_n \neq 0$ for only finitely many $n \in \mathbb{Z}$. In [14], Marley defined, for a finitely generated graded $R$-module $M$,

$$ g_\mathfrak{a}(M) := \sup\left\{k \in \mathbb{N}_0 : H^i_\mathfrak{a}(M) \text{ is finitely graded for all } i < k\right\}, $$

and he modified ideas of N. V. Trung and S. Ikeda in [16, Lemma 2.2] to prove that

$$ g_\mathfrak{a}(M) := \sup\left\{k \in \mathbb{N}_0 : R_+ \subseteq \sqrt{(0 :_R H^i_\mathfrak{a}(M))} \text{ for all } i < k\right\}; $$

he then used Faltings' Annihilator Theorem for local cohomology (see [5] and [4, Theorem 9.5.1]). In §5 below, we shall obtain some multi-graded analogues of some of Marley's results in this area.

## 1. Background results in multi-graded commutative algebra

Let $R = \bigoplus_{g\in G} R_g$ be a commutative Noetherian ring graded by a finitely generated, additively-written, torsion-free Abelian group $G$. Some aspects of the $G$-graded analogue of the theory of Bass numbers have been developed by S. Goto and K.-i. Watanabe [8, §§1.2, 1.3], and it is appropriate for us to review some of those here.

We shall denote by ${}^*\mathcal{C}^G(R)$ (or sometimes by ${}^*\mathcal{C}(R)$ when the grading group $G$ is clear) the category of all $G$-graded $R$-modules and $G$-homogeneous $R$-homomorphisms of degree $0_G$ between them. Projective (respectively injective) objects in the category ${}^*\mathcal{C}^G(R)$ will be referred to as *projective (respectively *injective) $G$-graded $R$-modules. Similarly, the attachment of '*' to other concepts indicates that they refer to the obvious interpretations of those concepts in the category ${}^*\mathcal{C}^G(R)$, although we shall sometimes use '$G$' instead of '*' in order to emphasize the grading group. However, the following comments about ${}^*\operatorname{Hom}_R$ and the ${}^*\operatorname{Ext}^i_R$ $(i \geq 0)$ may be helpful.

1.1. **Reminders.** Let $M = \bigoplus_{g\in G} M_g$ and $N = \bigoplus_{g\in G} N_g$ be $G$-graded $R$-modules.

(i) Let $a \in G$. We say that an $R$-homomorphism $f : M \longrightarrow N$ is *$G$-homogeneous of degree $a$* precisely when $f(M_g) \subseteq N_{g+a}$ for all $g \in G$. Such a $G$-homogeneous homomorphism of degree $0_G$ is simply called *$G$-homogeneous*. We denote by ${}^*\operatorname{Hom}_R(M,N)_a$ the $R_{0_G}$-submodule of $\operatorname{Hom}_R(M,N)$ consisting of all $G$-homogeneous $R$-homomorphisms from $M$ to $N$ of degree $a$. Then the sum $\sum_{a\in G} {}^*\operatorname{Hom}_R(M,N)_a$ is direct, and we set

$$ {}^*\operatorname{Hom}_R(M,N) := \sum_{a\in G} {}^*\operatorname{Hom}_R(M,N)_a = \bigoplus_{a\in G} {}^*\operatorname{Hom}_R(M,N)_a. $$

This is an $R$-submodule of $\operatorname{Hom}_R(M,N)$, and the above direct decomposition provides it with a structure as $G$-graded $R$-module. It is straightforward to check that

$$ {}^*\operatorname{Hom}_R(\,\bullet\,,\,\bullet\,) : {}^*\mathcal{C}^G(R) \times {}^*\mathcal{C}^G(R) \longrightarrow {}^*\mathcal{C}^G(R) $$

is a left exact, additive functor.

(ii) If $M$ is finitely generated, then $\operatorname{Hom}_R(M,N)$ is actually equal to ${}^*\operatorname{Hom}_R(M,N)$ with its $G$-grading forgotten.

(iii) For $i \in \mathbb{N}_0$, the functor ${}^*\operatorname{Ext}^i_R$ is the $i$th right derived functor in ${}^*\mathcal{C}^G(R)$ of ${}^*\operatorname{Hom}_R$. We make two comments here about the case where $M$ is finitely generated. In that case $\operatorname{Ext}^i_R(M,N)$ is actually equal to ${}^*\operatorname{Ext}^i_R(M,N)$ with its $G$-grading forgotten, and, second, one can calculate the ${}^*\operatorname{Ext}^i_R(M,N)$ by applying the functor ${}^*\operatorname{Hom}_R(M,\,\bullet\,)$ to a (deleted) *injective resolution of $N$ in the category ${}^*\mathcal{C}^G(R)$ and then taking cohomology of the resulting complex.

For $a \in G$, we shall denote the *$a$th shift functor* by $(\,\bullet\,)(a) : {}^*\mathcal{C}^G(R) \longrightarrow {}^*\mathcal{C}^G(R)$: thus, for a $G$-graded $R$-module $M = \bigoplus_{g\in G} M_g$, we have $(M(a))_g = M_{g+a}$ for all $g \in G$; also, $f(a)\lceil_{(M(a))_g} = f\lceil_{M_{g+a}}$ for each morphism $f$ in ${}^*\mathcal{C}^G(R)$ and all $g \in G$.

1.2. **Theorem** (S. Goto and K.-i. Watanabe [8, §1.3])**.** *Let $M$ be $G$-graded $R$-module, and denote by ${}^*\operatorname{Spec}(R)$ the set of $G$-graded prime ideals of $R$. We denote by ${}^*E(M)$ or ${}^*E_R(M)$ 'the' *injective envelope of $M$, and by ${}^*E^i(M)$ or ${}^*E^i_R(M)$ 'the' $i$th term in 'the' minimal *injective resolution of $M$ (for each $i \geq 0$).*

(i) $\operatorname{Ass}_R {}^*E_R(M) = \operatorname{Ass}_R M$.

(ii) *We have that $M$ is a *indecomposable *injective $G$-graded $R$-module if and only if $M$ is isomorphic (in the category ${}^*\mathcal{C}^G(R)$) to ${}^*E(R/\mathfrak{q})(a)$ for some $\mathfrak{q} \in {}^*\operatorname{Spec}(R)$ and $a \in G$. In this case, $\operatorname{Ass}_R M = \{\mathfrak{q}\}$ and $\mathfrak{q}$ is uniquely determined by $M$.*

(iii) *Let $(M_\lambda)_{\lambda\in\Lambda}$ be a non-empty family of $G$-graded $R$-modules. Then $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is *injective if and only if $M_\lambda$ is *injective for all $\lambda \in \Lambda$.*

(iv) *Each *injective $G$-graded $R$-module $M$ is a direct sum of *indecomposable *injective $G$-graded submodules, and this decomposition is uniquely determined by $M$ up to isomorphisms.*

(v) *Let $i$ be a non-negative integer. In view of part* (iv) *above, there is a family $(\mathfrak{p}_\alpha)_{\alpha\in\Lambda_i}$ of $G$-graded prime ideals of $R$ and a family $(g_\alpha)_{\alpha\in\Lambda_i}$ of elements of $G$ for which there is a $G$-homogeneous* *isomorphism*

$$ {}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha\in\Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-g_\alpha). $$

*Let $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$. Then the cardinality of the set $\{\alpha \in \Lambda_i : \mathfrak{p}_\alpha = \mathfrak{p}\}$ is equal to the ordinary Bass number $\mu^i(\mathfrak{p}, M)$ (that is, to $\dim_{k(\mathfrak{p})} \operatorname{Ext}^i_R(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p})$, where $k(\mathfrak{p})$ denotes the residue field of the local ring $R_\mathfrak{p}$).*

A significant part of §2 of this paper is concerned with the shifts '$-g_\alpha$' in the statement of part (v) of Theorem 1.2. (The minus signs are inserted for notational convenience.) In [15], the second author obtained some results about such shifts in the special case in which $R$ is graded by the semigroup $\mathbb{N}_0$ of non-negative integers, and in §2 below, we shall establish some multi-graded analogues.

We shall employ the following device used by Huy Tài Hà [9, §2].

1.3. **Definition.** Let $\phi : G \longrightarrow H$ be a homomorphism of finitely generated torsion-free Abelian groups, and let $R = \bigoplus_{g\in G} R_g$ be a $G$-graded commutative Noetherian ring.

For each $h \in H$, set $R^\phi_h := \bigoplus_{g\in\phi^{-1}(\{h\})} R_g$; then

$$ R^\phi := \bigoplus_{h\in H} R^\phi_h = \bigoplus_{h\in H}\left(\bigoplus_{g\in\phi^{-1}(\{h\})} R_g\right) $$

provides an $H$-grading on $R$, and we denote $R$ by $R^\phi$ when considering it as an $H$-graded ring in this way.

Furthermore, for each $G$-graded $R$-module $M = \bigoplus_{g\in G} M_g$, set $M^\phi_h := \bigoplus_{g\in\phi^{-1}(\{h\})} M_g$ and $M^\phi := \bigoplus_{h\in H} M^\phi_h$; then $M^\phi$ is an $H$-graded $R^\phi$-module. Also, if $f : M \longrightarrow N$ is a $G$-homogeneous homomorphism of $G$-graded $R$-modules, then the same map $f$ becomes an $H$-homogeneous homomorphism of $H$-graded $R^\phi$-modules $f^\phi : M^\phi \longrightarrow N^\phi$.

In this way, $(\,\bullet\,)^\phi$ becomes an exact additive covariant functor from ${}^*\mathcal{C}^G(R)$ to ${}^*\mathcal{C}^H(R)$.

1.4. **Notation.** We shall use $\mathbb{N}$ and $\mathbb{N}_0$ to denote the sets of positive and non-negative integers, respectively, and $r$ will denote a fixed positive integer. Throughout the remainder of the paper, $R := \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} R_\mathbf{n}$ will denote a commutative Noetherian ring, graded by the additively-written finitely generated free Abelian group $\mathbb{Z}^r$ (with its usual addition). For $\mathbf{n} = (n_1, \ldots, n_r)$, $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$, we shall write

$$ \mathbf{n} \le \mathbf{m} \quad \text{if and only if} \quad n_i \le m_i \text{ for all } i = 1, \ldots, r; $$

furthermore, $\mathbf{n} < \mathbf{m}$ will mean that $\mathbf{n} \le \mathbf{m}$ and $\mathbf{n} \ne \mathbf{m}$. The zero element of $\mathbb{Z}^r$ will be denoted by $\mathbf{0}$, and, for each $i = 1, \ldots, r$, we shall use $\mathbf{e}_i$ to denote the element of $\mathbb{Z}^r$ which has 1 in the $i$th spot and all other components zero. Also, $\mathbf{1}$ will denote $(1, \ldots, 1) \in \mathbb{Z}^r$. Thus $\mathbf{1} = \sum_{i=1}^r \mathbf{e}_i$, and $R_{\mathbf{e}_1}R_{\mathbf{e}_2}\ldots R_{\mathbf{e}_r} \subseteq R_\mathbf{1}$.

We shall sometimes denote the $i$th component of a general member $\mathbf{w}$ of $\mathbb{Z}^r$ by $w_i$ without additional explanation.

Comments made above that apply to the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$ will be used without further comment. For example, we shall say that a graded ideal of $R$ is **maximal* if it is maximal among the set of proper $\mathbb{Z}^r$-graded ideals of $R$, and that $R$ is **local* if it has a unique *maximal ideal. We shall use ${}^*\operatorname{Max}(R)$ to denote the set of *maximal ideals of $R$.

We shall use ${}^*\operatorname{Spec}(R)$ to denote the set of $\mathbb{Z}^r$-graded prime ideals of $R$; for a $\mathbb{Z}^r$-graded ideal $\mathfrak{a}$ of $R$, we shall set ${}^*\operatorname{Var}(\mathfrak{a}) := \{\mathfrak{p} \in {}^*\operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$.

The next three lemmas are multi-graded analogues of preparatory results in [15, §1].

1.5. **Lemma.** *Let $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$ and let $a$ be an $\mathbb{Z}^r$-homogeneous element of degree $\mathbf{n}$ in $R \setminus \mathfrak{p}$. Then multiplication by $a$ provides a $\mathbb{Z}^r$-homogeneous automorphism of degree $\mathbf{n}$ of ${}^*E(R/\mathfrak{p})$. Also, each element of ${}^*E(R/\mathfrak{p})$ is annihilated by some power of $\mathfrak{p}$.*

*Consequently, if $S$ is a multiplicatively closed subset of ${\mathbb{N}_0}^r$-homogeneous elements of $R$ such that $S \cap \mathfrak{p} \ne \emptyset$, then $S^{-1}\left({}^*E(R/\mathfrak{p})\right) = 0$.*

*Proof.* Multiplication by $a$ provides a $\mathbb{Z}^r$-homogeneous $R$-homomorphism

$$\mu_a : {}^*E(R/\mathfrak{p}) \longrightarrow {}^*E(R/\mathfrak{p})(\mathbf{n}).$$

Since $\operatorname{Ker}\mu_a$ has zero intersection with $R/\mathfrak{p}$, it follows that $\mu_a$ is injective. In view of Theorem 1.2(ii), $\operatorname{Im}\mu_a$ is a non-zero *injective $\mathbb{Z}^r$-graded submodule of the *indecomposable *injective $\mathbb{Z}^r$-graded $R$-module ${}^*E(R/\mathfrak{p})(\mathbf{n})$. Hence $\mu_a$ is surjective.

The fact that each element of ${}^*E(R/\mathfrak{p})$ is annihilated by some power of $\mathfrak{p}$ follows from Theorem 1.2(i), which shows that $\mathfrak{p}$ is the only associated prime ideal of each non-zero cyclic submodule of ${}^*E(R/\mathfrak{p})$. The final claim is then immediate. □

The next two lemmas below can be proved by making obvious modifications to the proofs of the (well-known) 'ungraded' analogues.

1.6. **Lemma.** *Let $f : L \longrightarrow M$ be a $\mathbb{Z}^r$-homogeneous homomorphism of $\mathbb{Z}^r$-graded $R$-modules such that $M$ is a *essential extension of $\operatorname{Im} f$. Let $S$ be a multiplicatively closed subset of $\mathbb{Z}^r$-homogeneous elements of $R$. Then $S^{-1}M$ is a *essential extension of its $\mathbb{Z}^r$-graded submodule $\operatorname{Im}(S^{-1}f)$.*

*Proof.* Modify the proof of [4, 11.1.5] in the obvious way. □

1.7. **Lemma.** *Let $S$ be a multiplicatively closed subset of $\mathbb{Z}^r$-homogeneous elements of $R$, and let $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$ be such that $\mathfrak{p} \cap S = \emptyset$. Then*

(i) *the natural map ${}^*E_R(R/\mathfrak{p}) \longrightarrow S^{-1}({}^*E_R(R/\mathfrak{p}))$ is a $\mathbb{Z}^r$-homogeneous $R$-isomorphism, so that ${}^*E_R(R/\mathfrak{p})$ has a natural structure as a $\mathbb{Z}^r$-graded $S^{-1}R$-module;*

(ii) *there is a $\mathbb{Z}^r$-homogeneous isomorphism (in ${}^*\mathcal{C}(S^{-1}R)$)*

$$ {}^*E_R(R/\mathfrak{p}) \cong {}^*E_{S^{-1}R}(S^{-1}R/S^{-1}\mathfrak{p});$$

(iii) *${}^*E_{S^{-1}R}(S^{-1}R/S^{-1}\mathfrak{p})$, when considered as a $\mathbb{Z}^r$-graded $R$-module by means of the natural homomorphism $R \longrightarrow S^{-1}R$, is $\mathbb{Z}^r$-homogeneously isomorphic to ${}^*E_R(R/\mathfrak{p})$;*

(iv) *for each $\mathbf{n} \in \mathbb{Z}^r$, there is a $\mathbb{Z}^r$-homogeneous isomorphism (in ${}^*\mathcal{C}(S^{-1}R)$)*

$$S^{-1}\big({}^*E_R(R/\mathfrak{p})(\mathbf{n})\big) \cong {}^*E_{S^{-1}R}(S^{-1}R/S^{-1}\mathfrak{p})(\mathbf{n});$$

(v) *if $I$ is a *injective $\mathbb{Z}^r$-graded $R$-module, then the $\mathbb{Z}^r$-graded $S^{-1}R$-module $S^{-1}I$ is *injective.*

*Proof.* (i) This is immediate from 1.5.

(ii) One can make the obvious modifications to the proof of [4, 10.1.11] to see that, as a $\mathbb{Z}^r$-graded $S^{-1}R$-module, ${}^*E_R(R/\mathfrak{p})$ is *injective; it is also $\mathbb{Z}^r$-homogeneously isomorphic, as a $\mathbb{Z}^r$-graded $S^{-1}R$-module, to $S^{-1}({}^*E_R(R/\mathfrak{p}))$. One can use 1.6 to see that $S^{-1}({}^*E_R(R/\mathfrak{p}))$ is a *essential extension of $S^{-1}R/S^{-1}\mathfrak{p}$. The claim follows.

(iii), (iv) These are now easy.

(v) This can now be proved by making the obvious modifications to the proof of [4, 10.1.13(ii)]. □

## 2. A multi-graded analogue of anchor point theory

2.1. **Definition.** We shall say that $R$ is *positively graded* precisely when $R_{\mathbf{n}} = 0$ for all $\mathbf{n} \not\geq \mathbf{0}$. When that is the case, we say that $R$ (as in 1.4) is *standard* precisely when $R = R_{\mathbf{0}}[R_{\mathbf{e}_1}, \ldots, R_{\mathbf{e}_r}]$.

The main results of this paper will concern the case where $R$ is positively graded and standard.

2.2. **Lemma.** *Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard. If $\mathfrak{a}$ is an $\mathbb{N}_0{}^r$-graded ideal of $R$ such that $\mathfrak{a} \supseteq R_{\mathbf{t}}$ for some $\mathbf{t} \in \mathbb{N}_0{}^r$, then $\mathfrak{a} \supseteq R_{\mathbf{n}}$ for each $\mathbf{n} \in \mathbb{N}_0{}^r$ with $\mathbf{n} \geq \mathbf{t}$.*

*Proof.* Since $R$ is standard, $R_{\mathbf{n}} = R_{\mathbf{t}}R_{\mathbf{n}-\mathbf{t}}$, and so is contained in $\mathfrak{a}$. □

2.3. **Definition.** Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard. Let $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$. The set $\left\{j \in \{1, \ldots, r\} : R_{\mathbf{e}_j} \subseteq \mathfrak{p}\right\}$ will be called *the set of $\mathfrak{p}$-directions* and will be denoted by $\operatorname{dir}(\mathfrak{p})$.

Observe that, if $i \in \operatorname{dir}(\mathfrak{p})$, then $\mathfrak{p} \supseteq R_{\mathbf{1}}$ by 2.2. Conversely, if $\mathfrak{p} \supseteq R_{\mathbf{1}}$, then, since $R_{\mathbf{1}} = R_{\mathbf{e}_1} \ldots R_{\mathbf{e}_r}$, there exists $i \in \{1, \ldots, r\}$ such that $R_{\mathbf{e}_i} \subseteq \mathfrak{p}$, and $i \in \operatorname{dir}(\mathfrak{p})$. Thus $\operatorname{dir}(\mathfrak{p}) \neq \emptyset$ if and only if $\mathfrak{p} \supseteq R_{\mathbf{1}}$.

More generally, let $\mathfrak{b}$ be an $\mathbb{N}_0{}^r$-graded ideal of $R$. We define *the set of $\mathfrak{b}$-directions* to be

$$\operatorname{dir}(\mathfrak{b}) := \left\{ j \in \{1, \ldots, r\} : R_{\mathbf{e}_j} \subseteq \sqrt{\mathfrak{b}} \right\}.$$

The members of the set $\{1, \ldots, r\} \setminus \operatorname{dir}(\mathfrak{b})$ are called the *non-$\mathfrak{b}$-directions*. It is easy to see that $\operatorname{dir}(\mathfrak{b}) = \bigcap_{\mathfrak{p} \in \operatorname{Min}(\mathfrak{b})} \operatorname{dir}(\mathfrak{p})$, where $\operatorname{Min}(\mathfrak{b})$ denotes the set of minimal prime ideals of $\mathfrak{b}$.

2.4. *Remark.* It follows from Lemma 2.2 that, in the situation of Definition 2.3, each ${\mathbb{N}_0}^r$-homogeneous element of $R \setminus \mathfrak{p}$ has degree with $i$th component 0 for all $i \in \operatorname{dir}(\mathfrak{p})$.

2.5. **Proposition.** *Suppose that $R := \bigoplus_{\mathbf{n} \in {\mathbb{N}_0}^r} R_{\mathbf{n}}$ is positively graded and standard. Let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$. For notational convenience, suppose that $\operatorname{dir}(\mathfrak{p}) = \{1, \ldots, m\}$, where $0 < m \leq r$. For each $i \in \{1, \ldots, r\} \setminus \operatorname{dir}(\mathfrak{p}) = \{m+1, \ldots, r\}$, select $u_i \in R_{\mathbf{e}_i} \setminus \mathfrak{p}$.*

*Let $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m$. For $\mathbf{c} = (c_{m+1}, \ldots, c_r) \in \mathbb{Z}^{r-m}$, we shall denote by $\mathbf{a}|\mathbf{c}$ the element $(a_1, \ldots, a_m, c_{m+1}, \ldots, c_r)$ of $\mathbb{Z}^r$ obtained by juxtaposition.*

(i) *For all choices of $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^{r-m}$, there is an isomorphism of $R_{\mathbf{0}}$-modules*
$$({}^*E_R(R/\mathfrak{p}))_{\mathbf{a}|\mathbf{c}} \cong ({}^*E_R(R/\mathfrak{p}))_{\mathbf{a}|\mathbf{d}}.$$
*(Note that this does not say anything of interest if $m = r$.)*

(ii) *If $({}^*E_R(R/\mathfrak{p}))_{\mathbf{a}|\mathbf{c}} \neq 0$ for any $\mathbf{c} \in \mathbb{Z}^{r-m}$, then $\mathbf{a} \leq \mathbf{0}$.*

(iii) *Let $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$, where $R_{(\mathfrak{p})}$ is the $\mathbb{Z}^r$-homogeneous localization of $R$ at $\mathfrak{p}$. Then*
  (a) *$T$ is a simple $\mathbb{Z}^r$-graded ring in the sense of* [8, Definition 1.1.1]*;*
  (b) *$T_{\mathbf{0}}$ is a field;*
  (c) *for each $\mathbf{c} = (c_{m+1}, \ldots, c_r) \in \mathbb{Z}^{r-m}$,*
$$T_{\mathbf{a}|\mathbf{c}} = \begin{cases} 0 & \text{if } \mathbf{a} \neq \mathbf{0}, \\ T_{\mathbf{0}}(\overline{u_{m+1}/1})^{c_{m+1}} \ldots (\overline{u_r/1})^{c_r} & \text{if } \mathbf{a} = \mathbf{0} \end{cases}$$
  *(where '$\overline{\phantom{xx}}$' is used to denote natural images of elements of $R_{(\mathfrak{p})}$ in $T$); and*
  (d) *every $\mathbb{Z}^r$-graded $T$-module is free.*

(iv) *We have $(0 :_{{}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})} \mathfrak{p}R_{(\mathfrak{p})}) = R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$.*

(v) *If $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^m$ and $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^{r-m}$, and there is a $\mathbb{Z}^r$-homogeneous isomorphism*
$$({}^*E_R(R/\mathfrak{p}))(\mathbf{a}|\mathbf{c}) \cong ({}^*E_R(R/\mathfrak{p}))(\mathbf{b}|\mathbf{d}),$$
*then $\mathbf{a} = \mathbf{b}$.*

*Note.* The obvious interpretation of the above statement is to be made in the case where $m = r$.

*Proof.* It will be convenient to write $\mathbf{v}$ for a general member of $\mathbb{Z}^m$ and $\mathbf{w}$ for a general member of $\mathbb{Z}^{r-m}$, and to use $\mathbf{v}|\mathbf{w}$ to indicate the element of $\mathbb{Z}^r$ obtained by juxtaposition.

(i) By Lemma 1.5, for each $i = m+1, \ldots, r$, multiplication by $u_i$ provides a $\mathbb{Z}^r$-homogeneous automorphism of ${}^*E_R(R/\mathfrak{p})$ of degree $\mathbf{e}_i$; the claim follows from this.

(ii) Set $\Delta := \{\mathbf{v} \in \mathbb{Z}^m : v_i > 0 \text{ for some } i \in \{1, \ldots, m\}\}$. Since $R_{\mathbf{e}_i} \subseteq \mathfrak{p}$ for all $i = 1, \ldots, m$, it follows from Lemma 2.2 that the $\mathbb{Z}^r$-graded $R$-module $R/\mathfrak{p}$ has $(R/\mathfrak{p})_{\mathbf{v}|\mathbf{w}} = 0$ for all choices of $\mathbf{v}|\mathbf{w} \in \mathbb{Z}^r$ with $\mathbf{v} \in \Delta$. Therefore the $\mathbb{Z}^r$-graded submodule
$$\bigoplus_{\substack{\mathbf{v} \in \Delta \\ \mathbf{w} \in \mathbb{Z}^{r-m}}} (R/\mathfrak{p})_{\mathbf{v}|\mathbf{w}}$$
of $R/\mathfrak{p}$ is zero. Since ${}^*E_R(R/\mathfrak{p})$ is a *essential extension of $R/\mathfrak{p}$, it follows that
$$\bigoplus_{\substack{\mathbf{v} \in \Delta \\ \mathbf{w} \in \mathbb{Z}^{r-m}}} ({}^*E_R(R/\mathfrak{p}))_{\mathbf{v}|\mathbf{w}} = 0.$$

(iii) By Remark 2.4, each ${\mathbb{N}_0}^r$-homogeneous element of $R \setminus \mathfrak{p}$ has degree $\mathbf{v}|\mathbf{w}$ with $\mathbf{v} = \mathbf{0}$. Also, $(R/\mathfrak{p})_{\mathbf{v}|\mathbf{w}} = 0$ for all $\mathbf{v} \in \mathbb{Z}^m$ with $\mathbf{v} > \mathbf{0}$. Now every non-zero $\mathbb{Z}^r$-homogeneous element of $T$ is a unit of $T$, so that $T$ is a simple $\mathbb{Z}^r$-graded ring. Furthermore, the subgroup
$$G := \{\mathbf{n} \in \mathbb{Z}^r : T_{\mathbf{n}} \text{ contains a unit of } T\}$$
is equal to $\{(n_1, \ldots, n_m, n_{m+1}, \ldots, n_r) \in \mathbb{Z}^r : n_1 = \cdots = n_m = 0\}$. The claims in parts (b), (c) and (d) now follow from [8, Lemma 1.1.2, Corollary 1.1.3 and Theorem 1.1.4].

(iv) Recall that $T = R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$. Now the $\mathbb{Z}^r$-graded $T$-module $(0 :_{{}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})} \mathfrak{p}R_{(\mathfrak{p})})$ contains its $\mathbb{Z}^r$-graded $T$-submodule $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$, and cannot be strictly larger, by *essentiality and the fact (see part (iii)) that every $\mathbb{Z}^r$-graded $T$-module is free.

(v) By Lemma 1.7(iv), there is a $\mathbb{Z}^r$-homogeneous isomorphism of $\mathbb{Z}^r$-graded $R_{(\mathfrak{p})}$-modules

$$\left({}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})\right)(\mathbf{a}|\mathbf{c}) \cong \left({}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})\right)(\mathbf{b}|\mathbf{d}).$$

Abbreviate ${}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})$ by $F$. It follows from part (iv) that

$$\begin{aligned} T(\mathbf{a}|\mathbf{c}) &= (0 :_F \mathfrak{p}R_{(\mathfrak{p})})(\mathbf{a}|\mathbf{c}) = (0 :_{F(\mathbf{a}|\mathbf{c})} \mathfrak{p}R_{(\mathfrak{p})}) \\ &\cong (0 :_{F(\mathbf{b}|\mathbf{d})} \mathfrak{p}R_{(\mathfrak{p})}) = (0 :_F \mathfrak{p}R_{(\mathfrak{p})})(\mathbf{b}|\mathbf{d}) \\ &= T(\mathbf{b}|\mathbf{d}), \end{aligned}$$

where the isomorphism is $\mathbb{Z}^r$-homogeneous. But, for $\mathbf{n} = (n_1, \ldots, n_m, n_{m+1}, \ldots, n_r) \in \mathbb{Z}^r$, we have

$$T(\mathbf{a}|\mathbf{c})_{\mathbf{n}} \neq 0 \quad \text{if and only if} \quad (n_1, \ldots, n_m) = -\mathbf{a}$$

(by part (iii)). Therefore $\mathbf{a} = \mathbf{b}$. □

2.6. *Remark.* Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard, and let $\mathfrak{b}$ be an $\mathbb{N}_0{}^r$-graded ideal of $R$ for which $\operatorname{dir}(\mathfrak{b}) \neq \emptyset$.

Write $\operatorname{dir}(\mathfrak{b}) = \{i_1, \ldots, i_m\}$, where $0 < m \leq r$ and $i_1 < \cdots < i_m$. Let $\phi(\mathfrak{b}) : \mathbb{Z}^r \longrightarrow \mathbb{Z}^m$ be the epimorphism of Abelian groups defined by

$$\phi(\mathfrak{b})((n_1, \ldots, n_r)) = (n_{i_1}, \ldots, n_{i_m}) \quad \text{for all } (n_1, \ldots, n_r) \in \mathbb{Z}^r.$$

We can think of $\phi(\mathfrak{b}) : \mathbb{Z}^r \longrightarrow \mathbb{Z}^m$ as the homomorphism which 'forgets the co-ordinates in the non-$\mathfrak{b}$-directions'.

Now let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$. The above defines an Abelian group homomorphism $\phi(\mathfrak{p}) : \mathbb{Z}^r \longrightarrow \mathbb{Z}^{\#\operatorname{dir}(\mathfrak{p})}$. (For a finite set $Y$, the notation $\#Y$ denotes the cardinality of the set $Y$.) In the case where $\mathfrak{b} \subseteq \mathfrak{p}$, we have $\operatorname{dir}(\mathfrak{b}) \subseteq \operatorname{dir}(\mathfrak{p})$, and we define the Abelian group homomorphism $\phi(\mathfrak{p};\mathfrak{b}) : \mathbb{Z}^{\#\operatorname{dir}(\mathfrak{p})} \longrightarrow \mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$ to be the unique $\mathbb{Z}$-homomorphism such that $\phi(\mathfrak{p};\mathfrak{b}) \circ \phi(\mathfrak{p}) = \phi(\mathfrak{b})$.

Now let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$ and $\#\operatorname{dir}(\mathfrak{p}) = m$; we use the notation of 1.3. Let $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$, and let $L$ be a $\mathbb{Z}^r$-graded $T$-module.

(i) By Proposition 2.5(iii), for each $\mathbf{a} \in \mathbb{Z}^m$ and each $\mathbf{n} \in \mathbb{Z}^r$,

$$(T(-\mathbf{n})^{\phi(\mathfrak{p})})_{\mathbf{a}} = \begin{cases} 0 & \text{if } \phi(\mathfrak{p})(\mathbf{n}) \neq \mathbf{a}, \\ (T^{\phi(\mathfrak{p})})_{\mathbf{0}} & \text{if } \phi(\mathfrak{p})(\mathbf{n}) = \mathbf{a}. \end{cases}$$

In particular, the $\mathbb{Z}^m$-graded ring $T^{\phi(\mathfrak{p})}$ is concentrated in degree $\mathbf{0} \in \mathbb{Z}^m$.

(ii) Each component of the $\mathbb{Z}^m$-graded $T^{\phi(\mathfrak{p})}$-module $L^{\phi(\mathfrak{p})}$ is a free $(T^{\phi(\mathfrak{p})})_{\mathbf{0}}$-submodule of $L^{\phi(\mathfrak{p})}$.

(iii) If $L$ is finitely generated, then

$$\operatorname{rank}_{T^{\phi(\mathfrak{p})}} L^{\phi(\mathfrak{p})} = \sum_{\mathbf{a}\in\mathbb{Z}^m} \operatorname{rank}_{(T^{\phi(\mathfrak{p})})_{\mathbf{0}}} \left(L^{\phi(\mathfrak{p})}\right)_{\mathbf{a}};$$

since the left-hand side of the above equation is finite, all except finitely many of the terms on the right-hand side are zero.

2.7. **Theorem.** *Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard. Let $M$ be a $\mathbb{Z}^r$-graded $R$-module, and let*

$$I^\bullet : 0 \longrightarrow {}^*E^0(M) \xrightarrow{d^0} {}^*E^1(M) \longrightarrow \cdots \longrightarrow {}^*E^i(M) \xrightarrow{d^i} {}^*E^{i+1}(M) \longrightarrow \cdots$$

*be the minimal *injective resolution of $M$. For each $i \in \mathbb{N}_0$, let*

$$\theta_i : {}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha\in\Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)$$

*be a $\mathbb{Z}^r$-homogeneous isomorphism, where $\mathfrak{p}_\alpha \in {}^*\operatorname{Spec}(R)$ and $\mathbf{n}_\alpha \in \mathbb{Z}^r$ for all $\alpha \in \Lambda_i$.*

*Let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$ and use the notation $\phi(\mathfrak{p}) : \mathbb{Z}^r \longrightarrow \mathbb{Z}^m$ and $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ of Remark 2.6, where $m$ is the number of $\mathfrak{p}$-directions.*

*Let $i \in \mathbb{N}_0$ and let $\mathbf{a} \in \mathbb{Z}^m$. Then the cardinality of the set $\{\alpha \in \Lambda_i : \mathfrak{p}_\alpha = \mathfrak{p}$ and $\phi(\mathfrak{p})(\mathbf{n}_\alpha) = \mathbf{a}\}$ is equal to*

$$\operatorname{rank}_{(T^{\phi(\mathfrak{p})})_{\mathbf{0}}}\left(\left(\left({}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})\right)^{\phi(\mathfrak{p})}\right)_{\mathbf{a}}\right).$$

*Proof.* By Lemmas 1.5, 1.6 and 1.7, there are $\mathbb{Z}^r$-homogeneous isomorphisms of graded $R_{(\mathfrak{p})}$-modules

$${}^*E^i_{R_{(\mathfrak{p})}}(M_{(\mathfrak{p})}) \cong \left({}^*E^i_R(M)\right)_{(\mathfrak{p})} \cong \bigoplus_{\substack{\alpha\in\Lambda_i\\ \mathfrak{p}_\alpha\subseteq\mathfrak{p}}} {}^*E(R_{(\mathfrak{p})}/\mathfrak{p}_\alpha R_{(\mathfrak{p})})(-\mathbf{n}_\alpha).$$

One can calculate ${}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})$ (up to isomorphism in the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R_{(\mathfrak{p})})$) by taking the $i$th cohomology module of the complex $(0 :_{(I^\bullet)_{(\mathfrak{p})}} \mathfrak{p}R_{(\mathfrak{p})})$. Note that, by Lemma 1.6, for each $j \in \mathbb{N}_0$, the inclusion $\operatorname{Ker}(d^j_{(\mathfrak{p})}) \subseteq {}^*E^j(M)_{(\mathfrak{p})}$ is *essential, so that the inclusion

$$\operatorname{Ker}(d^j_{(\mathfrak{p})}) \cap \left(0 :_{{}^*E^j(M)_{(\mathfrak{p})}} \mathfrak{p}R_{(\mathfrak{p})}\right) \subseteq \left(0 :_{{}^*E^j(M)_{(\mathfrak{p})}} \mathfrak{p}R_{(\mathfrak{p})}\right)$$

is also *essential. Because, by Proposition 2.5(iii)(d), each $\mathbb{Z}^r$-graded $T$-module is free, it follows that all the 'differentiation' maps in the complex $(0 :_{(I^\bullet)_{(\mathfrak{p})}} \mathfrak{p}R_{(\mathfrak{p})})$ are zero. Hence

$${}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \cong \bigoplus_{\substack{\alpha\in\Lambda_i\\ \mathfrak{p}_\alpha\subseteq\mathfrak{p}}} \left(0 :_{{}^*E(R_{(\mathfrak{p})}/\mathfrak{p}_\alpha R_{(\mathfrak{p})})(-\mathbf{n}_\alpha)} \mathfrak{p}R_{(\mathfrak{p})}\right) \quad \text{in } {}^*\mathcal{C}^{\mathbb{Z}^r}(R_{(\mathfrak{p})}).$$

For $\alpha \in \Lambda_i$ such that $\mathfrak{p}_\alpha \subset \mathfrak{p}$ (the symbol '$\subset$' is reserved to denote strict inclusion), there exists an ${\mathbb{N}_0}^r$-homogeneous element $u \in \mathfrak{p} \setminus \mathfrak{p}_\alpha$, and the fact (see Lemma 1.5) that multiplication by $u/1 \in R_{(\mathfrak{p})}$ provides an automorphism of ${}^*E(R_{(\mathfrak{p})}/\mathfrak{p}_\alpha R_{(\mathfrak{p})})$ ensures that

$$\left(0 :_{{}^*E(R_{(\mathfrak{p})}/\mathfrak{p}_\alpha R_{(\mathfrak{p})})(-\mathbf{n}_\alpha)} \mathfrak{p}R_{(\mathfrak{p})}\right) = 0.$$

If $\alpha \in \Lambda_i$ is such that $\mathfrak{p}_\alpha = \mathfrak{p}$, then, by Proposition 2.5(iv),

$$(0 :_{{}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})(-\mathbf{n}_\alpha)} \mathfrak{p}R_{(\mathfrak{p})}) = \left(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}\right)(-\mathbf{n}_\alpha)$$

and, by Proposition 2.5(iii)(d), this is a free $\mathbb{Z}^r$-graded $T$-module.

Therefore there is a $\mathbb{Z}^r$-homogeneous isomorphism of $\mathbb{Z}^r$-graded $T$-modules

$${}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \cong \bigoplus_{\substack{\alpha\in\Lambda_i\\ \mathfrak{p}_\alpha=\mathfrak{p}}} \left(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}\right)(-\mathbf{n}_\alpha).$$

Now apply the functor $(\,\bullet\,)^{\phi(\mathfrak{p})}$ to obtain a $\mathbb{Z}^m$-homogeneous isomorphism of $\mathbb{Z}^m$-graded $T^{\phi(\mathfrak{p})}$-modules

$$\left({}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})\right)^{\phi(\mathfrak{p})} \cong \bigoplus_{\substack{\alpha\in\Lambda_i\\ \mathfrak{p}_\alpha=\mathfrak{p}}} \left(\left(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}\right)(-\mathbf{n}_\alpha)\right)^{\phi(\mathfrak{p})}.$$

But, by Remark 2.6(i), for an $\alpha \in \Lambda_i$,

$$\left(\left(T(-\mathbf{n}_\alpha)\right)^{\phi(\mathfrak{p})}\right)_{\mathbf{a}} = \begin{cases} 0 & \text{if } \phi(\mathfrak{p})(\mathbf{n}_\alpha) \neq \mathbf{a},\\ \left(T^{\phi(\mathfrak{p})}\right)_{\mathbf{0}} & \text{if } \phi(\mathfrak{p})(\mathbf{n}_\alpha) = \mathbf{a}. \end{cases}$$

The desired result now follows from Remark 2.6(iii) □

2.8. **Definitions.** Let the situation and notation be as in Theorem 2.7, so that, in particular, $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$ and $m$ denotes the number of $\mathfrak{p}$-directions.

Let $i \in \mathbb{N}_0$. We say that $\mathbf{a} \in \mathbb{Z}^m$ is an *$i$th level anchor point of $\mathfrak{p}$ for $M$* if

$$\left(\left({}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})\right)^{\phi(\mathfrak{p})}\right)_{\mathbf{a}} \neq 0;$$

the set of all $i$th level anchor points of $\mathfrak{p}$ for $M$ is denoted by $\operatorname{anch}^i(\mathfrak{p}, M)$; also, we write

$$\operatorname{anch}(\mathfrak{p}, M) = \bigcup_{j\in\mathbb{N}_0} \operatorname{anch}^j(\mathfrak{p}, M),$$

and refer to this as the set of *anchor points of $\mathfrak{p}$ for $M$*.

Thus $\operatorname{anch}^i(\mathfrak{p}, M)$ is the set of $m$-tuples $\mathbf{a} \in \mathbb{Z}^m$ for which, when we decompose

$$ {}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha) $$

by means of a $\mathbb{Z}^r$-homogeneous isomorphism, there exists $\alpha \in \Lambda_i$ with $\mathfrak{p}_\alpha = \mathfrak{p}$ and $\phi(\mathfrak{p})(\mathbf{n}_\alpha) = \mathbf{a}$. Note that $\operatorname{anch}^i(\mathfrak{p}, M) = \emptyset$ if $\mu^i(\mathfrak{p}, M) = 0$, and that, if $M$ is finitely generated, then $\operatorname{anch}^i(\mathfrak{p}, M)$ is a finite set, by Remark 2.6(iii).

The details in our present multi-graded situation are more complicated (and therefore more interesting!) than in the singly-graded situation studied in [15] because there might exist a $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$ for which the set of $\mathfrak{p}$-directions is a proper subset of $\{1, \ldots, r\}$. This cannot happen when $r = 1$. It is worthwhile for us to draw attention to the simplifications that occur in the above theory when $\operatorname{dir}(\mathfrak{p}) = \{1, \ldots, r\}$, for that case provides a more-or-less exact analogue of the anchor point theory for the singly-graded case developed in [15].

2.9. **Example.** Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard. Let $M$ be a $\mathbb{Z}^r$-graded $R$-module, and let

$$ I^\bullet : 0 \longrightarrow {}^*E^0(M) \xrightarrow{d^0} {}^*E^1(M) \longrightarrow \cdots \longrightarrow {}^*E^i(M) \xrightarrow{d^i} {}^*E^{i+1}(M) \longrightarrow \cdots $$

be the minimal *injective resolution of $M$. For each $i \in \mathbb{N}_0$, let

$$ \theta_i : {}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha) $$

be a $\mathbb{Z}^r$-homogeneous isomorphism, where $\mathfrak{p}_\alpha \in {}^*\operatorname{Spec}(R)$ and $\mathbf{n}_\alpha \in \mathbb{Z}^r$ for all $\alpha \in \Lambda_i$.

Let $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$ be such that $\mathfrak{p} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} > \mathbf{0}$, so that $\operatorname{dir}(\mathfrak{p}) = \{1, \ldots, r\}$. In this case, $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ is concentrated in degree $\mathbf{0}$, and $T_{\mathbf{0}}$ is a field isomorphic to $k_{R_{\mathbf{0}}}(\mathfrak{p}_{\mathbf{0}})$.

Let $i \in \mathbb{N}_0$. Then $\operatorname{anch}^i(\mathfrak{p}, M)$ is the set of $r$-tuples $\mathbf{a} \in \mathbb{Z}^r$ for which there exists $\alpha \in \Lambda_i$ with $\mathfrak{p}_\alpha = \mathfrak{p}$ and $\mathbf{n}_\alpha = \mathbf{a}$. The cardinality of the set of such $\alpha$s is

$$ \dim_{k_{R_{\mathbf{0}}}(\mathfrak{p}_{\mathbf{0}})} \left( \left( {}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \right)_{\mathbf{a}} \right), $$

and we have

$$ \sum_{\mathbf{a} \in \mathbb{Z}^r} \dim_{k_{R_{\mathbf{0}}}(\mathfrak{p}_{\mathbf{0}})} \left( \left( {}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \right)_{\mathbf{a}} \right) = \mu^i(\mathfrak{p}, M). $$

In particular, if $M$ is finitely generated, then there are only finitely many $i$th level anchor points of $\mathfrak{p}$ for $M$.

This reflects rather well the singly-graded anchor point theory studied in [15].

Our next aim is to extend (in a sense) the final result in Example 2.9 (namely that, when $M$ (as in the example) is a finitely generated $\mathbb{Z}^r$-graded $R$-module and $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$ is such that $\mathfrak{p} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} > \mathbf{0}$, then, for each $i \in \mathbb{N}_0$, there are only finitely many $i$th level anchor points of $\mathfrak{p}$ for $M$) to *all* $\mathbb{N}_0{}^r$-graded primes of $R$ that contain $R_{\mathbf{1}}$.

2.10. *Remark.* Let $S$ be a multiplicatively closed set of $\mathbb{Z}^r$-homogeneous elements of $R$, and let $M$, $N$ be $\mathbb{Z}^r$-graded $R$-modules with $M$ finitely generated. Then, for each $i \in \mathbb{N}_0$, there is a $\mathbb{Z}^r$-homogeneous $S^{-1}R$-isomorphism

$$ S^{-1}\left({}^*\operatorname{Ext}^i_R(M, N)\right) \cong {}^*\operatorname{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N). $$

2.11. **Theorem.** *Assume that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard, and let $M$ be a $\mathbb{Z}^r$-graded $R$-module. Let $i \in \mathbb{N}_0$, and let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$. Then*

$$ \operatorname{anch}^i(\mathfrak{p}, M) = \operatorname{anch}^i(\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}), $$

*and so is finite if $M$ is finitely generated.*

*Proof.* Suppose, for ease of notation, that $\operatorname{dir}(\mathfrak{p}) = \{1, \ldots, m\}$, where $0 < m \leq r$. Note that $\mathfrak{p}^{\phi(\mathfrak{p})}$ is a $\mathbb{Z}^m$-graded prime ideal of the $\mathbb{Z}^m$-graded ring $R^{\phi(\mathfrak{p})}$, and that $\operatorname{dir}(\mathfrak{p}^{\phi(\mathfrak{p})}) = \{1, \ldots, m\}$ (by Lemma 2.2).

Set $E := {}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)$. Let $\mathbf{a} \in \mathbb{Z}^m$. In view of 2.10, the $m$-tuple $\mathbf{a}$ is an $i$th level anchor point of $\mathfrak{p}$ for $M$ if and only if $((E_{(\mathfrak{p})})^{\phi(\mathfrak{p})})_{\mathbf{a}} \neq 0$. Our initial task in this proof is to show that this is the case if and only if

$$\left(({}^*\operatorname{Ext}^i_{R^{\phi(\mathfrak{p})}}(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}))_{(\mathfrak{p}^{\phi(\mathfrak{p})})}\right)_{\mathbf{a}} \neq 0.$$

Now the $\mathbb{Z}^r$-graded $R$-module $E$ can be constructed by application of the functor ${}^*\operatorname{Hom}_R(\bullet, M)$ to a (deleted) *free resolution of $R/\mathfrak{p}$ by finitely generated *free $\mathbb{Z}^r$-graded modules in the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$ and then taking cohomology of the resulting complex. It follows that there is a $\mathbb{Z}^m$-homogeneous isomorphism of $\mathbb{Z}^m$-graded $R^{\phi(\mathfrak{p})}$-modules

$$E^{\phi(\mathfrak{p})} \cong {}^*\operatorname{Ext}^i_{R^{\phi(\mathfrak{p})}}(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}).$$

Suppose that $((E_{(\mathfrak{p})})^{\phi(\mathfrak{p})})_{\mathbf{a}} \neq 0$. Thus there exists $\mathbf{n} \in \mathbb{Z}^r$ such that $\phi(\mathfrak{p})(\mathbf{n}) = \mathbf{a}$ and $\xi \in (E_{(\mathfrak{p})})_{\mathbf{n}}$ such that $\xi \neq 0$. By Remark 2.4, there exists $\mathbf{n'} \in \mathbb{Z}^r$ such that $\phi(\mathfrak{p})(\mathbf{n'}) = \mathbf{a}$ and $e \in E_{\mathbf{n'}}$ which is not annihilated by any $\mathbb{Z}^r$-homogeneous element of $R \setminus \mathfrak{p}$. Now any $\mathbb{Z}^m$-homogeneous element of $R^{\phi(\mathfrak{p})} \setminus \mathfrak{p}^{\phi(\mathfrak{p})}$ will, when written as a sum of $\mathbb{Z}^r$-homogeneous elements of $R$, have at least one component outside $\mathfrak{p}$, and so $0 \neq e/1 \in (E^{\phi(\mathfrak{p})})_{(\mathfrak{p}^{\phi(\mathfrak{p})})}$. Hence $\left((E^{\phi(\mathfrak{p})})_{(\mathfrak{p}^{\phi(\mathfrak{p})})}\right)_{\mathbf{a}} \neq 0$, so that

$$\left(({}^*\operatorname{Ext}^i_{R^{\phi(\mathfrak{p})}}(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}))_{(\mathfrak{p}^{\phi(\mathfrak{p})})}\right)_{\mathbf{a}} \neq 0.$$

Now suppose that $\left(({}^*\operatorname{Ext}^i_{R^{\phi(\mathfrak{p})}}(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}))_{(\mathfrak{p}^{\phi(\mathfrak{p})})}\right)_{\mathbf{a}} \neq 0$. Then $\left((E^{\phi(\mathfrak{p})})_{(\mathfrak{p}^{\phi(\mathfrak{p})})}\right)_{\mathbf{a}} \neq 0$. Since every $\mathbb{Z}^m$-homogeneous element of $R^{\phi(\mathfrak{p})} \setminus \mathfrak{p}^{\phi(\mathfrak{p})}$ has degree $\mathbf{0} \in \mathbb{Z}^m$, it follows that there exists $e \in (E^{\phi(\mathfrak{p})})_{\mathbf{a}}$ that is not annihilated by any $\mathbb{Z}^m$-homogeneous element of $R^{\phi(\mathfrak{p})} \setminus \mathfrak{p}^{\phi(\mathfrak{p})}$. In particular, $e$ is not annihilated by any $\mathbb{Z}^r$-homogeneous element of $R \setminus \mathfrak{p}$. Therefore $0 \neq e/1 \in ((E_{(\mathfrak{p})})^{\phi(\mathfrak{p})})_{\mathbf{a}}$.

This proves that $\operatorname{anch}^i(\mathfrak{p}, M) = \operatorname{anch}^i(\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})})$. Finally, since $\operatorname{dir}(\mathfrak{p}^{\phi(\mathfrak{p})}) = \{1, \ldots, m\}$, it follows from Example 2.9 that $\operatorname{anch}^i(\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})})$ is finite when $M$ is finitely generated. $\square$

The aim of the remainder of this section is to establish a multi-graded analogue of a result of Bass [1, Lemma 3.1]. However, there are some subtleties which mean that our generalization of [15, Lemma 1.8] is not completely straightforward.

2.12. **Theorem.** *Assume that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard, and let $M$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module. Let $\mathfrak{p}, \mathfrak{q} \in {}^*\operatorname{Spec}(R)$ be such that $R_{\mathbf{1}}R \subseteq \mathfrak{p} \subset \mathfrak{q}$ (we reserve the symbol '$\subset$' to denote strict inclusion) and that there is no $\mathbb{Z}^r$-graded prime ideal strictly between $\mathfrak{p}$ and $\mathfrak{q}$. Note that $\operatorname{dir}(\mathfrak{p}) \subseteq \operatorname{dir}(\mathfrak{q})$: suppose, for ease of notation, that $\operatorname{dir}(\mathfrak{p}) = \{1, \ldots, m\}$ and $\operatorname{dir}(\mathfrak{q}) = \{1, \ldots, m, m+1, \ldots, h\}$, where $0 < m \leq h \leq r$.*

*Let $i \in \mathbb{N}_0$. Then, for each $\mathbf{a} = (a_1, \ldots, a_m) \in \operatorname{anch}^i(\mathfrak{p}, M)$, there exists*

$$\mathbf{b} = (b_1, \ldots, b_m, b_{m+1}, \ldots, b_h) \in \operatorname{anch}^{i+1}(\mathfrak{q}, M)$$

*such that $(b_1, \ldots, b_m) = (a_1, \ldots, a_m) = \mathbf{a}$.*

*Proof.* There exists an $\mathbb{N}_0{}^r$-homogeneous element $b \in \mathfrak{q} \setminus \mathfrak{p}$. By Remark 2.4, each $\mathbb{N}_0{}^r$-homogeneous element of $R \setminus \mathfrak{p}$ has degree with first $m$ components 0. In particular, $\deg(b) = \mathbf{0}|\mathbf{v} \in \mathbb{Z}^m \times \mathbb{Z}^{r-m}$ for some $\mathbf{v} \in \mathbb{Z}^{r-m}$.

Since $\mathbf{a} \in \operatorname{anch}^i(\mathfrak{p}, M)$, there exists $\mathbf{w} \in \mathbb{Z}^{r-m}$ such that $\left({}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})\right)_{\mathbf{a}|\mathbf{w}} \neq 0$. Set $E := {}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)$. In view of Remark 2.10, we must have $(E_{(\mathfrak{p})})_{\mathbf{a}|\mathbf{w}} \neq 0$. Since each $\mathbb{N}_0{}^r$-homogeneous element of $R \setminus \mathfrak{p}$ has degree with first $m$ components 0, this means that there exists a homogeneous element $e \in E$, with $\deg(e) = \mathbf{a}|\mathbf{w'}$ for some $\mathbf{w'} \in \mathbb{Z}^{r-m}$, that is not annihilated by any $\mathbb{N}_0{}^r$-homogeneous element of $R \setminus \mathfrak{p}$. But $R \setminus \mathfrak{q} \subseteq R \setminus \mathfrak{p}$, and so it follows that $(E_{(\mathfrak{q})})_{\mathbf{a}|\mathbf{w'}} \neq 0$. By Remark 2.10 again, $\left({}^*\operatorname{Ext}^i_{R_{(\mathfrak{q})}}(R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})}, M_{(\mathfrak{q})})\right)_{\mathbf{a}|\mathbf{w'}} \neq 0$. Write $F := {}^*\operatorname{Ext}^i_{R_{(\mathfrak{q})}}(R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})}, M_{(\mathfrak{q})})$.

There is an exact sequence

$$0 \longrightarrow \left(R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})}\right)(-(\mathbf{0}|\mathbf{v})) \xrightarrow{b/1} R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})} \longrightarrow R_{(\mathfrak{q})}/\left(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}\right) \longrightarrow 0$$

in ${}^*\mathcal{C}^{\mathbb{Z}^r}(R_{(\mathfrak{q})})$, and this induces an exact sequence

$$F \xrightarrow{b/1} F(\mathbf{0}|\mathbf{v}) \longrightarrow {}^*\operatorname{Ext}^{i+1}_{R_{(\mathfrak{q})}}(R_{(\mathfrak{q})}/(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}), M_{(\mathfrak{q})}).$$

Recall that $\deg(b) = \mathbf{0}|\mathbf{v}$. We claim that there exists $\mathbf{y} \in \mathbb{Z}^{r-m}$ such that $F_{\mathbf{a}|\mathbf{y}} \neq (b/1)F_{\mathbf{a}|\mathbf{y}-\mathbf{v}}$. To see this, note that $b/1 \in \mathfrak{q}R_{(\mathfrak{q})}$, the unique *maximal ideal of the homogeneous localization $R_{(\mathfrak{q})}$, and if we had $F_{\mathbf{a}|\mathbf{y}} = (b/1)F_{\mathbf{a}|\mathbf{y}-\mathbf{v}}$ for every $\mathbf{y} \in \mathbb{Z}^{r-m}$, then we should have $F_{\mathbf{a}|\mathbf{w}'} \subseteq \bigcap_{n\in\mathbb{N}}(b/1)^n F$, which is zero by the multi-graded version of Krull's Intersection Theorem. (One can show that $G := \bigcap_{n\in\mathbb{N}}(b/1)^n F$ satisfies $G = (b/1)G$, and then use the multi-graded version of Nakayama's Lemma.) Thus there exists $\mathbf{y} \in \mathbb{Z}^{r-m}$ such that $F_{\mathbf{a}|\mathbf{y}} \neq (b/1)F_{\mathbf{a}|\mathbf{y}-\mathbf{v}}$, and therefore, in view of the last exact sequence,

$$\left({}^*\operatorname{Ext}^{i+1}_{R_{(\mathfrak{q})}}(R_{(\mathfrak{q})}/(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}), M_{(\mathfrak{q})})\right)_{\mathbf{a}|\mathbf{y}} \neq 0.$$

Now $R_{(\mathfrak{q})}/\left(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}\right)$ is concentrated in $\mathbb{Z}^r$-degrees whose first $m$ components are all zero. Therefore all its $\mathbb{Z}^r$-graded $R$-homomorphic images and all its $\mathbb{Z}^r$-graded submodules are also concentrated in $\mathbb{Z}^r$-degrees whose first $m$ components are all zero.

The only $\mathbb{Z}^r$-graded prime ideal of $R_{(\mathfrak{q})}$ that contains the ideal $\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}$ is $\mathfrak{q}R_{(\mathfrak{q})}$, and so $\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}$ is $\mathfrak{q}R_{(\mathfrak{q})}$-primary. It follows that there is a chain of $\mathbb{Z}^r$-graded ideals of $R_{(\mathfrak{q})}$ from $\mathfrak{q}R_{(\mathfrak{q})}$ to $\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}$ with the property that each subquotient is $R_{(\mathfrak{q})}$-isomorphic to $\left(R_{(\mathfrak{q})}/\mathfrak{q}R_{(\mathfrak{q})}\right)(\mathbf{0}|\mathbf{z})$ for some $\mathbf{z} \in \mathbb{Z}^{r-m}$. It therefore follows from the half-exactness of ${}^*\operatorname{Ext}^{i+1}_{R_{(\mathfrak{q})}}$ that there exists $\mathbf{y}' \in \mathbb{Z}^{r-m}$ such that

$$\left({}^*\operatorname{Ext}^{i+1}_{R_{(\mathfrak{q})}}(R_{(\mathfrak{q})}/\mathfrak{q}R_{(\mathfrak{q})}, M_{(\mathfrak{q})})\right)_{\mathbf{a}|\mathbf{y}'} \neq 0.$$

The claim then follows from Theorem 2.7. □

2.13. **Corollary.** *Assume that $R = \bigoplus_{\mathbf{n}\in\mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard, and let $M$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module. Let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$, and suppose, for ease of notation, that* $\operatorname{dir}(\mathfrak{p}) = \{1, \ldots, m\}$.

*Let $\mathbf{a} \in \operatorname{anch}(\mathfrak{p}, M)$. Then there exists $\mathfrak{q} \in {}^*\operatorname{Spec}(R)$ such that $\mathfrak{q} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{N}_0{}^r$ with $\mathbf{n} > \mathbf{0}$ and $\mathbf{b} = (b_1, \ldots, b_m, b_{m+1}, \ldots, b_r) \in \operatorname{anch}(\mathfrak{q}, M)$ such that $\mathbf{a} = (b_1, \ldots, b_m)$.*

*Proof.* There exists a saturated chain $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_t = \mathfrak{q}$ of $\mathbb{Z}^r$-graded prime ideals of $R$ such that $\mathfrak{q}$ is *maximal. Since $\mathfrak{q}$ is contained in the $\mathbb{Z}^r$-graded prime ideal

$$(\mathfrak{q} \cap R_{\mathbf{0}}) \bigoplus \bigoplus_{\mathbf{n}>\mathbf{0}} R_{\mathbf{n}},$$

these two $\mathbb{Z}^r$-graded prime ideals must be the same; we therefore see that $\mathfrak{q} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{N}_0{}^r$ with $\mathbf{n} > \mathbf{0}$. The claim is now immediate from Theorem 2.12. □

## 3. The ends of certain multi-graded local cohomology modules

We begin with a combinatorial lemma.

3.1. **Lemma.** *Let $\mathbf{a} := (a_1, \ldots, a_r) \in \mathbb{Z}^r$ and let $\Sigma$ be a non-empty subset of $\mathbb{Z}^r$ such that $\mathbf{n} \leq \mathbf{a}$ for all $\mathbf{n} \in \Sigma$. Then $\Sigma$ has only finitely many maximal elements.*

*Note.* We are grateful to the referee for drawing our attention to the following proof, which is shorter than our original.

*Proof.* The set $\Delta := \mathbf{a} - \Sigma := \{\mathbf{a} - \mathbf{n} : \mathbf{n} \in \Sigma\}$ is a non-empty subset of $\mathbb{N}_0{}^r$. Now $\mathbb{N}_0{}^r$ is a Noetherian monoid with respect to addition, by [11, Proposition 1.3.5], for example. (All terminology concerning monoids in this proof is as in [11, Chapter 1].) Therefore the monoideal $(\Delta)$ of $\mathbb{N}_0{}^r$ generated by $\Delta$ can be generated by finitely many elements of $\Delta$, say by $\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(s)} \in \Delta$. Therefore

$$\Delta \subseteq (\Delta) = \left(\mathbf{m}^{(1)} + \mathbb{N}_0{}^r\right) \cup \cdots \cup \left(\mathbf{m}^{(s)} + \mathbb{N}_0{}^r\right),$$

from which it follows that any minimal member of $\Delta$ must belong to the set $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(s)}\}$. Therefore any maximal member of $\Sigma$ must belong to the set $\{\mathbf{a} - \mathbf{m}^{(1)}, \ldots, \mathbf{a} - \mathbf{m}^{(s)}\}$. □

3.2. **Notation.** Let $\Sigma, \Delta \subseteq \mathbb{Z}^r$. We shall denote by $\max(\Sigma)$ the set of maximal members of $\Sigma$. (If $\Sigma$ has no maximal member, then we interpret $\max(\Sigma)$ as the empty set.)

We shall write $\Sigma \preccurlyeq \Delta$ to indicate that, for each $\mathbf{n} \in \Sigma$, there exists $\mathbf{m} \in \Delta$ such that $\mathbf{n} \leq \mathbf{m}$; moreover, we shall describe this situation by the terminology '$\Delta$ *dominates* $\Sigma$'. We shall use obvious variants of this terminology. Observe that, if $\Sigma \preccurlyeq \Delta$ and $\Delta \preccurlyeq \Sigma$, then $\max(\Sigma) = \max(\Delta)$, and $\Sigma \preccurlyeq \max(\Sigma)$ if and only if $\Delta \preccurlyeq \max(\Delta)$.

3.3. *Remark* (Huy Tài Hà [9, §2]). Let $\phi : \mathbb{Z}^r \longrightarrow \mathbb{Z}^m$, where $m$ is a positive integer, be a homomorphism of Abelian groups. We use the notation $R^\phi$, *etcetera*, of Definition 1.3. Let $\mathfrak{a}$ be a $\mathbb{Z}^r$-graded ideal of $R$. Then $\big((H^i_{\mathfrak{a}}(\,\bullet\,))^\phi)\big)_{i\in\mathbb{N}_0}$ and $\big((H^i_{\mathfrak{a}^\phi}(\,\bullet\,^\phi))\big)_{i\in\mathbb{N}_0}$ are both negative strongly connected sequences of covariant functors from ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$ to ${}^*\mathcal{C}^{\mathbb{Z}^m}(R^\phi)$; moreover, the 0th members of these two connected sequences are the same functor, and, whenever, $I$ is a $^*$-injective $\mathbb{Z}^r$-graded $R$-module and $i > 0$, we have $H^i_{\mathfrak{a}}(I) = 0$ when all gradings are forgotten, so that $(H^i_{\mathfrak{a}}(I))^\phi = 0$ and $H^i_{\mathfrak{a}^\phi}(I^\phi) = 0$. Consequently, the two above-mentioned connected sequences are isomorphic. Hence, for each $\mathbb{Z}^r$-graded $R$-module $M$, there is a $\mathbb{Z}^m$-homogeneous isomorphism of $\mathbb{Z}^m$-graded $R^\phi$-modules

$$(H^i_{\mathfrak{a}}(M))^\phi \cong H^i_{\mathfrak{a}^\phi}(M^\phi) \quad \text{ for each } i \in \mathbb{N}_0.$$

3.4. **Notation.** Throughout this section, we shall be concerned with the situation where

$$R = \bigoplus_{\mathbf{n}\in{\mathbb{N}_0}^r} R_{\mathbf{n}}$$

is positively graded; we shall only assume that $R$ is standard when this is explicitly stated.

We shall be greatly concerned with the ${\mathbb{N}_0}^r$-graded ideal

$$\mathfrak{c} := \mathfrak{c}(R) := \bigoplus_{\substack{\mathbf{n}\in{\mathbb{N}_0}^r\\ \mathbf{n}>\mathbf{0}}} R_{\mathbf{n}}.$$

We shall accord $R_+$ its usual meaning (see E. Hyry [10, p. 2215]), so that

$$R_+ := \bigoplus_{\substack{\mathbf{n}\in{\mathbb{N}_0}^r\\ \mathbf{n}\geq\mathbf{1}}} R_{\mathbf{n}} = \bigoplus_{\mathbf{n}\in\mathbb{N}^r} R_{\mathbf{n}}.$$

Observe that, when $r = 1$, we have $\mathfrak{c} = R_+$. However, in general this is not the case when $r > 1$.

3.5. **Definition.** Suppose that $R = \bigoplus_{\mathbf{n}\in{\mathbb{N}_0}^r} R_{\mathbf{n}}$ is positively graded and standard; let $M = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module, and let $j \in \mathbb{N}_0$.

Let $\mathfrak{b}$ be an ${\mathbb{N}_0}^r$-graded ideal such that $\operatorname{dir}(\mathfrak{b}) \neq \emptyset$, and let $i \in \operatorname{dir}(\mathfrak{b})$; consider the Abelian group homomorphism $\phi_i : \mathbb{Z}^r \longrightarrow \mathbb{Z}$ for which $\phi_i((n_1, \dots, n_r)) = n_i$ for all $(n_1, \dots, n_r) \in \mathbb{Z}^r$, which is just the $i$th coordinate function.

By Lemma 2.2, since $R_{\mathbf{e}_i} \subseteq \sqrt{\mathfrak{b}}$, we have

$$(R^{\phi_i})_+ = \bigoplus_{\substack{\mathbf{n}\in{\mathbb{N}_0}^r\\ n_i>0}} R_{\mathbf{n}} \subseteq \sqrt{\mathfrak{b}}^{\phi_i}.$$

It therefore follows from [15, Corollary 2.5], with the notation of that paper, that the $\mathbb{N}_0$-graded $R^{\phi_i}$-module $(H^j_{\mathfrak{b}}(M))^{\phi_i} \cong H^j_{\mathfrak{b}^{\phi_i}}(M^{\phi_i})$, if non-zero, has finite end satisfying

$$\operatorname{end}((H^j_{\mathfrak{b}}(M))^{\phi_i}) \leq a^*(M^{\phi_i}) = \sup\{\operatorname{end}(H^k_{(R^{\phi_i})_+}(M^{\phi_i})) : k \in \mathbb{N}_0\} = \sup\{a^k_{(R^{\phi_i})_+}(M^{\phi_i}) : k \in \mathbb{N}_0\}.$$

(Note that, in these circumstances, the invariant $a^*(M^{\phi_i})$ is an integer.) Thus, if $\mathbf{n} := (n_1, \dots, n_r) \in \mathbb{Z}^r$ is such that $H^j_{\mathfrak{b}}(M)_{\mathbf{n}} \neq 0$, then $n_i \leq a^*(M^{\phi_i})$. Thus there exists $\mathbf{a} \in \mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$ such that, for all $\mathbf{n} := (n_1, \dots, n_r) \in \mathbb{Z}^r$ with $H^j_{\mathfrak{b}}(M)_{\mathbf{n}} \neq 0$, we have $\phi(\mathfrak{b})(\mathbf{n}) \leq \mathbf{a}$. We define the *end of* $H^j_{\mathfrak{b}}(M)$ by

$$\operatorname{end}(H^j_{\mathfrak{b}}(M)) := \max\left\{\phi(\mathfrak{b})(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^r \text{ and } H^j_{\mathfrak{b}}(M)_{\mathbf{n}} \neq 0\right\}.$$

By Lemma 3.1, if $H^j_{\mathfrak{b}}(M) \neq 0$ and $\operatorname{dir}(\mathfrak{b}) \neq \emptyset$, then this end is a non-empty finite set of points of $\mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$. Note that the end of $H^j_{\mathfrak{b}}(M)$ dominates $\phi(\mathfrak{b})(\mathbf{n})$ for every $\mathbf{n} \in \mathbb{Z}^r$ for which $H^j_{\mathfrak{b}}(M)_{\mathbf{n}} \neq 0$.

We draw the reader's attention to the fact that, when $r > 1$ and $R_{\mathbf{e}_i} \neq 0$ for all $i \in \{1, \ldots, r\}$, the ideal $R_+ = \bigoplus_{\substack{\mathbf{n}\in\mathbb{N}_0{}^r\\ \mathbf{n}\geq\mathbf{1}}} R_{\mathbf{n}}$ has empty set of directions; consequently, we have not defined the end of the $i$th local cohomology module $H^i_{R_+}(M)$ of $M$ with respect to $R_+$. Thus we are not, in this paper, making any contribution to the theory of multi-graded Castelnuovo regularity, and, in particular, we are not proposing an alternative definition of $a$-invariant or $a^*$-invariant (see [9, Definitions 3.1.1 and 3.1.2]).

With this definition of the ends of (certain) multi-graded local cohomology modules, we can now establish multi-graded analogues of some results in [15, §2].

3.6. **Theorem.** *Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard. Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module, and let*

$$I^\bullet : 0 \longrightarrow {}^*E^0(M) \xrightarrow{d^0} {}^*E^1(M) \longrightarrow \cdots \longrightarrow {}^*E^i(M) \xrightarrow{d^i} {}^*E^{i+1}(M) \longrightarrow \cdots$$

*be the minimal ${}^*$injective resolution of $M$.*

*Let $\mathfrak{b}$ be an $\mathbb{N}_0{}^r$-graded ideal such that* $\operatorname{dir}(\mathfrak{b}) \neq \emptyset$, *and let $j \in \mathbb{N}_0$. Then*

$$\max\left(\bigcup_{i=0}^{j} \operatorname{end}(H^j_{\mathfrak{b}}(M))\right) = \max\left\{\phi(\mathfrak{b})(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^r \text{ and } (\Gamma_{\mathfrak{b}}({}^*E^i(M)))_{\mathbf{n}} \neq 0 \text{ for some } i \in \{0, \ldots, j\}\right\}$$

$$= \max\left(\bigcup_{i=0}^{j} \bigcup_{\mathfrak{p}\in{}^*\operatorname{Var}(\mathfrak{b})} \phi(\mathfrak{p};\mathfrak{b})(\operatorname{anch}^i(\mathfrak{p}, M))\right).$$

*Proof.* Let $i \in \mathbb{N}_0$ and set

$$\Delta_i := \left\{\phi(\mathfrak{b})(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^r \text{ and } H^i_{\mathfrak{b}}(M)_{\mathbf{n}} \neq 0\right\}, \quad \Sigma_i := \left\{\phi(\mathfrak{b})(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^r \text{ and } (\Gamma_{\mathfrak{b}}({}^*E^i(M)))_{\mathbf{n}} \neq 0\right\}$$

and

$$\Phi_i := \bigcup_{\mathfrak{p}\in{}^*\operatorname{Var}(\mathfrak{b})} \phi(\mathfrak{p};\mathfrak{b})(\operatorname{anch}^i(\mathfrak{p}, M)).$$

Also, let

$$\theta_i : {}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha\in\Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)$$

be a $\mathbb{Z}^r$-homogeneous isomorphism, where $\mathfrak{p}_\alpha \in {}^*\operatorname{Spec}(R)$ and $\mathbf{n}_\alpha \in \mathbb{Z}^r$ for all $\alpha \in \Lambda_i$.

We shall first show that $\Delta_i \preccurlyeq \Sigma_i \preccurlyeq \Phi_i$. Now $H^i_{\mathfrak{b}}(M)$ is a homomorphic image, by a $\mathbb{Z}^r$-homogeneous epimorphism, of

$$\operatorname{Ker}\left(\Gamma_{\mathfrak{b}}(d^i) : \Gamma_{\mathfrak{b}}({}^*E^i(M)) \longrightarrow \Gamma_{\mathfrak{b}}({}^*E^{i+1}(M))\right).$$

Therefore, if $\mathbf{n} \in \mathbb{Z}^r$ is such that $H^i_{\mathfrak{b}}(M)_{\mathbf{n}} \neq 0$, then $(\Gamma_{\mathfrak{b}}({}^*E^i(M)))_{\mathbf{n}} \neq 0$. This proves that $\Delta_i \subseteq \Sigma_i$, so that $\Delta_i \preccurlyeq \Sigma_i$.

Furthermore, given $\mathbf{n} \in \mathbb{Z}^r$ such that $(\Gamma_{\mathfrak{b}}({}^*E^i(M)))_{\mathbf{n}} \neq 0$, we can see from the isomorphism $\theta_i$ that there must exist $\alpha \in \Lambda_i$ such that $\mathfrak{b} \subseteq \mathfrak{p}_\alpha$ and $({}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha))_{\mathbf{n}} \neq 0$. It now follows from Proposition 2.5(ii) that $\phi(\mathfrak{p}_\alpha)(\mathbf{n}) \leq \phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)$, so that

$$\phi(\mathfrak{p}_\alpha;\mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n})) \leq \phi(\mathfrak{p}_\alpha;\mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)).$$

Now $\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)$ is an $i$th level anchor point of $\mathfrak{p}_\alpha$ for $M$, and $\phi(\mathfrak{p}_\alpha;\mathfrak{b}) \circ \phi(\mathfrak{p}_\alpha) = \phi(\mathfrak{b})$. This is enough to prove that $\Sigma_i \preccurlyeq \Phi_i$.

In particular, we have proved that $\Delta_0 \preccurlyeq \Sigma_0 \preccurlyeq \Phi_0$. We shall prove the desired result by induction on $j$. We show next that $\Phi_0 \preccurlyeq \Delta_0$, and this, together with the above, will prove the claim in the case where $j = 0$. Let $\mathbf{m} \in \Phi_0$. Thus $\mathbf{m} \in \mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$ and there exists $\alpha \in \Lambda_0$ such that $\mathfrak{p}_\alpha \in {}^*\operatorname{Var}(\mathfrak{b})$ and $\mathbf{m} = \phi(\mathfrak{p}_\alpha;\mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha))$. Now the image of

$$\bigoplus_{\substack{\mathbf{n}\in\mathbb{Z}^r\\ \phi(\mathfrak{p}_\alpha)(\mathbf{n})\geq\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)}} ({}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha))_{\mathbf{n}}$$

under $\theta_0^{-1}$ is a non-zero $\mathbb{Z}^r$-graded submodule of $\Gamma_{\mathfrak{b}}({}^*E^0(M))$; as the latter is a *essential extension of $\Gamma_{\mathfrak{b}}(M)$, it follows that there exists $\mathbf{n} \in \mathbb{Z}^r$ with $\phi(\mathfrak{p}_\alpha)(\mathbf{n}) \geq \phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)$ such that $(\Gamma_{\mathfrak{b}}(M))_{\mathbf{n}} \neq 0$. Moreover,

$$\phi(\mathfrak{b})(\mathbf{n}) = \phi(\mathfrak{p}_\alpha;\mathfrak{b})\left(\phi(\mathfrak{p}_\alpha)(\mathbf{n})\right) \geq \phi(\mathfrak{p}_\alpha;\mathfrak{b})\left(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)\right) = \mathbf{m}.$$

It follows that $\Phi_0 \preccurlyeq \Delta_0$, so that $\max(\Delta_0) = \max(\Sigma_0) = \max(\Phi_0)$, and the desired result has been proved when $j = 0$.

Now suppose that $j > 0$ and make the obvious inductive assumption. As we have already proved that $\Delta_i \preccurlyeq \Sigma_i$ and $\Sigma_i \preccurlyeq \Phi_i$ for all $i = 0, \ldots, j$, it will be enough, in order to complete the inductive step, for us to prove that $\Phi_j \preccurlyeq \bigcup_{k=0}^{j} \Delta_k$. So consider $\alpha \in \Lambda_j$ such that $\mathfrak{p}_\alpha \in {}^*\operatorname{Var}(\mathfrak{b})$; we shall show that $\phi(\mathfrak{p}_\alpha;\mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha))$ is dominated by a member of $\Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{j-1} \cup \Delta_j$.

Now the image of

$$\bigoplus_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ \phi(\mathfrak{p}_\alpha)(\mathbf{n}) \geq \phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)}} ({}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha))_{\mathbf{n}}$$

under $\theta_j^{-1}$ is a non-zero $\mathbb{Z}^r$-graded submodule of $\Gamma_{\mathfrak{b}}({}^*E^j(M))$; as the latter is a *essential extension of $\operatorname{Ker}\Gamma_{\mathfrak{b}}(d^j)$, it follows that there exists $\mathbf{n} \in \mathbb{Z}^r$ with $\phi(\mathfrak{p}_\alpha)(\mathbf{n}) \geq \phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)$ such that $\left(\operatorname{Ker}\Gamma_{\mathfrak{b}}(d^j)\right)_{\mathbf{n}} \neq 0$. There is an exact sequence

$$0 \longrightarrow \operatorname{Im}\Gamma_{\mathfrak{b}}(d^{j-1}) \longrightarrow \operatorname{Ker}\Gamma_{\mathfrak{b}}(d^j) \longrightarrow H^j_{\mathfrak{b}}(M) \longrightarrow 0$$

of graded $\mathbb{Z}^r$-modules and homogeneous homomorphisms. Therefore either $H^j_{\mathfrak{b}}(M)_{\mathbf{n}} \neq 0$ or

$$\left(\operatorname{Im}\Gamma_{\mathfrak{b}}(d^{j-1})\right)_{\mathbf{n}} \neq 0.$$

In the first case, $\phi(\mathfrak{p}_\alpha;\mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n})) = \phi(\mathbf{b})(\mathbf{n}) \in \Delta_j$. In the second case, $(\Gamma_{\mathfrak{b}}({}^*E^{j-1}(M)))_{\mathbf{n}} \neq 0$, whence $\phi(\mathbf{b})(\mathbf{n}) \in \Sigma_{j-1}$, so that, by the inductive hypothesis, $\phi(\mathbf{b})(\mathbf{n})$ is dominated by an element of $\Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{j-1}$; thus, in this case also, $\phi(\mathfrak{p}_\alpha;\mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha))$ is dominated by an element of $\bigcup_{k=0}^{j} \Delta_k$. This is enough to complete the inductive step. $\square$

3.7. **Notation.** Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard, and let $M$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module. Let $\mathcal{Q}$ be a non-empty subset of $\{1, \ldots, r\}$. Define $\mathfrak{c}^{\mathcal{Q}} := \sum_{i\in\mathcal{Q}} R_{\mathbf{e}_i}R$. Then $\operatorname{dir}(\mathfrak{c}^{\mathcal{Q}}) \supseteq \mathcal{Q}$, and $\mathfrak{c}^{\mathcal{Q}}$ is the smallest ideal (up to radical) with set of directions containing $\mathcal{Q}$. We also define the *$\mathcal{Q}$-bound* $\operatorname{bnd}^{\mathcal{Q}}(M)$ of $M$ by

$$\operatorname{bnd}^{\mathcal{Q}}(M) := \max\left(\bigcup_{i\in\mathbb{N}_0} \operatorname{end}(H^i_{\mathfrak{c}^{\mathcal{Q}}}(M))\right).$$

Observe that $\operatorname{bnd}^{\mathcal{Q}}(M)$ is a finite set of points in $\mathbb{Z}^{\#\operatorname{dir}(\mathfrak{c}^{\mathcal{Q}})}$, because $H^i_{\mathfrak{c}^{\mathcal{Q}}}(M) = 0$ whenever $i$ exceeds the arithmetic rank of $\mathfrak{c}^{\mathcal{Q}}$.

For consistency with our earlier notation in 3.4, we abbreviate $\mathfrak{c}^{\{1,\ldots,r\}} = \sum_{\mathbf{n}>\mathbf{0}} R_{\mathbf{n}}$ by $\mathfrak{c}$. Note that $\operatorname{bnd}^{\{1,\ldots,r\}}(M) = \max\left(\bigcup_{i\in\mathbb{N}_0} \operatorname{end}(H^i_{\mathfrak{c}}(M))\right)$ is a finite set of points in $\mathbb{Z}^r$.

The following corollaries, which are multi-graded analogues of [15, Corollaries 2.5, 2.6], can now be deduced immediately from Theorem 3.6.

3.8. **Corollary.** *Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard. Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module, and let*

$$I^\bullet : 0 \longrightarrow {}^*E^0(M) \xrightarrow{d^0} {}^*E^1(M) \longrightarrow \cdots \longrightarrow {}^*E^i(M) \xrightarrow{d^i} {}^*E^{i+1}(M) \longrightarrow \cdots$$

*be the minimal ${}^*$injective resolution of $M$.*

*Let $\mathfrak{b}$ be an ${\mathbb{N}_0}^r$-graded ideal of $R$ such that* $\operatorname{dir}(\mathfrak{b}) \neq \emptyset$, *and let* $j \in \mathbb{N}_0$. *Then*

$$\begin{aligned}\max\left(\bigcup_{i=0}^{j} \operatorname{end}(H^j_{\mathfrak{b}}(M))\right) &\preccurlyeq \max\left(\bigcup_{i=0}^{j} \bigcup_{\mathfrak{p} \in {}^*\operatorname{Var}(\mathfrak{c}^{\operatorname{dir}(\mathfrak{b})})} \phi(\mathfrak{p}; \mathfrak{c}^{\operatorname{dir}(\mathfrak{b})})(\operatorname{anch}^i(\mathfrak{p}, M))\right)\\ &= \max\left\{\phi(\mathfrak{b})(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^r \text{ and } (\Gamma_{\mathfrak{c}^{\operatorname{dir}(\mathfrak{b})}}({}^*E^i(M)))_{\mathbf{n}} \neq 0 \text{ for an } i \in \{0, \ldots, j\}\right\}\\ &= \max\left(\bigcup_{i=0}^{j} \operatorname{end}(H^j_{\mathfrak{c}^{\operatorname{dir}(\mathfrak{b})}}(M))\right) \preccurlyeq \operatorname{bnd}^{\operatorname{dir}(\mathfrak{b})}(M).\end{aligned}$$

3.9. **Corollary.** *Suppose that $R := \bigoplus_{\mathbf{n} \in {\mathbb{N}_0}^r} R_{\mathbf{n}}$ is positively graded and standard. Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module.*

*Let $\mathfrak{b}$ be an ${\mathbb{N}_0}^r$-graded ideal of $R$ of arithmetic rank $t$ such that* $\operatorname{dir}(\mathfrak{b}) \neq \emptyset$, *and let $k \in \mathbb{N}$ with $k > t$. Then*

$$\begin{aligned}\max\left(\bigcup_{i=0}^{t} \bigcup_{\mathfrak{p} \in {}^*\operatorname{Var}(\mathfrak{b})} \phi(\mathfrak{p}; \mathfrak{b})(\operatorname{anch}^i(\mathfrak{p}, M))\right) &= \max\left(\bigcup_{i=0}^{t} \operatorname{end}(H^i_{\mathfrak{b}}(M))\right) = \max\left(\bigcup_{i=0}^{k} \operatorname{end}(H^i_{\mathfrak{b}}(M))\right)\\ &= \max\left(\bigcup_{i=0}^{k} \bigcup_{\mathfrak{p} \in {}^*\operatorname{Var}(\mathfrak{b})} \phi(\mathfrak{p}; \mathfrak{b})(\operatorname{anch}^i(\mathfrak{p}, M))\right).\end{aligned}$$

*Consequently, for a $\mathfrak{p} \in {}^*\operatorname{Var}(\mathfrak{b})$ and $\mathbf{a} \in \operatorname{anch}(\mathfrak{p}, M)$, we can conclude that $\phi(\mathfrak{p}; \mathfrak{b})(\mathbf{a})$ is dominated by*

$$\max\left(\bigcup_{i=0}^{t} \bigcup_{\mathfrak{p} \in {}^*\operatorname{Var}(\mathfrak{b})} \phi(\mathfrak{p}; \mathfrak{b})(\operatorname{anch}^i(\mathfrak{p}, M))\right),$$

*a set of points of $\mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$ which arises from consideration of just the $0$th, $1$st, …, $(t-1)$th and $t$th terms of the minimal ${}^*$injective resolution of $M$.*

Our next aim is the establishment of multi-graded analogues of [15, Corollaries 3.1 and 3.2].

3.10. **Lemma.** *Suppose that $R := \bigoplus_{\mathbf{n} \in {\mathbb{N}_0}^r} R_{\mathbf{n}}$ is positively graded, and let $\mathfrak{m}$ be a ${}^*$maximal ideal of $R$. Then $\mathfrak{m}_{\mathbf{0}} := \mathfrak{m} \cap R_{\mathbf{0}}$ is a maximal ideal of $R_{\mathbf{0}}$ and $\mathfrak{m} = \mathfrak{m}_{\mathbf{0}} \oplus \mathfrak{c}$, where $\mathfrak{c}$ is as defined in Notation 3.4.*

*Proof.* Recall that

$$\mathfrak{c} := \bigoplus_{\substack{\mathbf{n} \in {\mathbb{N}_0}^r \\ \mathbf{n} > \mathbf{0}}} R_{\mathbf{n}}.$$

Since $\mathfrak{m}_{\mathbf{0}} \in \operatorname{Spec}(R_{\mathbf{0}})$, it follows that $R \supset \mathfrak{m}_{\mathbf{0}} \bigoplus \mathfrak{c} \supseteq \mathfrak{m}$, so that $\mathfrak{m} = \mathfrak{m}_{\mathbf{0}} \bigoplus \mathfrak{c}$. Furthermore, $\mathfrak{m}_{\mathbf{0}}$ must be a maximal ideal of $R_{\mathbf{0}}$. □

3.11. **Corollary.** *Suppose that $R := \bigoplus_{\mathbf{n} \in {\mathbb{N}_0}^r} R_{\mathbf{n}}$ is positively graded and standard. Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module; let $\mathfrak{b}$ be an ${\mathbb{N}_0}^r$-graded ideal of $R$ such that* $\operatorname{dir}(\mathfrak{b}) \neq \emptyset$. *Then*

$$\max\left(\bigcup_{i \in \mathbb{N}_0} \operatorname{end}(H^i_{\mathfrak{b}}(M))\right) = \max\left(\bigcup_{\mathfrak{m} \in {}^*\operatorname{Var}(\mathfrak{b}) \cap {}^*\operatorname{Max}(R)} \bigcup_{i \in \mathbb{N}_0} \phi(\mathfrak{m}; \mathfrak{b}) \operatorname{end}(H^i_{\mathfrak{m}}(M))\right).$$

*Proof.* Let $\mathfrak{m} \in {}^*\operatorname{Var}(\mathfrak{b}) \cap {}^*\operatorname{Max}(R)$. By Lemma 3.10, $\operatorname{dir}(\mathfrak{m}) = \{1, \ldots, r\}$; therefore, by Theorem 3.6, $\max\left(\bigcup_{i \in \mathbb{N}_0} \operatorname{end}(H^i_{\mathfrak{m}}(M))\right) = \max\left(\bigcup_{i \in \mathbb{N}_0} \operatorname{anch}^i(\mathfrak{m}, M)\right)$. Another use of Theorem 3.6 therefore shows that

$$\begin{aligned}\max\left(\bigcup_{i\in\mathbb{N}_0}\phi(\mathfrak{m};\mathfrak{b})(\operatorname{end}(H^i_{\mathfrak{m}}(M)))\right) &= \max\left(\bigcup_{i\in\mathbb{N}_0}\phi(\mathfrak{m};\mathfrak{b})(\operatorname{anch}^i(\mathfrak{m},M))\right)\\ &\preccurlyeq \max\left(\bigcup_{i\in\mathbb{N}_0}\bigcup_{\mathfrak{p}\in{}^*\operatorname{Var}(\mathfrak{b})}\phi(\mathfrak{p};\mathfrak{b})(\operatorname{anch}^i(\mathfrak{p},M))\right)\\ &= \max\left(\bigcup_{i\in\mathbb{N}_0}\operatorname{end}(H^i_{\mathfrak{b}}(M))\right).\end{aligned}$$

We have thus proved that

$$\max\left(\bigcup_{i\in\mathbb{N}_0}\operatorname{end}(H^i_{\mathfrak{b}}(M))\right)\succcurlyeq\max\left(\bigcup_{\mathfrak{m}\in{}^*\operatorname{Var}(\mathfrak{b})\cap{}^*\operatorname{Max}(R)}\bigcup_{i\in\mathbb{N}_0}\phi(\mathfrak{m};\mathfrak{b})(\operatorname{end}(H^i_{\mathfrak{m}}(M)))\right).$$

Now let $\mathbf{n}\in\mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$ be a maximal member of $\bigcup_{i\in\mathbb{N}_0}\operatorname{end}(H^i_{\mathfrak{b}}(M))$. By Theorem 3.6, there exist $s\in\mathbb{N}_0$ and $\mathfrak{p}\in{}^*\operatorname{Var}(\mathfrak{b})$ such that $\mathbf{n}=\phi(\mathfrak{p};\mathfrak{b})(\mathbf{w})$ for some $s$th level anchor point $\mathbf{w}$ of $\mathfrak{p}$ for $M$. Now use Theorem 2.12 repeatedly, in conjunction with a saturated chain (of length $t$ say) of ${\mathbb{N}_0}^r$-graded prime ideals of $R$ with $\mathfrak{p}$ as its smallest term and a *maximal ideal $\mathfrak{m}$ as its largest term: the conclusion is that there exists $\mathbf{v}\in\operatorname{anch}^{s+t}(\mathfrak{m},M)$ such that $\phi(\mathfrak{m};\mathfrak{p})(\mathbf{v})=\mathbf{w}$. Now

$$\mathbf{n}=\phi(\mathfrak{p};\mathfrak{b})(\mathbf{w})=\phi(\mathfrak{p};\mathfrak{b})(\phi(\mathfrak{m};\mathfrak{p})(\mathbf{v}))=\phi(\mathfrak{m};\mathfrak{b})(\mathbf{v}).$$

But, by Theorem 3.6 again, $\mathbf{v}$ is dominated by $\max\left(\bigcup_{i\in\mathbb{N}_0}\operatorname{end}(H^i_{\mathfrak{m}}(M))\right)$; it follows that

$$\max\left(\bigcup_{i\in\mathbb{N}_0}\operatorname{end}(H^i_{\mathfrak{b}}(M))\right)\preccurlyeq\max\left(\bigcup_{\mathfrak{m}\in{}^*\operatorname{Var}(\mathfrak{b})\cap{}^*\operatorname{Max}(R)}\bigcup_{i\in\mathbb{N}_0}\phi(\mathfrak{m};\mathfrak{b})(\operatorname{end}(H^i_{\mathfrak{m}}(M)))\right).$$

The desired conclusion follows. □

3.12. **Corollary.** *Let the situation be as in Corollary* 3.11*, but assume in addition that* $(R_{\mathbf{0}},\mathfrak{m}_{\mathbf{0}})$ *is local and that* $\mathfrak{b}$ *is proper; set* $\mathfrak{m}:=\mathfrak{m}_{\mathbf{0}}\oplus\mathfrak{c}$*, where* $\mathfrak{c}$ *is as defined in Notation* 3.4*. Then*

$$\max\left(\bigcup_{i\in\mathbb{N}_0}\operatorname{end}(H^i_{\mathfrak{b}}(M))\right)=\max\left(\bigcup_{i\in\mathbb{N}_0}\phi(\mathfrak{m};\mathfrak{b})(\operatorname{end}(H^i_{\mathfrak{m}}(M)))\right).$$

*In particular,*

$$\max\left(\bigcup_{i\in\mathbb{N}_0}\operatorname{end}(H^i_{\mathfrak{c}}(M))\right)=\max\left(\bigcup_{i\in\mathbb{N}_0}\operatorname{end}(H^i_{\mathfrak{m}}(M))\right).$$

## 4. Some vanishing results for multi-graded components of local cohomology modules

It is well known that, when $r=1$, if $M$ is a finitely generated $\mathbb{Z}$-graded $R$-module, then there exists $t\in\mathbb{Z}$ such that $H^i_{R_+}(M)_n=0$ for all $i\in\mathbb{N}_0$ and all $n\geq t$; it then follows from [15, Corollary 2.5] that, if $\mathfrak{b}$ is any graded ideal of $R$ with $\mathfrak{b}\supseteq R_+$, then $H^i_{\mathfrak{b}}(M)_n=0$ for all $i\in\mathbb{N}_0$ and all $n\geq t$. One of the aims of this section is to establish a multi-graded analogue of this result.

4.1. **Notation.** Throughout this section, we shall be concerned with the situation where

$$R=\bigoplus_{\mathbf{n}\in{\mathbb{N}_0}^r}R_{\mathbf{n}}$$

is positively graded; we shall only assume that $R$ is standard when this is explicitly stated.

We shall be concerned with the ${\mathbb{N}_0}^r$-graded ideal $R_+$ of $R$ given (see Notation 3.4) by

$$R_+:=\bigoplus_{\substack{\mathbf{n}\in{\mathbb{N}_0}^r\\ \mathbf{n}\geq\mathbf{1}}}R_{\mathbf{n}}.$$

Although it is well known (see Hyry [10, Theorem 1.6]) that, if $M$ is a finitely generated $\mathbb{Z}^r$-graded $R$-module, then $H^i_{R_+}(M)_{(n_1,\ldots,n_r)} = 0$ for all $n_1, \ldots, n_r \gg 0$, we have not been able to find in the literature a proof of the corresponding statement with $R_+$ replaced by an ${\mathbb{N}_0}^r$-graded ideal $\mathfrak{b}$ that contains $R_+$. We present such a proof below, because we think it is of interest in its own right.

4.2. **Theorem.** *Suppose that $R = \bigoplus_{\mathbf{n}\in{\mathbb{N}_0}^r} R_{\mathbf{n}}$ is positively graded; let $M$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module. Let $\mathfrak{b}$ be an ${\mathbb{N}_0}^r$-graded ideal of $R$ such that $\mathfrak{b} \supseteq R_+$. Then there exists $t \in \mathbb{Z}$ such that*

$$H^i_{\mathfrak{b}}(M)_{\mathbf{n}} = 0 \quad \textit{for all } i \in \mathbb{N}_0 \textit{ and all } \mathbf{n} \geq (t, t, \ldots, t).$$

*Proof.* We shall prove this by induction on $r$. In the case where $r = 1$ the result follows from [15, Corollary 2.5], as was explained in the introduction to this section.

Now suppose that $r > 1$ and that the claim has been proved for smaller values of $r$. We define three more ${\mathbb{N}_0}^r$-graded ideals $\mathfrak{a}$, $\mathfrak{c}$ and $\mathfrak{d}$ of $R$, as follows. Set

$$\mathfrak{a} := \bigoplus_{\mathbf{n}=(n_1,\ldots,n_r)\in{\mathbb{N}_0}^r} \mathfrak{a}_{\mathbf{n}} \quad \text{where } \mathfrak{a}_{\mathbf{n}} = \begin{cases} \mathfrak{b}_{\mathbf{n}} & \text{if } n_r = 0, \\ R_{\mathbf{n}} & \text{if } n_r > 0; \end{cases}$$

$$\mathfrak{c} := \bigoplus_{\mathbf{n}=(n_1,\ldots,n_r)\in{\mathbb{N}_0}^r} \mathfrak{c}_{\mathbf{n}} \quad \text{where } \mathfrak{c}_{\mathbf{n}} = \begin{cases} \mathfrak{b}_{\mathbf{n}} & \text{if } (n_1, \ldots, n_{r-1}) \not\geq (1, \ldots, 1), \\ R_{\mathbf{n}} & \text{if } (n_1, \ldots, n_{r-1}) \geq (1, \ldots, 1); \end{cases}$$

and $\mathfrak{d} := \mathfrak{a} + \mathfrak{c}$.

Consider $\mathfrak{a} \cap \mathfrak{c}$: for each $\mathbf{n} = (n_1, \ldots, n_r) \in {\mathbb{N}_0}^r$, the $\mathbf{n}$th component $(\mathfrak{a} \cap \mathfrak{c})_{\mathbf{n}}$ satisfies

$$(\mathfrak{a} \cap \mathfrak{c})_{\mathbf{n}} = \mathfrak{a}_{\mathbf{n}} \cap \mathfrak{c}_{\mathbf{n}} = \begin{cases} \mathfrak{b}_{\mathbf{n}} & \text{if } n_r = 0 \text{ or } (n_1, \ldots, n_{r-1}) \not\geq (1, \ldots, 1), \\ R_{\mathbf{n}} & \text{if } n_r > 0 \text{ and } (n_1, \ldots, n_{r-1}) \geq (1, \ldots, 1). \end{cases}$$

Since $\mathfrak{b} \supseteq R_+$, we see that $\mathfrak{a} \cap \mathfrak{c} = \mathfrak{b}$.

Let $\sigma : \mathbb{Z}^r \longrightarrow \mathbb{Z}^{r-1}$ be the group homomorphism defined by

$$\sigma((n_1, \ldots, n_r)) = (n_1 + n_r, \ldots, n_{r-1} + n_r) \quad \text{for all } (n_1, \ldots, n_r) \in \mathbb{Z}^r.$$

Note that, for $(n_1, \ldots, n_r) \in {\mathbb{N}_0}^r$, we have $(n_1 + n_r, \ldots, n_{r-1} + n_r) \geq \mathbf{1}$ in $\mathbb{Z}^{r-1}$ if and only if $n_r \geq 1$ or $(n_1, \ldots, n_{r-1}) \geq \mathbf{1}$; furthermore, if $n_r \geq 1$, then $\mathfrak{a}_{\mathbf{n}} = R_{\mathbf{n}}$, and if $(n_1, \ldots, n_{r-1}) \geq \mathbf{1}$, then $\mathfrak{c}_{\mathbf{n}} = R_{\mathbf{n}}$. Let $\mathbf{m} \in \mathbb{Z}^{r-1}$ with $\mathbf{m} \geq \mathbf{1}$. Therefore, in the ${\mathbb{N}_0}^{r-1}$-graded ring $R^\sigma$, we have

$$(\mathfrak{d}^\sigma)_{\mathbf{m}} = \bigoplus_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ \sigma(\mathbf{n})=\mathbf{m}}} (\mathfrak{a}_{\mathbf{n}} + \mathfrak{c}_{\mathbf{n}}) = \bigoplus_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ \sigma(\mathbf{n})=\mathbf{m}}} R_{\mathbf{n}} = (R^\sigma)_{\mathbf{m}}.$$

Thus $\mathfrak{d}^\sigma \supseteq \bigoplus_{\mathbf{m}\geq\mathbf{1}} (R^\sigma)_{\mathbf{m}} = (R^\sigma)_+$.

It therefore follows from the inductive hypothesis that there exists $\widetilde{t} \in \mathbb{Z}$ such that $(H^j_{\mathfrak{d}^\sigma}(M^\sigma))_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (\widetilde{t}, \ldots, \widetilde{t})$ in $\mathbb{Z}^{r-1}$. In view of Remark 3.3, this means that $((H^j_{\mathfrak{d}}(M))^\sigma)_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (\widetilde{t}, \ldots, \widetilde{t})$ in $\mathbb{Z}^{r-1}$, so that, for all $j \in \mathbb{N}_0$,

$$H^j_{\mathfrak{d}}(M)_{(n_1,\ldots,n_r)} = 0 \quad \text{whenever } (n_1, \ldots, n_{r-1}, n_r) \geq \left(\tfrac{1}{2}\widetilde{t}, \ldots, \tfrac{1}{2}\widetilde{t}, \tfrac{1}{2}\widetilde{t}\right) \text{ in } \mathbb{Z}^r.$$

We now give two similar, but simpler, arguments. Let $\pi : \mathbb{Z}^r \longrightarrow \mathbb{Z}$ be the group homomorphism given by projection onto the $r$th co-ordinate. Note that, for $\mathbf{n} \in {\mathbb{N}_0}^r$, if $\pi(\mathbf{n}) \geq 1$, then $\mathfrak{a}_{\mathbf{n}} = R_{\mathbf{n}}$. Therefore $\mathfrak{a}^\pi \supseteq (R^\pi)_+$. It therefore follows from the case where $r = 1$ that there exists $\overline{t} \in \mathbb{Z}$ such that $(H^j_{\mathfrak{a}^\pi}(M^\pi))_n = 0$ for all $j \in \mathbb{N}_0$ and all $n \geq \overline{t}$. In view of Remark 3.3, this means that $((H^j_{\mathfrak{a}}(M))^\pi)_n = 0$ for all $j \in \mathbb{N}_0$ and all $n \geq \overline{t}$, that is,

$$H^j_{\mathfrak{a}}(M)_{(n_1,\ldots,n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } n_r \geq \overline{t}.$$

Next, let $\theta : \mathbb{Z}^r \longrightarrow \mathbb{Z}^{r-1}$ be the group homomorphism defined by

$$\theta((n_1, \ldots, n_r)) = (n_1, \ldots, n_{r-1}) \quad \text{for all } (n_1, \ldots, n_r) \in \mathbb{Z}^r.$$

Note that, if $\mathbf{n} \in \mathbb{Z}^r$ has $\theta(\mathbf{n}) \geq \mathbf{1}$ in $\mathbb{Z}^{r-1}$, then $\mathfrak{c}_{\mathbf{n}} = R_{\mathbf{n}}$. Therefore, for $\mathbf{m} \in \mathbb{Z}^{r-1}$ with $\mathbf{m} \geq \mathbf{1}$, we have $(\mathfrak{c}^\theta)_{\mathbf{m}} = (R^\theta)_{\mathbf{m}}$. This means that, in the ${\mathbb{N}_0}^{r-1}$-graded ring $R^\theta$, we have $\mathfrak{c}^\theta \supseteq \bigoplus_{\mathbf{m}\geq\mathbf{1}} (R^\theta)_{\mathbf{m}} = (R^\theta)_+$.

It therefore follows from the inductive hypothesis that there exists $\widehat{t} \in \mathbb{Z}$ such that $(H^j_{\mathfrak{c}^\theta}(M^\theta))_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (\widehat{t}, \ldots, \widehat{t})$ in $\mathbb{Z}^{r-1}$. In view of Remark 3.3, this means that $((H^j_{\mathfrak{c}}(M))^\theta)_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (\widehat{t}, \ldots, \widehat{t})$ in $\mathbb{Z}^{r-1}$, so that

$$H^j_{\mathfrak{c}}(M)_{(n_1,\ldots,n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } (n_1, \ldots, n_{r-1}) \geq (\widehat{t}, \ldots, \widehat{t}).$$

We recall that $\mathfrak{a} \cap \mathfrak{c} = \mathfrak{b}$. There is an exact Mayer–Vietoris sequence (in the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$)

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0_{\mathfrak{d}}(M) & \longrightarrow & H^0_{\mathfrak{c}}(M) \oplus H^0_{\mathfrak{a}}(M) & \longrightarrow & H^0_{\mathfrak{b}}(M) \\
 & \longrightarrow & H^1_{\mathfrak{d}}(M) & \longrightarrow & H^1_{\mathfrak{c}}(M) \oplus H^1_{\mathfrak{a}}(M) & \longrightarrow & H^1_{\mathfrak{b}}(M) \\
 & \longrightarrow & \cdots & & & & \cdots \\
 & \longrightarrow & H^i_{\mathfrak{d}}(M) & \longrightarrow & H^i_{\mathfrak{c}}(M) \oplus H^i_{\mathfrak{a}}(M) & \longrightarrow & H^i_{\mathfrak{b}}(M) \\
 & \longrightarrow & H^{i+1}_{\mathfrak{d}}(M) & \longrightarrow & \cdots . & &
\end{array}$$

It now follows from this Mayer–Vietoris sequence that, if we set $t := \max\{\frac{1}{2}\widetilde{t}, \widehat{t}, \overline{t}\}$, then

$$H^j_{\mathfrak{b}}(M)_{(n_1,\ldots,n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } (n_1, \ldots, n_r) \geq (t, \ldots, t).$$

This completes the inductive step, and the proof. □

We can deduce from the above Theorem 4.2 a vanishing result for multi-graded components of local cohomology modules with respect to a multi-graded ideal that has both directions and non-directions.

4.3. **Corollary.** *Suppose that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard; let $M$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module. Let $\mathfrak{b}$ be an ${\mathbb{N}_0}^r$-graded ideal of $R$ that has some directions and some non-directions: to be precise, and for ease of notation, suppose that $\operatorname{dir}(\mathfrak{b}) = \{m+1, \ldots, r\}$, where $1 \leq m < r$. Then there exists $t \in \mathbb{Z}$ such that, for all $j \in \mathbb{N}_0$, and for all $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$ for which $(n_1, \ldots, n_m, n_{m+1} + \cdots + n_r) \geq (t, \ldots, t)$ in $\mathbb{Z}^{m+1}$, we have $H^j_{\mathfrak{b}}(M)_{\mathbf{n}} = 0$.*

*Note.* As $\mathfrak{b}$ has some directions and $R$ is standard, it follows from Lemma 2.2 that $R_+ \subseteq \mathfrak{b}$, so that Theorem 4.2 yields a $t' \in \mathbb{Z}$ such that $H^i_{\mathfrak{b}}(M)_{\mathbf{n}} = 0$ for all $\mathbf{n} \geq (t', \ldots, t')$. Thus, when $m = r - 1$, the conclusion of Corollary 4.3 already follows from Theorem 4.2.

*Proof.* Without loss of generality, we can, and do, assume that $\mathfrak{b} = \sqrt{\mathfrak{b}}$.

Let $\phi : \mathbb{Z}^r \longrightarrow \mathbb{Z}^{m+1}$ be the group homomorphism defined by

$$\phi((n_1, \ldots, n_r)) = (n_1, \ldots, n_m, n_{m+1} + \cdots + n_r) \quad \text{for all } (n_1, \ldots, n_r) \in \mathbb{Z}^r.$$

Let $\mathbf{n} = (n_1, \ldots, n_r) \in {\mathbb{N}_0}^r$ be such that $\phi(\mathbf{n}) \geq \mathbf{1}$ in $\mathbb{Z}^{m+1}$. Then $n_{m+1} + \cdots + n_r \geq 1$, so that one of $n_{m+1}, \ldots, n_r$ is positive. Now $R_{\mathbf{e}_i} \subseteq \sqrt{\mathfrak{b}} = \mathfrak{b}$ for all $i = m+1, \ldots, r$, and since $\mathbf{n} \geq \mathbf{e}_i$ for one of these $i$s, it follows from Lemma 2.2 that $\mathfrak{b} \supseteq R_{\mathbf{n}}$. It therefore follows that, in the ${\mathbb{N}_0}^{m+1}$-graded ring $R^\phi$, we have $\mathfrak{b}^\phi \supseteq \bigoplus_{\mathbf{m} \geq \mathbf{1}} (R^\phi)_{\mathbf{m}} = (R^\phi)_+$.

We can now appeal to Theorem 4.2 to deduce that there exists $t \in \mathbb{Z}$ such that $(H^j_{\mathfrak{b}^\phi}(M^\phi))_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (t, \ldots, t)$ in $\mathbb{Z}^{m+1}$. In view of Remark 3.3, this means that $((H^j_{\mathfrak{b}}(M))^\phi)_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (t, \ldots, t)$ in $\mathbb{Z}^{m+1}$, so that

$$H^j_{\mathfrak{b}}(M)_{(n_1,\ldots,n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } (n_1, \ldots, n_m, n_{m+1} + \cdots + n_r) \geq (t, \ldots, t).$$

□

One of the reasons why we consider that Theorem 4.2 is of interest in its own right concerns the structure of the (multi-)graded components $H^i_{\mathfrak{b}}(M)_{\mathbf{n}}$ ($\mathbf{n} \in \mathbb{Z}^r$) as modules over $R_{\mathbf{0}}$ (the hypotheses and notation here are as in Theorem 4.2). The example in [4, Exercise 15.1.7] shows that these graded components need not be finitely generated $R_{\mathbf{0}}$-modules; however, it is always the case that (for a finitely generated $\mathbb{Z}^r$-graded $R$-module $M$) the (multi-)graded components $H^i_{R_+}(M)_{\mathbf{n}}$ ($\mathbf{n} \in \mathbb{Z}^r$) of the $i$th local cohomology module of $M$ with respect to $R_+$ are finitely generated $R_{\mathbf{0}}$-modules (for all $i \in \mathbb{N}_0$), as we now show.

4.4. **Theorem.** *Suppose that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded; let $M$ be a finitely generated $\mathbb{Z}^r$-graded $R$-module. Then $H^i_{R_+}(M)_{\mathbf{n}}$ is a finitely generated $R_{\mathbf{0}}$-module, for all $i \in \mathbb{N}_0$ and all $\mathbf{n} \in \mathbb{Z}^r$.*

*Note.* In the case where $r = 1$, this result is well known: see [4, Proposition 15.1.5].

*Proof.* We use induction on $i$. When $i = 0$, the claim is immediate from the fact that $H^0_{R_+}(M)$ is isomorphic to a submodule of $M$, and so is finitely generated. So suppose that $i > 0$ and that the claim has been proved for smaller values of $i$, for all finitely generated $\mathbb{Z}^r$-graded $R$-modules.

Recall that all the associated prime ideals of $M$ are $\mathbb{N}_0{}^r$-graded. Set $B(M) := \operatorname{Ass}_R(M) \setminus {}^*\operatorname{Var}(R_+)$, and denote $\#B(M)$ by $b(M)$; we shall argue by induction on $b(M)$. If $b(M) = 0$, then $M$ is $R_+$-torsion, so that $H^i_{R_+}(M) = 0$ and the desired result is clear in this case.

Now suppose that $b(M) = 1$: let $\mathfrak{p}$ be the unique member of $B(M)$. Set $\overline{M} := M/\Gamma_{R_+}(M)$. We can use the long exact sequence of local cohomology modules induced by the exact sequence

$$0 \longrightarrow \Gamma_{R_+}(M) \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0,$$

together with the fact that $H^j_{R_+}(\Gamma_{R_+}(M)) = 0$ for all $j \in \mathbb{N}$, to see that, in order to complete the proof in this case, it is sufficient for us to prove the result for $\overline{M}$. Now $\overline{M}$ is $R_+$-torsion-free, and $\operatorname{Ass}(\overline{M}) = \{\mathfrak{p}\}$. (See [4, Exercise 2.1.12].) There exists a $\mathbb{Z}^r$-homogeneous element $a \in R_+ \setminus \mathfrak{p}$; note that $a$ is a non-zero-divisor on $\overline{M}$. Let the degree of $a$ be $\mathbf{v} = (v_1, \ldots, v_r)$, and note that $v_j > 0$ for all $j = 1, \ldots, r$. By Theorem 4.2, there exists $t \in \mathbb{Z}$ such that $H^j_{R_+}(\overline{M})_{\mathbf{n}} = 0$ for all $\mathbf{n} \geq (t, t, \ldots, t)$.

Let $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$. Since $v_j > 0$ for all $j = 1, \ldots, r$, there exists $w \in \mathbb{N}$ such that $n_j + v_j w \geq t$ for all $j = 1, \ldots, r$. The exact sequence

$$0 \longrightarrow \overline{M} \xrightarrow{a^w} \overline{M}(w\mathbf{v}) \longrightarrow \left(\overline{M}/a^w\overline{M}\right)(w\mathbf{v}) \longrightarrow 0$$

induces an exact sequence of $R_{\mathbf{0}}$-modules

$$H^{i-1}_{R_+}(\overline{M}/a^w\overline{M})_{\mathbf{n}+w\mathbf{v}} \longrightarrow H^i_{R_+}(\overline{M})_{\mathbf{n}} \longrightarrow H^i_{R_+}(\overline{M})_{\mathbf{n}+w\mathbf{v}},$$

and since $w$ was chosen to ensure that the rightmost term in this sequence is zero, it follows from the inductive hypothesis that $H^i_{R_+}(\overline{M})_{\mathbf{n}}$ is a finitely generated $R_{\mathbf{0}}$-module. This completes the proof in the case where $b(M) = 1$.

Now suppose that $b(M) = b > 1$ and that it has been proved that all the graded components of $H^i_{R_+}(L)$ are finitely generated $R_{\mathbf{0}}$-modules for all choices of finitely generated $\mathbb{Z}^r$-graded $R$-module $L$ with $b(L) < b$. Let $\mathfrak{p}, \mathfrak{q} \in B(M)$ with $\mathfrak{p} \neq \mathfrak{q}$: suppose, for the sake of argument, that $\mathfrak{p} \not\subseteq \mathfrak{q}$. Consider the $\mathfrak{p}$-torsion submodule $\Gamma_{\mathfrak{p}}(M)$ of $M$. By [4, Exercise 2.1.12], $\operatorname{Ass}(\Gamma_{\mathfrak{p}}(M))$ and $\operatorname{Ass}(M/\Gamma_{\mathfrak{p}}(M))$ are disjoint and $\operatorname{Ass} M = \operatorname{Ass}(\Gamma_{\mathfrak{p}}(M)) \cup \operatorname{Ass}(M/\Gamma_{\mathfrak{p}}(M))$. Now $\mathfrak{p} \in \operatorname{Ass}(\Gamma_{\mathfrak{p}}(M))$ and $\mathfrak{q} \notin \operatorname{Ass}(\Gamma_{\mathfrak{p}}(M))$; hence $b(\Gamma_{\mathfrak{p}}(M)) < b$ and $b(M/\Gamma_{\mathfrak{p}}(M)) < b$. Therefore, by the inductive hypothesis, both $H^i_{R_+}(\Gamma_{\mathfrak{p}}(M))_{\mathbf{n}}$ and $H^i_{R_+}(M/\Gamma_{\mathfrak{p}}(M))_{\mathbf{n}}$ are finitely generated $R_{\mathbf{0}}$-modules, for all $\mathbf{n} \in \mathbb{Z}^r$. We can now use the long exact sequence of local cohomology modules (with respect to $R_+$) induced from the exact sequence $0 \longrightarrow \Gamma_{\mathfrak{p}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{p}}(M) \longrightarrow 0$ to deduce that $H^i_{R_+}(M)_{\mathbf{n}}$ is a finitely generated $R_{\mathbf{0}}$-module for all $\mathbf{n} \in \mathbb{Z}^r$. The result follows. □

## 5. A multi-graded analogue of Marley's work on finitely graded local cohomology modules

As was mentioned in the Introduction, the purpose of this section is to obtain some multi-graded analogues of results about finitely graded local cohomology modules that were proved, in the case where $r = 1$, by Marley in [14]. We shall present a multi-graded analogue of one of Marley's results and some extensions of that analogue.

5.1. **Notation.** Throughout this section, we shall be concerned with the situation where $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0{}^r} R_{\mathbf{n}}$ is positively graded and standard, and we shall let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a $\mathbb{Z}^r$-graded $R$-module. Also, $\mathfrak{b}$ will always denote an $\mathbb{N}_0{}^r$-graded ideal of $R$.

For $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}_0{}^r$, we shall denote $\{i \in \{1, \ldots, r\} : n_i \neq 0\}$ by $\mathcal{P}(\mathbf{n})$.

5.2. **Definition.** An $r$-tuple $\mathbf{n} \in \mathbb{Z}^r$ is called a *supporting degree of* $M$ precisely when $M_{\mathbf{n}} \neq 0$; we denote the set of all supporting degrees of $M$ by $\mathcal{S}(M)$.

Note that Theorem 4.2 imposes substantial restrictions on $\mathcal{S}(H^i_{\mathfrak{b}}(M))$ when ($i \in \mathbb{N}_0$ and) $\mathfrak{b} \supseteq R_+$. The example below is included as motivation for the introduction of some notation.

5.3. **Example.** Let $k$ be an algebraically closed field and let $A = k \oplus A_1 \oplus \cdots \oplus A_m \oplus \cdots$ and $B = k \oplus B_1 \oplus \cdots \oplus B_n \oplus \cdots$ be two normal Noetherian standard $\mathbb{N}_0$-graded $k$-algebra domains with $w := \dim A > 1$ and $v := \dim B > 1$. We consider the ${\mathbb{N}_0}^2$-graded $k$-algebra

$$R := A \otimes_k B = \bigoplus_{(m,n)\in {\mathbb{N}_0}^2} A_m \otimes_k B_n.$$

Clearly $R = k[R_{(1,0)}, R_{(0,1)}]$ is positively graded and standard, and, as a finitely generated $k$-algebra, is Noetherian. By [17, Chapter III, §15, Theorem 40, Corollary 1], $R$ is again an integral domain. Observe that $R_+ = R_{(1,1)}R = A_+ \otimes_k B_+$. As $A$ and $B$ are normal and their dimensions exceed 1, we have $H^i_{A_+}(A) = H^i_{B_+}(B) = 0$ for $i = 0, 1$. The Künneth relations for tensor products (see [7] or [13, Theorem 10.1]) yield, for each $i \in \mathbb{N}_0$, an isomorphism of $\mathbb{Z}^2$-graded $R$ modules

$$H^i_{R_+}(R) \cong \Big(A \otimes_k H^i_{B_+}(B)\Big) \oplus \Big(H^i_{A_+}(A) \otimes_k B\Big) \oplus \left(\bigoplus_{\substack{j,l\in\mathbb{N}\setminus\{1\}\\ j+l=i+1}} \Big(H^j_{A_+}(A) \otimes_k H^l_{B_+}(B)\Big)\right).$$

As $\mathcal{S}(A) = \mathcal{S}(B) = \mathbb{N}_0$, it follows that, for each $i \in \mathbb{N}_0$,

$$\mathcal{S}(H^i_{R_+}(R)) = \Big(\mathbb{N}_0 \times \mathcal{S}(H^i_{B_+}(B))\Big) \cup \Big(\mathcal{S}(H^i_{A_+}(A)) \times \mathbb{N}_0\Big) \cup \left(\bigcup_{\substack{j,l\in\mathbb{N}\setminus\{1\}\\ j+l=i+1}} \Big(\mathcal{S}(H^j_{A_+}(A)) \times \mathcal{S}(H^l_{B_+}(B))\Big)\right).$$

Observe, in particular, that $H^i_{R_+}(R) = 0$ for $i = 0, 1$ and for all $i \geq w + v$.

Appropriate choices for $A$ and $B$ yield many examples for $R$. We shall just concentrate on a class of examples obtained by this procedure when $A$ and $B$ are chosen in a particular way, which we now describe. We can use [2, Proposition (2.13)], in conjunction with the Serre–Grothendieck correspondence (see [4, 20.4.4]), to choose the algebra $A$ (as above) so that, for a prescribed set $W \subseteq \{2, \ldots, w-1\}$, we have

$$\mathcal{S}(H^i_{A_+}(A)) = \begin{cases} \emptyset & \text{for all } i \in \mathbb{N}_0 \setminus (W \cup \{w\}), \\ \{0\} & \text{for all } i \in W, \\ \{k \in \mathbb{Z} : k < 0\} & \text{for } i = w. \end{cases}$$

Similarly, for a prescribed set $V \subseteq \{2, \ldots, v-1\}$, we choose $B$ (as above) so that

$$\mathcal{S}(H^i_{B_+}(B)) = \begin{cases} \emptyset & \text{for all } i \in \mathbb{N}_0 \setminus (V \cup \{v\}), \\ \{0\} & \text{for all } i \in V, \\ \{k \in \mathbb{Z} : k < 0\} & \text{for } i = v. \end{cases}$$

With such a choice of $A$ for $w = 5$ and $W = \{2\}$, and such a choice of $B$ for $v = 5$ and $V = \{3\}$, the sets of supporting degrees $\mathcal{S}(H^i_{R_+}(R))$ for $i = 2, 3, 4, 5$ are as in Figure 1 below.

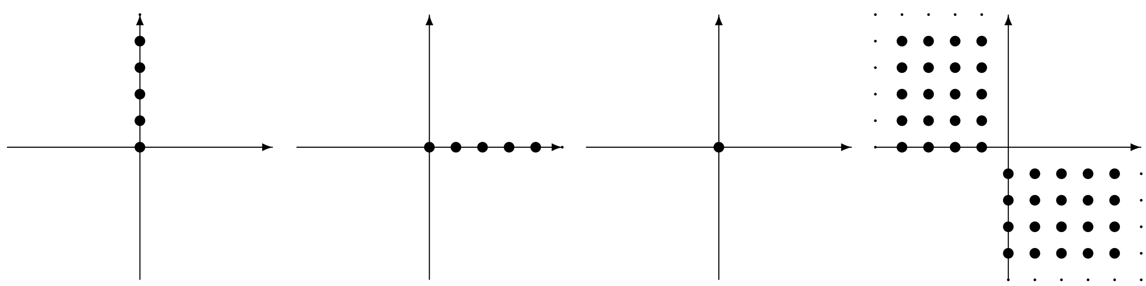

FIGURE 1. $\mathcal{S}(H^i_{R_+}(R))$ for $i = 2, 3, 4, 5$ respectively

In view of Theorem 4.2, the supporting set $\mathcal{S}(H^5_{R_+}(R))$ seems unremarkable. The local cohomology module $H^4_{R_+}(R)$ is finitely graded. Although neither $H^3_{R_+}(R)$ nor $H^2_{R_+}(R)$ is finitely graded, both have sets of supporting degrees that are quite restricted.

We now return to the general situation described in Notation 5.1. In the case where $r = 1$, one way of recording that a local cohomology module $H^i_{\mathfrak{b}}(M)$ is finitely graded is to state that there exist $s, t \in \mathbb{Z}$ with $s < t$ such that

$$\mathcal{S}(H^i_{\mathfrak{b}}(M)) = \left\{n \in \mathbb{Z} : H^i_{\mathfrak{b}}(M)_n \neq 0\right\} \subseteq \{n \in \mathbb{Z} : s \le n < t\}.$$

One might expect the natural generalization to our multi-graded situation to involve conditions such as

$$\mathcal{S}(H^i_{\mathfrak{b}}(M)) = \left\{\mathbf{n} \in \mathbb{Z}^r : H^i_{\mathfrak{b}}(M)_{\mathbf{n}} \neq 0\right\} \subseteq \{\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r : s_i \le n_i < t_i \text{ for all } i = 1, \ldots, r\},$$

where $\mathbf{s} = (s_1, \ldots, s_r)$, $\mathbf{t} = (t_1, \ldots, t_r) \in \mathbb{Z}^r$ satisfy $\mathbf{s} \le \mathbf{t}$. However, in the light of evidence like that provided by Example 5.3 above, and other examples, we introduce the following.

5.4. **Notation.** Let $\mathbf{s} = (s_1, \ldots, s_r)$, $\mathbf{t} = (t_1, \ldots, t_r) \in \mathbb{Z}^r$ with $\mathbf{s} \le \mathbf{t}$. We set

$$\mathbb{X}(\mathbf{s}, \mathbf{t}) := \{\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r : \text{ there exists } i \in \{1, \ldots, r\} \text{ such that } s_i \le n_i < t_i\}.$$

5.5. **Example.** Figure 2 below illustrates, in the case where $r = 2$, the set $\mathbb{X}((-2, 1), (0, 2))$.

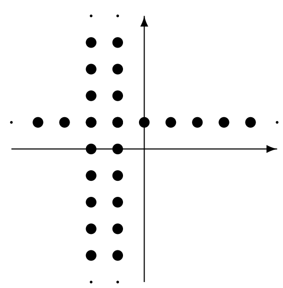

FIGURE 2. The set $\mathbb{X}((-2, 1), (0, 2))$ in $\mathbb{Z}^2$

5.6. *Remark.* Let $\mathbf{s}, \mathbf{s}', \mathbf{s}'', \mathbf{t}, \mathbf{t}', \mathbf{t}'' \in \mathbb{Z}^r$ with $\mathbf{s} \le \mathbf{t}$, $\mathbf{s}' \le \mathbf{t}'$ and $\mathbf{s}'' \le \mathbf{t}''$. Let $\mathbf{m} \in {\mathbb{N}_0}^r \setminus \{\mathbf{0}\}$.

(i) Clearly $(\mathbf{s} + {\mathbb{N}_0}^r) \setminus (\mathbf{t} + {\mathbb{N}_0}^r) \subseteq \mathbb{X}(\mathbf{s}, \mathbf{t})$.

(ii) Suppose that $\mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{P}(\mathbf{m})$. Let $\mathbf{w} \in \mathbb{Z}^r$ be such that $\mathcal{P}(\mathbf{w}) \subseteq \{1, \ldots, r\} \setminus \mathcal{P}(\mathbf{m})$. Then

$$\mathbb{X}(\mathbf{s} + \mathbf{w}, \mathbf{t} + \mathbf{w}) = \mathbb{X}(\mathbf{s}, \mathbf{t}) = \{\mathbf{n} \in \mathbb{Z}^r : \text{ there exists } i \in \mathcal{P}(\mathbf{m}) \text{ such that } s_i \le n_i < t_i\}.$$

(iii) Clearly $\mathbb{X}(\mathbf{s}', \mathbf{t}') \cup \mathbb{X}(\mathbf{s}'', \mathbf{t}'') \subseteq \mathbb{X}(\min\{\mathbf{s}', \mathbf{s}''\}, \max\{\mathbf{t}', \mathbf{t}''\})$.

(iv) Assume that $\mathcal{P}(\mathbf{t}' - \mathbf{s}') \subseteq \mathcal{P}(\mathbf{m})$ and $\mathcal{P}(\mathbf{t}'' - \mathbf{s}'') \subseteq \mathcal{P}(\mathbf{m})$. For each $i \in \{1, \ldots, r\}$, set

$$\widetilde{s}_i := \min\{s'_i, s''_i\} \quad \text{and} \quad \widetilde{t}_i := \begin{cases} \max\{t'_i, t''_i\} & \text{if } i \in \mathcal{P}(\mathbf{m}), \\ \widetilde{s}_i & \text{if } i \in \{1, \ldots, r\} \setminus \mathcal{P}(\mathbf{m}). \end{cases}$$

Set $\widetilde{\mathbf{s}} := (\widetilde{s}_1, \ldots, \widetilde{s}_r)$ and $\widetilde{\mathbf{t}} := (\widetilde{t}_1, \ldots, \widetilde{t}_r)$. Then

$$\widetilde{\mathbf{s}} \le \widetilde{\mathbf{t}}, \quad \mathcal{P}(\widetilde{\mathbf{t}} - \widetilde{\mathbf{s}}) \subseteq \mathcal{P}(\mathbf{m}) \quad \text{and} \quad \mathbb{X}(\mathbf{s}', \mathbf{t}') \cup \mathbb{X}(\mathbf{s}'', \mathbf{t}'') \subseteq \mathbb{X}(\widetilde{\mathbf{s}}, \widetilde{\mathbf{t}}).$$

The next lemma provides a small hint about the importance of the sets $\mathbb{X}(\mathbf{s}, \mathbf{t})$ of Notation 5.4 for our work.

5.7. **Lemma.** *Let $\mathbf{m} \in {\mathbb{N}_0}^r \setminus \{\mathbf{0}\}$. Assume that $M$ is finitely generated and that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$. Then there exist $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^r$ such that $\mathbf{s} \le \mathbf{t}$, $\mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{P}(\mathbf{m})$ and $\mathcal{S}(M) \subseteq (\mathbf{s} + {\mathbb{N}_0}^r) \setminus (\mathbf{t} + {\mathbb{N}_0}^r)$, so that $\mathcal{S}(M) \subseteq \mathbb{X}(\mathbf{s}, \mathbf{t})$ in view of Remark* 5.6(i).

*Proof.* As $M$ is finitely generated, there exist $\mathbf{s}, \mathbf{w} \in \mathbb{Z}^r$ such that $\mathbf{s} \leq \mathbf{w}$ and $M = \sum_{\mathbf{s} \leq \mathbf{n} \leq \mathbf{w}} RM_{\mathbf{n}}$. In particular, $\mathcal{S}(M) \subseteq \mathbf{s} + {\mathbb{N}_0}^r$.

Moreover, there exists $u \in \mathbb{N}$ such that $(R_{\mathbf{m}})^u \subseteq (0 :_R M)$; since $R$ is standard, $(R_{\mathbf{m}})^u = R_{u\mathbf{m}}$; hence $R_{u\mathbf{m}} M_{\mathbf{n}} = 0$ for all $\mathbf{n} \in \mathbb{Z}^r$.

Let $\mathbf{t} = \mathbf{s} + \sum_{i \in \mathcal{P}(\mathbf{m})} (w_i - s_i + um_i)\mathbf{e}_i$. Now, let $\mathbf{h} = (h_1, \ldots, h_r) \in \mathbf{t} + {\mathbb{N}_0}^r$. Our proof will be complete once we have shown that $M_{\mathbf{h}} = 0$. For each $i \in \mathcal{P}(\mathbf{m})$, we have $h_i \geq t_i = w_i + um_i$. Moreover,

$$M_{\mathbf{h}} = \sum_{\mathbf{n} \in \mathcal{T}} R_{\mathbf{h}-\mathbf{n}} M_{\mathbf{n}}, \quad \text{where } \mathcal{T} = \{\mathbf{n} \in \mathbb{Z}^r : \mathbf{s} \leq \mathbf{n} \leq \mathbf{w},\ \mathbf{n} \leq \mathbf{h}\}.$$

Let $\mathbf{n} = (n_1, \ldots, n_r) \in \mathcal{T}$. If $i \in \mathcal{P}(\mathbf{m})$, then $n_i + um_i \leq w_i + um_i \leq h_i$; if $i \in \{1, \ldots, r\} \setminus \mathcal{P}(\mathbf{m})$, then $n_i + um_i = n_i \leq h_i$. Consequently $\mathbf{n} + u\mathbf{m} \leq \mathbf{h}$ . Therefore $u\mathbf{m} \leq \mathbf{h} - \mathbf{n}$ for all $\mathbf{n} \in \mathcal{T}$, and hence

$$M_{\mathbf{h}} = \sum_{\mathbf{n} \in \mathcal{T}} R_{\mathbf{h}-\mathbf{n}} M_{\mathbf{n}} = \sum_{\mathbf{n} \in \mathcal{T}} R_{\mathbf{h}-\mathbf{n}-u\mathbf{m}} R_{u\mathbf{m}} M_{\mathbf{n}} = 0.$$

□

5.8. **Definition.** Let $\mathcal{Q} \subseteq \{1, \ldots, r\}$. By a *$\mathcal{Q}$-domain in $\mathbb{Z}^r$* we mean a set of the form

$$\mathbb{X}(\mathbf{s}, \mathbf{t}) \quad \text{with } \mathbf{s}, \mathbf{t} \in \mathbb{Z}^r,\ \mathbf{s} \leq \mathbf{t} \text{ and } \mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{Q}.$$

5.9. *Remarks.* The following statements are immediate from the definition.

(i) A $\emptyset$-domain in $\mathbb{Z}^r$ is empty.
(ii) If $\mathcal{Q} \subseteq \mathcal{Q}' \subseteq \{1, \ldots, r\}$ and if $\mathbb{X}$ is a $\mathcal{Q}$-domain in $\mathbb{Z}^r$, then $\mathbb{X}$ is a $\mathcal{Q}'$-domain in $\mathbb{Z}^r$.
(iii) If $\mathbb{X}$ is a $\mathcal{Q}$-domain in $\mathbb{Z}^r$ and $\mathbf{w} \in \mathbb{Z}^r$, then $\mathbf{w} + \mathbb{X} := \{\mathbf{w} + \mathbf{n} : \mathbf{n} \in \mathbb{X}\}$ is a $\mathcal{Q}$-domain in $\mathbb{Z}^r$.
(iv) If $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^r$ with $\mathbf{s} \leq \mathbf{t}$ and $\mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{Q}$, then $(\mathbf{s} + {\mathbb{N}_0}^r) \setminus (\mathbf{t} + {\mathbb{N}_0}^r)$ is contained in a $\mathcal{Q}$-domain in $\mathbb{Z}^r$, by Remark 5.6(i).
(v) If $\mathbb{X}$ is a $\mathcal{Q}$-domain in $\mathbb{Z}^r$ and $\mathbf{w} \in \mathbb{Z}^r$ is such that $\mathcal{P}(\mathbf{w}) \cap \mathcal{Q} = \emptyset$, then $\mathbb{X} = \mathbf{w} + \mathbb{X}$, by Remark 5.6(ii).
(vi) By Remark 5.6(iv), the union of finitely many $\mathcal{Q}$-domains in $\mathbb{Z}^r$ is contained in a $\mathcal{Q}$-domain in $\mathbb{Z}^r$.

5.10. **Lemma.** *Let $\mathbf{m}, \mathbf{k} \in {\mathbb{N}_0}^r \setminus \{\mathbf{0}\}$, and let $T$ be a $\mathbb{Z}^r$-graded $R$-module such that $R_{\mathbf{m}} T = 0$. Let $y \in R_{\mathbf{k}}$, and let $K$ denote the kernel of the homogeneous $R$-homomorphism $T \longrightarrow T(\mathbf{k})$ given by multiplication by $y$.*

(i) *If $\mathcal{P}(\mathbf{m}) \subseteq \mathcal{P}(\mathbf{k})$, then there exists $v \in \mathbb{N}_0$ such that $\mathcal{S}(T) \subseteq \bigcup_{j=0}^{v} (\mathcal{S}(K) - j\mathbf{k})$.*
(ii) *If $\mathcal{P}(\mathbf{m}) \not\subseteq \mathcal{P}(\mathbf{k})$, if multiplication by $y$ provides an isomorphism $T \xrightarrow{\cong} T(\mathbf{k})$, and if $T$ considered as an $R_y$-module is finitely generated, then $\mathcal{S}(T)$ is contained in a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$-domain in $\mathbb{Z}^r$.*

*Proof.* Write $\mathbf{m} = (m_1, \ldots, m_r)$ and $\mathbf{k} = (k_1, \ldots, k_r)$. Let $u \in \mathbb{N}$ be such that $m_i \leq uk_i$ for all $i \in \mathcal{P}(\mathbf{k})$. Set $\mathbf{h} := \sum_{i \in \{1, \ldots, r\} \setminus \mathcal{P}(\mathbf{k})} m_i \mathbf{e}_i$. Then, if $i \in \mathcal{P}(\mathbf{k})$, we have $(u\mathbf{k} + \mathbf{h})_i = uk_i \geq m_i$, whereas, if $i \in \{1, \ldots, r\} \setminus \mathcal{P}(\mathbf{k})$, we have $(u\mathbf{k} + \mathbf{h})_i = uk_i + m_i \geq m_i$. Therefore $\mathbf{m} \leq u\mathbf{k} + \mathbf{h}$.

Now, let $z \in R_{\mathbf{h}}$. Then, because $R$ is standard, $y^u z \in R_{u\mathbf{k}+\mathbf{h}} = R_{u\mathbf{k}+\mathbf{h}-\mathbf{m}} R_{\mathbf{m}}$. As $R_{\mathbf{m}} T = 0$, it follows that $y^u z T = 0$. Therefore $y^u R_{\mathbf{h}} T = 0$.

(i) Assume that $\mathcal{P}(\mathbf{m}) \subseteq \mathcal{P}(\mathbf{k})$. Then $\mathcal{P}(\mathbf{h}) = \mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}) = \emptyset$, so that $\mathbf{h} = \mathbf{0}$. Hence $y^u T = y^u R_{\mathbf{0}} T = 0$.

Now let $\mathbb{K} := \bigcup_{j=0}^{u-1} (\mathcal{S}(K) - j\mathbf{k})$, and let $\mathbf{n} \in \mathbb{Z}^r \setminus \mathbb{K}$. If we show that $T_{\mathbf{n}} = 0$, then we shall have proved part (i). Now $\mathbf{n} + j\mathbf{k} \notin \mathcal{S}(K)$ for all $j \in \{0, \ldots, u-1\}$, and so the $R_{\mathbf{0}}$-homomorphism $y^u : T_{\mathbf{n}} \longrightarrow T_{\mathbf{n}+u\mathbf{k}}$, which is the composition of the $R_{\mathbf{0}}$-homomorphisms $y : T_{\mathbf{n}+j\mathbf{k}} \longrightarrow T_{\mathbf{n}+(j+1)\mathbf{k}}$ for $j = 0, \ldots, u-1$, is injective. But $y^u T_{\mathbf{n}} = 0$, and so $T_{\mathbf{n}} = 0$.

(ii) Now assume that $\mathcal{P}(\mathbf{m}) \not\subseteq \mathcal{P}(\mathbf{k})$, that multiplication by $y$ provides an isomorphism $T \xrightarrow{\cong} T(\mathbf{k})$, and that $T$ considered as an $R_y$-module is finitely generated. As $y^u R_{\mathbf{h}} T = 0$, it follows that $R_{\mathbf{h}} T = 0$.

As $T$ is finitely generated over $R_y$, there are finitely many $r$-tuples $\mathbf{g}^{(1)}, \ldots, \mathbf{g}^{(q)} \in \mathbb{Z}^r$ such that $T = \sum_{j=1}^{q} R_y T_{\mathbf{g}^{(j)}}$. Now, for $i \in \{1, \ldots, r\}$, set

$$s_i := \begin{cases} 0 & \text{if } i \notin \mathcal{P}(\mathbf{h}), \\ \min\{g_i^{(j)} : j = 1, \ldots, q\} & \text{if } i \in \mathcal{P}(\mathbf{h}), \end{cases} \qquad t_i := \begin{cases} 0 & \text{if } i \notin \mathcal{P}(\mathbf{h}), \\ \max\{g_i^{(j)} : j = 1, \ldots, q\} + h_i & \text{if } i \in \mathcal{P}(\mathbf{h}), \end{cases}$$

and put $\mathbf{s} = (s_1, \ldots, s_r)$, $\mathbf{t} = (t_1, \ldots, t_r)$. Then $\mathbf{s} \leq \mathbf{t}$ and $\mathcal{P}(\mathbf{t} - \mathbf{s}) = \mathcal{P}(\mathbf{h}) = \mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k})$. Let $\mathbf{n} \in \mathbb{Z}^r \setminus \mathbb{X}(\mathbf{s}, \mathbf{t})$. If we show that $T_{\mathbf{n}} = 0$, then we shall have proved part (ii). Let $\alpha \in T_{\mathbf{n}}$. There exist integers $v_1, \ldots, v_q$ such that $\alpha \in \sum_{j=1}^{q} y^{v_j} R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} T_{\mathbf{g}^{(j)}}$.

Note that, for each $i \in \mathcal{P}(\mathbf{h}) = \mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k})$, we have either $n_i < s_i$ or $t_i \leq n_i$ (because $\mathbf{n} \notin \mathbb{X}(\mathbf{s}, \mathbf{t})$).

Assume first that there is some $i \in \mathcal{P}(\mathbf{h})$ with $n_i < s_i$. As $i \notin \mathcal{P}(\mathbf{k})$, it follows that

$$(\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)})_i = n_i - v_j k_i - g_i^{(j)} = n_i - g_i^{(j)} < s_i - g_i^{(j)} \leq 0,$$

for all $j \in \{1, \ldots, q\}$, so that $R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} = 0$ and $\alpha = 0$.

Therefore, we can, and do, assume that $t_i \leq n_i$ for all $i \in \mathcal{P}(\mathbf{h})$. In this case, for each $i \in \mathcal{P}(\mathbf{h})$ and each $j \in \{1, \ldots, q\}$, we have

$$(\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)})_i = n_i - v_j k_i - g_i^{(j)} = n_i - g_i^{(j)} \geq t_i - g_i^{(j)} \geq h_i.$$

Therefore, for each $j \in \{1, \ldots, q\}$, either $\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)} \geq \mathbf{h}$, or $\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}$ has a negative component and $R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} = 0$. This means that

$$\alpha \in \sum_{j=1}^{q} y^{v_j} R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} T_{\mathbf{g}^{(j)}} = \sum_{\substack{j=1 \\ \mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)} \geq \mathbf{0}}}^{q} y^{v_j} R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)} - \mathbf{h}} R_{\mathbf{h}} T_{\mathbf{g}^{(j)}} = 0.$$

It follows that $T_{\mathbf{n}} = 0$, as required. □

5.11. **Lemma.** *Let $\mathbf{m} \in {\mathbb{N}_0}^r \setminus \{\mathbf{0}\}$ and $\mathbf{k} \in {\mathbb{N}_0}^r$. Assume that $M$ is finitely generated and that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$. Let $y \in R_{\mathbf{k}}$. Then there exists a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$-domain $\mathbb{X}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H_{yR}^1(M)) \subseteq \mathbb{X}$.*

*Proof.* Assume first that $\mathbf{k} = \mathbf{0}$. Then $\mathcal{P}(\mathbf{k}) = \emptyset$ and, by the multi-graded analogue of [4, Lemma 13.1.10], there are $R_{\mathbf{0}}$-isomorphisms $H_{yR}^1(M)_{\mathbf{n}} \cong H_{yR_{\mathbf{0}}}^1(M_{\mathbf{n}})$ for all $\mathbf{n} \in \mathbb{Z}^r$. Therefore $\mathcal{S}(H_{yR}^1(M)) \subseteq \mathcal{S}(M)$, and the claim follows in this case from Lemma 5.7.

We now deal with the remaining case, where $\mathbf{k} \neq \mathbf{0}$. Since (by the multi-graded analogue of [4, 12.4.2]) there is a $\mathbb{Z}^r$-homogeneous epimorphism of $\mathbb{Z}^r$-graded $R$-modules $D_{yR}(M) \longrightarrow H_{yR}^1(M)$, it suffices for us to show that $\mathcal{S}(D_{yR}(M))$ is contained in a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$-domain in $\mathbb{Z}^r$.

Recall that there is a homogeneous isomorphism $D_{yR}(M) \cong M_y$, and so the multiplication map $y : D_{yR}(M) \longrightarrow D_{yR}(M)(\mathbf{k})$ is an isomorphism, and $D_{yR}(M)$ is finitely generated as an $R_y$-module. Since $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$, there exists $u \in \mathbb{N}$ such that $R_{u\mathbf{m}} M = 0$, so that $R_{u\mathbf{m}} M_y = 0$ and $R_{u\mathbf{m}} D_{yR}(M) = 0$. Observe that $\mathcal{P}(u\mathbf{m}) = \mathcal{P}(\mathbf{m})$. We now apply Lemma 5.10, with $D_{yR}(M)$ as the module $T$ and $u\mathbf{m}$ in the rôle of $\mathbf{m}$: if $\mathcal{P}(u\mathbf{m}) = \mathcal{P}(\mathbf{m}) \subseteq \mathcal{P}(\mathbf{k})$, then part (i) of Lemma 5.10 yields that $\mathcal{S}(D_{yR}(M)) = \emptyset$, while if $\mathcal{P}(u\mathbf{m}) = \mathcal{P}(\mathbf{m}) \not\subseteq \mathcal{P}(\mathbf{k})$, then it follows from part (ii) of Lemma 5.10 that $\mathcal{S}(D_{yR}(M))$ is contained in a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$-domain in $\mathbb{Z}^r$. □

5.12. **Lemma.** *Let $\mathbf{m} \in {\mathbb{N}_0}^r \setminus \{\mathbf{0}\}$. Assume that $M$ is finitely generated and that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$. Then there exists a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H_{\mathfrak{b}}^i(M)) \subseteq \mathbb{X}$ for all $i \in \mathbb{N}_0$.*

*Proof.* Since $H_{\mathfrak{b}}^i(M) = 0$ for all $i > \operatorname{ara}(\mathfrak{b})$, it follows from Remark 5.9(vi) that it is sufficient for us to show that, for each $i \in \mathbb{N}_0$, there exists a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}_i$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H_{\mathfrak{b}}^i(M)) \subseteq \mathbb{X}_i$. For $i = 0$, this is immediate from Lemma 5.7.

Let $y_1, \ldots, y_s$ be ${\mathbb{N}_0}^r$-homogeneous elements of $R$ that generate $\mathfrak{b}$. We argue by induction on $s$. When $s = 1$ and $i = 1$, the desired result follows from Lemma 5.11; as we have already dealt, in the preceding paragraph, with the case where $i = 0$, and as $H_{y_1 R}^i(M) = 0$ for all $i > 1$, we have established the desired result in all cases when $s = 1$.

So suppose now that $s > 1$ and that the desired result has been proved in all cases where $\mathfrak{b}$ can be generated by fewer than $s$ ${\mathbb{N}_0}^r$-homogeneous elements. Again, we have already dealt with the case where $i = 0$. For $i \in \mathbb{N}$, there is an exact Mayer–Vietoris sequence (in the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$)

$$\cdots \longrightarrow H^{i-1}_{(y_1y_s,\ldots,y_{s-1}y_s)R}(M) \longrightarrow H^i_{\mathfrak{b}}(M) \longrightarrow H^i_{(y_1,\ldots,y_{s-1})R}(M) \oplus H^i_{y_sR}(M) \longrightarrow \cdots .$$

By the inductive hypothesis, there exist $\mathcal{P}(\mathbf{m})$-domains $\mathbb{X}'_i, \mathbb{X}''_i, \mathbb{X}'''_i$ in $\mathbb{Z}^r$ such that

$$\mathcal{S}(H^{i-1}_{(y_1y_s,\ldots,y_{s-1}y_s)R}(M)) \subseteq \mathbb{X}'_i, \quad \mathcal{S}(H^i_{(y_1,\ldots,y_{s-1})R}(M)) \subseteq \mathbb{X}''_i \quad \text{and} \quad \mathcal{S}(H^i_{y_sR}(M)) \subseteq \mathbb{X}'''_i.$$

Therefore $\mathcal{S}(H^i_{\mathfrak{b}}(M)) \subseteq \mathbb{X}'_i \cup \mathbb{X}''_i \cup \mathbb{X}'''_i$, and so the desired result follows from Remark 5.9(vi). □

5.13. **Lemma.** *Let $\mathbf{m} \in {\mathbb{N}_0}^r \setminus \{\mathbf{0}\}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals of $R$ such that $R_{\mathbf{m}} \not\subseteq \mathfrak{p}_i$ for each $i = 1, \ldots, n$. Then there exists $u \in \mathbb{N}$ such that $R_{u\mathbf{m}} \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i$*

*Proof.* Consider the (Noetherian) $\mathbb{N}_0$-graded ring $R_{\mathbf{0}}[R_{\mathbf{m}}] = \bigoplus_{j\in\mathbb{N}_0} R_{j\mathbf{m}}$ (in which $R_{j\mathbf{m}}$ is the component of degree $j$, for all $j \in \mathbb{N}_0$). Apply the ordinary Homogeneous Prime Avoidance Lemma (see [4, Lemma 15.1.2]) to the graded ideal $R_{\mathbf{m}}R_{\mathbf{0}}[R_{\mathbf{m}}] = \bigoplus_{j\in\mathbb{N}} R_{j\mathbf{m}}$ and the prime ideals $\mathfrak{p}_i \cap R_{\mathbf{0}}[R_{\mathbf{m}}]$ ($i = 1, \ldots, n$). □

5.14. **Lemma.** *Let $\mathbf{m} \in {\mathbb{N}_0}^r \setminus \{\mathbf{0}\}$ and let $\mathbb{X}$ be a $\mathcal{P}(\mathbf{m})$-domain in $\mathbb{Z}^r$. Then there exists $u \in \mathbb{N}$ such that, for each $\mathbf{w} \in \mathbb{Z}^r$, there is some $j \in \{0, \ldots, \#\mathcal{P}(\mathbf{m})\}$ with $\mathbf{w} + ju\mathbf{m} \notin \mathbb{X}$.*

*Proof.* There exist $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^r$ with $\mathbf{s} \le \mathbf{t}$ and $\mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{P}(\mathbf{m})$ for which $\mathbb{X} = \mathbb{X}(\mathbf{s}, \mathbf{t})$. Choose $u \in \mathbb{N}$ such that $u\mathbf{m} \ge \mathbf{t} - \mathbf{s}$.

For an arbitrary $\mathbf{w} \in \mathbb{Z}^r$, set $\mathcal{I}(\mathbf{w}) = \{i \in \{1, \ldots, r\} : s_i \le w_i < t_i\}$, and observe that $\mathcal{I}(\mathbf{w}) \subseteq \mathcal{P}(\mathbf{m})$, and that $\mathbf{w} \in \mathbb{X}$ if and only if $\mathcal{I}(\mathbf{w}) \neq \emptyset$. Note also that, for $i \in \mathcal{I}(\mathbf{w})$ and $j \in \mathbb{N}$, we have

$$(\mathbf{w} + ju\mathbf{m})_i = w_i + jum_i \ge s_i + um_i \ge s_i + t_i - s_i = t_i,$$

so that $i \notin \mathcal{I}(\mathbf{w} + ju\mathbf{m})$. So, for each $i \in \mathcal{P}(\mathbf{m})$, if there is a $j' \in \mathbb{N}_0$ with $i \in \mathcal{I}(\mathbf{w} + j'u\mathbf{m})$, then $i \notin \mathcal{I}(\mathbf{w} + ju\mathbf{m})$ for all $j > j'$. This means that, for each $i \in \mathcal{P}(\mathbf{m})$, there is at most one $j' \in \mathbb{N}_0$ with $i \in \mathcal{I}(\mathbf{w} + j'u\mathbf{m})$. By the pigeon-hole principle, it is therefore possible to choose a $j \in \{0, \ldots, \#\mathcal{P}(\mathbf{m})\}$ for which $\mathcal{I}(\mathbf{w} + ju\mathbf{m}) \cap \mathcal{P}(\mathbf{m}) = \emptyset$, and then $\mathbf{w} + ju\mathbf{m} \notin \mathbb{X}$. □

The concept introduced in the next definition can be regarded as a multi-graded analogue of one defined by Marley in [14, §2].

5.15. **Definition.** Let $\mathcal{Q} \subseteq \{1, \ldots, r\}$, and let $\mathfrak{b}$ be an ${\mathbb{N}_0}^r$-graded ideal of $R$. We define the *$\mathcal{Q}$-finiteness dimension $g^{\mathcal{Q}}_{\mathfrak{b}}(M)$ of $M$ with respect to* $\mathfrak{b}$ by

$$g^{\mathcal{Q}}_{\mathfrak{b}}(M) := \sup\bigl\{k \in \mathbb{N}_0 : \text{ for all } i < k, \text{ there exists a } \mathcal{Q}\text{-domain } \mathbb{X}_i \text{ in } \mathbb{Z}^r \text{ with } \mathcal{S}(H^i_{\mathfrak{b}}(M)) \subseteq \mathbb{X}_i\bigr\},$$

if this supremum exists, and $\infty$ otherwise.

5.16. **Example.** For $R$ as in Example 5.3, we have

$$g^{\emptyset}_{R_+}(R) = 2, \qquad g^{\{1\}}_{R_+}(R) = 3, \qquad g^{\{2\}}_{R_+}(R) = 2, \qquad g^{\{1,2\}}_{R_+}(R) = 5.$$

5.17. *Remarks.* The first three of the statements below are immediate from Remarks 5.9(i),(ii),(iii) respectively.

(i) In the case where $\mathcal{Q} = \emptyset$, we have $g^{\emptyset}_{\mathfrak{b}}(M) = \inf\bigl\{i \in \mathbb{N}_0 : H^i_{\mathfrak{b}}(M) \neq 0\bigr\}$ (with the usual convention that the infimum of the empty set of integers is interpreted as $\infty$).

(ii) If $\mathcal{Q} \subseteq \mathcal{Q}' \subseteq \{1, \ldots, r\}$, then $g^{\mathcal{Q}}_{\mathfrak{b}}(M) \le g^{\mathcal{Q}'}_{\mathfrak{b}}(M)$.

(iii) For $\mathbf{n} \in \mathbb{Z}^r$, we have $g^{\mathcal{Q}}_{\mathfrak{b}}(M(\mathbf{n})) = g^{\mathcal{Q}}_{\mathfrak{b}}(M)$.

(iv) Let $(\mathcal{Q}_\lambda)_{\lambda\in\Lambda}$ be a family of subsets of $\{1, \ldots, r\}$. Set

$$\Omega := \left\{ \bigcap_{\lambda\in\Lambda} \mathbb{X}_\lambda : \mathbb{X}_\lambda \text{ is a } \mathcal{Q}_\lambda\text{-domain in } \mathbb{Z}^r \text{ for all } \lambda \in \Lambda \right\}.$$

It is straightforward to check that

$$\inf\Bigl\{g^{\mathcal{Q}_\lambda}_{\mathfrak{b}}(M) : \lambda \in \Lambda\Bigr\} = \sup\bigl\{k \in \mathbb{N}_0 : \text{ for all } i < k, \text{ there exists } \mathbb{Y}_i \in \Omega \text{ with } \mathcal{S}(H^i_{\mathfrak{b}}(M)) \subseteq \mathbb{Y}_i\bigr\}.$$

(v) Since a subset of $\mathbb{Z}^r$ is finite if and only if it is contained in a set of the form $\bigcap_{j=1}^r \mathbb{X}_j$, where $\mathbb{X}_j$ is a $\{j\}$-domain in $\mathbb{Z}^r$ for all $j \in \{1, \ldots, r\}$, it therefore follows from part (iv) that

$$\min\left\{ g_{\mathfrak{b}}^{\{1\}}(M), \ldots, g_{\mathfrak{b}}^{\{r\}}(M) \right\} = \sup\left\{ k \in \mathbb{N}_0 : \mathcal{S}(H_{\mathfrak{b}}^i(M)) \text{ is finite for all } i < k \right\}.$$

Thus we can say that $\min\left\{ g_{\mathfrak{b}}^{\{1\}}(M), \ldots, g_{\mathfrak{b}}^{\{r\}}(M) \right\}$ identifies the smallest integer $i$ (if there be any) for which $H_{\mathfrak{b}}^i(M)$ is not finitely graded.

5.18. **Proposition.** *Let $\mathbf{m} \in {\mathbb{N}_0}^r \setminus \{\mathbf{0}\}$, and let $f \in \mathbb{N}$. Assume that $M$ is finitely generated. The following statements are equivalent:*

(i) *$R_{\mathbf{m}} \subseteq \sqrt{(0 :_R H_{\mathfrak{b}}^i(M))}$ for all integers $i < f$;*
(ii) *for each integer $i < f$, there is a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}_i$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H_{\mathfrak{b}}^i(M)) \subseteq \mathbb{X}_i$, that is $f \leq g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M)$;*
(iii) *there is a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H_{\mathfrak{b}}^i(M)) \subseteq \mathbb{X}$ for all integers $i < f$.*

*Proof.* (ii) $\Leftrightarrow$ (iii) This is immediate from Remark 5.9(vi).

(iii) $\Rightarrow$ (i) Assume that statement (iii) holds. By Lemma 5.14, there exist $u, v := \#\mathcal{P}(\mathbf{m}) \in \mathbb{N}$ such that, for each $\mathbf{n} \in \mathbb{Z}^r$, there exists $j(\mathbf{n}) \in \{0, \ldots, v\}$ with $\mathbf{n} + j(\mathbf{n})u\mathbf{m} \notin \mathbb{X}$. So, for each $\mathbf{n} \in \mathbb{Z}^r$ and each integer $i < f$, we have $H_{\mathfrak{b}}^i(M)_{\mathbf{n}+j(\mathbf{n})u\mathbf{m}} = 0$ and

$$R_{vu\mathbf{m}} H_{\mathfrak{b}}^i(M)_{\mathbf{n}} = R_{vu\mathbf{m}-j(\mathbf{n})u\mathbf{m}} R_{j(\mathbf{n})u\mathbf{m}} H_{\mathfrak{b}}^i(M)_{\mathbf{n}} \subseteq R_{vu\mathbf{m}-j(\mathbf{n})u\mathbf{m}} H_{\mathfrak{b}}^i(M)_{\mathbf{n}+j(\mathbf{n})u\mathbf{m}} = 0.$$

Therefore $R_{vu\mathbf{m}} H_{\mathfrak{b}}^i(M) = 0$ for all integers $i < f$, and hence

$$(R_{\mathbf{m}})^{vu} \subseteq R_{vu\mathbf{m}} \subseteq (0 :_R H_{\mathfrak{b}}^i(M)) \quad \text{for all } i < f.$$

(i) $\Rightarrow$ (ii) Assume that statement (i) holds. We argue by induction on $f$. When $f = 1$, the desired conclusion is immediate from Lemma 5.7 (applied to $H_{\mathfrak{b}}^0(M)$).

So assume now that $f > 1$ and that statement (ii) has been proved for smaller values of $f$. This inductive hypothesis implies that there exist $\mathcal{P}(\mathbf{m})$-domains $\mathbb{X}_0, \ldots, \mathbb{X}_{f-2}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H_{\mathfrak{b}}^i(M)) \subseteq \mathbb{X}_i$ for all $i \in \{0, \ldots, f-2\}$. It thus remains to find a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}_{f-1}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H_{\mathfrak{b}}^{f-1}(M)) \subseteq \mathbb{X}_{f-1}$.

Set $\overline{M} := M/\Gamma_{R_{\mathbf{m}}R}(M)$, and observe that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R \Gamma_{R_{\mathbf{m}}R}(M))}$. It therefore follows from Lemma 5.12 that there is a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}'$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H_{\mathfrak{b}}^{f-1}(\Gamma_{R_{\mathbf{m}}R}(M))) \subseteq \mathbb{X}'$. In view of the exact sequence of $\mathbb{Z}^r$-graded $R$-modules

$$H_{\mathfrak{b}}^{f-1}(\Gamma_{R_{\mathbf{m}}R}(M)) \longrightarrow H_{\mathfrak{b}}^{f-1}(M) \longrightarrow H_{\mathfrak{b}}^{f-1}(\overline{M})$$

and Remark 5.9(vi), it is now enough for us to show that $\mathcal{S}(H_{\mathfrak{b}}^{f-1}(\overline{M}))$ is contained in a $\mathcal{P}(\mathbf{m})$-domain in $\mathbb{Z}^r$.

As $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R H_{\mathfrak{b}}^j(\Gamma_{R_{\mathbf{m}}R}(M)))}$ for all $j \in \mathbb{N}_0$, the exact sequence

$$H_{\mathfrak{b}}^i(M) \longrightarrow H_{\mathfrak{b}}^i(\overline{M}) \longrightarrow H_{\mathfrak{b}}^{i+1}(\Gamma_{R_{\mathbf{m}}R}(M))$$

shows that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R H_{\mathfrak{b}}^i(\overline{M}))}$ for all integers $i < f$. Set $\mathrm{Ass}_R(\overline{M}) =: \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$. As $R_{\mathbf{m}}R$ does not consist entirely of zero-divisors on $\overline{M}$, we have $R_{\mathbf{m}} \nsubseteq \mathfrak{p}_i$ for each $i = 1, \ldots, k$. Therefore, by Lemma 5.13, there exists $u' \in \mathbb{N}$ such that $R_{u'\mathbf{m}} \nsubseteq \bigcup_{i=1}^k \mathfrak{p}_i$, and hence there exists $y' \in R_{u'\mathbf{m}}$ which is not a zero-divisor on $\overline{M}$. We can now take a sufficiently high power $y$ of $y'$ to find $u \in \mathbb{N}$ and $y \in R_{u\mathbf{m}}$ such that $R_{u\mathbf{m}} H_{\mathfrak{b}}^{f-1}(\overline{M}) = 0$ and $y$ is a non-zero-divisor on $\overline{M}$, so that there is a short exact sequence of $\mathbb{Z}^r$-graded $R$ modules

$$0 \longrightarrow \overline{M}(-u\mathbf{m}) \xrightarrow{y} \overline{M} \longrightarrow \overline{M}/y\overline{M} \longrightarrow 0.$$

It now follows from the long exact sequence of local cohomology modules induced from the above short exact sequence that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R H_{\mathfrak{b}}^i(\overline{M}/y\overline{M}))}$ for all integers $i < f - 1$. Therefore, by the inductive hypothesis, there is a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}''$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H_{\mathfrak{b}}^{f-2}(\overline{M}/y\overline{M})) \subseteq \mathbb{X}''$. Let $K$ be the kernel of the map $H_{\mathfrak{b}}^{f-1}(\overline{M}) \longrightarrow H_{\mathfrak{b}}^{f-1}(\overline{M})(u\mathbf{m})$ provided by multiplication by $y$. The long exact sequence of local cohomology modules induced from the last-displayed short exact sequence now shows that $\mathcal{S}(K) \subseteq \mathbb{X}'' - u\mathbf{m}$.

We now apply Lemma 5.10(i) to $H^{f-1}_{\mathfrak{b}}(\overline{M})$, with $u\mathbf{m}$ playing the rôles of both $\mathbf{m}$ and $\mathbf{k}$: the conclusion is that there exists $v \in \mathbb{N}_0$ such that

$$\mathcal{S}(H^{f-1}_{\mathfrak{b}}(\overline{M})) \subseteq \bigcup_{j=0}^{v} (\mathcal{S}(K) - ju\mathbf{m}) \subseteq \bigcup_{j=0}^{v} (\mathbb{X}'' - u\mathbf{m} - ju\mathbf{m}).$$

We can now use Remarks 5.9(iii),(vi) to deduce the existence of a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}_{f-1}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H^{f-1}_{\mathfrak{b}}(\overline{M})) \subseteq \mathbb{X}_{f-1}$. With this, the proof is complete. $\square$

We now connect the concept of $\mathcal{Q}$-finiteness dimension of $M$ with respect to $\mathfrak{b}$, introduced in Definition 5.17, with the concept of $\mathfrak{a}$-finiteness dimension of $M$ relative to $\mathfrak{b}$ (where $\mathfrak{a}$ is a second ideal of $R$), studied by Faltings in [5]. (See also [4, Chapter 9].)

5.19. **Reminder.** Assume that $M$ is finitely generated, and let $\mathfrak{a}, \mathfrak{d}$ be ideals of $R$ (not necessarily graded).

The $\mathfrak{a}$-*finiteness dimension* $f^{\mathfrak{a}}_{\mathfrak{d}}(M)$ *of* $M$ *relative to* $\mathfrak{d}$ is defined by

$$f^{\mathfrak{a}}_{\mathfrak{d}}(M) = \inf\left\{ i \in \mathbb{N}_0 : \mathfrak{a} \not\subseteq \sqrt{(0 : H^i_{\mathfrak{d}}(M))} \right\}$$

and the $\mathfrak{a}$-*minimum* $\mathfrak{d}$-*adjusted depth* $\lambda^{\mathfrak{a}}_{\mathfrak{d}}(M)$ *of* $M$ is defined by

$$\lambda^{\mathfrak{a}}_{\mathfrak{d}}(M) := \inf\{\operatorname{depth} M_{\mathfrak{p}} + \operatorname{ht}(\mathfrak{d} + \mathfrak{p})/\mathfrak{p} : \mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(\mathfrak{a})\}.$$

(Here, $\operatorname{Var}(\mathfrak{a})$ denotes the *variety* $\{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$ *of* $\mathfrak{a}$.) It is always the case that $f^{\mathfrak{a}}_{\mathfrak{d}}(M) \leq \lambda^{\mathfrak{a}}_{\mathfrak{d}}(M)$; Faltings' (Extended) Annihilator Theorem [5] states that if $R$ admits a dualizing complex or is a homomorphic image of a regular ring, then $f^{\mathfrak{a}}_{\mathfrak{d}}(M) = \lambda^{\mathfrak{a}}_{\mathfrak{d}}(M)$. (See [3, Corollary 3.8] for an account of the extended version of Faltings' Annihilator Theorem.)

5.20. *Remark.* Let the situation be as in Reminder 5.19, let $K \subseteq R$, and let $(K_j)_{j\in J}$ be a family of subsets of $R$.

(i) It is easy to deduce from the definition that $f^{KR}_{\mathfrak{d}}(M) = \inf\{f^{aR}_{\mathfrak{d}}(M) : a \in K\}$.

(ii) We can then deduce from part (i) that $f^{(\bigcup_{j\in J} K_j)R}_{\mathfrak{d}}(M) = \inf\left\{f^{K_jR}_{\mathfrak{d}}(M) : j \in J\right\}$.

(iii) Similarly, it is easy to deduce from the definition that $\lambda^{KR}_{\mathfrak{d}}(M) = \inf\{\lambda^{aR}_{\mathfrak{d}}(M) : a \in K\}$.

(iv) We can then deduce from part (iii) that $\lambda^{(\bigcup_{j\in J} K_j)R}_{\mathfrak{d}}(M) = \inf\left\{\lambda^{K_jR}_{\mathfrak{d}}(M) : j \in J\right\}$.

5.21. **Theorem.** *Assume that $M$ is finitely generated, and let $\emptyset \neq \mathcal{T} \subseteq {\mathbb{N}_0}^r$.*

(i) *We have*

$$\begin{aligned}
&\sup\{k \in \mathbb{N}_0 : \textit{for all } i < k \textit{ and all } \mathbf{m} \in \mathcal{T}, \textit{ there exists a } \mathcal{P}(\mathbf{m})\textit{-domain } \mathbb{X}^{(\mathbf{m})}_i \textit{ in } \mathbb{Z}^r \\
&\qquad\qquad \textit{such that } \mathcal{S}(H^i_{\mathfrak{b}}(M)) \subseteq \mathbb{X}^{(\mathbf{m})}_i\Big\} \\
&\qquad = \inf\left\{g^{\mathcal{P}(\mathbf{m})}_{\mathfrak{b}}(M) : \mathbf{m} \in \mathcal{T}\right\} \\
&\qquad = f^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}_{\mathfrak{b}}(M) \leq \lambda^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}_{\mathfrak{b}}(M).
\end{aligned}$$

(ii) *If $R$ admits a dualizing complex or is a homomorphic image of a regular ring, then we can replace the inequality in part* (i) *by equality.*

*Proof.* Apply Remark 5.17(iv) to the family $(\mathcal{P}(\mathbf{m}))_{\mathbf{m}\in\mathcal{T}}$ of subsets of $\{1, \ldots, r\}$ to conclude that

$$\begin{aligned}
&\sup\{k \in \mathbb{N}_0 : \text{for all } i < k \text{ and all } \mathbf{m} \in \mathcal{T}, \text{ there exists a } \mathcal{P}(\mathbf{m})\text{-domain } \mathbb{X}^{(\mathbf{m})}_i \text{ in } \mathbb{Z}^r \\
&\qquad\qquad \text{such that } \mathcal{S}(H^i_{\mathfrak{b}}(M)) \subseteq \mathbb{X}^{(\mathbf{m})}_i\Big\} \\
&\qquad = \inf\left\{g^{\mathcal{P}(\mathbf{m})}_{\mathfrak{b}}(M) : \mathbf{m} \in \mathcal{T}\right\}.
\end{aligned}$$

By Proposition 5.18, we have $g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) = f_{\mathfrak{b}}^{R_{\mathbf{m}}R}(M)$ for all $\mathbf{m} \in \mathcal{T}$. Therefore, on use of Remark 5.20(ii), we deduce that

$$\inf\Big\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) : \mathbf{m} \in \mathcal{T}\Big\} = \inf\Big\{f_{\mathfrak{b}}^{R_{\mathbf{m}}R}(M) : \mathbf{m} \in \mathcal{T}\Big\} = f_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M).$$

We can now use Faltings' (Extended) Annihilator Theorem [5] (see Reminder 5.19) to complete the proof of part (i) and to obtain the statement in part (ii). □

5.22. **Corollary.** *Assume that $M$ is finitely generated.*

(i) *For each non-empty set $\mathcal{T} \subseteq \mathbf{1} + {\mathbb{N}_0}^r$, we have*

$$f_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M) = g_{\mathfrak{b}}^{\{1,\ldots,r\}}(M).$$

(ii) *For each set $\mathcal{T} \subseteq {\mathbb{N}_0}^r \setminus \{\mathbf{0}\}$ such that $\mathbb{N}\mathbf{e}_i \cap \mathcal{T} \neq \emptyset$ for all $i \in \{1,\ldots,r\}$, we have*

$$\begin{aligned} f_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M) &= \sup\big\{k \in \mathbb{N}_0 : \mathcal{S}(H_{\mathfrak{b}}^i(M)) \textit{ is finite for all } i < k\big\} \\ &= \sup\big\{k \in \mathbb{N}_0 : H_{\mathfrak{b}}^i(M) \textit{ is finitely graded for all } i < k\big\}. \end{aligned}$$

(iii) *If $M \neq \mathfrak{b}M$, then $f_{\mathfrak{b}}^R(M) = g_{\mathfrak{b}}^{\emptyset}(M) = \operatorname{grade}_M \mathfrak{b}$.*

*Note.* If, in the case where $r = 1$, we take $\mathcal{T} = \mathbb{N}$, so that $\sum_{m\in\mathcal{T}} R_m R = R_+$, then the statement in part (ii) becomes

$$f_{\mathfrak{b}}^{R_+}(M) = \sup\big\{k \in \mathbb{N}_0 : H_{\mathfrak{b}}^i(M) \text{ is finitely graded for all } i < k\big\},$$

a result proved by Marley in [14, Proposition 2.3].

*Proof.* (i) By Theorem 5.21(i), we have $f_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M) = \inf\Big\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) : \mathbf{m} \in \mathcal{T}\Big\}$. But $\mathcal{P}(\mathbf{m}) = \{1,\ldots,r\}$ for all $\mathbf{m} \in \mathbf{1} + {\mathbb{N}_0}^r$.

(ii) By Theorem 5.21(i), we have

$$f_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M) = \inf\Big\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) : \mathbf{m} \in \mathcal{T}\Big\}.$$

By the hypothesis, for each $i \in \{1,\ldots,r\}$, there exists $\mathbf{m}_i \in \mathcal{T}$ with $\mathcal{P}(\mathbf{m}_i) = \{i\}$. It therefore follows from Remark 5.17(ii) that $\inf\Big\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) : \mathbf{m} \in \mathcal{T}\Big\} = \min\Big\{g_{\mathfrak{b}}^{\{1\}}(M),\ldots,g_{\mathfrak{b}}^{\{r\}}(M)\Big\}$. However, we noted in Remark 5.17(v) that

$$\min\Big\{g_{\mathfrak{b}}^{\{1\}}(M),\ldots,g_{\mathfrak{b}}^{\{r\}}(M)\Big\} = \sup\big\{k \in \mathbb{N}_0 : \mathcal{S}(H_{\mathfrak{b}}^i(M)) \text{ is finite for all } i < k\big\}.$$

(iii) Since $R = R_{\mathbf{0}}R$, we can deduce from Theorem 5.21(i) and Remark 5.17(i) that

$$f_{\mathfrak{b}}^R(M) = f_{\mathfrak{b}}^{R_{\mathbf{0}}R}(M) = g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{0})}(M) = g_{\mathfrak{b}}^{\emptyset}(M) = \sup\big\{k \in \mathbb{N}_0 : H_{\mathfrak{b}}^i(M) = 0 \text{ for all } i < k\big\} = \operatorname{grade}_M \mathfrak{b}.$$

□

## References


1. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.
2. M. Brodmann, *Cohomological invariants of coherent sheaves over projective schemes: a survey*, in *Local cohomology and its applications (Guanajuato, 1999)*, Lecture Notes in Pure and Appl. Math. **226**, Dekker, New York, 2002, pp. 91–120.
3. M. P. Brodmann, Ch. Rotthaus and R. Y. Sharp, *On annihilators and associated primes of local cohomology modules*, J. Pure and Applied Algebra **153** (2000), 197–227.
4. M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics **60**, Cambridge University Press, 1998.
5. G. Faltings, *Über die Annulatoren lokaler Kohomologiegruppen*, Archiv der Math. **30** (1978), 473–476.
6. R. Fossum and H.-B. Foxby, *The category of graded modules*, Math. Scand. **35** (1974), 288–300.
7. S. Fumasoli, *Die Künnethrelation in abelschen Kategorien und ihre Anwendungen auf die Idealtransformation*, Diplomarbeit, Universität Zürich, 1999.
8. S. Goto and K.-i. Watanabe, *On graded rings* II *($\mathbb{Z}^n$-graded rings)*, Tokyo J. Math. **1** (1978), 237–261.
9. Huy Tài Hà, *Multigraded regularity, $a^*$-invariant and the minimal free resolution*, J. Algebra **310** (2007), 156–179.
10. E. Hyry, *The diagonal subring and the Cohen-Macaulay property of a multigraded ring*, Trans. Amer. Math. Soc. **351** (1999), 2213–2232.

11. M. Kreuzer and L. Robbiano, *Computational commutative algebra* 1, Springer, Berlin, 2000.
12. D. Maclagan and G. G. Smith, *Uniform bounds on multigraded regularity*, J. Algebraic Geom. **14** (2005), 137–164.
13. S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften **114**, Springer, Berlin, 1963.
14. T. Marley, *Finitely graded local cohomology modules and the depths of graded algebras*, Proc. American Math. Soc. **123** (1995), 3601–3607.
15. R. Y. Sharp, *Bass numbers in the graded case, a-invariant formulas, and an analogue of Faltings' Annihilator Theorem*, J. Algebra **222** (1999), 246–270.
16. N. V. Trung and S. Ikeda, *When is the Rees algebra Cohen–Macaulay?*, Communications in Algebra **17** (1989) 2893–2922.
17. O. Zariski and P. Samuel, *Commutative algebra*, Volume I, The University Series in Higher Mathematics, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1958.

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