FIRST LECTURES ON LOCAL COHOMOLOGY

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Contents

Preface

These notes are based on *introductory courses on Local Cohomology* given at the *University of Zürich* in 1997/1998, 2002/2003, 2005/2006, 2007 and on a series of lectures held at the *Pedagogical University of Quy Nhon* in September 1999 under the title *"Local Cohomology—Basic Notions and Applications to Algebraic Varieties"*.

The aim of these courses and lectures was to give an initiation to local cohomology which could be the basis for deeper penetration in the subject along the lines of the book [B-S]. They have been designed for an audience having some basic background in commutative algebra as presented in [S]. Most of the needed material is recalled in combined reminders and exercises. In the same way we develop the necessary basic notions from homological algebra. This way of presentation corresponds to what has shown to be successful in our courses and lectures. There are only a few basic results from algebra and algebraic geometry used without previous proof, notably:

- the Lemma of Artin-Rees cf. 1.3);
- the Nullstellensatz (cf. 6.4);
- the graded version and the sheaf theoretic version of the Lemma of Eckmann-Schopf (cf. 8.13 and 12.6);
- the fact that sheaf cohomology can be calculated using flasque resolutions (cf. 12.8).

Our suggested references are:

- commutative algebra: [S];
- homological algebra: [R];
- algebraic geometry: [H1];
- local cohomology: [B-S].

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1. Torsion Modules and Torsion Functors

Torsion modules are a straight forward generalization of torsion subgroups (of Abelian groups) and shall lead us to the concept of torsion functors. We prove results on the behaviour of associated primes and of localizations under the formation of torsion submodules (cf. 1.9, 1.11). We introduce the concept of torsion functors and we give some basic facts on it (cf. 1.15, 1.19).

1.0. **Notation.** All rings are assumed to be commutative. Throughout this chapter, let R be a ring and let $\mathfrak{a} \subseteq R$ be an ideal.

1.1. **Notation.** For an R-module M and a submodule $N \subseteq M$ let

$$(N :_M \mathfrak{a}) := \{m \in M \mid \mathfrak{a}m \subseteq N\}.$$

Observe that $(N :_M \mathfrak{a})$ is a submodule of M and that $N \subseteq (N :_M \mathfrak{a})$.

1.2. **Definition.** For an R-module M, the $\mathfrak{a}$-*torsion of* M is defined by

$$\Gamma_{\mathfrak{a}}(M) := \bigcup_{n \in \mathbb{N}} (0 :_M \mathfrak{a}^n).$$

Observe that $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M.

1.3. **Reminder.** Let R be Noetherian, let M be a finitely generated R-module and let $N \subseteq M$ be a submodule. The *Lemma of Artin-Rees* (cf. [M, Theorem 8.5]) says that there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$\mathfrak{a}^n M \cap N = \mathfrak{a}^{n-n_0}(N \cap \mathfrak{a}^{n_0} M).$$

1.4. **Remarks and Exercise.** A) Let M be an R-module and let $\mathfrak{b} \subseteq R$ be an ideal. Then:

a) $\Gamma_0(M) = M$ and $\Gamma_R(M) = 0$;
b) If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\Gamma_{\mathfrak{b}}(M) \subseteq \Gamma_{\mathfrak{a}}(M)$;
c) If $\mathfrak{a}$ and $\mathfrak{b}$ are finitely generated and $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$, then $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{b}}(M)$;
d) $\Gamma_{\mathfrak{a}+\mathfrak{b}}(M) = \Gamma_{\mathfrak{a}}(M) \cap \Gamma_{\mathfrak{b}}(M)$.

B) Moreover, for an R-module M we have:

a) If N is an R-module and $h : M \to N$ is a homomorphism of R-modules, then $h(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$;
b) If $\mathfrak{a}$ is finitely generated or M is Noetherian, then $\Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$.
c) If M is Noetherian, then there exists $n \in \mathbb{N}$ such that $\Gamma_{\mathfrak{a}}(M) = (0 :_M \mathfrak{a}^n)$;
d) If R is Noetherian and M is finitely generated, then there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n M \cap \Gamma_{\mathfrak{a}}(M) = 0$;

(Hint for "d)": use 1.3.)

1.5. **Notation.** For an R-module M, let $\mathrm{ZD}_R(M)$ denote the set of *zero divisors of R on M* and let $\mathrm{NZD}_R(M)$ denote the set of *non-zero divisors of R on M*, thus:

$$\begin{aligned}\mathrm{ZD}_R(M) &:= \{x \in R \mid \exists\, m \in M \setminus \{0\} : xm = 0\};\\ \mathrm{NZD}_R(M) &:= R \setminus \mathrm{ZD}_R(M).\end{aligned}$$

1.6. **Notation and Reminder.** A) Let M be an R-module and let $U, V \subseteq M$ be submodules. We write

$$(U :_R V) := \{x \in R \mid xV \subseteq U\}.$$

Observe that this set is an ideal in R. Remember, that

$$(0 :_R V) = \{x \in R \mid xV = 0\}$$

is the *annihilator of V*.

B) Let M be an R-module and let $\mathfrak{p} \in \mathrm{Spec}(R)$ be a prime ideal of R. Then, $\mathfrak{p}$ is said to be *associated to M* if it is the annihilator of a cyclic submodule of M, i.e. if

$$\mathfrak{p} = (0 :_R Rv) \text{ for some } v \in M.$$

If this is the case, we must have $v \neq 0$.

We denote by $\mathrm{Ass}_R(M)$ the set of associated primes of M. Thus:

$$\mathrm{Ass}_R(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \exists\, v \in M : \mathfrak{p} = (0 :_R Rv)\}.$$

Let us recall the following facts:

a) If R is Noetherian, then $\mathrm{ZD}_R(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}$;

b) If $N \subseteq M$ is a submodule, then

$$\mathrm{Ass}_R(N) \subseteq \mathrm{Ass}_R(M) \subseteq \mathrm{Ass}_R(M/N) \cup \mathrm{Ass}_R(N);$$

c) If M is Noetherian, then $\sharp\,\mathrm{Ass}_R(M) < \infty$;

d) If R is Noetherian, then $\mathrm{Ass}_R(M) = \emptyset$ is equivalent to $M = 0$.

1.7. **Lemma.** *Let M be an R-module. Then:*

a) If $\Gamma_{\mathfrak{a}}(M) \neq 0$, then $\mathfrak{a} \subseteq \mathrm{ZD}_R(M)$.

b) Let R be Noetherian and let M be finitely generated. If $\mathfrak{a} \subseteq \mathrm{ZD}_R(M)$, then $\Gamma_{\mathfrak{a}}(M) \neq 0$.

Proof. “a)”: Is easy and left as an exercise.

“b)”: By 1.6 B) c) the set $\mathrm{Ass}_R(M)$ is finite, so that we can write $\mathrm{Ass}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Using 1.6 B) a) we thus get $\mathfrak{a} \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r$. So, by prime avoidance there is some $i \in \{1, \ldots, r\}$ with $\mathfrak{a} \subseteq \mathfrak{p}_i$. As $\mathfrak{p}_i \in \mathrm{Ass}_R(M)$, there is some $v \in M \setminus 0$ with $\mathfrak{p}_i = (0 :_R Rv)$. It follows $\mathfrak{a}v \subseteq \mathfrak{p}_i v = 0$, thus $v \in \Gamma_{\mathfrak{a}}(M) \setminus 0$. $\square$

1.8. **Notations and Reminder.** A) The *variety of* $\mathfrak{a}$ shall be denoted by $\mathrm{Var}(\mathfrak{a})$, thus:

$$\mathrm{Var}(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

B) If $\mathcal{P} \subseteq \mathrm{Spec}(R)$ is a set of primes, we denote by $\mathrm{Min}(\mathcal{P})$ the set of all minimal members of $\mathcal{P}$ with respect to inclusion, thus:

$$\mathrm{Min}(\mathcal{P}) := \{\mathfrak{p} \in \mathcal{P} \mid \forall \mathfrak{q} \in \mathcal{P} : (\mathfrak{q} \subseteq \mathfrak{p} \Rightarrow \mathfrak{p} = \mathfrak{q})\}.$$

So $\mathrm{Min}(\mathrm{Var}(\mathfrak{a}))$ coincides with the set $\min(\mathfrak{a})$ of all *minimal primes of* $\mathfrak{a}$:

$$\mathrm{Min}\big(\mathrm{Var}(\mathfrak{a})\big) = \min(\mathfrak{a}).$$

C) Remember the following facts:

a) If $\mathfrak{p} \in \mathrm{Var}(\mathfrak{a})$, then there is a $\mathfrak{q} \in \min(\mathfrak{a})$ with $\mathfrak{q} \subseteq \mathfrak{p}$;
b) $\mathrm{Ass}_R(M) \subseteq \mathrm{Var}(0 :_R M)$;
c) If R is Noetherian and M is finitely generated, then

$$\mathrm{Min}(\mathrm{Ass}_R(M)) = \min(0 :_R M).$$

1.9. **Proposition.** *Let M be a Noetherian R-module. Then:*

a) $\mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(M) \cap \mathrm{Var}(\mathfrak{a})$.

b) If R is Noetherian, then $\mathrm{Ass}_R(M/\Gamma_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(M) \setminus \mathrm{Var}(\mathfrak{a})$.

Proof. "a)": "$\subseteq$": Let $\mathfrak{p} \in \mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M))$. As $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M, we have $\mathfrak{p} \in \mathrm{Ass}_R(M)$ (cf. 1.6 B) b)). As M is Noetherian, there is an $n \in \mathbb{N}$ with $\mathfrak{a}^n\Gamma_{\mathfrak{a}}(M) = 0$ (cf. 1.4 B) c)) so that $\mathfrak{a}^n \subseteq (0 :_R \Gamma_{\mathfrak{a}}(M))$. By 1.8 C) b) it follows $\mathfrak{a}^n \subseteq \mathfrak{p}$, thus $\mathfrak{a} \subseteq \mathfrak{p}$, hence $\mathfrak{p} \in \mathrm{Var}(\mathfrak{a})$.

"$\supseteq$": Let $\mathfrak{p} \in \mathrm{Ass}_R(M) \cap \mathrm{Var}(\mathfrak{a})$. Then, there is some $v \in M$ with $(0 :_R Rv) = \mathfrak{p}$. As $\mathfrak{a} \subseteq \mathfrak{p}$ we have $\mathfrak{a}v = 0$, thus $v \in \Gamma_{\mathfrak{a}}(M)$. It follows $\mathfrak{p} \in \mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M))$.

"b)": "$\supseteq$": Let $\mathfrak{p} \in \mathrm{Ass}_R(M)$. We have $\mathfrak{p} \in \mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M)) \cup \mathrm{Ass}_R(M/\Gamma_{\mathfrak{a}}(M))$ by 1.6 B) b). If $\mathfrak{p} \notin \mathrm{Var}(\mathfrak{a})$, statement a) gives $\mathfrak{p} \notin \mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M))$, hence $\mathfrak{p} \in \mathrm{Ass}_R(M/\Gamma_{\mathfrak{a}}(M))$.

"$\subseteq$": Let $\mathfrak{p} \in \mathrm{Ass}_R(M/\Gamma_{\mathfrak{a}}(M))$. As $\Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$ (cf. 1.4 B) b)) and $M/\Gamma_{\mathfrak{a}}(M)$ is Noetherian, there is some $x \in \mathrm{NZD}_R(M/\Gamma_{\mathfrak{a}}(M)) \cap \mathfrak{a}$ (cf. 1.7 b)). As $\mathfrak{p} \subseteq \mathrm{ZD}_R(M/\Gamma_{\mathfrak{a}}(M))$ (cf. 1.6 B) a)), it follows $x \notin \mathfrak{p}$. By our choice of $\mathfrak{p}$ we find an element $\bar{v} \in M/\Gamma_{\mathfrak{a}}(M)$ with $(0 :_R R\bar{v}) = \mathfrak{p}$. Let $v \in M$ be such that $\bar{v} = v + \Gamma_{\mathfrak{a}}(M)$. Then $\mathfrak{p}\bar{v} = 0$ implies $\mathfrak{p}v \subseteq \Gamma_{\mathfrak{a}}(M)$. As M is Noetherian, there is some $n \in \mathbb{N}$ with $\mathfrak{a}^n\Gamma_{\mathfrak{a}}(M) = 0$ (cf. 1.4 B) c)). It follows $\mathfrak{p}(Rx^nv) = x^n\mathfrak{p}v \subseteq \mathfrak{a}^n\Gamma_{\mathfrak{a}}(M) = 0$, thus $\mathfrak{p} \subseteq (0 :_R Rx^nv)$. Conversely, let $a \in (0 :_R Rx^nv)$. It follows $(ax^n)v = a(x^nv) = 0 \in \Gamma_{\mathfrak{a}}(M)$, thus $ax^n\bar{v} = 0$, hence $ax^n \in (0 :_R R\bar{v}) = \mathfrak{p}$. As $x \notin \mathfrak{p}$ we get $a \in \mathfrak{p}$. Altogether we have shown that $(0 :_R Rx^nv) = \mathfrak{p}$ and hence that $\mathfrak{p} \in \mathrm{Ass}_R(M)$. As $\mathfrak{p} \not\ni x \in \mathfrak{a}$, we also have $\mathfrak{p} \notin \mathrm{Var}(\mathfrak{a})$. $\square$

1.10. **Notation.** A subset $S \subseteq R$ is called *multiplicatively closed,* if $S \neq \emptyset$ and if $st \in S$ for $s, t \in S$. In this case, we can define the ring of fractions $S^{-1}R$ and for an R-module M the module of fractions $S^{-1}M$.

1.11. **Proposition.** *Let $S \subseteq R$ be multiplicatively closed and let M be an R-module. If $\mathfrak{a}$ is finitely generated, then there is an isomorphism of $S^{-1}R$-modules*

$$\rho = \rho_{\mathfrak{a},M} : S^{-1}\Gamma_{\mathfrak{a}}(M) \xrightarrow{\cong} \Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}M),$$

defined by $\frac{m}{s} \mapsto \frac{m}{s}$ for all $m \in \Gamma_{\mathfrak{a}}(M)$ and all $s \in S$.

Proof. As $\Gamma_{\mathfrak{a}}(M) \subseteq M$ is a submodule, there is an injective homomorphism $\rho : S^{-1}\Gamma_{\mathfrak{a}}(M) \to S^{-1}M$ of $S^{-1}R$-modules given by $\frac{m}{s} \mapsto \frac{m}{s}$ for all $m \in \Gamma_{\mathfrak{a}}(M)$ and all $s \in S$. It thus suffices to show that $\rho(S^{-1}\Gamma_{\mathfrak{a}}(M)) = \Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}M)$.

To prove the inclusion "$\subseteq$", let $u \in \rho(S^{-1}\Gamma_{\mathfrak{a}}(M))$. We may write $u = \frac{m}{s}$ with $m \in \Gamma_{\mathfrak{a}}(M)$ and $s \in S$. In particular, there is some $n \in \mathbb{N}$ with $\mathfrak{a}^n m = 0$. It follows immediately that $(\mathfrak{a}S^{-1}R)^n u = \mathfrak{a}^n S^{-1}R\frac{m}{s} = S^{-1}R\frac{\mathfrak{a}^n m}{s} = 0$.

To prove the inclusion "$\supseteq$", let $u \in \Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}M)$. We write $u = \frac{m}{s}$ with $m \in M$ and $s \in S$. There is some $n \in \mathbb{N}$ with $(\mathfrak{a}S^{-1}R)^n \frac{m}{s} = 0$. Let $(x_1, \ldots, x_r)$ be a finite system of generators of $\mathfrak{a}^n$. For each $i \in \{1, \ldots, r\}$ it follows $\frac{sx_i m}{s^2} = \frac{sx_i}{s}\frac{m}{s} = 0$ and hence $t_i x_i m = 0$ for some $t_i \in S$. Let $t := \prod_{i=1}^r t_i$. Then $t, st \in S$ and $x_i tm = 0$ for all $i \in \{1, \ldots, r\}$. It follows $\mathfrak{a}^n tm = 0$, hence $tm \in \Gamma_{\mathfrak{a}}(M)$ and thus $u = \frac{m}{s} = \frac{tm}{ts} = \rho(\frac{tm}{ts}) \in \rho(S^{-1}\Gamma_{\mathfrak{a}}(M))$. $\square$

1.12. **Reminders.** A) Let M, N be R-modules and let $\mathrm{Hom}_R(M, N)$ denote the set of all homomorphisms of R-modules $h : M \to N$. This set carries a natural structure of R-module given for $h, l \in \mathrm{Hom}_R(M, N)$ and $a \in R$ by

$$(h + l)(m) := h(m) + l(m) \text{ and } (ah)(m) := a(h(m)) \text{ for } m \in M.$$

B) Let R' be a second ring. By an *additive (covariant) functor from (the category of) R-modules to (the category of) R'-modules* we mean an assignment

$$F = F(\bullet) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(F(M) \xrightarrow{F(h)} F(N)\big)$$

which, to each R-module M assigns an R'-module $F(M)$ and to each homomorphism $h : M \to N$ of R-modules assigns a homomorphism of R'-modules $F(h) : F(M) \to F(N)$, such that the following properties hold:

(A1) $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$ for each R-module M;

(A2) $F(h \circ l) = F(h) \circ F(l)$, whenever $l : M \to N$ and $h : N \to P$ are homomorphisms of R-modules;

(A3) $F(h) + F(l) = F(h + l)$, whenever $h, l : M \to N$ are homomorphisms of R-modules.

C) Let $f : R \to R'$ be a homomorphism of rings. By a *linear (covariant) functor from (the category of) R-modules to (the category of) R'-modules (with* *respect to f)* we mean an additive covariant functor from the category of R-modules to the category of R'-modules $F = F(\bullet)$ with the condition

(A4) $F(ah) = f(a)F(h)$ for each $a \in R$ and each homomorphism of R-modules $h : M \to N$.

In the special case, where $f = \mathrm{id}_R : R \to R$ we say that F is a *linear (covariant) functor (in the category) of R-modules.* Clearly in this case, the condition (A4) may be written in the form

(A4′) $F(ah) = aF(h)$ for each $a \in R$ and each homomorphism of R-modules $h : M \to N$.

D) Next, let $g : R' \to R$ be a homomorphism of rings. By a *linear (covariant) functor from (the category of) R-modules to (the category of) R'-modules (with respect to g)* we mean an additive covariant functor from the category of R-modules to the category of R'-modules $F = F(\bullet)$ which—instead of (A4)—satisfies

(A4″) $aF(h) = F(g(a)h)$ for each $a \in R'$ and each homomorphism of R-modules $h : M \to N$.

1.13. **Remark and Exercise.** Let R' be a second ring, let F be an additive functor from R-modules to R'-modules and let $h : M \to N$ be a homomorphism of R-modules. Then:

a) If h is an isomorphism, then $F(h)$ is an isomorphism and $F(h^{-1}) = F(h)^{-1}$;
b) If $h = 0$, then $F(h) = 0$;
c) If $M = 0$, then $F(M) = 0$.

1.14. **Examples and Exercise.** A) The assignment

$$\mathrm{Id} : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(M \xrightarrow{h} N\big)$$

defines a linear functor of R-modules—the so called *identity functor on R-modules* $\mathrm{Id} = \mathrm{Id}(\bullet)$.

B) Let $S \subseteq R$ be multiplicatively closed and let $\eta_S : R \to S^{-1}R$ be the canonical homomorphism defined by $x \mapsto \frac{sx}{s}$ for all $x \in R$ and any $s \in S$. Then, the assignment

$$S^{-1} : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(S^{-1}M \xrightarrow{S^{-1}h} S^{-1}N\big)$$

(in which $S^{-1}h$ is defined by $\frac{m}{s} \mapsto \frac{h(m)}{s}$ for all $m \in M$ and all $s \in S$) defines a linear functor from R-modules to $S^{-1}R$-modules with respect to η_S. This functor $S^{-1} = S^{-1}\bullet$ is called the *localization functor with respect to S.*

C) Let $f : R \to R'$ be a homomorphism of rings and let M' be an R'-module. By means of the scalar multiplication defined by $am' = f(a)m'$ for all $a \in R$ and all $m' \in M'$, M' becomes an R-module. If we view M' in this way, we denote it by $M' \upharpoonright_f$ or by $M' \upharpoonright_R$. If $h' : M' \to N'$ is a homomorphism of R'-modules it then also becomes a homomorphism of R-modules. If we view h' in this way, we denote it by $h' \upharpoonright_f$ or by $h' \upharpoonright_R$. Now, the assignment

$$\upharpoonright_R : \big(M' \xrightarrow{h'} N'\big) \rightsquigarrow \big(M' \upharpoonright_R \xrightarrow{h' \upharpoonright_R} N' \upharpoonright_R\big)$$

defines a linear functor $\upharpoonright_R$ from R'-modules to R-modules with respect to f. This functor $\upharpoonright_R = \bullet \upharpoonright_R$ is called the *scalar restriction functor from R' to R (by means of f).*

D) Fix some R-module U. If $h : M \to N$ is a homomorphism of R-modules, we define a homomorphism of R-modules

$$\mathrm{Hom}_R(U, h) : \mathrm{Hom}_R(U, M) \to \mathrm{Hom}_R(U, N),\ l \mapsto h \circ l.$$

Then, it is easy to verify that the assignment

$$\mathrm{Hom}_R(U, \bullet) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(\mathrm{Hom}_R(U, M) \xrightarrow{\mathrm{Hom}_R(U,h)} \mathrm{Hom}_R(U, N)\big)$$

defines a linear functor of R-modules.

1.15. **Remark and Definition.** A) If $h : M \to N$ is a homomorphism of R-modules, we have $h(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$ (cf. 1.4 B) a)). So, we may define a homomorphism of R-modules

$$\Gamma_{\mathfrak{a}}(h) : \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(N),\ m \mapsto h(m).$$

B) Now, it is easy to see that the assignment

$$\Gamma_{\mathfrak{a}} : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(\Gamma_{\mathfrak{a}}(M) \xrightarrow{\Gamma_{\mathfrak{a}}(h)} \Gamma_{\mathfrak{a}}(N)\big)$$

defines a linear functor of R-modules. This functor $\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{a}}(\bullet)$ is called the $\mathfrak{a}$-*torsion functor.*

1.16. **Reminders.** A) A sequence of R-modules (and homomorphisms of such)

$$\cdots \to M_{i-1} \xrightarrow{h_{i-1}} M_i \xrightarrow{h_i} M_{i+1} \to \cdots$$

is said to be *exact at the place i,* if $\mathrm{Ker}(h_i) = \mathrm{Im}(h_{i-1})$. A sequence of R-modules is said to be *exact,* if it is exact at each "inner place" $\cdots \to \bullet \to \cdots$ i.e. at each place which is the source and the target of a homomorphism in it.

B) A sequence of R-modules of the shape $0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ is exact if and only if h is injective, l is surjective and $\mathrm{Ker}(l) = \mathrm{Im}(h)$, and then it is called a *short exact sequence.* An exact sequence of the form $0 \to N \xrightarrow{h} M \xrightarrow{l} P$ is called a *short left exact sequence.* Notice that the sequence $0 \to N \xrightarrow{h} M \xrightarrow{l} P$ is exact if and only if h is injective and $\mathrm{Ker}(l) = \mathrm{Im}(h)$. Similarly, an exact sequence of the form $N \xrightarrow{h} M \xrightarrow{l} P \to 0$ is called a *short right exact sequence.*

C) Let R' be a second ring, and let F be an additive functor from R-modules to R'-modules. Remember that the functor F is said to be *exact,* if it preserves the property of being a short exact sequence, i.e.: If $0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ is an exact sequence of R-modules, then $0 \to F(N) \xrightarrow{F(h)} F(M) \xrightarrow{F(l)} F(P) \to 0$ is an exact sequence of R'-modules.

Remember, that F is said to be *left exact* if it preserves the property of being a short left exact sequence, i.e.: If $0 \to N \xrightarrow{h} M \xrightarrow{l} P$ is an exact sequence of R-modules, then $0 \to F(N) \xrightarrow{F(h)} F(M) \xrightarrow{F(l)} F(P)$ is an exact sequence of R'-modules.

Finally, F is said to be *right exact* if it preserves the property of being a short right exact sequence, i.e.: If $N \xrightarrow{h} M \xrightarrow{l} P \to 0$ is an exact sequence of R-modules, then $F(N) \xrightarrow{F(h)} F(M) \xrightarrow{F(l)} F(P) \to 0$ is an exact sequence of R'-modules.

1.17. **Remark and Exercise.** Let R' be a second ring, and let F be an additive functor from R-modules to R'-modules. Then, if F is exact, it preserves the exactness of arbitrary exact sequences, i.e.:

a) If F is exact and if

$$\cdots \to M_{i-1} \xrightarrow{h_{i-1}} M_i \xrightarrow{h_i} M_{i+1} \to \cdots$$

is an exact sequence of R-modules, then

$$\cdots \to F(M_{i-1}) \xrightarrow{F(h_{i-1})} F(M_i) \xrightarrow{F(h_i)} F(M_{i+1}) \to \cdots$$

is an exact sequence of R'-modules.

Also keep in mind that

b) F is exact if and only if F is left exact and right exact.

1.18. **Examples and Exercise.** A) The identity functor Id of 1.14 A) is exact.

B) Let $S \subseteq R$ be multiplicatively closed. Then, the localization functor S^{-1} (cf. 1.14 B)) is exact.

C) Let $f : R \to R'$ be a homomorphism of rings. Then the functor $\lceil_R$ of scalar restriction from R' to R (by means of f) (cf. 1.14 C)) is exact.

D) Fix an R-module U. Then, the functor $\mathrm{Hom}_R(U, \bullet)$ of 1.14 D) is left exact.

Let $R = \mathbb{Z}$ and $U = \mathbb{Z}/2\mathbb{Z}$. Then, it is easy to see that the exact sequence $0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ does not remain exact, if one applies the functor $\mathrm{Hom}_R(U, \bullet)$. So, the functor $\mathrm{Hom}_R(U, \bullet)$ is not exact in general.

1.19. **Proposition.** *The $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ is left exact.*

Proof. Left as an exercise. □

1.20. **Example and Exercise.** Let $R = \mathbb{Z}$ and $\mathfrak{a} = 2\mathbb{Z}$. Use the sequence $0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ to show that the functor $\Gamma_{\mathfrak{a}}$ is not exact in general.

In a series of reminders and exercises we now present a few basic facts on exact sequences, additive functors and direct sums.

1.21. **Reminder and Exercise.** A) Let $h : N \to M$ and $l : M \to P$ be homomorphisms of R-modules. Show that the following statements are equivalent:

(i) $N \xrightarrow{h} M \xrightarrow{l} P \to 0$ is exact and there is a homomorphism of R-modules $r : M \to N$ such that $r \circ h = \mathrm{id}_N$;

(ii) $0 \to N \xrightarrow{h} M \xrightarrow{l} P$ is exact and there is a homomorphism of R-modules $s : P \to M$ such that $l \circ s = \mathrm{id}_P$;

(iii) $l \circ h = 0$ and there are homomorphisms of R-modules $r : M \to N$ and $s : P \to M$ such that $r \circ h = \mathrm{id}_N$, $l \circ s = \mathrm{id}_P$ and $s \circ l + h \circ r = \mathrm{id}_M$;

(iv) There is a commutative diagram with exact first row

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "h"] & M \arrow[r, "l"] & P \arrow[r] & 0 \\
 & N \arrow[u, equal] \arrow[r, "i"] & N \oplus P \arrow[u, "\cong"'] \arrow[r, "p"] & P \arrow[u, equal] &
\end{tikzcd}$$

in which i and p are the canonical homomorphisms defined by $x \mapsto (x, 0)$ and $(x, y) \mapsto y$ respectively.

If the equivalent statements (i)–(iv) hold, one says that the short exact sequence $0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ *splits.*

B) Now, let R' be a second ring, and let F be an additive functor from R-modules to R'-modules. Show (on use of statement A) (iii) for example):

a) If $0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ is a splitting exact sequence of R-modules, then $0 \to F(N) \xrightarrow{F(h)} F(M) \xrightarrow{F(l)} F(P) \to 0$ is a splitting exact sequence of R'-modules.

Hence:

b) If the short exact sequence of R-modules $0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ splits, the sequence $0 \to F(N) \xrightarrow{F(h)} F(M) \xrightarrow{F(l)} F(P) \to 0$ is exact.

1.22. **Reminder and Exercise.** A) Let R' be a second ring, and let F be an additive functor from R-modules to R'-modules. Let M and N be R-modules. Consider the natural homomorphisms of R-modules

$$\begin{aligned}
i &: M \to M \oplus N,\ x \mapsto (x, 0),\\
j &: N \to M \oplus N,\ y \mapsto (0, y),\\
p &: M \oplus N \to M,\ (x, y) \mapsto x,\\
q &: M \oplus N \to N,\ (x, y) \mapsto y,
\end{aligned}$$

and the homomorphism of R'-modules

$$\iota^{M,N} : F(M) \oplus F(N) \to F(M \oplus N),\ (u,v) \mapsto F(i)(u) + F(j)(v).$$

Show that for all $u \in F(M)$, all $v \in F(N)$ and all $w \in F(M \oplus N)$

$$F(p) \circ \iota^{M,N}(u,v) = u,$$
$$F(q) \circ \iota^{M,N}(u,v) = v,$$
$$\iota^{M,N}(F(p)(w), F(q)(w)) = w.$$

Conclude that $\iota^{M,N} : F(M) \oplus F(N) \xrightarrow{\cong} F(M \oplus N)$ is an isomorphism.

B) Keep the notations and hypotheses of part A). Let $h : M \to \bar{M}$, $l : N \to \bar{N}$ be two homomorphisms of R-modules. Show that we have the commutative diagram

$$\begin{tikzcd} F(M) \oplus F(N) \arrow[r, "\iota^{M,N}", "\cong"'] \arrow[d, "F(h)\oplus F(l)"'] & F(M \oplus N) \arrow[d, "F(h\oplus l)"] \\ F(\bar{M}) \oplus F(\bar{N}) \arrow[r, "\iota^{\bar{M},\bar{N}}", "\cong"'] & F(\bar{M} \oplus \bar{N}) \end{tikzcd}$$

in which the vertical homomorphisms are defined in the obvious way.

2. Local Cohomology Functors

We now introduce local cohomology functors as "right derived functors" of torsion functors. The notion of right derived functors shall be mostly developed in exercises.

2.0. **Notation.** Throughout this chapter, let R be a ring and let $\mathfrak{a} \subseteq R$ be an ideal.

2.1. **Remark and Reminder.** A) By a *cocomplex of R-modules* we mean a sequence of R-modules and of homomorphisms of R-modules

$$\cdots \to M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} M^{i+2} \to \cdots$$

such that $\operatorname{Ker}(d^i) \supseteq \operatorname{Im}(d^{i-1})$ for all $i \in \mathbb{Z}$, and we will denote such a cocomplex by $(M^\bullet, d^\bullet)$.

B) Let $(M^\bullet, d^\bullet), (N^\bullet, e^\bullet)$ be two cocomplexes of R-modules. By a *homomorphism of cocomplexes (of R-modules)*

$$h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$$

we mean a family $(h^i)_{i\in\mathbb{Z}}$ of homomorphisms $h^i : M^i \to N^i$ of R-modules such that for all $i \in \mathbb{Z}$ we have $h^{i+1} \circ d^i = e^i \circ h^i$, so that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "h^{i-1}"] & M^i \arrow[r, "d^i"] \arrow[d, "h^i"] & M^{i+1} \arrow[r] \arrow[d, "h^{i+1}"] & \cdots \\
\cdots \arrow[r] & N^{i-1} \arrow[r, "e^{i-1}"] & N^i \arrow[r, "e^i"] & N^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

C) Let $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ and $l^\bullet : (N^\bullet, e^\bullet) \to (P^\bullet, f^\bullet)$ be two homomorphisms of cocomplexes of R-modules. Then, the family $(l^i \circ h^i)_{i\in\mathbb{Z}}$ defines a homomorphism of cocomplexes of R-modules

$$l^\bullet \circ h^\bullet : (M^\bullet, d^\bullet) \to (P^\bullet, f^\bullet),$$

the so called *composition of $l^\bullet$ with $h^\bullet$.*

Observe furthermore, that the family $(\mathrm{id}_{M_i})_{i\in\mathbb{Z}}$ defines a homomorphism of cocomplexes of R-modules

$$\mathrm{id}_{(M^\bullet,d^\bullet)} : (M^\bullet, d^\bullet) \to (M^\bullet, d^\bullet)$$

and that we have the *composition laws*

a) $\mathrm{id}_{(N^\bullet,e^\bullet)} \circ h^\bullet = h^\bullet \circ \mathrm{id}_{(M^\bullet,d^\bullet)} = h^\bullet$;
b) $k^\bullet \circ (l^\bullet \circ h^\bullet) = (k^\bullet \circ l^\bullet) \circ h^\bullet$;

where $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$, $l^\bullet : (N^\bullet, e^\bullet) \to (P^\bullet, f^\bullet)$ and $k^\bullet : (P^\bullet, f^\bullet) \to (Q^\bullet, g^\bullet)$ are homomorphisms of cocomplexes.

D) If $(M^\bullet, d^\bullet)$ and $(N^\bullet, e^\bullet)$ are two cocomplexes of R-modules, we write $\mathrm{Hom}_R((M^\bullet, d^\bullet), (N^\bullet, e^\bullet))$ or just $\mathrm{Hom}_R(M^\bullet, N^\bullet)$ for the set of all homomorphisms of cocomplexes of R-modules $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$. This set carries a natural structure of R-module given by

$$h^\bullet + l^\bullet := (h^i + l^i)_{i\in\mathbb{Z}}, \qquad ah^\bullet := (ah^i)_{i\in\mathbb{Z}},$$

for all $h^\bullet = (h^i)_{i\in\mathbb{Z}}$, $l^\bullet = (l^i)_{i\in\mathbb{Z}} \in \mathrm{Hom}_R(M^\bullet, N^\bullet)$ and all $a \in R$.

2.2. **Reminder.** By a *linear (covariant) functor from (the category of) cocomplexes of R-modules to (the category of) R-modules* we mean an assignment

$$F = F(\bullet) : \big((M^\bullet, d^\bullet) \xrightarrow{h^\bullet} (N^\bullet, e^\bullet)\big) \rightsquigarrow \big(F(M^\bullet, d^\bullet) \xrightarrow{F(h^\bullet)} F(N^\bullet, e^\bullet)\big)$$

which to a cocomplex of R-modules $(M^\bullet, d^\bullet)$ assigns an R-module $F(M^\bullet, d^\bullet)$ and to a homomorphism $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ of cocomplexes of R-modules assigns a homomorphism of R-modules $F(h^\bullet) : F(M^\bullet, d^\bullet) \to F(N^\bullet, e^\bullet)$ such that the following properties hold:

(A1$^\bullet$) $F(\mathrm{id}_{(M^\bullet, d^\bullet)}) = \mathrm{id}_{F(M^\bullet, d^\bullet)}$ for each cocomplex of R-modules $(M^\bullet, d^\bullet)$;
(A2$^\bullet$) $F(h^\bullet \circ l^\bullet) = F(h^\bullet) \circ F(l^\bullet)$, whenever $l^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ and $h^\bullet : (N^\bullet, e^\bullet) \to (P^\bullet, f^\bullet)$ are homomorphisms of cocomplexes of R-modules;
(A3$^\bullet$) $F(h^\bullet) + F(l^\bullet) = F(h^\bullet + l^\bullet)$, whenever $h^\bullet, l^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ are homomorphisms of cocomplexes of R-modules;
(A4$^\bullet$) $F(ah^\bullet) = aF(h^\bullet)$ for each $a \in R$ and each homomorphism of cocomplexes of R-modules $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$.

2.3. **Remark and Reminder.** A) Fix $n \in \mathbb{Z}$, and let $(M^\bullet, d^\bullet)$ be a cocomplex of R-modules. Then, the *n-th cohomology of* $(M^\bullet, d^\bullet)$ is defined by

$$H^n(M^\bullet, d^\bullet) = H^n(M^\bullet) := \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1}).$$

B) Let $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ be a homomorphism of cocomplexes of R-modules. Then, an easy calculation shows that $h^n(\mathrm{Ker}(d^n)) \subseteq \mathrm{Ker}(e^n)$ and that $h^n(\mathrm{Im}(d^{n-1})) \subseteq \mathrm{Im}(e^{n-1})$. So, one can define a homomorphism of R-modules

$$\begin{array}{ccc}
H^n(M^\bullet, d^\bullet) & \xrightarrow{\quad H^n(h^\bullet) \quad} & H^n(N^\bullet, e^\bullet) \\
\| & & \| \\
\mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1}) & & \mathrm{Ker}(e^n)/\mathrm{Im}(e^{n-1}) \\
\cup & & \cup \\
m + \mathrm{Im}(d^{n-1}) & \longmapsto & h^n(m) + \mathrm{Im}(e^{n-1})
\end{array}$$

which is called the *homomorphism induced by $h^\bullet$ in n-th cohomology.*

2.4. **Remark and Exercise.** A) Fix $n \in \mathbb{Z}$. Then, using the notation of 2.1 and 2.3 we have

a) $H^n(\mathrm{id}_{(M^\bullet,d^\bullet)}) = \mathrm{id}_{H^n(M^\bullet,d^\bullet)}$;
b) $H^n(l^\bullet \circ h^\bullet) = H^n(l^\bullet) \circ H^n(h^\bullet)$;

where $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ and $l^\bullet : (N^\bullet, e^\bullet) \to (P^\bullet, f^\bullet)$ are homomorphisms of cocomplexes.

B) Moreover, for any two homomorphisms of cocomplexes $h^\bullet$, $l^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ and $a \in R$ we have

a) $H^n(h^\bullet + l^\bullet) = H^n(h^\bullet) + H^n(l^\bullet)$;
b) $H^n(ah^\bullet) = aH^n(h^\bullet)$.

C) In view of the observations made in A) and B) we can say (cf. 2.2) that the assignment

$$H^n = H^n(\bullet) : \big((M^\bullet, d^\bullet) \xrightarrow{h^\bullet} (N^\bullet, e^\bullet)\big) \rightsquigarrow \big(H^n(M^\bullet, d^\bullet) \xrightarrow{H^n(h^\bullet)} H^n(N^\bullet, e^\bullet)\big)$$

defines a linear functor from cocomplexes of R-modules to R-modules, the *n-th cohomology functor.*

2.5. **Reminder and Exercise.** A) Let $h^\bullet$, $l^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ be two homomorphisms of cocomplexes. A *homotopy from $h^\bullet$ to $l^\bullet$* is a family $(t_i)_{i\in\mathbb{Z}}$ of homomorphisms of R-modules $t_i : M^i \to N^{i-1}$ such that for all $i \in \mathbb{Z}$ we have

$$h^i - l^i = t_{i+1} \circ d^i + e^{i-1} \circ t_i.$$

B) If there is a homotopy $(t_i)_{i\in\mathbb{Z}}$ from $h^\bullet$ to $l^\bullet$, we say that *$h^\bullet$ is homotopic to $l^\bullet$* and we write $h^\bullet \sim l^\bullet$. This defines an equivalence relation on $\mathrm{Hom}_R((M^\bullet, d^\bullet), (N^\bullet, e^\bullet))$, i.e.:

a) $h^\bullet \sim h^\bullet$;
b) $h^\bullet \sim l^\bullet \Leftrightarrow l^\bullet \sim h^\bullet$;
c) $h^\bullet \sim l^\bullet, l^\bullet \sim k^\bullet \Rightarrow h^\bullet \sim k^\bullet$ (for a further homomorphism of cocomplexes of R-modules $k^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$).

C) A most important feature is that "homotopic homomorphisms of cocomplexes are cohomologueous": If $h^\bullet, l^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ are homomorphisms of cocomplexes of R-modules with $h^\bullet \sim l^\bullet$, then $H^n(h^\bullet) = H^n(l^\bullet)$ for all $n \in \mathbb{Z}$.

2.6. **Remark and Exercise.** A) Let R' be a second ring, and let F be an additive functor from R-modules to R'-modules. Let $(M^\bullet, d^\bullet)$ be a cocomplex of R-modules. Then we obtain a cocomplex of R'-modules

$$(F(M^\bullet), F(d^\bullet)) : \cdots \to F(M^{i-1}) \xrightarrow{F(d^{i-1})} F(M^i) \xrightarrow{F(d^i)} F(M^{i+1}) \to \cdots.$$

B) Keep the notation and hypotheses of part A). If

$$h^\bullet = (h^i)_{i\in\mathbb{Z}} : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$$

is a homomorphism of cocomplexes of R-modules, then the family $(F(h^i))_{i\in\mathbb{Z}}$ defines a homomorphism of cocomplexes of R'-modules

$$F(h^\bullet) : (F(M^\bullet), F(d^\bullet)) \to (F(N^\bullet), F(e^\bullet)).$$

C) Now, let $h^\bullet$, $l^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ be homomorphisms of cocomplexes of R-modules. Then we have:

a) If $(t_i)_{i\in\mathbb{Z}}$ is a homotopy from $h^\bullet$ to $l^\bullet$, then $(F(t_i))_{i\in\mathbb{Z}}$ is a homotopy from $F(h^\bullet)$ to $F(l^\bullet)$.

This fact implies in particular:

b) If $h^\bullet \sim l^\bullet$, then $F(h^\bullet) \sim F(l^\bullet)$.

Combining this statement with 2.5 C) we get

c) If $h^\bullet \sim l^\bullet$, then $H^n(F(h^\bullet)) = H^n(F(l^\bullet))$ for all $n \in \mathbb{Z}$.

2.7. **Reminder and Exercise.** A) An R-module I is said to be *injective,* if for each monomorphism $i : N \rightarrowtail M$ of R-modules and each homomorphism $h : N \to I$ of R-modules there is a homomorphism $l : M \to I$ of R-modules such that $h = l \circ i$. In diagrammatic form:

$$\begin{tikzcd} & I \\ N \arrow[ur, "h"] \arrow[r, tail, "i"'] & M. \arrow[u, dotted, "l"'] \end{tikzcd}$$

B) Show the following properties of injective modules:

a) If I and J are injective R-modules, so is $I \oplus J$;

b) If $\mathbb{S} : 0 \to I \xrightarrow{h} M \xrightarrow{l} P \to 0$ is an exact sequence of R-modules in which I is injective, then $\mathbb{S}$ splits (cf. 1.21 A));

c) If I is an injective submodule of an R-module M, then $M \cong I \oplus M/I$;

d) If I is an injective submodule of an injective R-module J, then J/I is an injective R-module, too.

The next two exercises yield fundamental results about injective modules. The first is the *Baer Criterion,* a useful tool to test whether an R-module is injective. The second is the *Lemma of Eckmann-Schopf* which says that (in some sense) “there are enough injective R-modules”.

2.8. **Exercise.** A) Let I and M be R-modules, let $N \subseteq M$ be a submodule and let $f : N \to I$ be a homomorphism of R-modules. Consider the set

$$\mathbb{M} := \{(E, g) | E \subseteq M \text{ is a submodule}, N \subseteq E, g \in \mathrm{Hom}_R(E, I), g \upharpoonright_N = f\}$$

and define on it a binary relation $\leq$ by setting $(E,g) \leq (E',g')$ if and only if $E \subseteq E'$ and $g' \restriction_E = g$. Show that $(\mathbb{M}, \leq)$ is an inductive ordered set, and conclude by Zorn's Lemma that there exists a maximal element in $\mathbb{M}$.

B) Keep all the hypotheses of A) and let $(E,g) \in \mathbb{M}$ with $E \neq M$. Choose an element $x \in M \setminus E$ and consider the ideal $\mathfrak{b} := \{r \in R | rx \in E\}$. Assume that there exists a homomorphism of R-modules $\psi : R \to I$ such that $\psi(r) = g(rx)$ for all $r \in \mathfrak{b}$. Show that there can be defined a homomorphism of R-modules $h : E + Rx \to I$ such that $h(e + ax) = g(e) + \psi(a)$ for $e \in E$ and $a \in R$, and conclude that (E,g) cannot be maximal in $\mathbb{M}$.

C) By A) and B) it follows: Assume that for each ideal $\mathfrak{b} \subseteq R$ and each homomorphism of R-modules $h : \mathfrak{b} \to I$ there is an element $e \in I$ such that $h(a) = ae$ for all $a \in \mathfrak{b}$. Then I is injective. (*Baer Criterion*)

2.9. **Exercise.** A) Let R be a domain. An R-module M is called *divisible* if the multiplication homomorphism $a\cdot : M \to M$ is surjective for each $a \in R \setminus \{0\}$. Prove the following statements:

a) If R is a principal ideal domain, then an R-module M is injective if and only if it is divisible;
b) If N is a submodule of a divisible R-module M, then M/N is divisible;
c) If K is an extension field of R, then the R-module K is divisible;
d) If $(M_i)_{i \in I}$ is a familiy of divisible R-modules, then $\bigoplus_{i \in I} M_i$ is divisible.

B) Let R be a principal ideal domain and let M be an R-module. Show that there exists an injective R-module I and a monomorphism of R-modules $M \rightarrowtail I$.

(Hint: Consider an isomorphism $M \cong (\bigoplus_{m \in M} R)/U$ and the quotient field Q of R.)

C) Let G be an Abelian group and let M be an R-module. Show the following statements:

a) The Abelian group $\mathrm{Hom}_{\mathbb{Z}}(M,G)$ of homomorphisms of groups from the additive group of M to G can be turned into an R-module defining $(a\varphi)(m) := \varphi(am)$ for $a \in R$, $m \in M$ and $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(M,G)$;
b) There is an isomorphism $\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R,G)) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(M,G)$ of R-modules;
c) If G is divisible as $\mathbb{Z}$-module, then the R-module $\mathrm{Hom}_{\mathbb{Z}}(R,G)$ is injective;
d) If M is an R-module and if $f : M \to G$ is a monomorphism of groups, then there is a monomorphism of R-modules $M \rightarrowtail \mathrm{Hom}_{\mathbb{Z}}(R,G)$.

D) Conclude on use of B) and C): For each R-module M there is an injective R-module I together with a monomorphism $M \rightarrowtail I$ of R-modules. (*Lemma of Eckmann-Schopf*)

2.10. **Reminder.** A) Let M be an R-module. A *right resolution of* M consists of a cocomplex of R-modules $(E^\bullet, e^\bullet)$ and a homomorphism $b : M \to E^0$ such that $E^i = 0$ for all $i < 0$ and that the sequence of R-modules

$$0 \to M \xrightarrow{b} E^0 \xrightarrow{e^0} E^1 \xrightarrow{e^1} E^2 \to \cdots$$

is exact. Such a right resolution will be denoted by $((E^\bullet, e^\bullet); b)$. Then, $(E^\bullet, e^\bullet)$ is called a *resolving cocomplex for* M and b is called a *coaugmentation.*

B) Let M and N be two R-modules, let $((D^\bullet, d^\bullet); a)$ be a right resolution of M and let $((E^\bullet, e^\bullet); b)$ a right resolution of N. Let $h : M \to N$ be a homomorphism of R-modules. Then, a *(right) resolution of* h *(between* $((D^\bullet, d^\bullet); a)$ *and* $((E^\bullet, e^\bullet); b)$*)* is a homomorphism of cocomplexes

$$h^\bullet : (D^\bullet, d^\bullet) \to (E^\bullet, e^\bullet)$$

such that $h^0 \circ a = b \circ h$, i.e. such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "a"] \arrow[d, "h"] & D^0 \arrow[r, "d^0"] \arrow[d, "h^0"] & D^1 \arrow[r, "d^1"] \arrow[d, "h^1"] & D^2 \arrow[r, "d^2"] \arrow[d, "h^2"] & D^3 \arrow[r] \arrow[d, "h^3"] & \cdots \\
0 \arrow[r] & N \arrow[r, "b"] & E^0 \arrow[r, "e^0"] & E^1 \arrow[r, "e^1"] & E^2 \arrow[r, "e^2"] & E^3 \arrow[r] & \cdots .
\end{tikzcd}$$

C) Let M be an R-module. An *injective resolution of* M is a right resolution $((I^\bullet, d^\bullet); a)$ of M such that all R-modules I^i are injective. So, in this case, we have an exact sequence of R-modules

$$0 \to M \xrightarrow{a} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \to \cdots$$

with injective R-modules I^0, I^1, I^2, I^3, $\ldots$

2.11. **Remark and Exercise.** A) By recursion on n and using the Lemma of Eckmann-Schopf (cf. 2.9), we may construct injective R-modules $I^0, I^1, \ldots$ and homomorphisms of R-modules $M \xrightarrow{a} I^0$, $I^0 \xrightarrow{d^0} I^1$, $I^1 \xrightarrow{d^1} I^2$, $\ldots$ such that the sequence

$$0 \to M \xrightarrow{a} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \to \cdots$$

is exact. So we can say: Each R-module M has an injective resolution $((I^\bullet, d^\bullet); a)$.

B) Concerning resolutions of homomorphisms we have: Let $M \xrightarrow{h} N$ be a homomorphism of R-modules, let $((E^\bullet, e^\bullet); b)$ be a right resolution of M and let $((I^\bullet, d^\bullet); a)$ be an injective resolution of N. Then h admits a resolution $h^\bullet : (E^\bullet, e^\bullet) \to (I^\bullet, d^\bullet)$.

(Here again, the homomorphisms h^0, h^1, h^2, $\ldots$ are constructed recursively, making use of the fact that the modules I^0, I^1, I^2, $\ldots$ are injective.)

C) Finally, let us notice: Let $M \xrightarrow{h} N$, $((E^\bullet, e^\bullet); b)$ and $((I^\bullet, d^\bullet); a)$ be as in B). Moreover let $h^\bullet$, $l^\bullet : (E^\bullet, e^\bullet) \to (I^\bullet, d^\bullet)$ be resolutions of h. Then $h^\bullet \sim l^\bullet$.

(Again by recursion one can construct a sequence of homomorphisms of R-modules $t_0 : E^0 \to I^{-1} = 0$, $t_1 : E^1 \to I^0, \ldots, t_i : E^i \to I^{i-1}, \ldots$ such that $h^i - l^i = d^{i-1} \circ t_i + t_{i+1} \circ e^i$ for all $i \in \mathbb{N}_0$.)

2.12. **Reminder and Exercise.** A) Let R' be a second ring, and let F be an additive functor from R-modules to R'-modules. Let $M \xrightarrow{h} N$ be a homomorphism of R-modules, let $((E^\bullet, e^\bullet); b)$ be a right resolution of M and let $((I^\bullet, d^\bullet); a)$ be an injective resolution of N. Let $h^\bullet,\ l^\bullet : (E^\bullet, e^\bullet) \to (I^\bullet, d^\bullet)$ be two resolutions of h. Let $n \in \mathbb{N}_0$. Then the two homomorphisms of R-modules

$$H^n(F(h^\bullet)),\ H^n(F(l^\bullet)) : H^n(F(E^\bullet), F(e^\bullet)) \to H^n(F(I^\bullet), F(d^\bullet))$$

are equal:

$$H^n(F(h^\bullet)) = H^n(F(l^\bullet)).$$

(This follows easily by 2.11 C) and 2.6 C) c).)

B) Let M be an R-module, let $((I^\bullet, d^\bullet); a)$ and $((J^\bullet, e^\bullet); b)$ be two injective resolutions of M and let $i^\bullet : (I^\bullet, d^\bullet) \to (J^\bullet, e^\bullet)$ be a resolution of $\mathrm{id}_M : M \to M$. Then, for each $n \in \mathbb{N}$ we have an isomorphism

$$H^n(F(i^\bullet)) : H^n(F(I^\bullet), F(d^\bullet)) \xrightarrow{\cong} H^n(F(J^\bullet), F(e^\bullet)).$$

(By 2.11 B) id_M also admits a resolution $j^\bullet : (J^\bullet, e^\bullet) \to (I^\bullet, d^\bullet)$. So $\mathrm{id}^\bullet = \mathrm{id}_{(I^\bullet, d^\bullet)}$ and $j^\bullet \circ i^\bullet : (I^\bullet, d^\bullet) \to (I^\bullet, d^\bullet)$ are both resolutions of id_M. By part A) it follows that $H^n(F(\mathrm{id}^\bullet)) = H^n(F(j^\bullet \circ i^\bullet))$. Using 2.4 A) we conclude that $H^n(F(j^\bullet)) \circ H^n(F(i^\bullet)) = \mathrm{id}_{H^n(F(I^\bullet), F(d^\bullet))}$. The claim follows now by symmetry.)

C) Let M, $((I^\bullet, d^\bullet); a)$ and $((J^\bullet, e^\bullet); b)$ be as in B). Let

$$i^\bullet,\ j^\bullet : (I^\bullet, d^\bullet) \to (J^\bullet, e^\bullet)$$

be resolutions of id_M. Then, the two isomorphisms

$$H^n(F(i^\bullet)),\ H^n(F(j^\bullet)) : H^n(F(I^\bullet), F(d^\bullet)) \xrightarrow{\cong} H^n(F(J^\bullet), F(e^\bullet))$$

are the same: $H^n(F(i^\bullet)) = H^n(F(j^\bullet))$. (This follows by another use of part A).)

2.13. **Construction and Exercise.** A) Let R' be a second ring, and let F be an additive functor from R-modules to R'-modules. For each R-module M we can choose an injective resolution $\mathbb{I}_M = ((I^\bullet_M, d^\bullet_M); a_M)$ (cf. 2.11 A)). So, for each M we have an exact sequence

$$0 \to M \xrightarrow{a_M} I^0_M \xrightarrow{d^0_M} I^1_M \xrightarrow{d^1_M} I^2_M \xrightarrow{d^2_M} I^3_M \to \cdots$$

in which all the modules I^n_M are injective. We write $\mathbb{I}_*$ for the assignment

$$M \leadsto \mathbb{I}_M = ((I^\bullet_M, d^\bullet_M); a_M)$$

and call $\mathbb{I}_*$ a *choice of injective resolutions (of R-modules).*

Now, for $n \in \mathbb{N}_0$ we define

$$\mathcal{R}^n_{\mathbb{I}_*} F(M) := H^n(F(I^\bullet_M), F(d^\bullet_M)),$$

i.e. we consider the cocomplex of R'-modules

$$\cdots \to 0 \xrightarrow{F(d_M^{-1})} F(I_M^0) \xrightarrow{F(d_M^0)} F(I_M^1) \xrightarrow{F(d_M^1)} F(I_M^2) \to \cdots$$

and the R'-modules (cf. 2.3 A)):

$$\mathcal{R}^n_{\mathbb{I}_*}F(M) = \operatorname{Ker}(F(d_M^n))/\operatorname{Im}(F(d_M^{n-1})).$$

B) Let $h : M \to N$ be a homomorphism of R-modules. Then h admits a resolution $h^\bullet : (I_M^\bullet, d_M^\bullet) \to (I_N^\bullet, d_N^\bullet)$ (cf. 2.11 B)), so that we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "a_M"] \arrow[d, "h"] & I_M^0 \arrow[r, "d_M^0"] \arrow[d, "h^0"] & I_M^1 \arrow[r, "d_M^1"] \arrow[d, "h^1"] & I_M^2 \arrow[r, "d_M^2"] \arrow[d, "h^2"] & I_M^3 \arrow[r] \arrow[d, "h^3"] & \cdots \\
0 \arrow[r] & N \arrow[r, "a_N"] & I_N^0 \arrow[r, "d_N^0"] & I_N^1 \arrow[r, "d_M^1"] & I_N^2 \arrow[r, "d_N^2"] & I_N^3 \arrow[r] & \cdots .
\end{tikzcd}$$

Now, by 2.12 A), the homomorphism

$$H^n(F(h^\bullet)) : H^n(F(I_M^\bullet), F(d_M^\bullet)) \to H^n(F(I_N^\bullet), F(d_N^\bullet))$$

is the same for each such resolution $h^\bullet$ of h. So, extending the definition of part A), we also may define for all $n \in \mathbb{N}_0$ a homomorphism of R'-modules:

$$\mathcal{R}^n_{\mathbb{I}_*}F(h) : \mathcal{R}^n_{\mathbb{I}_*}F(M) \xrightarrow{H^n(F(h^\bullet))} \mathcal{R}^n_{\mathbb{I}_*}F(N).$$

Thus, in view of 2.3 B) we can write:

$$\begin{array}{ccc}
\mathcal{R}^n_{\mathbb{I}_*}F(M) & \xrightarrow{\qquad \mathcal{R}^n_{\mathbb{I}_*}F(h) \qquad} & \mathcal{R}^n_{\mathbb{I}_*}F(N) \\
\| & & \| \\
\operatorname{Ker}(F(d_M^n))/\operatorname{Im}(F(d_M^{n-1})) & & \operatorname{Ker}(F(d_N^n))/\operatorname{Im}(F(d_N^{n-1})) \\
\uplus & & \uplus \\
m + \operatorname{Im}(F(d_M^{n-1})) & \longmapsto & F(h^n)(m) + \operatorname{Im}(F(d_N^{n-1})).
\end{array}$$

C) Now, fix some $n \in \mathbb{N}_0$. Then one can verify:

a) The assignment

$$\mathcal{R}^n_{\mathbb{I}_*}F = \mathcal{R}^n_{\mathbb{I}_*}F(\bullet) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(\mathcal{R}^n_{\mathbb{I}_*}F(M) \xrightarrow{\mathcal{R}^n_{\mathbb{I}_*}F(h)} \mathcal{R}^n_{\mathbb{I}_*}F(N)\big)$$

defines an additive functor from R-modules to R'-modules.

b) If F is linear with respect to a homomorphism of rings f between R and R', then $\mathcal{R}^n_{\mathbb{I}_*}F$ is linear with respect to f.

(To prove the composition law 1.12 B) (A2) for the assignment $\mathcal{R}^n_{\mathbb{I}_*}F$, one needs 2.4 A) b) and 2.12 A). To prove the properties 1.12 B) (A1), (A3) and C) (A4) one can use 2.4 A) a), B) a), b).)

The functor $\mathcal{R}^n_{\mathbb{I}_*}F = \mathcal{R}^n_{\mathbb{I}_*}F(\bullet)$ is called the *n-th right derived functor of F with respect to $\mathbb{I}_*$.*

D) Next, consider a second choice of injective resolutions $\mathbb{J}_*$ (of R-modules), which to each R-module M assigns an injective resolution $\mathbb{J}_M = ((J^\bullet_M, e^\bullet_M); b_M)$ of M. Now, fix an R-module M and let $i^\bullet : (I^\bullet_M, d^\bullet_M) \to (J^\bullet_M, e^\bullet_M)$ be a resolution of id_M (which exists by 2.11 B)). By 2.12 B) we have isomorphisms of R'-modules

$$H^n(F(i^\bullet)) : H^n(F(I^\bullet_M), F(d^\bullet_M)) \xrightarrow{\cong} H^n(F(J^\bullet_M), F(e^\bullet_M)).$$

If we fix n, the isomorphism $H^n(F(i^\bullet))$ is the same for each resolution $i^\bullet : (I^\bullet_M, d^\bullet_M) \to (J^\bullet_M, e^\bullet_M)$ of id_M (cf. 2.12 A)) and so it defines an isomorphism of R'-modules (cf. part A))

a) $$\varepsilon^{n,M}_{\mathbb{I}_*,\mathbb{J}_*} : \mathcal{R}^n_{\mathbb{I}_*}F(M) \xrightarrow[\cong]{H^n(F(i^\bullet))} \mathcal{R}^n_{\mathbb{J}_*}F(M)$$

which depends only on $\mathbb{I}_M$ and $\mathbb{J}_M$.

Let $h : M \to N$ be a homomorphism of R-modules and let $h^\bullet : (I^\bullet_M, d^\bullet_M) \to (I^\bullet_N, d^\bullet_N)$ and $l^\bullet : (J^\bullet_M, e^\bullet_M) \to (J^\bullet_N, e^\bullet_N)$ be resolutions of h. Let $j^\bullet : (I^\bullet_N, d^\bullet_N) \to (J^\bullet_N, e^\bullet_N)$ be a resolution of id_N. Then $j^\bullet \circ h^\bullet$, $l^\bullet \circ i^\bullet : (I^\bullet_M, d^\bullet_M) \to (J^\bullet_N, e^\bullet_N)$ are both resolutions of h, so that

$$H^n(F(j^\bullet \circ h^\bullet)) = H^n(F(l^\bullet \circ i^\bullet))$$

for all $n \in \mathbb{N}_0$ (cf. 2.12 A)). From this, we conclude:

b) If $h : M \to N$ is a homomorphism of R-modules,

$$\varepsilon^{n,N}_{\mathbb{I}_*,\mathbb{J}_*} \circ \mathcal{R}^n_{\mathbb{I}_*}F(h) = \mathcal{R}^n_{\mathbb{J}_*}F(h) \circ \varepsilon^{n,M}_{\mathbb{I}_*,\mathbb{J}_*},$$

and hence there is a commutative diagram

$$\begin{tikzcd}
\mathcal{R}^n_{\mathbb{I}_*}F(N) \arrow[r, "\varepsilon^{n,N}_{\mathbb{I}_*,\mathbb{J}_*}", "\cong"'] & \mathcal{R}^n_{\mathbb{J}_*}F(N) \\
\mathcal{R}^n_{\mathbb{I}_*}F(M) \arrow[u, "\mathcal{R}^n_{\mathbb{I}_*}F(h)"] \arrow[r, "\varepsilon^{n,M}_{\mathbb{I}_*,\mathbb{J}_*}", "\cong"'] & \mathcal{R}^n_{\mathbb{J}_*}F(M). \arrow[u, "\mathcal{R}^n_{\mathbb{J}_*}F(h)"']
\end{tikzcd}$$

In view of this observation we do not have to care about the choice of an injective resolution $\mathbb{I}_*$, when dealing with right derived functors. We thus write

$$\mathcal{R}^nF := \mathcal{R}^n_{\mathbb{I}_*}F$$

and call the functor $\mathcal{R}^nF$ simply the *n-th right derived functor of F.*

2.14. **Definition and Remark.** A) Let $n \in \mathbb{N}_0$. We define the *n-th local cohomology functor $H^n_{\mathfrak{a}} = H^n_{\mathfrak{a}}(\bullet)$ with respect to* $\mathfrak{a}$ as the n-th right derived functor $\mathcal{R}^n\Gamma_{\mathfrak{a}} = \mathcal{R}^n\Gamma_{\mathfrak{a}}(\bullet)$ of the $\mathfrak{a}$-torsion functor, thus

$$H^n_{\mathfrak{a}} := \mathcal{R}^n\Gamma_{\mathfrak{a}}.$$

B) Let $n \in \mathbb{N}_0$. Let M be an R-module. The *n-th local cohomology module of M with respect to $\mathfrak{a}$* is defined as the R-module $H^n_{\mathfrak{a}}(M)$. According to the construction done in 2.13, the module $H^n_{\mathfrak{a}}(M)$ may be obtained as follows: First choose an injective resolution $((I^\bullet, d^\bullet); a)$ of M, so that we have an exact sequence

$$0 \to M \xrightarrow{a} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \to \cdots$$

with injective R-modules I^i. Then, apply the functor $\Gamma_{\mathfrak{a}}$ to the resolving cocomplex $(I^\bullet, d^\bullet)$:

$$\cdots \to 0 \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \to \cdots$$

in order to obtain the cocomplex $(\Gamma_{\mathfrak{a}}(I^\bullet), \Gamma_{\mathfrak{a}}(d^\bullet))$:

$$\cdots \to 0 \xrightarrow{\Gamma_{\mathfrak{a}}(d^{-1})} \Gamma_{\mathfrak{a}}(I^0) \xrightarrow{\Gamma_{\mathfrak{a}}(d^0)} \Gamma_{\mathfrak{a}}(I^1) \xrightarrow{\Gamma_{\mathfrak{a}}(d^1)} \Gamma_{\mathfrak{a}}(I^2) \to \cdots .$$

Then, apply n-th cohomology to this cocomplex to end up with

$$H^n_{\mathfrak{a}}(M) = H^n(\Gamma_{\mathfrak{a}}(I^\bullet), \Gamma_{\mathfrak{a}}(d^\bullet)) = \operatorname{Ker}(\Gamma_{\mathfrak{a}}(d^n))/\operatorname{Im}(\Gamma_{\mathfrak{a}}(d^{n-1})).$$

C) Now, let $h : M \to N$ be a homomorphism of R-modules. The *homomorphism induced by h in n-th local cohomology with respect to $\mathfrak{a}$* is defined as the homomorphism of R-modules

$$H^n_{\mathfrak{a}}(h) : H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(N).$$

It may be obtained by choosing an injective resolution of $((I^\bullet_M, d^\bullet_M); a_M)$ and an injective resolution $((I^\bullet_N, d^\bullet_N); a_N)$ of N, by choosing a resolution $h^\bullet$: $(I^\bullet_M, d^\bullet_M) \to (I^\bullet_N, d^\bullet_N)$ of h and by defining $H^n_{\mathfrak{a}}(h)$ as indicated below:

$$\begin{array}{ccc}
H^n_{\mathfrak{a}}(M) & \xrightarrow{\qquad H^n_{\mathfrak{a}}(h) \qquad} & H^n_{a}(N) \\
\| & & \| \\
\operatorname{Ker}(\Gamma_{\mathfrak{a}}(d^n_M))/\operatorname{Im}(\Gamma_{\mathfrak{a}}(d^{n-1}_M)) & & \operatorname{Ker}(\Gamma_{\mathfrak{a}}(d^n_N))/\operatorname{Im}(\Gamma_{\mathfrak{a}}(d^{n-1}_N)) \\
\Cup & & \Cup \\
m + \operatorname{Im}(\Gamma_{\mathfrak{a}}(d^{n-1}_M)) & \longmapsto & h^n(m) + \operatorname{Im}(\Gamma_{\mathfrak{a}}(d^{n-1}_N)).
\end{array}$$

2.15. **Remark and Exercise.** Let R' be a second ring, and let F be an additive functor from R-modules to R'-modules. Then:

a) If I is an injective R-module, then $\mathcal{R}^n F(I) = 0$ for all $n > 0$.

In particular:

b) If I is an injective R-module, then $H^n_{\mathfrak{a}}(I) = 0$ for all $n > 0$.

Finally:

c) If F is exact, then $\mathcal{R}^n F(M) = 0$ for each R-module M and all $n > 0$.

(This can be shown on use of 1.17 a)).

3. Basic Properties of Local Cohomology

Some of the fundamental properties of local cohomology modules shall be presented, notably the cohomology sequence associated to a short exact sequence (cf. 3.9), and the behaviour of local cohomology with respect to torsion (cf. 3.13, 3.17).

3.0. **Notation.** Throughout this chapter, let R be a ring and let $\mathfrak{a} \subseteq R$ be an ideal.

3.1. **Reminder and Exercise.** A) Let R' be a second ring, and let F and G be additive functors from R-modules to R'-modules. A *natural transformation from F to G* is an assignment

$$\beta : M \leadsto (\beta_M : F(M) \to G(M))$$

which to each R-module M assigns a homomorphism of R'-modules $\beta_M : F(M) \to G(M)$ such that for each homomorphism of R-modules $h : M \to N$ we have

$$G(h) \circ \beta_M = \beta_N \circ F(h),$$

so that the diagram

$$\begin{tikzcd} F(M) \arrow[r, "\beta_M"] \arrow[d, "F(h)"] & G(M) \arrow[d, "G(h)"'] \\ F(N) \arrow[r, "\beta_N"] & G(N) \end{tikzcd}$$

is commutative. Such a natural transformation is denoted by $\beta : F \to G$.

B) A natural transformation $\beta : F \to G$ is called a *natural equivalence from F to G* if $\beta_M : F(M) \to G(M)$ is an isomorphism for all R-modules M.

If there is a natural equivalence β from F to G we write $\beta : F \xrightarrow{\cong} G$ or just $F \sim G$ and say that F and G are *naturally equivalent.* Natural equivalence of functors is indeed an equivalence relation: Namely, if F, G, H are additive functors from R-modules to R'-modules, then

a) $F \sim F$;

b) $F \sim G \Leftrightarrow G \sim F$;

c) $F \sim G$, $G \sim H \Rightarrow F \sim H$.

3.2. **Examples and Exercise.** A) For each R-module M, let $i_M : \Gamma_{\mathfrak{a}}(M) \to M$ denote the inclusion map and let Id denote the identity functor on R-modules. Then the assignment

$$M \leadsto (i_M : \Gamma_{\mathfrak{a}}(M) \to M)$$

defines a natural transformation

$$i : \Gamma_{\mathfrak{a}} \to \mathrm{Id}\,.$$

B) Let R', F, $\mathbb{I}_*$ and $\mathbb{J}_*$ be as in 2.13 D). Then, for each $n \in \mathbb{N}_0$, the functors $\mathcal{R}^n_{\mathbb{I}_*}F$ and $\mathcal{R}^n_{\mathbb{J}^*}F$ are naturally equivalent.

3.3. **Remark and Exercise.** A) Let R' be a second ring, and let F be an additive functor from R-modules to R'-modules. Let M be an R-module and let $((I^\bullet, d^\bullet); a)$ be an injective resolution of M. Observe that we can write (cf. 1.13 b))

$$\begin{aligned}\mathcal{R}^0F(M) &= H^0(F(I^\bullet), F(d^\bullet)) = \operatorname{Ker}(F(d^0))/\operatorname{Im}(F(d^{-1}))\\ &= \operatorname{Ker}(F(d^0))/\operatorname{Im}(F(0)) = \operatorname{Ker}(F(d^0))/0\\ &= \operatorname{Ker}(F(d^0)).\end{aligned}$$

As $d^0 \circ a = 0$ we have $F(d^0) \circ F(a) = F(d^0 \circ a) = F(0) = 0$ (cf. 1.13 b)) so that $\operatorname{Im}(F(a)) \subseteq \operatorname{Ker}(F(d^0))$. Therefore we get a homomorphism of R'-modules

$$\alpha^F_M : F(M) \to \mathcal{R}^0F(M) = \operatorname{Ker}(F(d^0))$$

defined by $m \mapsto F(a)(m)$ for all $m \in F(M)$. In particular we have a commutative diagram

$$\begin{tikzcd} F(M) \arrow[rr, "\alpha^F_M"] \arrow[dr, "F(a)"'] & & \mathcal{R}^0F(M) = \operatorname{Ker}(F(d^0)) \arrow[dl, hook'] \\ & F(I^0). & \end{tikzcd}$$

B) It is easy to verify, that the construction of α^F_M is natural. More precisely:

a) If $h : M \to N$ is a homomorphism of R-modules, then

$$\mathcal{R}^0F(h) \circ \alpha^F_M = \alpha^F_N \circ F(h),$$

so that we have a commutative diagram

$$\begin{tikzcd} F(M) \arrow[r, "\alpha^F_M"] \arrow[d, "F(h)"'] & \mathcal{R}^0F(M) \arrow[d, "\mathcal{R}^0F(h)"] \\ F(N) \arrow[r, "\alpha^F_N"] & \mathcal{R}^0F(N). \end{tikzcd}$$

Hence, the assignment

$$M \rightsquigarrow (\alpha^F_M : F(M) \to \mathcal{R}^0F(M))$$

defines a natural transformation $\alpha^F : F \to \mathcal{R}^0F$.

Moreover, if F is left exact, we have an exact sequence

$$0 \to F(M) \xrightarrow{F(a)} F(I^0) \xrightarrow{F(d^0)} F(I^1)$$

showing that $\operatorname{Ker}(F(d^0)) = \operatorname{Im}(F(a))$. By the diagram of part A) we thus see:

b) If F is left exact, then $\alpha^F_M : F(M) \xrightarrow{\cong} \mathcal{R}^0F(M)$ is an isomorphism.

In view of statement a) we now get:

c) If F is left exact, the natural transformation $\alpha^F : F \to \mathcal{R}^0 F$ is a natural equivalence.

C) Let F be left exact. In view of B) c) one usually identifies the functors F and $\mathcal{R}^0 F$ by means of the natural equivalence α^F, thus:

$$\mathcal{R}^0 F = F, \text{ if } F \text{ is left exact.}$$

3.4. **Proposition.** $\Gamma_{\mathfrak{a}}(\bullet) = H^0_{\mathfrak{a}}(\bullet)$.

Proof. Clear by 1.19 and 3.3 C). □

Our next aim is to introduce one of the most basic tools in local cohomology: the cohomology sequence with respect to an ideal $\mathfrak{a} \subseteq R$ associated to a short exact sequence $\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ of R-modules (cf. 3.9). We begin with an excursion in homological algebra and first introduce the cohomology sequence associated to a short exact sequence of cocomplexes of R-modules.

3.5. **Reminder and Exercise.** A) A *short exact sequence of cocomplexes of R-modules*

$$0 \to (N^\bullet, e^\bullet) \xrightarrow{h^\bullet} (M^\bullet, d^\bullet) \xrightarrow{l^\bullet} (P^\bullet, f^\bullet) \to 0,$$

or just written as $0 \to N^\bullet \xrightarrow{h^\bullet} M^\bullet \xrightarrow{l^\bullet} P^\bullet \to 0$, is given by two homomorphisms of cocomplexes of R-modules $h^\bullet : (N^\bullet, e^\bullet) \to (M^\bullet, d^\bullet)$ and $l^\bullet : (M^\bullet, d^\bullet) \to (P^\bullet, f^\bullet)$ such that for each $n \in \mathbb{Z}$ one has the short exact sequence

$$0 \to N^n \xrightarrow{h^n} M^n \xrightarrow{l^n} P^n \to 0.$$

So, for each $n \in \mathbb{Z}$ one has the commutative diagram of R-modules

$$\begin{tikzcd}
0 \arrow[r] & N^{n-1} \arrow[r, "h^{n-1}"] \arrow[d, "e^{n-1}"] & M^{n-1} \arrow[r, "l^{n-1}"] \arrow[d, "d^{n-1}"] & P^{n-1} \arrow[r] \arrow[d, "f^{n-1}"] & 0 \\
0 \arrow[r] & N^{n} \arrow[r, "h^{n}"] \arrow[d, "e^{n}"] & M^{n} \arrow[r, "l^{n}"] \arrow[d, "d^{n}"] & P^{n} \arrow[r] \arrow[d, "f^{n}"] & 0 \\
0 \arrow[r] & N^{n+1} \arrow[r, "h^{n+1}"] \arrow[d, "e^{n+1}"] & M^{n+1} \arrow[r, "l^{n+1}"] \arrow[d, "d^{n+1}"] & P^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & 0 \\
0 \arrow[r] & N^{n+2} \arrow[r, "h^{n+2}"] & M^{n+2} \arrow[r, "l^{n+2}"] & P^{n+2} \arrow[r] & 0
\end{tikzcd}$$

with exact rows and such that the composition of two consecutive maps in any column is 0.

B) Let $\mathbb{S}^\bullet : 0 \to (N^\bullet, e^\bullet) \xrightarrow{h^\bullet} (M^\bullet, d^\bullet) \xrightarrow{l^\bullet} (P^\bullet, f^\bullet) \to 0$ be a short exact sequence of cocomplexes of R-modules. Fix $n \in \mathbb{Z}$. We define a relation "$\looparrowright$" on $P^n \times N^{n+1}$ by setting for $x \in P^n$, $y \in N^{n+1}$:

$$x \looparrowright y :\Leftrightarrow \exists\, x' \in M^n : l^n(x') = x \wedge d^n(x') = h^{n+1}(y).$$

Show the following facts:

a) If $x \looparrowright y$, then $x \in \mathrm{Ker}(f^n)$ and $y \in \mathrm{Ker}(e^{n+1})$;
b) If $x \in \mathrm{Ker}(f^n)$, then there exists $y \in \mathrm{Ker}(e^{n+1})$ such that $x \looparrowright y$;
c) If $x \in \mathrm{Im}(f^{n-1})$, then $x \looparrowright 0$;
d) If $0 \looparrowright y$, then $y \in \mathrm{Im}(e^n)$;
e) If $x \looparrowright y$ and $\bar{x} \looparrowright \bar{y}$, then $x + \bar{x} \looparrowright y + \bar{y}$;
f) If $x \looparrowright y$ and $a \in R$, then $ax \looparrowright ay$;
g) If $x \looparrowright y$ and $x \looparrowright \bar{y}$, then $y - \bar{y} \in \mathrm{Im}(e^n)$;
h) If $x \looparrowright y$ and $\bar{x} \looparrowright \bar{y}$ with $x - \bar{x} \in \mathrm{Im}(f^{n-1})$, then $y - \bar{y} \in \mathrm{Im}(e^n)$.

Use statements a), b) and h) to show that there is a map

$$\delta^n_{\mathbb{S}^\bullet} : H^n(P^\bullet) \to H^{n+1}(N^\bullet), \quad \left(\begin{array}{r} x + \mathrm{Im}(f^{n-1}) \mapsto y + \mathrm{Im}(e^n) \\ \text{if } x \looparrowright y \end{array} \right).$$

Use statements e) and f) to show that $\delta^n_{\mathbb{S}^\bullet}$ is a homomorphism of R-modules, the so called *n-th connecting homomorphism associated to* $\mathbb{S}^\bullet$.

C) Keep the hypotheses and notations of part B). Show the following facts:

a) If $x' \in \mathrm{Ker}(d^n)$ and $l^n(x') \in \mathrm{Im}(f^{n-1})$, then $x' \in \mathrm{Im}(d^{n-1}) + \mathrm{Im}(h^n)$;
b) If $x' \in \mathrm{Ker}(d^n)$, then $l^n(x') \looparrowright 0$;
c) If $x \looparrowright 0$, then $x \in l^n(\mathrm{Ker}(d^n))$;
d) If $x \looparrowright y$ and $y \in \mathrm{Im}(e^n)$, then $x \looparrowright 0$;
e) If $x \looparrowright y$, then $h^{n+1}(y) \in \mathrm{Im}(d^n)$;
f) If $y \in N^{n+1}$ and $h^{n+1}(y) \in \mathrm{Im}(d^n)$, then there exists $x \in \mathrm{Ker}(f^n)$ with $x \looparrowright y$.

Use statement a) to show that the sequence

$$H^n(N^\bullet) \xrightarrow{H^n(h^\bullet)} H^n(M^\bullet) \xrightarrow{H^n(l^\bullet)} H^n(P^\bullet)$$

is exact. Use statements b), d) and c) to show that

$$H^n(M^\bullet) \xrightarrow{H^n(l^\bullet)} H^n(P^\bullet) \xrightarrow{\delta^n_{\mathbb{S}^\bullet}} H^{n+1}(N^\bullet)$$

is exact. Use statements e) and f) to show that the sequence

$$H^n(P^\bullet) \xrightarrow{\delta^n_{\mathbb{S}^\bullet}} H^{n+1}(N^\bullet) \xrightarrow{H^{n+1}(h^\bullet)} H^{n+1}(M^\bullet)$$

is exact. Conclude that there is an exact sequence of R-modules

g)
$$\begin{array}{l} \qquad\qquad \cdots \longrightarrow H^{n-1}(M^\bullet) \xrightarrow{H^{n-1}(l^\bullet)} H^{n-1}(P^\bullet) \\ \xrightarrow{\delta^{n-1}_{\mathbb{S}^\bullet}} H^n(N^\bullet) \xrightarrow{H^n(h^\bullet)} H^n(M^\bullet) \xrightarrow{H^n(l^\bullet)} H^n(P^\bullet) \\ \xrightarrow{\delta^n_{\mathbb{S}^\bullet}} H^{n+1}(N^\bullet) \longrightarrow \cdots \end{array}$$

the so called *(long exact) cohomology sequence associated to* $\mathbb{S}^\bullet$.

D) Now consider a second short exact sequence of cocomplexes of R-modules

$$\mathbb{T}^\bullet : 0 \to (\bar{N}^\bullet, \bar{e}^\bullet) \xrightarrow{\bar{h}^\bullet} (\bar{M}^\bullet, \bar{d}^\bullet) \xrightarrow{\bar{l}^\bullet} (\bar{P}^\bullet, \bar{f}^\bullet) \to 0$$

together with three homomorphisms of cocomplexes $u^\bullet, v^\bullet, w^\bullet$ such that we have a commutative diagram of cocomplexes of R-modules

$$\begin{tikzcd}
0 \arrow[r] & (N^\bullet, e^\bullet) \arrow[r, "h^\bullet"] \arrow[d, "u^\bullet"] & (M^\bullet, d^\bullet) \arrow[r, "l^\bullet"] \arrow[d, "v^\bullet"] & (P^\bullet, f^\bullet) \arrow[r] \arrow[d, "w^\bullet"] & 0 \\
0 \arrow[r] & (\bar{N}^\bullet, \bar{e}^\bullet) \arrow[r, "\bar{h}^\bullet"] & (\bar{M}^\bullet, \bar{d}^\bullet) \arrow[r, "\bar{l}^\bullet"] & (\bar{P}^\bullet, \bar{f}^\bullet) \arrow[r] & 0.
\end{tikzcd}$$

So, for each $n \in \mathbb{Z}$ we get the following diagram with exact rows and with commutative squares

$$\begin{tikzcd}
& 0 \arrow[r] & N^n \arrow[rr, "h^n"] \arrow[dl, "e^n"'] \arrow[dd, "u^n", near start] & & M^n \arrow[rr, "l^n"] \arrow[dl, "d^n"'] \arrow[dd, "v^n", near start] & & P^n \arrow[r] \arrow[dl, "f^n"'] \arrow[dd, "w^n"] & 0 \\
0 \arrow[r] & N^{n+1} \arrow[rr, "h^{n+1}", near start] \arrow[dd, "u^{n+1}"'] & & M^{n+1} \arrow[rr, "l^{n+1}", near start] \arrow[dd, "v^{n+1}"', near start] & & P^{n+1} \arrow[rr] \arrow[dd, "w^{n+1}"', near start] & & 0 \\
0 \arrow[rr] & & \bar{N}^n \arrow[rr, "\bar{h}^n"'] \arrow[dl, "\bar{e}^n"'] & & \bar{M}^n \arrow[rr, "\bar{l}^n"'] \arrow[dl, "\bar{d}^n"'] & & \bar{P}^n \arrow[r] \arrow[dl, "\bar{f}^n"'] & 0 \\
0 \arrow[r] & \bar{N}^{n+1} \arrow[rr, "\bar{h}^{n+1}"'] & & \bar{M}^{n+1} \arrow[rr, "\bar{l}^{n+1}"'] & & \bar{P}^{n+1} \arrow[r] & 0.
\end{tikzcd}$$

Show that for $x \in P^n$ and $y \in N^{n+1}$ with $x \looparrowright y$, we have $w^n(x) \looparrowright u^{n+1}(y)$.

Conclude that we have the commutative diagram

$$\begin{tikzcd}
H^n(P^\bullet) \arrow[r, "\delta^n_{\mathbb{S}^\bullet}"] \arrow[d, "H^n(w^\bullet)"'] & H^{n+1}(N^\bullet) \arrow[d, "H^{n+1}(u^\bullet)"] \\
H^n(\bar{P}^\bullet) \arrow[r, "\delta^n_{\mathbb{T}^\bullet}"] & H^{n+1}(\bar{N}^\bullet)
\end{tikzcd}$$

and hence the commutative diagram

$$\begin{tikzcd}
H^{n-1}(P^\bullet) \arrow[r, "\delta^{n-1}_{\mathbb{S}^\bullet}"] \arrow[d, "H^{n-1}(w^\bullet)"] & H^n(N^\bullet) \arrow[r, "H^n(h^\bullet)"] \arrow[d, "H^n(u^\bullet)"] & H^n(M^\bullet) \arrow[r, "H^n(l^\bullet)"] \arrow[d, "H^n(v^\bullet)"] & H^n(P^\bullet) \arrow[r, "\delta^n_{\mathbb{S}^\bullet}"] \arrow[d, "H^n(w^\bullet)"] & H^{n+1}(N^\bullet) \arrow[d, "H^{n+1}(u^\bullet)"] \\
H^{n-1}(\bar{P}^\bullet) \arrow[r, "\delta^{n-1}_{\mathbb{T}^\bullet}"] & H^n(\bar{N}^\bullet) \arrow[r, "H^n(\bar{h}^\bullet)"] & H^n(\bar{M}^\bullet) \arrow[r, "H^n(\bar{l}^\bullet)"] & H^n(\bar{P}^\bullet) \arrow[r, "\delta^n_{\mathbb{T}^\bullet}"] & H^{n+1}(\bar{N}^\bullet).
\end{tikzcd}$$

This diagram expresses the fact that the formation of cohomology sequence (of short exact sequences of cocomplexes of R-modules) is natural.

3.6. **Reminder and Exercise.** A) Consider a short exact sequence of R-modules

$$\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0.$$

An *injective resolution of* $\mathbb{S}$ is given by an injective resolution $((J^\bullet, e^\bullet); b)$ of N, an injective resolution $((I^\bullet, d^\bullet); a)$ of M, an injective resolution $((L^\bullet, f^\bullet); c)$ of P, and moreover by resolutions

$$h^\bullet : (J^\bullet, e^\bullet) \to (I^\bullet, d^\bullet),\ l^\bullet : (I^\bullet, d^\bullet) \to (L^\bullet, f^\bullet)$$

of $h : N \to M$ and of $l : M \to P$ respectively, such that we have a short exact sequence of cocomplexes of R-modules

$$\mathbb{S}^\bullet : 0 \to (J^\bullet, e^\bullet) \xrightarrow{h^\bullet} (I^\bullet, d^\bullet) \xrightarrow{l^\bullet} (L^\bullet, f^\bullet) \to 0.$$

In this situation we say that

$$(\mathbb{S}^\bullet; (b, a, c))$$

is an *injective resolution of* $\mathbb{S}$ *with resolving (short exact) sequence (of cocomplexes)* $\mathbb{S}^\bullet$ and with *coaugmentation* (b, a, c).

B) Let $\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ be a short exact sequence of R-modules. Let J and L be injective R-modules such that we have the following diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "i"] & J \oplus L \arrow[r, "p"] & L \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[u, "\beta"] \arrow[r, "h"] & M \arrow[r, "l"] & P \arrow[u, "\gamma"'] \arrow[r] & 0 \\
 & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

in which i and p are the canonical maps, given by $i(x) := (x, 0)$ and $p(x, y) := y$.

Show that:

a) There is a homomorphism $\sigma : M \to J$ such that $\sigma \circ h = \beta$.

b) There is a homomorphism $\alpha : M \to J \oplus L$ (to be defined by means of σ) such that we have the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "i"] & J \oplus L \arrow[r, "p"] & L \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[u, "\beta"] \arrow[r, "h"] & M \arrow[u, "\alpha"] \arrow[r, "l"] & P \arrow[u, "\gamma"'] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0. \arrow[u] &
\end{tikzcd}$$

c) There is an exact sequence

$$0 \to \mathrm{Coker}(\beta) \xrightarrow{\bar{i}} \mathrm{Coker}(\alpha) \xrightarrow{\bar{p}} \mathrm{Coker}(\gamma) \to 0$$

in which $\bar{i}$ and $\bar{p}$ are the maps canonically induced by i and p respectively.

(Hint for "c)": consider the diagram of statement b) as part of a short exact sequence of cocomplexes $\mathbb{S}^\bullet$ with $\mathbb{S}^i = 0$ for all $i \neq 0, 1$ and then apply the cohomology sequence 3.5 C) g).)

C) Let $\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ be a short exact sequence of R-modules. Let $((J^\bullet, e^\bullet); b)$ be an injective resolution of N and let $((L^\bullet, f^\bullet); c)$ be an injective resolution of P. For each $i \in \mathbb{N}_0$, let

$$J^i \xrightarrow{h^i} J^i \oplus L^i \xrightarrow{l^i} L^i$$

be the canonical maps given by $x \mapsto (x, 0)$ and $(x, y) \mapsto y$ respectively.

Apply what is said in part B) to show that there is a commutative diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & J^0 \arrow[r, "h^0"] & J^0 \oplus L^0 \arrow[r, "l^0"] & L^0 \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[u, "b"] \arrow[r, dash] & M \arrow[u, "a"] \arrow[r, dash] & P \arrow[u, "c"] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and an exact sequence

$$0 \to \mathrm{Coker}(b) \xrightarrow{\bar{h}^0} \mathrm{Coker}(a) \xrightarrow{\bar{l}^0} \mathrm{Coker}(c) \to 0$$

in which $\bar{h}^0$ and $\bar{l}^0$ are canonically induced by h^0 and l^0. Then, apply what is said in part B) to the following diagram with exact rows and columns (!)

$$\begin{tikzcd}
0 \arrow[r] & J^1 \arrow[r, "h^1"] & J^1 \oplus L^1 \arrow[r, "l^1"] & L^1 \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Coker}(b) \arrow[u, "\bar{e}^0"] \arrow[r, "\bar{h}^0"] & \mathrm{Coker}(a) \arrow[r, "\bar{l}^0"] & \mathrm{Coker}(c) \arrow[u, "\bar{f}^0"] \arrow[r] & 0 \\
 & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

in which $\bar{e}^0$ and $\bar{f}^0$ are given by $x + \mathrm{Im}(b) \mapsto e^0(x)$ and $y + \mathrm{Im}(c) \mapsto f^0(y)$ respectively: you get an injective homomorphism

$$\bar{d}^0 : \mathrm{Coker}(a) \to J^1 \oplus L^1$$

which completes the above diagram commutatively and an induced short exact sequence

$$0 \to \mathrm{Coker}(\bar{e}^0) \xrightarrow{\bar{h}^1} \mathrm{Coker}(\bar{d}^0) \xrightarrow{\bar{l}^1} \mathrm{Coker}(\bar{f}^0) \to 0.$$

Go on recursively and conclude that the injective resolutions $((J^\bullet, e^\bullet); b)$ and $((L^\bullet, f^\bullet); c)$ of N resp. P "may be extended to an injective resolution of $\mathbb{S}$":

There is an injective resolution of $\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ which has the shape

$$((\mathbb{S}^\bullet : 0 \to (J^\bullet, e^\bullet) \xrightarrow{h^\bullet} (I^\bullet, d^\bullet) \xrightarrow{l^\bullet} (L^\bullet, f^\bullet) \to 0); (b, a, c)).$$

In particular, each short exact sequence of R-modules has an injective resolution.

3.7. **Reminder and Exercise.** A) Consider the following commutative diagram of R-modules with exact rows

$$\begin{tikzcd} \mathbb{S} : 0 \arrow[r] & N \arrow[r, "h"] \arrow[d, "u"] & M \arrow[r, "l"] \arrow[d, "v"] & P \arrow[r] \arrow[d, "w"] & 0 \\ \overline{\mathbb{S}} : 0 \arrow[r] & \bar{N} \arrow[r, "\bar{h}"] & \bar{M} \arrow[r, "\bar{l}"] & \bar{P} \arrow[r] & 0. \end{tikzcd}$$

Let $\mathbb{I}_* : L \leadsto \mathbb{I}_L = ((I_L^\bullet, d_L^\bullet); a_L)$ be a choice of injective resolutions of R-modules. Let $u^\bullet$ be a resolution of u between $\mathbb{I}_N$ and $\mathbb{I}_{\bar{N}}$, and let $w^\bullet$ be a resolution of w between $\mathbb{I}_P$ and $\mathbb{I}_{\bar{P}}$. Let $((J^\bullet, e^\bullet), b)$ and $((\bar{J}^\bullet, \bar{e}^\bullet), \bar{b})$ be injective resolutions of M and $\bar{M}$ respectively and let

$$((\mathbb{S}^\bullet : 0 \to (I_N^\bullet, d_N^\bullet) \xrightarrow{h^\bullet} (J^\bullet, e^\bullet) \xrightarrow{l^\bullet} (I_P^\bullet, d_P^\bullet) \to 0); (a_N, b, a_P))$$

and

$$((\overline{\mathbb{S}}^\bullet : 0 \to (I_{\bar{N}}^\bullet, d_{\bar{N}}^\bullet) \xrightarrow{\bar{h}^\bullet} (\bar{J}^\bullet, \bar{e}^\bullet) \xrightarrow{\bar{l}^\bullet} (I_{\bar{P}}^\bullet, d_{\bar{P}}^\bullet) \to 0); (a_{\bar{N}}, \bar{b}, a_{\bar{P}}))$$

be injective resolutions of $\mathbb{S}$ and $\overline{\mathbb{S}}$ respectively (cf. 3.6 C)).

Show the following:

a) For each $n \in \mathbb{N}_0$ there are homomorphisms of R-modules $r^n : J^n \to I_N^n$ and $\bar{s}^n : I_{\bar{P}}^n \to \bar{J}^n$ such that $r^n \circ h^n = \mathrm{id}_{I_N^n}$ and $\bar{l}^n \circ \bar{s}^n = \mathrm{id}_{I_{\bar{P}}^n}$.

For $n \in \mathbb{N}_0$ set $\tilde{v}^n := \bar{h}^n \circ u^n \circ r^n + \bar{s}^n \circ w^n \circ l^n : J^n \to \bar{J}^n$ and show:

b) $\tilde{v}^n \circ h^n = \bar{h}^n \circ u^n$ and $\bar{l}^n \circ \tilde{v}^n = w^n \circ l^n$ for each $n \in \mathbb{N}_0$.

B) Show the following statements:

a) $\mathrm{Im}(\tilde{v}^0 \circ b - \bar{b} \circ v) \subseteq \mathrm{Im}(\bar{h}^0)$;
b) There is a uniquely determined homomorphism of R-modules $\alpha : M \to I_{\bar{N}}^0$ such that $\bar{h}^0 \circ \alpha = \tilde{v}^0 \circ b - \bar{b} \circ v$;
c) $b^{-1}(\mathrm{Im}(h^0)) \subseteq \mathrm{Ker}(\alpha)$;
d) There are homomorphisms of R-modules
$$\bar{\alpha} : M/b^{-1}(\mathrm{Im}(h^0)) \to I_{\bar{N}}^0,\ m + b^{-1}(\mathrm{Im}(h^0)) \mapsto \alpha(m),$$
$$\tilde{b} : M/b^{-1}(\mathrm{Im}(h^0)) \to J^0/\,\mathrm{Im}(h^0),\ m + b^{-1}(\mathrm{Im}(h^0)) \mapsto b(m) + \mathrm{Im}(h^0)$$
and $\bar{\beta} : J^0/\,\mathrm{Im}(h^0) \to I_{\bar{N}}^0$ such that $\bar{\beta} \circ \tilde{b} = \bar{\alpha}$;
e) There is a homomorphism of R-modules $\beta : J^0 \to I_{\bar{N}}^0,\ t \mapsto \tilde{\beta}(t + \mathrm{Im}(h^0))$ such that $\beta \circ b = \alpha$ and $\beta \circ h^0 = 0$.

Set $v^0 := \tilde{v}^0 - \bar{h}^0 \circ \beta : J^0 \to \bar{J}^0$ and show:

f) $v^0 \circ b = \bar{b} \circ v$, $v^0 \circ h^0 = \bar{h}^0 \circ u^0$ and $\bar{l}^0 \circ v^0 = w^0 \circ l^0$.

C) Let $n \in \mathbb{N}$. For $k \in \mathbb{N}$ with $k < n$, let $v^k : J^k \to \bar{J}^k$ be a homomorphism of R-modules such that $v^k \circ h^k = \bar{h}^k \circ u^k$, $w^n \circ l^k = \bar{l}^k \circ v^k$, and $v^k \circ e^k = \bar{e}^k v^{k-1}$. Show that:

a) $\operatorname{Im}(\tilde{v}^n \circ e^{n-1} - \bar{e}^{n-1} \circ v^{n-1}) \subseteq \operatorname{Im}(\bar{h}^n)$;
b) There is a homomorphism of R-modules $\alpha^n : J^{n-1} \to I^n_{\bar{N}}$ such that $\bar{h}^n \circ \alpha^n = \tilde{v}^n \circ e^{n-1} - \bar{e}^{n-1} \circ v^{n-1}$;
c) $(e^{n-1})^{-1}(\operatorname{Im}(h^n)) \subseteq \operatorname{Im}(h^{n-1}) + \operatorname{Ker}(e^{n-1})$;
d) $\alpha^n \circ h^{n-1} = 0$;
e) $\alpha^n((e^{n-1})^{-1}(\operatorname{Im}(h^n))) \subseteq \alpha(\operatorname{Ker}(e^{n-1}))$;
f) $\alpha^n(\operatorname{Ker}(e^{n-1})) = 0$;
g) $(e^{n-1})^{-1}(\operatorname{Im}(h^n)) \subseteq \operatorname{Ker}(\alpha^n)$;
h) There are a homomorphism of R-modules

$$\bar{\alpha}^n : J^{n-1}/(e^{n-1})^{-1}(\operatorname{Im}(h^n)) \to I^n_{\bar{N}},\ x + (e^{n-1})^{-1}(\operatorname{Im}(h^n)) \mapsto \alpha^n(x),$$

an injective homomorphism of R-modules

$$\tilde{e}^{n-1} : J^{n-1}/(e^{n-1})^{-1}(\operatorname{Im}(h^n)) \to J^n/\operatorname{Im}(h^n),$$

$$x + (e^{n-1})^{-1}(\operatorname{Im}(h^n)) \mapsto e^{n-1}(x) + \operatorname{Im}(h^n)$$

and a homomorphism of R-modules

$$\bar{\beta}^n : J^n/\operatorname{Im}(h^n) \to I^n_{\bar{N}}$$

such that $\bar{\beta}^n \circ \tilde{e}^{n-1} = \bar{\alpha}^n$.

Set $\beta^n : J^n \to I^n_{\bar{N}},\ t \mapsto \bar{\beta}^n(t + \operatorname{Im}(h^n))$ and $v^n := \tilde{v}^n - \bar{h}^n \circ \beta^n : J^n \to \bar{J}^n$, and show:

i) $\beta^n \circ e^{n-1} = \alpha^n$ and $\beta^n \circ h^n = 0$;
j) $v^n \circ e^{n-1} = \bar{e}^{n-1} \circ v^{n-1}$, $v^n \circ h^n = \bar{h}^n \circ u^n$ and $\bar{l}^n \circ v^n = w^n \circ l^n$.

D) Conclude by recursion and on use of B) e) and C) j) that there is a homomorphism $v^\bullet : (J^\bullet, e^\bullet) \to (\bar{J}^\bullet, \bar{e}^\bullet)$ of cocomplexes which is a resolution of v between $((J^\bullet, e^\bullet), b)$ and $((\bar{J}^\bullet, \bar{e}^\bullet), \bar{b})$ such that the following diagram of cocomplexes commutes:

$$\begin{array}{ccccccccc}
\mathbb{S}^\bullet : 0^\bullet & \longrightarrow & (I^\bullet_N, d^\bullet_N) & \xrightarrow{h^\bullet} & (J^\bullet, e^\bullet) & \xrightarrow{l^\bullet} & (I^\bullet_P, d^\bullet_P) & \longrightarrow & 0^\bullet \\
 & & \downarrow{\scriptstyle u^\bullet} & & \downarrow{\scriptstyle v^\bullet} & & \downarrow{\scriptstyle w^\bullet} & & \\
\overline{\mathbb{S}}^\bullet : 0^\bullet & \longrightarrow & (I^\bullet_{\bar{N}}, d^\bullet_{\bar{N}}) & \xrightarrow{\bar{h}^\bullet} & (\bar{J}^\bullet, \bar{e}^\bullet) & \xrightarrow{\bar{l}^\bullet} & (I^\bullet_{\bar{P}}, d^\bullet_{\bar{P}}) & \longrightarrow & 0^\bullet.
\end{array}$$

3.8. **Construction and Exercise.** A) Let R' be a second ring, and let F be an additive functor from R-modules to R'-modules. Let

$$\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$$

be an exact sequence of R-modules. Let $(\mathbb{S}^\bullet; (b, a, c))$ be an injective resolution of $\mathbb{S}$ (cf. 3.6) with resolving sequence

$$\mathbb{S}^\bullet : 0 \to (J^\bullet, e^\bullet) \xrightarrow{h^\bullet} (I^\bullet, d^\bullet) \xrightarrow{l^\bullet} (L^\bullet, f^\bullet) \to 0.$$

Show that for each $n \in \mathbb{N}_0$ we get the exact sequence (cf. 2.7 B) b) and 1.21 B))

$$0 \to F(J^n) \xrightarrow{F(h^n)} F(I^n) \xrightarrow{F(l^n)} F(L^n) \to 0$$

and thus obtain a short exact sequence of cocomplexes

$$F(\mathbb{S}^\bullet) : 0 \to (F(J^\bullet), F(e^\bullet)) \xrightarrow{F(h^\bullet)} (F(I^\bullet), F(d^\bullet)) \xrightarrow{F(l^\bullet)} (F(L^\bullet), F(f^\bullet)) \to 0.$$

Now, we may form the cohomology sequence associated to $F(\mathbb{S}^\bullet)$ (cf. 3.5) and end up with an exact sequence

$$\begin{aligned} 0 \to & H^0(F(J^\bullet)) \xrightarrow{H^0(F(h^\bullet))} H^0(F(I^\bullet)) \xrightarrow{H^0(F(l^\bullet))} H^0(F(L^\bullet)) \xrightarrow{\delta^0_{F(\mathbb{S}^\bullet)}} \\ & H^1(F(J^\bullet)) \xrightarrow{H^1(F(h^\bullet))} H^1(F(I^\bullet)) \xrightarrow{\qquad} \cdots \end{aligned}$$

Next, let $(\overline{\mathbb{S}}^\bullet, (\bar{b}, \bar{a}, \bar{c}))$ be a second injective resolution of $\mathbb{S}$ with resolving sequence

$$\overline{\mathbb{S}}^\bullet : 0 \to (\bar{J}^\bullet, \bar{e}^\bullet) \xrightarrow{\bar{h}^\bullet} (\bar{I}^\bullet, \bar{d}^\bullet) \xrightarrow{\bar{l}^\bullet} (\bar{L}^\bullet, \bar{f}^\bullet) \to 0.$$

Again we get a short exact sequence of cocomplexes

$$F(\overline{\mathbb{S}}^\bullet) : 0 \to (F(\bar{J}^\bullet), F(\bar{e}^\bullet)) \xrightarrow{F(\bar{h}^\bullet)} (F(\bar{I}^\bullet), F(\bar{d}^\bullet)) \xrightarrow{F(\bar{l}^\bullet)} (F(\bar{L}^\bullet), F(\bar{f}^\bullet)) \to 0$$

and thus may form the cohomology sequence associated to $F(\overline{\mathbb{S}}^\bullet)$. Finally let $u^\bullet : (J^\bullet, e^\bullet) \to (\bar{J}^\bullet, \bar{e}^\bullet)$ and $w^\bullet : (L^\bullet, f^\bullet) \to (\bar{L}^\bullet, f^\bullet)$ be resolutions of id_N and id_P respectively. Show on use of 3.7 D), that for each $n \in \mathbb{N}_0$ we get a commutative diagram

$$\begin{tikzcd} H^n(F(L^\bullet)) \arrow[r, "\delta^n_{F(\mathbb{S}^\bullet)}"] \arrow[d, "H^n(F(w^\bullet))"'] & H^{n+1}(F(J^\bullet)) \arrow[d, "H^{n+1}(F(u^\bullet))"] \\ H^n(F(\bar{L}^\bullet)) \arrow[r, "\delta^n_{F(\overline{\mathbb{S}}^\bullet)}"] & H^{n+1}(F(\bar{J}^\bullet)). \end{tikzcd}$$

B) Keep the notations and hypotheses of part A). To each short exact sequence of R-modules $\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ assign an injective resolution of $\mathbb{S}$, say $(\mathbb{S}^\bullet; (b_{\mathbb{S}}, a_{\mathbb{S}}, c_{\mathbb{S}}))$ with resolving sequence

$$\mathbb{S}^\bullet : 0 \to (J^\bullet_{\mathbb{S}}, e^\bullet_{\mathbb{S}}) \xrightarrow{h^\bullet_{\mathbb{S}}} (I^\bullet_{\mathbb{S}}, d^\bullet_{\mathbb{S}}) \xrightarrow{l^\bullet_{\mathbb{S}}} (L^\bullet_{\mathbb{S}}, f^\bullet_{\mathbb{S}}) \to 0$$

(cf. 3.6 C)). Fix such a *choice of injective resolutions of short exact sequences of R-modules.* Then, to each short exact sequence of R-modules

$$\$: 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$$

assign the family of connecting homomorphisms (cf. 3.5 B))

$$(\delta^n_{F(\$^\bullet)} : H^n(F(L^\bullet_\$)) \to H^{n+1}(F(J^\bullet_\$)))_{n\in\mathbb{N}_0}$$

associated to the short exact sequence of cocomplexes (cf. part A))

$$F(\$^\bullet) : 0 \to (F(J^\bullet_\$), F(e^\bullet_\$)) \xrightarrow{F(h^\bullet_\$)} (F(I^\bullet_\$), F(d^\bullet_\$)) \xrightarrow{F(l^\bullet_\$)} (F(L^\bullet_\$), F(f^\bullet_\$)) \to 0.$$

Use what has been said in part A) and in 2.13 D) to conclude that (setting $\delta^{n,F}_\$:= \delta^n_{F(\$^\bullet)}$) the assignment

$$\$ \leadsto (\delta^n_{F(\$^\bullet)} : H^n(F(L^\bullet_\$)) \to H^{n+1}(F(J^\bullet_\$)))_{n\in\mathbb{N}_0}$$

gives rise to an assignment

$$\delta^{\bullet,F}_* : (\$: 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0) \leadsto (\delta^{n,F}_\$: \mathcal{R}^n F(P) \to \mathcal{R}^{n+1}F(N))_{n\in\mathbb{N}_0},$$

which to each short exact sequence of R-modules

$$\$: 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$$

assigns a family $(\delta^{n,F}_\$: \mathcal{R}^n F(P) \to \mathcal{R}^{n+1}F(N))_{n\in\mathbb{N}_0}$ of homomorphisms of R-modules. Show that this gives rise to an exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{R}^0F(N) & \xrightarrow{\mathcal{R}^0F(h)} & \mathcal{R}^0F(M) & \xrightarrow{\mathcal{R}^0F(l)} & \mathcal{R}^0F(P) \\
 & \xrightarrow{\delta^{0,F}_\$} & \mathcal{R}^1F(N) & \xrightarrow{\mathcal{R}^1F(h)} & \mathcal{R}^1F(M) & \longrightarrow & \cdots \\
 & & & & \cdots & \longrightarrow & \mathcal{R}^{n-1}F(P) \\
 & \xrightarrow{\delta^{n-1,F}_\$} & \mathcal{R}^nF(N) & \xrightarrow{\mathcal{R}^nF(h)} & \mathcal{R}^nF(M) & \xrightarrow{\mathcal{R}^nF(l)} & \mathcal{R}^nF(P) \\
 & \xrightarrow{\delta^{n,F}_\$} & \mathcal{R}^{n+1}F(N) & \xrightarrow{\mathcal{R}^{n+1}F(h)} & \mathcal{R}^{n+1}F(M) & \longrightarrow & \cdots
\end{array}$$

for each exact sequence of R-modules $\$: 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$.

The homomorphism $\delta^{n,F}_\$: \mathcal{R}^nF(P) \to \mathcal{R}^{n+1}F(N)$ is called the *n-th connecting homomorphism with respect to F associated to* $\$$ and the sequence a) is called the *right derived sequence of F associated to* $\$$.

C) Show that—according to its construction—the assignment $\delta^{\bullet,F}_*$ of part B) has the following naturality property: For each commutative diagram of R-modules

$$\begin{tikzcd}
\$: 0 \arrow[r] & N \arrow[r, "h"] \arrow[d, "u"] & M \arrow[r, "l"] \arrow[d, "v"] & P \arrow[r] \arrow[d, "w"] & 0 \\
\$' : 0 \arrow[r] & N' \arrow[r, "h'"] & M' \arrow[r, "l'"] & P' \arrow[r] & 0
\end{tikzcd}$$

with exact rows $\mathbb{S}$ and $\mathbb{S}'$ and for all $n \in \mathbb{N}_0$ we have the commutative diagram

$$\begin{array}{ccc} \mathcal{R}^n F(P) & \xrightarrow{\delta_{\mathbb{S}}^{n,F}} & \mathcal{R}^{n+1} F(N) \\ {\scriptstyle \mathcal{R}^n F(w)}\downarrow & & \downarrow{\scriptstyle \mathcal{R}^{n+1} F(u)} \\ \mathcal{R}^n F(P') & \xrightarrow{\delta_{\mathbb{S}'}^{n,F}} & \mathcal{R}^{n+1} F(N'). \end{array}$$

3.9. **Definition and Remark.** A) Let $\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ be a short exact sequence of R-modules. Then we write $\delta_{\mathbb{S}}^{n,\mathfrak{a}}$ for the n-th connecting homomorphism with respect to the torsion functor $\Gamma_{\mathfrak{a}}$ associated to $\mathbb{S}$ (cf. 3.8):

$$\begin{array}{ccc} \mathcal{R}^n \Gamma_{\mathfrak{a}}(P) & \xrightarrow{\delta_{\mathbb{S}}^{n,\Gamma_{\mathfrak{a}}}} & \mathcal{R}^{n+1} \Gamma_{\mathfrak{a}}(N) \\ \Vert & & \Vert \\ H_{\mathfrak{a}}^n(P) & \xrightarrow{\delta_{\mathbb{S}}^{n,\mathfrak{a}}} & H_{\mathfrak{a}}^{n+1}(N). \end{array}$$

Sometimes we just write $\delta_{\mathbb{S}}^n$ or δ^n or δ for $\delta_{\mathbb{S}}^{n,\mathfrak{a}}$.

B) The right derived sequence of $\Gamma_{\mathfrak{a}}$ associated to $\mathbb{S}$ (cf. 3.8 B)) now takes the form

$$\begin{array}{lllll} 0 \longrightarrow & H_{\mathfrak{a}}^0(N) & \xrightarrow{H_{\mathfrak{a}}^0(h)} & H_{\mathfrak{a}}^0(M) & \xrightarrow{H_{\mathfrak{a}}^0(l)} H_{\mathfrak{a}}^0(P) \\ \xrightarrow{\delta_{\mathbb{S}}^{0,\mathfrak{a}}} & H_{\mathfrak{a}}^1(N) & \xrightarrow{H_{\mathfrak{a}}^1(h)} & H_{\mathfrak{a}}^1(M) & \longrightarrow \cdots \\ & & & \cdots & \longrightarrow H_{\mathfrak{a}}^{n-1}(P) \\ \xrightarrow{\delta_{\mathbb{S}}^{n-1,\mathfrak{a}}} & H_{\mathfrak{a}}^n(N) & \xrightarrow{H_{\mathfrak{a}}^n(h)} & H_{\mathfrak{a}}^n(M) & \xrightarrow{H_{\mathfrak{a}}^n(l)} H_{\mathfrak{a}}^n(P) \\ \xrightarrow{\delta_{\mathbb{S}}^{n,\mathfrak{a}}} & H_{\mathfrak{a}}^{n+1}(N) & \xrightarrow{H_{\mathfrak{a}}^{n+1}(h)} & H_{\mathfrak{a}}^{n+1}(M) & \longrightarrow \cdots . \end{array}$$

Usually this sequence is called the *(long exact) cohomology sequence with respect to* $\mathfrak{a}$ *and associated to* $\mathbb{S}$.

C) The naturality statement 3.8 C) here can be formulated as follows: Given a commutative diagram of R-modules

$$\begin{array}{ccccccccc} \mathbb{S} : 0 & \longrightarrow & N & \xrightarrow{h} & M & \xrightarrow{l} & P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ \mathbb{S}' : 0 & \longrightarrow & N' & \xrightarrow{h'} & M' & \xrightarrow{l'} & P' & \longrightarrow & 0 \end{array}$$

with exact rows $\mathbb{S}$ and $\mathbb{S}'$ we get a commutative diagram

$$\begin{array}{ccccccccc} \cdots H_{\mathfrak{a}}^n(N) & \xrightarrow{H_{\mathfrak{a}}^n(h)} & H_{\mathfrak{a}}^n(M) & \xrightarrow{H_{\mathfrak{a}}^n(l)} & H_{\mathfrak{a}}^n(P) & \xrightarrow{\delta_{\mathbb{S}}^{n,\mathfrak{a}}} & H_{\mathfrak{a}}^{n+1}(N) & \xrightarrow{H_{\mathfrak{a}}^{n+1}(h)} & H_{\mathfrak{a}}^{n+1}(M) \cdots \\ \downarrow{\scriptstyle H_{\mathfrak{a}}^n(u)} & & \downarrow{\scriptstyle H_{\mathfrak{a}}^n(v)} & & \downarrow{\scriptstyle H_{\mathfrak{a}}^n(w)} & & \downarrow{\scriptstyle H_{\mathfrak{a}}^{n+1}(u)} & & \downarrow{\scriptstyle H_{\mathfrak{a}}^{n+1}(v)} \\ \cdots H_{\mathfrak{a}}^n(N') & \xrightarrow{H_{\mathfrak{a}}^n(h')} & H_{\mathfrak{a}}^n(M') & \xrightarrow{H_{\mathfrak{a}}^n(l')} & H_{\mathfrak{a}}^n(P') & \xrightarrow{\delta_{\mathbb{S}'}^{n,\mathfrak{a}}} & H_{\mathfrak{a}}^{n+1}(N') & \xrightarrow{H_{\mathfrak{a}}^{n+1}(h')} & H_{\mathfrak{a}}^{n+1}(M') \cdots . \end{array}$$

3.10. **Remark and Exercise.** A) Let M be an R-module. Let $x \in R$. Then, the multiplication map

$$M \xrightarrow{x\cdot} M, \; m \mapsto xm$$

clearly is a homomorphism of R-modules. We also can write $x\cdot = x\,\mathrm{id}_M$. By 1.12 C) we thus see: If F is a linear functor of R-modules, then the homomorphism $F(x\cdot) : F(M) \to F(M)$ and the multiplication map $x\cdot : F(M) \to F(M)$ are the same.

B) Let M be an R-module. Observe that for $x \in R$ we have:

a) $x \in \mathrm{NZD}_R(M)$ if and only if $M \xrightarrow{x\cdot} M$ is injective.

So in particular :

b) If $x \in \mathrm{NZD}_R(M)$, then there is a short exact sequence of R-modules

$$0 \to M \xrightarrow{x\cdot} M \xrightarrow{p} M/xM \to 0.$$

C) Now, let M be an R-module and let $x \in \mathrm{NZD}_R(M)$. Then, the cohomology sequence with respect to $\mathfrak{a}$ and associated to the short exact sequence of B) b) takes the form (cf. A) a)):

$$\begin{cases} 0 \longrightarrow H^0_{\mathfrak{a}}(M) \xrightarrow{x\cdot} H^0_{\mathfrak{a}}(M) \xrightarrow{H^0_{\mathfrak{a}}(p)} H^0_{\mathfrak{a}}(M/xM) \xrightarrow{\delta^0} H^1_{\mathfrak{a}}(M) \longrightarrow \cdots \\ \\ \cdots \xrightarrow{\delta^{n-1}} H^n_{\mathfrak{a}}(M) \xrightarrow{x\cdot} H^n_{\mathfrak{a}}(M) \xrightarrow{H^n_{\mathfrak{a}}(p)} H^n_{\mathfrak{a}}(M/xM) \xrightarrow{\delta^n} H^{n+1}_{\mathfrak{a}}(M) \longrightarrow \cdots . \end{cases}$$

3.11. **Definition.** An R-module M is said to be $\mathfrak{a}$-*torsion,* if $M = \Gamma_{\mathfrak{a}}(M)$ or—equivalently—if for each element $m \in M$ there is some $n \in \mathbb{N}$ such that $\mathfrak{a}^n m = 0$.

3.12. **Remark and Exercise.** A) Observe that

a) $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) = \Gamma_{\mathfrak{a}}(M)$ for each R-module M.

As a consequence one sees:

b) If M is an R-module, then $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion (in the sense of 3.11).

B) Observe the following fact:

a) If M is $\mathfrak{a}$-torsion and $N \subseteq M$ is a submodule, then N and M/N are $\mathfrak{a}$-torsion.

More general:

b) If $0 \to N \to M \to P \to 0$ is a short exact sequence of R-modules and if $\mathfrak{a}$ is finitely generated or if N is Noetherian, then M is $\mathfrak{a}$-torsion if and only if N and P are $\mathfrak{a}$-torsion.

C) Moreover we can say:

a) A finitely generated R-module M is $\mathfrak{a}$-torsion if and only if there is some $n \in \mathbb{N}$ with $\mathfrak{a}^n M = 0$.
b) Let $x \in \mathfrak{a}$ and let M be an $\mathfrak{a}$-torsion R-module. Then $M \xrightarrow{x\cdot} M$ is injective if and only if $M = 0$.

3.13. **Proposition.** *Let $n \in \mathbb{N}_0$ and let M be an R-module. Then the local cohomology module $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion.*

Proof. This follows easily by 2.14 B) and on use of 3.12 B) a). $\square$

3.14. **Proposition.** *Let R be Noetherian and let I be an injective R-module. Then $\Gamma_{\mathfrak{a}}(I)$ is an injective R-module, too.*

Proof. Let $\mathfrak{b} \subseteq R$ be an ideal and let $h : \mathfrak{b} \to \Gamma_{\mathfrak{a}}(I)$ be a homomorphism of R-modules. By 2.8, it suffices to find some element $e \in \Gamma_{\mathfrak{a}}(I)$ such that $h(b) = be$ for all $b \in \mathfrak{b}$.

Since I is injective, there is a homomorphism of R-modules $l : R \to I$ such that $h = l \circ i$, where $i : \mathfrak{b} \rightarrowtail R$ is the inclusion map. Let $f := l(1)$. Then $h(b) = (l \circ i)(b) = l(i(b)) = l(b) = l(b \cdot 1) = bl(1) = bf$, hence $h(b) = bf$ for all $b \in \mathfrak{b}$. In particular, we have $h(\mathfrak{b}) = \mathfrak{b}f \subseteq Rf$. As $h(\mathfrak{b})$ is a submodule of $\Gamma_{\mathfrak{a}}(I)$, it is $\mathfrak{a}$-torsion (cf. 3.12 B) a)). Therefore obviously $h(\mathfrak{b}) \subseteq \Gamma_{\mathfrak{a}}(Rf)$. As R is Noetherian and Rf is a finitely generated R-module, there is some $n \in \mathbb{N}$ such that $\mathfrak{a}^n(Rf) \cap \Gamma_{\mathfrak{a}}(Rf) = 0$ (cf. 1.4 B) d)). It follows that $\mathfrak{a}^n(Rf) \cap h(\mathfrak{b}) = 0$, hence $\mathfrak{a}^n f \cap \mathfrak{b}f = 0$. This means that the sum $\mathfrak{a}^n f + \mathfrak{b}f$ is direct so that there is a homomorphism of R-modules $\varphi : \mathfrak{a}^n f + \mathfrak{b}f \to \mathfrak{b}f$ given by $uf + bf \mapsto bf$ for all $u \in \mathfrak{a}^n$ and all $b \in \mathfrak{b}$. As $(\mathfrak{a}^n + \mathfrak{b})f = \mathfrak{a}^n f + \mathfrak{b}f$ we thus may define a homomorphism of R-modules $\widetilde{h} : \mathfrak{a}^n + \mathfrak{b} \to \mathfrak{b}f$ by $v \mapsto \varphi(vf)$ for all $v \in \mathfrak{a}^n + \mathfrak{b}$. So, for all $u \in \mathfrak{a}^n$ and all $b \in \mathfrak{b}$ we get $\widetilde{h}(u + b) = bf$.

As I is injective, there is a homomorphism of R-modules $\widetilde{l} : R \to I$ such that $\widetilde{h} = \widetilde{l} \circ \widetilde{i}$, where $\widetilde{i} : (\mathfrak{a}^n + \mathfrak{b}) \rightarrowtail R$ is the inclusion map. Let $e = \widetilde{l}(1)$, so that for each $x \in R$ we have $\widetilde{l}(x) = \widetilde{l}(x \cdot 1) = x\widetilde{l}(1) = xe$, hence $\widetilde{l}(x) = xe$. For any $u \in \mathfrak{a}^n$ it follows $ue = \widetilde{l}(u) = \widetilde{l}(\widetilde{i}(u)) = (\widetilde{l} \circ \widetilde{i})(u) = \widetilde{h}(u) = \widetilde{h}(u + 0) = 0f = 0$. Therefore $\mathfrak{a}^n e = 0$ and hence $e \in \Gamma_{\mathfrak{a}}(I)$. Moreover, for any $b \in \mathfrak{b}$ we have $h(b) = bf = \widetilde{h}(0 + b) = \widetilde{h}(b) = (\widetilde{l} \circ \widetilde{i})(b) = \widetilde{l}(b) = be$. $\square$

3.15. **Corollary.** *Let R be Noetherian and let M be an $\mathfrak{a}$-torsion R-module. Then there is a monomorphism of R-modules $M \rightarrowtail I$ such that I is injective and $\mathfrak{a}$-torsion.*

Proof. By 2.9 there is some monomorphism $M \overset{i}{\rightarrowtail} J$ such that J is an injective R-module. As the functor $\Gamma_{\mathfrak{a}}$ is left exact, we obtain a monomorphism $\Gamma_{\mathfrak{a}}(M) \xrightarrow{\Gamma_{\mathfrak{a}}(i)} \Gamma_{\mathfrak{a}}(J)$. As M is $\mathfrak{a}$-torsion, we have $\Gamma_{\mathfrak{a}}(M) = M$. By 3.14 we see that $I := \Gamma_{\mathfrak{a}}(J)$ is injective. By 3.12 A) b) the R-module I is $\mathfrak{a}$-torsion. $\square$

3.16. **Corollary.** *Let R be Noetherian and let M be an R-module which is $\mathfrak{a}$-torsion. Then M has an injective resolution $((I^\bullet, d^\bullet); a)$ in which all the R-modules I^i are $\mathfrak{a}$-torsion.*

Proof. We must construct an exact sequence

$$0 \to M \xrightarrow{a} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \to \cdots$$

in which all the R-modules I^i are injective and $\mathfrak{a}$-torsion. By 3.15 we have already an exact sequence $0 \to M \xrightarrow{a} I^0$ in which I^0 is injective and $\mathfrak{a}$-torsion. Now, by 3.12 B) a), $\mathrm{Coker}(a) = I^0/\mathrm{Im}(a)$ is $\mathfrak{a}$-torsion. By 3.15 there is a monomorphism $\mathrm{Coker}(a) \overset{t^0}{\rightarrowtail} I^1$ such that I^1 is an injective $\mathfrak{a}$-torsion R-module. Now, let $d^0 : I^0 \to I^1$ be the homomorphism defined by $u \mapsto t^0(u + \mathrm{Im}(a))$. Then $\mathrm{Ker}(d^0) = \mathrm{Im}(a)$. So we have an exact sequence $0 \to M \xrightarrow{a} I^0 \xrightarrow{d^0} I^1$ in which I^0 and I^1 are injective and $\mathfrak{a}$-torsion. Going on recursively, we get the requested sequence. (This is a "late hint" how to prove 2.11 A)). $\square$

3.17. **Theorem.** *Let R be Noetherian and let M be an R-module which is $\mathfrak{a}$-torsion. Then $H^n_{\mathfrak{a}}(M) = 0$ for all $n > 0$.*

Proof. By 3.16 we know that M has an injective resolution $((I^\bullet, d^\bullet); a)$ in which all the R-modules I^i are $\mathfrak{a}$-torsion and hence satisfy $\Gamma_{\mathfrak{a}}(I^i) = I^i$. But this means, that the complexes $(\Gamma_{\mathfrak{a}}(I^\bullet), \Gamma_{\mathfrak{a}}(d^\bullet))$ and $(I^\bullet, d^\bullet)$ are equal. As the sequence

$$I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots$$

is exact, it follows that the (same) sequence

$$\Gamma_{\mathfrak{a}}(I^0) \xrightarrow{\Gamma_{\mathfrak{a}}(d^0)} \Gamma_{\mathfrak{a}}(I^1) \xrightarrow{\Gamma_{\mathfrak{a}}(d^1)} \Gamma_{\mathfrak{a}}(I^2) \to \cdots$$

is exact, too. So, for all $n > 0$ we have

$$H^n_{\mathfrak{a}}(M) = H^n(\Gamma_{\mathfrak{a}}(I^\bullet), \Gamma_{\mathfrak{a}}(d^\bullet)) = \mathrm{Ker}(\Gamma_{\mathfrak{a}}(d^n))/\,\mathrm{Im}(\Gamma_{\mathfrak{a}}(d^{n-1})) = 0.$$

$\square$

3.18. **Corollary.** *Let R be Noetherian. Let M be an R-module and let $N \subseteq M$ be a submodule such that N is $\mathfrak{a}$-torsion. Let $M \xrightarrow{p} M/N$ be the map defined by $m \mapsto m + N$. Then:*

a) The induced homomorphism $H^0_{\mathfrak{a}}(p) : H^0_{\mathfrak{a}}(M) \to H^0_{\mathfrak{a}}(M/N)$ is surjective.

b) For each $n > 0$, the induced homomorphism $H^n_{\mathfrak{a}}(p) : H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M/N)$ is an isomorphism.

Proof. Consider the short exact sequence $\mathbb{S} : 0 \to N \xrightarrow{i} M \xrightarrow{p} M/N \to 0$, in which i denotes the inclusion homomorphism. Then form the cohomology sequence with respect to $\mathfrak{a}$ and associated to $\mathbb{S}$ (cf. 3.9 B)) and observe that $H^n_{\mathfrak{a}}(N) = 0$ for all $n > 0$ (cf. 3.17). $\square$

4. Vanishing Results

Let R be a ring, let $\mathfrak{a} \subseteq R$ be an ideal and let M be an R-module. In this situation, two important cohomological invariants may be defined, which are both related to the vanishing of local cohomology of M with respect to $\mathfrak{a}$, namely the *cohomological $\mathfrak{a}$-depth of M,* given by

$$\mathrm{t}_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M) \neq 0\},$$

and the *cohomological dimension of M with respect to $\mathfrak{a}$,* given by

$$\mathrm{cd}_{\mathfrak{a}}(M) := \sup\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

We shall give a characterization of $\mathrm{t}_{\mathfrak{a}}(M)$ in non-cohomological terms if R is Noetherian and M is finitely generated (cf. 4.6, 4.7). Also we give two different upper bounds on $\mathrm{cd}_{\mathfrak{a}}(M)$ in terms of non-cohomological invariants if R is Noetherian and—for the first one—M is finitely generated (cf. 4.11, 4.21).

4.0. **Notation.** Throughout this chapter, let R be a ring and let $\mathfrak{a} \subseteq R$ be an ideal.

4.1. **Reminder and Exercise.** A) Let M be an R-module. A finite sequence $(x_1, \ldots, x_r)$ of elements of R is said to be an *M-sequence* if

$$x_i \in \mathrm{NZD}_R\big(M/\sum_{j=1}^{i-1} x_j M\big) \text{ for } i \in \{1, \ldots, r\}.$$

(We use the convention that $\Sigma_{j=1}^{0} x_j M = 0$.) So, this means that

$$x_1 \in \mathrm{NZD}_R(M),\ x_2 \in \mathrm{NZD}_R(M/x_1M),\ x_3 \in \mathrm{NZD}_R(M/(x_1M + x_2M)),\ \ldots$$

In particular $x \in R$ forms an M-sequence if and only if $x \in \mathrm{NZD}_R(M)$. Note that the empty sequence of elements of $\mathfrak{a}$ is an M-sequence.

If $(x_1, \ldots, x_r)$ is an M-sequence such that $x_i \in \mathfrak{a}$ for $i \in \{1, \ldots, r\}$, then $(x_1, \ldots, x_r)$ is called an *M-sequence in $\mathfrak{a}$*. If $(x_1, \ldots, x_r)$ is an M-sequence, r is called its *length.*

B) Let $x_1, \ldots, x_r \in R$, let $s \in \{1, \ldots, r-1\}$ and let M be an R-module. Then, one has the following composition property of M-sequences:

a) $(x_1, \ldots, x_r)$ is an M-sequence if and only if $(x_1, \ldots, x_s)$ is an M-sequence and $(x_{s+1}, \ldots, x_r)$ is an $M/\Sigma_{l=1}^{s} x_l M$-sequence.

(To prove this, one can fix s and make induction on r, observing the isomorphisms $M/\Sigma_{j=1}^{i} x_j M \cong (M/\Sigma_{l=1}^{s} x_l M)/\Sigma_{j=s+1}^{i} x_j (M/\Sigma_{l=1}^{s} x_l M)$ for all $i \in \{s, \ldots, r\}$.)

As a special case of statement a) we obtain:

b) $(x_1, \ldots, x_r)$ is an M-sequence if and only if $x_1 \in \mathrm{NZD}_R(M)$ and $(x_2, \ldots, x_r)$ is an M/x_1M-sequence.

4.2. **Lemma.** *Let M be an R-module. Let $(x_1, \dots, x_r)$ be an M-sequence in $\mathfrak{a}$. Then*

$$H^i_{\mathfrak{a}}(M) = 0 \textit{ for all } i < r.$$

Proof. (Induction on r.) Let $r = 1$. Then $x_1 \in \mathfrak{a} \cap \mathrm{NZD}_R(M)$ (cf. 4.1 A)), hence $\Gamma_{\mathfrak{a}}(M) = 0$ (cf. 1.7 a)). By 3.4 we obtain $H^0_{\mathfrak{a}}(M) = 0$. Let $r > 1$. Then clearly $(x_1, \dots, x_{r-1})$ is an M-sequence in $\mathfrak{a}$. Hence, by induction, $H^i_{\mathfrak{a}}(M) = 0$ for all $i < r - 1$. It remains to be shown that $H^{r-1}_{\mathfrak{a}}(M) = 0$. By 4.1 B) b) we know that $x_1 \in \mathrm{NZD}_R(M)$ and that $(x_2, \dots, x_r)$ is an M/x_1M-sequence in $\mathfrak{a}$. In particular, the cohomology sequence in 3.10 C) (applied with $x = x_1$) gives us an exact sequence of R-modules

$$H^{r-2}_{\mathfrak{a}}(M/x_1M) \xrightarrow{\delta^{r-2}} H^{r-1}_{\mathfrak{a}}(M) \xrightarrow{x_1\cdot} H^{r-1}_{\mathfrak{a}}(M).$$

As $(x_2, \dots, x_r)$ is an M/x_1M-sequence in $\mathfrak{a}$ we get by induction that

$$H^{r-2}_{\mathfrak{a}}(M/x_1M) = 0.$$

So, the multiplication homomorphism $x_1\cdot : H^{r-1}_{\mathfrak{a}}(M) \to H^{r-1}_{\mathfrak{a}}(M)$ is injective. As $H^{r-1}_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion (cf. 3.13), and as $x_1 \in \mathfrak{a}$, we get $H^{r-1}_{\mathfrak{a}}(M) = 0$ (cf. 3.12 C) b)). $\square$

4.3. **Proposition.** *Let R be Noetherian and let M be a finitely generated R-module. Let $r \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *There is an M-sequence of length r in $\mathfrak{a}$;*
(ii) *$H^i_{\mathfrak{a}}(M) = 0$ for all $i < r$.*

Proof. "(i) $\Rightarrow$ (ii)": Clear by 4.2.

"(ii) $\Rightarrow$ (i)": Let $H^i_{\mathfrak{a}}(M) = 0$ for $i \in \{0, \dots, r-1\}$. We have to find an M-sequence $(x_1, \dots, x_r)$ in $\mathfrak{a}$. We construct this sequence by induction on r. So, let $r = 1$. Then $H^0_{\mathfrak{a}}(M) = 0$ by our assumption (ii). By 3.4 we have $\Gamma_{\mathfrak{a}}(M) = 0$. By 1.7 b) it follows $\mathfrak{a} \nsubseteq \mathrm{ZD}_R(M)$ and hence $\mathfrak{a} \cap \mathrm{NZD}_R(M) \neq \emptyset$. So, we find an element $x \in \mathfrak{a} \cap \mathrm{NZD}_R(M)$. This proves the case $r = 1$.

Let $r > 1$. By the case $r = 1$, there is some $x_1 \in \mathfrak{a} \cap \mathrm{NZD}_R(M)$. The cohomology sequence 3.10 C) (applied with $x = x_1$) gives exact sequences

$$H^j_{\mathfrak{a}}(M) \to H^j_{\mathfrak{a}}(M/x_1M) \to H^{j+1}_{\mathfrak{a}}(M)$$

for $j \in \mathbb{N}_0$. These show that $H^j_{\mathfrak{a}}(M/x_1M) = 0$ for all $j < r - 1$. So—by induction—there is an M/x_1M-sequence $(x_2, \dots, x_r)$ in $\mathfrak{a}$. By 4.1 B) b) we now see, that $(x_1, x_2, \dots, x_r)$ is an M-sequence in $\mathfrak{a}$. So (i) is true. $\square$

4.4. **Notation.** For a subset $A \subseteq \mathbb{Z}$ we take the supremum and the infimum of A always in $\mathbb{Z} \cup \{-\infty, \infty\}$. Therefore, the supremum and the infimum of the empty set of integers is $-\infty$ and ∞ respectively.

4.5. **Reminder and Remark.** A) Let M be an R-module. We define the *grade of* $\mathfrak{a}$ *with respect to* M by

$$\mathrm{grade}_M(\mathfrak{a}) := \sup\{r \in \mathbb{N}_0 \mid \exists\, M\text{-sequence of length } r \text{ in } \mathfrak{a}\}.$$

B) Observe that:

a) $\mathrm{grade}_M(\mathfrak{a}) = 0 \Leftrightarrow \mathfrak{a} \subseteq \mathrm{ZD}_R(M)$;
b) If $\mathfrak{b} \subseteq R$ is a second ideal with $\mathfrak{b} \subseteq \mathfrak{a}$, then $\mathrm{grade}_M(\mathfrak{b}) \leq \mathrm{grade}_M(\mathfrak{a})$;
c) $\mathrm{grade}_0(\mathfrak{a}) = \infty$.

C) If the ring R is local with maximal ideal $\mathfrak{m}$, the *depth of* M is defined by

$$\mathrm{depth}_R(M) := \mathrm{grade}_M(\mathfrak{m}).$$

4.6. **Theorem.** *Let R be Noetherian and let M be a finitely generated R-module. Then*

$$\mathrm{grade}_M(\mathfrak{a}) = \inf\{i \in \mathbb{N}_0 \,\big|\, H^i_{\mathfrak{a}}(M) \neq 0\}.$$

Proof. Let $\rho := \inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M) \neq 0\} \in \mathbb{N}_0 \cup \{\infty\}$. Let $g \in \mathbb{N}_0$ with $g \leq \mathrm{grade}_M(\mathfrak{a})$. Then, there is an M-sequence $(x_1, \ldots, x_r)$ in $\mathfrak{a}$ with $r \geq g$. By 4.3 it follows that $H^i_{\mathfrak{a}}(M) = 0$ for all $i < r$, so that $\rho \geq r \geq g$. This proves that $\mathrm{grade}_M(\mathfrak{a}) \leq \rho$. Now, let $r \leq \rho$. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i < r$. By 4.3 there is an M-sequence $(x_1, \ldots, x_r)$ in $\mathfrak{a}$, so that $\mathrm{grade}_M(\mathfrak{a}) \geq r$. This proves $\mathrm{grade}_M(\mathfrak{a}) \geq \rho$. $\square$

4.7. **Proposition.** *Let R be Noetherian and let M be a finitely generated R-module. Then, the following statements are equivalent:*

(i) $\mathfrak{a}M = M$*;*
(ii) $H^i_{\mathfrak{a}}(M) = 0$ *for all* $i \in \mathbb{N}_0$*;*
(iii) $\mathrm{grade}_M(\mathfrak{a}) = \infty$.

Proof. By 4.6 it is sufficient to show that statements (i) and (ii) are equivalent.

"(i) $\Rightarrow$ (ii)": Assume that $\mathfrak{a}M = M$. As M is finitely generated, there is some $a \in \mathfrak{a}$ such that $(1-a)M = 0$. Hence, the multiplication homomorphism $M \xrightarrow{(1-a)\cdot} M$ is zero. So, for each $i \in \mathbb{N}_0$, the multiplication homomorphism $H^i_{\mathfrak{a}}(M) \xrightarrow{(1-a)\cdot} H^i_{\mathfrak{a}}(M)$ is zero, too (cf. 3.10 A), 1.13 b)). Now, fix some $i \in \mathbb{N}_0$ and assume that $H^i_{\mathfrak{a}}(M) \neq 0$. Choose $m \in H^i_{\mathfrak{a}}(M) \setminus \{0\}$. As $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion (cf. 3.13), there is some $n \in \mathbb{N}$ with $\mathfrak{a}^n m = 0$. As $a \in \mathfrak{a}$ we obtain $a^n m = 0$. Choose $n \in \mathbb{N}$ minimal with the property that $a^n m = 0$. Then we get $a^{n-1}m = a^{n-1}m - a^n m = a^{n-1}(1-a)m = 0$, a contradiction. Thus $H^i_{\mathfrak{a}}(M) = 0$.

"(ii) $\Rightarrow$ (i)": It suffices to show, that no finitely generated R-module M satisfies the condition

$$(*) \qquad \mathfrak{a}M \neq M \text{ and } H^i_{\mathfrak{a}}(M) = 0 \text{ for all } i \in \mathbb{N}_0.$$

Assume to the contrary, that there is a finitely generated R-module M which satisfies $(*)$. Let

$$\mathbb{M} := \{N \subseteq M \mid N \text{ submodule, } M/N \text{ satisfies } (*)\}.$$

As M satisfies $(*)$ the zero module 0 belongs to $\mathbb{M}$, so that $\mathbb{M} \neq \emptyset$. As M is Noetherian, $\mathbb{M}$ thus has a maximal member, say N. Then $\bar{M} := M/N$ satisfies $(*)$ but $\bar{M}/U$ does not satisfy $(*)$ for any non-zero submodule U of $\bar{M}$.

As $\bar{M}$ is finitely generated with $H^0_{\mathfrak{a}}(\bar{M}) = 0$, there is some $x \in \mathfrak{a} \cap \mathrm{NZD}_R(\bar{M})$ (cf. 4.3). As $\mathfrak{a}\bar{M} \neq \bar{M}$ we have $\bar{M} \neq 0$. As $x \in \mathrm{NZD}_R(\bar{M})$ it thus follows $x\bar{M} \neq 0$. Therefore, $\bar{M}/x\bar{M}$ does not satisfy $(*)$. As $x \in \mathfrak{a}$ and $\mathfrak{a}\bar{M} \neq \bar{M}$, we have $\mathfrak{a}(\bar{M}/x\bar{M}) \neq \bar{M}/x\bar{M}$. As $\bar{M}/x\bar{M}$ does not satisfy $(*)$, we thus must find some $i \in \mathbb{N}_0$ with $H^i_{\mathfrak{a}}(\bar{M}/x\bar{M}) \neq 0$. By 3.10 C) (applied with $\bar{M}$ instead of M) we get an exact sequence of R-modules

$$H^i_{\mathfrak{a}}(\bar{M}) \to H^i_{\mathfrak{a}}(\bar{M}/x\bar{M}) \to H^{i+1}_{\mathfrak{a}}(\bar{M}),$$

showing that $H^i_{\mathfrak{a}}(\bar{M}) \neq 0$ or $H^{i+1}_{\mathfrak{a}}(\bar{M}) \neq 0$. This contradicts the property $(*)$ of $\bar{M}$. So, M cannot satisfy $(*)$. $\square$

4.8. **Reminder.** Let M be an R-module. Then, a *maximal M-sequence in $\mathfrak{a}$* is an M-sequence $(x_1, \ldots, x_r)$ in $\mathfrak{a}$, such that there is no $x_{r+1} \in \mathfrak{a}$ for which $(x_1, \ldots, x_r, x_{r+1})$ is an M-sequence.

4.9. **Corollary.** *Let R be Noetherian and let M be a finitely generated R-module with $\mathfrak{a}M \neq M$. Then all maximal M-sequences in $\mathfrak{a}$ have the length* $\mathrm{grade}_M(\mathfrak{a})$.

Proof. Let $g := \mathrm{grade}_M(\mathfrak{a})$. By 4.7 we have $g < \infty$. If $g = 0$, $\mathfrak{a}$ contains only the empty M-sequences and so our claim is clear. Thus, let $g > 0$. Let $(x_1, \ldots, x_r)$ be an M-sequence in $\mathfrak{a}$. Then $r \leq g$. It therefore is sufficient to show, that the M-sequence $(x_1, \ldots, x_r)$ is not maximal if $r < g$. For each $t \in \{1, \ldots, r\}$ we have a short exact sequence of R-modules

$$0 \to M/\sum_{l=1}^{t-1} x_l M \xrightarrow{x_t \cdot} M/\sum_{l=1}^{t-1} x_l M \to M/\sum_{l=1}^{t} x_l M \to 0,$$

as $x_t \in \mathrm{NZD}_R(M/\Sigma_{l=1}^{t-1} x_l M)$ and

$$\big(M/\sum_{l=1}^{t-1} x_l M\big)/x_t\big(M/\sum_{l=1}^{t-1} x_l M\big) \cong M/\sum_{l=1}^{t} x_l M.$$

So, in cohomology we get exact sequences

$$(*) \qquad H^k_{\mathfrak{a}}\big(M/\Sigma_{l=1}^{t-1} x_l M\big) \to H^k_{\mathfrak{a}}\big(M/\Sigma_{l=1}^{t} x_l M\big) \to H^{k+1}_{\mathfrak{a}}\big(M/\Sigma_{l=1}^{t-1} x_l M\big)$$

for $t \in \{1, \ldots, r\}$ and $k \in \mathbb{N}_0$. As $H^i_{\mathfrak{a}}(M) = 0$ for all $i < g$ (cf. 4.6) and as $M \cong M/\Sigma_{l=1}^{0} x_l M$, the sequences $(*)$ show that $H^i_{\mathfrak{a}}(M/x_1 M) = 0$ for all $i < g - 1$. But then the sequences $(*)$ show that $H^i_{\mathfrak{a}}\big(M/(x_1 M + x_2 M)\big) = 0$ for $i < g - 2$. Inductively we see that $H^i_{\mathfrak{a}}(M/\Sigma_{l=1}^{r} x_l M) = 0$ for all $i < g - r$. If $r < g$, this means that $H^0_{\mathfrak{a}}(M/\Sigma_{l=1}^{r} x_l M) = 0$. So, there is some element

$x_{r+1} \in \mathfrak{a} \cap \mathrm{NZD}_R(M/\Sigma_{l=1}^r x_l M)$ (cf. 4.3). But then $(x_1, \ldots, x_r, x_{r+1})$ is an M-sequence in $\mathfrak{a}$. So, $(x_1, \ldots, x_r)$ was not a maximal M-sequence in $\mathfrak{a}$. $\square$

4.10. **Reminder and Exercise.** A) Let M be a finitely generated R-module. Then, the *dimension of* M is defined as the supremum of lengths of chains of primes in the variety of the annihilator of M:

$$\dim(M) := \sup\{l \in \mathbb{N}_0 \mid \exists\, \mathfrak{p}_0, \ldots, \mathfrak{p}_l \in \mathrm{Var}(0 :_R M) : \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_l\}.$$

B) Observe the following facts:

a) $\dim(M) = -\infty \Leftrightarrow M = 0$;

b) If $N \subseteq M$ is a finitely generated submodule, then

$$\dim(N), \dim(M/N) \leq \dim(M);$$

c) If R is Noetherian and $x \in \mathrm{NZD}_R(M)$, then $\dim(M/xM) \leq \dim(M) - 1$.

(Hints: For b) observe that $(0 :_R N), (0 :_R M/N) \supseteq (0 :_R M)$. For c) observe that $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \min(0 :_R M)$ (cf. 1.6 B) a), 1.8 C) c).)

4.11. **Theorem.** *Let R be Noetherian and let M be a finitely generated R-module. Then*

$$H^i_{\mathfrak{a}}(M) = 0 \textit{ for all } i > \dim(M).$$

Proof. Let $d := \dim(M)$. For $M = 0$ our statement is clear by 1.13 c). So, let $M \neq 0$. Then, by 4.10 B) a), $d \geq 0$. Obviously we may assume that $d < \infty$. We proceed by induction on d. Let $d = 0$. We have to show, that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$. Let $\bar{M} := M/\Gamma_{\mathfrak{a}}(M)$. Then $H^i_{\mathfrak{a}}(\bar{M}) \cong H^i_{\mathfrak{a}}(M)$ for all $i > 0$ (cf. 3.18 b), 3.12 A) b)). So, it suffices to show that $H^i_{\mathfrak{a}}(\bar{M}) = 0$ for all $i > 0$. By 1.4 B) b) we have $\Gamma_{\mathfrak{a}}(\bar{M}) = 0$, hence $H^0_{\mathfrak{a}}(\bar{M}) = 0$ (cf. 3.4). So, there is some $x \in \mathfrak{a} \cap \mathrm{NZD}_R(\bar{M})$ (cf. 4.3). By 4.10 B) b) we have $\dim(\bar{M}) \leq d = 0$. So, by 4.10 B) c) $\dim(\bar{M}/x\bar{M}) \leq 0 - 1$, hence $\bar{M}/x\bar{M} = 0$ (cf. 4.10 B) a)) so that $x\bar{M} = \bar{M}$. As $x \in \mathfrak{a}$ it follows $\mathfrak{a}\bar{M} = \bar{M}$. By 4.7 we get $H^i_{\mathfrak{a}}(\bar{M}) = 0$ for all $i \in \mathbb{N}_0$, hence in particular $H^i_{\mathfrak{a}}(\bar{M}) = 0$ for all $i > 0$.

Now, let $d > 0$. We have to show that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > d$. Again, let $\bar{M} := M/\Gamma_{\mathfrak{a}}(M)$, so that $H^i_{\mathfrak{a}}(\bar{M}) \cong H^i_{\mathfrak{a}}(M)$ for all $i > 0$. It thus suffices to show that $H^i_{\mathfrak{a}}(\bar{M}) = 0$ for all $i > d$. By 4.10 B) b) we have $\dim(\bar{M}) \leq d$. If $\dim(\bar{M}) < d$, by induction, $H^i_{\mathfrak{a}}(\bar{M}) = 0$ for all $i > \dim(\bar{M})$ and hence for all $i > d$. So, assume that $\dim(\bar{M}) = d$. Again we see by 3.4, 1.4 B) b) and by 4.3 that there is some $x \in \mathfrak{a} \cap \mathrm{NZD}_R(\bar{M})$. By 4.10 B) c) we obtain $\dim(\bar{M}/x\bar{M}) \leq \dim(\bar{M}) - 1 = d - 1$. So, by induction, $H^i_{\mathfrak{a}}(\bar{M}/x\bar{M}) = 0$ for all $i > \dim(\bar{M}/x\bar{M})$, hence for all $i > d - 1$. On the other hand 3.10 C) gives us exact sequences

$$H^j_{\mathfrak{a}}(\bar{M}/x\bar{M}) \xrightarrow{\delta^j} H^{j+1}_{\mathfrak{a}}(\bar{M}) \xrightarrow{x\cdot} H^{j+1}_{\mathfrak{a}}(\bar{M})$$

for $j \in \mathbb{N}_0$ which show that $x\cdot : H^i_{\mathfrak{a}}(\bar{M}) \to H^i_{\mathfrak{a}}(\bar{M})$ is injective for all $i > d$. As $x \in \mathfrak{a}$ it follows that $H^i_{\mathfrak{a}}(\bar{M}) = 0$ for all $i > d$ (cf. 3.12 C) b), 3.13). $\square$

The above result is often called the *Vanishing Theorem of Grothendieck for Local Cohomology* (cf. [G]). It can be shown that its conclusion still holds if the R-module M is not finitely generated (cf. [B-S, Theorem 6.1.2]).

Now, in order to prove further vanishing results we have to develop another tool from homological algebra, the so called triad sequence. This will lead us to a fundamental exact sequence of local cohomology theory, the *Mayer-Vietoris sequence.* As usual, we perform these constructions in a series of combined reminders and guided exercises.

4.12. **Reminder and Exercise.** A) Let R' be a second ring, and let F and G be two additive functors from R-modules to R'-modules. Moreover, let

$$\mu : F \to G,\ M \leadsto (\mu_M : F(M) \to G(M))$$

be a natural transformation (cf. 3.1). Now, let M be a R-module with injective resolution $((I^\bullet, d^\bullet); a)$. Then:

a) The family $\mu_{I^\bullet} := (\mu_{I^n})_{n\in\mathbb{Z}}$ is a homomorphism of cocomplexes of R'-modules

$$\mu_{I^\bullet} : (F(I^\bullet), F(d^\bullet)) \to (G(I^\bullet), G(d^\bullet)).$$

Let N be a second R-module with injective resolution $((J^\bullet, e^\bullet); b)$, let $h : M \to N$ be a homomorphism of R-modules and let $h^\bullet : (I^\bullet, d^\bullet) \to (J^\bullet, e^\bullet)$ be a resolution of h. Then:

b) For all $n \in \mathbb{Z}$ we have the commutative diagram

$$\begin{tikzcd}
H^n(F(I^\bullet), F(d^\bullet)) \arrow[r, "H^n(\mu_{I^\bullet})"] \arrow[d, "H^n(F(h^\bullet))"'] & H^n(G(I^\bullet), G(d^\bullet)) \arrow[d, "H^n(G(h^\bullet))"] \\
H^n(F(J^\bullet), F(e^\bullet)) \arrow[r, "H^n(\mu_{J^\bullet})"] & H^n(G(J^\bullet), G(e^\bullet)).
\end{tikzcd}$$

B) Keep the notations and hypotheses of part A) and let

$$\mathbb{I}_* : M \leadsto \mathbb{I}_M = ((I^\bullet_M, d^\bullet_M); a_M)$$

be a choice of injective resolutions of R-modules. Use what has been said in part A) and in 2.13 D) to show that for each $n \in \mathbb{N}_0$, the assignment

$$M \leadsto \left(H^n(\mu_{I^\bullet_M}) : H^n(F(I^\bullet_M), F(d^\bullet_M)) \to H^n(G(I^\bullet_M), G(d^\bullet_M))\right)$$

defines a natural transformation

$$\mathcal{R}^n\mu : \mathcal{R}^nF \to \mathcal{R}^nG,\ M \leadsto (\mathcal{R}^n\mu_M := H^n(\mu_{I^\bullet_M})).$$

The transformation $\mathcal{R}^n\mu$ is called the *n-th right derived (transformation) of* μ.

C) Keep the above notations and hypotheses. Show that in the notation of 3.3:

a) For each R-module M, we have the commutative diagram

$$\begin{tikzcd} F(M) \arrow[r, "\mu_M"] \arrow[d, "\alpha^F_M"'] & G(M) \arrow[d, "\alpha^G_M"] \\ \mathcal{R}^0F(M) \arrow[r, "\mathcal{R}^0\mu_M"] & \mathcal{R}^0G(M). \end{tikzcd}$$

Also show that in the notation of 3.8 B):

b) For each short exact sequence $\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ of R-modules and each $n \in \mathbb{N}_0$ we have the commutative diagram

$$\begin{tikzcd} \mathcal{R}^nF(P) \arrow[r, "\delta^{n,F}_{\mathbb{S}}"] \arrow[d, "\mathcal{R}^n\mu_P"'] & \mathcal{R}^{n+1}F(N) \arrow[d, "\mathcal{R}^{n+1}\mu_N"] \\ \mathcal{R}^nG(P) \arrow[r, "\delta^{n,G}_{\mathbb{S}}"] & \mathcal{R}^{n+1}G(N). \end{tikzcd}$$

Statements a) and b) express the fact that the formation of right derived transformations is natural. Finally prove:

c) If μ is a natural equivalence, so are its right derived transformations $\mathcal{R}^n\mu$ for $n \in \mathbb{N}_0$.

4.13. **Reminder, Exercise and Construction.** A) Let R' be a second ring, and let F, G, H be three additive functors from R-modules to R'-modules. Also, let $\mu : F \to G$ and $\nu : G \to H$ be two natural transformations (cf. 3.1). We say that

$$\Delta : F \xrightarrow{\mu} G \xrightarrow{\nu} H$$

is an *(admissible) triad of (additive covariant) functors (from R-modules to R'-modules)* if the sequence

$$0 \to F(I) \xrightarrow{\mu_I} G(I) \xrightarrow{\nu_I} H(I) \to 0$$

is exact for each injective R-module I.

B) Assume that $\Delta : F \xrightarrow{\mu} G \xrightarrow{\nu} H$ is a triad of functors from R-modules to R'-modules. Let $\mathbb{I} = ((I^\bullet, d^\bullet); a)$ be an injective resolution of some fixed R-module M. Then, we get a short exact sequence of cocomplexes of R'-modules

$$\Delta_{\mathbb{I}} : 0 \to (F(I^\bullet), F(d^\bullet)) \xrightarrow{\mu_{I^\bullet}} (G(I^\bullet), G(d^\bullet)) \xrightarrow{\nu_{I^\bullet}} (H(I^\bullet), H(d^\bullet)) \to 0$$

in which $\mu_{I^\bullet}$ and $\nu_{I^\bullet}$ are the homomorphisms of cocomplexes defined according to 4.12 A) a). Now, we form the cohomology sequence associated to $\Delta_{\mathbb{I}}$ (cf. 3.5 C) g)) and end up with an exact sequence

$$\begin{aligned} 0 \to & H^0(F(I^\bullet)) \xrightarrow{H^0(\mu_{I^\bullet})} H^0(G(I^\bullet)) \xrightarrow{H^0(\nu_{I^\bullet})} H^0(H(I^\bullet)) \xrightarrow{\delta^0_{\Delta_{\mathbb{I}}}} \\ & H^1(F(I^\bullet)) \xrightarrow{H^1(\mu_{I^\bullet})} H^1(G(I^\bullet)) \longrightarrow \cdots , \end{aligned}$$

where $\delta^n_{\Delta_{\mathbb{I}}}$ denotes the n-th connecting homomorphism associated to the sequence $\Delta_{\mathbb{I}}$ (cf. 3.5 B)). Next, let $h : M \to N$ be a homomorphism of R-modules, let $\mathbb{J} = ((J^\bullet, e^\bullet); b))$ be an injective resolution of N and let $h^\bullet : (I^\bullet, d^\bullet) \to (J^\bullet, e^\bullet)$ be a resolution of h. We can form the short exact sequence of cocomplexes

$$\Delta_{\mathbb{J}} : 0 \to (F(J^\bullet), F(e^\bullet)) \xrightarrow{\mu_{J^\bullet}} (G(J^\bullet), G(e^\bullet)) \xrightarrow{\nu_{J^\bullet}} (H(J^\bullet), H(e^\bullet)) \to 0,$$

and consider the associated cohomology sequence and the associated connecting homomorphisms $\delta^n_{\Delta_{\mathbb{J}}}$. Then, for each $n \in \mathbb{N}_0$ we get the commutative diagram

$$\begin{array}{ccc}
H^n(H(I^\bullet)) & \xrightarrow{\delta^n_{\Delta_{\mathbb{I}}}} & H^{n+1}(F(I^\bullet)) \\
\downarrow{\scriptstyle H^n(H(h^\bullet))} & & \downarrow{\scriptstyle H^{n+1}(F(h^\bullet))} \\
H^n(H(J^\bullet)) & \xrightarrow{\delta^n_{\Delta_{\mathbb{J}}}} & H^{n+1}(F(J^\bullet)).
\end{array}$$

C) Keep the above notations and hypotheses. Let

$$\mathbb{I}_* : M \rightsquigarrow \mathbb{I}_M = ((I^\bullet_M, d^\bullet_M); a_M)$$

be a choice of injective resolutions of R-modules. Use what has been said in part B) and in 2.13 D) to show that (setting $\delta^{n,\Delta}_M := \delta^n_{\Delta_{\mathbb{I}_M}}$) the assignment

$$M \rightsquigarrow \left(\delta^n_{\Delta_{\mathbb{I}_M}} : H^n(H(I^\bullet_M)) \to H^{n+1}(F(I^\bullet_M))\right)$$

gives rise to an assignment

$$\delta^{\bullet,\Delta}_* : M \rightsquigarrow (\delta^{n,\Delta}_M : \mathcal{R}^n H(M) \to \mathcal{R}^{n+1} F(M))_{n \in \mathbb{N}_0},$$

and hence to an exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{R}^0 F(M) & \xrightarrow{\mathcal{R}^0 \mu_M} & \mathcal{R}^0 G(M) & \xrightarrow{\mathcal{R}^0 \nu_M} & \mathcal{R}^0 H(M) \\
 & \xrightarrow{\delta^{0,\Delta}_M} & \mathcal{R}^1 F(M) & \xrightarrow{\mathcal{R}^1 \mu_M} & \mathcal{R}^1 G(M) & \longrightarrow & \cdots \\
 & & & & \cdots & \longrightarrow & \mathcal{R}^{n-1} H(M) \\
 & \xrightarrow{\delta^{n-1,\Delta}_M} & \mathcal{R}^n F(M) & \xrightarrow{\mathcal{R}^n \mu_M} & \mathcal{R}^n G(M) & \xrightarrow{\mathcal{R}^n \nu_M} & \mathcal{R}^n H(M) \\
 & \xrightarrow{\delta^{n,\Delta}_M} & \mathcal{R}^{n+1} F(M) & \xrightarrow{\mathcal{R}^{n+1} \mu_M} & \mathcal{R}^{n+1} G(M) & \longrightarrow & \cdots
\end{array}$$

for each R-module M.

The homomorphism

$$\delta^{n,\Delta}_M : \mathcal{R}^n H(M) \to \mathcal{R}^{n+1} F(M)$$

is called the *n-th connecting homomorphism with respect to Δ associated to$M$*, and the above sequence is called the *right derived sequence of Δ associated to M*.

D) Show that—according to its suggested construction—the assignment of part C) has the following naturality property: If $h : M \to N$ is a homomorphism of R-modules, the diagrams

$$\begin{tikzcd} \mathcal{R}^n H(M) \arrow[r, "\delta_M^{n,\Delta}"] \arrow[d, "\mathcal{R}^n H(h)"'] & \mathcal{R}^{n+1} F(M) \arrow[d, "\mathcal{R}^{n+1} F(h)"] \\ \mathcal{R}^n H(N) \arrow[r, "\delta_N^{n,\Delta}"] & \mathcal{R}^{n+1} F(N) \end{tikzcd}$$

commute for all $n \in \mathbb{N}_0$.

After this fairly general excursion in homological algebra we now focus our interest to local cohomology with the goal to establish the Mayer-Vietoris sequence.

4.14. **Proposition.** *Let R be Noetherian, let $\mathfrak{b} \subseteq R$ be a second ideal and let I be an injective R-module. Then*

$$\Gamma_{\mathfrak{a}\cap\mathfrak{b}}(I) = \Gamma_{\mathfrak{a}}(I) + \Gamma_{\mathfrak{b}}(I).$$

Proof. "$\supseteq$": is obvious (cf. 1.4 A) b)).

"$\subseteq$": Let $z \in \Gamma_{\mathfrak{a}\cap\mathfrak{b}}(I)$. There is some $n \in \mathbb{N}$ with $(\mathfrak{a} \cap \mathfrak{b})^n z = 0$. Applying the Lemma of Artin-Rees (cf. 1.3) to the pair of R-modules $\mathfrak{b}^n \subseteq R$ and the ideal $\mathfrak{a}$, we find some $m_0 \in \mathbb{N}$ such that $\mathfrak{a}^m \cap \mathfrak{b}^n = \mathfrak{a}^m R \cap \mathfrak{b}^n \subseteq \mathfrak{a}^{m-m_0}\mathfrak{b}^n$ for all $m \geq m_0$. Choosing $m = n+m_0$ we obtain $\mathfrak{a}^m \cap \mathfrak{b}^m \subseteq \mathfrak{a}^m \cap \mathfrak{b}^n \subseteq \mathfrak{a}^n\mathfrak{b}^n \subseteq (\mathfrak{a}\cap\mathfrak{b})^n$ and hence $(\mathfrak{a}^m \cap \mathfrak{b}^m)z = 0$.

Now, consider the injective homomorphism of R-modules

$$\varepsilon : R/\mathfrak{a}^m \cap \mathfrak{b}^m \to R/\mathfrak{a}^m \oplus R/\mathfrak{b}^m,\ x + \mathfrak{a}^m \cap \mathfrak{b}^m \mapsto (x + \mathfrak{a}^m, x + \mathfrak{b}^m).$$

As $(\mathfrak{a}^m \cap \mathfrak{b}^m)z = 0$, there is a homomorphism of R-modules

$$h : R/\mathfrak{a}^m \cap \mathfrak{b}^m \to I,\ x + \mathfrak{a}^m \cap \mathfrak{b}^m \mapsto xz.$$

As I is injective, there is a homomorphism of R-modules l, which appears in the commutative diagram

$$\begin{tikzcd} & I \\ R/\mathfrak{a}^m \cap \mathfrak{b}^m \arrow[ur, "h"] \arrow[r, "\varepsilon"'] & R/\mathfrak{a}^m \oplus R/\mathfrak{b}^m. \arrow[u, "l"'] \end{tikzcd}$$

Consider the two elements

$$u := l(1 + \mathfrak{a}^m, 0),\ v := l(0, 1 + \mathfrak{b}^m) \in I.$$

For each $x \in \mathfrak{a}^m$ we have $xu = l(x(1 + \mathfrak{a}^m), 0) = l(0, 0) = 0$, so that $\mathfrak{a}^m u = 0$, and hence $u \in \Gamma_{\mathfrak{a}}(I)$. Similarly we have $v \in \Gamma_{\mathfrak{b}}(I)$. As $z = h(1 + \mathfrak{a}^m \cap \mathfrak{b}^m) = l(\varepsilon(1 + \mathfrak{a}^m \cap \mathfrak{b}^m)) = l(1 + \mathfrak{a}^m, 1 + \mathfrak{b}^m) = l((1 + \mathfrak{a}^m, 0) + (0, 1 + \mathfrak{b}^m)) = u + v$, we obtain $z \in \Gamma_{\mathfrak{a}}(I) + \Gamma_{\mathfrak{b}}(I)$. $\square$

4.15. **Remark and Exercise.** A) Let $M^\bullet = (M^\bullet, d^\bullet)$ and $N^\bullet = (N^\bullet, e^\bullet)$ be two cocomplexes of R-modules. Consider the *direct sum* of these cocomplexes

$$M^\bullet \oplus N^\bullet = (M^\bullet \oplus N^\bullet, d^\bullet \oplus e^\bullet) : \cdots \to M^n \oplus N^n \xrightarrow{d^n \oplus e^n} M^{n+1} \oplus N^{n+1} \to \cdots$$

in which $d^n \oplus e^n : M^n \oplus N^n \to M^{n+1} \oplus N^{n+1}$ is given by $(x, y) \mapsto (d^n(x), e^n(y))$.

Show, that for each $n \in \mathbb{Z}$ one has an isomorphism of R-modules

$$\iota^n_{M^\bullet, N^\bullet} : H^n(M^\bullet, d^\bullet) \oplus H^n(N^\bullet, e^\bullet) \xrightarrow{\cong} H^n(M^\bullet \oplus N^\bullet, d^\bullet \oplus e^\bullet),$$

given by $(x + \mathrm{Im}(d^{n-1}), y + \mathrm{Im}(e^{n-1})) \mapsto (x, y) + \mathrm{Im}(d^{n-1} \oplus e^{n-1})$.

Observe that these canonical isomorphisms have the expected naturality property: If $h^\bullet : (M^\bullet, d^\bullet) \to (\bar{M}^\bullet, \bar{d}^\bullet)$ and $l^\bullet : (N^\bullet, e^\bullet) \to (\bar{N}^\bullet, \bar{e}^\bullet)$ are homomorphisms of cocomplexes of R-modules, we get a homomorphism of cocomplexes of R-modules

$$h^\bullet \oplus l^\bullet := (h^n \oplus l^n)_{n \in \mathbb{Z}} : (M^\bullet \oplus N^\bullet, d^\bullet \oplus e^\bullet) \to (\bar{M}^\bullet \oplus \bar{N}^\bullet, \bar{d}^\bullet \oplus \bar{e}^\bullet).$$

Then, for each $n \in \mathbb{Z}$, we have the commutative diagram

$$\begin{tikzcd}
H^n(M^\bullet, d^\bullet) \oplus H^n(N^\bullet, e^\bullet) \arrow[r, "\iota^n_{M^\bullet, N^\bullet}"] \arrow[d, "H^n(h^\bullet) \oplus H^n(l^\bullet)"'] & H^n(M^\bullet \oplus N^\bullet, d^\bullet \oplus e^\bullet) \arrow[d, "H^n(h^\bullet \oplus l^\bullet)"] \\
H^n(\bar{M}^\bullet, \bar{d}^\bullet) \oplus H^n(\bar{N}^\bullet, \bar{e}^\bullet) \arrow[r, "\iota^n_{\bar{M}^\bullet, \bar{N}^\bullet}"] & H^n(\bar{M}^\bullet \oplus \bar{N}^\bullet, \bar{d}^\bullet \oplus \bar{e}^\bullet).
\end{tikzcd}$$

B) Let R' be a second ring, and let F and G be two additive functors from R-modules to R'-modules. Then we may define a new additive functor from R-modules to R'-modules

$$F \oplus G : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(F(M) \oplus G(M) \xrightarrow{F(h) \oplus G(h)} F(N) \oplus G(N)\big),$$

the *direct sum of F and G*. If F and G are linear with respect to some homomorphism of rings between R and R', then so is $F \oplus G$.

Now, let $\mathbb{I}_* : M \rightsquigarrow \mathbb{I}_M = ((I^\bullet_M, d^\bullet_M); a_M)$ be a choice of injective resolutions of R-modules. Fix $n \in \mathbb{N}_0$. Use what has been said in part A) and in 2.13 D) to show that the assignment

$$M \rightsquigarrow \big(\iota^n_{F(I^\bullet_M), G(I^\bullet_M)} : H^n(F(I^\bullet_M)) \oplus H^n(G(I^\bullet_M)) \xrightarrow{\cong} H^n(F(I^\bullet_M) \oplus G(I^\bullet_M))\big)$$

gives rise to a natural equivalence of functors

$$\iota^{n;F,G} : \mathcal{R}^n F \oplus \mathcal{R}^n G \xrightarrow{\cong} \mathcal{R}^n(F \oplus G).$$

In particular, if $\mathfrak{b} \subseteq R$ is a second ideal, we get a natural equivalence

$$\iota^n_{\mathfrak{a},\mathfrak{b}} := \iota^{n;\Gamma_\mathfrak{a},\Gamma_\mathfrak{b}} : H^n_\mathfrak{a} \oplus H^n_\mathfrak{b} \xrightarrow{\cong} \mathcal{R}^n(\Gamma_\mathfrak{a} \oplus \Gamma_\mathfrak{b})$$

for each $n \in \mathbb{N}_0$.

4.16. **Construction and Exercise.** A) Fix a second ideal $\mathfrak{b} \subseteq R$. Then, for each R-module M we may define homomorphisms of R-modules

$$\mu_M^{\mathfrak{a},\mathfrak{b}} : \Gamma_{\mathfrak{a}+\mathfrak{b}}(M) \to \Gamma_{\mathfrak{a}}(M) \oplus \Gamma_{\mathfrak{b}}(M),\ m \mapsto (m, m);$$
$$\nu_M^{\mathfrak{a},\mathfrak{b}} : \Gamma_{\mathfrak{a}}(M) \oplus \Gamma_{\mathfrak{b}}(M) \to \Gamma_{\mathfrak{a}\cap\mathfrak{b}}(M),\ (m, n) \mapsto m - n.$$

Observe, that in this way we get two natural transformations

$$\mu^{\mathfrak{a},\mathfrak{b}} : \Gamma_{\mathfrak{a}+\mathfrak{b}} \to \Gamma_{\mathfrak{a}} \oplus \Gamma_{\mathfrak{b}};$$
$$\nu^{\mathfrak{a},\mathfrak{b}} : \Gamma_{\mathfrak{a}} \oplus \Gamma_{\mathfrak{b}} \to \Gamma_{\mathfrak{a}\cap\mathfrak{b}}.$$

Show that:

a) For each R-module M we have the exact sequence

$$0 \to \Gamma_{\mathfrak{a}+\mathfrak{b}}(M) \xrightarrow{\mu_M^{\mathfrak{a},\mathfrak{b}}} \Gamma_{\mathfrak{a}}(M) \oplus \Gamma_{\mathfrak{b}}(M) \xrightarrow{\nu_M^{\mathfrak{a},\mathfrak{b}}} \Gamma_{\mathfrak{a}\cap\mathfrak{b}}(M).$$

Use 4.14 to show:

b) If R is Noetherian and I is an injective R-module, the homomorphism

$$\nu_I^{\mathfrak{a},\mathfrak{b}} : \Gamma_{\mathfrak{a}}(I) \oplus \Gamma_{\mathfrak{b}}(I) \to \Gamma_{\mathfrak{a}\cap\mathfrak{b}}(I)$$

is surjective.

Conclude that we have the triad of functors

$$\Delta_{\mathfrak{a},\mathfrak{b}} : \Gamma_{\mathfrak{a}+\mathfrak{b}} \xrightarrow{\mu^{\mathfrak{a},\mathfrak{b}}} \Gamma_{\mathfrak{a}} \oplus \Gamma_{\mathfrak{b}} \xrightarrow{\nu^{\mathfrak{a},\mathfrak{b}}} \Gamma_{\mathfrak{a}\cap\mathfrak{b}}.$$

B) Keep the notations and hypotheses of part A), and let R be Noetherian. Fix an R-module M. Then, we may form the right derived sequence of $\Delta_{\mathfrak{a},\mathfrak{b}}$ associated to M (cf. 4.13 C)) and identify the R-modules $H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M)$ with $\mathcal{R}^n(\Gamma_{\mathfrak{a}} \oplus \Gamma_{\mathfrak{b}})(M)$ by means of the natural equivalence

$$\iota^n_{\mathfrak{a},\mathfrak{b}} : H^n_{\mathfrak{a}} \oplus H^n_{\mathfrak{b}} \xrightarrow{\cong} \mathcal{R}^n(\Gamma_{\mathfrak{a}} \oplus \Gamma_{\mathfrak{b}})$$

(cf. 4.15 B)). Then, setting

$$\mu_M^{n;\mathfrak{a},\mathfrak{b}} := \mathcal{R}^n \mu_M^{\mathfrak{a},\mathfrak{b}},\ \nu_M^{n;\mathfrak{a},\mathfrak{b}} := \mathcal{R}^n \nu_M^{\mathfrak{a},\mathfrak{b}} \text{ and } \delta_M^{n;\mathfrak{a},\mathfrak{b}} := \delta_M^{n,\Delta_{\mathfrak{a},\mathfrak{b}}}$$

for all $n \in \mathbb{N}_0$, we end up with an exact sequence

$$\begin{array}{llllll}
0 \longrightarrow & H^0_{\mathfrak{a}+\mathfrak{b}}(M) & \xrightarrow{\mu_M^{0;\mathfrak{a},\mathfrak{b}}} & H^0_{\mathfrak{a}}(M) \oplus H^0_{\mathfrak{b}}(M) & \xrightarrow{\nu_M^{0;\mathfrak{a},\mathfrak{b}}} & H^0_{\mathfrak{a}\cap\mathfrak{b}}(M) \\
\xrightarrow{\delta_M^{0;\mathfrak{a},\mathfrak{b}}} & H^1_{\mathfrak{a}+\mathfrak{b}}(M) & \xrightarrow{\mu_M^{1;\mathfrak{a},\mathfrak{b}}} & H^1_{\mathfrak{a}}(M) \oplus H^1_{\mathfrak{b}}(M) & \longrightarrow & \cdots \\
 & & & \cdots & \longrightarrow & H^{n-1}_{\mathfrak{a}\cap\mathfrak{b}}(M) \\
\xrightarrow{\delta_M^{n-1;\mathfrak{a},\mathfrak{b}}} & H^n_{\mathfrak{a}+\mathfrak{b}}(M) & \xrightarrow{\mu_M^{n;\mathfrak{a},\mathfrak{b}}} & H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M) & \xrightarrow{\nu_M^{n;\mathfrak{a},\mathfrak{b}}} & H^n_{\mathfrak{a}\cap\mathfrak{b}}(M) \\
\xrightarrow{\delta_M^{n;\mathfrak{a},\mathfrak{b}}} & H^{n+1}_{\mathfrak{a}+\mathfrak{b}}(M) & \xrightarrow{\mu_M^{n+1;\mathfrak{a},\mathfrak{b}}} & H^{n+1}_{\mathfrak{a}}(M) \oplus H^{n+1}_{\mathfrak{b}}(M) & \longrightarrow & \cdots ,
\end{array}$$

the *Mayer-Vietoris sequence with respect to* $\mathfrak{a}$ *and* $\mathfrak{b}$ *associated to* M.

C) Clearly, in the obvious sense we can say, that the formation of the Mayer-Vietoris sequence with respect to $\mathfrak{a}$ and $\mathfrak{b}$ is natural.

4.17. **Remark and Exercise.** Let $\mathfrak{b} \subseteq R$ be a second ideal. If $\mathfrak{a}$ and $\mathfrak{b}$ are finitely generated, as a consequence of 1.4 A) c) we have:

a) If $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$, then $H^n_{\mathfrak{a}}(\bullet) = H^n_{\mathfrak{b}}(\bullet)$ for all $n \in \mathbb{N}_0$;
b) $H^n_{\mathfrak{ab}}(\bullet) = H^n_{\mathfrak{a}\cap\mathfrak{b}}(\bullet)$ for all $n \in \mathbb{N}_0$.

4.18. **Lemma.** *Let R be Noetherian, let $a \in R$ and let M be an R-module. Then $H^i_{\langle a\rangle}(M) = 0$ for all $i > 1$.*

Proof. Let $\bar{M} := M/\Gamma_{\langle a\rangle}(M)$. Let $\eta_a : M \to M_a,\ m \mapsto \frac{m}{1}$ be the canonical homomorphism of R-modules. Then

$$\mathrm{Ker}(\eta_a) = \{m \in M \mid \exists n \in \mathbb{N} : a^n m = 0\} = \Gamma_{\langle a\rangle}(M).$$

Hence there is a short exact sequence of R-modules

$$0 \to \bar{M} \xrightarrow{\bar{\eta}} M_a \to M_a/\bar{\eta}(\bar{M}) \to 0,$$

in which $\bar{\eta}$ is given by $m + \Gamma_{\langle a\rangle}(M) \mapsto \eta_a(m)$. But now we can apply cohomology to this sequence and get exact sequences

$$H^j_{\langle a\rangle}(M_a/\bar{\eta}(\bar{M})) \xrightarrow{\delta^j} H^{j+1}_{\langle a\rangle}(\bar{M}) \to H^{j+1}_{\langle a\rangle}(M_a) \to H^{j+1}_{\langle a\rangle}(M_a/\bar{\eta}(\bar{M}))$$

for all $j \in \mathbb{N}_0$. Clearly $M_a/\bar{\eta}(\bar{M})$ is $\langle a\rangle$-torsion. Therefore $H^k_{\langle a\rangle}(M_a/\bar{\eta}(\bar{M})) = 0$ for all $k > 0$ (cf. 3.17). The above sequences now show that $H^i_{\langle a\rangle}(\bar{M}) \cong H^i_{\langle a\rangle}(M_a)$ for all $i > 1$. By 3.18 b) and 3.12 A) b) we also have $H^i_{\langle a\rangle}(M) \cong H^i_{\langle a\rangle}(\bar{M})$ for all $i > 0$. So, we only have to show that $H^i_{\langle a\rangle}(M_a) = 0$ for all $i > 1$. As $a\cdot : M_a \to M_a$ is an isomorphism, so is $a\cdot : H^i_{\langle a\rangle}(M_a) \to H^i_{\langle a\rangle}(M_a)$ (cf. 1.13 A) a), 3.10 A)). Now, we conclude by 3.13 and 3.12 C) b). $\square$

4.19. **Proposition.** *Let R be Noetherian, let $\mathfrak{a} = \langle a_1, \ldots, a_r\rangle$ and let M be an R-module. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i > r$.*

Proof. (Induction on r.) The case $r = 0$ is clear by 1.4 A) a) and 2.15 c), and the case $r = 1$ is clear by 4.18. So, let $r > 1$. Let $\mathfrak{b} := \langle a_1, \ldots, a_{r-1}\rangle$. As $\mathfrak{a} = \mathfrak{b} + \langle a_r\rangle$, the Mayer-Vietoris sequence with respect to $\mathfrak{b}$ and $\langle a_r\rangle$ associated to M gives us exact sequences

$$H^{i-1}_{\mathfrak{b}\cap\langle a_r\rangle}(M) \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{b}}(M) \oplus H^i_{\langle a_r\rangle}(M)$$

for all $i \in \mathbb{N}$ (cf. 4.16 B)). By 4.17 b) we can write $H^{i-1}_{\mathfrak{b}\cap\langle a_r\rangle}(M) = H^{i-1}_{\mathfrak{b}\langle a_r\rangle}(M) = H^{i-1}_{\langle a_1a_r,\ldots,a_{r-1}a_r\rangle}(M)$. Hence it follows by induction, that $H^{i-1}_{\mathfrak{b}\cap\langle a_r\rangle}(M) = 0$ whenever $i-1 > r-1$, thus for all $i > r$. By induction we also have $H^i_{\mathfrak{b}}(M) = 0$ for all $i > r-1$. By 4.18 we know that $H^i_{\langle a_r\rangle}(M) = 0$ for all $i > 1$. So the above sequences show that indeed $H^i_{\mathfrak{a}}(M) = 0$ for all $i > r$. $\square$

4.20. **Reminder.** Let $\mathfrak{a}$ be finitely generated. Then, the *arithmetic rank of* $\mathfrak{a}$ is defined as the minimal number of elements of R which generate an ideal radically equal to $\mathfrak{a}$. Thus

$$\operatorname{ara}(\mathfrak{a}) := \min\{r \in \mathbb{N}_0 \mid \exists\, a_1, \ldots, a_r \in R : \sqrt{\langle a_1, \ldots, a_r \rangle} = \sqrt{\mathfrak{a}}\}.$$

Observe that $\operatorname{ara}(\mathfrak{a}) = 0$ is equivalent to $\sqrt{\mathfrak{a}} = \sqrt{0}$.

4.21. **Theorem.** *Let R be Noetherian and let M be an R-module. Then*

$$H^i_{\mathfrak{a}}(M) = 0 \;\; \textit{for all} \;\; i > \operatorname{ara}(\mathfrak{a}).$$

Proof. Let $r := \operatorname{ara}(\mathfrak{a})$. We thus find elements $a_1, \ldots, a_r \in R$ such that $\sqrt{\mathfrak{a}} = \sqrt{\langle a_1, \ldots, a_r \rangle}$. By 4.17 a) and 4.19 we therefore obtain $H^i_{\mathfrak{a}}(M) = H^i_{\langle a_1, \ldots, a_r \rangle}(M) = 0$ for all $i > r$. $\square$

The above result is an algebraic version of a result shown by Hartshorne (cf. [H2]).

5. Localization and Finiteness

We first show that "taking local cohomology commutes with localization" (cf. 5.6). Then, we show that "the first non-finitely generated local cohomology module has finitely many associated primes" (cf. 5.11). As an application of this we prove *Faltings' Local-Global Principle for finiteness dimensions* (cf. 5.14).

5.0. **Notation.** Throughout this chapter, let R be a ring and let $\mathfrak{a} \subseteq R$ be an ideal.

5.1. **Proposition.** *Let R be Noetherian, let $S \subseteq R$ be multiplicatively closed and let I be an injective R-module. Then $S^{-1}I$ is an injective $S^{-1}R$-module.*

Proof. Let $\mathfrak{b} \subseteq S^{-1}R$ be an ideal and let $h : \mathfrak{b} \to S^{-1}I$ be a homomorphism of $S^{-1}R$-modules. By the Baer Criterion (cf. 2.8) we must find an element $e \in S^{-1}I$ such that $h(b) = be$ for all $b \in \mathfrak{b}$. Let $\eta : R \to S^{-1}R$ be the canonical ring homomorphism defined by $x \mapsto \frac{sx}{s}$ for all $x \in R$ and any $s \in S$. Let $\mathfrak{a} := \eta^{-1}(\mathfrak{b}) \subseteq R$. Then it is immediate that $\mathfrak{b} = \mathfrak{a}S^{-1}R$. As R is Noetherian, we find elements $a_1, \ldots, a_r$ in R with $\mathfrak{a} = \langle a_1, \ldots, a_r \rangle$.

Now, we consider $S^{-1}I$ as an R-module by means of η and introduce the homomorphism of R-modules $\tau : I \to S^{-1}I$ given by $x \mapsto \frac{sx}{s}$ for all $x \in I$ and any $s \in S$. Then, for each $i \in \{1, \ldots, r\}$ we find elements $b_i \in I, t_i \in S$ such that $h(\eta(a_i)) = \frac{1}{t_i}\tau(b_i)$. Let $t := \prod_{i=1}^{r} t_i \in S$ and $c_i := \prod_{j \neq i} t_j b_i \in I$ for $i \in \{1, \ldots, r\}$. Then clearly $th(\eta(a_i)) = \frac{t}{t_i}\tau(b_i) = \prod_{j \neq i} t_j \tau(b_i) = \tau(c_i)$ for all $i \in \{1, \ldots, r\}$.

Let $N := \sum_{i=1}^{r} Rc_i \subseteq I$. As N is a Noetherian R-module, $\mathrm{Ker}(\tau) \cap N$ is a finitely generated R-module. Write $\mathrm{Ker}(\tau) \cap N = \sum_{j=1}^{k} Rd_j$ with $d_1, \ldots, d_k \in \mathrm{Ker}(\tau) \cap N$. For each $j \in \{1, \ldots, k\}$ there is some $s_j \in S$ such that $s_j d_j = 0$. Let $s := \prod_{j=1}^{k} s_j \in S$. Then $s(\mathrm{Ker}(\tau) \cap N) = 0$. If we apply the Lemma of Artin-Rees (cf. 1.3) to the R-modules $\mathrm{Ker}(\tau) \cap N$ and N and the ideal $\langle s \rangle \subseteq R$ we find some $m_0 \in \mathbb{N}_0$ such that

$$\langle s \rangle^m N \cap (\mathrm{Ker}(\tau) \cap N) \subseteq \langle s \rangle^{m-m_0}(\mathrm{Ker}(\tau) \cap N)$$

for all $m > m_0$. Choosing $m = m_0 + 1$ we get $\langle s \rangle^m N \cap (\mathrm{Ker}(\tau) \cap N) \subseteq \langle s \rangle(\mathrm{Ker}(\tau) \cap N) = 0$, thus $s^m N \cap \mathrm{Ker}(\tau) = 0$. So, there is an isomorphism of R-modules

$$\sigma : s^m N \xrightarrow{\cong} \tau(s^m N),\ x \mapsto \tau(x).$$

Let $v := s^m t \in S$. As $h(\eta(v\mathfrak{a})) = s^m \sum_{i=1}^{r} Rth(\eta(a_i)) = s^m \sum_{i=1}^{r} R\tau(c_i) = \tau(s^m \sum_{i=1}^{r} Rc_i) = \tau(s^m N)$ we may define a homomorphism of R-modules

$$\tilde{h} : v\mathfrak{a} \to I,\ y \mapsto \sigma^{-1}(h(\eta(y))).$$

Let $i : v\mathfrak{a} \rightarrowtail R$ denote the inclusion homomorphism. As I is injective, there is a homomorphism of R-modules $\tilde{l} : R \to I$ with $\tilde{h} = \tilde{l} \circ i$. Set $e := \tau(\tilde{l}(1))$.

Now, let $b \in \mathfrak{b}$. Then, with appropriate $a \in \mathfrak{a}$ and $u \in S$ we may write $b = \frac{a}{u}$. It follows

$$\begin{aligned} h(b) &= h(\tfrac{a}{u}) = h(\tfrac{va}{vu}) = h(\tfrac{1}{vu}\eta(va)) = \tfrac{1}{vu}h(\eta(va)) \\ &= \tfrac{1}{vu}\sigma(\sigma^{-1}(h(\eta(va)))) = \tfrac{1}{vu}\sigma(\tilde{h}(va)) = \tfrac{1}{vu}\tau(\tilde{h}(va)) \\ &= \tfrac{1}{vu}\tau((\tilde{l}\circ i)(va)) = \tfrac{1}{vu}\tau(\tilde{l}(va)) = \tfrac{va}{vu}\tau(\tilde{l}(1)) = \tfrac{a}{u}e = be. \end{aligned}$$

□

To exploit this result for local cohomology, we need a few further facts on homological algebra which we shall develop now in a series of reminders and exercises.

5.2. **Exercise.** A) Let R' be a second ring, and let F be an exact additive functor from R-modules to R'-modules. Let $(M^\bullet, d^\bullet)$ be a cocomplex of R-modules. Let $i : \operatorname{Ker}(d^n) \rightarrowtail M^n$, $j : \operatorname{Im}(d^{n-1}) \rightarrowtail M^n$ and $k : \operatorname{Im}(d^{n-1}) \rightarrowtail \operatorname{Ker}(d^n)$ be the inclusion homomorphisms and let $p : \operatorname{Ker}(d^n) \twoheadrightarrow \operatorname{Ker}(d^n)/\operatorname{Im}(d^{n-1})$ be the canonical homomorphism of R-modules. Show that $F(k) : F(\operatorname{Im}(d^{n-1})) \to F(\operatorname{Ker}(d^n))$ is injective and that $\operatorname{Im}(F(i)) = \operatorname{Ker}(F(d^n))$ and $\operatorname{Im}(F(d^{n-1})) = \operatorname{Im}(F(j)) = \operatorname{Im}(F(i\circ k)) = F(i)(\operatorname{Im}(F(k)))$. Conclude that there are isomorphisms of R-modules

$$\begin{aligned} \varphi^n_{M^\bullet} : F(\operatorname{Ker}(d^n))/\operatorname{Im}(F(k)) &\xrightarrow{\cong} F(\operatorname{Ker}(d^n)/\operatorname{Im}(d^{n-1})), \\ x + \operatorname{Im}(F(k)) &\mapsto F(p)(x); \\ \psi^n_{M^\bullet} : F(\operatorname{Ker}(d^n))/\operatorname{Im}(F(k)) &\xrightarrow{\cong} \operatorname{Ker}(F(d^n))/\operatorname{Im}(F(d^{n-1})), \\ x + \operatorname{Im}(F(k)) &\mapsto F(i)(x) + \operatorname{Im}(F(d^{n-1})). \end{aligned}$$

B) Keep the hypotheses and notations of part A) to conclude (with $\gamma^{n,F}_{M^\bullet} := \psi^n_{M^\bullet}\circ(\varphi^n_{M^\bullet})^{-1}$):

a) There is an isomorphism of R'-modules

$$\gamma^{n,F}_{M^\bullet} : F(H^n(M^\bullet, d^\bullet)) \xrightarrow{\cong} H^n(F(M^\bullet), F(d^\bullet))$$

for each $n \in \mathbb{Z}$.

Show in addition, that the above isomorphisms are natural, namely:

b) If $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ is a homomorphism of cocomplexes of R-modules, we have the commutative diagram

$$\begin{tikzcd} F(H^n(M^\bullet, d^\bullet)) \arrow[r, "\gamma^{n,F}_{M^\bullet}", "\cong"'] \arrow[d, "F(H^n(h^\bullet))"'] & H^n(F(M^\bullet), F(d^\bullet)) \arrow[d, "H^n(F(h^\bullet))"] \\ F(H^n(N^\bullet, e^\bullet)) \arrow[r, "\gamma^{n,F}_{N^\bullet}", "\cong"'] & H^n(F(N^\bullet), F(e^\bullet)). \end{tikzcd}$$

5.3. **Reminder and Exercise.** A) Let R' and R'' be two further rings. Let F be an additive functor from R-modules to R'-modules and let F' be an additive functor from R'-modules to R''-modules. Then:

a) The assignment

$$F' \circ F = F'(F(\bullet)) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(F'(F(M)) \xrightarrow{F'(F(h))} F'(F(N))\big)$$

defines an additive functor from R-modules to R''-modules. If there are homomorphisms of rings f between R and R' and f' between R' and R'' such that f and f' may be composed and if F and F' are linear with respect to f and f' respectively, then $F' \circ F$ is linear with respect to the appropriate composition of f and f'.

The functor $F' \circ F = F'(F(\bullet))$ is called the *composition of F' with F.*

b) Composition of functors is associative and composition with the identity functor does not change anything.
c) If $F \sim G$ and $F' \sim G'$, then $F' \circ F \sim G' \circ G$.

B) Keep the notations and hypotheses of part A). Moreover let

$$\mathbb{I}_* : M \rightsquigarrow \mathbb{I}_M = ((I^\bullet_M, d^\bullet_M); a_M)$$

be a choice of injective resolutions of R-modules. Assume in addition that the functor F' is exact. Then, according to 5.2 B), for each $n \in \mathbb{N}_0$ there is an isomorphism of R''-modules

$$\gamma^{n,F'}_{F(I^\bullet_M)} : F'(H^n(F(I^\bullet_M), F(d^\bullet_M)) \xrightarrow{\cong} H^n(F' \circ F(I^\bullet_M), F' \circ F(d^\bullet_M)).$$

Use what is said in 5.2 B) b) and 2.13 D) to show that on use of the assignment $M \rightsquigarrow \gamma^{n,F',F}_M := \gamma^{n,F'}_{F(I^\bullet_M)}$ we can say: For each $n \in \mathbb{N}_0$, there is a natural equivalence of functors

$$\gamma^{n,F',F} : F' \circ (\mathcal{R}^n F) \xrightarrow{\cong} \mathcal{R}^n(F' \circ F).$$

5.4. **Exercise.** Let R', R'' and F and F' be as in 5.3 A). Assume that F is exact and that $F(I)$ is an injective R'-module for each injective R-module I. Let $\mathbb{I}_* : M \rightsquigarrow \mathbb{I}_M = ((I^\bullet_M, d^\bullet_M); a_M)$ be a choice of injective resolutions of R-modules. Use 2.13 D) to conclude that the assignment

$$M \rightsquigarrow \big(H^n(F' \circ F(I^\bullet_M), F' \circ F(d^\bullet_M)) \xrightarrow{\mathrm{id}} H^n(F'(F(I^\bullet_M)), F'(F(d^\bullet_M)))\big)$$

defines a natural equivalence of functors

$$\iota^{n,F,F'} : \mathcal{R}^n(F' \circ F) \xrightarrow{\cong} (\mathcal{R}^n F') \circ F$$

for each $n \in \mathbb{N}_0$.

Now, we focus our interest on local cohomology and localization. We begin with a preliminary remark.

5.5. **Remark.** Let $S \subseteq R$ be multiplicatively closed. Then the assignments

$$S^{-1}\Gamma_{\mathfrak{a}}(\bullet) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(S^{-1}\Gamma_{\mathfrak{a}}(M) \xrightarrow{S^{-1}\Gamma_{\mathfrak{a}}(h)} S^{-1}\Gamma_{\mathfrak{a}}(N)\big),$$
$$\Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}\bullet) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(\Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}M) \xrightarrow{\Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}h)} \Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}N)\big)$$

both define functors from R-modules to $S^{-1}R$-modules which are linear with respect to the canonical homomorphism of rings η_S (cf. 1.14). Indeed, these assignments may both be understood as composition of functors:

$$S^{-1}\Gamma_{\mathfrak{a}}(\bullet) = (S^{-1}\bullet) \circ \Gamma_{\mathfrak{a}}\,; \quad \Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}\bullet) = \Gamma_{\mathfrak{a}S^{-1}R} \circ (S^{-1}\bullet).$$

Now, if $\mathfrak{a}$ is finitely generated, the assignment

$$\rho_{\mathfrak{a}} : M \rightsquigarrow (\rho_{\mathfrak{a},M} : S^{-1}\Gamma_{\mathfrak{a}}(M) \xrightarrow{\cong} \Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}M))$$

in which $\rho_{\mathfrak{a},M}$ is defined according to 1.11, defines a natural equivalence of functors

$$\rho_{\mathfrak{a}} : S^{-1}\Gamma_{\mathfrak{a}}(\bullet) \xrightarrow{\cong} \Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}\bullet).$$

5.6. **Theorem.** *Let R be Noetherian and let $S \subseteq R$ be multiplicatively closed. Then, for each $n \in \mathbb{N}_0$, there is a natural equivalence of functors*

$$\rho^n_{\mathfrak{a}} : S^{-1}H^n_{\mathfrak{a}}(\bullet) \xrightarrow{\cong} H^n_{\mathfrak{a}S^{-1}R}(S^{-1}\bullet).$$

In particular, for each $n \in \mathbb{N}_0$ and each R-module M there is an isomorphism of $S^{-1}R$-modules

$$\rho^n_{\mathfrak{a},M} : S^{-1}H^n_{\mathfrak{a}}(M) \xrightarrow{\cong} H^n_{\mathfrak{a}S^{-1}R}(S^{-1}M).$$

Proof. By 5.5 and 4.12 C) c) we may form the n-th right derived transformation of $\rho_{\mathfrak{a}}$ (cf. 5.5) and obtain a natural equivalence

$$\mathcal{R}^n\rho_{\mathfrak{a}} : \mathcal{R}^n(S^{-1}\Gamma_{\mathfrak{a}}(\bullet)) \xrightarrow{\cong} \mathcal{R}^n(\Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}\bullet)).$$

If we apply 5.3 B) with $F = \Gamma_{\mathfrak{a}}$ and $F' = S^{-1}\bullet$, we get—observing 1.18 B)—a natural equivalence

$$S^{-1}H^n_{\mathfrak{a}} = (S^{-1}\bullet) \circ \mathcal{R}^n\Gamma_{\mathfrak{a}} \xrightarrow[\cong]{\gamma^{n,S^{-1}\bullet,\Gamma_{\mathfrak{a}}}} \mathcal{R}^n((S^{-1}\bullet) \circ \Gamma_{\mathfrak{a}}) = \mathcal{R}^n(S^{-1}\Gamma_{\mathfrak{a}}(\bullet)).$$

If we observe 5.1 and 1.18 B) and apply 5.4 with $F = S^{-1}\bullet$ and $F' = \Gamma_{\mathfrak{a}S^{-1}R}$ we get a natural equivalence

$$\mathcal{R}^n(\Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}\bullet)) \xrightarrow[\cong]{\iota^{n,S^{-1}\bullet,\Gamma_{\mathfrak{a}S^{-1}R}}} (\mathcal{R}^n\Gamma_{\mathfrak{a}S^{-1}R}) \circ (S^{-1}\bullet) = H^n_{\mathfrak{a}S^{-1}R}(S^{-1}\bullet).$$

Setting $\rho^n_{\mathfrak{a}} := (\iota^{n,S^{-1}\bullet,\Gamma_{\mathfrak{a}S^{-1}R}}) \circ \mathcal{R}^n\rho_{\mathfrak{a}} \circ (\gamma^{n,S^{-1}\bullet,\Gamma_{\mathfrak{a}}})$ we get our claim. □

5.7. **Remark.** Theorem 5.6 may be expressed by saying simply: Local cohomology commutes with localization. This is a special case of a far more general result, namely the so-called *Flat Base Change Property of Local Cohomology* (cf. [B-S, Chapter 4.3]).

After this rather general result of functorial nature we now shall establish the announced finiteness result for the set of associated primes of the "first non-finitely generated local cohomology module".

5.8. **Lemma.** *Let $M \xrightarrow{f} N \xrightarrow{g} P$ be an exact sequence of R-modules and let $U \subseteq P$ be a submodule. If M and U are Noetherian, then so is the submodule $g^{-1}(U)$ of N.*

Proof. Left as an exercise. $\square$

5.9. **Lemma.** *Let $M \xrightarrow{f} N \xrightarrow{g} P$ be an exact sequence of R-modules, let $U \subseteq N$ be a submodule such that $g(U) = 0$, and let $\mathfrak{p} \in \operatorname{Ass}_R(N/U) \setminus \operatorname{Ass}_R(M/f^{-1}(U))$. Then, there exists $n \in N$ such that $(0 :_R R(n+U)) = \mathfrak{p} \in \operatorname{Ass}_R(Rg(n))$.*

Proof. Let $p : N \twoheadrightarrow N/U$ be the canonical homomorphism of R-modules. There are an $n \in N$ such that $\mathfrak{p} = (0 :_R p(n)) = (0 :_R R(n+U))$ and a homomorphism of R-modules $\bar{f} : M/f^{-1}(U) \to N/U$ with $\bar{f}(m + f^{-1}(U)) = p(f(m))$ for $m \in M$. Setting $V := \bar{f}^{-1}(Rp(n))$, there is an exact sequence of R-modules

$$0 \to V \xrightarrow{\bar{f}} Rp(n) \xrightarrow{\bar{g}} P,$$

where $\bar{g}$ is defined by $\bar{g}(m+U) = g(m)$ for $m \in N$. As $\mathfrak{p} \in \operatorname{Ass}_R(Rp(n))$, we get $\mathfrak{p} \in \operatorname{Ass}_R(V) \cup \operatorname{Ass}_R(\bar{g}(Rp(n)))$ by 1.6 B) b). As $\operatorname{Ass}_R(V) \subseteq \operatorname{Ass}_R(M/f^{-1}(U)) \not\ni \mathfrak{p}$, it follows $\mathfrak{p} \in \operatorname{Ass}_R(\bar{g}(Rp(n))) = \operatorname{Ass}_R(Rg(n))$. $\square$

5.10. **Proposition.** *Let R be Noetherian and let M be a finitely generated R-module. Let $i \in \mathbb{N}_0$ be such that $H^j_{\mathfrak{a}}(M)$ is finitely generated for all $j \in \{0, \ldots, i-1\}$ and let $N \subseteq H^i_{\mathfrak{a}}(M)$ be a finitely generated submodule. Then*

$$\sharp \operatorname{Ass}_R(H^i_{\mathfrak{a}}(M)/N) < \infty.$$

Proof. (Induction on i.) The case $i = 0$ is clear as $H^0_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M) \subseteq M$ is finitely generated (cf. 1.6 B) c)). So, let $i > 0$ and set $\bar{M} := M/\Gamma_{\mathfrak{a}}(M)$. Then $H^0_{\mathfrak{a}}(\bar{M}) = \Gamma_{\mathfrak{a}}(\bar{M}) = 0$ (cf. 1.4 B) b)) and $H^k_{\mathfrak{a}}(\bar{M}) \cong H^k_{\mathfrak{a}}(M)$ for all $k > 0$ (cf. 3.12 A) b), 3.18 b)). Thus $H^j_{\mathfrak{a}}(\bar{M})$ is also finitely generated for $j \in \{0, \ldots, i-1\}$ and $H^i_{\mathfrak{a}}(\bar{M}) \cong H^i_{\mathfrak{a}}(M)$. So, we may replace M by $\bar{M}$ and hence assume that $H^0_{\mathfrak{a}}(M) = 0$. We thus find some element $y \in \mathfrak{a} \cap \operatorname{NZD}_R(M)$ (cf. 4.3). As N is finitely generated, there is some $m \in \mathbb{N}$ with $y^m N = 0$. We set $x := y^m$.

So, we may apply 3.10 C) in order to get the following commutative diagram with exact rows

$$\begin{tikzcd}
H^{i-1}_{\mathfrak{a}}(M) \arrow[r, "\varepsilon"] & H^{i-1}_{\mathfrak{a}}(M/xM) \arrow[r, "\delta"] \arrow[d] & H^i_{\mathfrak{a}}(M) \arrow[r, "x\cdot"] \arrow[d, "\rho"] & H^i_{\mathfrak{a}}(M) \arrow[d, equal] \\
0 \arrow[r] & H^{i-1}_{\mathfrak{a}}(M/xM)/\delta^{-1}(N) \arrow[r, "\bar{\delta}"] & H^i_{\mathfrak{a}}(M)/N \arrow[r, "\varphi"] & H^i_{\mathfrak{a}}(M)
\end{tikzcd}$$

in which $\varepsilon := H^{i-1}_{\mathfrak{a}}(p)$ is induced by the natural homomorphism $p : M \to M/xM$, ρ is the canonical homomorphism, δ is the connecting homomorphism, $\bar{\delta}$ is defined by $m+\delta^{-1}(N) \mapsto \delta(m)$ and φ is defined by $u+N \mapsto xu$. By 5.8 it follows, that $\delta^{-1}(N)$ is finitely generated. By 3.10 C) we have exact sequences

$$H^{j-1}_{\mathfrak{a}}(M) \to H^{j-1}_{\mathfrak{a}}(M/xM) \to H^{j}_{\mathfrak{a}}(M)$$

for all $j \in \mathbb{N}$ which show that $H^k_{\mathfrak{a}}(M/xM)$ is finitely generated for all $k < i-1$. By induction it thus follows, that $T := H^{i-1}_{\mathfrak{a}}(M/xM)/\delta^{-1}(N)$ has only finitely many associated primes. As N is finitely generated, we also have $\sharp \operatorname{Ass}_R(N) < \infty$ (cf. 1.6 B) c)). So, it suffices to show:

$$\operatorname{Ass}_R(H^i_{\mathfrak{a}}(M)/N) \subseteq \operatorname{Ass}_R(T) \cup \operatorname{Ass}_R(N).$$

To do so, let $\mathfrak{p} \in \operatorname{Ass}_R(H^i_{\mathfrak{a}}(M)/N) \setminus \operatorname{Ass}_R(T)$. By 5.9 we find an element $h \in H^i_{\mathfrak{a}}(M)$ such that $(0 :_R R\rho(h)) = \mathfrak{p} \in \operatorname{Ass}_R(Rxh)$. So, there is some $s \in R$ with $\mathfrak{p} = (0 :_R Rxsh)$.

As $x \in \mathfrak{a}$ and as $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion there is some $n \in \mathbb{N}$ with $x^n(xsh) = 0$, hence with $x^n \in (0 :_R Rxsh) = \mathfrak{p}$, and as $\mathfrak{p}$ is prime this implies $x \in \mathfrak{p} = (0 :_R R\rho(h))$. It follows $xh + N = \rho(xh) = x\rho(h) = 0$, hence $xh \in N$ and thus $xsh \in N$. As $\mathfrak{p} = (0 :_R Rxsh)$, it follows $\mathfrak{p} \in \operatorname{Ass}_R(N)$ and this proves our claim. □

5.11. **Theorem.** *Let R be Noetherian and let M be a finitely generated R-module. Let $i \in \mathbb{N}_0$ be such that $H^j_{\mathfrak{a}}(M)$ is finitely generated for all $j \in \{0, \dots, i-1\}$. Then $\sharp \operatorname{Ass}_R(H^i_{\mathfrak{a}}(M)) < \infty$.*

Proof. Apply 5.10 with $N = 0$. □

The results 5.10 and 5.11 are shown in [B-L]. Observe that if R is Noetherian and M is a finitely generated R-module, the sets $\operatorname{Ass}_R(H^i_{\mathfrak{a}}(M))$ need not be finite in general (cf. [K], [Si]).

Now, we have paved the way to attack the last main result of this section, the Local-Global Principle for finite generation of local cohomology modules. We begin with two auxiliary results.

5.12. **Lemma.** *Let R be Noetherian. Let L be an R-module such that $0 < \sharp \operatorname{Ass}_R(L) < \infty$. Assume that for each $\mathfrak{p} \in \operatorname{Ass}_R(L)$ there is some $n_{\mathfrak{p}} \in \mathbb{N}$ such that $(\mathfrak{a}^{n_{\mathfrak{p}}}L)_{\mathfrak{p}} = 0$. Let $n := \max\{n_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Ass}_R(L)\}$. Then $\mathfrak{a}^n L = 0$.*

Proof. Let $x \in L$ and let $t_1, \dots, t_r \in L$ be such that $\mathfrak{a}^n x = \sum_{i=1}^r Rt_i$. Let $\mathfrak{p} \in \operatorname{Ass}_R(L)$. By our choice of n we have $(\mathfrak{a}^n x)_{\mathfrak{p}} \subseteq (\mathfrak{a}^n L)_{\mathfrak{p}} \subseteq (\mathfrak{a}^{n_{\mathfrak{p}}} L)_{\mathfrak{p}} = 0$, hence $(\sum_{i=1}^r Rt_i)_{\mathfrak{p}} = (\mathfrak{a}^n x)_{\mathfrak{p}} = 0$. So, for each $i \in \{1, \dots, r\}$ there is some element $s_{i,\mathfrak{p}} \in R \setminus \mathfrak{p}$ with $s_{i,\mathfrak{p}} t_i = 0$. Let $s_{\mathfrak{p}} := \prod_{i=1}^r s_{i,\mathfrak{p}}$. Then $s_{\mathfrak{p}} \in R \setminus \mathfrak{p}$ and $s_{\mathfrak{p}} t_i = 0$ for $i \in \{1, \dots, r\}$. It follows $s_{\mathfrak{p}}\mathfrak{a}^n x = 0$. Now, consider the ideal $\mathfrak{b} := \sum_{\mathfrak{p} \in \operatorname{Ass}_R(L)} Rs_{\mathfrak{p}}$. Clearly $\mathfrak{b}\mathfrak{a}^n x = 0$. As $\mathfrak{b} \ni s_{\mathfrak{p}} \notin \mathfrak{p}$ we have $\mathfrak{b} \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(L)$. As the set $\operatorname{Ass}_R(L)$ is finite, prime avoidance gives $\mathfrak{b} \not\subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(L)} \mathfrak{p} = \operatorname{ZD}_R(L)$ (cf. 1.6 B) a)). So, there is some $b \in \mathfrak{b} \cap \operatorname{NZD}_R(L)$.

But now $b\mathfrak{a}^n x \subseteq \mathfrak{b}\mathfrak{a}^n x = 0$ shows that $\mathfrak{a}^n x = 0$. As $x \in L$ was arbitrary, this proves our claim. $\square$

5.13. **Lemma.** *Let R be Noetherian, let $r \in \mathbb{N}$ and let M be a finitely generated R-module. Then, the following statements are equivalent:*

(i) $H^i_{\mathfrak{a}}(M)$ *is finitely generated for all $i < r$;*
(ii) $\mathfrak{a} \subseteq \sqrt{(0 :_R H^i_{\mathfrak{a}}(M))}$ *for all $i < r$.*

Proof. "(i) $\Rightarrow$ (ii)": As $H^i_{\mathfrak{a}}(M)$ is always $\mathfrak{a}$-torsion, this implication is clear (cf. 3.12 C) a)).

"(ii) $\Rightarrow$ (i)": Assume that $\mathfrak{a} \subseteq \sqrt{(0 :_R H^i_{\mathfrak{a}}(M))}$ for all $i < r$, so that for each $i < r$ there is some $n_i \in \mathbb{N}$ with $\mathfrak{a}^{n_i} \subseteq (0 :_R H^i_{\mathfrak{a}}(M))$. Put $n := \max\{n_i \mid i \in \{0, \ldots, r-1\}\}$. Then clearly $\mathfrak{a}^n H^i_{\mathfrak{a}}(M) = 0$ for all $i < r$. By induction on r we prove that $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < r$. As $H^0_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M) \subseteq M$ is finitely generated, the case $r = 1$ is obvious and we may assume that $r > 1$ and restrict ourselves to show that $H^i_{\mathfrak{a}}(M)$ is finitely generated for $i \in \{1, \ldots, r-1\}$.

Let $\bar{M} := M/\Gamma_{\mathfrak{a}}(M)$. As observed many times we have $H^0_{\mathfrak{a}}(\bar{M}) = 0$ and $H^i_{\mathfrak{a}}(\bar{M}) \cong H^i_{\mathfrak{a}}(M)$ for all $i > 0$. In particular we have $\mathfrak{a}^n H^i_{\mathfrak{a}}(\bar{M}) = 0$ for all $i < r$. So, we may replace M by $\bar{M}$ and hence assume that $H^0_{\mathfrak{a}}(M) = 0$. But once more this means that there is some $x \in \mathfrak{a} \cap \mathrm{NZD}_R(M)$. It follows $x^n \in \mathfrak{a}^n \cap \mathrm{NZD}_R(M)$. As $x^n \in \mathfrak{a}^n$ we have $x^n H^i_{\mathfrak{a}}(M) = 0$ for all $i < r$. As $x^n \in \mathrm{NZD}_R(M)$, the sequence 3.10 C) thus gives exact sequences

$$0 \to H^{i-1}_{\mathfrak{a}}(M) \to H^{i-1}_{\mathfrak{a}}(M/x^n M) \xrightarrow{\delta^{i-1}} H^i_{\mathfrak{a}}(M) \to 0$$

for all $i \in \{1, \ldots, r-1\}$. These first show that $\mathfrak{a}^{2n} H^{i-1}_{\mathfrak{a}}(M/x^n M) = 0$ and hence $\mathfrak{a} \subseteq \sqrt{(0 :_R H^{i-1}_{\mathfrak{a}}(M/x^n M))}$ for all $i \in \{1, \ldots, r-1\}$. So, by induction, the modules $H^{i-1}_{\mathfrak{a}}(M/x^n M)$ are finitely generated for all $i \in \{1, \ldots, r-1\}$. Another use of the previous sequences shows that $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i \in \{1, \ldots, r-1\}$. $\square$

5.14. **Theorem.** *Let R be Noetherian and let $r \in \mathbb{N}$. Let M be a finitely generated R-module. Then, the following statements are equivalent:*

(i) $H^i_{\mathfrak{a}}(M)$ *is finitely generated for all $i < r$;*
(ii) The $R_{\mathfrak{p}}$-module $H^i_{\mathfrak{a}}(M)_{\mathfrak{p}}$ is finitely generated for all $i < r$ and all $\mathfrak{p} \in \mathrm{Spec}(R)$;
(iii) The $R_{\mathfrak{p}}$-module $H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is finitely generated for all $i < r$ and all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. "(i) $\Rightarrow$ (ii)": Clear by the basic properties of localization.

"(ii) $\Leftrightarrow$ (iii)": Clear by the isomorphism of $R_{\mathfrak{p}}$-modules $H^i_{\mathfrak{a}}(M)_{\mathfrak{p}} \cong H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ obtained from 5.6 applied with $S = R \setminus \mathfrak{p}$.

"(ii) $\Rightarrow$ (i)": (Induction on r.) The case $r = 1$ is obvious as $H^0_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M) \subseteq M$ is finitely generated. So, let $r > 1$. By induction we know that $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < r - 1$. So we have to show that $L := H^{r-1}_{\mathfrak{a}}(M)$ is finitely generated. By 5.13 it is equivalent to show that $\mathfrak{a} \subseteq \sqrt{(0 :_R L)}$, hence, to find an $n \in \mathbb{N}$ with $\mathfrak{a}^n L = 0$. By 5.11 we see that $\sharp \operatorname{Ass}_R(L) < \infty$. If $L = 0$, our claim is clear. Let $L \neq 0$ and let $\mathfrak{p} \in \operatorname{Ass}_R(L)$. By our hypothesis, $L_{\mathfrak{p}}$ is finitely generated as an $R_{\mathfrak{p}}$-module. As L is $\mathfrak{a}$-torsion, $L_{\mathfrak{p}}$ is $\mathfrak{a}R_{\mathfrak{p}}$-torsion. So there is some $n_{\mathfrak{p}} \in \mathbb{N}$ with $(\mathfrak{a}^{n_{\mathfrak{p}}} L)_{\mathfrak{p}} = \mathfrak{a}^{n_{\mathfrak{p}}} R_{\mathfrak{p}} L_{\mathfrak{p}} = (\mathfrak{a}R_{\mathfrak{p}})^{n_{\mathfrak{p}}} L_{\mathfrak{p}} = 0$ (cf. 3.12 C) a)). Now, we conclude by 5.12. $\square$

The previous result first has been proved by Faltings [F]. It can be formulated as a "Local-Global Principle" as follows.

5.15. **Definition.** The $\mathfrak{a}$-*finiteness dimension of* M is defined by

$$f_{\mathfrak{a}}(M) := \inf\{r \in \mathbb{N} \mid H^r_{\mathfrak{a}}(M) \text{ is not finitely generated}\}.$$

5.16. **Corollary.** *Let R be Noetherian and let M be a finitely generated R-module. Then*

$$f_{\mathfrak{a}}(M) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Proof. This follows easily from the equivalence "(i) $\Leftrightarrow$ (iii)" in 5.14. $\square$

5.17. **Exercise.** Let R and M be as in 5.16. Prove that

$$f_{\mathfrak{a}}(M) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Var}(\mathfrak{a}) \cap \operatorname{Var}(0 :_R M)\}.$$

5.18. **Remark.** Let R be Noetherian and let M be a finitely generated R-module. Let

$$\lambda_{\mathfrak{a}}(M) := \inf\{\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{ht}((\mathfrak{a} + \mathfrak{p})/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(\mathfrak{a})\}.$$

(Here, $\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is defined according to 4.5 C), whereas the *height* of an ideal $\mathfrak{b}$ of a ring A is defined by

$$\operatorname{ht}(\mathfrak{b}) := \inf\{\dim(A_{\mathfrak{q}}) \mid \mathfrak{q} \in \operatorname{Var}(\mathfrak{b})\}.)$$

Then it is known that

$$f_{\mathfrak{a}}(M) \leq \lambda_{\mathfrak{a}}(M).$$

Moreover, if R is a homomorphic image of a regular ring, equality holds. So, whenever R is a homomorphic image of a polynomial ring $K[X_1, \ldots, X_n]$ over a field, we have $f_{\mathfrak{a}}(M) = \lambda_{\mathfrak{a}}(M)$ (cf. [B-S, Chapter 9]). This finiteness criterion for the modules $H^i_{\mathfrak{a}}(M)$ is due to Grothendieck (cf. [G]).

6. Equations for Algebraic Varieties

The arithmetic rank of the vanishing ideal of an algebraic set is the minimal number of equations needed to define this set (cf. 6.6). We use this to translate the vanishing result 4.21 to the context of algebraic sets.

6.0. **Notation.** Throughout this chapter, let K be an algebraically closed field, let $n \in \mathbb{N}_0$ and let $R := K[X_1, \dots, X_n]$ denote the polynomial ring in n indeterminates over K.

6.1. **Reminder and Exercise.** A) We define the *set of zeros* of a set $\mathcal{M} \subseteq R$ by

$$V(\mathcal{M}) := \{p = (p_1, \dots, p_n) \in K^n \mid \forall f \in \mathcal{M} : f(p) = 0\}.$$

Note that $V(\emptyset) = K^n$. A set $V \subseteq K^n$ is said to be an *algebraic set* if there is a set of polynomials $\mathcal{M} \subseteq R$ such that $V = V(\mathcal{M})$.

B) For $\mathcal{M} \subseteq R$ let $\langle \mathcal{M} \rangle \subseteq R$ as usual denote the ideal generated by $\mathcal{M}$. Note that $\langle \emptyset \rangle = 0$. Then one has:

a) $V(\langle \mathcal{M} \rangle) = V(\mathcal{M})$ for $\mathcal{M} \subseteq R$.

An important consequence of this is:

b) Each algebraic set $V \subseteq K^n$ is the set of zeros of an ideal, hence of the form $V = V(\mathfrak{a})$, where $\mathfrak{a} \subseteq R$ is an ideal.

C) If $r \in \mathbb{N}$ and $f_1, \dots, f_r$ are finitely many polynomials in R we write

$$V(f_1, \dots, f_r) := V(\{f_1, \dots, f_r\}).$$

As R is Noetherian, it follows from B) b) that each algebraic set $V \subseteq K^n$ is the set of zeros of finitely many polynomials, hence of the form $V = V(f_1, \dots, f_r)$ where $f_1, \dots, f_r \in R$.

If $V = V(f_1, \dots, f_r)$ with $f_1, \dots, f_r \in R$, we say that $f_1 = 0, \dots, f_r = 0$ are *defining equations for* V and that V *can be defined by* r *equations.*

6.2. **Reminder and Exercise.** A) The following hold:

a) $V(0) = K^n$ and $V(R) = \emptyset$;
b) $V(\sum_{i \in I} \mathfrak{a}_i) = \bigcap_{i \in I} V(\mathfrak{a}_i)$, where $(\mathfrak{a}_i)_{i \in I}$ is a family of ideals of R;
c) $V(\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_r) = V(\mathfrak{a}_1) \cup \dots \cup V(\mathfrak{a}_r)$, where $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are finitely many ideals of R.

B) The above statements imply:

a) K^n and $\emptyset$ are algebraic sets;
b) Intersections of arbitrary families of algebraic sets are algebraic sets;
c) Unions of finite families of algebraic sets are algebraic sets.

But this means:

d) On K^n we can define a topology whose closed sets are precisely the algebraic sets $V \subseteq K^n$.

C) The topology mentioned in B) d) is called the *Zariski topology.* We write $\mathbb{A}^n(K)$ for the space K^n furnished with the Zariski topology. So keep in mind that instead of algebraic sets in K^n we can speak of closed sets in $\mathbb{A}^n(K)$.

D) A closed set $V \subseteq \mathbb{A}^n(K)$ is said to be *irreducible* if it is non-empty and if it may not be written as the union of two proper closed subsets. Irreducible closed sets $V \subseteq \mathbb{A}^n(K)$ are called *affine algebraic varieties.*

6.3. **Examples and Exercise.** A) Let $p \in \mathbb{A}^n(K)$. Then $\{p\}$ is an irreducible closed subset of $\mathbb{A}^n(K)$.

B) The closed subsets of $\mathbb{A}^1(K)$ are precisely $\mathbb{A}^1(K)$ and all finite sets, including $\emptyset$.

C) The Zariski topology on $\mathbb{A}^2(K)$ is not the same as the product topology on $\mathbb{A}^1(K) \times \mathbb{A}^1(K)$.

6.4. **Reminder and Exercise.** A) Let $V \subseteq \mathbb{A}^n(K)$ be closed. Then the set

$$I(V) := \{f \in R \mid \forall\, p \in V : f(p) = 0\}$$

is a radical ideal of R, i.e. an ideal which is equal to its radical. It is called the *vanishing ideal of* V. Note that $I(\emptyset) = R$.

B) It is easy to verify that

a) $V(I(V)) = V$ for each closed set $V \subseteq \mathbb{A}^n(K)$.

On the other hand, the *Nullstellensatz* (cf. [H1, Theorem 1.3A]) says:

b) $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for each ideal $\mathfrak{a} \subseteq R$.

C) Finally, let us notice: A closed set $V \subseteq \mathbb{A}^n(K)$ is irreducible if and only if $I(V) \subseteq R$ is a prime ideal.

Indeed, let $I(V)$ be prime and assume that V is not irreducible. As $I(V) \neq R$ we have $V \neq \emptyset$. So, we can write $V = V_1 \cup V_2$ with V_1, V_2 closed in $\mathbb{A}^n(K)$ and V_1, $V_2 \subsetneq V$. Clearly $I(V_1)$, $I(V_2) \supseteq I(V)$. By B) a) it follows that $I(V_1), I(V_2) \supsetneq I(V)$. So, there are polynomials $f_1 \in I(V_1) \setminus I(V)$ and $f_2 \in I(V_2) \setminus I(V)$. Obviously $f_1 f_2 \in I(V_1 \cup V_2) = I(V)$. As $I(V)$ is prime, this is a contradiction.

Conversely, assume that $I(V)$ is not prime. If $I(V) = R$, then B) a) and 6.2 A) a) show that $V = \emptyset$ and so V is not irreducible. So, let $I(V) \subsetneq R$. We then find polynomials $f_1, f_2 \in R \setminus I(V)$ with $f_1 f_2 \in I(V)$. Let $V_1 := V(I(V) + \langle f_1 \rangle)$ and $V_2 := V(I(V) + \langle f_2 \rangle)$. Then clearly V_1, V_2 are closed in $\mathbb{A}^n(K)$ with V_1, $V_2 \subsetneq V$. Now, let $p \in V$. As $f_1 f_2 \in I(V)$ we have $f_1(p) f_2(p) = 0$, thus $f_1(p) = 0$ or $f_2(p) = 0$. In the first case $p \in V_1$, in the second $p \in V_2$. This shows that $V = V_1 \cup V_2$. So V is not irreducible.

6.5. **Examples and Exercise.** A) Let $p = (p_1, \ldots, p_n) \in \mathbb{A}^n(K)$. Then $I(\{p\}) = \langle X_1 - p_1, \ldots, X_n - p_n \rangle$.

B) $I(\mathbb{A}^n(K)) = 0$. So $\mathbb{A}^n(K)$ is irreducible.

6.6. **Theorem.** *Let $V \subseteq \mathbb{A}^n(K)$ be a closed set. Then, the minimal number of equations needed to define V is equal to the arithmetic rank of $I(V)$, thus:*

$$\min\{r \in \mathbb{N}_0 \mid \exists\, f_1, \ldots, f_r \in R : V = V(f_1, \ldots, f_r)\} = \operatorname{ara}(I(V)).$$

Proof. On use of 6.4 B) a), b) we obtain

$$V = V(f_1, \ldots, f_r) \Rightarrow I(V) = I(V(f_1, \ldots, f_r)) = \sqrt{\langle f_1, \ldots, f_r \rangle}$$

for $f_1, \ldots, f_r \in R$ and moreover

$$I(V) = \sqrt{\langle f_1, \ldots, f_r \rangle} \Rightarrow V = V(I(V)) = V\big(\sqrt{\langle f_1, \ldots, f_r \rangle}\big).$$

By the relation $V\big(\sqrt{\langle f_1, \ldots, f_r \rangle}\big) = V(f_1, \ldots, f_r)$ we thus obtain

$$V = V(f_1, \ldots, f_r) \Leftrightarrow I(V) = \sqrt{\langle f_1, \ldots, f_r \rangle}.$$

This proves our claim. □

6.7. **Exercise.** A) Let M be a finitely generated module over a Noetherian ring A. Let $(x_1, \ldots, x_r)$ be an M-sequence in A such that $\langle x_1, \ldots, x_r \rangle M \neq M$. Show that

$$H^i_{\langle x_1, \ldots, x_r \rangle}(M) \neq 0 \Leftrightarrow i = r.$$

(Here is a hint: Use 4.3, 4.19 and 4.7.)

B) Let A be a ring, let $r \in \mathbb{N}_0$ and consider the polynomial ring $A[X_1, \ldots, X_r]$. Show that $(X_1, \ldots, X_r)$ is an $A[X_1, \ldots, X_r]$-sequence. Conclude that, if A is Noetherian,

$$H^i_{\langle X_1, \ldots, X_r \rangle}(A[X_1, \ldots, X_r]) \neq 0 \Leftrightarrow i = r.$$

6.8. **Example and Exercise.** A) Let $n = 4$, and consider the closed set

$$V := V(X_1X_3, X_1X_4, X_2X_3, X_2X_4) \subseteq \mathbb{A}^4(K).$$

The relations

$$(X_2X_3)^2 = (X_1X_4 + X_2X_3)X_2X_3 - X_1X_3X_2X_4,$$

$$(X_1X_4)^2 = (X_1X_4 + X_2X_3)X_1X_4 - X_1X_3X_2X_4$$

and 6.4 B) a), b) show that

$$V = V(X_1X_3, X_1X_4 + X_2X_3, X_2X_4),$$

hence V can be defined by 3 equations. But is it possible to define V by 2 equations only?

B) By 6.6 the above question finds its answer if we can determine $\mathrm{ara}(I(V))$. Again by 6.6 we see already that $\mathrm{ara}(I(V)) \leq 3$. On the other hand we have by the Nullstellensatz 6.4 B) b)

$$I(V) = \sqrt{\langle X_1X_3, X_1X_4, X_2X_3, X_2X_4\rangle} = \sqrt{\langle X_1, X_2\rangle\langle X_3, X_4\rangle}$$
$$= \langle X_1, X_2\rangle \cap \langle X_3, X_4\rangle.$$

The Mayer-Vietoris sequence with respect to the ideals $\langle X_1, X_2\rangle$ and $\langle X_3, X_4\rangle$ associated to the R-module R gives us an exact sequence

$$H^3_{I(V)}(R) \to H^4_{\langle X_1,X_2,X_3,X_4\rangle}(R) \to H^4_{\langle X_1,X_2\rangle}(R) \oplus H^4_{\langle X_3,X_4\rangle}(R)$$

(cf. 4.16 B)). By 4.19 and by 6.7 B) we see that $H^3_{I(V)}(R) \neq 0$. By 4.21 it follows $\mathrm{ara}(I(V)) \geq 3$ and hence $\mathrm{ara}(I(V)) = 3$. So one needs exactly 3 equations to define V.

C) Let $V_1 = V(X_1, X_2)$ and $V_2 = V(X_3, X_4)$. Then, V_1 and V_2 are planes in the 4-space $\mathbb{A}^4(K)$. Moreover

$$V_1 \cap V_2 = V(X_1, X_2, X_3, X_4) = \{0\}$$

and

$$V_1 \cup V_2 = V(\langle X_1, X_2\rangle \cap \langle X_3, X_4\rangle) = V(I(V)) = V$$

(cf. B), 6.2 A) c), 6.4 B) a)). So V is the union of two planes which intersect each other precisely at the origin. Unlike to what we have met in linear algebra, the "surface" $V \subseteq \mathbb{A}^4(K)$ cannot be defined by 2 equations. But V clearly is not irreducible. There are indeed examples of irreducible surfaces in $\mathbb{A}^4(K)$ which cannot be defined by two equations (cf. [B-S, Example 4.3.7]).

7. Extending Regular Functions

Let U be a (Zariski-) open subset of an affine algebraic variety V. We show that a certain first local cohomology module is the obstacle for the extension of regular functions from U to V (cf. 7.8, 7.9).

7.0. **Notation.** Throughout this chapter, let K be an algebraically closed field, let $n \in \mathbb{N}_0$, let $R := K[X_1, \ldots, X_n]$ denote the polynomial ring in n indeterminates over K and let $V \subseteq \mathbb{A}^n(K)$ be an affine algebraic variety.

7.1. **Convention and Notation.** Let $W \subseteq \mathbb{A}^n(K)$ and let $p \in W$. We always furnish W with its induced topology and use the following notations:

$$\begin{aligned} \mathbb{U}_W &:= \{U \subseteq W \mid U \text{ is open in } W\}; \\ \mathbb{U}_{W,p} &:= \{U \in \mathbb{U}_W \mid p \in U\}; \\ \mathring{\mathbb{U}}_W &:= \mathbb{U}_W \setminus \{\emptyset\}. \end{aligned}$$

7.2. **Reminder and Exercise.** A) As V is irreducible, one has:

a) If $U_1, U_2 \in \mathring{\mathbb{U}}_V$, then $U_1 \cap U_2 \in \mathring{\mathbb{U}}_V$.

Let $f, g \in R$. Then $\{p \in \mathbb{A}^n(K) \mid f(p) \neq g(p)\} = \mathbb{A}^n(K) \setminus V(f-g) \in \mathbb{U}_{\mathbb{A}^n(K)}$. As a consequence one has:

b) If $f, g \in R$, then $\{p \in V \mid f(p) \neq g(p)\} \in \mathbb{U}_V$.

As a consequence of a) and b) one gets:

c) Let $f, g \in R$ and let $U \in \mathring{\mathbb{U}}_V$ be such that $f(p) = g(p)$ for all $p \in U$. Then $f(p) = g(p)$ for all $p \in V$.

B) A function $h : V \to K$ is called a *polynomial function on* V if there is a polynomial $f \in R$ such that $h(p) = f(p)$ for all $p \in V$. The set

$$K[V] := \{h : V \to K \mid h \text{ is a polynomial function}\}$$

obviously is a subring of the ring of all functions $V \to K$. As K is algebraically closed we have $\sharp K = \infty$, and therefore we may identify

a) $K[\mathbb{A}^n(K)] = R = K[X_1, \ldots, X_n]$.

Now we can conclude:

b) There is a surjective ring homomorphism

$$\bullet \restriction_V : R \to K[V],\ f \mapsto f \restriction_V,$$

the *restriction homomorphism.* This homomorphism satisfies $\operatorname{Ker}(\bullet \restriction_V) = I(V)$. In particular $K[V] \cong R/I(V)$, so that

$$K[V] = K[X_1 \restriction_V, \ldots, X_n \restriction_V]$$

is a finitely generated extension domain of K and hence Noetherian.

The ring $K[V]$ of polynomial functions on V is also called the *coordinate ring of* V as it is generated over K by the restrictions $X_1\upharpoonright_V, \ldots, X_n\upharpoonright_V$ of the *coordinate functions* $X_1, \ldots, X_n \in K[\mathbb{A}^n(K)]$.

C) The quotient field of $K[V]$,

$$K(V) := \mathrm{Quot}(K[V]) = (K[V] \setminus \{0\})^{-1}K[V],$$

is called the *function field of* V. There is a canonical embedding $K[V] \subseteq K(V)$.

Now, for $f \in K[V]$ we set

$$U(f) = U_V(f) := \{p \in V \mid f(p) \neq 0\}.$$

As $U_V(f) = \{p \in V \mid f(p) \neq 0(p)\}$ we conclude from A) b):

a) If $f \in K[V] \setminus \{0\}$, then $U_V(f) \in \mathring{\mathbb{U}}_V$.

Now, fix some element $r \in K(V)$. We define:

$$C(r) := \big\{(f, g) \in K[V]^2 \mid g \neq 0 \wedge r = \tfrac{f}{g}\big\};$$
$$\mathcal{D}(r) := \bigcup_{(f,g)\in C(r)} U_V(g) \in \mathring{\mathbb{U}}_V.$$

By A) a) one easily sees:

b) Let $r \in K(V)$ and let $(f, g), (f', g') \in C(r)$. Then $U_V(g) \cap U_V(g') \neq \emptyset$ and for each $p \in U_V(g) \cap U_V(g')$ we have $\frac{f(p)}{g(p)} = \frac{f'(p)}{g'(p)}$.

So, we may define a function $\tilde{r} : \mathcal{D}(r) \to K$ such that $\tilde{r}(p) = \frac{f(p)}{g(p)}$ whenever $(f, g) \in C(r)$ with $p \in U_V(g)$. Moreover, by A) c) it follows in the previous notations:

c) Let $r, s \in K(V)$ such that there is some $U \in \mathring{\mathbb{U}}_V$ with $U \subseteq \mathcal{D}(r) \cap \mathcal{D}(s)$ and $\tilde{r}(p) = \tilde{s}(p)$ for all $p \in U$. Then $r = s$ (hence $\mathcal{D}(r) = \mathcal{D}(s)$ and $\tilde{r} = \tilde{s}$).

This means that r is determined by the function $\tilde{r}$. So we may identify r with $\tilde{r}$ and consider $r : \mathcal{D}(r) \to K$ as a function. If this is done, we say that r is a *rational function on* V. Then, $\mathcal{D}(r) \in \mathring{\mathbb{U}}_V$ is the *range of definition of* r and $V \setminus \mathcal{D}(r)$ is the *set of poles of* r.

7.3. **Reminder and Exercise.** A) If $Z \subseteq V$ is a closed subset, we may introduce the set

$$I_V(Z) := \{f \in K[V] \mid \forall p \in Z : f(p) = 0\}.$$

This is a radical ideal of $K[V]$, called the *vanishing ideal of* Z *in* V. Comparing with our earlier notion of vanishing ideal we can say:

a) $I_{\mathbb{A}^n(K)}(Z) = I(Z)$;
b) If $(\bullet\upharpoonright_V) : R \to K[V]$ denotes the restriction homomorphism, then $I(Z) = (\bullet\upharpoonright_V)^{-1}(I_V(Z))$ and $I_V(Z) = (\bullet\upharpoonright_V)(I(Z))$.

B) For $p \in V$, we can say:

a) $I_V(\{p\})$ is the kernel of the surjective ring homomorphism

$$\bullet(p) : K[V] \to K,\ f \mapsto f(p).$$

As a consequence of a), we get:

b) $I_V(\{p\})$ is a maximal ideal of $K[V]$.

Now, one can define the *local ring of* V *at* p by

$$\mathcal{O}_{V,p} := K[V]_{I_V(\{p\})} \subseteq K(V).$$

The unique *maximal ideal of* $\mathcal{O}_{V,p}$ is given by

$$\mathfrak{m}_{V,p} := I_V(\{p\})\mathcal{O}_{V,p}.$$

It is easy to verify:

c) Let $r \in K(V)$ be a rational function on V. Then, $p \in \mathcal{D}(r)$ if and only if $r \in \mathcal{O}_{V,p}$.

7.4. **Reminder and Exercise.** A) Let $U \in \mathring{\mathbb{U}}_V$. A *regular function on* U is a function $f : U \to K$ which is the restriction of a rational function on V. More precisely: $f : U \to K$ is a regular function on U if there is some $r \in K(V)$ such that $U \subseteq \mathcal{D}(r)$ and $f(p) = r(p)$ for all $p \in U$, i.e. if $f = r\upharpoonright_U$.

It follows easily from 7.2 C) c):

a) If $f : U \to K$ is a regular function and if $r \in K(V)$ is such that $U \subseteq \mathcal{D}(r)$ and $f = r\upharpoonright_U$, then r is uniquely determined by f.

So, in view of a) we may identify a regular function $f : U \to K$ with the uniquely determined rational function $r \in K(V)$ for which $U \subseteq \mathcal{D}(r)$ and $f = r\upharpoonright_U$. If we write

$$\mathcal{O}(U) := \{f : U \to K \mid f \text{ is a regular function}\}$$

we thus get

b) $\mathcal{O}(U) = \{r \in K(V) \mid U \subseteq \mathcal{D}(r)\}$.

It is easy to see that

c) $\mathcal{O}(U)$ is a subring of $K(V)$.

We thus refer to $\mathcal{O}(U)$ as the *ring of regular functions on* U. By 7.3 B) c) we may write:

d) $\mathcal{O}(U) = \bigcap_{p \in U} \mathcal{O}_{V,p}$.

B) Next, let us note:

a) If $\mathfrak{m} \subseteq K[V]$ is a maximal ideal then there is some $p \in V$ with $\mathfrak{m} = I_V(\{p\})$.

(Hint: Let $\tilde{\mathfrak{m}} := (\bullet\restriction_V)^{-1}(\mathfrak{m})$, where $\bullet\restriction_V$ is as in 7.2 B) b). Use 6.4 B) b) to show that there is a $p \in V(\tilde{\mathfrak{m}})$. Use $I(V) \subseteq \tilde{\mathfrak{m}}$ to show that $p \in V$. Show $\mathfrak{m} \subseteq I_V(\{p\})$ and use the maximality of $\mathfrak{m}$.)

As a consequence of this we see:

b) $\operatorname{Max}(K[V]) = \{I_V(\{p\}) \mid p \in V\}$.

This finally leads to

c) $K[V] = \mathcal{O}(V)$.

(Hint: Use b), A) d) and the fact that for any domain A we have $A = \bigcap_{\mathfrak{m}\in\operatorname{Max}(A)} A_{\mathfrak{m}}$.)

7.5. **Proposition.** *Let $U \in \mathring{\mathbb{U}}_V$ and let $\mathfrak{a} := I_V(V \setminus U)$. Then,*

$$\mathcal{O}(U) = \bigcup_{m\in\mathbb{N}} (K[V] :_{K(V)} \mathfrak{a}^m).$$

Proof. "$\subseteq$": Let $f \in \mathcal{O}(U)$. For each $p \in U$ we then have $f \in \mathcal{O}_{V,p} = K[V]_{I_V(\{p\})} \subseteq K(V)$ (cf. 7.4 A) d)). So, for each $p \in U$ there is some $g_p \in K[V] \setminus I_V(\{p\})$ such that $g_p f \in K[V]$. Now, consider the ideal $\mathfrak{b} := \sum_{p\in U} K[V]g_p$. Then clearly $\mathfrak{b}f \subseteq K[V]$.

Let $\bullet\restriction_V\colon R \to K[V]$ denote the restriction homomorphism (cf. 7.2 B) b)). Then, for each $p \in U$ we find a polynomial $\tilde{g}_p \in R$ with $\tilde{g}_p\restriction_V = g_p$. Consider the ideal $\tilde{\mathfrak{b}} := \sum_{p\in U} R\tilde{g}_p + I(V) \subseteq R$. As $I(V) = \operatorname{Ker}(\bullet\restriction_V)$ (cf. 7.2 B) b)) and as $\bullet\restriction_V$ is surjective, we may write $\tilde{\mathfrak{b}} = (\bullet\restriction_V)^{-1}(\mathfrak{b})$ and $\tilde{\mathfrak{b}}\restriction_V = (\bullet\restriction_V)(\tilde{\mathfrak{b}}) = \mathfrak{b}$.

Now, let $W := V(\tilde{\mathfrak{b}}) \subseteq \mathbb{A}^n(K)$. As $I(V) \subseteq \tilde{\mathfrak{b}}$, we have $W \subseteq V(I(V)) = V$ (cf. 6.4 B) a)). We want to show that $W \subseteq V \setminus U$. Assume to the contrary, that $W \nsubseteq V \setminus U$. As $W \subseteq V$ it follows that there is some $p \in W \cap U$. But then $\tilde{g}_p \in \tilde{\mathfrak{b}}$ shows that $g_p(p) = \tilde{g}_p(p) = 0$ and thus leads to the contradiction that $g_p \in I_V(\{p\})$. Therefore we have indeed $W \subseteq V \setminus U$. This implies in particular that $I(V \setminus U) \subseteq I(W)$.

But now, by 7.3 A) b) we can write

$$I(V \setminus U) = (\bullet\restriction_V)^{-1}(I_V(V \setminus U)) = (\bullet\restriction_V)^{-1}(\mathfrak{a}) =: \tilde{\mathfrak{a}}.$$

By 6.4 B) b) we have $I(W) = I(V(\tilde{\mathfrak{b}})) = \sqrt{\tilde{\mathfrak{b}}}$. Altogether we obtain $\tilde{\mathfrak{a}} \subseteq \sqrt{\tilde{\mathfrak{b}}}$. So, there is some $m \in \mathbb{N}$ with $\tilde{\mathfrak{a}}^m \subseteq \tilde{\mathfrak{b}}$. As $(\bullet\restriction_V)$ is surjective, it follows $\tilde{\mathfrak{a}}\restriction_V = \mathfrak{a}$ and hence $\mathfrak{a}^m = \tilde{\mathfrak{a}}^m\restriction_V \subseteq \tilde{\mathfrak{b}}\restriction_V = \mathfrak{b}$. So $\mathfrak{a}^m \subseteq \mathfrak{b}$ and hence $\mathfrak{a}^m f \subseteq \mathfrak{b}f \subseteq K[V]$ so that $f \in \bigcup_{m\in\mathbb{N}}(K[V] :_{K(V)} \mathfrak{a}^m)$.

"$\supseteq$": Let $f \in \bigcup_{m\in\mathbb{N}}(K[V] :_{K(V)} \mathfrak{a}^m)$. So, we find some $m \in \mathbb{N}$ with $\mathfrak{a}^m f \in K[V]$. Let $p \in U$. Again, let $\tilde{\mathfrak{a}} := (\bullet\restriction_V)^{-1}(\mathfrak{a}) = I(V \setminus U)$ (cf. 7.3 A) b)), so that $V(\tilde{\mathfrak{a}}) = V(I(V \setminus U)) = V \setminus U$. As $p \in U$ we have $p \notin V(\tilde{\mathfrak{a}})$ and so there is some polynomial $\tilde{h} \in \tilde{\mathfrak{a}}$ with $\tilde{h}(p) \neq 0$. Let $h := \tilde{h}\restriction_V$. Then $h \in \mathfrak{a}$ and

$h(p) \neq 0$. So, $h^m \in \mathfrak{a}^m \setminus I_V(\{p\})$. As $h^m f \in K[V]$ we find some $k \in K[V]$ with $h^m f = k$. It follows by $h^m \notin I_V(\{p\})$ that $f = \frac{k}{h^m} \in K[V]_{I_V(\{p\})} = \mathcal{O}_{V,p}$ (cf. 7.3 B)).

As $p \in U$ was arbitrary we get $f \in \bigcap_{p \in U} \mathcal{O}_{V,p}$. By 7.4 A) d) we therefore have $f \in \mathcal{O}(U)$. □

7.6. **Remark.** Let $U \in \mathring{\mathbb{U}}_V$. Then, by 7.4 B) c) and by 7.4 A) b) we get

a) $K[V] = \mathcal{O}(V) \subseteq \mathcal{O}(U)$.

In particular, we may consider $\mathcal{O}(U)$ as a module over the ring $\mathcal{O}(V) = K[V]$. Moreover, using the definition of regular function we may consider the inclusion map $\mathcal{O}(V) \hookrightarrow \mathcal{O}(U)$ as the *restriction homomorphism,* thus:

b) $\mathrm{res}_{VU} : \mathcal{O}(V) \overset{\mathrm{incl}}{\hookrightarrow} \mathcal{O}(U),\ f \mapsto f\restriction_U$.

7.7. **Lemma.** *Let $U \in \mathring{\mathbb{U}}_V$ and let $\mathfrak{a} := I_V(V \setminus U)$. Then:*

a) $\mathfrak{a} \neq 0$ and $H^0_{\mathfrak{a}}(\mathcal{O}(V)) = 0$.

b) $H^i_{\mathfrak{a}}(K(V)) = 0$ for all $i \in \mathbb{N}_0$.

Proof. "a)": As $U \neq \emptyset$, we have $V \setminus U \subsetneq V$, hence

$$V(I(V \setminus U)) = V \setminus U \neq V = V(I(V))$$

(cf. 6.4 B) a)) and hence $I(V \setminus U) \neq I(V)$. The relation $V \setminus U \subsetneq V$ implies $I(V \setminus U) \supsetneq I(V)$. Let $(\bullet\restriction_V) : R \to K[V]$ be the restriction homomorphism, so that $I(V) = \mathrm{Ker}(\bullet\restriction_V)$ (cf. 7.2 B) b)). Then, in view of 7.3 A) b) we have $\mathfrak{a} = I_V(V \setminus U) = (\bullet\restriction_V)(I(V \setminus U)) \neq 0$, hence $\mathfrak{a} \neq 0$.

As $\mathcal{O}(V) = K[V]$ is a domain, it follows $\Gamma_{\mathfrak{a}}(\mathcal{O}(V)) = 0$, hence $H^0_{\mathfrak{a}}(\mathcal{O}(V)) = 0$.

"b)": By 4.7 it suffices to show that $\mathfrak{a}K(V) = K(V)$. The inclusion "$\subseteq$" is trivial. Let $g \in K(V)$. By statement a), there is an element $f \in \mathfrak{a}$ such that $f \neq 0$. Hence, $g = \frac{fg}{f} = f \cdot \frac{g}{f} \in K(V)$. □

7.8. **Theorem.** *Let $U \in \mathring{\mathbb{U}}_V$ and let $\mathfrak{a} := I_V(V \setminus U)$. Then, there is a short exact sequence of $\mathcal{O}(V)$-modules*

$$0 \longrightarrow \mathcal{O}(V) \xrightarrow{\mathrm{res}_{VU}} \mathcal{O}(U) \longrightarrow H^1_{\mathfrak{a}}(\mathcal{O}(V)) \longrightarrow 0.$$

Proof. In view of 7.6 b) we may consider res_{VU} as the inclusion map $i : \mathcal{O}(V) \hookrightarrow \mathcal{O}(U)$. Consider the short exact sequence of $\mathcal{O}(V)$-modules

$$(1) \qquad 0 \longrightarrow \mathcal{O}(V) \xrightarrow{i} \mathcal{O}(U) \xrightarrow{p} \mathcal{O}(U)/\mathcal{O}(V) \longrightarrow 0$$

in which p is the canonical homomorphism. As $\mathcal{O}(V) = K[V]$ we may use 7.5 to write

$$\mathcal{O}(U)/\mathcal{O}(V) = \Big(\bigcup_{m \in \mathbb{N}} (K[V] :_{K(V)} \mathfrak{a}^m)\Big)/K[V].$$

This first shows that $\mathcal{O}(U)/\mathcal{O}(V)$ is $\mathfrak{a}$-torsion, thus (cf. 3.4):

$$\mathcal{O}(U)/\mathcal{O}(V) = \Gamma_{\mathfrak{a}}(\mathcal{O}(U)/\mathcal{O}(V)) = H^0_{\mathfrak{a}}(\mathcal{O}(U)/\mathcal{O}(V)). \tag{2}$$

Next, consider the short exact sequence of $\mathcal{O}(V)$-modules

$$0 \longrightarrow \mathcal{O}(U) \xrightarrow{j} K(V) \xrightarrow{q} K(V)/\mathcal{O}(U) \longrightarrow 0 \tag{3}$$

in which j is the inclusion map and q is the canonical homomorphism. We may use 7.5 to write

$$K(V)/\mathcal{O}(U) = K(V)/\big(\bigcup_{m\in\mathbb{N}}(K[V] :_{K(V)} \mathfrak{a}^m)\big).$$

Using the right hand side of this equality, it is easy to see that

$$K(V)/\mathcal{O}(U)$$

has no $\mathfrak{a}$-torsion, thus

$$H^0_{\mathfrak{a}}(K(V)/\mathcal{O}(U)) = \Gamma_{\mathfrak{a}}(K(V)/\mathcal{O}(U)) = 0.$$

By 7.7 b) we have $H^i_{\mathfrak{a}}(K(V)) = 0$ for all $i \in \mathbb{N}_0$. Now, the long exact cohomology sequence with respect to $\mathfrak{a}$ and associated to (3) shows that $H^0_{\mathfrak{a}}(\mathcal{O}(U)) = 0$ and gives us an isomorphism of $\mathcal{O}(V)$-modules

$$H^0_{\mathfrak{a}}(K(V)/\mathcal{O}(U)) \xrightarrow{\cong} H^1_{\mathfrak{a}}(\mathcal{O}(U)).$$

It follows that $H^1_{\mathfrak{a}}(\mathcal{O}(U)) = 0$. But now, the long exact cohomology sequence with respect to $\mathfrak{a}$ and associated to (1) gives us an isomorphism

$$H^0_{\mathfrak{a}}(\mathcal{O}(U)/\mathcal{O}(V)) \xrightarrow{\cong} H^1_{\mathfrak{a}}(\mathcal{O}(V)).$$

In view of (2) we thus get $H^1_{\mathfrak{a}}(\mathcal{O}(V)) \cong \mathcal{O}(U)/\mathcal{O}(V)$. But now, the sequence (1) gives our claim. □

7.9. **Corollary.** *Let $U \in \mathring{\mathbb{U}}_V$ and let $\mathfrak{a} := I_V(V \setminus U)$. Then, the following statements are equivalent:*

(i) *Each regular function $f \in \mathcal{O}(U)$ may be extended to a regular function $\tilde{f} \in \mathcal{O}(V)$;*
(ii) *The restriction map* $\mathrm{res}_{VU} : \mathcal{O}(V) \to \mathcal{O}(U)$ *is surjective;*
(iii) *The restriction map* $\mathrm{res}_{VU} : \mathcal{O}(V) \to \mathcal{O}(U)$ *is an isomorphism;*
(iv) $H^1_{\mathfrak{a}}(\mathcal{O}(V)) = 0$*;*
(v) $\mathrm{grade}_{\mathcal{O}(V)}(\mathfrak{a}) > 1$.

Proof. “(i) ⇔ (ii) ⇔ (iii) ⇔ (iv)” are clear from 7.8.

“(v) ⇒ (iv)” follows from 4.6

“(iv) ⇒ (v)” is clear from 4.6 and 7.7 a). □

To those readers who would like to see examples which illustrate the results of this section, we recommend [B-S, Example and Exercises 2.3.7–10].

8. Local Cohomology and Gradings

We now look at the case where our ring R carries a $\mathbb{Z}$-grading, $\mathfrak{a} \subseteq R$ is a graded ideal and M is a graded R-module. We shall establish the fact, that the local cohomology modules $H^i_{\mathfrak{a}}(M)$ carry a grading in this situation.

We start with a series of reminders and exercises about graded rings and modules and homomorphisms between them.

8.1. **Reminder and Exercise.** A) Let R be a ring. A *($\mathbb{Z}$-)grading on R* is a family $(R_n)_{n\in\mathbb{Z}}$ of additive subgroups $R_n \subseteq R$ such that the following properties hold:

(G1) For any $m, n \in \mathbb{Z}$, if $a \in R_m$ and $b \in R_n$, then $ab \in R_{m+n}$;

(G2) $R = \bigoplus_{n\in\mathbb{Z}} R_n$.

B) A ring R together with a grading $(R_n)_{n\in\mathbb{Z}}$ on R is called a *($\mathbb{Z}$-)graded ring.* We usually express this by saying "$R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a graded ring". If we just say "R is a graded ring", then we mean that we are given a grading on R, and we will denote this grading usually by $(R_n)_{n\in\mathbb{Z}}$.

C) Let R be a graded ring. It then follows from the axioms (G1) and (G2) that $1 \in R_0$. From this, we immediately obtain that R_0 is a subring of R and that R_n is an R_0-submodule of R for all $n \in \mathbb{Z}$.

D) Let R be a graded ring. Assume that there are finitely many elements $y_1, \ldots, y_r \in R$ such that $R = R_0[y_1, \ldots, y_r]$. We may write each of these elements as a sum of finitely many homogeneous elements, i.e. for $i \in \{1, \ldots, r\}$ we have $y_i = \sum_{i=1}^{n_i} y_{ij}$ with $n_i \in \mathbb{N}_0$ and $d_{ij} \in \mathbb{Z}$ and $y_{ij} \in R_{d_{ij}}$ for $j \in \{1, \ldots, n_i\}$. Then clearly $R = R_0[y_{11}, \ldots, y_{1n_1}, y_{21}, \ldots, y_{rn_r}]$. Therefore: As an R_0-algebra, R is finitely generated over R_0 if and only if R is generated by finitely many homogeneous elements of R.

8.2. **Reminder and Exercise.** A) Let R be a graded ring. Let M be an R-module. A *grading on M* is a family $(M_n)_{n\in\mathbb{Z}}$ of additive subgroups $M_n \subseteq M$ such that the following properties hold:

($\overline{\mathrm{G1}}$) For any $m, n \in \mathbb{Z}$, if $a \in R_m$ and $b \in M_n$, then $ab \in M_{m+n}$;

($\overline{\mathrm{G2}}$) $M = \bigoplus_{n\in\mathbb{Z}} M_n$.

B) An R-module M together with a grading $(M_n)_{n\in\mathbb{Z}}$ is called a *graded R-module.* We usually express this by saying "$M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a graded R-module". If we just say "M is a graded R-module", then we mean that we are given a grading on M, and we will denote this grading usually by $(M_n)_{n\in\mathbb{Z}}$.

C) Let M be a graded R-module. If $n \in \mathbb{Z}$, then clearly M_n is an R_0-submodule of M, and this R_0-module is called the *n-th graded component of M.*

D) Let M be a graded R-module. An element $x \in M$ is said to be *homogeneous,* if there is some $n \in \mathbb{Z}$ such that $x \in M_n$. If in this situation $x \neq 0$, then n

is unique and is called the *degree of* x. We then also write $\deg(x) := n$. We write $M^{\text{hom}} := \bigcup_{n\in\mathbb{Z}} M_n$ for the set of homogeneous elements of M.

E) Let M be a graded R-module. If $x \in M$, then there is a unique presentation $x = \sum_n x_n$ with $x_n \in M_n$ for $n \in \mathbb{Z}$ and $\sharp\{n \in \mathbb{Z} \mid x_n \neq 0\} < \infty$. In this situation, for $n \in \mathbb{Z}$ we call x_n the *n-th homogeneous component of x,* and the above presentation of x is called the *decomposition of x in homogeneous components.*

F) If M is a graded R-module and if $\bullet_n$ is used to denote n-th homogeneous components, for $n \in \mathbb{Z}$, $x, y \in M$ and $a \in R$ we have:

a) $(x + y)_n = x_n + y_n$;
b) $(ax)_n = \sum_{i+j=n} a_i x_j$.

G) Clearly R may be viewed as a graded R-module in a canonical way. So, the notions and observation made in C)–F) apply to R in the obvious way.

H) Let M be a finitely generated graded R-module. Then, there are finitely many elements $m_1, \ldots, m_r \in M$ such that $M = \sum_{i=1}^r Rm_i$. Decomposing each of these in homogeneous components, for $i \in \{1, \ldots, r\}$ we may write $m_i = \sum_{j=1}^{n_i} m_{ij}$ with $n_i \in \mathbb{N}_0$ and $d_{ij} \in \mathbb{Z}$ and $m_{ij} \in M_{d_{ij}}$ for $j \in \{1, \ldots, n_i\}$. It follows $M = \sum_{i=1}^r \sum_{j=1}^{n_i} Rm_{ij}$. So we can say: A graded R-module M is finitely generated if and only if it is generated by finitely many homogeneous elements.

8.3. **Reminder and Exercise.** A) Let R be a graded ring and let M be a graded R-module. Let $N \subseteq M$ be a submodule. Then, the following statements are equivalent:

(i) $(M_n \cap N)_{n\in\mathbb{Z}}$ is a grading on N;
(ii) N is generated by homogeneous elements;
(iii) If $y \in N$, then all homogeneous components of y belong to N.

If N satisfies the above conditions (i)–(iii), it is called a *graded submodule of* M. In this situation we always write $N_n := M_n \cap N$ for $n \in \mathbb{Z}$ and say that $N = \bigoplus_{n\in\mathbb{Z}} N_n$ is a graded submodule of M.

B) It is easy to check that for a family $(N^{(i)})_{i\in I}$ of graded submodules of M one has:

a) $\sum_{i\in I} N^{(i)}$ is a graded submodule of M such that $\left(\sum_{i\in I} N^{(i)}\right)_n = \sum_{i\in I} N_n^{(i)}$ for all $n \in \mathbb{Z}$;
b) $\bigcap_{i\in I} N^{(i)}$ is a graded submodule of M such that $\left(\bigcap_{i\in I} N^{(i)}\right)_n = \bigcap_{i\in I} N_n^{(i)}$ for all $n \in \mathbb{Z}$.

C) Let $N \subseteq M$ be a graded submodule. Then it is easy to check that the family $((M_n + N)/N)_{n\in\mathbb{Z}}$ is a grading on the R-module M/N. This grading is called the *canonical grading on M/N.* We write $M/N = \bigoplus_{n\in\mathbb{Z}}(M/N)_n$ with $(M/N)_n := (M_n+N)/N$ for $n \in \mathbb{Z}$ and keep in mind the natural isomorphisms of R_0-modules $(M/N)_n \cong M_n/N_n$ for $n \in \mathbb{Z}$.

D) A *graded ideal of* R is an ideal $\mathfrak{a} \subseteq R$ which is graded as a submodule of R. In this situation it is easy to verify that the canonical grading $((R_n+\mathfrak{a})/\mathfrak{a})_{n\in\mathbb{Z}} = ((R/\mathfrak{a})_n)_{n\in\mathbb{Z}}$ turns $R/\mathfrak{a}$ into a graded ring.

E) If $\mathfrak{a} \subseteq R$ is a graded ideal, then $\mathfrak{a}M$ is a graded submodule of M.

8.4. **Reminder.** A) Let R be a graded ring. Let M and N be graded R-modules. A homomorphism $h : M \to N$ of R-modules is said to be *graded* or a *homomorphism of graded R-modules* if $h(M_n) \subseteq N_n$ for all $n \in \mathbb{Z}$. If $n \in \mathbb{Z}$ and if $h : M \to N$ is a homomorphism of graded R-modules, the map

$$h_n := h\restriction_{M_n} : M_n \to N_n$$

is a homomorphism of R_0-modules and is called the *n-th graded component of h.*

B) If M is a graded R-module, the map $\mathrm{id}_M : M \to M$ is a homomorphism of graded R-modules. The composition of two homomorphisms of graded R-modules is again a homomorphism of graded R-modules.

C) A homomorphism of graded R-modules $h : M \to N$ is called an *isomorphism of graded R-modules* if there is a homomorphism of graded R-modules $l : N \to M$ with $h \circ l = \mathrm{id}_N$ and $l \circ h = \mathrm{id}_M$. Clearly, a homomorphism of graded R-modules is an isomorphism of graded R-modules if and only if it is an isomorphism of R-modules.

D) For each graded submodule $L \subseteq M$ the inclusion homomorphism $L \to M$ and the canonical homomorphism $M \to M/L$ are homomorphisms of graded R-modules.

E) Let $h : M \to N$ be a homomorphism of graded R-modules. Then clearly $\mathrm{Ker}(h) \subseteq M$ and $\mathrm{Im}(h) \subseteq N$ are graded submodules. So, in view of 8.3 C) we see that $\mathrm{Coker}(h) = N/\mathrm{Im}(h)$ is a graded R-module.

F) Clearly:

a) The sum $h + l$ of two homomorphisms of graded R-modules $h, l : M \to N$ is again a homomorphism of graded R-modules.
b) If $h : M \to N$ is a homomorphism of graded R-modules and $x \in R_0$, then xh is a homomorphism of graded R-modules.

8.5. **Reminder.** A) Let R be a graded ring, let M be a graded R-module and let $r \in \mathbb{Z}$. For $n \in \mathbb{Z}$ let $M(r)_n := M_{n+r}$. Then, $M(r) = \bigoplus_{n\in\mathbb{Z}} M(r)_n$ obviously is a graded R-module. This graded R-module is called the *r-th shift of M.* It is the same R-module as M but furnished with the grading which is obtained by "shifting the original grading $(M_n)_{n\in\mathbb{Z}}$ by r places to the left".

B) It is immediately checked that the identity map $\mathrm{id}_M : M \to M$ gives isomorphisms of graded R-modules

$$M(0) \cong M;$$
$$M(r+s) \cong M(r)(s) \cong M(s)(r) \text{ for } r, s \in \mathbb{Z}.$$

Finally we can say: If $N \subseteq M$ is a graded submodule, then $N(r)$ is a graded submodule of $M(r)$ and $(M/N)(r) = M(r)/N(r)$.

C) Let N be a second graded R-module and let $h : M \to N$ be a homomorphism of graded R-modules. If $h(r) : M(r) \to N(r)$ denotes the same map as h, then $h(r)$ is a homomorphism of graded R-modules and is called the *r-th shift of h*. For $n \in \mathbb{Z}$ we have $h(r)_n = h_{r+n} : M_{r+n} \to N_{r+n}$. Clearly, in these notations $h(0) = h$ and $h(r+s) = h(r)(s) = h(s)(r)$ for all $r, s \in \mathbb{Z}$.

D) We have $\mathrm{id}_M(r) = \mathrm{id}_{M(r)}$, and if $h : M \to N$ and $l : N \to P$ are homomorphisms of graded R-modules, we have $(l \circ h)(r) = l(r) \circ h(r)$. Moreover, if $h : M \to N$ is a homomorphism of graded R-modules, then $\mathrm{Ker}(h(r)) = \mathrm{Ker}(h)(r)$ and $\mathrm{Im}(h(r)) = \mathrm{Im}(h)(r)$.

E) Let $x \in R_r$. If $h : M \to N$ is a homomorphism of graded R-modules, then we have a homomorphism of graded R-modules $xh : M \to N(r),\ y \mapsto xh(y)$ (cf. 1.12 A)). Applying this with $t \in \mathbb{Z}$ and $h = \mathrm{id}_{M(t)}$ we get the graded multiplication homomorphisms $x\cdot : M(t) \to M(t+r),\ y \mapsto xy$.

8.6. **Definition and Remark.** A) Let R be a graded ring. By an *additive (covariant) functor (in the category) of graded R-modules* we mean an assignment

$$F = F(\bullet) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(F(M) \xrightarrow{F(h)} F(N)\big)$$

which to each graded R-module M assigns a graded R-module $F(M)$ and to each homomorphism of graded R-modules $h : M \to N$ assigns a homomorphism of graded R-modules $F(h) : F(M) \to F(N)$, such that the following properties hold:

(*A1) $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$ for each graded R-module M;
(*A2) $F(h \circ l) = F(h) \circ F(l)$, whenever $l : M \to N$ and $h : N \to P$ are homomorphisms of graded R-modules;
(*A3) $F(h) + F(l) = F(h + l)$, whenever $h, l : M \to N$ are homomorphisms of graded R-modules.

B) Let $r \in \mathbb{Z}$. Then, by

$$\bullet(r) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(M(r) \xrightarrow{h(r)} N(r)\big)$$

there is defined an additive functor of graded R-modules, the *functor of taking r-th shifts.*

C) We can define the composition of two additive functors of graded R-modules analogously to 5.3 A). Therefore, if F is an additive functor of graded R-modules, for any $r \in \mathbb{Z}$ the compositions $F \circ (\bullet(r))$ and $(\bullet(r)) \circ F$ are additive functors of graded R-modules.

D) By a *linear (covariant) functor (in the category) of graded R-modules* we mean an additive functor of R-modules

$$F = F(\bullet) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(F(M) \xrightarrow{F(h)} F(N)\big)$$

such that the following properties hold:

(*A4) $F \circ (\bullet(r)) = (\bullet(r)) \circ F$ for each $r \in \mathbb{Z}$;
(*A5) $F(xh) = xF(h)$ for $r \in \mathbb{Z}$, $x \in R_r$ and for each homomorphism of graded R-modules $h : M \to N$.

E) By an *additive (covariant) functor from (the category of) graded R-modules to (the category of) R_0-modules* we mean an assignment

$$F = F(\bullet) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(F(M) \xrightarrow{F(h)} F(N)\big)$$

which, to each graded R-module M assigns an R_0-module $F(M)$ and to each homomorphism of graded R-modules h assigns a homomorphism of R_0-modules $F(h)$ such that the following properties hold:

(*$\mathrm{A}_0$1) $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$ for each graded R-module M;
(*$\mathrm{A}_0$2) $F(h \circ l) = F(h) \circ F(l)$, whenever $l : M \to N$ and $h : N \to P$ are homomorphisms of graded R-modules;
(*$\mathrm{A}_0$3) $F(h) + F(l) = F(h + l)$, whenever $h, l : M \to N$ are homomorphisms of graded R-modules;

F) By a *linear (covariant) functor from (the category of) graded R-modules to (the category of) R_0-modules* we mean an additive functor from graded R-modules to R_0-modules

$$F = F(\bullet) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(F(M) \xrightarrow{F(h)} F(N)\big)$$

such that the following property holds:

(*$A_0$4) $F(xh) = xF(h)$, whenever $h : M \to N$ is a homomorphism of graded R-modules and $x \in R_0$.

G) For functors as defined in A) or in E), the notion of (left and right) exactness is defined in the obvious way (cf. 1.16).

8.7. **Examples.** A) Let R be a graded ring and let $r \in \mathbb{Z}$. Then, the functor of taking r-th shifts

$$\bullet(r) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(M(r) \xrightarrow{h(r)} N(r)\big)$$

(cf. 8.6 B)) is an exact linear functor of graded R-modules.

B) The *functor of taking r-th graded components*

$$\bullet_r : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(M_r \xrightarrow{h_r} N_r\big)$$

is an exact linear functor from graded R-modules to R_0-modules.

8.8. **Exercise and Definition.** A) Let R be a graded ring and let $\mathfrak{a} \subseteq R$ be a graded ideal. Show that, if M is a graded R-module, $\Gamma_{\mathfrak{a}}(M)$ is a graded submodule of M and $\Gamma_{\mathfrak{a}}(M(r)) = \Gamma_{\mathfrak{a}}(M)(r)$ for all $r \in \mathbb{Z}$. Show that by the assignment

$$\big(M \xrightarrow{h} N\big) \rightsquigarrow \big(\Gamma_{\mathfrak{a}}(M) \xrightarrow{\Gamma_{\mathfrak{a}}(h)} \Gamma_{\mathfrak{a}}(N)\big)$$

a left exact linear functor of graded R-modules is defined.

B) The linear functor of graded R-modules described in part A) is denoted by ${}^*\Gamma_{\mathfrak{a}}$ and called the *graded $\mathfrak{a}$-torsion functor.*

8.9. **Definition and Remark.** A) Let R be a graded ring, let F be an additive functor of R-modules and let ${}^*F$ be an additive functor of graded R-modules. F is said to be **equivalent to ${}^*F$ (by means of ι)* if for each graded R-module M there is an isomorphism of R-modules $\iota^M : {}^*F(M) \xrightarrow{\cong} F(M)$ such that for each homomorphism of graded R-modules $h : M \to N$ one has the commutative diagram

$$\begin{array}{ccc} {}^*F(M) & \xrightarrow{{}^*F(h)} & {}^*F(N) \\ \cong \big\downarrow \iota^M & & \cong \big\downarrow \iota^N \\ F(M) & \xrightarrow{F(h)} & F(N). \end{array}$$

B) If F is left exact and *equivalent to ${}^*F$, it is easy to see that ${}^*F$ is left exact too.

8.10. **Example.** By what we said in 8.8 we may conclude: If R is a graded ring and $\mathfrak{a} \subseteq R$ is a graded ideal, the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ is *equivalent to the graded $\mathfrak{a}$-torsion functor ${}^*\Gamma_{\mathfrak{a}}$.

(Hint: If M is a graded R-module, set $\iota^M : {}^*\Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(M),\ m \mapsto m$.)

8.11. **Reminder and Exercise.** Let R be a graded ring. Let $h : N \to M$ and $l : M \to P$ be homomorphisms of graded R-modules. Show that the following statements are equivalent:

- (i) $N \xrightarrow{h} M \xrightarrow{l} P \to 0$ is exact and there is a homomorphism of graded R-modules $r : M \to N$ such that $r \circ h = \mathrm{id}_N$;
- (ii) $0 \to N \xrightarrow{h} M \xrightarrow{l} P$ is exact and there is a homomorphism of graded R-modules $s : P \to M$ such that $l \circ s = \mathrm{id}_P$;
- (iii) $l \circ h = 0$ and there are homomorphisms of graded R-modules $r : M \to N$ and $s : P \to M$ such that $r \circ h = \mathrm{id}_N$, $l \circ s = \mathrm{id}_P$ and $s \circ l + h \circ r = \mathrm{id}_M$;

(iv) There is a commutative diagram of graded R-modules with exact first row

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "h"] \arrow[d, equal] & M \arrow[r, "l"] & P \arrow[r] \arrow[d, equal] & 0 \\ & N \arrow[r, "i"] & N \oplus P \arrow[r, "p"] \arrow[u, "\cong"'] & P & \end{tikzcd}$$

in which i and p are the canonical homomorphisms defined by $x \mapsto (x, 0)$ and $(x, y) \mapsto y$ respectively;

(v) The sequence $0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ is exact and splits.

8.12. **Definition, Exercise and Remark.** A) Let R be a graded ring. A graded R-module I is said to be **injective* if for each monomorphism of graded R-modules $i : N \rightarrowtail M$ and each homomorphism of graded R-modules $h : N \to I$ there is a homomorphism of graded R-modules $l : M \to I$ such that $h = l \circ i$. In diagrammatic form:

$$\begin{tikzcd} & I \\ N \arrow[ur, "h"] \arrow[r, tail, "i"] & M. \arrow[u, dotted, "l"'] \end{tikzcd}$$

Observe that (cf. 8.11, 2.7 and 8.5):

a) If $\mathbb{S} : 0 \to I \xrightarrow{h} M \xrightarrow{l} P \to 0$ is an exact sequence of graded R-modules in which I is *injective, then $\mathbb{S}$ splits.
b) If I is a *injective submodule of a graded R-module M, there is an isomorphism of graded R-modules $M \cong I \oplus M/I$.
c) If I is a *injective submodule of a *injective R-module J, then J/I is *injective.
d) If I is a *injective R-module and $r \in \mathbb{Z}$, then $I(r)$ is *injective.

B) By following the proof of 2.8, prove the graded version of the Baer Criterion: A graded R-module I is *injective, if for every $r \in \mathbb{Z}$, every graded ideal $\mathfrak{a} \subseteq R$ and every homomorphism of graded R-modules $h : \mathfrak{a} \to I(r)$ there is an element $e \in I(r)_0$ such that $h(a) = ae$ for all $a \in \mathfrak{a}$.

8.13. **Remark.** Also in the graded case, one has a version of the Lemma of Eckmann-Schopf (cf. [Br-H, Theorem 3.6.2]): For each graded R-module M there is a *injective R-module I together with a monomorphism of graded R-modules $M \rightarrowtail I$. So, each graded R-module M is a graded submodule of a *injective R-module I.

8.14. **Reminder and Remark.** A) Let R be a graded ring. By a *cocomplex of graded R-modules* we mean a sequence of graded R-modules and of homomorphisms of graded R-modules

$$\cdots \to M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} M^{i+2} \to \cdots$$

such that, disregarding the gradings, $(M^\bullet, d^\bullet)$ is a cocomplex of R-modules.

B) Let $(M^\bullet, d^\bullet)$, $(N^\bullet, e^\bullet)$ be two cocomplexes of graded R-modules. By a *homomorphism of cocomplexes of graded R-modules* $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ we mean a homomorphism of cocomplexes of R-modules $h^\bullet = (h^i)_{i\in\mathbb{Z}} : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ (cf. 2.1 B)) such that h^i is a homomorphism of graded R-modules for all $i \in \mathbb{Z}$.

C) What has been said in 2.1 C) holds literally for homomorphisms of cocomplexes of graded R-modules.

D) If $(M^\bullet, d^\bullet)$ and $(N^\bullet, e^\bullet)$ are cocomplexes of graded R-modules, if $h^\bullet, l^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ are homomorphisms of cocomplexes of graded R-modules and if $a \in R_0$, the homomorphisms of cocomplexes of R-modules

$$h^\bullet + l^\bullet, ah^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$$

(cf. 2.1 D)) are in fact homomorphisms of cocomplexes of graded R-modules.

E) Let $r \in \mathbb{Z}$. If $(M^\bullet, d^\bullet)$ is a cocomplex of graded R-modules, then so is

$$(M^\bullet, d^\bullet)(r) : \cdots \to M^{i-1}(r) \xrightarrow{d^{i-1}(r)} M^i(r) \to \cdots .$$

This cocomplex is called the *r-th shift of* $(M^\bullet, d^\bullet)$ and is sometimes denoted by $(M^\bullet(r), d^\bullet(r))$. If $(N^\bullet, e^\bullet)$ is a second cocomplex of graded R-modules, and if $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, d^\bullet)$ is a homomorphism of cocomplexes of graded R-modules, then so is

$$h^\bullet(r) := (h^i(r))_{i\in\mathbb{Z}} : (M^\bullet, d^\bullet)(r) \to (N^\bullet, e^\bullet)(r),$$

the *r-th shift of $h^\bullet$*.

F) Let $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ be a homomorphism of cocomplexes of graded R-modules. Let $r \in \mathbb{Z}$ and let $x \in R_r$. Then one has the homomorphism of cocomplexes of graded R-modules

$$xh^\bullet = (xh^i)_{i\in\mathbb{Z}} : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)(r)$$

(cf. 2.1 D), 8.5 E)).

8.15. **Reminder.** Let R be a graded ring. By a *linear (covariant) functor from (the category of) cocomplexes of graded R-modules to (the category of) graded R-modules* we mean an assignment

$$F = F(\bullet) : \big((M^\bullet, d^\bullet) \xrightarrow{h^\bullet} (N^\bullet, e^\bullet)\big) \rightsquigarrow \big(F(M^\bullet, d^\bullet) \xrightarrow{F(h^\bullet)} F(N^\bullet, e^\bullet)\big)$$

which to a cocomplex of graded R-modules $(M^\bullet, d^\bullet)$ assigns a graded R-module $F(M^\bullet, d^\bullet)$ and to a homomorphism $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ of cocomplexes of graded R-modules assigns a homomorphism of graded R-modules $F(h^\bullet) : F(M^\bullet, d^\bullet) \to F(N^\bullet, e^\bullet)$ such that the following properties hold:

(*A1$^\bullet$) $F(\mathrm{id}_{(M^\bullet,d^\bullet)}) = \mathrm{id}_{F(M^\bullet,d^\bullet)}$ for each cocomplex of graded R-modules $(M^\bullet, d^\bullet)$;

(*A2$^\bullet$) $F(h^\bullet \circ l^\bullet) = F(h^\bullet) \circ F(l^\bullet)$, whenever $l^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ and $h^\bullet : (N^\bullet, e^\bullet) \to (P^\bullet, f^\bullet)$ are homomorphisms of cocomplexes of graded R-modules;

(*A3$^\bullet$) $F(h^\bullet) + F(l^\bullet) = F(h^\bullet + l^\bullet)$, whenever $h^\bullet, l^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ are homomorphisms of cocomplexes of graded R-modules;

(*A4$^\bullet$) $F((M^\bullet, d^\bullet)(r)) = (F(M^\bullet, d^\bullet))(r)$ for each cocomplex of graded R-modules $(M^\bullet, d^\bullet)$ and each $r \in \mathbb{Z}$, and $F(h^\bullet(r)) = (F(h^\bullet))(r)$ for each homomorphism of cocomplexes of graded R-modules $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, d^\bullet)$ and each $r \in \mathbb{Z}$.

(*A5$^\bullet$) $F(xh^\bullet) = xF(h^\bullet)$ for $r \in \mathbb{Z}$, $x \in R_r$ and for each homomorphism of cocomplexes of graded R-modules $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$.

8.16. **Reminder and Remark.** A) Let R be a graded ring and let $n \in \mathbb{Z}$. Let $(M^\bullet, d^\bullet)$ be a cocomplex of graded R-modules. Then, according to 8.4 E), the n-th cohomology $H^n(M^\bullet, d^\bullet)$ of $(M^\bullet, d^\bullet)$ (cf. 2.3 A)) is a graded R-module in a canonical way. In this situation, we always furnish $H^n(M^\bullet, d^\bullet)$ with this grading.

B) Let $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ be a homomorphism of cocomplexes of graded R-modules. Then, an easy calculation shows that the induced homomorphism of R-modules $H^n(h^\bullet) : H^n(M^\bullet, d^\bullet) \to H^n(N^\bullet, e^\bullet)$ (cf. 2.3 B)) are graded, too.

C) Clearly, for each $r \in \mathbb{Z}$ we have:

a) $H^n(M^\bullet(r), d^\bullet(r)) = H^n(M^\bullet, d^\bullet)(r)$ for each cocomplex of graded R-modules $(M^\bullet, d^\bullet)$;
b) $H^n(h^\bullet(r)) = H^n(h^\bullet)(r)$ for each homomorphism of cocomplexes of graded R-modules $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$.

Moreover, statements A) a), b) and B) a) of 2.4 hold as well for cocomplexes of graded R-modules and homomorphism between such. Instead of 2.4 B) b) and in view of 8.14 F) we get for each $r \in \mathbb{Z}$ and each $x \in R_r$:

c) $H^n(xh^\bullet) = xH^n(h^\bullet) : H^n(M^\bullet, d^\bullet) \to H^n(N^\bullet, e^\bullet)(r)$,

where $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ is a homomorphism of cocomplexes of graded R-modules.

So, altogether, similar to what we said in 2.4 C), the assignment

$$H^n = H^n(\bullet) : \big((M^\bullet, d^\bullet) \xrightarrow{h^\bullet} (N^\bullet, e^\bullet)\big) \leadsto \big(H^n(M^\bullet, d^\bullet) \xrightarrow{H^n(h^\bullet)} H^n(N^\bullet, e^\bullet)\big)$$

defines a linear functor from cocomplexes of graded R-modules to graded R-modules.

8.17. **Reminder and Exercise.** A) Let R be a graded ring. Let $h^\bullet, l^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ be two homomorphisms of cocomplexes of graded R-modules. A *graded homotopy from $h^\bullet$ to $l^\bullet$* is a family $(t_i)_{i \in \mathbb{Z}}$ of homomorphisms of graded R-modules $t_i : M^i \to N^{i-1}$ which is a homotopy from $h^\bullet$ to $l^\bullet$ in the sense of 2.5 A). If such a graded homotopy exists, we say that $h^\bullet$ and $l^\bullet$ are *graded homotopic* and write $h^\bullet \overset{*}{\sim} l^\bullet$. Now, clearly, all what is said in 2.5 B), C) remains valid for graded homotopies.

B) If F is an additive functor of graded R-modules, then all what is said in 2.6 applies literally if one considers cocomplexes of graded R-modules, homomorphisms between such and graded homotopies of them. Also, if F is an additive functor from graded R-modules to R_0-modules, the statements of 2.6 hold in the "graded context".

8.18. **Remark.** A) Let R be a graded ring. Let M be a graded R-module. A *graded right resolution of* M consists of a cocomplex of graded R-modules $(E^\bullet, e^\bullet)$ and a homomorphism of graded R-modules $b : M \to E^0$ such that, disregarding the grading, $((E^\bullet, e^\bullet); b)$ is a right resolution of M (cf. 2.10 A)).

B) Let N be a second graded R-module, let $h : M \to N$ be a homomorphism of graded R-modules, let $((D^\bullet, d^\bullet); a)$ be a graded right resolution of M and let $((E^\bullet, e^\bullet); b)$ be a graded right resolution of N. A *graded (right) resolution of* h *(between* $((D^\bullet, d^\bullet); a)$ *and* $((E^\bullet, e^\bullet); b)$*)* is a homomorphism of cocomplexes of graded R-modules $h^\bullet : (D^\bullet, d^\bullet) \to (E^\bullet, e^\bullet)$ such that $h^0 \circ a = b \circ h$.

C) A **injective resolution of* M is a graded right resolution $((I^\bullet, d^\bullet); a)$ of M such that all R-modules I^i are *injective. Now, on use of 8.13 and following the hints of 2.11 one immediately sees that all the statements made in 2.11 also apply in an obvious way in the "graded context", if "injective" is replaced by "*injective".

D) Finally, let F be an additive functor of graded R-modules, or an additive functor from graded R-modules to R_0-modules. Then it is a mere matter of translation and an easy exercise to formulate and to establish the statements 2.12 A), B), C) in the "graded context".

8.19. **Reminder.** Let R be a graded ring. Let F and G be additive functors of graded R-modules or from graded R-modules to R_0-modules. A *natural transformation from* F *to* G is an assignment

$$\beta : M \leadsto (\beta_M : F(M) \to G(M))$$

which to each graded R-module M assigns a homomorphism of graded R-modules resp. of R_0-modules $\beta_M : F(M) \to G(M)$ such that for each homomorphism of graded R-modules $h : M \to N$ we have $G(h) \circ \beta_M = \beta_N \circ F(h)$. Such a natural transformation is called a *natural equivalence from* F *to* G if $\beta_M : F(M) \to G(M)$ is an isomorphism for all graded R-modules M. Clearly all the further notations and statements made in 3.1 apply in the "graded context".

8.20. **Reminder and Exercise.** A) Let R be a graded ring and let F be an additive functor of graded R-modules or from graded R-modules to R_0-modules. As in 2.13 A) one can choose a *injective resolution $\mathbb{I}_M = ((I_M^\bullet, d_M^\bullet); a_M)$ for each graded R-module M and thus define the notion of a *choice of *injective resolutions (of graded R-modules).* Then, one can formulate and perform the "graded analogues" of all what is done in 2.13 A)–D) to end up with the notion of the *n-th right derived functor of* F*,* denoted by $\mathcal{R}^n F$. Here, $\mathcal{R}^n F$

is an additive functor of graded R-modules resp. from graded R-modules to R_0-modules. If F is moreover linear, then so is $\mathcal{R}^n F$.

B) As in 3.3 A), for each graded R-module M one can define a homomorphism of graded R-modules resp. of R_0-modules

$$\alpha_M^F : F(M) \to \mathcal{R}^0 F(M) = \mathrm{Ker}(F(d^0)),\ m \mapsto F(a)(m),$$

where $\mathbb{I} = ((I^\bullet, d^\bullet); a)$ is a *injective resolution of M. Again, these homomorphisms α_M^F constitute a natural transformation $\alpha^F : F \to \mathcal{R}^0 F$. Moreover, again as in 3.3 B), we can say: If F is left exact, then $\alpha_M^F : F(M) \to \mathcal{R}^0 F(M)$ is an isomorphism for each graded R-module M. So, as in 3.3 C) we convene to identify F and $\mathcal{R}^0 F$ by means of the natural equivalence α^F, whenever the functor F is left exact.

C) As in 2.15 we obtain:

a) If I is a *injective R-module, then $\mathcal{R}^n F(I) = 0$ for all $n > 0$;
b) If F is exact, then $\mathcal{R}^n F(M) = 0$ for each graded R-module M and all $n > 0$.

D) As expected, 3.5, 3.6 and hence 3.8 “translate” to the “graded context”: So, finally there is again an assignment

$$\delta_*^{\bullet,F} : \big(\$: 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0\big) \mapsto \big(\delta_\$^{n,F} : \mathcal{R}^n F(P) \to \mathcal{R}^{n+1} F(N)\big)_{n\in\mathbb{N}_0},$$

which to each short exact sequence of graded R-modules

$$\$: 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$$

assigns a family $(\delta_\$^{n,F} : \mathcal{R}^n F(P) \to \mathcal{R}^{n+1} F(N))_{n\in\mathbb{N}_0}$ of homomorphisms of graded R-modules resp. of R_0-modules such that we have an exact sequence (cf. 3.8 B))

$$\begin{array}{lllll}
0 \longrightarrow & \mathcal{R}^0 F(N) & \xrightarrow{\mathcal{R}^0 F(h)} \mathcal{R}^0 F(M) & \xrightarrow{\mathcal{R}^0 F(l)} & \mathcal{R}^0 F(P) \\
\xrightarrow{\delta_\$^{0,F}} & \mathcal{R}^1 F(N) & \xrightarrow{\mathcal{R}^1 F(h)} \mathcal{R}^1 F(M) & \longrightarrow & \cdots \\
 & & \qquad\qquad \cdots & \longrightarrow & \mathcal{R}^{i-1} F(P) \\
\xrightarrow{\delta_\$^{i-1,F}} & \mathcal{R}^i F(N) & \xrightarrow{\mathcal{R}^i F(h)} \mathcal{R}^i F(M) & \xrightarrow{\mathcal{R}^i F(l)} & \mathcal{R}^i F(P) \\
\xrightarrow{\delta_\$^{i,F}} & \mathcal{R}^{i+1} F(N) & \xrightarrow{\mathcal{R}^{i+1} F(h)} \mathcal{R}^{i+1} F(M) & \xrightarrow{\mathcal{R}^{i+1} F(l)} & \cdots .
\end{array}$$

This sequence is called the *right derived sequence of F associated to* $\$$. The graded homomorphism $\delta_\$^{n,F}$ is called the *n-th (graded) connecting homomorphism with respect to F associated to* $\$$. Moreover, the assignment $\delta_*^{\bullet,F}$ again has the expected naturality properties (cf. 3.8 C)).

E) We now consider the case in which F is a linear functor of graded R-modules. Then, as $\mathcal{R}^n F$ is again such a functor, we have $\mathcal{R}^n F(M(r)) = \mathcal{R}^n F(M)(r)$ and $\mathcal{R}^n F(h(r)) = \mathcal{R}^n F(h)(r)$ for each $r \in \mathbb{Z}$ and each homomorphism of graded

R-modules $h : M \to N$. Moreover, if $r \in \mathbb{Z}$ and $x \in R_r$, then $\mathcal{R}^n F(xh) = x\mathcal{R}^n F(h)$ for each $h : M \to N$ as above.

An essential step toward the goal of this section is to have a criterion which allows to “lift” the *equivalence of two functors F and $^*F$ to their right derived functors $\mathcal{R}^i F$ and $\mathcal{R}^i\, {}^*F$. We now shall give such a criterion.

8.21. **Proposition.** *Let R be a graded ring, let F be a left exact additive functor of R-modules and let $^*F$ be an additive functor of graded R-modules. Assume that $\mathcal{R}^i F(I) = 0$ for each $i \in \mathbb{N}$ and each *injective R-module I, and assume that F is *equivalent to $^*F$. Then, for each $i \in \mathbb{N}_0$, the functor $\mathcal{R}^i F$ is *equivalent to $\mathcal{R}^i\, {}^*F$.*

Proof. For each graded R-module M there is an isomorphism of R-modules $\iota^M : {}^*F(M) \to F(M)$ such that F is *equivalent to $^*F$ by means of $\iota : (M \leadsto \iota^M)$. As $^*F$ is left exact by 8.9 B), we may identify $\mathcal{R}^0 F = F$ and $\mathcal{R}^0\, {}^*F = {}^*F$. Now, we show our claim by induction on i.

The case $i = 0$ is clear from the above. So, let $i > 0$. For each graded R-module M there is a short exact sequence of graded R-modules

$$\mathbb{S}_M : 0 \to M \xrightarrow{i_M} I_M \xrightarrow{p_M} K_M \to 0$$

in which I_M is *injective (cf. 8.13). Also, for each graded R-module M, by induction, there is some isomorphism $\lambda^M : \mathcal{R}^{i-1}\, {}^*F(M) \to \mathcal{R}^{i-1} F(M)$ such that $\mathcal{R}^{i-1} F$ is *equivalent to $\mathcal{R}^{i-1}\, {}^*F$ by means of $\lambda : (M \leadsto \lambda^M)$. We can assume that $\lambda = \iota$ if $i = 1$.

As I_M is *injective, by 8.20 C) a) and by our hypotheses we have $\mathcal{R}^j\, {}^*F(I_M) = \mathcal{R}^j F(I_M) = 0$ for all $j > 0$. We set $\delta^{j,*} := \delta^{j,\, {}^*F}_{\mathbb{S}_M}$ and $\delta^j := \delta^{j,F}_{\mathbb{S}_M}$. If $i = 1$, we get the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}^0\, {}^*F(M) \arrow[r] \arrow[d, "\iota^M"', "\cong"] & \mathcal{R}^0\, {}^*F(I_M) \arrow[r] \arrow[d, "\iota^{I_M}"', "\cong"] & \mathcal{R}^0\, {}^*F(K_M) \arrow[r, "\delta^{0,*}"] \arrow[d, "\iota^{K_M}"', "\cong"] & \mathcal{R}^1\, {}^*F(M) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{R}^0 F(M) \arrow[r] & \mathcal{R}^0 F(I_M) \arrow[r] & \mathcal{R}^0 F(K_M) \arrow[r, "\delta^0"] & \mathcal{R}^1 F(M) \arrow[r] & 0
\end{tikzcd}$$

and thus a unique isomorphism

$$\nu^M : \mathcal{R}^1\, {}^*F(M) \xrightarrow{\cong} \mathcal{R}^1 F(M)$$

which occurs in the commutative diagram

$$\begin{tikzcd}
\mathcal{R}^0\, {}^*F(K_M) \arrow[r, "\delta^{0,*}"] \arrow[d, "\iota^{K_M} = \lambda^{K_M}"', "\cong"] & \mathcal{R}^1\, {}^*F(M) \arrow[d, "\cong"', "\nu^M"] \\
\mathcal{R}^0 F(K_M) \arrow[r, "\delta^0"] & \mathcal{R}^1 F(M).
\end{tikzcd}$$

If $i > 1$, we get the isomorphisms

$$\begin{tikzcd}
\mathcal{R}^{i-1\,*}F(K_M) \arrow[r, "\delta^{i-1,*}", "\cong"'] \arrow[d, "\lambda^{K_M}"', "\cong"] & \mathcal{R}^{i\,*}F(M) \\
\mathcal{R}^{i-1}F(K_M) \arrow[r, "\delta^{i-1}", "\cong"'] & \mathcal{R}^{i}F(M)
\end{tikzcd}$$

and thus the isomorphism

$$\nu^M := \delta^{i-1} \circ \lambda^{K_M} \circ (\delta^{i-1,*})^{-1} : \mathcal{R}^{i\,*}F(M) \xrightarrow{\cong} \mathcal{R}^iF(M).$$

Our aim is to show, that $\mathcal{R}^iF$ is *equivalent to $\mathcal{R}^{i\,*}F$ by means of the assignment $\nu : (M \rightsquigarrow \nu^M)$. So, let $h : M \to N$ be a homomorphism of graded R-modules. We have to show, that the diagram

$$\begin{tikzcd}
\mathcal{R}^{i\,*}F(M) \arrow[r, "\nu^M", "\cong"'] \arrow[d, "\mathcal{R}^{i\,*}F(h)"'] & \mathcal{R}^{i}F(M) \arrow[d, "\mathcal{R}^{i}F(h)"] \\
\mathcal{R}^{i\,*}F(N) \arrow[r, "\nu^N", "\cong"'] & \mathcal{R}^{i}F(N)
\end{tikzcd}$$

commutes.

As I_N is *injective, we get the following commutative diagram of graded R-modules

$$\begin{tikzcd}
\$_M : 0 \arrow[r] & M \arrow[r] \arrow[d, "h"'] & I_M \arrow[r] \arrow[d, "u"] & K_M \arrow[r] \arrow[d, "\bar{u}"] & 0 \\
\$_N : 0 \arrow[r] & N \arrow[r] & I_N \arrow[r] & K_N \arrow[r] & 0.
\end{tikzcd}$$

Applying cohomology, we get a "cube" diagram:

$$\begin{tikzcd}
& \mathcal{R}^{i-1\,*}F(K_M) \arrow[rr, two heads, "\delta^{i-1,\,*F}_{\$_M}"] \arrow[dl, "\mathcal{R}^{i-1\,*}F(\bar{u})"'] \arrow[dd, "\lambda^{K_M}", "\cong"' near start] & & \mathcal{R}^{i\,*}F(M) \arrow[dl, "\mathcal{R}^{i\,*}F(h)"'] \arrow[dd, "\nu^M", "\cong"'] \\
\mathcal{R}^{i-1\,*}F(K_N) \arrow[rr, two heads, crossing over, "\delta^{i-1,\,*F}_{\$_N}" near end] \arrow[dd, "\lambda^{K_N}"', "\cong"] & & \mathcal{R}^{i\,*}F(N) & \\
& \mathcal{R}^{i-1}F(K_M) \arrow[rr, two heads, "\delta^{i-1,F}_{\$_M}" near start] \arrow[dl, "\mathcal{R}^{i-1}F(\bar{u})"] & & \mathcal{R}^{i}F(M) \arrow[dl, "\mathcal{R}^{i}F(h)"] \\
\mathcal{R}^{i-1}F(K_N) \arrow[rr, two heads, "\delta^{i-1,F}_{\$_N}"] & & \mathcal{R}^{i}F(N) \arrow[from=uu, crossing over, "\nu^N", "\cong"' near start] &
\end{tikzcd}$$

The top and the bottom square are commutative by the naturality of the right derived sequence. The back and the front square are commutative by the definition of ν_M resp. ν_N. The left hand side square is commutative as $\mathcal{R}^{i-1}F$ is *equivalent to $\mathcal{R}^{i-1\,*}F$ by means of λ. As $\delta^{i-1,\,*F}_{\$_M}$ is surjective, the right hand side square becomes commutative. $\square$

Our next step is to show that, for a graded ideal $\mathfrak{a}$ of a Noetherian graded ring R, our "lifting criterion" 8.21 applies to the functors $F = \Gamma_{\mathfrak{a}}$ and ${}^*F = {}^*\Gamma_{\mathfrak{a}}$ (cf. 8.8). We begin with a preparatory exercise.

8.22. **Exercise.** A) Let R be a graded ring. Let I be a graded R-module. Show that I is *injective if and only if $I(r)$ is *injective for all $r \in \mathbb{Z}$.

B) Let I be a *injective R-module and let $\mathfrak{a} \subseteq R$ be a graded ideal. Use 8.12 B) and follow the traces of the proof of 3.14 in order to show:

a) If R is Noetherian, then $\Gamma_{\mathfrak{a}}(I)$ is *injective.

Conclude that:

b) If R is Noetherian, then $I/\Gamma_{\mathfrak{a}}(I)$ is *injective.

C) Let I be a *injective R-module and let $r \in \mathbb{Z}$. Prove:

a) If $x \in R_r \cap \mathrm{NZD}_R(I)$, then the multiplication map $x\cdot : I \to I(r)$ is an isomorphism.

Conclude on use of B) b):

b) If R is Noetherian and if $x \in R_r$, then the multiplication map
$$x\cdot : I/\Gamma_{\langle x\rangle}(I) \to (I/\Gamma_{\langle x\rangle}(I))(r)$$
is an isomorphism.

D) Let I be a *injective R-module and let $\mathfrak{b} \subseteq R$ be a second graded ideal. Consider the canonical homomorphism of graded R-modules
$$\begin{aligned} i = i^I_{\mathfrak{a},\mathfrak{b}} : I/\Gamma_{\mathfrak{a}}(I) \cap \Gamma_{\mathfrak{b}}(I) &\to I/\Gamma_{\mathfrak{a}}(I) \oplus I/\Gamma_{\mathfrak{b}}(I), \\ y + \Gamma_{\mathfrak{a}}(I) \cap \Gamma_{\mathfrak{b}}(I) &\mapsto (y + \Gamma_{\mathfrak{a}}(I), y + \Gamma_{\mathfrak{b}}(I)). \end{aligned}$$
Let R be Noetherian. Show that there is a homomorphism of graded R-modules $\lambda = \lambda^I_{\mathfrak{a},\mathfrak{b}}$ which occurs in the following commutative diagram:

$$\begin{tikzcd} I/\Gamma_{\mathfrak{a}}(I) \cap \Gamma_{\mathfrak{b}}(I) \arrow[dd, "\mathrm{id}_{I/\Gamma_{\mathfrak{a}}(I)\cap\Gamma_{\mathfrak{b}}(I)}"'] \arrow[dr, "i"] & \\ & I/\Gamma_{\mathfrak{a}}(I) \oplus I/\Gamma_{\mathfrak{b}}(I) \arrow[dl, "\lambda"] \\ I/\Gamma_{\mathfrak{a}}(I) \cap \Gamma_{\mathfrak{b}}(I). & \end{tikzcd}$$

8.23. **Theorem.** *Let R be a Noetherian graded ring, let $\mathfrak{a} \subseteq R$ be a graded ideal and let I be a *injective R-module. Then, for each $i \in \mathbb{N}$ we have $H^i_{\mathfrak{a}}(I) = 0$.*

Proof. As R is Noetherian, there are $n \in \mathbb{N}$, $r_i \in \mathbb{Z}$ and $x_i \in R_{r_i}$ for $i \in \{1, \ldots, n\}$ such that $\mathfrak{a} = \langle x_1, \ldots, x_n\rangle$. Let $t \in \{1, \ldots, n\}$. We prove by induction on t, that for each $i \in \mathbb{N}$ we have

$$H^i_{\mathfrak{a}}(I/\Gamma_{\langle x_1,\ldots,x_t\rangle}(I)) = 0.$$

Assume first, that $t = 1$. Then, according to 8.22 C) b) we have an isomorphism of R-modules $x_1\cdot : I/\Gamma_{\langle x_1\rangle}(I) \xrightarrow{\cong} I/\Gamma_{\langle x_1\rangle}(I)$. So, for each $i \in \mathbb{N}$ we get an isomorphism

$$x_1\cdot = H^i_{\mathfrak{a}}(x_1\cdot) : H^i_{\mathfrak{a}}(I/\Gamma_{\langle x_1\rangle}(I)) \xrightarrow{\cong} H^i_{\mathfrak{a}}(I/\Gamma_{\langle x_1\rangle}(I)).$$

As $x_1 \in \mathfrak{a}$ and $H^i_{\mathfrak{a}}(I/\Gamma_{\langle x_1\rangle}(I))$ is $\mathfrak{a}$-torsion, it follows $H^i_{\mathfrak{a}}(I/\Gamma_{\langle x_1\rangle}(I)) = 0$.

So, let $t > 1$ and fix $i \in \mathbb{N}$. By induction we have

$$H^i_{\mathfrak{a}}(I/\Gamma_{\langle x_1,\ldots,x_{t-1}\rangle}(I)) = H^i_{\mathfrak{a}}(I/\Gamma_{\langle x_t\rangle}(I)) = 0$$

and hence (cf. 1.22 A))

$$(*) \qquad H^i_{\mathfrak{a}}((I/\Gamma_{\langle x_1,\ldots,x_{t-1}\rangle}(I)) \oplus (I/\Gamma_{\langle x_t\rangle}(I))) = 0.$$

As $\Gamma_{\langle x_1,\ldots,x_t\rangle}(I) = \Gamma_{\langle x_1,\ldots,x_{t-1}\rangle}(I) \cap \Gamma_{\langle x_t\rangle}(I)$ (cf. 1.4 A) d)), we can use 8.22 D) in order to get a commutative diagram of R-modules:

$$\begin{tikzcd}
I/\Gamma_{\langle x_1,\ldots,x_t\rangle}(I) \arrow[dd, "\mathrm{id}"'] \arrow[dr, "j"] & \\
& I/\Gamma_{\langle x_1,\ldots,x_{t-1}\rangle}(I) \oplus I/\Gamma_{\langle x_t\rangle}(I) \arrow[dl, "\lambda"] \\
I/\Gamma_{\langle x_1,\ldots,x_t\rangle}(I). &
\end{tikzcd}$$

Applying $H^i_{\mathfrak{a}}(\bullet)$ we end up with a commutative diagram

$$\begin{tikzcd}
H^i_{\mathfrak{a}}(I/\Gamma_{\langle x_1,\ldots,x_t\rangle}(I)) \arrow[dd, "H^i_{\mathfrak{a}}(\mathrm{id})=\mathrm{id}"'] \arrow[dr, "H^i_{\mathfrak{a}}(j)"] & \\
& H^i_{\mathfrak{a}}((I/\Gamma_{\langle x_1,\ldots,x_{t-1}\rangle}(I)) \oplus (I/\Gamma_{\langle x_t\rangle}(I))) \overset{(*)}{=} 0 \arrow[dl, "H^i_{\mathfrak{a}}(\lambda)"] \\
H^i_{\mathfrak{a}}(I/\Gamma_{\langle x_1,\ldots,x_t\rangle}(I)) &
\end{tikzcd}$$

It follows $H^i_{\mathfrak{a}}(I/\Gamma_{\langle x_1,\ldots,x_t\rangle}(I)) = 0$.

Applying this with $t = n$, we get $H^i_{\mathfrak{a}}(I/\Gamma_{\mathfrak{a}}(I)) = 0$. In view of the natural isomorphism $H^i_{\mathfrak{a}}(I) \xrightarrow{\cong} H^i_{\mathfrak{a}}(I/\Gamma_{\mathfrak{a}}(M))$ for each $i \in \mathbb{N}$ (cf. 3.18) we thus get our claim. $\square$

By 8.23 we now may apply the "lifting criterion" 8.21 to the functors $\Gamma_{\mathfrak{a}}$ and ${}^*\Gamma_{\mathfrak{a}}$ and use this to perform the final step of this section: To furnish the modules $H^i_{\mathfrak{a}}(M)$ with a grading, if $\mathfrak{a}$ is a graded ideal of a Noetherian graded ring R and M is a graded R-module.

8.24. **Corollary.** *Let R be a Noetherian graded ring, let $\mathfrak{a} \subseteq R$ be a graded ideal and let $i \in \mathbb{N}_0$. Then,*

$$H^i_{\mathfrak{a}} \text{ is } {}^*\text{equivalent to } {}^*H^i_{\mathfrak{a}} := \mathcal{R}^i\, {}^*\Gamma_{\mathfrak{a}}.$$

Proof. Clear by 8.10, 8.21 and 8.23. □

8.25. **Convention and Remark.** A) Let R be a Noetherian graded ring and let $\mathfrak{a} \subseteq R$ be a graded ideal. Then, for each $i \in \mathbb{N}_0$ we have an assignment $\iota^i : M \rightsquigarrow \iota^{i,M}$ which to each graded R-module M assigns an isomorphism of R-modules

$$\iota^{i,M} : {}^*H^i_{\mathfrak{a}}(M) \xrightarrow{\cong} H^i_{\mathfrak{a}}(M)$$

such that for each homomorphism of graded R-modules $h : M \to N$ we have the commutative diagram

$$\begin{tikzcd}
{}^*H^i_{\mathfrak{a}}(M) \arrow[r, "{}^*H^i_{\mathfrak{a}}(h)"] \arrow[d, "\iota^{i,M}"', "\cong"] & {}^*H^i_{\mathfrak{a}}(N) \arrow[d, "\iota^{i,N}"', "\cong"] \\
H^i_{\mathfrak{a}}(M) \arrow[r, "H^i_{\mathfrak{a}}(h)"] & H^i_{\mathfrak{a}}(N)
\end{tikzcd}$$

(cf. 8.9 and 8.24). In this diagram ${}^*H^i_{\mathfrak{a}}(h) : {}^*H^i_{\mathfrak{a}}(M) \to {}^*H^i_{\mathfrak{a}}(N)$ is a homomorphism of graded R-modules.

B) For each $i \in \mathbb{N}_0$ we once for all choose an assignment ι^i as described in part A). Then, for each graded R-module M we identify $H^i_{\mathfrak{a}}(M)$ with ${}^*H^i_{\mathfrak{a}}(M)$ by means of $\iota^{i,M}$. If $h : M \to N$ is a homomorphism of graded R-modules, we then may identify $H^i_{\mathfrak{a}}(h)$ in a natural way with ${}^*H^i_{\mathfrak{a}}(h)$. So, we can write

$$\begin{tikzcd}
H^i_{\mathfrak{a}}(M) \arrow[r] \arrow[d, equal, "\iota^{i,M}"] & H^i_{\mathfrak{a}}(N) \arrow[d, equal, "\iota^{i,N}"] \\
{}^*H^i_{\mathfrak{a}}(M) \arrow[r, "{}^*H^i_{\mathfrak{a}}(h)"] \arrow[ur, phantom, "H^i_{\mathfrak{a}}(h) \, \| "] & {}^*H^i_{\mathfrak{a}}(N).
\end{tikzcd}$$

In this way, for each graded R-module M the R-module $H^i_{\mathfrak{a}}(M)$ carries a grading (induced by ι^i) and for each homomorphism of graded R-modules $h : M \to N$ the induced homomorphism

$$H^i_{\mathfrak{a}}(h) : H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(N)$$

is graded, too.

C) If M is a graded R-module, then for $i \in \mathbb{N}_0$ and for each $n \in \mathbb{Z}$ the n-th graded component of $H^i_{\mathfrak{a}}(M)$ is given by

$$H^i_{\mathfrak{a}}(M)_n = \iota^{i,M}({}^*H^i_{\mathfrak{a}}(M)_n).$$

Correspondingly, if $h : M \to N$ is a homomorphism of graded R-modules, for $i \in \mathbb{N}_0$ and for each $n \in \mathbb{Z}$ we have

$$H^i_{\mathfrak{a}}(h)_n : H^i_{\mathfrak{a}}(M)_n \to H^i_{\mathfrak{a}}(N)_n,\ x \mapsto (\iota^{i,N} \circ {}^*H^i_{\mathfrak{a}}(h)_n \circ (\iota^{i,M})^{-1})(x).$$

D) Let M be a graded R-module, let $i \in \mathbb{N}_0$ and let

$$\kappa^i : M \rightsquigarrow (\kappa^{i,M} : {}^*H^i_{\mathfrak{a}}(M) \xrightarrow{\cong} H^i_{\mathfrak{a}}(M))$$

be such that $H^i_{\mathfrak{a}}(\bullet)$ becomes *equivalent to ${}^*H^i_{\mathfrak{a}}(\bullet)$ by means of κ^i. Then there are isomorphisms of R_0-modules

$$\iota^{i,M}({}^*H^i_{\mathfrak{a}}(M)_n) \cong \kappa^{i,M}({}^*H^i_{\mathfrak{a}}(M)_n)$$

for all $n \in \mathbb{Z}$. So, in view of C) we can say:

a) Up to an isomorphism of R_0-modules, the n-th graded component $H^i_{\mathfrak{a}}(M)_n$ of $H^i_{\mathfrak{a}}(M)$ is independent of the chosen assignment ι^i, for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.

Thus:

b) For all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$, the R_0-isomorphism class of $H^i_{\mathfrak{a}}(M)_n$ is an invariant of M, $\mathfrak{a}$, i and n.

E) Let M be a graded R-module and let $r \in \mathbb{Z}$. Then, by 8.20 E) we may write $H^i_{\mathfrak{a}}(M(r)) = H^i_{\mathfrak{a}}(M)(r)$ and $H^i_{\mathfrak{a}}(M(r))_n = H^i_{\mathfrak{a}}(M)_{n+r}$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.

8.26. **Remark.** A) Let R be a Noetherian graded ring and let $\mathfrak{a} \subseteq R$ be a graded ideal. Consider a short exact sequence of graded R-modules

$$\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0.$$

According to 8.20 D) we then get an exact sequence of graded R-modules

$$\begin{array}{lllll}
0 \longrightarrow & {}^*H^0_{\mathfrak{a}}(N) & \xrightarrow{{}^*H^0_{\mathfrak{a}}(h)} & {}^*H^0_{\mathfrak{a}}(M) & \xrightarrow{{}^*H^0_{\mathfrak{a}}(l)} {}^*H^0_{\mathfrak{a}}(P) \\
\xrightarrow{{}^*\delta^{0,\mathfrak{a}}_{\mathbb{S}}} & {}^*H^1_{\mathfrak{a}}(N) & \xrightarrow{{}^*H^1_{\mathfrak{a}}(h)} & {}^*H^1_{\mathfrak{a}}(M) & \longrightarrow \cdots \\
 & & & \cdots & \longrightarrow {}^*H^{i-1}_{\mathfrak{a}}(P) \\
\xrightarrow{{}^*\delta^{i-1,\mathfrak{a}}_{\mathbb{S}}} & {}^*H^i_{\mathfrak{a}}(N) & \xrightarrow{{}^*H^i_{\mathfrak{a}}(h)} & {}^*H^i_{\mathfrak{a}}(M) & \xrightarrow{{}^*H^i_{\mathfrak{a}}(l)} {}^*H^i_{\mathfrak{a}}(P) \\
\xrightarrow{{}^*\delta^{i,\mathfrak{a}}_{\mathbb{S}}} & {}^*H^{i+1}_{\mathfrak{a}}(N) & \xrightarrow{{}^*H^{i+1}_{\mathfrak{a}}(h)} & {}^*H^{i+1}_{\mathfrak{a}}(M) & \longrightarrow \cdots,
\end{array}$$

the right derived sequence of ${}^*\Gamma_{\mathfrak{a}}$ associated to $\mathbb{S}$. Performing the identifications of 8.25 B), we thus get an exact sequence of graded R-modules

$$\begin{array}{lllll}
0 \longrightarrow & H^0_{\mathfrak{a}}(N) & \xrightarrow{H^0_{\mathfrak{a}}(h)} & H^0_{\mathfrak{a}}(M) & \xrightarrow{H^0_{\mathfrak{a}}(l)} H^0_{\mathfrak{a}}(P) \\
\xrightarrow{{}^*\delta^{0,\mathfrak{a}}_{\mathbb{S}}} & H^1_{\mathfrak{a}}(N) & \xrightarrow{H^1_{\mathfrak{a}}(h)} & H^1_{\mathfrak{a}}(M) & \longrightarrow \cdots \\
 & & & \cdots & \longrightarrow H^{i-1}_{\mathfrak{a}}(P) \\
\xrightarrow{{}^*\delta^{i-1,\mathfrak{a}}_{\mathbb{S}}} & H^i_{\mathfrak{a}}(N) & \xrightarrow{H^i_{\mathfrak{a}}(h)} & H^i_{\mathfrak{a}}(M) & \xrightarrow{H^i_{\mathfrak{a}}(l)} H^i_{\mathfrak{a}}(P) \\
\xrightarrow{{}^*\delta^{i,\mathfrak{a}}_{\mathbb{S}}} & H^{i+1}_{\mathfrak{a}}(N) & \xrightarrow{H^{i+1}_{\mathfrak{a}}(h)} & H^{i+1}_{\mathfrak{a}}(M) & \longrightarrow \cdots.
\end{array}$$

This sequence is called the *(long exact) graded cohomology sequence with respect to* $\mathfrak{a}$ *and associated to* $\mathbb{S}$. The homomorphism ${}^*\delta_{\mathbb{S}}^{i,\mathfrak{a}}$ is called the *i-th graded connecting homomorphisms* and is denoted often by $\delta_{\mathbb{S}}^i$ or just by δ^i.

As the formation of right derived sequences of ${}^*\Gamma_{\mathfrak{a}}$ has the expected naturality property, the same is true for the formation of graded cohomology sequences with respect to the graded ideal $\mathfrak{a}$.

B) Let M be a graded R-module, let $t \in \mathbb{Z}$ and let $x \in R_t \cap \mathrm{NZD}_R(M)$. Then, "applying graded cohomology" to the short exact sequence of graded R-modules

$$0 \to M \xrightarrow{x\cdot} M(t) \to (M/xM)(t) \to 0$$

and on use of 8.20 E) and 8.25 B) we get an exact sequence of graded R-modules

$$\begin{array}{l} 0 \longrightarrow H^0_{\mathfrak{a}}(M) \xrightarrow{x\cdot} H^0_{\mathfrak{a}}(M)(t) \longrightarrow H^0_{\mathfrak{a}}(M/xM)(t) \\ \xrightarrow{\delta^0} H^1_{\mathfrak{a}}(M) \xrightarrow{x\cdot} H^1_{\mathfrak{a}}(M)(t) \longrightarrow \cdots \\ \qquad\qquad \cdots \longrightarrow H^{i-1}_{\mathfrak{a}}(M/xM)(t) \\ \xrightarrow{\delta^{i-1}} H^i_{\mathfrak{a}}(M) \xrightarrow{x\cdot} H^i_{\mathfrak{a}}(M)(t) \longrightarrow H^i_{\mathfrak{a}}(M/xM)(t) \\ \xrightarrow{\delta^i} H^{i+1}_{\mathfrak{a}}(M) \xrightarrow{x\cdot} H^{i+1}_{\mathfrak{a}}(M)(t) \longrightarrow \cdots . \end{array}$$

Passing to individual graded components, for each $n \in \mathbb{Z}$ we get an exact sequence of R_0-modules

$$\begin{array}{l} 0 \longrightarrow H^0_{\mathfrak{a}}(M)_n \xrightarrow{x} H^0_{\mathfrak{a}}(M)_{n+t} \longrightarrow H^0_{\mathfrak{a}}(M/xM)_{n+t} \\ \xrightarrow{\delta^0} H^1_{\mathfrak{a}}(M)_n \xrightarrow{x\cdot} H^1_{\mathfrak{a}}(M)_{n+t} \longrightarrow \cdots \\ \qquad\qquad \cdots \longrightarrow H^{i-1}_{\mathfrak{a}}(M/xM)_{n+t} \\ \xrightarrow{\delta^{i-1}} H^i_{\mathfrak{a}}(M)_n \xrightarrow{x\cdot} H^i_{\mathfrak{a}}(M)_{n+t} \longrightarrow H^i_{\mathfrak{a}}(M/xM)_{n+t} \\ \xrightarrow{\delta^i} H^{i+1}_{\mathfrak{a}}(M)_n \xrightarrow{x\cdot} H^{i+1}_{\mathfrak{a}}(M)_{n+t} \longrightarrow \cdots . \end{array}$$

9. Cohomological Hilbert Functions

We now consider the special case in which R is a homogeneously graded algebra over a field, $\mathfrak{a} = R_+$ is the so called irrelevant ideal of R and M is a graded R-module. In this context we can introduce the so called *cohomological Hilbert functions,* which provide important numerical invariants of graded R-modules.

9.0. **Notation.** Throughout this chapter, let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring.

9.1. **Reminder and Exercise.** A) The grading $(R_n)_{n\in\mathbb{Z}}$ on R is said to be *positive,* if $R_n = 0$ for all $n < 0$. We usually express this by saying "R is positively graded" or "$R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is graded". It is easy to see, that in this case the subgroup

$$R_+ := \bigoplus_{n\in\mathbb{N}} R_n = \bigoplus_{n>0} R_n \subseteq R = \bigoplus_{n\in\mathbb{N}_0} R_n$$

is a graded ideal of R. This ideal R_+ is called the *irrelevant ideal of R.*

B) If R be positively graded, then there is a canonical isomorphism of rings $R_0 \xrightarrow{\cong} R/R_+,\ x \mapsto x + R_+$.

C) Let R be positively graded and let $\mathcal{M} \subseteq R_+^{\mathrm{hom}}$ be a set of homogeneous elements of R. Then, $R = R_0[\mathcal{M}]$ if and only if $R_+ = \langle\mathcal{M}\rangle$. As a consequence of this, the following statements are equivalent (cf. 8.1 D) and 8.2 H)):

(i) R is Noetherian;
(ii) R_0 is Noetherian and the ideal R_+ is generated by finitely many (homogeneous) elements;
(iii) R_0 is Noetherian and the ring R is generated over R_0 by finitely many (homogeneous) elements.

9.2. **Reminder and Exercise.** A) Let R be positively graded. By 9.1 C) it follows that $R = R_0[R_1]$ if and only if $R_+ = \langle R_1\rangle$. If these equivalent conditions hold, the grading $(R_n)_{n\in\mathbb{Z}}$ is said to be *homogeneous.* We usually express this by saying "R is homogeneously graded" or "R is homogeneous".

If A is a ring, by saying "R is a homogeneous algebra over A" or "R is a homogeneous A-algebra" we mean that R is homogeneous and that $R_0 = A$. Then, the ring $R_0 = A$ is also called the *base ring of R.*

B) If R is positively graded, then the following statements are equivalent (cf. 9.1 C)):

(i) R is homogeneous and Noetherian;
(ii) R_0 is Noetherian and the ideal R_+ is generated by finitely many elements of R_1;
(iii) R_0 is Noetherian and the ring R is generated over R_0 by finitely many elements of R_1.

9.3. **Examples.** A) Let $R = R_0[\underline{\mathbf{x}}] = R_0[\mathbf{x}_0, \ldots, \mathbf{x}_r]$ be a polynomial ring over R_0. For each $n \in \mathbb{Z}$ set

$$R_n := \sum_{\substack{\nu_0,\ldots,\nu_r \in \mathbb{N}_0 \\ \nu_0+\cdots+\nu_r=n}} R_0 \mathbf{x}_0^{\nu_0} \cdots \mathbf{x}_r^{\nu_r}.$$

Then, the family $(R_n)_{n\in\mathbb{Z}}$ is a homogeneous grading on R, and hence together with this grading R is a homogeneous R_0-algebra. This grading is called the *standard grading of the polynomial ring* $R = R_0[\underline{\mathbf{x}}]$.

B) Let $\mathfrak{a} \subseteq R$ be a graded ideal and let the ring $R/\mathfrak{a}$ be furnished with its canonical grading (cf. 8.3 C), D)). If R is positively graded, then so is $R/\mathfrak{a}$; in this case

$$(R/\mathfrak{a})_+ = R_+(R/\mathfrak{a}) = (R_+ + \mathfrak{a})/\mathfrak{a}.$$

In particular: If R is homogeneous, then so is $R/\mathfrak{a}$.

9.4. **Exercise and Remark.** A) A *graded maximal ideal of* R is a proper graded ideal $\mathfrak{m} \subsetneq R$ such that for a proper graded ideal $\mathfrak{a} \subsetneq R$ the relation $\mathfrak{m} \subseteq \mathfrak{a}$ implies $\mathfrak{m} = \mathfrak{a}$. Note that a graded maximal ideal of R is not necessarily a maximal ideal of R.

B) Let R be positively graded. Then, the graded maximal ideals of R are precisely the ideals of the form $\mathfrak{m}_0 + R_+$ with $\mathfrak{m}_0$ a maximal ideal of R_0. In particular, the graded maximal ideals of R are maximal ideals of R. Moreover, each proper graded ideal of R is contained in such a graded maximal ideal.

C) Let R be positively graded and let R_0 be a field. Then R_+ is the unique graded maximal ideal of R. In particular each proper graded ideal $\mathfrak{a} \subsetneq R$ satisfies $\mathfrak{a} \subseteq R_+$.

9.5. **Notation.** If $\mathcal{A}(n)$ is a statement about an integer n, then by saying "$\mathcal{A}(n)$ holds for all $n \ll 0$" we mean that there exists $n_0 \in \mathbb{Z}$ such that $\mathcal{A}(n)$ holds for each $n \in \mathbb{Z}$ with $n \leq n_0$. The notation "$n \gg 0$" is defined analogously.

9.6. **Reminder and Exercise.** A) Let M be a graded R-module. If R is positively graded, it is easy to see that the R_0-module

$$M_{\geq m} := \bigoplus_{n \geq m} M_n$$

is a graded R-submodule of M for each $m \in \mathbb{Z}$.

B) Clearly we have: If R is positively graded and M is finitely generated, then $M_n = 0$ for all $n \ll 0$.

C) Assume that R is positively graded and in addition Noetherian. Then, there are positive integers $d_1, \ldots, d_r \in \mathbb{N}$ and homogeneous elements $x_i \in R_{d_i}$ for $i \in \{1, \ldots, d_r\}$ such that $R = R_0[x_1, \ldots, x_r]$ (cf. 9.1 C)). Moreover, if M is finitely generated, by 8.2 H) there are integers $s_1, \ldots, s_t \in \mathbb{Z}$ and elements

$m_j \in M_{s_j}$ for $j \in \{1, \dots, t\}$ such that $M = \sum_{j=1}^{t} Rm_j$. Now, for each $n \in \mathbb{Z}$

$$M_n = \sum_{j=1}^{t} \sum_{\substack{\nu_{1j},\dots,\nu_{rj} \in \mathbb{N}_0 \\ s_j + \sum_{i=1}^{r} \nu_{ij} d_i = n}} R_0 \big(\prod_{i=1}^{r} x_i^{\nu_{ij}}\big) m_j.$$

In particular, M_n is a finitely generated graded R_0-module. This shows: If R is positively graded and Noetherian and M is finitely generated, then the R_0-modules M_n are finitely generated for all $n \in \mathbb{Z}$.

D) We define the *generating degree of* M by

$$\operatorname{gendeg}(M) := \inf\big\{t \in \mathbb{Z} \bigm| M = \sum_{n \leq t} RM_n\big\} \in \mathbb{Z} \cup \{\pm\infty\}.$$

Clearly, if M is finitely generated, $\operatorname{gendeg}(M) < \infty$.

E) Let R be positively graded. For $m, n \in \mathbb{Z}$ we use the notation

$$R_m M_n := \sum_{\substack{y \in M_n \\ x \in R_m}} R_0 xy \subseteq M_{m+n}$$

to denote the R_0-submodule of M_{m+n} spanned by all products xy with $x \in R_m, y \in M_n$. In this notation, for $m, n \in \mathbb{Z}$ one has:

a) If R is homogeneous and $\operatorname{gendeg}(M) \leq m \leq n$ then $M_n = R_{n-m} M_m$.

In particular one can say:

b) If R is homogeneous and M is finitely generated, then the following are equivalent:
   (i) There exists $r \in \mathbb{N}_0$ such that $(R_+)^r M = 0$;
   (ii) $M_n = 0$ for all $n \gg 0$;
   (iii) M is R_+-torsion.

9.7. **Reminder.** A) Let R be Noetherian and positively graded and let M be a graded R-module. Then, according to 8.25, for each $i \in \mathbb{N}$ we may consider $H^i_{R_+}(M) = \bigoplus_{n \in \mathbb{Z}} H^i_{R_+}(M)_n$ with $H^i_{R_+}(M)_n \cong {}^*H^i_{R_+}(M)_n$ for $n \in \mathbb{Z}$ as a graded R-module such that for each homomorphism of graded R-modules $h : M \to N$ the induced homomorphism $H^i_{R_+}(h) : H^i_{R_+}(M) \to H^i_{R_+}(N)$ is graded.

B) We have $H^0_{R_+}(M/\Gamma_{R_+}(M)) = \Gamma_{R_+}(M/\Gamma_{R_+}(M)) = 0$ (cf. 1.4 B) b)). Moreover by 3.18, the homomorphism $H^i_{R_+}(p) : H^i_{R_+}(M) \to H^i_{R_+}(M/\Gamma_{R_+}(M))$ (induced by the canonical homomorphism $p : M \to M/\Gamma_{R_+}(M)$) is an epimorphism of graded R-modules if $i = 0$ and an isomorphism of graded R-modules if $i > 0$.

C) Let $x \in R_t \cap \mathrm{NZD}_R(M)$ for some $t \in \mathbb{N}_0$. Then, by 8.26 B), for each $n \in \mathbb{Z}$ we have the following exact sequences of R_0-modules:

$$
\begin{array}{l}
0 \longrightarrow H^0_{R_+}(M)_n \xrightarrow{x\cdot} H^0_{R_+}(M)_{n+t} \longrightarrow H^0_{R_+}(M/xM)_{n+t} \\
\longrightarrow H^1_{R_+}(M)_n \xrightarrow{x\cdot} H^1_{R_+}(M)_{n+t} \longrightarrow \qquad \cdots \\
\qquad\qquad\qquad\qquad \cdots \qquad \longrightarrow H^{i-1}_{R_+}(M/xM)_{n+t} \\
\longrightarrow H^i_{R_+}(M)_n \xrightarrow{x\cdot} H^i_{R_+}(M)_{n+t} \longrightarrow H^i_{R_+}(M/xM)_{n+t} \\
\longrightarrow H^{i+1}_{R_+}(M)_n \longrightarrow \qquad \cdots .
\end{array}
$$

D) According to 8.25 A) the above grading of the modules $H^i_{R_+}(M)$ is not necessarily unique, as it depends on a choice of an appropriate assignment ι^i. But in view of 8.25 D) b), the individual graded components $H^i_{R_+}(M)_n$ are unique up to isomorphisms of R_0-modules, and this is enough for our later purposes.

9.8. **Lemma.** *Let K be a field, let V be a finite dimensional K-vector space, and let $(V_i)_{i \in I}$ be a family of proper K-subspaces of V with $1 \le \#I < \#K$. Then:*

a) $\bigcup_{i \in I} V_i \subsetneq V$.

b) There exists a K-subspace $W \subseteq V$ with $W \cap (\bigcup_{i \in I} V_i) = \{0\}$ and with $\dim_K(W) = \min\{\operatorname{codim}_V(V_i) \mid i \in I\}$.

Proof. (Induction on $d := \dim_K(V)$.) As $1 \le \#I$ we have $d > 0$. If $d = 1$, then a) is trivial, and for b) take $W := V$. Thus, we assume that $d > 1$. It is easy to see that V has at least $\#K$ different subspaces of dimension $d-1$. Hence, there exists a subspace $V' \subseteq V$ of dimension $d-1$ such that $V' \neq V_i$ for all $i \in I$. For $i \in I$ set $V'_i := V \cap V_i$. By induction there exists a subspace $W' \subseteq V'$ with $W' \cap (\bigcup_{i \in I} V'_i) = \{0\}$ and with $\dim_K(W') = \min\{\operatorname{codim}_{V'}(V'_i) \mid i \in I\}$. In particular $W' \cap (\bigcup_{i \in I} V_i) = W' \cap V' \cap (\bigcup_{i \in I} V'_i) = W' \cap (\bigcup_{i \in I} V'_i) = \{0\}$ and thus $\bigcup_{i \in I} V_i \subsetneq V$. This shows a).

Set $m := \min\{\operatorname{codim}_V(V_i) \mid i \in I\}$. If $\dim_K(W') = m$, we set $W := W'$, and the proof of b) is finished. So, we assume that $\dim_K(W') \neq m$. For $i \in I$ we have $\operatorname{codim}_{V'}(V'_i) = \operatorname{codim}_V(V_i) - \operatorname{codim}_V(V_i + V')$. As

$$
\begin{aligned}
&\min\{\operatorname{codim}_{V'}(V'_i) | i \in I\} = \\
&\min\bigl\{\inf\{\operatorname{codim}_V(V_i) | i \in I, V_i \nsubseteq V'\}, \sup\{\operatorname{codim}_V(V_i) | i \in I, V_i \subseteq V'\} - 1\bigr\}
\end{aligned}
$$

we get $\dim_K(W') = m - 1$. For $i \in I$ we have $\dim_K((V_i + W')/W') = \dim_K(V_i/(V_i \cap W')) = \dim_K(V_i) \le d - m$ and $\dim_K(V/W') = d - m + 1 \le d$. Hence $(V_i + W')/W'$ is a proper subspace of V/W' for all $i \in I$ and we may apply a). Choose an element $u \in V$ such that $u + W' \notin \bigcup_{i \in I}(V_i + W')/W'$. Now, set $W := W' + \langle u \rangle_K$. Then $\dim_K(W) = \dim_K(W') + 1 = m$. In order to prove $W \cap (\bigcup_{i\in I} V_i) = \{0\}$, assume that there exists $v \in W \cap (\bigcup_{i\in I} V_i) \setminus \{0\}$. Let $w \in W'$, $k \in K$ and $i \in I$ such that $w + ku = v \in V_i$. Since $v \neq 0$ and $V_i \cap W' = \{0\}$, we conclude $k \neq 0$. Thus $u - k^{-1}v \in W'$ and $k^{-1}v \in V_i$, therefore $u + W' \in (V_i + W')/W'$, a contradiction. $\square$

9.9. **Lemma.** *Let K be a field, let R be a Noetherian homogeneous K-algebra and let M be a finitely generated R-module such that $\Gamma_{R_+}(M) = 0$ and that $\sharp\, \mathrm{Ass}_R(M) < \sharp K$. Then, there is an element $x \in R_1 \cap \mathrm{NZD}_R(M)$.*

Proof. As $\Gamma_{R_+}(M) = 0$ we have $R_+ \not\subseteq \mathrm{ZD}_R(M)$ (cf. 1.7 b)) and therefore $R_+ \not\subseteq \mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Ass}_R(M)$ (cf. 1.6 B) c), a)). As R is homogeneous we have $R_+ = \langle R_1 \rangle$ (cf. 9.2 A)). It follows that $\mathfrak{p} \cap R_1 \subsetneq R_1$ is a proper K-subspace for each $\mathfrak{p} \in \mathrm{Ass}_R(M)$. As $\sharp\, \mathrm{Ass}_R(M) < \sharp K$ it follows from 9.8 a) that $\bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(M)} (\mathfrak{p} \cap R_1) \subsetneq R_1$. So, there is some $x \in R_1 \setminus \bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(M)} \mathfrak{p}$. By 1.6 B) a) we have $x \in \mathrm{NZD}_R(M)$. $\square$

9.10. **Proposition.** *Let K be an infinite field, let R be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Then:*

a) For each $n \in \mathbb{Z}$, the K-vector space $H^i_{R_+}(M)_n$ is of finite dimension;

b) $H^i_{R_+}(M)_n = 0$ for all $n \gg 0$.

Proof. (Induction on i.) As $H^0_{R_+}(M) = \Gamma_{R_+}(M) \subseteq M$ is a finitely generated graded R-module, the claim for $i = 0$ follows from 9.6 C), E) b). Now, let $i > 0$. In view of the isomorphism of graded R-modules $H^i_{R_+}(M) \cong H^i_{R_+}(M/\Gamma_{R_+}(M))$ (cf. 9.7 B)) we may replace M by $M/\Gamma_{R_+}(M)$ and hence assume that $\Gamma_{R_+}(M) = 0$. So, by 9.9 there is an element $x \in R_1 \cap \mathrm{NZD}_R(M)$. Hence, for each $n \in \mathbb{Z}$ there is an exact sequence of K-vector spaces (cf. 9.7 C))

$$H^{i-1}_{R_+}(M/xM)_{n+1} \to H^i_{R_+}(M)_n \xrightarrow{x\cdot} H^i_{R_+}(M)_{n+1}.$$

By induction there is some $n_0 \in \mathbb{Z}$ such that $H^{i-1}_{R_+}(M/xM)_{n+1} = 0$ for all $n \geq n_0$ and hence such that the multiplication map

$$x\cdot : H^i_{R_+}(M)_n \to H^i_{R_+}(M)_{n+1}$$

is injective for all $n \geq n_0$. As $x \in R_1 \subseteq R_+$ and as $H^i_{R_+}(M)$ is R_+-torsion (cf. 3.13) it follows easily that $H^i_{R_+}(M)_n = 0$ for all $n \geq n_0$. This proves statement b). By induction we also know that $H^{i-1}_{R_+}(M/xM)_{n+1}$ is of finite dimension over K for all $n \in \mathbb{Z}$. Now statement a) follows from statement b) by descending induction on n on use of the above three term exact sequence. $\square$

9.11. **Remark.** Proposition 9.10 is a special case of the following much more general result (cf. [B-S, Proposition 15.1.5]): Let R be Noetherian and positively graded and let M be a finitely generated graded R-module. Then:

a) For each $i \in \mathbb{N}_0$ and each $n \in \mathbb{Z}$ the R_0-module $H^i_{R_+}(M)_n$ is finitely generated.

b) There is some $n_0 \in \mathbb{Z}$ such that $H^i_{R_+}(M)_n = 0$ for all $i \in \mathbb{N}_0$ and all $n \geq n_0$.

9.12. **Exercise.** Let R be Noetherian and homogeneous, and assume that the R_0-module R_1 is generated by r elements. Show that $H^i_{R_+}(M) = 0$ for each $i > r$ and each graded R-module M.

9.13. **Definition and Remark.** A) Let K be an infinite field, let R be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$. Then, according to 9.10 we may define the number

$$h^i_M(n) := \dim_K(H^i_{R_+}(M)_n) = \dim_K({}^*H^i_{R_+}(M)_n) \in \mathbb{N}_0$$

which—according to 9.7 D)—depends only on i, n and M, but not on the assignment ι^i of 8.25 A). So, $h^i_M(n)$ is a proper numerical invariant of M.

Correspondingly we may define the function

$$h^i_M : \mathbb{Z} \to \mathbb{N}_0, \ n \mapsto h^i_M(n),$$

the *i-th cohomological Hilbert function of M.*

B) By 9.10 and 9.12 we have:

a) $h^i_M = 0$ for all $i > \dim_K(R_1)$;
b) There exists $n_0 \in \mathbb{Z}$ such that for all $i \in \mathbb{N}_0$ and all $b \geq n_0$ we have $h^i_M(n) = 0$.

C) By 8.25 E) there are isomorphisms of graded R-modules

$$H^i_{R_+}(M(r)) \cong H^i_{R_+}(M)(r)$$

for all $r \in \mathbb{Z}$ and hence isomorphisms of K-vector spaces

$$H^i_{R_+}(M(r))_n \cong H^i_{R_+}(M)_{n+r}$$

for all $r, n \in \mathbb{Z}$. These clearly imply

$$h^i_{M(r)}(n) = h^i_M(n+r) \text{ for } r, n \in \mathbb{Z}.$$

D) By 9.7 B), for $i \in \mathbb{N}$ and $n \in \mathbb{Z}$ we have $h^i_M(n) = h^i_{M/\Gamma_{R_+}(M)}(n)$.

9.14. **Definition and Remark.** A) Let K be an infinite field, let R be a Noetherian homogeneous K-algebra, and let M be a finitely generated graded R-module. Then, according to 9.6 C) and 9.13 A), B) a), for each $n \in \mathbb{Z}$, we may define the number

$$\chi_M(n) := \dim_K(M_n) - \sum_{i \geq 0} (-1)^i h^i_M(n) \in \mathbb{Z}.$$

Correspondingly we may define the function

$$\chi_M : \mathbb{Z} \to \mathbb{Z}, \ n \mapsto \chi_M(n),$$

the *characteristic function of M.*

B) From 9.13 B) b) it follows immediately that $\chi_M(n) = \dim_K(M_n)$ for all $n \gg 0$.

C) One has the following equivalence:

$$\chi_M = 0 \Leftrightarrow M \text{ is } R_+\text{-torsion}.$$

Indeed, if $\chi_M = 0$ we conclude by part B) that $M_n = 0$ for all $n \gg 0$, and hence that M is R_+-torsion (cf. 9.6 E) b)). Conversely, if M is R_+-torsion we have $H^i_{R_+}(M) = 0$ for all $i > 0$ and $H^0_{R_+}(M) = M$ (cf. 3.4, 3.17) so that $h^i_M = 0$ for all $i > 0$ and $h^0_M(n) = \dim_K(M_n)$ for all $n \in \mathbb{Z}$. This implies $\chi_M = 0$.

D) By 9.13 C) one immediately gets $\chi_{M(r)}(n) = \chi_M(n+r)$ for $n, r \in \mathbb{Z}$.

9.15. **Theorem** (Additivity of the characteristic function)**.** *Let K be an infinite field, let R be a Noetherian homogeneous K-algebra and let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated graded R-modules. Then $\chi_M = \chi_L + \chi_N$.*

Proof. For each $n \in \mathbb{Z}$ one has an exact sequence of K-vector spaces (of finite dimension) $0 \to L_n \to M_n \to N_n \to 0$ which shows that

$$\text{(A)} \qquad \dim_K(M_n) = \dim_K(L_n) + \dim_K(N_n).$$

If we apply cohomology to the short exact sequence $0 \to L \to M \to N \to 0$ and then pass over to n-th graded components, by 8.26 A) we get an exact sequence of K-vector spaces

$$0 \longrightarrow H^0_{R_+}(L)_n \longrightarrow H^0_{R_+}(M)_n \longrightarrow H^0_{R_+}(N)_n \longrightarrow H^1_{R_+}(L)_n \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{i-1}_{R_+}(N)_n \longrightarrow H^i_{R_+}(L)_n \longrightarrow H^i_{R_+}(M)_n \longrightarrow \cdots .$$

By 9.10 all K-vector spaces in this sequence are of finite dimension. By 9.13 B) a) also $H^i_{R_+}(L)_n = H^i_{R_+}(M)_n = H^i_{R_+}(N)_n = 0$ for all $i > \dim_K(R_1)$. So using the additivity of K-vector space dimension we get

$$\begin{aligned}
&\sum_{i\geq 0}(-1)^i h^i_L(n) - \sum_{i\geq 0}(-1)^i h^i_M(n) + \sum_{i\geq 0}(-1)^i h^i_N(n) \\
&\quad = \dim_K(H^0_{R_+}(L)_n) - \dim_K(H^0_{R_+}(M)_n) + \dim_K(H^0_{R_+}(N)_n) \\
&\qquad - \dim_K(H^1_{R_+}(L)_n) + \dim_K(H^1_{R_+}(M)_n) - \dim_K(H^1_{R_+}(N)_n) + \cdots = 0
\end{aligned}$$

and hence

$$\text{(B)} \qquad \sum_{i\geq 0}(-1)^i h^i_M(n) = \sum_{i\geq 0}(-1)^i h^i_N(n) + \sum_{i\geq 0}(-1)^i h^i_L(n).$$

Combining (A) and (B) for all $n \in \mathbb{Z}$ we get our claim. $\square$

9.16. **Reminder.** A function $f : \mathbb{Z} \to \mathbb{Z}$ is said to be *presented by a polynomial* if there is a polynomial $P \in \mathbb{Q}[\mathbf{x}]$ with $P(n) = f(n)$ for all $n \in \mathbb{Z}$. As any polynomial in $\mathbb{Q}[\mathbf{x}]$ is determined by its values at infinitely many places, such a P is unique.

9.17. **Theorem.** *Let K be an infinite field, let R be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Then, the characteristic function of M is presented by a polynomial.*

Proof. In order to prove the claim, assume that χ_M is not presented by a polynomial. Let $\mathcal{S}$ be the set of all graded submodules $N \subseteq M$ such that $\chi_{M/N}$ is not presented by a polynomial. Then $0 \in \mathcal{S}$ and hence $\mathcal{S} \neq \emptyset$. As M is Noetherian, $\mathcal{S}$ has a maximal member, say N. So $\chi_{M/N}$ is not presented by a polynomial, whereas $\chi_{M/U}$ is presented by a polynomial for each graded submodule $U \subseteq M$ with $N \subsetneq U$. This means that with $\bar{M} := M/N$ the function $\chi_{\bar{M}}$ is not presented by a polynomial, whereas $\chi_{\bar{M}/\bar{U}}$ is presented by a polynomial for each non-zero graded submodule $\bar{U} \subseteq \bar{M}$. By replacing M with $\bar{M}$ we thus may assume that χ_M is not presented by a polynomial, whereas $\chi_{M/U}$ is presented by a polynomial for each non-zero graded submodule $U \subseteq M$.

If we apply 9.15 to the exact sequence

$$0 \to \Gamma_{R_+}(M) \to M \to M/\Gamma_{R_+}(M) \to 0$$

and keep in mind that $\chi_{\Gamma_{R_+}(M)} = 0$ (cf. 9.14 C)) we get $\chi_{M/\Gamma_{R_+}(M)} = \chi_M$. By our assumptions on M this implies $\Gamma_{R_+}(M) = 0$. So, by 9.9 there is some $x \in R_1 \cap \mathrm{NZD}_R(M)$. The exact sequence of graded R-modules

$$0 \to M(-1) \xrightarrow{x\cdot} M \to M/xM \to 0$$

shows that $\chi_M - \chi_{M(-1)} = \chi_{M/xM}$ (cf. 9.15). As χ_M is not presented by a polynomial, we have $M \neq 0$. As $x \in \mathrm{NZD}_R(M)$ it follows $xM \neq 0$. So, by our assumptions on M, there is some polynomial $Q \in \mathbb{Q}[\mathbf{x}]$ such that $\chi_{M/xM}(n) = Q(n)$ for all $n \in \mathbb{Z}$. On use of 9.14 D) we now get for all $n \in \mathbb{Z}$

$$\chi_M(n) - \chi_M(n-1) = \chi_M(n) - \chi_{M(-1)}(n) = Q(n).$$

But it is easy to see that this implies that χ_M is presented by a polynomial, a contradiction. □

9.18. **Definition.** Under the hypotheses of 9.17, the unique polynomial in $\mathbb{Q}[\mathbf{x}]$ which coincides on $\mathbb{Z}$ with χ_M is denoted by P_M and is called the *Hilbert-Serre polynomial of M.*

9.19. **Corollary.** *Let K be an infinite field, let R be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Then,* $\dim_K(M_n) = P_M(n)$ *for all $n \gg 0$.*

Proof. Use 9.14 B). □

9.20. **Remark.** The concepts developed in 9.13 and 9.14 can be worked out in a much more general context, namely for Noetherian homogeneous rings R whose base ring R_0 is of dimension 0. What is said in 9.15 and 9.17 then also holds in this more general setting (cf. [B-S, Chapter 17]), using the length of R_0-modules instead of the K-vector space dimension.

10. Left-Vanishing of Cohomology

Let R be a Noetherian homogeneous algebra over an infinite field and let M be a finitely generated graded R-module. According to 9.10 the cohomological Hilbert functions h^i_M are all right-vanishing, i.e. have the property that $h^i_M(n) = 0$ for all $n \gg 0$. In this section we are interested in the left-vanishing of these functions, hence the property that $h^i_M(n) = 0$ for all $n \ll 0$.

10.0. **Notation.** Throughout this chapter, let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring.

10.1. **Reminder and Notation.** A) The *graded spectrum of R,* denoted by ${}^*\mathrm{Spec}(R)$, is the set of graded prime ideals of R, i.e.

$$^*\mathrm{Spec}(R) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \text{ is graded}\}.$$

B) Let R be positively graded. A graded prime $\mathfrak{p} \in {}^*\mathrm{Spec}(R)$ is called *essential* if $\mathfrak{p} \not\supseteq R_+$. The *projective spectrum of R,* denoted by $\mathrm{Proj}(R)$, is defined to be the set of all essential graded primes of R, i.e.

$$\mathrm{Proj}(R) := {}^*\mathrm{Spec}(R) \setminus \mathrm{Var}(R_+).$$

C) Let R be positively graded. The *maximal projective spectrum of R,* denoted by $\mathrm{mProj}(R)$, is defined to be the set of all elements of $\mathrm{Proj}(R)$ which are maximal with respect to inclusion, i.e.

$$\mathrm{mProj}(R) := \{\mathfrak{p} \in \mathrm{Proj}(R) \mid \nexists \mathfrak{q} \in \mathrm{Proj}(R) : \mathfrak{p} \subsetneq \mathfrak{q}\}.$$

10.2. **Reminder and Exercise.** Let $\mathfrak{a} \subsetneq R$ be a proper graded ideal. Show that $\mathfrak{a}$ is a prime ideal if and only if $fg \notin \mathfrak{a}$ for $f, g \in R^{\mathrm{hom}} \setminus \mathfrak{a}$.

10.3. **Reminder and Exercise.** A) Let M be a graded R-module and let $m \in M \setminus \{0\}$. For each $n \in \mathbb{Z}$ let $m_n \in M_n$ be the n-th homogeneous component of m. Then, clearly $\nu(m) := \inf\{n \in \mathbb{Z} \mid m_n \neq 0\} \in \mathbb{Z}$, $\mu(m) := \sup\{n \in \mathbb{Z} \mid m_n \neq 0\} \in \mathbb{Z}$ and $\nu(m) \leq \mu(m)$ with equality if and only if $m \in M^{\mathrm{hom}}$. If $\nu(m) < \mu(m)$, then $m - m_{\nu(m)}, m - m_{\mu(m)} \neq 0$ and moreover

$$\nu(m) < \nu(m - m_{\nu(m)}),\ \mu(m) = \mu(m - m_{\nu(m)}),$$
$$\mu(m) > \mu(m - m_{\mu(m)}),\ \nu(m) = \nu(m - m_{\mu(m)}).$$

B) Let $\mathfrak{p} \in \mathrm{Spec}(R)$ be such that $\mathfrak{p} = (0 :_R Rm)$. Assume that $(0 :_R Rm_{\mu(m)}) \subseteq \mathfrak{p}$. Let $y \in \mathfrak{p}\setminus\{0\}$. Show that $y_{\mu(y)} \in (0 :_R Rm_{\mu(m)})$ and hence that $y - y_{\mu(y)} \in \mathfrak{p}$. Use this argument to show by induction on $\mu(y) - \nu(y)$ that in this case all homogeneous components of y belong to $\mathfrak{p}$. Conclude that $\mathfrak{p} \in {}^*\mathrm{Spec}(R)$.

Assume now that $(0 :_R Rm_{\mu(m)}) \nsubseteq \mathfrak{p}$ and let $s \in (0 :_R Rm_{\mu(m)})^{\mathrm{hom}} \setminus \mathfrak{p}$. Show that $\mu(sm) - \nu(sm) < \mu(m) - \nu(m)$ and $\mathfrak{p} = (0 :_R Rsm)$.

C) Let $\mathfrak{p}$ be as in part B). Show by induction on $\mu(m) - \nu(m)$ (and on use of the observations of part B)) that $\mathfrak{p}$ is graded and conclude:

If M is a graded R-module and $\mathfrak{p} \in \mathrm{Ass}_R(M)$, then $\mathfrak{p} \in {}^*\mathrm{Spec}(R)$ and $\mathfrak{p} = (0 :_R Rm)$ for some $m \in M^{\mathrm{hom}} \setminus \{0\}$.

10.4. **Reminder and Exercise.** A) Let K be a field and let R be a homogeneous K-algebra. Then (cf. 9.4 C)):

$$\mathrm{Proj}(R) = {}^*\mathrm{Spec}(R) \setminus \{R_+\} = \{\mathfrak{p} \in {}^*\mathrm{Spec}(R) \mid \mathfrak{p} \subsetneq R_+\}.$$

B) Let $\mathfrak{p} \in \mathrm{Spec}(R)$ be such that $\dim_K(R_1/\mathfrak{p} \cap R_1) = 1$ and let $\mathbf{x}$ be an indeterminate. Show that there is a surjective homomorphism of K-algebras $\Phi : K[\mathbf{x}] \to R/\mathfrak{p}$. Use the fact that $\Phi^{-1}(\mathfrak{q}/\mathfrak{p}) \in \mathrm{Spec}(K[\mathbf{x}])$ for each $\mathfrak{q} \in \mathrm{Var}(\mathfrak{p})$ to conclude that the dimension of the R-module $R/\mathfrak{p}$ is at most 1.

C) Use the observations of part A) and B) to show that if $\mathfrak{p} \in {}^*\mathrm{Spec}(R)$ with $\dim_K(R_1/\mathfrak{p} \cap R_1) = 1$, then $\mathfrak{p} \in \mathrm{mProj}(R)$.

10.5. **Lemma.** *Let K be a field, let R be a Noetherian homogeneous K-algebra and let $M \neq 0$ be a finitely generated graded R-module such that $\Gamma_{R_+}(M) = 0$, $\#\,\mathrm{Ass}_R(M) < \#K$ and $\mathrm{Ass}_R(M) \cap \mathrm{mProj}(R) = \emptyset$. Then, there is a K-subspace $L \subseteq R_1$ such that $\dim_K(L) = 2$ and $L \setminus \{0\} \subseteq \mathrm{NZD}_R(M)$.*

Proof. Let $\mathfrak{p} \in \mathrm{Ass}_R(M)$. By 10.3 C) we have $\mathfrak{p} \in {}^*\mathrm{Spec}(R)$. As $\Gamma_{R_+}(M) = 0$ we have $R_+ \nsubseteq \mathfrak{p}$ (cf. 1.6 B) a), 1.7 b)). As $R_+ = \langle R_1 \rangle$ it follows $\mathfrak{p} \cap R_1 \subsetneq R_1$ and hence $\dim_K(R_1/\mathfrak{p} \cap R_1) \geq 1$. As $\mathfrak{p} \notin \mathrm{mProj}(R)$ we see by 10.4 C) that $\dim_K(R_1/\mathfrak{p} \cap R_1) \geq 2$. As $1 \leq \#\,\mathrm{Ass}_R(M) < \#K$ it follows from Lemma 9.8, that there is a K-subspace $L \subseteq R_1$ with $\dim_K(L) = 2$ and such that $L \cap (\bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p} \cap R_1) = \{0\}$. In particular we have $L \setminus \{0\} \subseteq R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p} = \mathrm{NZD}_R(M)$ (cf. 1.6 B) a)). □

10.6. **Definition and Remark.** A) Let K be an infinite field, let R be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. For each $n \in \mathbb{Z}$ we set

$$d_M^0(n) := \dim_K(M_n) - h_M^0(n) + h_M^1(n),$$

so that (cf. 9.14 A))

$$\chi_M(n) = d_M^0(n) - \sum_{i \geq 2} (-1)^i h_M^i(n).$$

So, we may define a function

$$d_M^0 : \mathbb{Z} \to \mathbb{Z},\ n \mapsto d_M^0(n).$$

B) We have

$$\begin{aligned} & d^0_{M/\Gamma_{R_+}(M)}(n) \\ & \quad = \dim_K((M/\Gamma_{R_+}(M))_n) - h^0_{M/\Gamma_{R_+}(M)}(n) + h^1_{M/\Gamma_{R_+}(M)}(n) \\ & \quad = \dim_K(M_n) - \dim_K(\Gamma_{R_+}(M)_n) - h^0_{M/\Gamma_{R_+}(M)}(n) + h^1_{M/\Gamma_{R_+}(M)}(n). \end{aligned}$$

As $\Gamma_{R_+}(M) = H^0_{R_+}(M)$, $H^0_{R_+}(M/\Gamma_{R_+}(M)) = 0$ and $H^1_{R_+}(M/\Gamma_{R_+}(M)) \cong H^1_{R_+}(M)$ we conclude:

a) $d^0_{M/\Gamma_{R_+}(M)} = d^0_M$.

As $h^0_M(n) \leq \dim_K(M_n)$ for all $n \in \mathbb{Z}$ we conclude:

b) $0 \leq h^1_M(n) \leq d^0_M(n)$ for all $n \in \mathbb{Z}$.

10.7. **Exercise.** Let K be an algebraically closed field and let V, W be two K-vector spaces such that $0 < \dim_K(W) < \infty$. Let $f, g : V \to W$ be two K-linear maps. Show that if the map $\alpha f + \beta g : V \to W$ is injective for all $(\alpha, \beta) \in K^2 \setminus \{(0,0)\}$, then $\dim_K(V) < \dim_K(W)$.

(Hint: Assume that $\dim_K(V) \geq \dim_K(W)$, show that f and g are isomorphisms and use the fact that $g^{-1} \circ f$ has an eigenvalue in K).

10.8. **Proposition.** *Let K be an infinite field, let R be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Then, for $t \in \mathbb{Z}$:*

a) $d^0_M(n) \leq d^0_M(t)$ for all $n \leq t$.

b) If K is algebraically closed and $\mathrm{Ass}_R(M) \cap \mathrm{mProj}(R) = \emptyset$, then $d^0_M(n) = 0$ for all $n \leq t - 2d^0_M(t)$.

Proof. By 10.6 B) a) the function d^0_M does not change if we replace M by $M/\Gamma_{R_+}(M)$. As

$$\mathrm{Ass}_R(M/\Gamma_{R_+}(M)) = \mathrm{Ass}_R(M) \setminus \mathrm{Var}(R_+)$$

and $\mathrm{Var}(R_+) \cap \mathrm{Proj}(R) = \emptyset$ (cf. 1.9 b), 10.1 B)), the condition

$$\mathrm{Ass}_R(M) \cap \mathrm{mProj}(R) = \emptyset$$

is not affected, if we replace M by $M/\Gamma_{R_+}(M)$ and hence assume $H^0_{R_+}(M) = \Gamma_{R_+}(M) = 0$. So, in particular we then have $h^0_M = 0$ and hence

$$\text{(A)} \qquad d^0_M(n) = \dim_K(M_n) + h^1_M(n)$$

for all $n \in \mathbb{Z}$. Now, by 9.9, there is some $x \in R_1 \cap \mathrm{NZD}_R(M)$. For each such x the short exact sequence of graded R-modules

$$0 \to M \xrightarrow{x\cdot} M(1) \to (M/xM)(1) \to 0$$

and the induced exact sequences of K-vector spaces (cf. 9.7 C))

$$\text{(B)} \qquad 0 \to H^0_{R_+}(M/xM)_{n+1} \to H^1_{R_+}(M)_n \xrightarrow{x\cdot} H^1_{R_+}(M)_{n+1}$$

show that for all $n \in \mathbb{Z}$ we have

$$\text{(C)} \qquad \begin{cases} \dim_K(M_n) &= \dim_K(M_{n+1}) - \dim_K((M/xM)_{n+1}); \\ h^1_M(n) &\leq h^1_M(n+1) + h^0_{M/xM}(n+1). \end{cases}$$

As $H^0_{R_+}(M/xM) = \Gamma_{R_+}(M/xM) \subseteq M/xM$ we also have

$$\text{(D)} \qquad h^0_{M/xM}(n+1) \leq \dim_K((M/xM)_{n+1})$$

for all $n \in \mathbb{Z}$. Combining (A), (C) and (D) we thus get $d^0_M(n) \leq d^0_M(n+1)$ for all $n \in \mathbb{Z}$. This proves statement a).

Assume now that K is algebraically closed and that $\mathrm{Ass}_R(M) \cap \mathrm{mProj}(R) = \emptyset$. If $M = 0$, then $d_M^0 = 0$ and hence the claim is clear. So, let $M \neq 0$. By 10.5 there is a K-subspace $L \subseteq R_1$ of dimension 2 such that $L \setminus \{0\} \subseteq \mathrm{NZD}_R(M)$. Let $f, g \in L$ form a K-basis of L. Then $\alpha f + \beta g \in L \setminus \{0\} \subseteq \mathrm{NZD}_R(M)$ for all $(\alpha, \beta) \in K^2 \setminus \{(0,0)\}$. So, for each $n \in \mathbb{Z}$ and each pair (α, β) the linear map $\alpha f + \beta g : M_n \to M_{n+1}$ is injective. Hence by 10.7 we can say

$$\begin{aligned} &M_{n+1} \neq 0 \Rightarrow \dim_K(M_n) < \dim_K(M_{n+1}),\\ &M_{n+1} = 0 \Rightarrow M_n = 0. \end{aligned}$$

This implies that $M_n = 0$ for all $n \leq t - \dim_K(M_t)$. As $\dim_K(M_t) \leq \dim_K(M_t) + h_M^1(t) = d_M^0(t)$ it follows

(E) $$M_n = 0 \text{ for all } n \leq t - d_M^0(t).$$

In particular on use of (A) we get:

(F) $$d_M^0(n) = h_M^1(n) \text{ for all } n \leq t - d_M^0(t).$$

Let $(\alpha, \beta) \in K^2 \setminus \{(0,0)\}$. If we apply (D) with $x = \alpha f + \beta g$ we get

$$0 \leq h^0_{M/(\alpha f + \beta g)M}(n+1) \leq \dim_K((M/(\alpha f + \beta g)M)_{n+1}) \leq \dim_K(M_{n+1}),$$

so that by statement (E) we have $h^0_{M/(\alpha f+\beta g)M}(n+1) = 0$ for all $n < t - d_M^0(t)$. If we apply (B) with $x = \alpha f + \beta g$ we thus get a monomorphism of K-vector spaces $\alpha f + \beta g : H^1_{R_+}(M)_n \rightarrowtail H^1_{R_+}(M)_{n+1}$ for all $n < t - d_M^0(t)$ and all $(\alpha, \beta) \in K^2 \setminus \{(0,0)\}$. By 10.7 we thus can say for $n < t - d_M^0(t)$:

$$\begin{aligned} &h_M^1(n+1) \neq 0 \Rightarrow h_M^1(n) < h_M^1(n+1),\\ &h_M^1(n+1) = 0 \Rightarrow h_M^1(n) = 0. \end{aligned}$$

From this we conclude that $h_M^1(n) = 0$ for all $n \leq t - d_M^0(t) - h_M^1(t - d_M^0(t))$. By statement a) we have $h_M^1(t - d_M^0(t)) \leq d_M^0(t - d_M^0(t)) \leq d_M^0(t)$ (cf. 10.6 B) b)). So, in view of (F) we get $d_M^0(n) = h_M^1(n) = 0$ for all $n \leq t - 2d_M^0(t)$. $\square$

10.9. **Definition and Remark.** A) Let K be an infinite field, let R be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ set

$$B_M^k(n) := d_M^0(n) + \sum_{i=2}^{k} \binom{k-1}{i-1} h_M^i(n-i+1).$$

So, for $k \in \mathbb{N}$ we may define a function $B_M^k : \mathbb{Z} \to \mathbb{Z},\ n \mapsto B_M^k(n)$.

B) Clearly (cf. 10.6 B) b)):

a) $B_M^1 = d_M^0$;
b) $0 \leq d_M^0(n), h_M^2(n-1), \ldots, h_M^k(n-k+1) \leq B_M^k(n)$ for $k \in \mathbb{N}$ and $n \in \mathbb{Z}$.

Also, in view of 10.6 B) a) and as $h^i_{M/\Gamma_{R_+}(M)} = h^i_M$ for all $i > 0$ (cf. 9.13 D))

c) $B^k_{M/\Gamma_{R_+}(M)} = B^k_M$ for $k \in \mathbb{N}$.

10.10. **Lemma.** *Let K be an infinite field, let R be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module, let $x \in R_1 \cap \mathrm{NZD}_R(M)$ and let $k \geq 2$. Then, for $n \in \mathbb{Z}$ we have $B^{k-1}_{M/xM}(n) \leq B^k_M(n)$.*

Proof. As $x \in R_1 \cap \mathrm{NZD}_R(M)$, we have $H^0_{R_+}(M) = \Gamma_{R_+}(M) = 0$. The short exact sequence $0 \to M \xrightarrow{x\cdot} M(1) \to (M/xM)(1) \to 0$ of graded R-modules and the induced exact sequences of K-vector spaces

$$\cdots \to H^i_{R_+}(M)_{m+1} \to H^i_{R_+}(M/xM)_{m+1} \to H^{i+1}_{R_+}(M)_m \to \cdots$$

(cf. 9.7 C)) show that for all $n \in \mathbb{Z}$ and all $i \in \mathbb{N}$

(A) $\dim_K(M_n) = \dim_K(M_{n-1}) + \dim_K((M/xM)_n)$;

(B) $h^i_{M/xM}(n-i+1) \leq h^i_M(n-i+1) + h^{i+1}_M(n-i)$.

So, on use of (A) and of (B) (applied with $i = 1$) we first get

$$\begin{aligned} d^0_{M/xM}(n) &= \dim_K((M/xM)_n) - h^0_{M/xM}(n) + h^1_{M/xM}(n) \\ &\leq \dim_K((M/xM)_n) + h^1_{M/xM}(n) \\ &= \dim_K(M_n) - \dim_K(M_{n-1}) + h^1_{M/xM}(n) \\ &\leq \dim_K(M_n) + h^1_{M/xM}(n) \leq \dim_K(M_n) + h^1_M(n) + h^2_M(n-1). \end{aligned}$$

As $H^0_{R_+}(M) = 0$, we have $\dim_K(M_n) + h^1_M(n) = d^0_M(n)$. By the previous inequalities we get

(C) $B^1_{M/xM}(n) = d^0_{M/xM}(n) \leq d^0_M(n) + h^2_M(n-1) = B^2_M(n)$ for all $n \in \mathbb{Z}$.

This proves the case $k = 2$. So, let $k > 2$. On use of (B) and the Pascal formula for binomial coefficients, for each $n \in \mathbb{Z}$ we obtain

$$\begin{aligned} &\sum_{i=2}^{k-1} \binom{(k-1)-1}{i-1} h^i_{M/xM}(n-i+1) \\ &\qquad \leq \sum_{i=2}^{k-1} \binom{k-2}{i-1} (h^i_M(n-i+1) + h^{i+1}_M(n-i)) \\ &\qquad = (k-2)h^2_M(n-1) + h^k_M(n-k+1) \\ &\qquad\qquad + \sum_{j=3}^{k-1} \left(\binom{k-2}{j-2} + \binom{k-2}{j-1} \right) h^j_M(n-j+1) \\ &\qquad = (k-2)h^2_M(n-1) + \sum_{j=3}^{k} \binom{k-1}{j-1} h^j_M(n-j+1) \\ &\qquad \leq -h^2_M(n-1) + \sum_{i=2}^{k} \binom{k-1}{i-1} h^i_M(n-i+1). \end{aligned}$$

Adding this inequality and (C) we get $B^{k-1}_{M/xM}(n) \leq B^k_M(n)$. $\square$

10.11. **Reminder and Exercise.** A) Let A be a ring and let M be an A-module. If A is local with maximal ideal $\mathfrak{m}$, according to 4.5 C) we write $\operatorname{depth}_A(M) = \operatorname{grade}_M(\mathfrak{m})$.

B) Let A be Noetherian and local with maximal ideal $\mathfrak{m}$ and let M be finitely generated. Use 4.7 to show:

a) $\operatorname{depth}_A(M) = \infty \Leftrightarrow M = 0$.

Use 4.11 to show:

b) $\operatorname{depth}_A(M) \leq \dim(M) \Leftrightarrow M \neq 0$.

Use 4.9 to show:

c) If $x \in \mathfrak{m} \cap \operatorname{NZD}_A(M)$, then $\operatorname{depth}_A(M/xM) = \operatorname{depth}_A(M) - 1$.

C) Let A be Noetherian, let M be finitely generated and let $\mathfrak{p} \in \operatorname{Spec}(A)$. Use the fact that

$$\operatorname{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{q}A_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Ass}_A(M), \mathfrak{q} \subseteq \mathfrak{p}\}$$

together with prime avoidance to prove that $\mathfrak{p} \in \operatorname{Ass}_A(M)$ if and only if $\operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$.

10.12. **Definition and Remark.** A) Let R be positively graded and let M be a graded R-module. We define the *global subdepth of M* by

$$\delta(M) := \inf\{\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{mProj}(R)\}.$$

B) Let R be Noetherian and let M be finitely generated. From 10.11 C) it follows immediately:

a) $\delta(M) = 0 \Leftrightarrow \operatorname{Ass}_R(M) \cap \operatorname{mProj}(R) \neq \emptyset$.

As $\Gamma_{R_+}(M)_\mathfrak{p} = 0$ (and hence $(M/\Gamma_{R_+}(M))_\mathfrak{p} \cong M_\mathfrak{p}$) for $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(R_+)$ we clearly have:

b) $\delta(M/\Gamma_{R_+}(M)) = \delta(M)$.

10.13. **Exercise** (Homogeneous prime avoidance)**.** Let $\mathfrak{a} \subseteq R$ be a graded ideal which is generated by homogeneous elements of positive degrees. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in \operatorname{Spec}(R)$ such that $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ for $i \in \{1, \ldots, r\}$. Then, there is a homogeneous element of positive degree $x \in \mathfrak{a} \setminus \bigcup_{i=1}^r \mathfrak{p}_i$.

10.14. **Lemma.** *Let R be Noetherian and positively graded and let M be a finitely generated graded R-module. Then:*

a) $\delta(M) = \infty \Leftrightarrow M = \Gamma_{R_+}(M)$;

b) If $x \in \operatorname{NZD}_R(M)$ is homogeneous, then $\delta(M/xM) \geq \delta(M) - 1$;

c) Let R be homogeneous and let R_0 be a field. If $\Gamma_{R_+}(M) = 0$ and $\delta(M) > 0$, then there exists $x \in R_+^{\mathrm{hom}} \cap \operatorname{NZD}_R(M)$ such that $\delta(M/xM) = \delta(M) - 1$.

Proof. "a)": "$\Leftarrow$": Assume that $M = \Gamma_{R_+}(M)$. Then, by 10.12 C), A) and 10.11 B) a) we have $\delta(M) = \delta(M/\Gamma_{R_+}(M)) = \delta(0) = \infty$.

"$\Rightarrow$": Assume that $\delta(M) = \infty$ and $M \neq \Gamma_{R_+}(M)$. Let $\bar{M} := M/\Gamma_{R_+}(M)$. Then $\bar{M} \neq 0$ and therefore

$$\emptyset \neq \mathrm{Ass}_R(\bar{M}) = \mathrm{Ass}_R(M) \setminus \mathrm{Var}(R_+) \subseteq {}^*\mathrm{Spec}(R) \setminus \mathrm{Var}(R_+) = \mathrm{Proj}(R)$$

(cf. 1.9 b), 10.3 C), 1.6 B) d) and 10.1 B)). Now, let $\mathfrak{q} \in \mathrm{Ass}_R(\bar{M})$. Then $\mathrm{depth}_{R_\mathfrak{q}}(\bar{M}_\mathfrak{q}) = 0$ (cf. 10.11 C)) and hence $\bar{M}_\mathfrak{q} \neq 0$ (cf. 10.11 B) a)). On the other side $\mathfrak{q} \in \mathrm{Proj}(R)$ implies that there is some $\mathfrak{p} \in \mathrm{mProj}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$. As $\delta(\bar{M}) = \delta(M) = \infty$ (cf. 10.12 C)) we have $\bar{M}_\mathfrak{p} = 0$ (cf. 10.11 B) a)). As $\bar{M}$ is finitely generated, we find some $s \in R \setminus \mathfrak{p}$ with $s\bar{M} = 0$. As $s \notin \mathfrak{q}$ we get the contradiction that $\bar{M}_\mathfrak{q} = 0$.

"b)": Let $\mathfrak{p} \in \mathrm{mProj}(R)$. If $x \notin \mathfrak{p}$, $\frac{x}{1} \in R_\mathfrak{p}$ is a unit so that $xM_\mathfrak{p} = \frac{x}{1}M_\mathfrak{p} = M_\mathfrak{p}$ and hence $\mathrm{depth}_{R_\mathfrak{p}}((M/xM)_\mathfrak{p}) = \mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}/xM_\mathfrak{p}) = \mathrm{depth}_{R_\mathfrak{p}}(0) = \infty \geq \delta(M) - 1$.

If $x \in \mathfrak{p}$, then $\frac{x}{1} \in \mathfrak{p}R_\mathfrak{p} \cap \mathrm{NZD}_{R_\mathfrak{p}}(M_\mathfrak{p})$. So, on use of 10.11 B) c) we get in this case $\mathrm{depth}_{R_\mathfrak{p}}((M/xM)_\mathfrak{p}) = \mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}/\frac{x}{1}M_\mathfrak{p}) = \mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) - 1 \geq \delta(M) - 1$. Altogether this proves our claim.

"c)": If $\delta(M) = \infty$ statement a) gives $M = 0$ and any $x \in R_1$ will do. So, let $\delta(M) < \infty$. Then, there is some $\mathfrak{p} \in \mathrm{mProj}(R)$ with $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \delta(M)$. As $\Gamma_{R_+}(M) = 0$ we have $R_+ \notin \mathrm{Ass}_R(M)$ and so 10.3 C) and 10.4 A) imply that $\mathrm{Ass}_R(M) \subseteq \mathrm{Proj}(R)$. As $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \delta(M) > 0$ we have $\mathfrak{p} \notin \mathrm{Ass}_R(M)$ (cf. 10.11 C)). As $\mathfrak{p} \in \mathrm{mProj}(R)$ it follows that $\mathfrak{p} \not\subseteq \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Ass}_R(M)$. So, by 10.13 there exists $x \in \mathfrak{p}^{\mathrm{hom}} \setminus \bigcup_{\mathfrak{q} \in \mathrm{Ass}_R(M)} \mathfrak{q}$. As $\mathfrak{p} \subseteq R_+$ we have $x \in R_+$ and in view of 1.6 B) a) we also have $x \in \mathrm{NZD}_R(M)$.

So, statement b) gives $\delta(M/xM) \geq \delta(M) - 1$. Also, as $\frac{x}{1} \in \mathrm{NZD}_{R_\mathfrak{p}}(M_\mathfrak{p}) \cap \mathfrak{p}R_\mathfrak{p}$, as in the proof of statement b) we get $\mathrm{depth}_{R_\mathfrak{p}}((M/xM)_\mathfrak{p}) = \delta(M) - 1$ and hence $\delta(M/xM) \leq \delta(M) - 1$. $\square$

10.15. **Proposition.** *Let K be an algebraically closed field, let R be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Them, for $k \in \mathbb{N}$ such that $k \leq \delta(M)$:*

a) $d_M^0(n), h_M^1(n), h_M^2(n-1), \ldots, h_M^k(n-k+1) \leq \frac{1}{2}(2B_M^k(0))^{2^{k-1}}$ for all $n \leq 0$.

b) $B_M^k(n) = 0$ for all $n \leq -(2B_M^k(0))^{2^{k-1}}$.

Proof. (Induction on k.) We write $B := B_M^k(0)$. As $\delta(M) \geq 1$ we have $\mathrm{Ass}_R(M) \cap \mathrm{mProj}(R) = \emptyset$ (cf. 10.12 B) a)). So, in view of 10.9 B) b) and 10.8 we get $d_M^0(n) \leq d_M^0(0) \leq B$ for all $n \leq 0$ and hence $d_M^0(n)$ for all $n \leq -2B$. This proves in particular that

$$\text{(A)} \qquad \begin{cases} h_M^1(n), d_M^0(n) \leq \frac{1}{2}(2B)^{2^{k-1}} \text{ for all } n \leq 0; \\ h_M^1(n) = d_M^0(n) = 0 \text{ for all } n \leq -(2B)^{2^{k-1}} \end{cases}$$

(cf. 10.6 B) b)). Clearly, this proves the case $k = 1$ as we know that $B_M^1 = d_M^0$ by 10.9 B) a).

So, from now on, let $k > 1$. By 10.6 B) a), 10.9 B) c), 10.12 C) and 9.12 D) we may replace M by $M/\Gamma_{R_+}(M)$ and hence assume that $\Gamma_{R_+}(M) = 0$. If $M = 0$, our claim is clear. So, let $M \neq 0$. By 10.5 there is a K-subspace $L \subseteq R_1$ of dimension 2 such that $L \setminus \{0\} \subseteq \mathrm{NZD}_R(M)$. Let $f, g \in L$ form a K-basis of L. Let $(\alpha, \beta) \in K^2 \setminus \{(0,0)\}$. Then $x := \alpha f + \beta g \in R_1 \cap \mathrm{NZD}_R(M)$ and the short exact sequence of graded R-modules $0 \to M(-1) \xrightarrow{x\cdot} M \to M/xM \to 0$ induces exact sequences of K-vector spaces

$$\text{(B)} \qquad H^{i-1}_{R_+}(M/xM)_{n+1} \to H^i_{R_+}(M)_n \xrightarrow{x\cdot} H^i_{R_+}(M)_{n+1}$$

for $i \in \mathbb{N}$ and $n \in \mathbb{Z}$. In particular we have

$$\text{(C)} \qquad h^i_M(n) \leq h^{i-1}_{M/xM}(n+1) + h^i_M(n+1) \text{ for all } i \in \mathbb{N} \text{ and all } n \in \mathbb{Z}.$$

By 10.14 b) we have $k - 1 \leq \delta(M/xM)$.

Now, fix $i \in \{2, \ldots, k\}$ so that $i - 1 \in \{1, \ldots, k-1\}$. By induction and as $B^{k-1}_{M/xM}(0) \leq B$ (cf. 10.10) we have

$$h^{i-1}_{M/xM}(m - i + 2) \leq \frac{1}{2}(2B)^{2^{k-2}} \text{ for all } m \leq 0$$

and

$$B^{k-1}_{M/xM}(m) = 0 \text{ for all } m \leq -(2B)^{2^{k-2}},$$

and hence (cf. 10.9 B) b))

$$\text{(D)} \qquad h^{i-1}_{M/xM}(m - i + 2) = 0 \text{ for all } m \leq -(2B)^{2^{k-2}}.$$

Also, by 10.9 B) b) we have $h^i_M(-i+1) \leq B$. So, the inequalities (C) allow to conclude by induction on $-n$ that for all $n \leq 0$ we have

$$\begin{aligned} h^i_M(n-i+1) &\leq B + \sum_{m=n}^{-1} h^{i-1}_{M/xM}(m-i+2) \leq B + \big((2B)^{2^{k-2}} - 1\big) \cdot \frac{1}{2}(2B)^{2^{k-2}} \\ &= B - \frac{1}{2}(2B)^{2^{k-2}} + \frac{1}{2}\big((2B)^{2^{k-2}}\big)^2 \leq \frac{1}{2}\big((2B)^{2^{k-2}}\big)^2 = \frac{1}{2}(2B)^{2^{k-1}}. \end{aligned}$$

Together with (A) this proves statement a).

Next we prove statement b). If $B = 0$ we conclude by statement a) and the definition of $B^k_M(n)$ (cf. 10.9 A)). So, let $B > 0$. Then, by (B) and (D), for any $i \in \{2, \ldots, k\}$ and any $m \leq -(2B)^{2^{k-2}}$, we get an injection $H^i_{R_+}(M)_{m-i+1} \overset{x\cdot}{\rightarrowtail} H^i_{R_+}(M)_{m-i+2}$. So, by 10.7,

$$h^i_M(m-i+1) \leq \max\{0, h^i_M(m-i+2) - 1\} \text{ for all } m \leq -(2B)^{2^{k-2}}.$$

Consequently

$$h^i_M(n-i+1) = 0 \text{ for all } n \leq -(2B)^{2^{k-2}} - h^i_M\big(-(2B)^{2^{k-2}} - i + 2\big).$$

So, by statement a) we get that

$$h^i_M(n-i+1) = 0 \text{ for all } n \leq -((2B)^{2^{k-2}} + \frac{1}{2}(2B)^{2^{k-1}}).$$

As the right hand side term is greater or equal than $(2B)^{2^{k-1}}$, and in view of (A) we get $B^k_M(n) = d^0_M(n) + \sum_{i=2}^k \binom{k-1}{i-1} h^i_M(n-i+1) = d^0_M(n) = 0$ for $n \leq -(2B)^{2^{k-1}}$. This is statement b). $\square$

10.16. **Lemma.** *Let K be an infinite field, let R be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module with $\delta(M) = 0$. Then $h^1_M(n) \neq 0$ and $d^0_M(n) \neq 0$ for all $n \ll 0$.*

Proof. It suffices to show that $h^1_M(n) \neq 0$ for all $n \ll 0$ (cf. 10.6 B) b)). By 10.12 B) a) there is some $\mathfrak{p} \in \mathrm{Ass}_R(M) \cap \mathrm{mProj}(R)$. By 10.3 C) there is some $t \in \mathbb{Z}$ and some $m \in M_t \setminus \{0\}$ such that $\mathfrak{p} = (0 :_R Rm)$. So, the multiplication map $\cdot m : R \to M(t)$, $x \mapsto xm$ has kernel $\mathfrak{p}$ and hence gives rise to a short exact sequence of graded R-modules $0 \to R/\mathfrak{p} \to M(t) \to N \to 0$. Applying cohomology we get exact sequences of K-vector spaces

$$H^0_{R_+}(N)_n \to H^1_{R_+}(R/\mathfrak{p})_n \to H^1_{R_+}(M)_{n+t}$$

for all $n \in \mathbb{Z}$. As N is finitely generated, $H^0_{R_+}(N)_n = \Gamma_{R_+}(N)_n \subseteq N_n = 0$ for all $n \ll 0$. Therefore it is enough to show that $h^1_{R/\mathfrak{p}}(n) \neq 0$ for all $n \ll 0$. As $\mathfrak{p} \subsetneq R_+ = \langle R_1 \rangle$ we have $\mathfrak{p} \cap R_1 \subsetneq R_1$ and hence find some $x \in R_1 \setminus \mathfrak{p}$. As $\mathfrak{p}$ is prime, we have $x \in \mathrm{NZD}_R(R/\mathfrak{p})$ and hence get an exact sequence of graded R-modules

$$(*) \qquad 0 \to R/\mathfrak{p} \xrightarrow{x\cdot} (R/\mathfrak{p})(1) \to (R/(\mathfrak{p} + xR))(1) \to 0.$$

As $x \in R_+ \cap \mathrm{NZD}_R(R/\mathfrak{p})$ we have $H^0_{R_+}(R/\mathfrak{p}) = \Gamma_{R_+}(R/\mathfrak{p}) = 0$.

Now, let $\mathfrak{q} \in \min(\mathfrak{p} + xR)$. As $\mathfrak{p} + xR = (0 :_R R/(\mathfrak{p} + xR))$ it follows by 1.8 C) c) that $\mathfrak{q} \in \mathrm{Ass}_R(R/(\mathfrak{p} + xR))$. By 10.3 C) it follows $\mathfrak{q} \in {}^*\mathrm{Spec}(R)$. As $\mathfrak{p} \subsetneq \mathfrak{q} \subseteq R_+$ and as $\mathfrak{p} \in \mathrm{mProj}(R)$ we get $\mathfrak{q} = R_+$. Therefore $\min(\mathfrak{p} + xR) = \{R_+\}$ and thus $R_+ = \sqrt{\mathfrak{p} + xR}$. So, there is some $r \in \mathbb{N}$ with $(R_+)^r \subseteq \mathfrak{p} + xR$ so that $(R_+)^r(R/(\mathfrak{p} + xR)) = 0$. So $(R/(\mathfrak{p} + xR))(1)$ is R_+-torsion and hence $H^0_{R_+}((R/(\mathfrak{p} + xR))(1)) = (R/(\mathfrak{p} + xR))(1)$ and $H^1_{R_+}((R/(\mathfrak{p} + xR))(1)) = 0$ (cf. 3.17). So, if we apply cohomology to $(*)$ and pass to graded components, we get exact sequences of K-vector spaces for all $n \in \mathbb{Z}$

$$0 \to (R/(\mathfrak{p} + xR))_{n+1} \to H^1_{R_+}(R/\mathfrak{p})_n \xrightarrow{x\cdot} H^1_{R_+}(R/\mathfrak{p})_{n+1} \to 0.$$

As $(R/(\mathfrak{p} + xR))_0 \cong K \neq 0$ it follows that $h^1_{R_+}(R/\mathfrak{p})_n \neq 0$ for all $n < 0$. $\square$

10.17. **Theorem.** *Let K be an algebraically closed field, let R be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Then:*

a) $h^{\delta(M)+1}_M(n) \neq 0$ for all $n \ll 0$.

b) If $\delta(M) > 0$, then $B^{\delta(M)}_M(n) = 0$ for all $n \leq -\big(2B^{\delta(M)}_M(0)\big)^{2^{\delta(M)-1}}$.

c) If $\delta(M) > 0$, then $d_M^0(n) = 0$ and $h_M^i(n-i+1) = 0$ for all $n \leq -\big(2B_M^{\delta(M)}(0)\big)^{2^{\delta(M)-1}}$ and all $i \in \{1, \ldots, \delta(M)\}$.

Proof. b) and c) are clear from 10.15 b), 10.6 B) b) and 10.9 B) b) applied with $k = \delta(M)$.

We show statement a) by induction on $\delta := \delta(M)$. If $\delta = 0$ we conclude by 10.16. So, let $\delta > 0$. By 10.12 B) b) and as $h^{\delta+1}_{M/\Gamma_{R_+}(M)} = h_M^{\delta+1}$ (cf. 9.13 D)) we may replace M by $M/\Gamma_{R_+}(M)$ and hence assume that $\Gamma_{R_+}(M) = 0$. So, by 10.14 c) there is some $t \in \mathbb{N}$ and some $x \in R_t \cap \mathrm{NZD}_R(M)$ such that $\delta(M/xM) = \delta - 1$. Thus, by induction $h^\delta_{M/xM}(n) \neq 0$ for all $n \ll 0$.

By 9.7 C) we have exact sequences of K-vector spaces

$$H^\delta_{R_+}(M)_n \to H^\delta_{R_+}(M/xM)_n \to H^{\delta+1}_{R_+}(M)_{n-t}.$$

By statement c) the first term in these sequences vanishes for all $n \ll 0$, so that $h_M^{\delta+1}(n-t) \geq h^\delta_{M/xM}(n) > 0$ for all $n \ll 0$. This proves statement a). □

10.18. **Corollary.** *Let K be an algebraically closed field, let R be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Then:*

a) $h_M^i(n) = 0$ for all $n \ll 0$ and all $i \leq \delta(M)$.

b) $h_M^{\delta(M)+1}(n) \neq 0$ for all $n \ll 0$. □

This latter result is an algebraic version of a fundamental theorem on cohomology of projective varieties (or schemes): The *Vanishing Theorem of Severi-Enriques-Zariski-Serre* (cf. [H1, Chapter III], [B-S, Theorem 20.4.20], [Se] and also 10.19 D), 10.20 and 12.16 below).

10.19. **Remark.** A) Let K be an infinite field, let R be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. We write $d_M^i := h_M^{i+1}$ for all $i \in \mathbb{N}$. For $i \in \mathbb{N}_0$, the function

$$d_M^i : \mathbb{Z} \to \mathbb{N}_0,\ n \mapsto d_M^i(n)$$

is called the *i-th geometric cohomological Hilbert function of M.*

If $\dim(M) \leq 0$, we have $M_n = 0$ for all $n \gg 0$, so that M is R_+-torsion, and hence $d_M^0 = d_0^0 = 0$ (cf. 10.6 B) a)). Combining this with 4.11, for $i \geq \dim(M)$ we get $d_M^i = 0$.

Finally, for $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ we may write:

$$B_M^k(n) = \sum_{i=0}^{k-1} \binom{k-1}{i} d_M^i(n-i).$$

B) The sequence

$$\mathrm{diag}(M) := (d_M^i(-i))_{i=0}^{\dim(M)-1}$$

is called the *cohomology diagonal of M.* If $\delta \in \{0,\dots,\dim(M)-1\}$, the sequence

$$\operatorname{diag}(M)^{<\delta} := (d^i_M(-i))_{i=0}^{\delta-1}$$

is called the *cohomology diagonal of M below level δ.*

Now, fix $t \in \mathbb{N}_0$ and $\delta \in \{0,\dots,t\}$. We consider the polynomial

$$G^\delta = G^\delta(u_0,\dots,u_{\delta-1}) \in \mathbb{Z}[u_0,\dots,u_{\delta-1}]$$

given by

$$G^\delta(u_0,\dots,u_{\delta-1}) := \begin{cases} 0, & \text{if } \delta = 0,\\ \left(2\sum_{i=0}^{\delta-1}\binom{\delta-1}{i}u_i\right)^{2^{\delta-1}}, & \text{if } \delta > 0.\end{cases}$$

Observe in particular: If $\delta > 0$, then $G^\delta(\operatorname{diag}^{<\delta}(M)) = (2B^\delta_M(0))^{2^{\delta-1}}$.

C) We write $\mathfrak{C}_t$ for the class of all pairs (R,M) in which R is a Noetherian homogeneous algebra over an algebraically closed field and M is a finitely generated graded R-module with $\dim(M) = t+1$. In addition, we introduce the class $\mathfrak{C}^\delta_t := \{(R,M) \in \mathfrak{C}_t \mid \delta(M) = \delta\}$. Then, as an immediate consequence of 10.15 we get:

a) If $(R,M) \in \mathfrak{C}^\delta_t$ and $i \in \{0,\dots,\delta-1\}$, then:
   (α) $d^i_M(n) \leq \frac{1}{2}G^\delta(\operatorname{diag}^\delta(M))$ for all $n \leq -i$;
   (β) $d^i_M(n) = 0$ for all $n \leq -G^\delta(\operatorname{diag}^\delta(M)) - i$.

This may concisely be expressed as follows:

b) The cohomology diagonal below level $\delta(M)$ bounds cohomology left of the diagonal below level $\delta(M)$.

D) The above statement C) a), 10.15 and 10.17 are special cases of results which are established in [B-M-M]. We call the type of bounds which are given in these results *a priori bounds of Severi type.*

We speak of *a priori bounds* as these bounds apply to arbitrary finitely generated graded modules. We speak of *bounds of Severi type* as these bounds concern the numbers $d^i_M(n)$ in the range $i < \delta(M)$. This has historic reasons: In 1942, Severi proved a *completeness result for linear systems on smooth surfaces in complex projective three-space* which can be formulated algebraically as follows (the hypotheses in Severis original result were more restrictive, indeed):

a) Let $f \in \mathbb{C}[x_0,x_1,x_2,x_3]$ be a homogeneous polynomial of positive degree and let $R := \mathbb{C}[x_0,x_1,x_2,x_3]/\langle f\rangle$. Let M be a finitely generated graded R-module such that $M_\mathfrak{p} \cong R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{mProj}(R)$. Then, $d^1_M(n) = 0$ for all $n \ll 0$.

For us, this result is an exercise. Namely, for all $\mathfrak{p} \in \operatorname{mProj}(R)$ the ring $R_\mathfrak{p}$ is a CM-ring of dimension 2, so that $\delta(M) = 2$.

In 1949, Enriques generalized Severi's result to polynomial rings in more than 4 indeterminates. In 1952, this was again generalized by Zariski, who showed that the ring R of statement a) may be replaced by the homogeneous coordinate ring of an arbitrary normal projective variety X of dimension greater than 1 over an algebraically closed field. Using a bit of basic local algebraic geometry, this generalization is still an easy exercise for us now: We namely have $\operatorname{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \geq 2$ for all $\mathfrak{p} \in \operatorname{mProj}(R)$, as $R_{\mathfrak{p}}$ is a normal Noetherian domain of dimension greater than 1.

It was finally Serre, who proved 1955 in geometric terms (cf. [Se]):

b) For all $(R, M) \in \mathfrak{C}_t^{\delta}$ and all $i < \delta$ it holds $d_M^i(n) = 0$ for all $n \ll 0$.

This is in fact nothing else than statement 10.18 a). In view of the sketched genealogy Serre's result often is quoted as the *Vanishing Theorem of Severi-Enriques-Zariski-Serre,* as stated already above. Observe that 10.15 is a "quantitative version" of this latter theorem.

E) Let us also mention here (cf. [B-M-M], [B-S, Chapter 16]):

a) The cohomology diagonal bounds cohomology right of the diagonal: There is a polynomial $H^t(u_0, \ldots, u_t) \in \mathbb{Z}[u_0, \ldots, u_t]$ such that for all $i \in \{1, \ldots, t\}$ and all $(R, M) \in \mathfrak{C}_t$ it holds:
   (α) $d_M^i(n) \leq \frac{1}{2} H^t(\operatorname{diag}(M))$ for all $n \geq -i$;
   (β) $d_M^i(n) = 0$ for all $n \geq H^t(\operatorname{diag}(M)) - i$.

The bound of statement a) is referred to as an *a priori bound of Castelnuovo type.* This refers to *Castelnuovo's Regularity Bound for smooth space curves* of 1893. Let us rephrase this latter bound in purely algebraic terms:

b) Let $\mathfrak{q} \subseteq R := \mathbb{C}[x_0, x_1, x_2, x_3]$ be a graded prime ideal such that $R_{\mathfrak{p}}/\mathfrak{q}_{\mathfrak{p}}$ is a principal ideal domain for each $\mathfrak{p} \in \operatorname{mProj}(R) \cap \operatorname{Var}(\mathfrak{q})$. Let $e \in \mathbb{N}$ be the leading coefficient of the Hilbert-Serre polynomial $P_{R/\mathfrak{q}}(x) \in \mathbb{Q}[x]$ of $R/\mathfrak{q}$. Let $i \in \{1, 2, 3\}$. Then $d_{\mathfrak{q}}^i(n) = 0$ for all $n \geq e - i + 1$.

Contrary to the classical reference bounds of Severi-Enriques-Zariski, the above result is not recovered by our a priori bounds. These are too general to be sharp enough in the very particular situation of statement b).

On the other hand, 9.10 yields

c) For each $(R, M) \in \mathfrak{C}$, for each $i \in \mathbb{N}$ and all $n \gg 0$ it holds $d_M^i(n) = 0$.

This result was shown by Serre [Se] in geometric terms. In view of its historic roots it sometimes is called the *Vanishing Theorem of Castelnuovo-Serre.*

10.20. **Remark and Exercise.** A) The striking fact around 10.15, 10.17 and 10.18 is that these results relate the *local behaviour of a finitely generated graded R-module M along* $\operatorname{Proj}(R)$ (measured in terms of the invariant $\delta(M)$)

to the *global cohomological behaviour of* M (expressed in terms of the left-vanishing of the geometric cohomological Hilbert functions d^i_M).

B) To illustrate the previous statement, we mention an application of 10.18, the *cohomological criterion for vector bundles.* We keep the notations of 10.19. Let $(R, M) \in \mathfrak{C}_t$. We say that M *defines a vector bundle* or that M is *locally free along* $\mathrm{Proj}(R)$ if the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is free for all $\mathfrak{p} \in \mathrm{Proj}(R)$. We say that $\mathrm{Proj}(R)$ is *smooth* if the local ring $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \mathrm{Proj}(R)$. Now, the announced criterion takes the following form:

Let $(R, M) \in \mathfrak{C}_t$ be such that R is a domain with $\dim(R) = \dim(M)$ and $\mathrm{Proj}(R)$ is smooth. Then, M defines a vector bundle if and only if the function d^i_M is left-vanishing for all $i < t$.

This follows by 10.18 and the fact that $\dim(R_\mathfrak{p}) = t$ for all $\mathfrak{p} \in \mathrm{mProj}(R)$ and the formula of Auslander-Buchsbaum.

11. Local Families of Fractions

In this section we develop the basic concepts which allow to link local cohomology to sheaf cohomology. We restrict ourselves to a fairly particular situation. For a general and more conceptual introduction of the theme we recommend to consult [B-S, Chapter 20].

11.0. **Notation.** Throughout this chapter, let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring.

11.1. **Reminder and Exercise.** A) Let M be a graded R-module. Fix $n \in \mathbb{Z}$ and let $U \subseteq M_n$ be an R_0-submodule. Then it is easy to verify that $(\sum_{u\in U} Ru) \cap M_n = U$. As a consequence of this we see:

a) If $U^{(0)} \subsetneq U^{(1)} \subsetneq \ldots \subsetneq U^{(r)}$ is a strictly ascending sequence of R_0-submodules of M_n, then $\sum_{u\in U^{(0)}} Ru \subsetneq \sum_{u\in U^{(1)}} Ru \subsetneq \ldots \subsetneq \sum_{u\in U^{(r)}} Ru$ is a strictly ascending sequence of R-submodules of M;
b) If $U^{(0)} \supsetneq U^{(1)} \supsetneq \cdots \supsetneq U^{(r)}$ is a strictly descending sequence of R_0-submodules of M_n, then $\sum_{u\in U^{(0)}} Ru \supsetneq \sum_{u\in U^{(1)}} Ru \supsetneq \cdots \supsetneq \sum_{u\in U^{(r)}} Ru$ is a strictly descending sequence of R-submodules of M.

B) For each $n \in \mathbb{Z}$ we may conclude from A) a), b):

a) If M is Noetherian, then M_n is a Noetherian R_0-module;
b) If M is Artinian, then M_n is an Artinian R_0-module.

C) Applying the previous statements to $M = R$ with $n = 0$, we get:

a) If R is Noetherian, then R_0 is a Noetherian ring;
b) If R is Artinian, then R_0 is an Artinian ring.

As a consequence of a) and B) a) we now get a generalization of 9.6 C):

c) If R is Noetherian and M is finitely generated, then M_n is a finitely generated R_0-module for each $n \in \mathbb{Z}$.

11.2. **Reminder and Exercise.** A) Let $S \subseteq R^{\mathrm{hom}}$ be a multiplicatively closed set of homogeneous elements of R. Let M be a graded R-module and let $n \in \mathbb{Z}$. We define the set of *homogeneous fractions of degree n with numerator in M and denominator in S* by

$$(S^{-1}M)_n := \{y \in S^{-1}M \mid \exists\, t \in \mathbb{Z}\, \exists\, s \in S_t\, \exists\, m \in M_{n+t} : y = \tfrac{m}{s}\}.$$

It is easy to verify that for each $n \in \mathbb{Z}$, the set $(S^{-1}M)_n$ is an additive subgroup of $S^{-1}M$. Show that $S^{-1}M = \bigoplus_{n\in\mathbb{Z}} (S^{-1}M)_n$ and that for $m, n \in \mathbb{Z}$, $y \in (S^{-1}R)_m$ and $z \in (S^{-1}M)_n$ we have $yz \in (S^{-1}M)_{m+n}$.

B) Applying the statements of A) to $M = R$ we now get that the family $((S^{-1}R)_n)_{n\in\mathbb{Z}}$ defines a grading on $S^{-1}R$, so that $S^{-1}R = \bigoplus_{n\in\mathbb{Z}} (S^{-1}R)_n$ is a graded ring. If we apply the above statements to M, we obtain that $S^{-1}M =$

$\bigoplus_{n\in\mathbb{Z}}(S^{-1}M)_n$ is a graded $S^{-1}R$-module. Moreover $S^{-1}(M(r)) = (S^{-1}M)(r)$ for all $r \in \mathbb{Z}$.

C) It is easy to verify that if $h : M \to N$ is a homomorphism of graded R-modules, then the induced homomorphism $S^{-1}h : S^{-1}M \to S^{-1}N$ is a homomorphism of graded $S^{-1}R$-modules. Finally, as the functors $S^{-1}\bullet$ of localization with respect to S and $\bullet_n$ of taking n-th graded components are both exact, we can say: If $0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ is an exact sequence of graded R-modules, then $0 \to (S^{-1}N)_n \xrightarrow{(S^{-1}h)_n} (S^{-1}M)_n \xrightarrow{(S^{-1}l)_n} (S^{-1}P)_n \to 0$ is an exact sequence of $(S^{-1}R)_0$-modules for each $n \in \mathbb{Z}$.

D) If R is Noetherian, then so is $S^{-1}R$. By 11.1 C) a) it thus follows that if R is Noetherian, then $(S^{-1}R)_0$ is a Noetherian ring. If M is finitely generated, then $S^{-1}M$ is a finitely generated $S^{-1}R$-module. By 11.1 C) c) we therefore obtain that if R is Noetherian and M is finitely generated, then the $(S^{-1}R)_0$-module $(S^{-1}M)_n$ is finitely generated for all $n \in \mathbb{Z}$.

11.3. **Remark and Definition.** A) Let R be positively graded and let $\mathfrak{p} \in \operatorname{Proj}(R)$. Then, $S(\mathfrak{p}) := (\bigcup_{n\in\mathbb{N}_0} R_n) \setminus \mathfrak{p}$ is a multiplicatively closed set of homogeneous elements of R. So, according to 11.2 B) we may consider the graded ring $S(\mathfrak{p})^{-1}R = \bigoplus_{n\in\mathbb{Z}}(S(\mathfrak{p})^{-1}R)_n$ and its 0-th graded component $R_{(\mathfrak{p})} := (S(\mathfrak{p})^{-1}R)_0$. The ring $R_{(\mathfrak{p})}$ is called the *homogeneous localization of R at $\mathfrak{p}$.* Keep in mind that

$$R_{(\mathfrak{p})} = \{y \in S(\mathfrak{p})^{-1}R \mid \exists\, t \in \mathbb{N}_0\, \exists\, z \in R_t \setminus \mathfrak{p}\, \exists\, x \in R_t : y = \tfrac{x}{z}\}.$$

B) Let M be a graded R-module. Then, according to 11.2 B) we may consider the graded $S(\mathfrak{p})^{-1}R$-module $S(\mathfrak{p})^{-1}M = \bigoplus_{n\in\mathbb{Z}}(S(\mathfrak{p})^{-1}M)_n$ and its 0-th graded component, that is the $R_{(\mathfrak{p})}$-module $M_{(\mathfrak{p})} := (S(\mathfrak{p})^{-1}M)_0$. The $R_{(\mathfrak{p})}$-module $M_{(\mathfrak{p})}$ is called the *homogeneous localization of M at $\mathfrak{p}$.* Keep in mind that

$$M_{(\mathfrak{p})} = \{w \in S(\mathfrak{p})^{-1}M \mid \exists\, t \in \mathbb{N}_0\, \exists\, z \in R_t \setminus \mathfrak{p}\, \exists\, m \in M_t : w = \tfrac{m}{z}\}.$$

C) Let $h : M \to N$ be a homomorphism of graded R-modules. According to 11.2 C) we have an induced homomorphism of graded $S(\mathfrak{p})^{-1}R$-modules

$$S(\mathfrak{p})^{-1}h : S(\mathfrak{p})^{-1}M \to S(\mathfrak{p})^{-1}N$$

and thus may consider its 0-th graded component

$$h_{(\mathfrak{p})} := (S(\mathfrak{p})^{-1}h)_0 : M_{(\mathfrak{p})} \to N_{(\mathfrak{p})},$$

which we call correspondingly the *homogeneous localization of h at $\mathfrak{p}$.* According to 11.2 C) we can say: If $0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$ is an exact sequence of graded R-modules, then $0 \to N_{(\mathfrak{p})} \xrightarrow{h_{(\mathfrak{p})}} M_{(\mathfrak{p})} \xrightarrow{l_{(\mathfrak{p})}} P_{(\mathfrak{p})} \to 0$ is an exact sequence of $R_{(\mathfrak{p})}$-modules.

D) Let $\mathfrak{a} \subseteq R$ be a graded ideal and let $i : \mathfrak{a} \rightarrowtail R$ be the inclusion homomorphism. By C) we have a monomorphism of $R_{(\mathfrak{p})}$-modules $i_{(\mathfrak{p})} : \mathfrak{a}_{(\mathfrak{p})} \rightarrowtail R_{(\mathfrak{p})}$. So, we may identify $\mathfrak{a}_{(\mathfrak{p})}$ with its image in $R_{(\mathfrak{p})}$ and thus write:

a) $\mathfrak{a}_{(\mathfrak{p})} = \{y \in R_{(\mathfrak{p})} \mid \exists\, t \in \mathbb{N}_0\, \exists\, z \in R_t \setminus \mathfrak{p}\, \exists\, x \in \mathfrak{a}_t : y = \frac{x}{z}\}$.

Using this convention we can say:

b) $R_{(\mathfrak{p})}$ is a local ring with maximal ideal $\mathfrak{p}_{(\mathfrak{p})}$.

Indeed, as $S(\mathfrak{p}) \cap \mathfrak{p} = \emptyset$, we have $S(\mathfrak{p})^{-1}\mathfrak{p} \subsetneq S(\mathfrak{p})^{-1}R$ and consequently (as $1_{S(\mathfrak{p})^{-1}R} \in (S(\mathfrak{p})^{-1}R)_0$)

$$\mathfrak{p}_{(\mathfrak{p})} = (S(\mathfrak{p})^{-1}\mathfrak{p})_0 \subsetneq (S(\mathfrak{p})^{-1}R)_0 = R_{(\mathfrak{p})}.$$

So $\mathfrak{p}_{(\mathfrak{p})}$ is a proper ideal of $R_{(\mathfrak{p})}$. Now, let $y \in R_{(\mathfrak{p})} \setminus \mathfrak{p}_{(\mathfrak{p})}$. With appropriate $t \in \mathbb{N}_0$, $z \in R_t \setminus \mathfrak{p}$ and $x \in R_t$ we have $y = \frac{x}{z}$. According to a) we have $x \notin \mathfrak{p}$. It follows $\frac{z}{x} \in R_{(\mathfrak{p})}$ and $y\frac{z}{x} = 1$, hence $y \in (R_{(\mathfrak{p})})^*$. Therefore $R_{(\mathfrak{p})} \setminus \mathfrak{p}_{(\mathfrak{p})} \subseteq (R_{(\mathfrak{p})})^*$ and this proves our claim.

E) According to 11.2 D) we can say:

a) If R is Noetherian, then $R_{(\mathfrak{p})}$ is Noetherian.
b) If R is Noetherian and M is finitely generated, then $M_{(\mathfrak{p})}$ is finitely generated over $R_{(\mathfrak{p})}$.

11.4. **Remark and Exercise.** A) Let R be positively graded. We furnish $\mathrm{Proj}(R)$ with its *Zariski topology* so that the closed sets of $\mathrm{Proj}(R)$ are precisely the sets of the form $Z \cap \mathrm{Proj}(R)$, where $Z \subseteq \mathrm{Spec}(R)$ is closed. If $W \subseteq \mathrm{Spec}(R)$, let $\overline{W}^{\mathrm{Spec}(R)}$ denote the topological closure of W in $\mathrm{Spec}(R)$. Then, we may say: If $Z \subseteq \mathrm{Proj}(R)$ is closed, then $Z = \mathrm{Proj}(R) \cap \overline{Z}^{\mathrm{Spec}(R)}$.

B) Keep in mind that a topological space X is said to be *Noetherian* if any open subset $U \subseteq X$ is quasi-compact. It is equivalent that any descending sequence $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$ of closed subsets eventually becomes stationary, that is we find an $n_0 \in \mathbb{N}_0$ such that $Z_n = Z_{n_0}$ for all $n \geq n_0$. Moreover keep in mind that $\mathrm{Spec}(R)$ is a Noetherian topological space, whenever R is a Noetherian ring. So, it follows from A) that if R is Noetherian, then $\mathrm{Proj}(R)$ is a Noetherian topological space.

C) If $\mathfrak{a} \subseteq R$ is a graded ideal, we set $\mathcal{Z}(\mathfrak{a}) := \mathrm{Var}(\mathfrak{a}) \cap \mathrm{Proj}(R)$ and $\mathcal{U}(\mathfrak{a}) := \mathrm{Proj}(R) \setminus \mathcal{Z}(\mathfrak{a})$. Then:

a) If $Z \subseteq \mathrm{Proj}(R)$ is closed, then there is a graded ideal $\mathfrak{a} \subseteq R_+$ of R such that $Z = \mathcal{Z}(\mathfrak{a})$.

Indeed, there is some ideal $\mathfrak{b} \subseteq R$ such that $Z = \mathrm{Var}(\mathfrak{b}) \cap \mathrm{Proj}(R)$. Let $\mathfrak{b}'$ be the *graded hull of* $\mathfrak{b}$, i.e. $\mathfrak{b}' := \bigcap\limits_{\substack{\mathfrak{c} \subseteq R \text{ graded ideal} \\ \mathfrak{b} \subseteq \mathfrak{c}}} \mathfrak{c}$. Then by 8.3 B) b) $\mathfrak{b}' \subseteq R$ is a graded ideal such that $\mathfrak{b} \subseteq \mathfrak{b}'$ and $\mathrm{Var}(\mathfrak{b}') \cap \mathrm{Proj}(R) = Z$. Now, let $\mathfrak{a} := R_+ \cap \mathfrak{b}'$. Then clearly $Z \subseteq \mathcal{Z}(\mathfrak{a})$. Conversely, if $\mathfrak{p} \in \mathcal{Z}(\mathfrak{a})$, then $R_+ \cap \mathfrak{b}' \subseteq \mathfrak{p}$ and hence $R_+ \subseteq \mathfrak{p}$ or $\mathfrak{b}' \subseteq \mathfrak{p}$. As $\mathfrak{p} \in \mathrm{Proj}(R)$ we have $R_+ \not\subseteq \mathfrak{b}$ and hence $\mathfrak{b}' \subseteq \mathfrak{p}$, thus $\mathfrak{p} \in \mathrm{Var}(\mathfrak{b}') \cap \mathrm{Proj}(R) = Z$. So $Z = \mathcal{Z}(\mathfrak{a})$.

Now, obviously we have:

b) If $U \subseteq \mathrm{Proj}(R)$ is open, then there is a graded ideal $\mathfrak{a} \subseteq R_+$ of R such that $U = \mathcal{U}(\mathfrak{a})$.

11.5. **Remark, Exercise and Definition.** A) Let R be positively graded and let $U \subseteq \mathrm{Proj}(R)$ be an open set. Let M be a graded R-module. Consider the direct products $\prod_{\mathfrak{p}\in U} R_{(\mathfrak{p})}$ and $\prod_{\mathfrak{p}\in U} M_{(\mathfrak{p})}$, which by convention are 0 if $U = \emptyset$. Then, clearly $\prod_{\mathfrak{p}\in U} R_{(\mathfrak{p})}$ is a ring and $\prod_{\mathfrak{p}\in U} M_{(\mathfrak{p})}$ is a modul over this ring in a natural way.

B) A family $(z_\mathfrak{p})_{\mathfrak{p}\in U} \in \prod_{\mathfrak{p}\in U} M_{(\mathfrak{p})}$ is called a *local family of homogeneous fractions over U with numerators in M* if for each $\mathfrak{p} \in U$ there are an open neighbourhood $W \subseteq U$ of $\mathfrak{p}$ in U, an integer $t \in \mathbb{N}_0$ and homogeneous elements $m \in M_t$, $s \in R_t$ such that for each $\mathfrak{q} \in W$ we have $s \notin \mathfrak{q}$ and $z_\mathfrak{q} = \frac{m}{s}$.

We write $\widetilde{M}(U)$ for the set of all local families of homogeneous fractions over U with numerators in M. We may apply this concept in the case $M = R$ and thus consider the set $\widetilde{R}(U)$ of all local families of homogeneous fractions over U with numerators in R. Now, it is not hard to verify:

a) $\widetilde{R}(U)$ is a subring of $\prod_{\mathfrak{p}\in U} R_{(\mathfrak{p})}$;

b) $\widetilde{M}(U)$ is an $\widetilde{R}(U)$-submodule of $\prod_{\mathfrak{p}\in U} M_{(\mathfrak{p})}$.

C) There is a homomorphism of Abelian groups

$$\eta_M = \eta_M^U : M_0 \to \widetilde{M}(U),\ m \mapsto (\tfrac{m}{1})_{\mathfrak{p}\in U}.$$

Moreover, η_R^U is a homomorphism of rings. From now on, we view $\widetilde{M}(U)$ as an R_0-module be means of the homomorphism of rings η_R^U. Then, it is easy to prove that $\eta_M^U : M_0 \to \widetilde{M}(U)$ is a homomorphism of R_0-modules.

D) Let $h : M \to N$ be a homomorphism of graded R-modules. Then, it is not hard to see that there is a homomorphism of R_0-modules

$$\widetilde{h}(U) : \widetilde{M}(U) \to \widetilde{N}(U),\ (z_\mathfrak{p})_{\mathfrak{p}\in U} \mapsto (h_{(\mathfrak{p})}(z_\mathfrak{p}))_{\mathfrak{p}\in U}.$$

Now, without any further difficulties but with a bit of work one may verify:

a) The assignment $\widetilde{\bullet}(U) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(\widetilde{M}(U) \xrightarrow{\widetilde{h}(U)} \widetilde{N}(U)\big)$ defines a linear functor from graded R-modules to R_0-modules.

Moreover:

b) The functor $\widetilde{\bullet}(U)$ is left exact: If $0 \to N \xrightarrow{h} M \xrightarrow{l} P$ is an exact sequence of graded R-modules, then $0 \to \widetilde{N}(U) \xrightarrow{\widetilde{h}(U)} \widetilde{M}(U) \xrightarrow{\widetilde{l}(U)} \widetilde{P}(U)$ is an exact sequence of R_0-modules.

Indeed, by the exactness of the sequences $0 \to N_{(\mathfrak{p})} \xrightarrow{h_{(\mathfrak{p})}} M_{(\mathfrak{p})} \xrightarrow{l_{(\mathfrak{p})}} P_{(\mathfrak{p})}$ for all $\mathfrak{p} \in U$ it is easy to see that the homomorphism $\widetilde{h}(U)$ is injective and that $\widetilde{l}(U) \circ \widetilde{h}(U) = 0$. It remains to show that $\mathrm{Ker}(\widetilde{l}(U)) \subseteq \mathrm{Im}(\widetilde{h}(U))$. So, let $z = (z_{\mathfrak{p}})_{\mathfrak{p} \in U} \in \mathrm{Ker}(\widetilde{l}(U))$. Then $z_{\mathfrak{p}} \in \mathrm{Ker}(l_{(\mathfrak{p})})$ for all $\mathfrak{p} \in U$. As $\mathrm{Ker}(l_{(\mathfrak{p})}) = \mathrm{Im}(h_{(\mathfrak{p})})$ there is some $y_{\mathfrak{p}} \in N_{(\mathfrak{p})}$ with the property that $h_{(\mathfrak{p})}(y_{\mathfrak{p}}) = z_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$. It suffices to show that the family of fractions $(y_{\mathfrak{p}})_{\mathfrak{p} \in U} \in \prod_{\mathfrak{p} \in U} M_{(\mathfrak{p})}$ is local (in the sense of B)). So, fix $\mathfrak{p} \in U$. As $z \in \widetilde{M}(U)$, there is an open neighbourhood $W \subseteq U$ of $\mathfrak{p}$, an integer $t \in \mathbb{N}_0$ and a pair of elements $(m, s) \in M_t \times R_t$ such that for each $\mathfrak{q} \in W$ we have $s \notin \mathfrak{q}$ and $z_{\mathfrak{q}} = \frac{m}{s} \in M_{(\mathfrak{q})}$. As $\frac{l(m)}{s} = l_{(\mathfrak{p})}(z_{\mathfrak{p}}) = 0$ in $P_{(\mathfrak{p})}$, there is an integer $r \in \mathbb{N}_0$ and a homogeneous element $u \in R_r \setminus \mathfrak{p}$ such that $ul(m) = 0$, hence $l(um) = 0$. By the exactness of our original sequence we thus find some element $n \in N_{r+t}$ with $h(n) = um$. Now, clearly $V := W \cap \mathcal{U}(uR)$ is an open neighbourhood of $\mathfrak{p}$ in U and for all $\mathfrak{q} \in V$ we have $us \notin \mathfrak{q}$ and $h_{(\mathfrak{q})}(y_{\mathfrak{q}}) = z_{\mathfrak{q}} = \frac{m}{s} = \frac{um}{us} = \frac{h(n)}{us} = h_{(\mathfrak{q})}(\frac{n}{us})$. By the injectivity of $h_{(\mathfrak{q})}$ it follows $y_{\mathfrak{q}} = \frac{n}{us}$ for all $\mathfrak{q} \in V$.

E) If $h : M \to N$ is a homomorphism of graded R-modules, then there is a commutative diagram

$$\begin{tikzcd} M_0 \arrow[r, "\eta_M^U"] \arrow[d, "h_0"'] & \widetilde{M}(U) \arrow[d, "\widetilde{h}(U)"] \\ N_0 \arrow[r, "\eta_N^U"] & \widetilde{N}(U). \end{tikzcd}$$

Hence, the assignment $\eta = \eta^U : M \rightsquigarrow \big(M_0 \xrightarrow{\eta_M^U} \widetilde{M}(U)\big)$ is a natural transformation of functors (from graded R-modules to R_0-modules) $\eta^U : \bullet_0 \to \widetilde{\bullet}(U)$.

11.6. **Remark and Exercise.** Let R be Noetherian and let $\mathfrak{a}, \mathfrak{b} \subseteq R$ be graded ideals. Let I be a *injective R-module. Then $\Gamma_{\mathfrak{a} \cap \mathfrak{b}}(I) = \Gamma_{\mathfrak{a}}(I) + \Gamma_{\mathfrak{b}}(I)$. We leave the proof of this equality as an exercise, which can be done along the traces of the proof of 4.14.

11.7. **Exercise.** Let R be positively graded and let $\mathfrak{b}, \mathfrak{c} \subseteq R_+$ be graded ideals of R. Show that $\mathcal{U}(\mathfrak{b}) \cap \mathcal{U}(\mathfrak{c}) = \mathcal{U}(\mathfrak{b} \cap \mathfrak{c})$.

11.8. **Proposition.** *Let R be Noetherian and positively graded, let $\mathfrak{a} \subseteq R$ be a graded ideal with $\mathfrak{a} \subseteq R_+$, and let $U := \mathcal{U}(\mathfrak{a}) \subseteq \mathrm{Proj}(R)$ be the open set defined by $\mathfrak{a}$. Then:*

a) For each graded R-module M we have $\mathrm{Ker}(\eta_M^U : M_0 \to \widetilde{M}(U)) = \Gamma_{\mathfrak{a}}(M)_0$.

b) For each *injective R-module M the homomorphism $\eta_I^U : I_0 \to \widetilde{I}(U)$ is surjective.

Proof. a): "$\subseteq$": Let $m \in \mathrm{Ker}(\eta_M^U)$, so that $\frac{m}{1} = 0$ in $M_{(\mathfrak{p})}$ for all $\mathfrak{p} \in U$. So, for each $\mathfrak{p} \in U$ there is some $s_{\mathfrak{p}} \in S(\mathfrak{p})$ with $s_{\mathfrak{p}} m = 0$. Consider the graded ideal $\mathfrak{b} := \sum_{\mathfrak{p} \in U} Rs_{\mathfrak{p}}$ of R. It follows $\mathfrak{b}m = 0$.

Now, let $\mathfrak{q} \in \min(\mathfrak{b})$. We aim to show that $\mathfrak{a} \subseteq \mathfrak{q}$. Assume to the contrary that $\mathfrak{a} \nsubseteq \mathfrak{q}$. As $\mathfrak{a} \subseteq R_+$ it follows $R_+ \nsubseteq \mathfrak{q}$. As $\mathfrak{q} \in \min(\mathfrak{b}) \subseteq \mathrm{Ass}_R(R/\mathfrak{b}) \subseteq {}^*\mathrm{Spec}(R)$ (cf. 1.8 C) c) and 10.3 C)), it follows $\mathfrak{q} \in \mathrm{Proj}(R)$. But now $\mathfrak{a} \nsubseteq \mathfrak{q}$ implies that $\mathfrak{q} \in U$ and we get the contradiction that $\mathfrak{q} \not\ni s_\mathfrak{q} \in \mathfrak{b} \subseteq \mathfrak{q}$. So, indeed $\mathfrak{a} \subseteq \mathfrak{q}$ whenever $\mathfrak{q} \in \min(\mathfrak{b})$. But this yields $\min(\mathfrak{b}) \subseteq \mathrm{Var}(\mathfrak{a})$ and hence

$$\mathfrak{a} \subseteq \bigcap_{\mathfrak{p} \in \mathrm{Var}(\mathfrak{a})} \mathfrak{p} \subseteq \bigcap_{\mathfrak{q} \in \min(\mathfrak{b})} \mathfrak{q} = \sqrt{\mathfrak{b}}.$$

So, there is some $n \in \mathbb{N}$ with $\mathfrak{a}^n \subseteq \mathfrak{b}$ and it follows $\mathfrak{a}^n m \subseteq \mathfrak{b} m = 0$, whence $m \in \Gamma_\mathfrak{a}(M)$.

"$\supseteq$": Let $m \in \Gamma_\mathfrak{a}(M)_0$. Then, there is some $n \in \mathbb{N}$ with $\mathfrak{a}^n m = 0$. Let $\mathfrak{p} \in U = \mathcal{U}(\mathfrak{a})$. As $\mathfrak{p} \not\supseteq \mathfrak{a}$ there is some element $s_\mathfrak{p} \in \mathfrak{a} \setminus \mathfrak{p}$. At least one homogeneous component of $s_\mathfrak{p}$ is not contained in $\mathfrak{p}$. As $\mathfrak{a}$ is graded, this component is yet contained in $\mathfrak{a}$. This allows to assume that $s_\mathfrak{p}$ is homogeneous. It now follows that $s_\mathfrak{p}^n m = 0$ and hence that $\frac{m}{1} = \frac{s_\mathfrak{p}^n m}{s_\mathfrak{p}^n} = \frac{0}{s_\mathfrak{p}^n} = 0 \in M_{(\mathfrak{p})}$. As $\mathfrak{p}$ was arbitrary it follows $m \in \mathrm{Ker}(\eta_M^U)$.

b) Let $(z_\mathfrak{p})_{\mathfrak{p} \in U} \in \widetilde{I}(U)$. For each $\mathfrak{p} \in U$ there is an open neighbourhood $W_\mathfrak{p} \subseteq U$ of $\mathfrak{p}$, an integer $t_\mathfrak{p} \in \mathbb{N}_0$ and a pair of elements $(s_\mathfrak{p}, m_\mathfrak{p}) \in R_{t_\mathfrak{p}} \times I_{t_\mathfrak{p}}$ such that for each $\mathfrak{q} \in W_\mathfrak{p}$ we have $s_\mathfrak{p} \notin \mathfrak{q}$ and $z_\mathfrak{q} = \frac{m_\mathfrak{p}}{s_\mathfrak{p}}$. Now, fix $\mathfrak{p} \in U$. According to 8.22 C) b) the multiplication homomorphism $s_\mathfrak{p} \cdot : I/\Gamma_{(s_\mathfrak{p})}(I) \to (I/\Gamma_{(s_\mathfrak{p})}(I))(t_\mathfrak{p})$ is an isomorphism of graded R-modules. So, in particular the homomorphism $I_0 \to I_{t_\mathfrak{p}}/\Gamma_{(s_\mathfrak{p})}(I)_{t_\mathfrak{p}}$ given by $m \mapsto s_\mathfrak{p} m + \Gamma_{(s_\mathfrak{p})}(I)_{t_\mathfrak{p}}$ is surjective. Therefore, we have elements $n_\mathfrak{p} \in I_0$ and $y_\mathfrak{p} \in \Gamma_{(s_\mathfrak{p})}(I)_{t_\mathfrak{p}}$ such that $m_\mathfrak{p} = s_\mathfrak{p} n_\mathfrak{p} + y_\mathfrak{p}$. Now, there is some $\nu_\mathfrak{p} \in \mathbb{N}$ with $s_\mathfrak{p}^{\nu_\mathfrak{p}} y_\mathfrak{p} = 0$. It thus follows for all $\mathfrak{q} \in W_\mathfrak{p}$

$$z_\mathfrak{q} = \frac{m_\mathfrak{p}}{s_\mathfrak{p}} = \frac{s_\mathfrak{p}^{\nu_\mathfrak{p}} m_\mathfrak{p}}{s_\mathfrak{p}^{\nu_\mathfrak{p}+1}} = \frac{s_\mathfrak{p}^{\nu_\mathfrak{p}}(s_\mathfrak{p} n_\mathfrak{p} + y_\mathfrak{p})}{s_\mathfrak{p}^{\nu_\mathfrak{p}+1}} = \frac{s_\mathfrak{p}^{\nu_\mathfrak{p}+1} n_\mathfrak{p}}{s_\mathfrak{p}^{\nu_\mathfrak{p}+1}} = \frac{n_\mathfrak{p}}{1}.$$

So, for each $\mathfrak{p} \in U$ there is some $n_\mathfrak{p} \in I_0$ such that for all $\mathfrak{q} \in W_\mathfrak{p}$ we have $z_\mathfrak{q} = \frac{n_\mathfrak{p}}{1}$.

As $\mathrm{Proj}(R)$ is a Noetherian topological space (cf. 11.4 B)) there are finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in U$ such that $W_{\mathfrak{p}_1} \cup \cdots \cup W_{\mathfrak{p}_r} = U$. We now prove our claim by induction on r.

If $r = 0$ the claim is clear. Let $r > 0$. We set $V := W_{\mathfrak{p}_1} \cup \cdots \cup W_{\mathfrak{p}_{r-1}}$. By induction, the homomorphism $\eta_I^V : I_0 \to \widetilde{I}(V)$ is surjective. Observe that $z\restriction_V := (z_\mathfrak{q})_{\mathfrak{q} \in V} \in \widetilde{I}(V)$. So, there is some $m \in I_0$ with $\eta_I^V(m) = z\restriction_V$ and hence $\frac{m}{1} = z_\mathfrak{q}$ for all $\mathfrak{q} \in V$.

We set $\mathfrak{p} := \mathfrak{p}_r$ and $W := W_\mathfrak{p}$. Then, by the above observation there is some $n = n_\mathfrak{p} \in I_0$ such that $z_\mathfrak{q} = \frac{n}{1}$ for all $\mathfrak{q} \in W$. According to 11.4 C) b) there are graded ideals $\mathfrak{b}, \mathfrak{c} \subseteq R_+$ of R such that $V = \mathcal{U}(\mathfrak{b})$, $W = \mathcal{U}(\mathfrak{c})$. Now, by 11.7 we have $V \cap W = \mathcal{U}(\mathfrak{b}) \cap \mathcal{U}(\mathfrak{c}) = \mathcal{U}(\mathfrak{b} \cap \mathfrak{c})$. For each $\mathfrak{q} \in \mathcal{U}(\mathfrak{b} \cap \mathfrak{c}) = V \cap W$ we have $\frac{m}{1} = z_\mathfrak{q} = \frac{n}{1}$ so that $\eta_I^{V \cap W}(m) = \eta_I^{V \cap W}(n)$, hence $m - n \in \mathrm{Ker}(\eta_I^{V \cap W})$. By a) it follows $m-n \in \Gamma_{\mathfrak{b}\cap\mathfrak{c}}(I)_0$. According to 11.6 we have $\Gamma_{\mathfrak{b}\cap\mathfrak{c}}(I) = \Gamma_{\mathfrak{b}}(I) + \Gamma_{\mathfrak{c}}(I)$ so that

$$m-n \in (\Gamma_{\mathfrak{b}}(I) + \Gamma_{\mathfrak{c}}(I))_0 = \Gamma_{\mathfrak{b}}(I)_0 + \Gamma_{\mathfrak{c}}(I)_0.$$

So, there are elements $p \in \Gamma_{\mathfrak{b}}(I)_0$ and $q \in \Gamma_{\mathfrak{c}}(I)_0$ such that $m-n = p-q$, hence $m-p = n-q =: y$. According to a) we have $\eta_I^V(p) = 0$ and hence $\frac{p}{1} = 0 \in I_{(\mathfrak{q})}$ for all $\mathfrak{q} \in V$. Similarly $\frac{q}{1} = 0 \in I_{(\mathfrak{q})}$ for all $\mathfrak{q} \in W$. Therefore, for each $\mathfrak{q} \in V$ we have

$$\frac{y}{1} = \frac{m-p}{1} = \frac{m}{1} - \frac{p}{1} = \frac{m}{1} - 0 = \frac{m}{1} = z_{\mathfrak{q}}$$

in $I_{(\mathfrak{q})}$. Similarly, for each $\mathfrak{q} \in W$ we have $\frac{y}{1} = z_{\mathfrak{q}}$ in $I_{(\mathfrak{q})}$. It follows that $\frac{y}{1} = z_{\mathfrak{q}}$ for all $\mathfrak{q} \in V \cup W = U$. This shows that $\eta_I^U(y) = (z_{\mathfrak{q}})_{\mathfrak{q}\in U}$. It follows that η_I^U is surjective. $\square$

11.9. **Remark and Exercise.** A) Let F and G be two additive functors from graded R-modules to R_0-modules. Let $\mu : M \rightsquigarrow \big(F(M) \xrightarrow{\mu_M} G(M)\big)$ be a natural transformation of functors from graded R-modules to R_0-modules (cf. 8.19). Let $\mathbb{I}_* : M \rightsquigarrow ((I_M^\bullet, d_M^\bullet); a_M)$ be a choice of *injective resolutions on graded R-modules (cf. 8.20 A)). Now, we can form the right derived functors $\mathcal{R}^n F := \mathcal{R}^n_{\mathbb{I}_*} F$ and $\mathcal{R}^n G := \mathcal{R}^n_{\mathbb{I}_*} G$ (cf. 8.20 A)). Now, completely analogue as in 4.12 A), B) we may define the *n-th right derived (transformation) of* μ for each $n \in \mathbb{N}_0$, that is

$$\mathcal{R}^n\mu : \mathcal{R}^n F \to \mathcal{R}^n G,\ M \rightsquigarrow \mathcal{R}^n\mu_M$$

with $\mathcal{R}^n\mu_M := H^n(\mu_{I_M^\bullet}) : H^n(F(I_M^\bullet), F(d_M^\bullet)) \to H^n(G(I_M^\bullet), G(d_M^\bullet))$.

B) Let H be a third additive functor from graded R-modules to R_0-modules. Let $\nu : G \to H$ be a natural transformation. In accordance with 4.13 A) we say that $\Delta : F \xrightarrow{\mu} G \xrightarrow{\nu} H$ is an *(admissible) triad of (additive covariant) functors (from graded R-modules to R_0-modules)* if the sequence

$$0 \to F(I) \xrightarrow{\mu_I} G(I) \xrightarrow{\nu_I} H(I) \to 0$$

is exact for each *injective R-module I.

Assume now, that $\Delta : F \xrightarrow{\mu} G \xrightarrow{\nu} H$ is such a triad of functors. Then, we may perform the construction of 4.13 B), C) and end up again (for each graded R-module M) with a (natural) exact sequence of R_0-modules

$$\begin{array}{rlllll}
0 \longrightarrow & \mathcal{R}^0F(M) & \xrightarrow{\mathcal{R}^0\mu_M} & \mathcal{R}^0G(M) & \xrightarrow{\mathcal{R}^0\nu_M} & \mathcal{R}^0H(M) \\
\xrightarrow{\delta_M^{0,\Delta}} & \mathcal{R}^1F(M) & \xrightarrow{\mathcal{R}^1\mu_M} & \mathcal{R}^1G(M) & \longrightarrow & \cdots \\
 & & & \cdots & \longrightarrow & \mathcal{R}^{n-1}H(M) \\
\xrightarrow{\delta_M^{n-1,\Delta}} & \mathcal{R}^nF(M) & \xrightarrow{\mathcal{R}^n\mu_M} & \mathcal{R}^nG(M) & \xrightarrow{\mathcal{R}^n\nu_M} & \mathcal{R}^nH(M) \\
\xrightarrow{\delta_M^{n,\Delta}} & \mathcal{R}^{n+1}F(M) & \xrightarrow{\mathcal{R}^{n+1}\mu_M} & \mathcal{R}^{n+1}G(M) & \longrightarrow & \cdots .
\end{array}$$

Again, we call this sequence the *right derived sequence of* Δ *associated to* M.

11.10. **Notation and Remark.** A) Let R be positively graded and let $\mathfrak{a} \subseteq R_+$ be a graded ideal of R. Let $i^{\mathfrak{a}}_M : \Gamma_{\mathfrak{a}}(M)_0 \rightarrowtail M_0$ denote the inclusion map for each graded R-module M. It is easy to verify that the assignment $i^{\mathfrak{a}} : M \rightsquigarrow \big(\Gamma_{\mathfrak{a}}(M)_0 \xrightarrow{i^{\mathfrak{a}}_M} M_0\big)$ defines a natural transformation of functors from graded R-modules to R_0-modules

$$i^{\mathfrak{a}} : \Gamma_{\mathfrak{a}}(\bullet)_0 \to \bullet_0.$$

B) Let $U := \mathcal{U}(\mathfrak{a}) \subseteq \mathrm{Proj}(R)$ and consider the linear functor

$$\widetilde{\bullet}(U) : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(\widetilde{M}(U) \xrightarrow{\widetilde{h}(U)} \widetilde{N}(U)\big)$$

from graded R-modules to R_0-modules (cf. 11.5 D) a)).

For each $n \in \mathbb{N}_0$ let $\mathcal{R}^n(\widetilde{\bullet}(U))$ be the n-th right derived functor of $\widetilde{\bullet}(U)$, which is again a linear functor from graded R-modules to R_0-modules (cf. 8.20 A)). We use the notation $H^n(U, \widetilde{\bullet}) := \mathcal{R}^n(\widetilde{\bullet}(U))$, so that $\mathcal{R}^n(\widetilde{\bullet}(U))$ is given by the assignment

$$\big(M \xrightarrow{h} N\big) \rightsquigarrow \big(H^n(U, \widetilde{M}) \xrightarrow{H^n(U,\widetilde{h})} H^n(U, \widetilde{N})\big).$$

As the linear functor $\widetilde{\bullet}(U)$ is left exact (cf. 11.5 D) b)) we can and do identify $H^0(U, \widetilde{M}) = \mathcal{R}^0(\widetilde{M}(U)) = \widetilde{M}(U)$ for each graded R-module M (cf. 8.20 B)).

C) Next consider the linear functor $\Gamma_{\mathfrak{a}}(\bullet)_0$ from graded R-modules to R_0-modules and its right derived functors $\mathcal{R}^n(\Gamma_{\mathfrak{a}}(\bullet)_0)$ for $n \in \mathbb{N}_0$ (cf. 8.20 A)). Observe that $\Gamma_{\mathfrak{a}}(\bullet)_0$ is the composition of the linear functor of graded R-modules ${}^*\Gamma_{\mathfrak{a}}$ (cf. 8.8 B)) with the exact linear functor from graded R-modules to R_0-modules $\bullet_0$ (cf. 8.7 B)), i.e. $\Gamma_{\mathfrak{a}}(\bullet)_0 = (\bullet_0) \circ {}^*\Gamma_{\mathfrak{a}}$. We leave it as an exercise to show along the traces of 5.3 B), that for each $n \in \mathbb{N}_0$ we can say:

a) There is a natural equivalence of functors from graded R-modules to R_0-modules $\gamma^n : (\bullet_0) \circ (\mathcal{R}^n\, {}^*\Gamma_{\mathfrak{a}}) \xrightarrow{\cong} \mathcal{R}^n(\Gamma_{\mathfrak{a}}(\bullet)_0)$.

So if R is Noetherian, writing ${}^*H^n_{\mathfrak{a}} = \mathcal{R}^n({}^*\Gamma_{\mathfrak{a}})$ (cf. 8.24), and on use of the gradings described in 8.25 B) we get:

b) For each graded R-module M and each $n \in \mathbb{N}_0$ there are isomorphisms of R_0-modules $\mathcal{R}^n(\Gamma_{\mathfrak{a}}(\bullet)_0)(M) \cong {}^*H^n_{\mathfrak{a}}(M)_0 \cong H^n_{\mathfrak{a}}(M)_0$.

11.11. **Corollary.** *Let R be Noetherian and positively graded, let $\mathfrak{a} \subseteq R$ be a graded ideal with $\mathfrak{a} \subseteq R_+$, and let $U := \mathcal{U}(\mathfrak{a}) \subseteq \mathrm{Proj}(R)$ be the open set defined by $\mathfrak{a}$. Then,*

$$\Delta^{\mathfrak{a}} : \Gamma_{\mathfrak{a}}(\bullet)_0 \xrightarrow{i^{\mathfrak{a}}} (\bullet_0) \xrightarrow{\eta^U} (\widetilde{\bullet}(U))$$

is an admissible triad of linear covariant functors from graded R-modules to R_0-modules.

Proof. Clear by 11.8. $\square$

11.12. **Theorem.** *Let R be Noetherian and positively graded, let $\mathfrak{a} \subseteq R$ be a graded ideal with $\mathfrak{a} \subseteq R_+$, and let $U := \mathcal{U}(\mathfrak{a}) \subseteq \mathrm{Proj}(R)$ be the open set defined by $\mathfrak{a}$. Then, for each graded R-module M we have:*

a) There is an exact sequence of R_0-modules

$$0 \to H^0_{\mathfrak{a}}(M)_0 \to M_0 \to H^0(U, \widetilde{M}) \to H^1_{\mathfrak{a}}(M)_0 \to 0.$$

b) For each $i \in \mathbb{N}$ there is an isomorphism of R_0-modules

$$H^i(U, \widetilde{M}) \cong H^{i+1}_{\mathfrak{a}}(M)_0.$$

Proof. Consider the right derived sequence of the triad

$$\Delta^{\mathfrak{a}} : \Gamma_{\mathfrak{a}}(\bullet)_0 \xrightarrow{i^{\mathfrak{a}}} (\bullet_0) \xrightarrow{\eta^U} (\widetilde{\bullet}(U))$$

associated to the graded R-module M (cf. 11.9 B)):

$$\begin{array}{lllll}
0 \longrightarrow & \mathcal{R}^0(\Gamma_{\mathfrak{a}}(\bullet)_0)(M) & \xrightarrow{\mathcal{R}^0 i^{\mathfrak{a}}_M} \mathcal{R}^0(\bullet_0)(M) & \xrightarrow{\mathcal{R}^0 \eta^U_M} \mathcal{R}^0(\widetilde{\bullet}(U))(M) \\
\xrightarrow{\delta^{0,\Delta^{\mathfrak{a}}}_M} & \mathcal{R}^1(\Gamma_{\mathfrak{a}}(\bullet)_0)(M) & \xrightarrow{\mathcal{R}^1 i^{\mathfrak{a}}_M} \mathcal{R}^1(\bullet_0)(M) & \longrightarrow \quad \cdots \\
 & & \cdots & \longrightarrow \mathcal{R}^{n-1}(\widetilde{\bullet}(U))(M) \\
\xrightarrow{\delta^{n-1,\Delta^{\mathfrak{a}}}_M} & \mathcal{R}^n(\Gamma_{\mathfrak{a}}(\bullet)_0)(M) & \xrightarrow{\mathcal{R}^n i^{\mathfrak{a}}_M} \mathcal{R}^n(\bullet_0)(M) & \xrightarrow{\mathcal{R}^n \eta^U_M} \mathcal{R}^n(\widetilde{\bullet}(U))(M) \\
\xrightarrow{\delta^{n,\Delta^{\mathfrak{a}}}_M} & \mathcal{R}^{n+1}(\Gamma_{\mathfrak{a}}(\bullet)_0)(M) & \xrightarrow{\mathcal{R}^{n+1} i^{\mathfrak{a}}_M} \mathcal{R}^{n+1}(\bullet_0)(M) & \longrightarrow \quad \cdots .
\end{array}$$

Then, use the isomorphisms

$$\mathcal{R}^n(\Gamma_{\mathfrak{a}}(\bullet)_0)(M) \cong H^n_{\mathfrak{a}}(M)_0$$

(cf. 11.10 C) b)) and the notation $H^n(U, \widetilde{\bullet}) = \mathcal{R}^n(\widetilde{\bullet}(U))$ (cf. 11.10 B)). Moreover, observe that the functor $\bullet_0$ is exact, so that $\mathcal{R}^i(\bullet_0)(M) = 0$ for all $i \in \mathbb{N}$ (cf. 8.20 C) b)). $\square$

11.13. **Corollary.** *Let R be Noetherian and positively graded, let $\mathfrak{a} \subseteq R$ be a graded ideal with $\mathfrak{a} \subseteq R_+$ and let $U := \mathcal{U}(\mathfrak{a}) \subseteq \mathrm{Proj}(R)$ be the open set defined by $\mathfrak{a}$. Let M be a graded R-module and let $t \in \mathbb{Z}$. Then:*

a) There is an exact sequence of R_0-modules

$$0 \to H^0_{\mathfrak{a}}(M)_t \to M_t \to H^0(U, \widetilde{M(t)}) \to H^1_{\mathfrak{a}}(M)_t \to 0.$$

b) For each $i \in \mathbb{N}$ there is an isomorphism of R_0-modules

$$H^i(U, \widetilde{M(t)}) \cong H^{i+1}_{\mathfrak{a}}(M)_t.$$

Proof. Apply 11.12 to the graded R-module $M(t)$ and observe that $M(t)_0 = M_t$ (cf. 8.5 A)) and $H^i_{\mathfrak{a}}(M(t))_0 = (H^i_{\mathfrak{a}}(M)(t))_0 = H^i_{\mathfrak{a}}(M)_t$ for all $i \in \mathbb{N}_0$ (cf. 8.25 E)). $\square$

11.14. **Corollary.** *Let R be Noetherian and positively graded, let M be a graded R-module, let $t \in \mathbb{Z}$ and let $X := \operatorname{Proj}(R)$. Then:*

a) There is an exact sequence of R_0-modules

$$0 \to H^0_{R_+}(M)_t \to M_t \to H^0(X, \widetilde{M(t)}) \to H^1_{R_+}(M)_t \to 0.$$

b) For each $i \in \mathbb{N}$ there is an isomorphism of R_0-modules

$$H^i(X, \widetilde{M(t)}) \cong H^{i+1}_{R_+}(M)_t.$$

Proof. Apply 11.13 and observe that $X = \mathcal{U}(R_+)$. $\square$

11.15. **Remark.** Let the notations and hypotheses be as in 11.12. We will see in the next section that $H^i(U, \widetilde{M})$ is the *i-th cohomology module of U with coefficients in the sheaf $\widetilde{M}$ induced by M on* $\operatorname{Proj}(R)$ (cf. [H1, Chapter III]). Then we may conclude that 11.12 (and its corollaries 11.13 and 11.14) establishes a correspondence between *sheaf cohomology* and *local cohomology.* This correspondence sometimes is referred to as the *Serre-Grothendieck correspondence.*

12. Cohomology of Projective Schemes

We shall give a few applications of the results of the previous section to cohomology of projective schemes. We first briefly mention a few basic facts on sheaves over projective schemes. For the corresponding details we refer to [H1, Chapters II, III].

12.1. **Reminder and Exercise.** A) Let X be a topological space. We write $\mathbb{U}_X$ for the set of open subsets $U \subseteq X$. A *presheaf of Abelian groups over X,* denoted by $\mathcal{F}$, is given by an assignment

$$U \leadsto \mathcal{F}(U)$$

which to each open set $U \in \mathbb{U}_X$ assigns an Abelian group $\mathcal{F}(U)$, and an assignment

$$(U, V) \leadsto \big(\rho^{\mathcal{F}}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)\big)$$

which to each pair of open subsets $(U, V) \in \mathbb{U}_X^2$ with $V \subseteq U$ assigns a homomorphism of Abelian groups $\rho^{\mathcal{F}}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that:

(PS$_1$) $\rho^{\mathcal{F}}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ for all $U \in \mathbb{U}_X$;
(PS$_2$) If U, V, $W \in \mathbb{U}_X$ with $U \supseteq V \supseteq W$, one has the commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[rr, "\rho^{\mathcal{F}}_{U,W}"] \arrow[dr, "\rho^{\mathcal{F}}_{U,V}"'] & & \mathcal{F}(W) \\ & \mathcal{F}(V) \arrow[ur, "\rho^{\mathcal{F}}_{V,W}"'] & \end{tikzcd} .$$

In this case $\mathcal{F}(U)$ is called the *group of sections of $\mathcal{F}$ over U.* The homomorphism $\rho^{\mathcal{F}}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the *restriction homomorphism from (sections of $\mathcal{F}$ over) U to (sections of $\mathcal{F}$ over) V.*

B) Let $\mathcal{F}$ be a presheaf of Abelian groups over X. Let $\mathbb{U} \subseteq \mathbb{U}_X$ and let $(f_U)_{U\in\mathbb{U}} \in \prod_{U\in\mathbb{U}} \mathcal{F}(U)$ be a family of sections (of $\mathcal{F}$ over $\mathbb{U}$). This family is called *compatible* if

$$\rho^{\mathcal{F}}_{U,U\cap V}(f_U) = \rho^{\mathcal{F}}_{V,U\cap V}(f_V) \text{ for all } U, V \in \mathbb{U}_X.$$

A preasheaf $\mathcal{F}$ of Abelian groups over X is called a *sheaf,* if it satisfies the following "gluing axiom":

(S) For any set $\mathbb{U} \subseteq \mathbb{U}_X$ and any compatible family of sections $(f_U)_{U\in\mathbb{U}} \in \prod_{U\in\mathbb{U}} \mathcal{F}(U)$ there exists a unique section $f \in \mathcal{F}(\bigcup_{U\in\mathbb{U}} U)$ such that $\rho^{\mathcal{F}}_{\bigcup_{U\in\mathbb{U}} U,W}(f) = f_W$ for all $W \in \mathbb{U}$.

12.2. **Reminder and Exercise.** A) Let R be a positively graded ring and let $X = \mathrm{Proj}(R)$. Let M be a graded R-module. For each open set $U \in \mathbb{U}_X$ we may consider the R_0-module $\widetilde{M}(U)$ (cf. 11.5 B), C)), and in particular its additive group. Moreover, for any pair $(U,V) \in \mathbb{U}_X^2$ with $V \subseteq U$ there is a map obtained by restriction:

$$\rho_{U,V}^{\widetilde{M}} : \widetilde{M}(U) \to \widetilde{M}(V),\ z = (z_\mathfrak{p})_{\mathfrak{p}\in U} \mapsto z\restriction_V := (z_\mathfrak{p})_{\mathfrak{p}\in V}.$$

Clearly, this map is a homomorphism of R_0-modules, hence in particular of Abelian groups. Now, it is easy to verify that the assignments

$$U \leadsto \widetilde{M}(U) \text{ for} U \in \mathbb{U}_X$$

and

$$(U,V) \leadsto \big(\rho_{U,V}^{\widetilde{M}} : \widetilde{M}(U) \to \widetilde{M}(V)\big) \text{ for } U,V \in \mathbb{U}_X \text{ with } V \subseteq U$$

define a sheaf $\widetilde{M}$ of Abelian groups over X.

As the groups $\widetilde{M}(U)$ are actually R_0-modules and the homomorphisms $\rho_{U,V}^{\widetilde{M}}$ are actually homomorphisms of R_0-modules, we say that $\widetilde{M}$ is a *sheaf of* R_0-*modules.* This sheaf $\widetilde{M}$ is called the *sheaf (over X) induced by M.*

B) We now may apply what was done above to the graded R-module R and consider the sheaf $\mathcal{O}_X = \mathcal{O}_{\mathrm{Proj}(R)} := \widetilde{R}$ induced by R. This special sheaf is called the *structure sheaf of* X, and the pair $(X,\mathcal{O}_X) = (\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$ is called the *projective scheme induced by R.*

Observe that here all the R_0-modules $\mathcal{O}_X(U)$ are indeed R_0-algebras (hence rings) and that the maps $\rho_{U,V}^{\mathcal{O}_X}$ are indeed homomorphisms of R_0-algebras (and hence of rings). So, we say that the structure sheaf $\mathcal{O}_X$ is a *sheaf of* R_0*-algebras* or a *sheaf of rings.*

The *notion of a scheme* is in fact defined in a much more general context. For this and for a proof of the fact that $(\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$ is indeed a scheme, we refer to [H1, Section II.2]. For our purposes it is sufficient to take the wording "$(\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$ is the projective scheme induced by R" as a definition.

C) A *sheaf of* $\mathcal{O}_X$*-modules* is a sheaf $\mathcal{F}$ of Abelian groups over X such that:

- (M_1) $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for all $U \in \mathbb{U}_X$;
- (M_2) $\rho_{U,V}^{\mathcal{F}} : \mathcal{F}(U) \to \mathcal{F}(V)\restriction_{\mathcal{O}_X(U)}$ is a homomorphism of $\mathcal{O}_X(U)$-modules for all $U,V \in \mathbb{U}_X$ with $V \subseteq U$.

In (M_2), scalar restriction is understood with respect to the homomorphism $\rho_{U,V}^{\mathcal{O}_X} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$.)

Now, one easily sees that if M is a graded R-module, then $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules.

D) We say that a sheaf of $\mathcal{O}_X$-modules is *quasicoherent* if it is induced by a graded R-module, that is of the form $\widetilde{M}$ for some graded R-module M.

Let R be Noetherian and homogeneous. Then, we say that a sheaf of $\mathcal{O}_X$-modules is *coherent* if it is induced by a finitely generated graded R-module, that is of the form $\widetilde{M}$ for some finitely generated graded R-module M.

If one uses the standard definition of quasicoherent and coherent sheaves, the above characterizations of these types of sheaves need to be proved. For this we refer to [H1, Section II.5].

E) A sheaf $\mathcal{F}$ of Abelian groups over a topological space is said to be *flasque* if all its restriction maps $\rho^{\mathcal{F}}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ are surjective. Now, it follows form 11.8 b):

If R is a Noetherian, positively graded ring and I is a *injective R-module, then the induced sheaf $\widetilde{I}$ over $\operatorname{Proj}(R)$ is flasque.

12.3. **Reminder and Exercise.** A) Let X be a topological space and let $x \in X$. We consider the set

$$\mathbb{U}_{X,x} := \{U \in \mathbb{U}_X \mid x \in U\}$$

of open neighbourhoods of x in X. Moreover, we consider a sheaf $\mathcal{F}$ of Abelian groups over X. We consider the set

$$\mathcal{S}^{\mathcal{F}}_x := \{(U, f) \mid U \in \mathbb{U}_{X,x},\ f \in \mathcal{F}(U)\}.$$

On this set, we introduce a binary relation $\sim_x$ by

$$(U, f) \sim_x (V, g) \ :\Leftrightarrow\ \exists\, W \in \mathbb{U}_{X,x} : W \subseteq U \cap V \text{ and } \rho^{\mathcal{F}}_{U,W}(f) = \rho^{\mathcal{F}}_{V,W}(g)$$

for (U, f), $(V, g) \in \mathcal{S}^{\mathcal{F}}_x$. It is easy to verify, that $\sim_x$ is an equivalence relation on $\mathcal{S}^{\mathcal{F}}_x$. So we may consider the set of equivalence classes

$$\mathcal{F}_x := \mathcal{S}^{\mathcal{F}}_x / \sim_x .$$

This set $\mathcal{F}_x$ is called the *stalk of $\mathcal{F}$ at x*. Also, whenever $U \in \mathbb{U}_{X,x}$, we may introduce the canonical map

$$\rho^{\mathcal{F}}_{U,x} : \mathcal{F}(U) \to \mathcal{F}_x,\ f \mapsto f_x := (U, f)/ \sim_x .$$

In this situation $f_x = \rho^{\mathcal{F}}_{U,x}(f)$ is called the *germ of f at x*. So in particular:

a) Let U, $V \in \mathbb{U}_{X,x}$ and $f \in \mathcal{F}(U)$, $g \in \mathcal{F}(V)$. Then $f_x = g_x$ if and only if there is some $W \in \mathbb{U}_{X,x}$ with $W \subseteq U \cap V$ and $\rho^{\mathcal{F}}_{U,W}(f) = \rho^{\mathcal{F}}_{V,W}(g)$.

Moreover, by the gluing axiom 12.1 B) (S) one has the following local criterion for the equality of sections:

b) Let $U \in \mathbb{U}_X$ and let f, $g \in \mathcal{F}(U)$. Then $f = g$ if and only if $f_x = g_x$ for all $x \in U$.

B) Let $x \in X$. If (U, f), (U', f'), (V, g), $(V', g') \in \mathcal{S}^{\mathcal{F}}_x$ with $f_x = f'_x$ and $g_x = g'_x$, and if W, $W' \in \mathbb{U}_{X,x}$ with $W \subseteq U \cap V$ and $W' \subseteq U' \cap V'$, then

$$(\rho^{\mathcal{F}}_{U,W}(f) + \rho^{\mathcal{F}}_{V,W}(g))_x = (\rho^{\mathcal{F}}_{U',W'}(f') + \rho^{\mathcal{F}}_{V',W'}(g'))_x.$$

This allows to define a binary operation $+$ on $\mathcal{F}_x$ by

$$f_x + g_x := (\rho^{\mathcal{F}}_{U,W}(f) + \rho^{\mathcal{F}}_{V,W}(g))_x$$

where $U, V, W \in \mathbb{U}_{X,x}$ with $W \subseteq U \cap V$, $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$. It is easy to check now that $\mathcal{F}_x$ furnished with the above binary operation $+$ becomes an Abelian group, and then the map of taking germs $\rho^{\mathcal{F}}_{U,x} : \mathcal{F}(U) \to \mathcal{F}_x$ is a homomorphism of Abelian groups. Similarly, if $\mathcal{F}$ is a sheaf of rings over X, the stalks $\mathcal{F}_x$ all carry a canonical structure of rings and the maps of taking germs are all homomorphisms of rings.

12.4. **Reminder and Exercise.** A) Let R be a positively graded ring and let $X = \mathrm{Proj}(R)$. Let $x \in X$. Then, the stalk $\mathcal{O}_{X,x} := (\mathcal{O}_X)_x$ is a ring, and for each $U \in \mathbb{U}_{X,x}$ the map $\rho^{\mathcal{O}_X}_{U,x} : \mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is a homomorphism of rings. The stalk $\mathcal{O}_{X,x}$ is called the *local ring of X at x.*

Now, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then it is obvious how to define a canonical structure of $\mathcal{O}_{X,x}$-module on the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x. Then, the map of taking germs $\rho^{\mathcal{F}}_{U,x} : \mathcal{F}(U) \to \mathcal{F}_x \upharpoonright_{\mathcal{O}_X(U)}$ becomes a homomorphism of $\mathcal{O}_X(U)$-modules for all $U \in \mathbb{U}_{X,x}$.

B) Let M be a graded R-module and let $x \in X$. Then:

a) There is a bijective map $\varepsilon^M_x : \widetilde{M}_x \to M_{(x)}$ such that for each $U \in \mathbb{U}_{X,x}$ and each $z = (z_x)_{x \in U} \in \widetilde{M}(U)$ we have $\varepsilon^M_x(\rho^{\widetilde{M}}_{U,x}(z)) = z_x$;
b) The map $\varepsilon^R_x : \mathcal{O}_{X,x} \to R_{(x)}$ is an isomorphism of rings.

As a consequence we get (cf. 11.3 D) b)):

c) $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_{X,x} := (\varepsilon^R_x)^{-1}(x_{(x)})$.

This ideal $\mathfrak{m}_{X,x}$ of $\mathcal{O}_{X,x}$ is called the *(local) maximal ideal (of X) at x.*

We now identify $\mathcal{O}_{X,x}$ and $R_{(x)}$ by means of the isomorphism ε^R_x. Then, it follows readily:

d) For each graded R-module M the map $\varepsilon^M_x : \widetilde{M}_x \to M_{(x)}$ is an isomorphism of $\mathcal{O}_{X,x}$-modules.

We thus again identify $\widetilde{M}_x$ and $M_{(x)}$ by means of ε^M_x. Now, as a consequence of 11.3 E) we get:

e) If R is Noetherian, then
   (i) For each $x \in X$, the local ring $\mathcal{O}_{X,x}$ is Noetherian;
   (ii) For each $x \in X$ and each finitely generated graded R-module M, the stalk $\widetilde{M}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module;
   (iii) If in addition R is homogeneous, for each $x \in X$ and each coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, the stalk $\mathcal{F}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module.

12.5. **Construction and Exercise.** A) Let R be a homogeneously graded ring, so that $R = R_0[R_1]$, and let $X = \text{Proj}(R)$. If $x \in X$ we have $R_1 \nsubseteq x$ and thus find some $l \in R_1 \setminus x$. It follows that $x \in \mathcal{U}(l) := \mathcal{U}(Rl)$ (cf. 11.4 C)). Now, for each $U \in \mathbb{U}_X$ and each $l \in R_1$ let $U_l := U \cap \mathcal{U}(l)$. Then, by the above observation we have $U = \bigcup_{l \in R_1} U_l$ for all $U \in \mathbb{U}_X$.

Let $U \in \mathbb{U}_X$, let $l,\, h \in R_1$ and let $x \in U_l \cap U_h$. Then $\frac{l}{h}$ is a unit in the local ring $\mathcal{O}_{X,x}$. Now, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules and let $n \in \mathbb{Z}$. We consider a family of sections

$$(f_l)_{l \in R_1} \in \prod_{l \in R_1} \mathcal{F}(U_l),$$

where $f_l \in \mathcal{F}(U_l)$ for each $l \in R_1$. We say that this family is a *family of n-sections (of $\mathcal{F}$ over U)* if

$$(\mathrm{S}_n)\ \forall\, h, l \in R_1\, \forall\, x \in U_h \cap U_l : (\tfrac{h}{l})^n f_{h,x} = f_{l,x} \in \mathcal{F}_x.$$

(Here, $f_{h,x}$ and $f_{l,x}$ are the germs of the sections f_h and f_l at the point x.)

We now set

$$\mathcal{F}(n)(U) := \{(f_l)_{l \in R_1} \in \prod_{l \in R_1} \mathcal{F}(U_l) \mid (f_l)_{l \in R_1} \text{ is a family of } n\text{-sections}\}.$$

Observe that the additive group $\prod_{l \in R_1} \mathcal{F}(U_l)$ actually carries a canonical structure of an $\mathcal{O}_X(U)$-module with scalar multiplication given by

$$f \cdot (f_l)_{l \in R_1} := (\rho^{\mathcal{O}_X}_{U,U_l}(f) \cdot f_l)_{l \in R_1}$$

for $f \in \mathcal{O}_X(U)$ and $(f_l)_{l \in R_1} \in \prod_{l \in R_1} \mathcal{F}(U_l)$. Now, it is easy to verify that $\mathcal{F}(n)(U)$ is an $\mathcal{O}_X(U)$-submodule of $\prod_{l \in R_1} \mathcal{F}(U_l)$.

Also, if $V \subseteq U$ is a second open set, there is defined a homomorphism of Abelian groups

$$\rho^{\mathcal{F}(n)}_{U,V} : \mathcal{F}(n)(U) \to \mathcal{F}(n)(V),\ (f_l)_{l \in R_1} \mapsto (\rho^{\mathcal{F}}_{U_l,V_l}(f_l))_{l \in R_1}.$$

Now, one verifies without further difficulties that the assignments $U \rightsquigarrow \mathcal{F}(n)(U)$ for $U \in \mathbb{U}_X$ and $(U,V) \rightsquigarrow \big(\rho^{\mathcal{F}(n)}_{U,V} : \mathcal{F}(n)(U) \to \mathcal{F}(n)(V)\big)$ for $U, V \in \mathbb{U}_X$ with $V \subseteq U$ define a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}(n)$. This sheaf $\mathcal{F}(n)$ of $\mathcal{O}_X$-modules is called the *n-th twist of $\mathcal{F}$.*

B) Observe that the property of being a 0-section over U means nothing else than that the family of sections $(f_l)_{l \in R_1} \in \prod_{l \in R_1} \mathcal{F}(U_l)$ is compatible. So, by the gluing axiom (S) of 12.1 B) we get a canonical bijection

$$\iota^{\mathcal{F}}_U : \mathcal{F}(U) \to \mathcal{F}(0)(U),\ f \mapsto (\rho^{\mathcal{F}}_{U,U_l}(f))_{l \in R_1}.$$

In fact, $\iota_U^{\mathcal{F}}$ is an isomorphism of $\mathcal{O}_X(U)$-modules. If $V \subseteq U$ is a second open set we have in addition the commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\iota_U^{\mathcal{F}}", "\cong"'] \arrow[d, "\rho_{U,V}^{\mathcal{F}}"'] & \mathcal{F}(0)(U) \arrow[d, "\rho_{U,V}^{\mathcal{F}(0)}"] \\ \mathcal{F}(V) \arrow[r, "\iota_V^{\mathcal{F}}", "\cong"'] & \mathcal{F}(0)(V). \end{tikzcd}$$

We use this to identify $\mathcal{F}(0) = \mathcal{F}$.

C) Let $n \in \mathbb{Z}$, let M be a graded R-module and let $x \in X$. Then, according to the definition of the grading of the $S(x)^{-1}R$-module $S(x)^{-1}M$ (cf. 11.3 B)) and according to the identification 12.4 B) we have

a) $\widetilde{M(n)}_x = (M(n))_x = (S(x)^{-1}M)_n$.

Now, let $U \in \mathbb{U}_X$ and $l \in R_1$, and assume that $x \in U_l$. Then $\frac{l^n}{1} \in (S(x)^{-1}R)_n$ is a unit in the ring $S(x)^{-1}R$. Therefore, we have two homomorphisms of $(S(x)^{-1}R)_0 = R_{(x)} = \mathcal{O}_{X,x}$-modules

$$\frac{l^n}{1}\cdot\ : \widetilde{M}_x = M_{(x)} = (S(x)^{-1}M)_0 \to (S(x)^{-1}M)_n,$$

$$\frac{1}{l^n}\cdot\ : (S(x)^{-1}M)_n \to (S(x)^{-1}M)_0 = M_{(x)} = \widetilde{M}_x,$$

which are inverse to each other. So, we obtain an isomorphism of $\mathcal{O}_{X,x}$-modules

b) $\frac{l^n}{1}\cdot\ : \widetilde{M}_x \xrightarrow{\cong} \widetilde{M(n)}_x$.

Now, it is easy to verify that these isomorphisms give rise to an isomorphism of $\mathcal{O}_X(W)$-modules

c) $l^n\cdot\ : \widetilde{M}(W) \xrightarrow{\cong} \widetilde{M(n)}(W),\ (z_x)_{x\in W} \mapsto (\frac{l^n}{1}z_x)_{x\in W}$

for any open set $W \subseteq U_l$. In particular, if h, $l \in R_1$ and $g \in \widetilde{M(n)}(U)$, then

d) $(\frac{h}{l})^n((h^n\cdot)^{-1}(\rho_{U,U_h}^{\widetilde{M(n)}}(g)))_x = ((l^n\cdot)^{-1}(\rho_{U,U_l}^{\widetilde{M(n)}}(g)))_x$ for all $x \in U_h \cap U_l$.

But this means:

e) $\varphi^U(g) := ((l^n\cdot)^{-1}(\rho_{U,U_l}^{\widetilde{M(n)}}(g)))_{l\in R_1} \in \prod_{l\in R_1}\widetilde{M}(U_l)$ is a family of n-sections over U in $\widetilde{M}$ for each $g \in \widetilde{M(n)}(U)$.

It now is easy to check that the induced map

$$\varphi^U : \widetilde{M(n)}(U) \to \widetilde{M}(n)(U),\ g \mapsto \varphi^U(g)$$

is a homomorphism of $\mathcal{O}_X(U)$-modules.

Conversely let $f = (f_l)_{l \in R_1} \in \widetilde{M}(n)(U)$, so that $f_l \in \widetilde{M}(U)$ for all $l \in R_1$ and condition (S_n) of part A) is satisfied. Then clearly, by c), $l^n \cdot (f_l) \in \widetilde{M(n)}(U_l)$ for all $l \in R_1$. Moreover, if $x \in U_h \cap U_l$ for some elements $h,\ l \in R_1$, by statement c) and the mentioned condition (S_n) we have

$$(l^n \cdot (f_l))_x = \tfrac{l^n}{1} \cdot f_{l,x} = \tfrac{l^n}{1} \cdot (\tfrac{h}{l})^n f_{h,x} = \tfrac{h^n}{1} \cdot f_{h,x} = (h^n \cdot (f_h))_x.$$

But this means that $(l^n \cdot (f_l))_{l \in R_1} \in \prod \widetilde{M(n)}(U_l)$ is a compatible family of sections over U and thus defines a unique section $\psi^U(f) \in \widetilde{M(n)}(U)$, given as a local family by

$$\psi^U(f) = (\psi^U(f)_x)_{x \in U} = (l^n \cdot (f_l))_x \text{ for } x \in U_l \text{ and } l \in R_1.$$

Now, it is easy to check that the induced map

$$\psi^U : \widetilde{M}(n)(U) \to \widetilde{M(n)}(U),\ f \mapsto \psi^U(f)$$

is a homomorphism of $\mathcal{O}_X(U)$-modules. Also, φ^U and ψ^U are inverse to each other. So, we finally end up with an isomorphism of $\mathcal{O}_X(U)$-modules

f) $\varphi^U : \widetilde{M(n)}(U) \xrightarrow{\cong} \widetilde{M}(n)(U)$.

If $V \subseteq U$ is a second open set, we have the commutative diagram

$$\begin{array}{ccc}
\widetilde{M(n)}(U) & \xrightarrow[\cong]{\varphi^U} & \widetilde{M}(n)(U) \\
{\scriptstyle \rho^{\widetilde{M(n)}}_{U,V}} \downarrow & & \downarrow {\scriptstyle \rho^{\widetilde{M}(n)}_{U,V}} \\
\widetilde{M(n)}(V) & \xrightarrow[\cong]{\varphi^V} & \widetilde{M}(n)(V).
\end{array}$$

We use this to identify $\widetilde{M(n)} = \widetilde{M}(n)$.

D) As an easy consequence of this we now may conclude:

a) If $\mathcal{F}$ is a quasicoherent sheaf of $\mathcal{O}_X$-modules, then so is $\mathcal{F}(n)$ for all $n \in \mathbb{Z}$.
b) If R is Noetherian and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, then $\mathcal{F}(n)$ is a coherent sheaf of $\mathcal{O}_X$-modules for all $n \in \mathbb{Z}$.

12.6. **Remark and Exercise.** A) Let X be a topological space and let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves of Abelian groups over X. A *homomorphism of sheaves (of Abelian groups over X)* $h : \mathcal{F} \to \mathcal{G}$ is given by an assignment

$$U \rightsquigarrow \big(h(U) : \mathcal{F}(U) \to \mathcal{G}(U)\big)$$

which to each open set U of X assigns a homomorphism of Abelian groups $h(U)$ such that for any pair $(U, V) \in \mathbb{U}_X^2$ with $V \subseteq U$ there is a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "h(U)"] \arrow[d, "\rho^{\mathcal{F}}_{U,V}"] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "h(V)"] & \mathcal{G}(V). \end{tikzcd}$$

If $\mathcal{F}$ is a sheaf of Abelian groups over X, the *identity homomorphism* $\mathrm{id}_{\mathcal{F}}$ is given by the assignment

$$U \leadsto \big(\mathrm{id}_{\mathcal{F}}(U) := \mathrm{id}_{\mathcal{F}(U)} : \mathcal{F}(U) \to \mathcal{F}(U)\big) \text{ for } U \in \mathbb{U}_X.$$

Moreover, if $h : \mathcal{F} \to \mathcal{G}$ and $l : \mathcal{G} \to \mathcal{H}$ are homomorphisms of sheaves of Abelian groups over X, the *composition* $l \circ h : \mathcal{F} \to \mathcal{H}$ of these is the homomorphism of sheaves given by the assignment

$$U \leadsto \big((l \circ h)(U) := l(U) \circ h(U) : \mathcal{F}(U) \to \mathcal{H}(U)\big) \text{ for } U \in \mathbb{U}_X.$$

B) Let $x \in X$ and let $h : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of Abelian groups over X. Then, there is a homomorphism of Abelian groups

$$h_x : \mathcal{F}_x \to \mathcal{G}_x,\ f_x \mapsto h(U)(f)_x \text{ for any } U \in \mathbb{U}_{X,x} \text{ and any } f \in \mathcal{F}(U).$$

This homomorphism is called the *homomorphism induced by h in the stalks over x.* Clearly we have $(\mathrm{id}_{\mathcal{F}})_x = \mathrm{id}_{\mathcal{F}_x}$ and $(h \circ l)_x = h_x \circ l_x$.

Now, a homomorphism of sheaves $h : \mathcal{F} \to \mathcal{G}$ is called *injective* respectively *surjective* if all the induced homomorphisms $h_x : \mathcal{F}_x \to \mathcal{G}_x$ have this property. A sequence of homomorphisms of sheaves

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is said to be *exact* if the induced sequence of Abelian groups

$$\mathcal{F}_x \xrightarrow{f_x} \mathcal{G}_x \xrightarrow{g_x} \mathcal{H}_x$$

is exact for all $x \in X$.

C) In particular, in the above context one can speak again of *cocomplexes of sheaves (of Abelian groups over X),* that is of sequences

$$(\mathcal{F}^\bullet, d^\bullet) : \cdots \to \mathcal{F}^{n-1} \xrightarrow{d^{n-1}} \mathcal{F}^n \xrightarrow{d^n} \mathcal{F}^{n+1} \to \cdots$$

of homomorphisms of sheaves such that for all $n \in \mathbb{Z}$ and for each $x \in X$ one has $d^n_x \circ d^{n-1}_x = 0$.

Moreover, as in the case of modules, one may define the notion of a *right resolution* of a sheaf. Such a right resolution $((\mathcal{G}^\bullet, d^\bullet); a)$ is said to be *flasque* if the sheaves $\mathcal{G}^n$ are flasque (cf. 12.2 D)) for all $n \in \mathbb{N}_0$.

By the same diagrammatic condition as in the case of modules, one may define the notion of *injective sheaves of Abelian groups over X.* It is a nice exercise to show:

a) Injective sheaves are flasque.

It is more than just an exercise to show that for any sheaf $\mathcal{F}$ there is an injective homomorphism of sheaves $\mathcal{F} \to \mathcal{I}$ such that $\mathcal{I}$ is injective (cf. [H1, Corollary III.3.2]). As a consequence of this sheaf theoretic version of the Lemma of Eckmann-Schopf one gets:

b) Each sheaf of Abelian groups over X admits an *injective resolution,* that is a right resolution $((\mathcal{I}^\bullet, d^\bullet); a)$ in which the sheaves $\mathcal{I}^n$ are injective for all $n \in \mathbb{N}_0$.

12.7. **Reminders.** A) Let X be a topological space and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of Abelian groups over X. If $h, g : \mathcal{F} \to \mathcal{G}$ are homomorphisms of sheaves, we define their sum $h+g$ by the assignment $U \rightsquigarrow (h(U) + g(U))$.

B) Let X be a topological space. By an *additive (covariant) functor from (the category of) sheaves over X to (the category of) Abelian groups* we mean an assignment

$$F : \big(\mathcal{F} \xrightarrow{h} \mathcal{G}\big) \rightsquigarrow \big(F(\mathcal{F}) \xrightarrow{F(h)} F(\mathcal{G})\big)$$

which to each sheaf $\mathcal{F}$ of Abelian groups over X assigns an Abelian group $F(\mathcal{F})$ and to each homomorphism $h : \mathcal{F} \to \mathcal{G}$ of sheaves of Abelian groups over$X$ assigns a homomorphism of Abelian groups $F(h) : F(\mathcal{F}) \to F(\mathcal{G})$ such that the following properties hold:

($\widetilde{\mathrm{A}}$1) $F(\mathrm{id}_{\mathcal{F}}) = \mathrm{id}_{F(\mathcal{F})}$ for each sheaf $\mathcal{F}$ of Abelian groups over X;

($\widetilde{\mathrm{A}}$2) $F(h \circ l) = F(h) \circ F(l)$, whenever $l : \mathcal{F} \to \mathcal{G}$ and $h : \mathcal{G} \to \mathcal{H}$ are homomorphisms of sheaves of Abelian groups over X;

($\widetilde{\mathrm{A}}$3) $F(h) + F(l) = F(h+l)$, whenever $h, l : \mathcal{F} \to \mathcal{G}$ are homomorphisms of sheaves of Abelian groups over X.

C) Let R be a graded ring and let X be a topological space. By an *additive (covariant) functor from (the category of) graded R-modules to (the category of) sheaves of Abelian groups over X* we mean an assignment

$$F : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(F(M) \xrightarrow{F(h)} F(N)\big)$$

which to each graded R-module M assigns a sheaf $F(M)$ of Abelian groups over X and to each homomorphism $h : M \to N$ of graded R-modules assigns a homomorphism $F(h) : F(M) \to F(N)$ of sheaves of Abelian groups over X such that the following properties hold:

($^*\widetilde{\mathrm{A}}$1) $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$ for each graded R-module M;

($^*\widetilde{\mathrm{A}}$2) $F(h \circ l) = F(h) \circ F(l)$, whenever $l : M \to N$ and $h : N \to P$ are homomorphisms of graded R-modules;

($^*\widetilde{\mathrm{A}}$3) $F(h) + F(l) = F(h+l)$, whenever $h, l : M \to N$ are homomorphisms of graded R-modules.

D) For functors as defined in B) and C), there is an obvious notion of exactness and of left and right exactness (cf. 1.16).

12.8. **Remark and Exercise.** A) Let X be a topological space. Let $U \in \mathbb{U}_X$. Then it is easy to verify that the assignment

$$\Gamma(U, \bullet) = \bullet(U) : \big(\mathcal{F} \xrightarrow{h} \mathcal{G}\big) \rightsquigarrow \big(\mathcal{F}(U) \xrightarrow{h(U)} \mathcal{G}(U)\big)$$

defines an additive functor from sheaves of Abelian groups over X to Abelian groups. This functor is called the *functor of taking sections over U.* It is not hard to verify that the functor $\Gamma(U, \bullet)$ is left exact.

Now, similar as we did this for modules, one now has the concept of *right derived functor.* In particular, we may define the i-th right derived functor of $\Gamma(U, \bullet)$, that is

$$H^i(U, \bullet) := \mathcal{R}^i(\Gamma(U, \bullet)).$$

This functor is called the *i-th cohomology functor of U with coefficients in sheaves of Abelian groups*—or shorter—the *i-th sheaf cohomology functor over U.* (Note the use of the same notation here and in 11.10 B), which is deliberate (cf. 12.9 C) b)).)

So, if $\mathcal{F}$ is a sheaf of Abelian groups, $H^i(U, \mathcal{F})$ is calculated by first choosing an injective resolution $((\mathcal{I}^\bullet, d^\bullet); a)$ of $\mathcal{F}$ (cf. 12.6 C) b)), then applying the functor $\Gamma(U, \bullet)$ to the resolving cocomplex $(\mathcal{I}^\bullet, d^\bullet)$ and then taking the i-th cohomology group $H^i(\Gamma(U, \mathcal{I}^\bullet), \Gamma(U, d^\bullet))$ of the resulting cocomplex $(\Gamma(U, \mathcal{I}^\bullet), \Gamma(U, d^\bullet))$ of Abelian groups.

B) For us it is important to notice that in the above calculation the injective resolution $((\mathcal{I}^\bullet, d^\bullet); a)$ of $\mathcal{F}$ may be replaced by an arbitrary flasque resolution (cf. [H1, Corollary III.3.2]): If $((\mathcal{E}^\bullet, d^\bullet); a)$ is a flasque resolution of a sheaf of Abelian groups $\mathcal{F}$, then for all $i \in \mathbb{N}_0$

$$H^i(U, \mathcal{F}) \cong H^i(\Gamma(U, \mathcal{E}^\bullet), \Gamma(U, d^\bullet)).$$

12.9. **Remark and Exercise.** A) Let R be a positively graded ring and let $X = \mathrm{Proj}(R)$. Then it is clear that the assignment

$$\widetilde{\bullet} : \big(M \xrightarrow{h} N\big) \rightsquigarrow \big(\widetilde{M} \xrightarrow{\widetilde{h}} \widetilde{N}\big)$$

defines a functor from graded R-modules to sheaves of Abelian groups over X, the *functor of taking induced sheaves.* On use of 11.3 C) and 12.4 B) d) it is not hard to prove that the functor $\widetilde{\bullet}$ of taking induced sheaves is exact.

B) Let $U \in \mathbb{U}_X$. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and any $i \in \mathbb{N}_0$ we may consider the i-th cohomology group $H^i(U, \mathcal{F})$ of U with coefficients in $\mathcal{F}$, defined according to 12.8 A). In fact, $H^i(U, \mathcal{F})$ is not only an Abelian group, but also carries a natural structure of an R_0-module. Indeed, for any $a \in R_0$ and any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ (in fact even of R_0-modules) there is a multiplication homomorphism $a\cdot : \mathcal{F} \to \mathcal{F}$, given by $a\cdot(W) : \mathcal{F}(W) \xrightarrow{a\cdot} \mathcal{F}(W)$ for each $W \in \mathbb{U}_X$. Clearly $a\cdot : \mathcal{F} \to \mathcal{F}$ is a homomorphism of sheaves of Abelian groups. So, we may define an R_0-operation on $H^i(U, \mathcal{F})$ by

$$ah := H^i(U, a\cdot)(h)$$

for $a \in R_0$ and $h \in H^i(U, \mathcal{F})$. We leave it as an exercise to show that the above R_0-operation turns $H^i(U, \mathcal{F})$ into an R_0-module.

C) Let $U \in \mathbb{U}_X$. We leave it as a slightly more involved exercise to prove by means of 12.2 E), of 12.8 B) and of the previous part A):

a) If M is a graded R-module and $((I^\bullet, d^\bullet); a)$ is a *injective resolution of M, then $H^i(U, \widetilde{M}) \cong H^i(\Gamma(U, \widetilde{I^\bullet}), \Gamma(U, \widetilde{d^\bullet}))$.

(Here $(\widetilde{I^\bullet}, \widetilde{d^\bullet})$ denotes the cocomplex $\cdots \to \widetilde{I^{n-1}} \xrightarrow{\widetilde{d^{n-1}}} \widetilde{I^n} \to \cdots$ of sheaves of $\mathcal{O}_X$-modules.)

As a consequence of this we get:

b) If M is a graded R-module, the R_0-modules $H^i(U, \widetilde{M})$ are the same (up to isomorphism) if calculated according to 11.10 B) or according to 12.8 A):

$$\mathcal{R}^i(\widetilde{\bullet}(U))(M) \cong \mathcal{R}^i(\Gamma(U, \bullet))(\widetilde{M}) \text{ for all } i \in \mathbb{N}_0.$$

12.10. **Proposition.** *Let K be an infinite field, let R be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module, let $X =$ Proj(R) and let $\mathcal{F} := \widetilde{M}$. Then:*

a) For all $n \in \mathbb{Z}$ there is an exact sequence of K-vector spaces

$$0 \to H^0_{R_+}(M)_n \to M_n \to H^0(X, \mathcal{F}(n)) \to H^1_{R_+}(M)_n \to 0.$$

b) For all $n \in \mathbb{Z}$ and all $i \in \mathbb{N}$ there is an isomorphism of K-vector spaces

$$H^i(X, \mathcal{F}(n)) \cong H^{i+1}_{R_+}(M)_n.$$

Proof. By 12.9 C) b) and 12.5 C) we can write

$$H^i(X, \mathcal{F}(n)) = H^i(X, \widetilde{M}(n)) = H^i(X, \widetilde{M(n)}).$$

Therefore 11.14 yields the claim. □

12.11. **Theorem.** *Let K be an infinite field, let R be a Noetherian homogeneous K-algebra, let $X =$ Proj(R) and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then:*

a) For each $i \in \mathbb{N}_0$ and each $n \in \mathbb{Z}$ the K-vector space $H^i(X, \mathcal{F}(n))$ is of finite dimension.

b) For each $i \in \mathbb{N}$ an all $n \gg 0$ we have $H^i(X, \mathcal{F}(n)) = 0$.

Proof. We conclude by 12.10 and 9.10. □

12.12. **Remark and Definition.** A) Let K be an infinite field, let R be a Noetherian homogeneous K-algebra, let $X =$ Proj(R) and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, according to 12.11 a) the K-vector space

$H^i(X, \mathcal{F}(n))$ is of finite dimension for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. Therefore, we may define the number

$$h^i(X, \mathcal{F}(n)) := \dim_K H^i(X, \mathcal{F}(n)) \in \mathbb{N}_0.$$

For $i \in \mathbb{N}_0$, the function

$$h^i_{\mathcal{F}} : \mathbb{Z} \to \mathbb{N}_0,\ n \mapsto h^i(X, \mathcal{F}(n))$$

is called the *i-th cohomological Hilbert function of (X with respect to the coherent sheaf of coefficients) $\mathcal{F}$.* According to 12.11 b) we may say: If $i > 0$, the cohomological Hilbert function $h^i_{\mathcal{F}} : \mathbb{Z} \to \mathbb{N}_0$ is right-vanishing.

B) Let M be a finitely generated graded R-module such that $\mathcal{F} = \widetilde{M}$. Then, 12.10 yields (in the notations of 10.6 A) and 10.19 A))

a) $h^i(X, \mathcal{F}(n)) = d^i_M(n)$ for $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$,

and hence $h^i_{\mathcal{F}} = d^i_M$ for $i \in \mathbb{N}_0$. This justifies the expression of geometric cohomological Hilbert function used in 10.19 A). Moreover, we now see that 12.11 b) is the proper geometric (and original) form of the *Vanishing Theorem of Castelnuovo-Serre,* whose algebraic version we mentioned in 10.19 E) c).

Finally, it follows by statement a), that in the notation of 9.14 A) we have

b) $\chi_M(n) = \sum_{i\geq 0}(-1)^i d^i_M(n) = \sum_{i\geq 0}(-1)^i h^i(X, \mathcal{F}(n))$.

Writing

$$\chi_{\mathcal{F}}(n) := \sum_{i\geq 0}(-1)^i h^i(X, \mathcal{F}(n))$$

we thus have

c) $\chi_M = \chi_{\mathcal{F}}$.

The function $\chi_{\mathcal{F}} : \mathbb{Z} \to \mathbb{Z},\ n \mapsto \chi_{\mathcal{F}}(n)$ is called the *characteristic function of $\mathcal{F}$.*

12.13. **Theorem.** *Let K be an infinite field, let R be a Noetherian homogeneous K-algebra, let $X = \mathrm{Proj}(R)$ and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, the characteristic function of $\mathcal{F}$ is presented by a polynomial.*

Proof. Let M be a finitely generated graded R-module such that $\mathcal{F} = \widetilde{M}$. Conclude by 12.12 B) c) and 9.17. $\square$

12.14. **Definition and Remark.** Let K be an infinite field, let R be a Noetherian homogeneous K-algebra, let $X = \mathrm{Proj}(R)$ and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, the unique polynomial in $\mathbb{Q}[\mathbf{x}]$ which coincides on $\mathbb{Z}$ with $\chi_{\mathcal{F}}$ is denoted by $P_{\mathcal{F}} \in \mathbb{Q}[x]$ and is called the *Serre polynomial of $\mathcal{F}$.* It follows immediately from 12.11 b) that $P_{\mathcal{F}}(n) = \chi_{\mathcal{F}}(n) = h^0(X, \mathcal{F}(n))$ for $n \gg 0$.

We assign the name of Serre to the polynomial $P_{\mathcal{F}}$ as the above result 12.13 is due to him (cf. [Se]).

12.15. **Remark, Exercise and Definition.** A) Let R be a Noetherian homogeneous algebra over a field K and let $X = \mathrm{Proj}(R)$. Let M be a finitely generated graded R-module. It is a somehow involved exercise in commutative algebra to show:

a) $\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \mathrm{depth}_{R_{(\mathfrak{p})}}(M_{(\mathfrak{p})})$ for $\mathfrak{p} \in X$.

Now, let $\mathcal{F} = \widetilde{M}$ be the coherent sheaf of $\mathcal{O}_X$-modules induced by M. It now follows easily from the above equality and from 12.4 B) that

b) $\mathrm{depth}_{R_x}(M_x) = \mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$ for all $x \in X$.

B) We set $\mathrm{m}X := \mathrm{mProj}(R)$. Then, $\mathrm{m}X$ is the set of "closed points of X". More precisely, denoting topological closure in X by $\overline{\bullet}$, for $x \in X$ we have $x \in \mathrm{m}X$ if and only if $\{x\} = \overline{\{x\}}$.

If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules we define the *subdepth of* $\mathcal{F}$ by

$$\delta(\mathcal{F}) := \inf\{\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \mid x \in \mathrm{m}X\}.$$

According to A) b) we can say: If M is a finitely generated graded R-module with $\mathcal{F} = \widetilde{M}$, then $\delta(\mathcal{F}) = \delta(M)$.

12.16. **Theorem.** *Let K be an algebraically closed field, let R be a Noetherian homogeneous K-algebra, let $X = \mathrm{Proj}(R)$ and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then:*

a) $h^i(X, \mathcal{F}(n)) = 0$ for all $i < \delta(\mathcal{F})$ and all $n \ll 0$.

b) $h^{\delta(\mathcal{F})}(X, \mathcal{F}(n)) \neq 0$ for all $n \ll 0$.

Proof. There is a finitely generated graded R-module M such that $\widetilde{M} = \mathcal{F}$. Observe that $\delta(M) = \delta(\mathcal{F})$ (cf. 12.15 B)). Then conclude by 12.10 and by 10.18. ☐

12.17. **Remark.** A) The above result 12.16 is nothing else than (essentially) Serre's original form of the Vanishing Theorem of Severi-Enriques-Zariski-Serre (cf. [Se]). In a purely algebraic context we already have spoken on this result after 10.18 has been established, in 10.19 D) and in 10.20.

B) We now leave it to the reader to formulate and to prove the bounding results 10.17, 10.19 C) in the purely sheaf theoretic context.

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Index