ON CANONICAL COHEN-MACAULAY MODULES

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Abstract Let $(R, \mathfrak{m})$ be a Noetherian local ring which is a homomorphic image of a local Gorenstein ring and let M be a finitely generated R-module of dimension $d > 0$. According to Schenzel (2004) [Sc3], M is called a *canonical Cohen-Macaulay* module (CCM module for short) if the canonical module $K(M)$ of M is Cohen-Macaulay. We give another characterization of CCM modules. We describe the non-canonical Cohen-Macaulay locus $\mathrm{nCCM}(M)$ of M. If $d \leqslant 4$ then $\mathrm{nCCM}(M)$ is closed in $\mathrm{Spec}(R)$. For each $d \geq 5$ there are reduced geometric local rings R of dimension d such that $\mathrm{nCCM}(R)$ is not stable under specialization [1].

1 Introduction

We first fix a few notations and recall a few notions.

Notation 1.1. Throughout this paper, $(R, \mathfrak{m})$ is a Noetherian local ring which is a quotient of a n-dimensional local Gorenstein ring $(R', \mathfrak{m}')$. Let M be a finitely generated R-module with $\dim M = d$. For each $i \in \mathbb{N}_0$, let $K^i_R(M) = K^i(M)$ denote the i-th *deficiency module* of M that is the finitely generated R-module $\mathrm{Ext}^{n-i}_{R'}(M, R')$. Observe that the formation of deficiency modules is base-ring independent in the following sense: if $\mathfrak{a} \subset R$ is an ideal such that $\mathfrak{a}M = 0$, then the R-modules $K^i_R(M)$ and $K^i_{R/\mathfrak{a}}(M)$ are equal for each $i \in \mathbb{N}_0$. If $E(R/\mathfrak{m})$ denotes the injective envelope of $R/\mathfrak{m}$, the Local Duality Theorem gives an isomorphism $H^i_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(K^i(M), E(R/\mathfrak{m}))$ for all $i \in \mathbb{N}_0$. By $K(M)$ we denote the *canonical module* $K^d(M)$ of M. Throughout our paper, we use the standard convention that $\mathrm{depth}_R(0) = \infty > 0$.

In Section 2 we shall prove a lifting result for the CCM-property of M and deduce two new characterizations of CCM modules. To make this more precise, let us recall, that Schenzel [Sc3] did prove that for $d > 0$, the property of being a CCM module on M is inherited by M/xM if $x \in \mathfrak{m}$ is a so-called strict f-element with respect to M, (see Definition 2.2). Our first main result proves that for $d \geq 4$ the CCM-property also lifts from M/xM to M for

[1] *Date: 11.September 2012.*
Key words and phrases: Canonical Cohen-Macaulay modules, Non-canonical Cohen-Macaulay locus.
2000 Subject Classification: 13D45, 13H10.
The second author was supported by the Viet Nam National Foundation for Science and Technology Development (Nafosted) under grant number 101.01-2011-20.

such elements x, (see Theorem 2.5). For $d = 3$, this lifting property need not hold any more, (see Remark 2.8). As a consequence of this we get that for $d \geq 3$ and for each so-called strict f-sequence $x_1, \ldots, x_{d-3}$ with respect to M (see Definition 2.2) the module M is CCM if and only if $K^2(M/\sum_{i=1}^{d-3} x_i M)$ is of depth > 0 – which includes the case that the latter module vanishes, (see Corollary 2.6). As a further consequence we get that a generalized Cohen-Macaulay module of dimension ≥ 3 is CCM if and only if the local cohomology modules $H^i_{\mathfrak{m}}(M)$ vanish for all $i = 2, \ldots, d-1$ or – equivalently – the $\mathfrak{m}$-transform $D_{\mathfrak{m}}(M)$ of M is a (finitely generated) Cohen-Macaulay module, (see Corollary 2.7).

In Section 3 we study the relation between the CCM-property and the polynomial type $p(M)$ of the module M (see Reminder 3.1). We first consider the case $p(M) = 1$ and prove that in this situation M is CCM if and only if it satisfies the two equivalent conditions, in which $M_{[1]} \subseteq M$ denotes the largest submodule of dimension ≤ 1 (see Proposition 3.7):

(ii) $H^i_{\mathfrak{m}}(M) = 0$ for all $i = 3, \ldots, d-1$ and $K^2(M)$ is 0 or Cohen-Macaulay of dimension 1.

(iii) The ideal transform $D_{\mathfrak{a}}(M^{[1]})$ of $M^{[1]} := M/M_{[1]}$ with respect to $\mathfrak{a} := (0 :_R H^2_{\mathfrak{m}}(M))$ is a (finitely generated) Cohen-Macaulay module.

In a second step, we also admit that $p(M) > 1$. This will lead us to another characterization of CCM modules, which involves once more strict f-sequences, (see Theorem 3.10): namely, if $d \geq 3$, if the polynomial type $p(M) =: k$ of M is positive and if $x_1, \ldots, x_{k-1}$ is a strict f-sequence with respect to M, then M is CCM if and only if either

(a) $k \leqslant d-2$, $K^i(M) = 0$ for $i = k+2, \ldots, d-1$ and $K^2(M/\sum_{j=1}^{k-1} x_j M)$ is 0 or Cohen-Macaulay of dimension 1; or

(b) $k = d-1$ and $K^2(M/\sum_{j=1}^{d-3} x_j M)$ is of dimension 2 and of depth > 0.

In Section 4 we finally study the non-CCM-locus of M, which is defined as the set of primes $\mathrm{nCCM}(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \text{ is not CCM}\}$. We show that under a certain "unmixedness" condition on the support $\mathrm{Supp}_R M$ of M, the set $\mathrm{nCCM}(M)$ coincides with the non-CM-locus $\mathrm{nCM}(K(M))$ and hence with the union of all pseudo supports $\mathrm{Psupp}^i(K(M))$ of the canonical module $K(M)$ of M, (see Proposition 4.4). It follows in particular that $\mathrm{nCCM}(M)$ is a closed set if $d \leqslant 4$. We also compare the locus $\mathrm{nCCM}(M)$ with the union of all pseudo supports of the canonical modules of the components of the dimension filtration of M, (see Proposition 4.6). Finally, on use of a construction found in Evans-Griffith [EG] we show that for each $d \geq 5$ and each field K there is a reduced local ring of $(R, \mathfrak{m})$ of dimension d, essentially of finite type over K, satisfying $R/\mathfrak{m} \cong K$ and such that $\mathrm{nCCM}(R)$ consists of single prime $\mathfrak{p}$ which is in addition different from $\mathfrak{m}$, (see Proposition 4.9). Hence $\mathrm{nCCM}(R)$ is not stable under specialization in this case, (see Corollary 4.10).

2 Strict f-Sequences and Canonical Cohen-Macaulay Modules

Definition 2.1. (See [Sc3, Definition 3.1]). The finitely generated R-module M is called *canonical Cohen-Macaulay* (CCM for short) if the canonical module $K(M)$ of M is Cohen-Macaulay (CM for short).

In view of the previously mentioned base-ring independence of deficiency modules, the property of being CCM is also base-ring independent. More precisely, if $\mathfrak{a} \subset R$ is an ideal such that $\mathfrak{a}M = 0$, then M is CCM as an R-module if and only if it is as an $R/\mathfrak{a}$-module.

There is a number of sufficient conditions for a module M to be CCM. If $\dim M \leqslant 2$ then M is CCM. It is clear that any CM module is CCM. The concept of *sequentially Cohen-Macaulay module* (sequentially CM module for short) was introduced by Stanley [St, p. 87] for graded modules. This notion was studied in the local case starting with the work of Schenzel [Sc2]. It is easy to see that any sequentially CM module is CCM. The notion of *pseudo Cohen-Macaulay module* (pseudo CM module for short) was introduced in [CN]. By [CN, Theorem 3.1] the module M is pseudo CM if and only if M/N is CM, where $N \subset M$ is the largest submodule of dimension less than d. As $K(M) \cong K(M/N)$, any pseudo CM module is CCM.

Our first main result shows that for certain elements $x \in R$ (under an obvious restriction on the dimension), the R-module M is CCM if and only if M/xM is. That the CCM-property of M implies the CCM-property of M/xM was actually proved already by Schenzel [Sc3, 3.3 (b)].

Definition 2.2. (See [CMN1]). Following I. G. Macdonald [Mac], we denote the set of attached primes of an Artinian R-module A by $\operatorname{Att}_R A$. An element $x \in \mathfrak{m}$ is called a *strict filter regular element* (strict f-element for short) with respect to M if $x \notin \mathfrak{p}$ for all

$$\mathfrak{p} \in \bigcup_{i=1}^{d} \operatorname{Att}_R H^i_{\mathfrak{m}}(M) \setminus \{\mathfrak{m}\}.$$

A sequence $x_1, \ldots, x_t$ of elements in $\mathfrak{m}$ is called a *strict filter regular sequence* (strict f-sequence for short) with respect to M if x_{j+1} is a strict f-element with respect to the R-module $M/(x_1, \ldots, x_j)M$ for all $j = 0, \ldots, t-1$.

Note that for each integer $t > 0$, by Prime Avoidance, there always exists a strict f-sequence of length t with respect to M.

Remark 2.3. (a) By [BS, 11.3.3] we have $\operatorname{Ass}_R M \subseteq \bigcup_{i=0}^{d} \operatorname{Att}_R H^i_{\mathfrak{m}}(M)$. Therefore, if x is a strict f-element with respect to M, it is a filter regular element (f-element for short) with respect to M, i.e. $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R M \setminus \{\mathfrak{m}\}$. Hence each strict f-sequence with respect to M is a filter regular sequence (f-sequence for short) with respect to M in sense of Cuong-Schenzel-Trung [CST].

(b) By [S, Theorem 2.3] we have $\operatorname{Ass}_R K^i(M) = \operatorname{Att}_R H^i_{\mathfrak{m}}(M)$ for all i. Therefore x is a strict f-element with respect to M if and only if x is an f-element with respect to $K^i(M)$ for all $i \geq 0$, thus if and only if $\ell(0 :_{K^i(M)} x) < \infty$ for all $i \geq 0$. In particular, if $d > 0$ and x is a strict f-element with respect to M then x is $K(M)$-regular.

We begin with the following auxiliary result. The sequence occurring in statement (a) may be found already in [Sc3].

Lemma 2.4. *Let $x \in \mathfrak{m}$ and let $i \in \mathbb{N}_0$.*

(a) *If x is a strict* f*-element with respect to M, there is an exact sequence*

$$0 \to K^{i+1}(M)/xK^{i+1}(M) \to K^i(M/xM) \to (0 :_{K^i(M)} x) \to 0.$$

(b) *If x is a strict* f*-element with respect to M, then $K^i(M/xM) = 0$ if and only if $K^{i+1}(M) = 0$ and x is $K^i(M)$-regular.*

(c) *If x is an* f*-element with respect to M such that* $\operatorname{depth}(M/xM) > 0$*, then x is M-regular.*

(d) *If x is a strict* f*-element with respect to M such that* $\operatorname{depth}(K^i(M/xM)) > 0$*, then* $\operatorname{depth}(K^{i+1}(M)) > 1$.

(e) *If x is a strict* f*-element with respect to M such that either* $\dim K^{i+1}(M) > 0$ *or* $\dim K^i(M/xM) > 0$*, then* $\dim K^i(M/xM) = \dim K^{i+1}(M) - 1$.

Proof. (a): We form the $K^*(\bullet) = \operatorname{Ext}^{n-*}_{R'}(\bullet, R')$-sequence associated to the exact sequence $0 \to M/(0 :_M x) \xrightarrow{\iota} M \to M/xM \to 0$, where ι is induced by $x\mathrm{Id}_M$. This gives an exact sequence $0 \to \operatorname{Coker}(K^{i+1}(\iota)) \to K^i(M/xM) \to \operatorname{Ker}(K^i(\iota)) \to 0$. Consider the short exact sequence $0 \to (0 :_M x) \to M \xrightarrow{\varrho} M/(0 :_M x) \to 0$, in which ϱ is the canonical map. As x is an f-element with respect to M, the module $(0 :_M x)$ is of finite length, so that $K^j(0 :_M x) = 0$ for all $j > 0$. This shows that $K^j(\varrho) : K^j(M/(0 :_M x)) \to K^j(M)$ is an isomorphism for $j > 0$ and a monomorphism for $j = 0$. As $\iota \circ \varrho = x\mathrm{Id}_M$ our claim follows.
(b): This follows immediate from statement (a) on use of Nakayama.
(c): As x is an f-element with respect to M we have $xM \cap H^0_{\mathfrak{m}}(M) = x(H^0_{\mathfrak{m}}(M) :_M x) = xH^0_{\mathfrak{m}}(M)$. So, by Nakayama, $H^0_{\mathfrak{m}}(M) \subseteq xM$ implies that $H^0_{\mathfrak{m}}(M) = 0$. This proves our claim as $\operatorname{depth}(M/xM) > 0$ implies that $H^0_{\mathfrak{m}}(M) \subseteq xM$.
(d): By statement (a) we get $\operatorname{depth}\big(K^{i+1}(M)/xK^{i+1}(M)\big) > 0$. As x is an f-element with respect to $K^{i+1}(M)$, our claim follows on use of statement (c).
(e): This is clear by statement (a) as $(0 :_{K^i(M)} x)$ is of finite length. □

Now, we can give the announced first main result.

Theorem 2.5. *Assume that* $\dim M = d \geq 4$ *and let $x \in \mathfrak{m}$ be a strict* f*-element with respect to M. Then M is* CCM *if and only if M/xM is.*

Proof. By [Sc3, 3.3(b)] we know that $M_1 := M/xM$ is CCM if M is CCM.
Conversely, assume that M_1 is CCM. As x is an f-element with respect to M we have $\dim(M_1) = d-1$. Let $y \in \mathfrak{m}$ be a strict f-element with respect to M_1. Since M_1 is CCM, the module $K^{d-1}(M_1) = K(M_1)$ is CM of dimension $d-1$. As y is a strict f-element with respect to M_1, it is $K(M_1)$-regular and so $K^{d-1}(M_1)/yK_{d-1}(M_1)$ is CM of dimension $d-2 \geq 2$. As $\dim(M_1/yM_1) = d-2 \geq 2$ we have $\operatorname{depth}(K^{d-2}(M_1/yM_1)) = \operatorname{depth}(K(M_1/yM_1)) \geq 2$. So, by the exact sequence of Lemma 2.4 (a), applied with M_1, y and $d-2$ instead of M, x and i respectively, the module $(0 :_{K^{d-2}(M_1)} y)$ has depth > 0 and hence vanishes. It follows that y is $K^{d-2}(M_1)$-regular and thus $K^{d-2}(M_1)$ is of positive depth.
Our next claim is that x is $K^{d-1}(M)$-regular. If $K^{d-2}(M/xM) = K^{d-2}(M_1) = 0$, this is immediate by 2.4 (b). So, let $K^{d-2}(M_1) \neq 0$. As $K^{d-2}(M_1)$ is of positive depth, the sequence of 2.4 (a), applied with $i = d-2$, shows that $K^{d-1}(M)/xK^{d-1}(M)$ is of positive depth, too. On application of 2.4 (c) with $K^{d-1}(M)$ instead of M it follows that x is indeed $K^{d-1}(M)$-regular.

Now, if we apply the exact sequence of 2.4 (a) with $i = d-1$ we get an isomorphism $K(M_1) \cong K(M)/xK(M)$. Since $K(M_1)$ is CM and x is $K(M)$-regular, $K(M)$ is CM, and hence M is CCM. $\square$

Corollary 2.6. *Suppose that $d \geq 3$ and let $x_1, \ldots, x_{d-3}$ is a strict* f*-sequence with respect to M. Then M is* CCM *if and only if* $\operatorname{depth}\big(K^2(M/\sum_{i=1}^{d-3} x_i M)\big) > 0$.

Proof. We proceed by induction on d. First, let $d = 3$. Let $D^\bullet$ be a dualizing complex of R and let $C^\bullet(M)$ be the corresponding complex of deficiency of M, as defined by Schenzel (see [Sc3, Definition 4.1]). Then by [Sc3, Proposition 4.2], the 3-dimensional module M is CCM if and only if the cohomology module $H^2(\operatorname{Hom}\big(C^\bullet(M), D^\bullet)\big)$ vanishes. As in the the proof of [Sc3, Proposition 4.2, pg. 760], we get an isomorphism

$$K^0(K^2(M)) \cong H^0(\operatorname{Hom}\big(K^2(M), D^\bullet)\big) \cong H^2\big(\operatorname{Hom}(C^\bullet(M), D^\bullet)\big),$$

so that M is CCM if and only if $\operatorname{depth} K^2(M) > 0$.
Now let $d > 3$. By Theorem 2.5, M is CCM if and only if $M_1 := M/x_1M$ is. Observe that $x_2, \ldots, x_{d-3}$ is a strict f-sequence with respect to M_1. So, by induction M_1 is CCM if and only if $K^2(M_1/\sum_{i=2}^{d-3} x_i M_1) = K^2(M/\sum_{i=1}^{d-3} x_i M)$ is of depth > 0. Altogether, this completes our proof. $\square$

Recall that according to [CST] the R-module M is said to be *generalized Cohen-Macaulay* (generalized CM for short) if $H^i_{\mathfrak{m}}(M)$ is of finite length for all $i < \dim M$. For each ideal $\mathfrak{a} \subseteq R$ we use $D_{\mathfrak{a}}(M)$ to denote the $\mathfrak{a}$-transform $\varinjlim_{n} \operatorname{Hom}_R(\mathfrak{a}^n, M)$ of M. As a further application of Theorem 2.5 we get the following criterion for the CCM-property of generalized CM modules.

Corollary 2.7. *Suppose that $d \geq 3$ and M is generalized* CM. *Then the following statements are equivalent.*

(i) *M is* CCM.

(ii) *$H^i_{\mathfrak{m}}(M) = 0$ for all $i = 2, \ldots, d-1$.*

(iii) *$D_{\mathfrak{m}}(M)$ is a (finitely generated)* CM *module.*

Proof. (i) $\Leftrightarrow$ (ii): As M is generalized CM, the modules $K^i(M)$ are of finite length for $0 \leq i < d$. Now Schenzel's result [Sc1, Korollar 1.4] – which holds also if the ring R is replaced by the R-module M – yields isomorphisms

$$H^i_{\mathfrak{m}}(K(M)) \cong K^0(K^{d-i+1}(M)), \text{ for all } i = 2, \ldots, d-1.$$

As $H^i_{\mathfrak{m}}(K(M)) = 0$ for $i = 1, 2$ and by local duality it follows, that $K(M)$ is CM if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i = 2, \ldots, d-1$. This proves the stated equivalence.
(ii) $\Leftrightarrow$ (iii): This is clear by the short exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to D_{\mathfrak{m}}(M) \to H^1_{\mathfrak{m}}(M) \to 0$$

and the relations $H^i_{\mathfrak{m}}(D_{\mathfrak{m}}(M)) = 0$ for $i = 0, 1$ and $H^j_{\mathfrak{m}}(D_{\mathfrak{m}}(M)) \cong H^j_{\mathfrak{m}}(M)$ for all $j > 1$ (see [BS, Chapter 2]). $\square$

Remark 2.8. Note that the conclusion of Theorem 2.5 is not true for $d = 3$. In fact, let M be a generalized CM module with $\dim M = 3$ and $H^2_{\mathfrak{m}}(M) \neq 0$. (For example take $M = R := A_{A_+}$, where K is a field, $B := K[x, y, z]/(x^3 + y^3 + z^3)$ and A is the Segre product ring $B \times B = \bigoplus_{n \in \mathbb{N}_0} B_n \otimes_K B_n$.) Then M is not CCM by Corollary 2.6. However, since $\dim(M/xM) = 2$ for any strict f-element x with respect to M, it follows that M/xM is CCM.

3 Polynomial Type and Canonical Cohen-Macaulay Modules

Reminder 3.1. The concept of polynomial type was introduced by N.T. Cuong [C]. Let $\underline{x} = (x_1, \ldots, x_d)$ be a system of parameters for M and $\underline{n} = (n_1, \ldots, n_d)$ a family of positive integers. Set

$$I_{M,\underline{x}}(\underline{n}) := \ell\big(M/(x_1^{n_1}, \ldots, x_d^{n_d})M\big) - n_1 \ldots n_d e(\underline{x}; M),$$

where $e(\underline{x}; M)$ is the multiplicity of M with respect to $\underline{x}$. Consider $I_{M,\underline{x}}(\underline{n})$ as a function in $\underline{n}$. In general this function is not a polynomial for $n_1, \ldots, n_d \gg 0$, but it always takes non-negative values for $n_1, \ldots, n_d \gg 0$ and is bounded from above by polynomials. Especially, the least degree of all polynomials in $\underline{n}$ which bound from above the function $I_{M,\underline{x}}(\underline{n})$ is independent of the choice of $\underline{x}$, (see [C, Theorem 2.3]). This least degree is called the *polynomial type* of M and denoted by $p(M)$.

Remark 3.2. If we stipulate that the degree of the zero polynomial is $-\infty$, then M is CM if and only if $p(M) = -\infty$. Moreover M is generalized CM if and only if $p(M) \leqslant 0$. In general

$$p(M) = \max_{0 \leqslant i \leqslant d-1} \dim\big(R/\operatorname{Ann}_R H^i_{\mathfrak{m}}(M)\big) = \max_{0 \leqslant i \leqslant d-1} \dim K^i(M) \leqslant d - 1$$

(see [C, Theorem 3.1(i)] and observe that by Local Duality $\operatorname{Ann}_R H^i_{\mathfrak{m}}(M) = \operatorname{Ann}_R K^i(M)$). If M is equidimensional then $p(M) = \dim \mathrm{nCM}(M)$, where $\mathrm{nCM}(M)$ is the non-CM-locus of M, (see [C, Theorem 3.1(ii)]).

Lemma 3.3. ([CMN, Lemma 3.1]). *Let $x \in \mathfrak{m}$ be a strict* f*-element with respect to M. If $p(M) > 0$ then $p(M/xM) = p(M) - 1$.*

In this section, we give a characterization of CCM modules which depends on the polynomial type. First we consider the case where $p(M) = 1$. In this case, a certain ideal transform will play a crucial role, and so we first prove the two following auxiliary results.

Lemma 3.4. *Set $\mathfrak{a} := (0 :_R H^2_{\mathfrak{m}}(M))$. Assume that $\ell(H^1_{\mathfrak{m}}(M)) < \infty$ and $\dim(R/\mathfrak{a}) \leqslant 1$. Then $D_{\mathfrak{a}}(M)$ is a finitely generated R-module such that $H^i_{\mathfrak{m}}(D_{\mathfrak{a}}(M)) = 0$ for $i = 0, 1$ and the R-module $H^2_{\mathfrak{m}}(D_{\mathfrak{a}}(M)))$ is of finite length. Moreover, $H^i_{\mathfrak{m}}(D_{\mathfrak{a}}(M)) \cong H^i_{\mathfrak{m}}(M)$ for all $i \geq 3$.*

Proof. According to our hypothesis we have

$$f^{\mathfrak{a}}_{\mathfrak{m}}(M) := \inf\{i \in \mathbb{N}_0 \mid \mathfrak{a} \not\subseteq \operatorname{rad}(0 :_R H^i_{\mathfrak{m}}(M))\} \geq 3.$$

As $\operatorname{ht}((\mathfrak{p} + \mathfrak{m})/\mathfrak{p}) = \dim(R/\mathfrak{p})$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$, it follows by [BS, (9.3.5)] that

$$3 \leqslant \lambda^{\mathfrak{a}}_{\mathfrak{m}}(M) := \inf\{\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(\mathfrak{a})\}.$$

As R is catenary, each $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(\mathfrak{a})$ satisfies $\operatorname{ht}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p}) \geq \dim(R/\mathfrak{p}) - \dim(R/\mathfrak{a}) \geq \dim(R/\mathfrak{p}) - 1$ and it follows that for each such $\mathfrak{p}$ it holds $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \operatorname{ht}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p}) \geq \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \dim(R/\mathfrak{p}) - 1 \geq \lambda_\mathfrak{m}^\mathfrak{a}(M) - 1 \geq 3 - 1 = 2$.
Therefore $\lambda_\mathfrak{a}(M) := \min\{\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \operatorname{ht}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(\mathfrak{a})\} \geq 2$. As R is a homomorphic image of a local Gorenstein ring it is universally catenary and all its formal fibers are CM. Hence, by [BS, (9.6.7)] we get

$$f_\mathfrak{a}(M) := \inf\{i \in \mathbb{N}_0 \mid H^i_\mathfrak{a}(M) \text{ is not finitely generated}\} = \lambda_\mathfrak{a}(M) \geq 2.$$

In particular, $H^1_\mathfrak{a}(M)$ is finitely generated. So the four-term exact sequence (cf. [BS, (2.2.4)(i)])

$$0 \to H^0_\mathfrak{a}(M) \to M \to D_\mathfrak{a}(M) \to H^1_\mathfrak{a}(M) \to 0$$

shows that $N := D_\mathfrak{a}(M)$ is finitely generated.
Moreover, $H^i_\mathfrak{a}(N) = 0$ for $i = 0, 1$ (cf. [BS, (2.2.8)(iv)]). Therefore $\operatorname{grade}_N(\mathfrak{a}) \geq 2$. As $\mathfrak{a} \subseteq \mathfrak{m}$, it follows that $H^i_\mathfrak{m}(N) = 0$ for $i = 0, 1$.

Our next aim is to show that $H^2_\mathfrak{m}(N) = H^2_\mathfrak{m}(D_\mathfrak{a}(M))$ is of finite length. To this end let $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. If $\mathfrak{p} \in \operatorname{Var}(\mathfrak{a})$ then $\dim(R/\mathfrak{p}) = 1$ and $\operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) \geq \operatorname{grade}_N(\mathfrak{a}) \geq 2$ and hence $\operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) + \dim(R/\mathfrak{p}) \geq 3$.
If $\mathfrak{p} \notin \operatorname{Var}(\mathfrak{a})$, then $H^0_\mathfrak{a}(M)_\mathfrak{p} = H^1_\mathfrak{a}(M)_\mathfrak{p} = 0$ and hence the above four-term exact sequence yields $N_\mathfrak{p} \cong M_\mathfrak{p}$, so that $\operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) + \dim(R/\mathfrak{p}) = \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \dim(R/\mathfrak{p}) \geq \lambda^\mathfrak{a}_\mathfrak{m}(M) \geq 3$.
Altogether we obtain $\lambda_\mathfrak{m}(N) := \inf\{\operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) + \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}\} \geq 3$. So, by [BS, (9.6.7)] we get $f_\mathfrak{m}(N) := \inf\{i \in \mathbb{N} \mid H^i_\mathfrak{m}(N) \text{ is not finitely generated}\} = \lambda_\mathfrak{m}(M) \geq 3$. Therefore $H^2_\mathfrak{m}(N)$ is finitely generated and thus of finite length.

By our hypothesis the finitely generated R-modules $H^0_\mathfrak{a}(M)$ and $H^1_\mathfrak{a}(M)$ are both of dimension $\leqslant 1$. Another use of the above four-term exact sequence now yields that $H^i_\mathfrak{m}(N) \cong H^i_\mathfrak{m}(M)$ for all $i \geq 3$. □

Lemma 3.5. *Let the notations and the assumptions be as in Lemma 3.4. Then $H^2_\mathfrak{m}(D_\mathfrak{a}(M)) = 0$ if and only if $\mathfrak{m} \notin \operatorname{Att}_R(H^2_\mathfrak{m}(M))$.*

Proof. Set $N := D_\mathfrak{a}(M)$ and let $\overline{M} := M/H^0_\mathfrak{a}(M)$. As $\dim(H^0_\mathfrak{a}(M)) \leqslant \dim(R/\mathfrak{a}) \leqslant 1$, we have $H^2_\mathfrak{m}(M) \cong H^2_\mathfrak{m}(\overline{M})$.
Assume first that $\mathfrak{m} \notin \operatorname{Att}_R(H^2_\mathfrak{m}(M))$. Then there is some $x \in \mathfrak{m}$ which avoids all members of $\operatorname{Att}_R(H^2_\mathfrak{m}(M))$ and hence multiplication by x on $H^2_\mathfrak{m}(\overline{M})$ is surjective. Since $\dim(H^1_\mathfrak{a}(M)) \leqslant \dim(R/\mathfrak{a}) \leqslant 1$, it holds $H^2_\mathfrak{m}(H^1_\mathfrak{a}(M)) = 0$. Therefore, the short exact sequence (cf. [BS, (2.2.4)(i)])

$$0 \to \overline{M} \to N \to H^1_\mathfrak{a}(M) \to 0$$

gives rise to an epimorphism $H^2_\mathfrak{m}(\overline{M}) \twoheadrightarrow H^2_\mathfrak{m}(N)$. Hence, multiplication by x on $H^2_\mathfrak{m}(N)$ is surjective. As $H^2_\mathfrak{m}(N)$ is finitely generated, Nakayama yields $H^2_\mathfrak{m}(N) = 0$.

Conversely, assume that $H^2_\mathfrak{m}(N) = 0$. Then, the above exact sequence gives rise to an epimorphism $H^1_\mathfrak{m}(H^1_\mathfrak{a}(M)) \twoheadrightarrow H^2_\mathfrak{m}(\overline{M})$. Let $x \in \mathfrak{m}$ be such that $\operatorname{rad}(\mathfrak{a} + xR) = \mathfrak{m}$. Then $H^1_\mathfrak{m}(H^1_\mathfrak{a}(M)) \cong H^1_{xR}(H^1_\mathfrak{a}(M))$, so that multiplication by x on $H^1_\mathfrak{m}(H^1_\mathfrak{a}(M))$ is an epimorphism. Hence multiplication by x on $H^2_\mathfrak{m}(\overline{M}) \cong H^2_\mathfrak{m}(M)$ is an epimorphism, so that $\mathfrak{m} \notin \operatorname{Att}_R(H^2_\mathfrak{m}(M))$. □

For each $j \in \{0, \ldots, d\}$ let $M^{[j]} := M/M_{[j]}$, where $M_{[j]} \subset M$ is the largest submodule of M whose dimension is $\leqslant j$. Keep in mind that we can write $M_{[j]}$ as the torsion submodule $\Gamma_{\mathfrak{B}}(M) := \{m \in M \mid \exists \mathfrak{b} \in \mathfrak{B}$ such that $\mathfrak{b}m = 0\}$ of M with respect to the multiplicatively closed set of ideals $\mathfrak{B} := \{\mathfrak{b} = \text{ideal in } R \mid \dim(R/\mathfrak{b}) \leqslant j\}$.

Lemma 3.6. *Suppose that $d \geq 3$ and let $p(M) = 1$. Let $\mathfrak{a} := (0 :_R H^2_{\mathfrak{m}}(M))$. Then the following statements are equivalent:*

(i) *$H^i_{\mathfrak{m}}(M) = 0$ for all $i = 3, \ldots, d-1$ and $K^2(M)$ is* 0 *or* CM *of dimension* 1.

(ii) *$D_{\mathfrak{a}}(M^{[1]})$ is a (finitely generated)* CM *module.*

Proof. As $\dim(M_{[1]}) \leqslant 1$, it is clear that $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(M^{[1]})$ for all $i \geq 2$ and that $K^2(M) \cong K^2(M^{[1]})$. Observe that by our hypotheses and by Remark 3.2 we have $\dim(R/\mathfrak{a}) \leqslant 1$ and hence also $\dim(K^2(M)) \leqslant 1$.
Our next claim is that $\ell(H^1_{\mathfrak{m}}(M^{[1]})) < \infty$. Suppose that this is not the case. Then $\dim H^1_{\mathfrak{m}}(M^{[1]}) = 1$. Hence $\dim(R/\mathfrak{p}) = 1$ for some $\mathfrak{p} \in \operatorname{Att}_R H^1_{\mathfrak{m}}(M^{[1]})$. So, $\mathfrak{p} \in \operatorname{Ass}_R(M^{[1]})$ by [BS, 11.3.3], this is a contradiction as $\operatorname{Ass}_R M^{[1]} = \{\mathfrak{q} \in \operatorname{Ass}_R M \mid \dim(R/\mathfrak{q}) > 1\}$.
So, by Lemma 3.4 the module $N := D_{\mathfrak{a}}(M^{[1]})$ is finitely generated and $H^i_{\mathfrak{m}}(N) \cong H^i_{\mathfrak{m}}(M^{[1]}) \cong H^i_{\mathfrak{m}}(M)$ for all $i \geq 3$.
Therefore N is CM if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i \in \{3, \ldots, d-1\}$ and $K^2(N) = 0$. It thus remains to show that $K^2(N) = 0$ if and only if $K^2(M)$ is of depth > 0 or equivalently, if and only if $\mathfrak{m} \notin \operatorname{Att}_R(H^2_{\mathfrak{m}}(M))$. But this is clear by Lemma 3.5. □

Proposition 3.7. *Suppose that $d \geq 3$ and let $p(M) = 1$. Let $\mathfrak{a} := (0 :_R H^2_{\mathfrak{m}}(M))$. Then the following statements are equivalent:*

(i) *M is* CCM.

(ii) *$H^i_{\mathfrak{m}}(M) = 0$ for all $i = 3, \ldots, d-1$ and $K^2(M)$ is either* 0 *or* CM *of dimension* 1.

(iii) *$D_{\mathfrak{a}}(M^{[1]})$ is a (finitely generated)* CM *module.*

Proof. (i) $\Rightarrow$ (ii): Let $x \in \mathfrak{m}$ be a strict f-element with respect to M. Since M is CCM so is $M_1 := M/xM$ by Theorem 2.5. As $p(M) = 1$, we get by Lemma 3.3 that M_1 is generalized CM. By Corollary 2.7, we have $K^i(M_1) = 0$ for all $i = 2, \ldots, d-2$. By Lemma 2.4 (b) we thus see that $K^i(M) = 0$ for all $i = 3, \ldots, d-1$ and that $x \in \mathfrak{m}$ is $K^2(M)$-regular. Since $p(M) = 1$, we have $\dim K^2(M) \leqslant 1$ by Remark 3.2. Therefore $K^2(M)$ is 0 or CM of dimension 1.

(ii) $\Rightarrow$ (i): Let $x \in \mathfrak{m}$ and M_1 be as above. Assume first that $d = 3$. Since $\operatorname{depth}(K^2(M)) > 0$ we have $(0 :_{K^2(M)} x) = 0$. If we apply the sequence of Lemma 2.4 (a) with $i = 2$ we obtain $K(M)/xK(M) \cong K(M_1)$. As $\dim M_1 = 2$, the module $K(M_1)$ is CM. As x is $K(M)$-regular it follows that $K(M)$ is CM.
Now, let $d > 3$. Since $K^i(M) = 0$ for all $i = 3, \ldots, d-1$, it follows by the exact sequences of Lemma 2.4 (a) that $K^i(M_1) = 0$ for all $i = 3, \ldots, d-2$. As $K^2(M)$ is 0 or CM of dimension 1, we have $(0 :_{K^2(M)} x) = 0$. Since $d > 3$, we have $K^3(M) = 0$. Therefore another use of the mentioned exact sequences for $i = 2$ gives that $K^2(M_1) = 0$. Note that M_1 is generalized CM of dimension $d-1$ by Lemma 3.3. Hence M_1 is CCM by Corollary 2.7, and so M is CCM by Theorem 2.5.

(ii) $\Leftrightarrow$ (iii): This is clear by Lemma 3.6. □

The main result of this section is the extension of the equivalence (i) $\Leftrightarrow$ (ii) of Proposition 3.7 which applies for all possible values of $p(M)$ and which again involves strict f-sequences. To pave the way to this, we first prove two auxiliary results.

Lemma 3.8. *Let $p(M) =: k > 0$ and let $x_1, \ldots, x_{k-1}$ be a strict* f*-sequence with respect to M. Set $M_i = M/\sum_{j=1}^{i} x_j M$ for all $i = 0, \ldots, k-1$. Then, the following statements are equivalent:*

(i) *M is* CCM *with $k \leqslant d-2$.*

(ii) *$K^i(M) = 0$ for all $i = k+2, \ldots, d-1$ and $K^2(M_{k-1})$ is* 0 *or* CM *of dimension* 1*.*

Proof. (i)$\Rightarrow$(ii): We proceed by induction on k. The case $k = 1$ follows by Proposition 3.7. So, let $k \geq 2$ and note that M_1 is CCM by Theorem 2.5. By Lemma 3.3 we have $p(M_1) = k-1 \leqslant \dim M_1 - 2$. As $x_2, \ldots, x_{k-1}$ is a strict f-sequence with respect to M_1, we get by induction that $K^2(M_{k-1}) = K^2(M_1/\sum_{j=2}^{k-1} x_j M_1)$ is 0 or CM of dimension 1 and that $K^i(M_1) = 0$ for all $i = k+1, \ldots, d-2$. By Lemma 2.4 (b) the latter implies that $K^i(M) = 0$ for all $i = k+2, \ldots, d-1$.

(ii)$\Rightarrow$(i): We proceed again by induction on k. The case $k = 1$ is immediately clear by Proposition 3.7. So, let $k \geq 2$. Observe that $(x_2, \ldots, x_{k-1})$ is a strict f-sequence with respect to M_1. By Lemma 3.3 we also have $p(M_1) = k-1$. As $K^i(M) = 0$ for all $i = k+2, \ldots, d-1$, the exact sequences of Lemma 2.4 (a), applied with $x := x_1$, imply that $K^i(M_1) = 0$ for all $i = k+2, \ldots, d-2$.
Our next aim is to show that $K^{k+1}(M_1) = 0$. Keep in mind that x_i is a strict f-element with respect to M_{i-1} and that $M_{i-1}/x_i M_{i-1} \cong M_i$, for each $i = 1, \ldots, k-1$. Observe also that $K^2(M_{k-1})$ is 0 or CM of dimension 1 by assumption (ii).
Assume first that $K^2(M_{k-1}) = 0$. Then, applying Lemma 2.4 (b) with $j+2$ instead of i, with M_{k-j} instead of M and with x_{k-j} instead of x for $j = 1, \ldots k-1$ we inductively get that $K^{2+j}(M_{k-j-1}) = 0$ so that finally $K^{k+1}(M) = 0$. By the hypothesis (ii) we also have $K^{k+2}(M) = 0$. So, the sequence of Lemma 2.4 (a), applied with $i = k+1$ and $x = x_1$ gives indeed $K^{k+1}(M_1) = 0$ in this case, as requested.
Now, we consider the case where $K^2(M_{k-1})$ is CM of dimension 1. In this situation we have $\operatorname{depth}(K^2(M_{k-1})) > 0$. Applying 2.4 (d) for $i = 2, \ldots, k$ to M_{k-i} instead of M and x_{k-i+1} instead of x, we inductively get that that $\operatorname{depth}(K^{i+1}(M_{k-i})) > 0$ and hence in particular that $\operatorname{depth}(K^{k+1}(M)) > 0$. Since x_1 is an f-element with respect to $K^{k+1}(M)$, this implies that $0 :_{K^{k+1}(M)} x_1 = 0$. By (ii) we have $K^{k+2}(M) = 0$ and hence on use of the exact sequence of Lemma 2.4 (a) with $i = k+1$ and $x = x_1$ it follows that again $K^{k+1}(M_1) = 0$ in this second case.
Moreover, by (ii) the module $K^2(M_1/\sum_{j-2}^{k-1} x_j M_1) = K^2(M_{k-1})$ is 0 or CM of dimension 1. So, M_1 satisfies the hypothesis of induction. Therefore M_1 is CCM and $k-1 = p(M)-1 = p(M_1) \leqslant d-3$, so that $p(M) = k \leqslant d-2$. Since $k \geq 2$, we have $d \geq 4$ and thus M is CCM by Theorem 2.5. $\square$

Lemma 3.9. *Let $d \geq 3$ and let $x_1, \ldots, x_{d-3}$ be a strict* f*-sequence with respect to M. Then, the following statements are equivalent:*

(i) *M is* CCM *with $p(M) = d-1$.*

(ii) *$K^2(M/\sum_{j=1}^{d-3} x_j M)$ is of dimension* 2 *and of positive depth.*

Proof. Set $M_i := M/\sum_{j=1}^{i} x_j M$ for all $i = 1, \dots, d-3$.

(i)$\Rightarrow$(ii): By Corollary 2.6 we have $\operatorname{depth}(K^2(M_{d-3})) > 0$. Note that $\dim K^i(M) \leqslant i$ for all i, (see [Sc3, Proposition 2.3 (a)]). Since $p(M) = d-1$, Remark 3.2 implies that $\dim(K^{d-1}(M)) = d-1$. Assume first that $d = 3$. Then $\dim(K^2(M)) = 2$ and the result follows.
So, let $d > 3$. Then $\dim(K^{d-1}(M)) = d-1$. On use of Lemma 2.4 (e) with $i = d-j$, M_{j-1} instead of M, x_j instead of x for $j = 2, \dots d-2$, we inductively get that $\dim K^{d-j}(M_{j-1}) = d-j$. Hence in particular $\dim K^2(M_{d-3}) = 2$.

(ii)$\Rightarrow$(i): By Corollary 2.6, the module M is CCM. By (ii) we have $\dim K^2(M_{d-3}) = 2$. By Lemma 2.4 (e) applied with M_{d-i-1} instead of M and x_{d-i} instead of x for $i = 1, \dots d-2$ we inductively get that $\dim K^{i+1}(M_{d-i-2}) = i+1$, so that $\dim(K^{d-1}(M)) = d-1$. Therefore by Remark 3.2 we have $p(M) = d-1$. $\square$

Now, we are ready to give the announced main result of the present section.

Theorem 3.10. *Assume that* $\dim M = d \geq 3$*,* $p(M) = k > 0$ *and that* $x_1, \dots, x_{k-1}$ *is a strict* f*-sequence with respect to* M*. For all* $i = 1, \dots, k-1$ *set* $M_i := M/\sum_{j=1}^{i} x_j M$*.*

(a) *If* $k \leqslant d-2$*, then* M *is* CCM *if and only if* $K^i(M) = 0$ *for all* $i = k+2, \dots, d-1$ *and* $K^2(M_{k-1})$ *is* 0 *or* CM *of dimension* 1*.*

(b) *If* $k = d-1$ *then* M *is* CCM *if and only if* $K^2(M_{d-3})$ *is of dimension* 2 *and of positive depth.*

Proof. The proof is immediate by Lemma 3.8 and Lemma 3.9. $\square$

4 Non-Canonical Cohen-Macaulay Loci

Definition 4.1. The *non-canonical Cohen-Macaulay locus* (non-CCM locus for short) of M, denoted by $\operatorname{nCCM}(M)$, is defined by

$$\operatorname{nCCM}(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \text{ is not CCM}\}.$$

Before describing the non-CCM locus of M, we recall the notion of pseudo support introduced in [BS1].

Definition 4.2. The *i-th pseudo support* of M, denoted by $\operatorname{Psupp}_R^i(M)$, is defined by

$$\operatorname{Psupp}_R^i(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid H_{\mathfrak{p} R_{\mathfrak{p}}}^{i-\dim(R/\mathfrak{p})}(M_{\mathfrak{p}}) \neq 0\}.$$

Remark 4.3. (a) Since R is a quotient of a Gorenstein ring

$$\operatorname{Psupp}_R^i(M) = \operatorname{Var}(\operatorname{Ann}_R H_{\mathfrak{m}}^i(M)) = \operatorname{Supp}_R K^i(M),$$

which is a closed subset of $\operatorname{Spec}(R)$, (see [BS1]).
(b) Keep in mind that

$$\operatorname{Ass}_R K(M) = \operatorname{Att}_R H_{\mathfrak{m}}^d(M) = \{\mathfrak{q} \in \operatorname{Ass}_R M \mid \dim(R/\mathfrak{q}) = d\},$$

so that $K(M)$ is equidimensional. Therefore, by [CNN, Corollary 3.2] we have

$$\mathrm{nCM}(K(M)) = \bigcup_{i=0,\dots,d-1} \mathrm{Psupp}^i(K(M)).$$

(c) Observe also that

$$\mathrm{Supp}_R K(M) = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \dim(R/\mathfrak{p}) + \dim M_{\mathfrak{p}} = d\}.$$

In particular $K(M)_{\mathfrak{p}} \cong K(M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \mathrm{Supp}_R K(M)$.

Proposition 4.4. *If* $\dim(R/\mathfrak{q}) = d$ *or* $\dim(R/\mathfrak{q}) \leqslant 3$ *for all* $\mathfrak{q} \in \min \mathrm{Ass}_R M$ *then*

$$\mathrm{nCCM}(M) = \mathrm{nCM}(K(M)) = \bigcup_{i=0,\dots,d-1} \mathrm{Psupp}^i_R(K(M)),$$

which is a closed subset of $\mathrm{Spec}(R)$. *In particular, if* $\dim M \leqslant 4$ *then* $\mathrm{nCCM}(M)$ *is closed.*

Proof. By Remark 4.3 (b) it suffices to show that $\mathrm{nCCM}(M) = \mathrm{nCM}(K(M))$. First, let $\mathfrak{p} \in \mathrm{Supp}_R K(M)$. It follows by Remark 4.3 (c) that $K(M_{\mathfrak{p}}) \cong (K(M))_{\mathfrak{p}}$. Therefore $\mathfrak{p} \in \mathrm{nCCM}(M)$ if and only if $(K(M))_{\mathfrak{p}}$ is not CM, hence if and only if $\mathfrak{p} \in \mathrm{nCM}(K(M))$. This shows that $\mathrm{Supp}_R K(M) \cap \mathrm{nCCM}(M) = \mathrm{nCM}(K(M))$.
Next, assume that $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \mathrm{Supp}_R K(M)$. Then $\dim(R/\mathfrak{q}) < d$ for all $\mathfrak{q} \in \mathrm{Ass}\, M$ with $\mathfrak{q} \subseteq \mathfrak{p}$. In particular $\mathfrak{p} \neq \mathfrak{m}$. So by our hypothesis we have $\dim M_{\mathfrak{p}} \leqslant 2$. Hence $K(M_{\mathfrak{p}})$ is CM, so that $\mathfrak{p} \notin \mathrm{nCCM}(M)$. □

Reminder 4.5. The notion of dimension filtration was introduced by P. Schenzel [Sc2]. A sequence of submodules $H^0_{\mathfrak{m}}(M) = M_0 \subset M_1 \subset \ldots \subset M_t = M$ is called a *dimension filtration* of M if M_i is the largest submodule $(M_{i+1})_{[\dim(M_{i+1})-1]}$ of M_{i+1} such that $\dim M_i < \dim M_{i+1}$ for all $i = 0, \dots, t-1$. Note that there exists precisely one dimension filtration of M.

Proposition 4.6. *Let* $H^0_{\mathfrak{m}}(M) = M_0 \subset M_1 \subset \ldots \subset M_t = M$ *be the dimension filtration of* M. *Set* $d_k = \dim M_k$. *Then*

$$\mathrm{nCCM}(M) \subseteq \bigcup_{\substack{k=1,\dots,t \\ i=0,\dots,d_k-1}} \mathrm{Psupp}^i(K(M_k)) = \bigcup_{k=1,\dots,t} \mathrm{nCM}(K(M_k)).$$

Proof. We proceed by induction on t. If $t = 0$ then $\mathrm{nCCM}(M) = \emptyset$ and the result is true. Let $t = 1$. Then $\dim(R/\mathfrak{p}) = d$ for all $\mathfrak{p} \in \mathrm{Ass}_R M \setminus \{\mathfrak{m}\}$. Therefore Proposition 4.4 yields $\mathrm{nCCM}(M) = \mathrm{nCM}(K(M)) = \bigcup_{i=0,\dots,d} \mathrm{Psupp}^i(K(M))$. Thus, the result is true for $t = 1$.
Now, let $t > 1$ and let $\mathfrak{p} \in \mathrm{nCCM}(M)$. Assume first that $\mathfrak{p} \in \mathrm{Supp}_R K(M)$. By Remark 4.3 (c) we have $K(M_{\mathfrak{p}}) \cong (K(M))_{\mathfrak{p}}$, and hence $\mathfrak{p} \in \mathrm{nCM}(K(M)) \subseteq \bigcup_{i=0,\dots,d} \mathrm{Psupp}^i(K(M))$, by another use of Remark 4.3 (c).
Suppose now, that $\mathfrak{p} \notin \mathrm{Supp}_R K(M)$. Then $\mathfrak{q} \not\subseteq \mathfrak{p}$ for all $\mathfrak{q} \in \mathrm{Ass}_R M$ with $\dim(R/\mathfrak{q}) = d$ by Remark 4.3 (c). Since

$$\mathrm{Ass}_R(M/M_{t-1}) = \{\mathfrak{q} \in \mathrm{Ass}_R M \mid \dim(R/\mathfrak{q}) = d\},$$

we have $\mathfrak{p} \notin \mathrm{Supp}(M/M_{t-1})$. So, from the exact sequence

$$0 \to (M_{t-1})_{\mathfrak{p}} \to M_{\mathfrak{p}} \to (M/M_{t-1})_{\mathfrak{p}} \to 0$$

we get $M_{\mathfrak{p}} \cong (M_{t-1})_{\mathfrak{p}}$, Hence $\mathfrak{p} \in \mathrm{nCCM}(M_{t-1})$. So by induction

$$\mathfrak{p} \in \bigcup_{k=1,\dots,t-1} \mathrm{nCM}(K(M_k)) = \bigcup_{\substack{k=1,\dots,t-1\\ i=0,\dots,d_k-1}} \mathrm{Psupp}^i(K(M_k)).$$

□

Finally, we give an example to show that for $d \geq 5$ the set $\mathrm{nCCM}(M)$ need not be closed, indeed even not stable under specialization. We first prove the following lemma.

Lemma 4.7. *Let K be a field, let n be an integer such that $n \geq 3$. Consider the polynomial ring $U = K[x_1, \dots, x_{n+2}]$, furnished with its standard grading. Then there is a graded prime ideal $\mathfrak{t} \subseteq U$ of height 2 such that $\mathfrak{t} \cap U_1 = 0$ and such that $A := U/\mathfrak{t}$ is a normal domain (of dimension n) with homogeneous maximal ideal $A_+ = U_+/\mathfrak{t}$ and*

$$H^i_{A_+}(A) = \begin{cases} 0, & \text{if } i \neq n-1, n \\ A/A_+, & \text{if } i = n-1. \end{cases}$$

Proof. See [EG, Theorem 4.13] or [MNP, Theorem 2.3 and Remark 2.4(ii)]. □

Notation and Construction 4.8. Let $n, U, \mathfrak{t}$ and A be as in Lemma 4.7. Consider the polynomial ring $S := U[x_{n+3}] = K[x_1, \dots, x_{n+3}]$ and set $R := \Big(S/\mathfrak{t}S \cap x_{n+3}S\Big)_{S_+}$, $\mathfrak{m} := S_+R$, $\mathfrak{r} := x_{n+3}R$, $\mathfrak{q} := \mathfrak{t}R$ and $\mathfrak{p} := U_+R$.

Proposition 4.9. *$(R, \mathfrak{m})$ is a reduced local ring of dimension $n+2$, essentially of finite type over K with $R/\mathfrak{m} \cong K$ and $\dim_K(\mathfrak{m}/\mathfrak{m}^2) = n+3$. Moreover,*

(a) $\min \mathrm{Ass}\, R = \{\mathfrak{r}, \mathfrak{q}\}$ *and* $\mathfrak{p} \in \mathrm{Var}(\mathfrak{q}) \setminus \mathrm{Var}(\mathfrak{r})$.

(b) *$R/\mathfrak{r}$ is regular local of dimension $n+2$, $R/\mathfrak{q}$ is normal of dimension $n+1$ and $R_{\mathfrak{p}}$ is normal of dimension n.*

(c) $H^i_{\mathfrak{p} R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = \begin{cases} 0, & \text{if } i \neq n-1, n \\ K(\mathfrak{p}), & \text{if } i = n-1. \end{cases}$

(d) $\mathrm{nCCM}(R) = \{\mathfrak{p}\}$.

Proof. The claims in the preamble of our proposition follow immediately by the previous construction.

(a): This is also an easy consequence of our construction.

(b): Observe that according to our construction

$$R/\mathfrak{r} \cong U_{U_+} \cong K[x_1, \dots, x_{n+2}]_{(x_1,\dots,x_{n+2})} \quad \text{and} \quad R/\mathfrak{q} \cong A[x_{n+3}]_{(A_+, x_{n+3})}.$$

Since $\mathfrak{p} \in \mathrm{Var}(\mathfrak{q}) \setminus \mathrm{Var}(\mathfrak{r})$, we have

$$R_{\mathfrak{p}} \cong (R/\mathfrak{q})_{\mathfrak{p}} \cong A[x_{n+3}]_{A_+ . A[x_{n+3}]}.$$

From this, all claims in statement (b) are immediate.

(c): The last isomorphism in the previous paragraph and the A-flatness of $A[x_{n+3}]_{A_+.A[x_{n+3}]}$ yield

$$\begin{aligned} H^i_{\mathfrak{p} R_\mathfrak{p}}(R_\mathfrak{p}) &\cong H^i_{A_+.A[x_{n+3}]_{A_+.A[x_{n+3}]}}\big(A[x_{n+3}]_{A_+.A[x_{n+3}]}\big) \\ &\cong H^i_{A_+}(A) \otimes_A A[x_{n+3}]_{A_+.A[x_{n+3}]} \\ &\cong H^i_{A_+}(A) \otimes_A R_\mathfrak{p}. \end{aligned}$$

As $A_+ R_\mathfrak{p} = \mathfrak{p} R_\mathfrak{p}$, this proves our claim.

(d): According to statement (c), the ring $R_\mathfrak{p}$ is generalized CM but not CCM. Therefore $\mathfrak{p} \in \mathrm{nCCM}(R)$. By statement (b), the unmixed part $U(R) := R^{[\dim R-1]}$ of R is given by $U(R) = R/\mathfrak{r}$ and as $R/\mathfrak{r}$ is regular, it follows that $K(R) = K(U(R)) = R/\mathfrak{r}$ and hence $\mathrm{Var}(\mathfrak{r}) \cap \mathrm{nCCM}(R) = \emptyset$.
It remains to show that $K(R_\mathfrak{s})$ is CM for all $\mathfrak{s} \in \mathrm{Spec}(R) \setminus \big(\mathrm{Var}(\mathfrak{r}) \cup \{\mathfrak{p}\}\big)$. Fix such an $\mathfrak{s}$ and consider the canonical map $A \to R$. As $\mathrm{Var}(A_+R) = \{\mathfrak{p}, \mathfrak{m}\}$, we must have that $\mathfrak{n} := \mathfrak{s} \cap A \subset A_+$ and $\mathfrak{n} \neq A_+$, so that $A_\mathfrak{n}$ is CM. In view of the last isomorphism in the proof of (b), the ring $R_\mathfrak{s}$ is isomorphic to a localization of $A_\mathfrak{n}[x_{n+3}]$ and hence is CM, so that $K(R_\mathfrak{s})$ is CM. □

Corollary 4.10. *For each field K and each integer $d \geq 5$ there is a reduced local ring $(R, \mathfrak{m})$ of dimension d which is essentially of finite type over K, with $R/\mathfrak{m} \cong K$ and such that* $\mathrm{nCCM}(R)$ *is not stable under specialization.*

Proof. Apply Proposition 4.9 with $n = d - 2$. □

References

[BS] M. Brodmann and R. Y. Sharp, "Local cohomology: an algebraic introduction with geometric applications", Cambridge University Press, 1998.

[BS1] M. Brodmann and R. Y. Sharp, On the dimension and multiplicity of local cohomology modules, *Nagoya Math. J.*, **167** (2002), 217-233.

[C] N. T. Cuong, On the least degree of polynomials bounding above the differences between lengths and multiplicities of certain systems of parameters in local rings. *Nagoya Math. J.*, **125** (1992), 105-114.

[CMN] N. T. Cuong, M. Morales and L. T. Nhan, On the length of generalized fractions, *J. Algebra*, **265** (2003), 100-113.

[CMN1] N. T. Cuong, M. Morales and L. T. Nhan, The finiteness of certain sets of attached prime ideals and the length of generalized fractions, *J. Pure Appl. Algebra*, **(1-3) 189**, (2004), 109-121.

[CN] N. T. Cuong and L. T. Nhan, On pseudo Cohen-Macaulay and pseudo generalized Cohen-Macaulay modules, *J. Algebra*, **267** (2003), 156-177.

[CNN] N. T. Cuong, L. T. Nhan, N. T. K. Nga, On pseudo supports and non Cohen-Macaulay locus of a finitely generated module, *Journal of Algebra*, **323** (2010), 3029-3038.

[CST] N. T. Cuong, P. Schenzel and N. V. Trung, Verallgemeinerte Cohen - Macaulay Moduln, *Math. Nachr.*, **85** (1978), 57-75.

[EG] E. G. Evans and P. Griffith, “Syzygies”, London mathematical Society Lecture Notes Series 106, Cambridge University Press, 1985.

[Mac] I. G. Macdonald, Secondary representation of modules over a commutative ring, *Symposia Mathematica*, **11** (1973), 23-43.

[MNP] J. Migliore, U. Nagel, C. Peterson, Constructing schemes with prescribed cohomology in arbitrary codimension, *J. Pure and Applied Algebra*, **152** (2000), 245-251.

[Sc1] P. Schenzel, Zur lokalen Kohomologie des kanonischen Moduls, *Mathematische Zeitschrift*, **165** (1979), no. 3, 223 - 230.

[Sc2] P. Schenzel, On the dimension filtration and Cohen-Macaulay filtered modules, *In: Proc. of the Ferrara meeting in honour of Mario Fiorentini, University of Antwerp Wilrijk, Belgium,* (1998), 245-264.

[Sc3] P. Schenzel, On birational Macaulayfications and Cohen-Macaulay canonical modules, *J. Algebra*, **275** (2004), 751-770.

[S] R. Y. Sharp, Some results on the vanishing of local cohomology modules, *Proc. London Math. Soc.*, **30** (1975), 177-195.

[St] R. P. Stanley, “Combinatorics and Commutative algebra”, Second edition, Birkhäuser Boston-Basel-Berlin, 1996.