A guide to (étale) motivic sheaves

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Abstract. We recall the construction, following the method of Morel and Voevodsky, of the triangulated category of étale motivic sheaves over a base scheme. We go through the formalism of Grothendieck's six operations for these categories. We mention the relative rigidity theorem. We discuss some of the tools developed by Voevodsky to analyze motives over a base field. Finally, we discuss some long-standing conjectures.

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1. Introduction

The (co)homological invariants associated to an algebraic variety fall into two classes:

(a) the *algebro-geometric invariants* such as higher Chow groups (measuring the complexity of algebraic cycles inside the variety) and Quillen K-theory groups (measuring the complexity of vector bundles over the variety);

(b) the class of *transcendental invariants* such as Betti cohomology (with its mixed Hodge structure) and ℓ-adic cohomology (with its Galois representation).

The distinction between these two classes is extreme.

- The algebro-geometric invariants are abstract Abelian groups, often of infinite rank, carrying no extra structure.[1] They vary chaotically in families and are not computable in any reasonable sense.

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[1] To avoid confusion, we mention that the kind of extra structures we have in mind are those that can be given by the action of some group of symmetries such as the Galois group of the base field or, more generally, the fundamental group of a Tannakian category such as the category of mixed Hodge structures. It should be mentioned here that higher Chow groups are expected to carry a filtration, the conjectural Bloch–Beilinson filtration, with quite remarkable properties.

- On the other hand, transcendental invariants are concrete groups of finite rank (over some coefficient ring) carrying a rich extra structure. Together with their extra structure, they vary "continuously" in families.

Nevertheless, all these invariants are expected to be shadows of some master invariants, called the *motives* of the algebraic variety. The algebro-geometric invariants are expected to be groups of morphisms, extensions and higher extensions between these motives and other basic ones (such as Tate motives), while each of these motives *realizes* (i.e., gives rise) to a multitude of transcendental invariants of different types that, a priori, look poorly related.

One of the ultimate goals of the theory of motives is to serve as a bridge *between the above two classes of cohomological invariants.*

Until now, establishing a fully satisfactory theory of motives has defied all attempts. Thinking about it as a bridge between (a) and (b), one can describe the present status of the theory as a broken bridge or, better, as a union of two half-bridges that, for the moment, fail to meet.

- The first half bridge, the one starting from (a), is a theory of motives that gives a satisfactory framework for understanding the algebro-geometric invariants.

- The second half-bridge, the one starting form (b), is a theory of motives that encapsulates the transcendental invariants and endows them with universal extra structures.

Concerning the second half-bridge, we just mention few highlights. In the *pure* case, i.e., for smooth and proper varieties, an approach was pioneered by Grothendieck [20]. Roughly speaking, Grothendieck's idea was to "decompose" smooth and proper varieties into "cohomological atoms" called *pure numerical motives* using certain *algebraic cycles* whose existence would be guaranteed by his (yet unproven) *Standard Conjectures* [12]. Later on, Deligne [11] and then André [2] made Grothendieck's approach unconditional by replacing algebraic cycles with *absolute Hodge cycles* and *motivated cycles* respectively. In the *mixed* case, i.e., for possibly open and singular varieties, an approach was invented by Nori (unpublished, but see [23, §5.3.3] for an account) based on his weak Tannakian reconstruction theorem which is an abstract devise yielding an Abelian category out of a representation of a diagram (aka., quiver). The main geometric ingredient behind most results about Nori's motives is the so-called *Basic Lemma* which can be considered as an enhanced form of the Lefschetz hyperplane theorem. In all these approaches (in the pure and mixed cases), the outcome is a *Tannakian* (and hence *Abelian*) category of motives whose fundamental group is the so-called *motivic Galois group*. It is also important to note here a crucial drawback: except the original construction of Grothendieck which is conditional on the Standard Conjectures, all available unconditional constructions of Abelian categories of motives depend on transcendental data (namely, a Weil cohomology theory such as Betti cohomology or ℓ-adic cohomology). For this reason, the existence of the "true"

Abelian category of motives is still considered to be an open question.[2]

The present article is mainly concerned with the first half-bridge, i.e., the one starting from (a). Here the outcome of the theory is a *triangulated* category of motives whose groups of morphisms are blends of the algebro-geometric invariants of algebraic varieties (and more precisely, their higher Chow groups). If the existence of such categories was part of the Grothendieck motivic picture, it was probably Beilinson and Deligne who first expressed the hope that such categories might be easier to construct than their Abelian counterparts. And indeed, three different constructions of triangulated categories of motives appeared in the nineties by Hanamura [13, 14, 15], Levine [22] and Voevodsky [29] (see also its precursor [28]). Although, the three categories were found to be equivalent, Voevodsky's construction [29] attracted most attention due to its beauty, simplicity and potential.

Nearly a decade later, it was realized (based on work of Morel and Cisinski–Déglise) that a mild modification of Voevodsky's construction, yields an even simpler (and certainly as beautiful) construction of the same (up to equivalence) triangulated category of motives at least if torsion is neglected or, more precisely, if descent for the étale topology is imposed (which is the right thing to do for many questions concerning integral motives such as the Hodge and Tate conjectures, existence of a motivic t-structure, etc; see §5.2). This simplified construction is more in the spirit of the construction of Morel–Voevodsky $\mathbb{A}^1$-homotopy category [25] (and more precisely its stabilization that was worked out by Jardine [19]) and has the advantage of giving the correct triangulated categories over any base scheme.[3] These triangulated categories are denoted by $\mathbf{DA}^{\text{ét}}(S;\Lambda)$, where S is the base scheme and Λ is the ring of coefficients, and their objects are called *motivic sheaves* over S or simply S-*motives*;[4] they are the subject of this paper.

The organization is as follows. In §2 we give the details of the construction of $\mathbf{DA}^{\text{ét}}(S;\Lambda)$. We hope to convince the reader that this construction is simple and natural. In §3 we explain the basic operations that one can do on motivic sheaves; the story here is parallel to what one has in the context of étale and ℓ-adic sheaves although the construction of the operations follows a different route. One should consider the formalism of the six operations as a tool to reduce questions about motivic sheaves over general bases to questions about motives over a point (i.e., the spectrum of a field). In order for this formalism to be of any use, one needs information about motives over fields. In §4 we start discussing results about the internal structure of the category of motives over a field. More precisely, we give a concrete description of the group of morphisms between certain motives; such groups are usually called *motivic cohomology.* Here all the results are due to Voevodsky and this is the place where the extra complexity in his original construction

[2] Over a field of characteristic zero, it can be shown that if the "true" Abelian category of mixed motives exists, then it must be equivalent to Nori's category, and its subcategory of semi-simple objects must be equivalent to André's category. (The equivalence between André's and Deligne's categories is another story as it would require a weak form of the *Hodge Conjecture.*)

[3] The original construction of Voevodsky is also known to give the correct triangulated categories when the base scheme is normal. However, the question remains open for more general base schemes (but see Remark 4.6).

[4] It is common to use the terminology "étale motivic sheaves". However, as the main article concerns motives in the étale topology, we use the shorthand "motivic sheaves".

pays off. In particular, we recall the original construction of Voevodsky in §4.1 and explain in §4.2 how it permits the computation of motivic cohomology. In §5 we list some of the big open questions concerning motives. It is these conjectures that need to be solved for having a satisfactory theory of motivic sheaves and filling the gap between the two half-bridges discussed above.

2. Construction

In this section, we go through the construction of the categories $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ of *étale motivic sheaves* (or *motivic sheaves* for short) over a base scheme S and with coefficients in a commutative ring Λ. This construction is a slight variation of Voevodsky's original construction of his $\mathbf{DM}^{\text{ét}}(S;\Lambda)$ [29, 24] (see Remark 4.3 for more precisions). In fact, it is really a *simplification* of the latter as sheaves with transfers get replaced by ordinary sheaves. The category $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ should be also considered as the linearized counterpart of the Morel–Voevodsky stable $\mathbb{A}^1$-homotopy category in the étale topology $\mathbf{SH}^{\text{ét}}(S)$ [25, 19]. In fact, both categories $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ and $\mathbf{SH}^{\text{ét}}(S)$ are constructed in a uniform way in [6, Chapitre 4] as special cases of categories $\mathbf{SH}^{T}_{\mathfrak{M}}(S)$ by choosing $\mathfrak{M}$ to be the category of Λ-modules or the category of simplicial symmetric spectra.

In order to keep the technicalities as low as possible, we will be using Verdier localization of triangulated categories [27] instead of the more natural/satisfactory Bousfield localization of model categories [16] which is usually employed in this context. We start by recalling Verdier localization.

2.1. A technical tool: Verdier localization. Recall that a *triangulated category* $\mathcal{T}$ is an additive category endowed with an autoequivalence $A \mapsto A[1]$ and a class of *distinguished triangles* which are diagrams of the form

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1] \tag{1}$$

satisfying a list of axioms. In particular, given a distinguished triangle as above, one has $\beta \circ \alpha = 0$ and $\gamma \circ \alpha = 0$. Moreover, the distinguished triangle (1) is determined by the map $\alpha : A \to B$ up to an isomorphism, which is in general not unique. Nevertheless, it will be sometimes convenient to abuse notation by writing $C = \mathrm{Cone}(\alpha)$ (and thus pretending that C depends canonically on α). Of course, this notation is inspired from topology: one thinks about a distinguished triangle (1) as an abstract version of a cofibre sequence. An important fact to keep in mind is the following: *α is an isomorphism if and only if* $\mathrm{Cone}(\alpha)$ *is zero.*

Now, let $\mathcal{T}$ be a triangulated category and $\mathcal{E} \subset \mathcal{T}$ a full subcategory closed under suspensions and desuspensions (i.e., under application of the powers $[n]$, positive and negative, of the autoequivalence [1]) and under cones. (Such an $\mathcal{E}$ is called a *triangulated subcategory* of $\mathcal{T}$.) In this situation, we have (see [27, Théorème 2.2.6]):

Proposition 2.1. *There exists a triangulated category $\mathcal{T}/\mathcal{E}$, called the* Verdier quotient *of $\mathcal{T}$ by $\mathcal{E}$, which is universal for the following two properties.*

(i) There is a canonical triangulated functor $\mathcal{T} \to \mathcal{T}/\mathcal{E}$ which is the identity on objects (in particular $\mathcal{T}$ and $\mathcal{T}/\mathcal{E}$ have the same class of objects).

(ii) For every $A \in \mathcal{E}$, one has $A \simeq 0$ in $\mathcal{T}/\mathcal{E}$.

Remark 2.2. The construction of $\mathcal{T}/\mathcal{E}$ goes as follows. Consider the class of arrows $S_{\mathcal{E}}$ in $\mathcal{T}$ given by

$$S_{\mathcal{E}} = \{\alpha : A \to B \mid \mathrm{Cone}(\alpha) \in \mathcal{E}\}.$$

The axioms satisfied by the class of distinguished triangles imply that $S_{\mathcal{E}}$ admits a "calculus of fractions". The Verdier quotient is then defined by

$$\mathcal{T}/\mathcal{E} := \mathcal{T}[(S_{\mathcal{E}})^{-1}].$$

In words, $\mathcal{T}/\mathcal{E}$ is the category obtained by formally inverting the arrows in $S_{\mathcal{E}}$.[5] This explains why the Verdier quotient is also called a *localization*.

2.2. An almost correct construction in two steps.

The category $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ is obtained from the derived category of étale sheaves on smooth S-schemes by formally forcing two simple properties. In this subsection, we discuss these properties and explain how to force them successively. This yields a slightly naive notion of motivic sheaves. The correct notion will be given in §2.3.

2.2.1. Some notation. From now on, Λ will always denote a commutative ring that we call the *ring of coefficients*. (In practice, Λ is $\mathbb{Z}$, $\mathbb{Q}$, a subring of $\mathbb{Q}$ or a quotient of $\mathbb{Z}$. However, it is sometimes useful to take for Λ a number ring, a number field, a local field, etc.) Given a set E, we denote by $\Lambda \otimes E = \bigoplus_{e \in E} \Lambda \cdot e$ the free Λ-module generated by E.

For simplicity, all schemes will be separated and the reader will not loose much by assuming that all schemes are also Noetherian of finite Krull dimension.

Let S be a base scheme. We denote by Sm/S the category of smooth S-schemes.[6] We endow Sm/S with the étale topology ([3, Exposé VII]) and we denote by $\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda)$ the category of étale sheaves with values in Λ-modules. If no confusion can arise, objects of $\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda)$ will be simply called *étale sheaves* on Sm/S. Given a smooth S-scheme X, we denote by $\Lambda_{\text{ét}}(X) := \mathrm{a}_{\text{ét}}(\Lambda \otimes X)$ the étale sheaf associated to the presheaf $U \in \mathrm{Sm}/S \mapsto \Lambda \otimes \mathrm{Hom}_S(U,X)$. This gives a Yoneda functor

$$\Lambda_{\text{ét}} : \mathrm{Sm}/S \to \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda) \tag{2}$$

which one should consider as the first/obvious linearization of the category of smooth S-schemes, a necessary step for passing from S-schemes to S-motives.

The following lemma is left as an exercise and will not be used elsewhere. It shows that étale sheaves on Sm/S have transfers along finite étale covers.

[5] Needless to say that we are ignoring some set-theoretical issues here.

[6] Recall that *smooth* implies in particular *locally of finite presentation*. One may also restrict to smooth quasi-projective S-schemes and even to smooth quasi-affine S-schemes as these will define equivalent sites for the étale topology.

Lemma 2.3. *Let X and U be smooth S-schemes and assume that S is normal. Then $\Lambda_{\text{ét}}(X)(U)$ is the free Λ-module generated by closed integral subschemes $Z \subset U \times_S X$ such that the normalization of Z is étale and finite over U.*

The category $\text{Shv}_{\text{ét}}(\text{Sm}/S; \Lambda)$ possesses a monoidal structure. If $\mathcal{M}$ and $\mathcal{N}$ are étale sheaves on Sm/S, then $\mathcal{M} \otimes_\Lambda \mathcal{N}$ is simply the étale sheaf associated to the presheaf $U \in \text{Sm}/S \mapsto \mathcal{M}(U) \otimes_\Lambda \mathcal{N}(U)$. If there is no risk of confusion, we will write $- \otimes -$ instead of $- \otimes_\Lambda -$ for the tensor product of Λ-modules and sheaves of Λ-modules. Given two smooth S-schemes X and Y, it follows readily from the definitions that

$$\Lambda_{\text{ét}}(X) \otimes \Lambda_{\text{ét}}(Y) \simeq \Lambda_{\text{ét}}(X \times_S Y).$$

Said differently, the functor $\Lambda_{\text{ét}}$ is monoidal (when Sm/S is endowed with its Cartesian monoidal structure).

2.2.2. First step: $\mathbb{A}^1$-localization. To motivate what follows, we note that, for a scheme U, the projection $\mathbb{A}^1 \times U \to U$ (where $\mathbb{A}^1 = \text{Spec}(\mathbb{Z}[t])$ is the affine line) induces isomorphisms in most cohomology theories (for instance, in Betti cohomology if $U \in \text{Sm}/\mathbb{C}$, in ℓ-adic cohomology if ℓ is invertible on U, in algebraic K-theory if U is regular, etc). Thus, it is natural to expect the motives of U and $\mathbb{A}^1 \times U$ to be isomorphic.

To impose this in a "homologically correct" way, consider the derived category $\mathbf{D}(\text{Shv}_{\text{ét}}(\text{Sm}/S; \Lambda))$ of the Abelian category $\text{Shv}_{\text{ét}}(\text{Sm}/S; \Lambda)$. Let $\mathcal{T}_{\mathbb{A}^1}$ be the smallest triangulated subcategory of $\mathbf{D}(\text{Shv}_{\text{ét}}(\text{Sm}/S; \Lambda))$ which is closed under arbitrary direct sums and containing the 2-terms complexes

$$[\ldots \to 0 \to \Lambda_{\text{ét}}(\mathbb{A}^1 \times U) \to \Lambda_{\text{ét}}(U) \to 0 \to \ldots] \tag{3}$$

for all smooth S-schemes U. (In the above complex, the nonzero map is induced by the obvious projection $\mathbb{A}^1 \times U \to U$.) Then define $\mathbf{DA}^{\text{eff, ét}}(S; \Lambda)$ to be the Verdier quotient of $\mathbf{D}(\text{Shv}_{\text{ét}}(\text{Sm}/S))$ by $\mathcal{T}_{\mathbb{A}^1}$:

$$\mathbf{DA}^{\text{eff, ét}}(S; \Lambda) := \mathbf{D}(\text{Shv}_{\text{ét}}(\text{Sm}/S; \Lambda))/\mathcal{T}_{\mathbb{A}^1}.$$

The categories $\mathbf{DA}^{\text{eff, ét}}(S; \Lambda)$ and $\mathbf{D}(\text{Shv}_{\text{ét}}(\text{Sm}/S; \Lambda))$ have the same objects, that is complexes of étale sheaves on Sm/S; however, a morphism in $\mathbf{D}(\text{Shv}_{\text{ét}}(\text{Sm}/S; \Lambda))$ whose cone belongs to $\mathcal{T}_{\mathbb{A}^1}$ gets inverted in $\mathbf{DA}^{\text{eff, ét}}(S; \Lambda)$. As a matter of fact, the map $\Lambda_{\text{ét}}(\mathbb{A}^1 \times U) \to \Lambda_{\text{ét}}(U)$, whose cone is the complex (3), is an isomorphism in $\mathbf{DA}^{\text{eff, ét}}(S; \Lambda)$.

Definition 2.4. An object of $\mathbf{DA}^{\text{eff, ét}}(S; \Lambda)$ is called an *effective motivic sheaf* over S (or simply an *effective S-motive*). Given a smooth S-scheme X, then $\Lambda_{\text{ét}}(X)$, viewed as an object of $\mathbf{DA}^{\text{eff, ét}}(S; \Lambda)$, is called the *effective homological motive* of X and will be denoted by $\text{M}^{\text{eff}}(X)$.

Definition 2.5. We denote by $\mathbf{DA}^{\text{eff, ét}}_{\text{ct}}(S; \Lambda)$ the smallest triangulated subcategory of $\mathbf{DA}^{\text{eff, ét}}(S; \Lambda)$ closed under direct summands and containing the motives $\text{M}^{\text{eff}}(X)$ for $X \in \text{Sm}/S$ of finite presentation. Effective motivic sheaves in $\mathbf{DA}^{\text{eff, ét}}_{\text{ct}}(S; \Lambda)$ are called *constructible*.

Remark 2.6. The category $\mathbf{DA}^{\mathrm{eff},\,\text{ét}}(S;\Lambda)$ (as well as $\mathbf{D}(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda))$) inherits the monoidal structure of $\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda)$. If $\mathcal{M}_\bullet$ and $\mathcal{N}_\bullet$ are complexes of étale sheaves on Sm/S (i.e., objets of $\mathbf{DA}^{\mathrm{eff},\,\text{ét}}(S;\Lambda)$), then their tensor product $(\mathcal{M}\otimes\mathcal{N})_\bullet$ is the total complex associated to the bi-complex $\mathcal{M}_\bullet\otimes\mathcal{N}_\bullet$.

2.2.3. Second step: naive stabilization. In this subsection, we give a low-tech (and slightly naive) construction yielding the category $\mathbf{DA}^{\text{ét},\,\mathrm{naive}}(S;\Lambda)$ which, nevertheless, captures the essence of the category $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ (see Remark 2.7).

The stabilization here refers to the process of rendering the Tate motive invertible for the tensor product.

To motivate this process, we need to explain another simple fact about the cohomology of algebraic varieties. To fix ideas, we consider ℓ-adic cohomology H^*_ℓ for schemes over an algebraically closed field k in which ℓ is invertible. The reduced cohomology of the pointed (by infinity) projective line $(\mathbb{P}^1_k,\infty)$ is given by

$$H^*_\ell(\mathbb{P}^1_k,\infty)\simeq\mathbb{Z}_\ell(-1)[-2]$$

where, as usual, $\mathbb{Z}_\ell(-1)$ is the dual of the Tate module $\mathbb{Z}_\ell(1)=\mathrm{Lim}_{n\in\mathbb{N}}\,\mu_{\ell^n}(k)$. Hence, seen as an object of the derived category $\mathbf{D}(\mathbb{Z}_\ell)$, the complex $H^*_\ell(\mathbb{P}^1_k,\infty)$ has total rank one and, equivalently, is invertible for the tensor product. It is the latter property that we want to impose on the motivic level.

To this effect, let $L:=\Lambda_{\text{ét}}(\mathbb{P}^1_S,\infty_S)$ be the étale sheaf on Sm/S given by the cokernel of the inclusion $\Lambda(\infty_S)\hookrightarrow\Lambda_{\text{ét}}(\mathbb{P}^1_S)$. Seen as an object of $\mathbf{DA}^{\mathrm{eff},\,\text{ét}}(S;\Lambda)$, L is the reduced effective homological S-motive of the pointed S-scheme $(\mathbb{P}^1_S,\infty_S)$. We will refer to L as the *Lefschetz motive*; it is the motive that corresponds to the constant complex of ℓ-adic sheaves $\mathbb{Z}_\ell(1)[2]$ over S (for ℓ invertible in $\mathcal{O}_S$).[7] However, it is easy to see that L is not an invertible object of $\mathbf{DA}^{\mathrm{eff},\,\text{ét}}(S;\Lambda)$. Therefore, one is lead to invert it formally by considering

$$\mathbf{DA}^{\text{ét},\,\mathrm{naive}}(S;\Lambda):=\mathbf{DA}^{\mathrm{eff},\,\text{ét}}(S;\Lambda)[L^{-1}].$$

An objet of $\mathbf{DA}^{\text{ét},\,\mathrm{naive}}(S;\Lambda)$ consists of a pair (M,m) where $M\in\mathbf{DA}^{\mathrm{eff},\,\text{ét}}(S;\Lambda)$ and $m\in\mathbb{Z}$. The group $\hom_{\mathbf{DA}^{\text{ét},\,\mathrm{naive}}(S;\Lambda)}((M,m),(N,n))$ of morphisms between two such objets is given by

$$\varinjlim_{r\geq-\min(m,n)}\hom_{\mathbf{DA}^{\mathrm{eff},\,\text{ét}}(S;\Lambda)}(M\otimes L^{r+m},N\otimes L^{r+n}).\qquad(4)$$

With this definition, it is easy to see that the endofunctor $-\otimes L$ on $\mathbf{DA}^{\mathrm{eff},\,\text{ét}}(S;\Lambda)$ corresponds to the functor $(M,m)\mapsto(M,m+1)$ on $\mathbf{DA}^{\text{ét},\,\mathrm{naive}}(S;\Lambda)$ which is an equivalence of categories with inverse $(M,m)\mapsto(M,m-1)$.

The formula (4) is reminiscent to the formula computing stable homotopy groups ot a topological space. This analogy suggests already that, as in topology, it is technically more convenient to use the formalism of spectra for inverting L. This is indeed the right method and will be explained in §2.3.

[7] This is consistent with what we said before: the ℓ-adic cohomology of $(\mathbb{P}^1_k,\infty)$ is $\mathbb{Z}_\ell(-1)[-2]$ and hence its ℓ-adic homology is $\mathbb{Z}_\ell(1)[2]$; it is the latter that should corresponds to the homological motive of $(\mathbb{P}^1_k,\infty)$.

Remark 2.7. The category $\mathbf{DA}^{\text{ét, naive}}(S;\Lambda)$ suffers many technical defects. For instance, it is not a triangulated category and it doesn't have arbitrary direct sums. However, modulo these technical defects, $\mathbf{DA}^{\text{ét, naive}}(S;\Lambda)$ is essentially the right category of S-motives. More precisely, under some technical assumptions,[8] its full subcategory $\mathbf{DA}^{\text{ét, naive}}_{\text{ct}}(S;\Lambda)$ consisting of pairs (M,m) with $M \in \mathbf{DA}^{\text{eff, ét}}_{\text{ct}}(S;\Lambda)$, is equivalent to the category $\mathbf{DA}^{\text{ét}}_{\text{ct}}(S;\Lambda)$ of constructible motives (see Definition 2.11 below), which is certainly the most interesting part of $\mathbf{DA}^{\text{ét}}(S;\Lambda)$.

2.3. The definitive construction.

This subsection can be skipped by the reader who is satisfied by the almost correct construction explained in §2.2. The goal here is to invert in a "homologically correct" manner the Lefschetz motive $L = \Lambda_{\text{ét}}(\mathbb{P}^1_S, \infty_S)$ for the tensor product. In fact, we will treat the localization (§2.2.2) and the stabilization (§2.2.3) in one single step!

We will borrow the machinery developed by topologists in the context of stable homotopy theory [1, 30] for inverting the (pointed) 1-dimensional sphere S^1 for the smash product. The only difference is that, instead of considering S^1-spectra (for the smash product), we will consider L-spectra (for the tensor product).

Definition 2.8. An *L-spectrum* (of étale sheaves on Sm/S) is a pair

$$\mathcal{E} = ((\mathcal{E}_n)_{n\in\mathbb{N}}, (\gamma_n)_{n\in\mathbb{N}})$$

where $\mathcal{E}_n$ is an étale sheaf on Sm/S and $\gamma_n : L \otimes \mathcal{E}_n \to \mathcal{E}_{n+1}$ is a morphism of sheaves called the n-th *assembly map*. We refer to the sheaf $\mathcal{E}_n$ as the n-th *level* of the L-spectrum $\mathcal{E}$.

A morphism of L-spectra $f : \mathcal{E} \to \mathcal{E}'$ is a collection of morphisms of sheaves $f_n : \mathcal{E}_n \to \mathcal{E}'_n$ that commute with the assembly maps, i.e., such that $f_{n+1} \circ \gamma_n = \gamma'_n \circ (\mathrm{id}_L \otimes f_n)$ for all $n \in \mathbb{N}$. We denote by $\mathrm{Spt}_L(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda))$ the category of L-spectra. This is an Abelian category.

Remark 2.9. The functor $\mathrm{Ev}_p : \mathcal{E} \mapsto \mathcal{E}_p$, sending an L-spectrum to its p-th level admits a left adjoint

$$\mathrm{Sus}^p_L : \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda) \to \mathrm{Spt}_L(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda)).$$

If $\mathcal{F}$ is a complex of sheaves on Sm/S, then $\mathrm{Sus}^p_L\mathcal{F}$ is given by

$$(\mathrm{Sus}^p_L\mathcal{F})_n = \begin{cases} 0 & \text{if} \quad n \leq p-1, \\ L^{\otimes n-p} \otimes \mathcal{F} & \text{if} \quad n \geq p, \end{cases}$$

with the obvious assembly maps. Usually, Sus^0_L is called the *infinite suspension* functor and is denoted by Σ^∞_L.

We will define $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ as a Verdier localization of the derived category $\mathbf{D}(\mathrm{Spt}_L(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda)))$ of L-spectra over Sm/S. For this, we consider the smallest triangulated subcategory $\mathcal{T}_{\mathbb{A}^1\text{-st}}$ ("st" stands for "stable") of the latter closed

[8] Such as S being Noetherian, of finite Krull dimension and of pointwise finite ℓ-cohomological dimension for very prime ℓ which is not invertible in Λ.

under arbitrary direct sums and containing the complexes

$$[\ldots \to 0 \to \mathrm{Sus}^p_L \Lambda_{\text{ét}}(\mathbb{A}^1 \times U) \to \mathrm{Sus}^p_L \Lambda_{\text{ét}}(U) \to 0 \to \ldots] \tag{5}$$

$$[\ldots \to 0 \to \mathrm{Sus}^{p+1}_L (L \otimes \Lambda_{\text{ét}}(U)) \to \mathrm{Sus}^p_L \Lambda_{\text{ét}}(U) \to 0 \to \ldots] \tag{6}$$

for all smooth S-schemes U and all $p \in \mathbb{N}$. (In the first complex above, the nonzero map is induced by the projection to the second factor; in the second complex above, the nonzero map is the map of L-spectra given by the identity starting from level $p+1$.) We now define a new triangulated category as a Verdier quotient

$$\mathbf{DA}^{\text{ét}}(S; \Lambda) := \mathbf{D}(\mathrm{Spt}_L(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S; \Lambda)))/\mathcal{T}_{\mathbb{A}^1\text{-st}}.$$

Definition 2.10. An object of $\mathbf{DA}^{\text{ét}}(S; \Lambda)$ is called a *motivic sheaf* over S (or simply an S-*motive*). Given a smooth S-scheme X, then $\Sigma^\infty_L \Lambda_{\text{ét}}(X)$, viewed as an object of $\mathbf{DA}^{\text{ét}}(S; \Lambda)$, is called the *homological motive* of X and will be denoted by $\mathrm{M}(X)$.

Definition 2.11. We denote by $\mathbf{DA}^{\text{ét}}_{\text{ct}}(S; \Lambda)$ the smallest triangulated subcategory of $\mathbf{DA}^{\text{ét}}(S; \Lambda)$ closed under direct summands and containing the motives $\mathrm{M}(X)(-p)[-2p] := \mathrm{Sus}^p_L \Lambda_{\text{ét}}(X)$ for $p \in \mathbb{N}$ and $X \in \mathrm{Sm}/S$ of finite presentation. Motivic sheaves in $\mathbf{DA}^{\text{ét}}_{\text{ct}}(S; \Lambda)$ are called *constructible*.

Remark 2.12. It can be shown that $\mathbf{DA}^{\text{ét}}(S; \Lambda)$ is a triangulated category admitting arbitrary direct sums. Therefore, the construction via L-spectra resolves the technical defects of the category $\mathbf{DA}^{\text{ét, naive}}(S; \Lambda)$ constructed in §2.2.3.

Definition 2.13. For $p \in \mathbb{N}$, we denote by $\Lambda_S(p)$ (or simply $\Lambda(p)$) the S-motive $\mathrm{Sus}^0_L(L^{\otimes p})[-2p]$ and $\Lambda_S(-p)$ (or simply $\Lambda(-p)$) the S-motive $\mathrm{Sus}^p(\Lambda)[2p]$. These are the Tate motives over S. We also define

$$H^p_{\mathcal{L}}(S; \Lambda(q)) := \hom_{\mathbf{DA}^{\text{ét}}(S;\Lambda)}(\Lambda_S(0), \Lambda_S(q)[p])$$

for p, $q \in \mathbb{Z}$. These groups are called the *étale* (or *Lichtenbaum*) *motivic cohomology groups* of S (with coefficients in Λ).

2.4. Complements.

From Definition 2.10, a motivic sheaf over S is simply a complex of L-spectra on Sm/S, i.e., essentially a sequence of complexes of étale sheaves on Sm/S. This is of course deceiving and slightly misleading. The point is that every complex of L-spectra is *isomorphic* in $\mathbf{DA}^{\text{ét}}(S; \Lambda)$ to a *stably* $\mathbb{A}^1$-*local* complex of L-spectra and it is the latter that deserves better to be called a motivic sheaf. Our goal in this paragraph is to explain this in some detail. We start with the effective case. (Below, $H^i_{\text{ét}}(-; A)$ stands for the étale hyper-cohomology with coefficients in a complex of étale sheaves A.)

Definition 2.14. Let $\mathcal{F}$ be a complex of étale sheaves on Sm/S. We say that $\mathcal{F}$ is $\mathbb{A}^1$-*local* if for all $U \in \mathrm{Sm}/S$ and $i \in \mathbb{Z}$, the map

$$H^i_{\text{ét}}(U; \mathcal{F}) \to H^i_{\text{ét}}(\mathbb{A}^1 \times U; \mathcal{F}),$$

induced by the projection to the second factor, is an isomorphism.

Remark 2.15. $\mathbb{A}^1$-locality is important for the following reason. Let $\mathcal{E}$ and $\mathcal{F}$ be two complexes of étale sheaves on Sm/S. Then, if $\mathcal{F}$ is $\mathbb{A}^1$-local, the natural homomorphism

$$\hom_{\mathbf{D}(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda))}(\mathcal{E},\mathcal{F}) \to \hom_{\mathbf{DA}^{\mathrm{eff,\,ét}}(S;\Lambda)}(\mathcal{E},\mathcal{F})$$

is an isomorphism. In words, computing morphisms between effective motivic sheaves can be performed in the more familiar derived category of étale sheaves when the target is $\mathbb{A}^1$-local. The next result gives, in theory, a way to reduce to this favorable case.

Lemma 2.16. *There is, up to a unique isomorphism, an endofunctor* $\mathrm{Loc}_{\mathbb{A}^1}$ *of* $\mathbf{D}(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda))$ *endowed with a natural transformation* $\mathrm{id} \to \mathrm{Loc}_{\mathbb{A}^1}$ *such that the following two properties are satisfied for every complex* $\mathcal{F}$ *of étale sheaves on* Sm/S*:*

- $\mathrm{Loc}_{\mathbb{A}^1}(\mathcal{F})$ *is* $\mathbb{A}^1$*-local, and*
- $\mathcal{F} \to \mathrm{Loc}_{\mathbb{A}^1}(\mathcal{F})$ *is an* $\mathbb{A}^1$-weak equivalence *(i.e., becomes an isomorphism in* $\mathbf{DA}^{\mathrm{eff,\,ét}}(S;\Lambda)$*).*

$\mathrm{Loc}_{\mathbb{A}^1}$ *is called the* $\mathbb{A}^1$*-localization functor.*

Remark 2.17. If one adopts the convention that an "effective S-motive" is an $\mathbb{A}^1$-local complex of sheaves on Sm/S, then the effective motive of a smooth S-scheme X would be given by $\mathrm{Loc}_{\mathbb{A}^1}(\Lambda_{\text{ét}}(X))$. Therefore, understanding the $\mathbb{A}^1$-localization functor is of utmost importance in the theory of motives!

Remark 2.18. One of the drawback of the abstract construction is that it gives no information about the $\mathbb{A}^1$-localization functor. We will explain in §4.2 how Voevodsky is able to overcome this crucial difficulty (sadly, only when S is the spectrum of a field) using his theory of *homotopy invariant presheaves with transfers*.

We now turn to the stable setting.

Definition 2.19. Let $\mathcal{K} = ((\mathcal{K}_n)_{n\in\mathbb{N}}, (\gamma_n)_{n\in\mathbb{N}})$ be a complex of L-spectra of étale sheaves on Sm/S. We say that $\mathcal{K}$ is *stably* $\mathbb{A}^1$*-local* if the following two properties are satisfied for all $U \in \mathrm{Sm}/S$, $i \in \mathbb{Z}$ and $n \in \mathbb{N}$:

(i) the map
$$H^i_{\text{ét}}(U;\mathcal{K}_n) \to H^i_{\text{ét}}(\mathbb{A}^1 \times U;\mathcal{K}_n),$$
induced by the projection to the second factor, is an isomorphism;

(ii) the map
$$H^i_{\text{ét}}(U;\mathcal{K}_n) \to H^{i+2}_{\text{ét}}((\mathbb{P}^1,\infty) \times U;\mathcal{K}_{n+1}),$$
induced by the n-th assembly map, is an isomorphism.

Remark 2.20. Stably $\mathbb{A}^1$-local complexes of L-spectra are important for the same reason as the one explained in Remark 2.15.

Remark 2.21. Let $\mathcal{K}$ be a stably $\mathbb{A}^1$-local complex of L-spectra. Writing $\mathcal{K}(n)$ for the complex $\mathcal{K}_n[-2n]$, the two properties in Definition 2.19 gives the familiar isomorphisms:

(i) $H^*_{\text{ét}}(\mathbb{A}^1 \times U; \mathcal{K}(n)) = H^*_{\text{ét}}(U; \mathcal{K}(n))$;

(ii) $H^*_{\text{ét}}((\mathbb{A}^1 \smallsetminus 0) \times U; \mathcal{K}(n)) \simeq H^*_{\text{ét}}(U; \mathcal{K}(n)) \oplus H^{*-1}_{\text{ét}}(U; \mathcal{K}(n-1))$.

Lemma 2.22. *There is, up to a unique isomorphism, an endofunctor* $\mathrm{Loc}_{\mathbb{A}^1\text{-st}}$ *of* $\mathbf{D}(\mathrm{Spt}_L(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S; \Lambda)))$ *endowed with a natural transformation* $\mathrm{id} \to \mathrm{Loc}_{\mathbb{A}^1\text{-st}}$ *such that the following two properties are satisfied for every complex of L-spectra $\mathcal{K}$:*

- $\mathrm{Loc}_{\mathbb{A}^1\text{-st}}(\mathcal{K})$ *is stably $\mathbb{A}^1$-local, and*
- $\mathcal{K} \to \mathrm{Loc}_{\mathbb{A}^1\text{-st}}(\mathcal{K})$ *is a* stable $\mathbb{A}^1$-weak equivalence *(i.e., becomes an isomorphism in* $\mathbf{DA}^{\text{ét}}(S; \Lambda)$*).*

Remark 2.23. As in the effective case, if one adopts the convention that an "S-motive" is a stably $\mathbb{A}^1$-local complex of L-spectra, then the motive of a smooth S-scheme X would be given by $\mathrm{Loc}_{\mathbb{A}^1\text{-st}}(\Sigma^\infty_T \Lambda_{\text{ét}}(X))$.

2.5. Relative rigidity theorem.

When the characteristic of Λ is nonzero, the category $\mathbf{DA}^{\text{ét}}(S; \Lambda)$ has a very simple description. Indeed, one has the following (see [9, Théorème 4.1]):

Theorem 2.24. *Let $n \in \mathbb{N} \smallsetminus \{0\}$ be an integer invertible in $\mathcal{O}(S)$. If Λ is a $\mathbb{Z}/n\mathbb{Z}$-algebra (and S satisfies some mild technical hypothesis*[9]*), then there is an equivalence of categories*

$$\mathbf{DA}^{\text{ét}}(S; \Lambda) \simeq \mathbf{D}(S_{\text{ét}}; \Lambda)$$

where $\mathbf{D}(S_{\text{ét}}; \Lambda)$ is the derived category of étale sheaves on $S_{\text{ét}}$ (the small étale site of S).

Remark 2.25. Theorem 2.24 is a relative version of a well-known result of Suslin–Voevodsky [29, Proposition 3.3.3 of Chapter 5] stating the same conclusion for the category $\mathbf{DM}^{\text{ét}}(S; \Lambda)$ when S is a field.

Remark 2.26. From a certain perspective, Theorem 2.24 is disappointing. Indeed, its shows that the categories $\mathbf{DA}^{\text{ét}}(S; \Lambda)$ are too simple to capture the complexity of the torsion in Chow groups. This is not so surprising as it is well-known that higher Chow groups do not satisfy étale descent. A way around this is to replace in the construction "étale" by "Nisnevich" which yields the categories $\mathbf{DA}(S; \Lambda)$. The latter "see" the higher Chow groups integrally (but also other things like oriented Chow groups).

[9] These hypothesis are satisfied when S is excellent.

Remark 2.27. From another perspective, Theorem 2.24 is encouraging. Indeed, it is also well-known that integrality in Chow groups is chaotic in general. For instance, there are famous counterexamples (the first ones by Atiyah–Hirzebruch [4, Theorem 6.5] and Kollár [21, page 134–135]) to the integral Hodge and Tate conjectures. Imposing étale descent forces a better organization in the integral structure of higher Chow groups. As a matter of fact, it has been shown recently by Rosenschon–Srinivas [26] that the Hodge and Tate conjectures can be "corrected" integrally by replacing the Chow groups by their étale version.[10] See also Remark 5.7 below for another (but related) reason to be happy about Theorem 2.24.

3. Operations on motivic sheaves

In this section, we review the functorialities of the categories of motivic sheaves. As for the classical "cohomological coefficients" (in the sense of Grothendieck), one has for motivic sheaves the Grothendieck six operations formalism and Verdier's duality. One also has the nearby cycles formalism, but this will not be discussed here (see [6, Chapitre 4] and [9]).

3.1. Operations associated to morphisms of schemes.

In this subsection, we will recall the construction of the formalism of the four operations f^*, f_*, $f_!$ and $f^!$, associated to a morphism of schemes f, in the context of motivic sheaves.

3.1.1. Ordinary inverse and direct images. Let $f : T \to S$ be a morphism of schemes. Then f induces a pair of adjoint functors:

$$f^* : \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S; \Lambda) \rightleftarrows \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/T; \Lambda) : f_*. \tag{7}$$

The functor f_* is easy to understand; given an étale sheaf $\mathcal{G}$ over Sm/T, one has $f_*\mathcal{G}(U) := \mathcal{G}(T \times_S U)$ for all $U \in \mathrm{Sm}/S$. The functor f^* is characterized by its property of commuting with arbitrary colimits and by the formula

$$f^*\Lambda_{\text{ét}}(U) \simeq \Lambda_{\text{ét}}(T \times_S U) \tag{8}$$

for all $U \in \mathrm{Sm}/S$.

The adjunction (7) can be derived yielding an adjunction on the level of effective motivic sheaves

$$\mathrm{L}f^* : \mathbf{DA}^{\mathrm{eff},\,\text{ét}}(S; \Lambda) \rightleftarrows \mathbf{DA}^{\mathrm{eff},\,\text{ét}}(T; \Lambda) : \mathrm{R}f_*. \tag{9}$$

[10] For a smooth algebraic variety X over a field k, the étale Chow groups of X can be defined by the formula (see Definition 2.13)

$$\mathrm{CH}^n_{\text{ét}}(X) := H^{2n}_{\mathcal{L}}(X; \mathbb{Z}(n)) = \hom_{\mathbf{DA}^{\text{ét}}(k;\mathbb{Z})}(\mathrm{M}(X), \mathbb{Z}(n)[2n])$$

(or, equivalently, using $\mathbf{DM}^{\text{ét}}(k;\mathbb{Z})$ instead of $\mathbf{DA}^{\text{ét}}(k;\mathbb{Z})$). When $k = \mathbb{C}$, Rosenschon and Srinivas construct in [26] a cycle map $\mathrm{CH}^n_{\text{ét}}(X) \to H^{2n}(X(\mathbb{C}), \mathbb{Z})$ and show that if the Hodge conjecture holds for the rational Chow groups (i.e., for $\mathrm{CH}^n_{\mathbb{Q}}(X) := \mathrm{CH}^n(X) \otimes \mathbb{Q}$) then it also holds integrally for the étale Chow groups. They also show a similar statement for the Tate conjecture.

It can also be extended to L-spectra and then derived yielding an adjunction on the level of motivic sheaves

$$\mathrm{L}f^* : \mathbf{DA}^{\text{ét}}(S;\Lambda) \rightleftarrows \mathbf{DA}^{\text{ét}}(T;\Lambda) : \mathrm{R}f_*. \tag{10}$$

These functors are triangulated.

Remark 3.1. The formula (8) still holds for the left derived functors $\mathrm{L}f^*$ in (9) and (10). In words, $\mathrm{L}f^*$ takes the homological motive of an S-scheme U to the homological motive of the T-scheme $T \times_S U$ (in the effective and non-effective settings).

Lemma 3.2. *Assume that f is smooth. Then, the functor f^* admits a left adjoint*

$$f_\sharp : \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/T;\Lambda) \to \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda).$$

If $V \in \mathrm{Sm}/T$, then $f_\sharp \Lambda_{\text{ét}}(V/T) = \Lambda_{\text{ét}}(V/S)$. Moreover, $f_\sharp$ can be left derived yielding left adjoints to $\mathrm{L}f^*$ on the level of motivic sheaves:

$$\mathrm{L}f_\sharp : \mathbf{DA}^{\text{eff},\,\text{ét}}(T;\Lambda) \to \mathbf{DA}^{\text{eff},\,\text{ét}}(S;\Lambda) \ \textit{ and } \ \mathrm{L}f_\sharp : \mathbf{DA}^{\text{ét}}(T;\Lambda) \to \mathbf{DA}^{\text{ét}}(S;\Lambda).$$

Remark 3.3. The existence of a left adjoint to f^*, when f is smooth, is part of the formalism of the six operations of Grothendieck. However, in the classical setting, this property is one of the deepest, whereas for motivic sheaves one has it for free!

3.1.2. A list of axioms. From now on, we will drop the "L" and "R" when dealing with the operations $\mathrm{L}f^*$, $\mathrm{L}f_\sharp$ and $\mathrm{R}f_*$.

Let SCH be the category of all schemes and $\mathfrak{TR}$ the 2-category of triangulated categories. Then, the 2-functor

$$\begin{array}{rcccc} \mathbf{DA}^{\text{ét}}(-;\Lambda) & : & \mathrm{SCH} & \to & \mathfrak{TR} \\ & & f & \mapsto & f^* \end{array}$$

satisfies the following list of axioms. (Only one of these axioms fails to hold for $\mathbf{DA}^{\text{eff},\,\text{ét}}(-,\Lambda)$, namely the sixth!)

1. $\mathbf{DA}^{\text{ét}}(\emptyset;\Lambda)$ is equivalent to the zero triangulated category.
2. For every morphism of schemes $f : T \to S$, the functor $f^* : \mathbf{DA}^{\text{ét}}(S;\Lambda) \to \mathbf{DA}^{\text{ét}}(T;\Lambda)$ admits a right adjoint f_*.
3. For every *smooth* morphism $f : T \to S$, the functor $f^* : \mathbf{DA}^{\text{ét}}(S;\Lambda) \to \mathbf{DA}^{\text{ét}}(T;\Lambda)$ admits a left adjoint $f_\sharp$. Moreover, given a cartesian square

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S, \end{array}$$

the natural exchange morphism $f'_\sharp \circ g'^* \to g^* \circ f_\sharp$ is an isomorphism.

4. For every closed immersion i with complementary open immersion j, the pair (i^*, j^*) is conservative (i.e., if a motive M satisfies $i^*M \simeq 0$ and $j^*M \simeq 0$, then $M \simeq 0$). Moreover, the counit of the adjunction $i^* \circ i_* \to \mathrm{id}$ is an isomorphism.

5. If $p : V \to S$ is the projection of a vector bundle, then the unit of the adjunction $\mathrm{id} \to p_*p^*$ is an isomorphism.

6. If $f : T \to S$ is smooth and $s : S \to T$ is a section of f (i.e., $f \circ s = \mathrm{id}_S$), then the functor $f_\sharp \circ s_*$ is an autoequivalence of $\mathbf{DA}^{\text{ét}}(S; \Lambda)$.

We will call such a 2-functor an *extended stable homotopical* 2*-functor*.

Remark 3.4. Except the fourth axiom, all these axioms follow readily from the construction. For instance, the fifth axiom is a consequence of the $\mathbb{A}^1$-localization and the sixth axiom follows from inverting the Lefschetz motive (for the tensor product).

The fourth axiom (aka., the locality axiom) is due to Morel–Voevodsky [25, Theorem 2.21 of §3.2]. (In loc. cit., only the non-Abelian setting is considered but their proof can be adapted to the additive setting without much difficulties; see [6, §4.5.3].) It is the proof of this axiom that dictates some of the choices that were made by Morel–Voevodsky (and repeated in §2) such as considering sheaves on *smooth* S-schemes instead of sheaves on larger categories of S-schemes.

Remark 3.5. That these axioms suffices to derive the full formalism of the four operations is due to Voevodsky (unpublished). The details of the verifications were carried on in [5, Chapitre 1].

For later use, we make the following definition.

Definition 3.6. Given an $\mathcal{O}_S$-module $\mathcal{M}$ on a scheme S, we set $\mathrm{Th}(\mathcal{M}) = p_\sharp \circ s_*$ where $p : \mathrm{V}(\mathcal{M}) \to S$ is the projection of the associated vector bundle and s is its zero section. By the sixth axiom, $\mathrm{Th}(\mathcal{M})$ is an autoequivalence of $\mathbf{DA}^{\text{ét}}(S; \Lambda)$, called the *Thom equivalence*. Its inverse is denoted by $\mathrm{Th}^{-1}(\mathcal{M})$.

Remark 3.7. It is customary to denote $\mathrm{Th}(\mathcal{O}_S^{\oplus r})(-)[-2r]$ by $(-)(r)$ and to call it the r-th *Tate twist* (extended to negative integers in the usual way).

If $\mathcal{M}$ has constant rank r, it can be shown that $\mathrm{Th}(\mathcal{M})[-2r]$ is canonically equivalent to $(-)(r)$ (see [9, Remarque 11.3]). This is a special property of $\mathbf{DA}^{\text{ét}}(-; \Lambda)$ called *orientation*.

3.1.3. The proper base change theorem. One of the most surprising fact here is that the axioms of §3.1.2 imply quite formally the so-called *proper base change theorem*. (All the axioms are used in the proof of this theorem; as a matter of fact, this theorem fails for the categories $\mathbf{DA}^{\text{eff, ét}}(-; \Lambda)$.)

Theorem 3.8. *Given a cartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

with f proper, the exchange morphism $g^* \circ f_*(\mathcal{M}) \to f'_* \circ g'^*(\mathcal{M})$ is an isomorphism for every motivic sheaf $\mathcal{M} \in \mathbf{DA}^{\text{ét}}(Y;\Lambda)$.

To prove Theorem 3.8, it is enough to treat the case where f is the projection $p_n : \mathbb{P}^n_X \to X$. (This reduction is easy and classical; it appears for example in [3, Exposé XII].) To treat the case of p_n, one needs a completely different approach than the one used in [3, Exposés XII et XIII]. Here is a sketch of the proof following [5, Chapitre 1]:

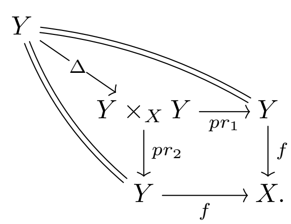
Proof. In contrast with the étale formalism, here we define the extraordinary push-forward functors $f_!$ before knowing the validity of the proper base change theorem. That this can be done relies on the (easy) existence of a left adjoint $h_\sharp$ to h^* when h is smooth. Indeed, assuming that f is smoothable, i.e., can be written as $f = h \circ i$ with h smooth and i a closed immersion, one sets

$$f_! := h_\sharp \circ \mathrm{Th}^{-1}(\Omega_h) \circ i_* \qquad \text{(and dually} \quad f^! := i^! \circ \mathrm{Th}(\Omega_h) \circ h^*).$$

A big deal of effort in [5, Chapitre 1] is devoted to showing that these definitions are independent (up to natural isomorphisms) of the choice of the factorization $f = h \circ i$ and that there are coherent choices of isomorphisms $(f \circ f')_! \simeq f_! \circ f'_!$, for composable smoothable morphisms, etc. Assuming this is granted, it is then easy to explain the strategy of the proof of Theorem 3.8.

From the third axiom in §3.1.2 and the definition of the extraordinary direct image, it is quite easy to see that one has an exchange isomorphism $g^* \circ f_! \simeq f'_! \circ g'^*$ (without any condition on f beside being smoothable).

On the other hand, one can construct a natural transformation $\alpha_f : f_! \to f_*$ (which is reminiscent to the obvious morphism from cohomology with support to ordinary cohomology). It is defined as follows. Consider the commutative diagram



From the square, one gets a natural exchange morphism $f_! \circ pr_{1*} \to f_* \circ pr_{2!}$ (deduced by adjunction from the exchange isomorphism given by the third axiom of §3.1.2). Applying this to $\Delta_* = \Delta_!$ and using the identifications $pr_{1*} \circ \Delta_* = \mathrm{id}$ and $pr_{2!} \circ \Delta_! = \mathrm{id}$, one gets the promised natural transformation.

This is said, we are left to showing that $p_{n!} \to p_{n*}$ is an isomorphism for $p_n : \mathbb{P}^n \times X \to X$. This is done by induction on n using a rather tricky argument. The point is to realize that it suffices to show that

$$p_{n!} \circ p_n^* \to p_{n*} \circ p_n^* \qquad \text{and} \qquad p_{n!} \circ p_n^! \to p_{n*} \circ p_n^!$$

are both isomorphisms. Indeed, assuming this, one can then define two maps $p_n^* \to p_n^!$ by the compositions of

$$p_{n*} \xrightarrow{\eta} p_{n*} \circ p_n^* \circ p_{n*} \simeq p_{n!} \circ p_n^* \circ p_{n*} \xrightarrow{\delta} p_{n!}$$

$$p_{n*} \xrightarrow{\eta} p_{n*} \circ p_n^! \circ p_{n!} \simeq p_{n!} \circ p_n^! \circ p_{n!} \xrightarrow{\delta} p_{n!}$$

A direct computation shows that these morphisms give respectively left and right inverses to the canonical morphism $p_{n!} \to p_{n*}$. See [5, §1.7.2] for the complete proof. □

3.1.4. Extraordinary direct and inverse images. As said in the sketch of the proof of Theorem 3.8, one has, for f smoothable (and, in particular, for f quasi-projective), two extraordinary operations $f^!$ and $f_!$.

Once the proper base change theorem is established, it is possible to extend the extraordinary operations to the case where f is of finite presentation (but not necessarily smoothable) following the receipt of [3, Exposé XVII]. Indeed, by Nagata's compactification, we may factor $f = \overline{f} \circ j$ where $\overline{f}$ is proper and j is an open immersion. Then, one sets $f_! := \overline{f}_* \circ j_\sharp$. The proper base change theorem implies that this is independent of the choice of the compactification.[11]

In any case, one has an adjunction $(f_!, f^!)$ for every finite type separated morphism. (The existence of $f^!$ is local over the source of f and hence, one may reduce to the case where f is quasi-projective.)

Theorem 3.9. *For every cartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

with f of finite type and g arbitrary, one has exchange isomorphisms

$$g^* f_! \simeq f'_! g'^* \qquad \textit{and} \qquad f^! g_* \simeq g'_* f'^!.$$

3.2. Closed monoidal structures and Verdier duality.

The category $\mathbf{DA}^{\text{ét}}(S; \Lambda)$, as constructed in §2.3, possesses a monoidal structure. However, as it is the case for the smash product of spectra in topology, it is not possible to define the tensor product directly on the category $\mathrm{Spt}_L(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S; \Lambda))$ of L-spectra. Different ways around this difficulty have been developed in topology. One of these ways is via the notion of *symmetric spectra* [18] that had been greatly generalized in [17].

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[11] It is worth noting here that checking that $\overline{f}_* \circ j_\sharp$ is independent of the factorization $f = \overline{f} \circ j$ is easier than checking that $h_\sharp \circ \mathrm{Th}(\Omega_h) \circ i_*$ is independent of the factorization $f = h \circ i$. The reason for this is that "the category of compactifications" is filtered whereas the "category of smoothifications" is not.

More specifically, one considers the Abelian category $\mathrm{Spt}_L^{\Sigma}(\mathrm{Shv}_{\text{ét}}(S;\Lambda))$ of symmetric L-spectra of étale sheaves on Sm/S. A symmetric L-spectrum is an L-spectrum $\mathcal{E}$ endowed with an action of the n-th symmetric group Σ_n on its n-th level $\mathcal{E}_n$ and such that the assembly maps are equivariant in an appropriate sense.

The point is that the extra symmetry that symmetric L-spectra possess permits to define a symmetric and associative tensor product on $\mathrm{Spt}_L^{\Sigma}(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda))$. The latter induces a tensor product on $\mathbf{D}(\mathrm{Spt}_L^{\Sigma}(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S;\Lambda)))$ and its localization with respect to its triangulated subcategory $\mathcal{T}_{\mathbb{A}^1-\mathrm{st}}^{\Sigma}$ defined similarly as in §2.3. Finally, one can show that this localization yields an equivalent category to $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ inducing a monoidal structure on the latter.

Unfortunately, the details of this story are quite technical and boring. We refer the interested reader to [6, Chapitre 4] for a complete (and self-contained) account (using however the language of model categories).

Theorem 3.10. *The categories $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ are symmetric monoidal and closed (i.e., $A\otimes -$ admits a right adjoint $\underline{Hom}(A,-)$ for every S-motive A). The operations f^* are monoidal functors. One also has the usual formulas*

$$f_!(-)\otimes - \simeq f_!(-\otimes f^*(-)),\quad f^!\underline{Hom}(-,-)\simeq \underline{Hom}(f^*(-),f^!(-)),$$

$$f_*\underline{Hom}(f^*(-),-)\simeq \underline{Hom}(-,f_*(-)),\quad \underline{Hom}(f_!(-),-)\simeq f_*\underline{Hom}(-,f^!(-)),\ \textit{etc.}$$

Finally, assuming that S is of finite type over a characteristic zero field k and denoting π_S to projection to the point, there is a dualizable objet in $\mathbf{DA}^{\text{ét}}_{\mathrm{ct}}(S;\Lambda)$ given by $\pi_S^!\Lambda(0)$.

Another important result to mention here is:

Theorem 3.11. *If X is a proper and smooth S-scheme of pure relative dimension d, then $\mathrm{M}(X)$ admits a strong dual given by $\mathrm{M}(X)(-d)[-2d]$.*

Proof. This follows from Theorem 3.10 using that

$$\mathrm{M}(X)\simeq(\pi_X)_!(\pi_X)^!\Lambda_S(0)\qquad\text{and}\qquad \mathrm{M}(X)(-d)[-2d]\simeq(\pi_X)_*(\pi_X)^*\Lambda_S(0)$$

where $\pi_X : X\to S$ is the structural morphism. □

4. Motives over a base field

The formalism of Grothendieck's six operations is a powerful tool for reducing questions about general sheaves to questions about lisse sheaves and, ultimately, to questions about (germs of) sheaves on generic points of varieties. For this formalism to be of any use in the context of motivic sheaves, one needs informations about motives over fields.

In this section we list some of what is known concerning motives over a field; everything here is essentially due to Voevodsky. When dealing with Voevodsky's motives, we mostly work over a base field k except for the construction §4.1.1 and the comparison theorem §4.1.2 where this restriction is irrelevant. The use of the étale topology results in inverting automatically the exponent-characteristic of k.[12] Therefore, there is no need in assuming k perfect in quoting [24, 29].

4.1. Voevodsky's motives.

Many theorems about motives over a field and morphisms between them are obtained by using a slightly more complicated construction than the one explained in §2. The extra complication is the requirement of having transfers and is the key for many concrete computations.

4.1.1. The construction. The construction of Voevodsky's category $\mathbf{DM}^{\text{ét}}(k;\Lambda)$ follows exactly the same pattern as the construction given in §2 with only one difference: one uses the Abelian category of étale *sheaves with transfers* instead of the Abelian category of ordinary étale sheaves. To expand on this, we need some notation.

Let S be a base scheme that we assume to be Noetherian. In [29, Chapter 2], a category of finite correspondences SmCor/S was constructed. This is an additive category whose objects are smooth S-schemes. Given two smooth S-schemes U and V, the group of morphisms from U to V in SmCor/S is denoted by $\text{Cor}_S(U,V)$. When S is regular, this group is freely generated by integral and closed subschemes $Z \subset U \times_S V$ such that the projection $Z \to U$ is finite and surjective over a connected component of U. Moreover, the composition of finite correspondences is then given by the usual formula involving Serre's multiplicities.

Definition 4.1. A *presheaf with transfers* on Sm/S is a contravariant additive functor from SmCor/S to the category of Λ-modules. An *étale sheaf with transfers* is a presheaf with transfers Sm/S which is, after forgetting transfers, a sheaf for the étale topology. Étale sheaves with transfers form an Abelian category that we denote by $\text{Str}_{\text{ét}}(\text{Sm}/S;\Lambda)$.

Example 4.2. For a smooth S-scheme X, we denote by $\Lambda_{\text{tr}}(X)$ the presheaf with transfers on Sm/S represented by X, i.e., given by $\Lambda_{\text{tr}}(X)(U) = \text{Cor}_S(U,X) \otimes_{\mathbb{Z}} \Lambda$ for all $U \in \text{Sm}/S$. In fact, $\Lambda_{\text{tr}}(X)$ is an étale sheaf with transfers on Sm/S. After forgetting transfers, one has an inclusion of étale sheaves $\Lambda_{\text{ét}}(X) \subset \Lambda_{\text{tr}}(X)$.

As said before, replacing everywhere "$\text{Shv}_{\text{ét}}(\text{Sm}/S;\Lambda)$" by "$\text{Str}_{\text{ét}}(\text{Sm}/S;\Lambda)$" in §2 yields Voevodsky's triangulated categories of S-motives. More precisely, one obtains two versions.

- The category of *effective Voevodsky S-motives* given by

$$\mathbf{DM}^{\text{eff, ét}}(S;\Lambda) := \mathbf{D}(\text{Str}_{\text{ét}}(\text{Sm}/S;\Lambda))/\mathcal{T}^{\text{tr}}_{\mathbb{A}^1}$$

[12] This is well-known and easy. Indeed, if $k = \mathbb{F}_p$, then the Artin–Schreier exact sequence of étale sheaves on $\text{Sm}/\mathbb{F}_p$:

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathcal{O} \overset{(-)^p}{\to} \mathcal{O} \to 0,$$

and the fact that $\mathcal{O}$ is $\mathbb{A}^1$-contractible, show that the constant étale sheaf $\mathbb{Z}/p\mathbb{Z}$ is also $\mathbb{A}^1$-contractible. From this, it is easy to deduce that multiplication by p is invertible in $\mathbf{DA}^{\text{ét}}(\mathbb{F}_p;\Lambda)$ and more generally in $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ for every $\mathbb{F}_p$-scheme S. The same holds true for $\mathbf{DM}^{\text{ét}}(\mathbb{F}_p;\Lambda)$ and $\mathbf{DM}^{\text{ét}}(S;\Lambda)$.

where $\mathcal{T}^{\mathrm{tr}}_{\mathbb{A}^1}$ is defined similarly as $\mathcal{T}_{\mathbb{A}^1}$ in §2.2.2 (writing "Λ_{tr}" instead of "$\Lambda_{\mathrm{\acute{e}t}}$" in (3)).

- The category of (non-effective) *Voevodsky S-motives* given by

$$\mathbf{DM}^{\mathrm{\acute{e}t}}(S;\Lambda) := \mathbf{D}(\mathrm{Spt}_{L_{\mathrm{tr}}}(\mathrm{Str}_{\mathrm{\acute{e}t}}(\mathrm{Sm}/S;\Lambda)))/\mathcal{T}^{\mathrm{tr}}_{\mathbb{A}^1\text{-st}}$$

  where $L_{\mathrm{tr}} = \Lambda_{\mathrm{tr}}(\mathbb{P}^1_S, \infty_S)$ and $\mathcal{T}^{\mathrm{tr}}_{\mathbb{A}^1\text{-st}}$ is defined similarly as $\mathcal{T}_{\mathbb{A}^1\text{-st}}$ in §2.3 (writing "Λ_{tr}" and "L_{tr}" instead of "$\Lambda_{\mathrm{\acute{e}t}}$" and "$L$" in (5) and (6)).

Remark 4.3. Strictly speaking, Voevodsky [24] considered categories

$$\mathbf{DM}^{\mathrm{eff},\,\mathrm{\acute{e}t}}_{-}(S;\Lambda) \qquad \text{and} \qquad \mathbf{DM}^{\mathrm{\acute{e}t}}_{\mathrm{gm}}(S;\Lambda)$$

for S the spectrum of a perfect field (with finite cohomological dimension). The category $\mathbf{DM}^{\mathrm{eff},\,\mathrm{\acute{e}t}}_{-}(S;\Lambda)$ is the triangulated subcategory of $\mathbf{DM}^{\mathrm{eff},\,\mathrm{\acute{e}t}}(S;\Lambda)$ consisting of complexes that are bounded on the right. The category $\mathbf{DM}^{\mathrm{eff},\,\mathrm{\acute{e}t}}_{\mathrm{gm}}(S;\Lambda)$ is the triangulated subcategory of $\mathbf{DM}^{\mathrm{eff},\,\mathrm{\acute{e}t}}(S;\Lambda)$ generated by $\Lambda_{\mathrm{tr}}(X)$ for $X \in \mathrm{Sm}/S$ of finite type. Finally, $\mathbf{DM}^{\mathrm{\acute{e}t}}_{\mathrm{gm}}(S;\Lambda)$ is obtained from $\mathbf{DM}^{\mathrm{eff},\,\mathrm{\acute{e}t}}_{\mathrm{gm}}(S;\Lambda)$ by formally inverting tensoring by the Lefschetz motive L_{tr} (i.e., using the naive construction as in §2.2.3); it is also the triangulated subcategory of $\mathbf{DM}^{\mathrm{\acute{e}t}}(S;\Lambda)$ generated by S-motives of finite type smooth S-schemes and their negative Tate twists.

4.1.2. The comparison theorem. There is a pair of adjoint functors:

$$\mathrm{a}_{\mathrm{tr}} : \mathrm{Shv}_{\mathrm{\acute{e}t}}(\mathrm{Sm}/S;\Lambda) \rightleftarrows \mathrm{Str}_{\mathrm{\acute{e}t}}(\mathrm{Sm}/S;\Lambda) : \mathrm{o}_{\mathrm{tr}}. \tag{11}$$

The functor o_{tr} is a forgetful functor: it takes an étale sheaf with transfers to its underlying étale sheaf. The functor a_{tr} is characterized by its property of commuting with arbitrary colimits and by the formula

$$\mathrm{a}_{\mathrm{tr}}(\Lambda_{\mathrm{\acute{e}t}}(U)) \simeq \Lambda_{\mathrm{tr}}(U)$$

for all $U \in \mathrm{Sm}/S$. The adjunction (11) can be derived yielding an adjunction on the level of effective S-motives:

$$\mathrm{La}_{\mathrm{tr}} : \mathbf{DA}^{\mathrm{eff},\,\mathrm{\acute{e}t}}(S;\Lambda) \rightleftarrows \mathbf{DM}^{\mathrm{eff},\,\mathrm{\acute{e}t}}(S;\Lambda) : \mathrm{Ro}_{\mathrm{tr}}. \tag{12}$$

It can also be extended to spectra and then derived yielding an adjunction on the level of (non-effective) S-motives:

$$\mathrm{La}_{\mathrm{tr}} : \mathbf{DA}^{\mathrm{\acute{e}t}}(S;\Lambda) \rightleftarrows \mathbf{DM}^{\mathrm{\acute{e}t}}(S;\Lambda) : \mathrm{Ro}_{\mathrm{tr}}. \tag{13}$$

Theorem 4.4. *If S is normal (and some technical assumptions are satisfied), the functors in* (13) *are equivalences of categories.*

Proof. When Λ is a $\mathbb{Q}$-algebra, Theorem 4.4 was proved by Morel, for S the spectrum of a field, and was generalized later by Cisinski–Déglise. [13] In [9, Annexe B], we simplified the proof of Cisinski–Déglise and extended their result to more general coefficient rings using Theorem 2.24. □

[13] In fact, Morel and Cisinski–Déglise prove a stronger result where the étale topology is replaced by the Nisnevich topology. Indeed, they prove that $\mathbf{DM}(k;\mathbb{Q})$ is equivalent to a direct summand $\mathbf{DA}(S;\mathbb{Q})_+$ of $\mathbf{DA}(S;\mathbb{Q})$ whose complement vanishes when étale descent is imposed.

Remark 4.5. If the normal scheme S has characteristic zero and if Λ is a $\mathbb{Q}$-algebra, then the functors in (12) are also known to be equivalences of categories by [8, Théorème B.1]. (This is indeed a stronger statement!)

Remark 4.6. It is unknown if Theorem 4.4 holds for general base schemes (e.g., reducible). This is because the theory of finite correspondences over non-normal schemes is quite complicated. A related (and probably equivalent) open question is to know if the 2-functor $\mathbf{DM}^{\text{ét}}(-;\Lambda)$ satisfies the localization axiom (i.e., the fourth axiom in §3.1.2). In fact, this is the only missing property that prevents one to promote $\mathbf{DM}^{\text{ét}}(-;\Lambda)$ into an extended stable homotopical 2-functor. But, in our opinion, these questions have minor impact for the following reasons:

1. A stable homotopical 2-functor H, say over quasi-projective S-schemes with S regular, is essentially determined by its values on smooth S-schemes. Indeed, if X is a quasi-projective S-scheme, one can choose an embedding $i : X \hookrightarrow Y$ with Y a smooth S-scheme. Then, thanks to the locality axiom, $\mathsf{H}(X)$ can be described as the subcategory of $\mathsf{H}(Y)$ consisting of those objects supported on X, i.e., those objects that vanish when pulled back along the complement of i. Therefore, Theorem 4.4 tells that $\mathbf{DA}^{\text{ét}}(-;\Lambda)$ is, up to an equivalence, the unique stable homotopical 2-functor that extends Voevodsky's category of motives over regular bases.

2. A stressed before, the construction of $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ is really simpler than $\mathbf{DM}^{\text{ét}}(S;\Lambda)$. Moreover, the advantage of using transfers in defining motivic sheaves disappears when the base scheme S has dimension ≥ 1. Indeed, all the results that will be explained in §4.2 require the base to be a field.

Remark 4.7. The reader might wonder which construction of categories of motives is better. The answer is that both $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ and $\mathbf{DM}^{\text{ét}}(S;\Lambda)$ have their advantages and disadvantages.

- $\mathbf{DA}^{\text{ét}}(S;\Lambda)$ is simpler[14] and is the correct category of motivic sheaves for any S. On the other hand, one does not have a concrete model for the $\mathbb{A}^1$-localization functor when S is the spectrum of a field.

- Over a field, one has the theory of homotopy invariant presheaves with transfers which is a powerful tool to study the category $\mathbf{DM}^{\text{ét}}(k;\Lambda)$. However, over a curve and higher dimensional bases, this advantage disappears as the theory of homotopy invariant presheaves with transfers breaks down completely. Moreover, it is unclear if $\mathbf{DM}^{\text{ét}}(S;\Lambda)$ is the correct category when S is not normal.

4.2. Homotopy invariant presheaves with transfers. Let $\mathcal{F}$ be a presheaf on Sm/k. We say that $\mathcal{F}$ is homotopy invariant if $\mathcal{F}(U) \to \mathcal{F}(\mathbb{A}^1 \times U)$ is an isomorphism for all $U \in \mathrm{Sm}/k$. For simplicity, we assume that the exponent

[14] For instance, it is very convenient not to have to worry about transfers when discussing realizations!

characteristic of k is invertible in Λ. A basic theorem of Voevodsky [29, Chapter 3] states the following.[15]

Theorem 4.8. *Let $\mathcal{F}$ be a homotopy invariant presheaf with transfers on* Sm/k *(with values in Λ-modules). Then $\mathrm{a}_{\text{ét}}(\mathcal{F})$, the étale sheaf associated to $\mathcal{F}$, is an $\mathbb{A}^1$-local object of* $\mathbf{D}(\mathrm{Str}_{\text{ét}}(\mathrm{Sm}/k;\Lambda))$. *More concretely,*

$$H^i_{\text{ét}}(U;\mathrm{a}_{\text{ét}}(\mathcal{F})) \to H^i_{\text{ét}}(\mathbb{A}^1\times U;\mathrm{a}_{\text{ét}}(\mathcal{F}))$$

is an isomorphism for all $i \in \mathbb{N}$ and $U \in \mathrm{Sm}/k$.

Remark 4.9. All the hypothesis in this theorem are necessary. For instance, the theorem is wrong for presheaves without transfers. It is also wrong if k is replaced by a curve or a higher dimensional base.

One reason why this theorem is important is that it enables one to construct very easily the $\mathbb{A}^1$-localization of any complex of étale sheaves with transfers. To explain this, we need some notation.

Definition 4.10. For $n \in \mathbb{N}$, set

$$\Delta^n = \mathrm{Spec}(\mathbb{Z}[t_0,\ldots,t_n]/(t_0+\ldots+t_n-1)).$$

These schemes form a cosimplicial scheme $\Delta^\bullet$. Given a complex of presheaves with transfers $\mathcal{K}_\bullet$, we define $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K})$ to be the total complex of the double complex $\underline{\mathrm{hom}}(\Delta^\bullet;\mathcal{K}_\bullet)$. (Recall that $\underline{\mathrm{hom}}(\Delta^n,\mathcal{F})(U)=\mathcal{F}(\Delta^n\times U)$ for any presheaf $\mathcal{F}$ and any $U\in\mathrm{Sm}/k$.) The functor $\mathrm{Sing}^{\mathbb{A}^1}$ is called the *Suslin–Voevodsky construction.*

Corollary 4.11. *Let $\mathcal{K}$ be a complex of étale sheaves with transfers. Then* $\mathrm{Loc}_{\mathbb{A}^1}(\mathcal{K})$ *is given by the Suslin–Voevodsky construction* $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K})$.

Proof. It follows formally from the construction that the canonical map $\mathcal{K}\to\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K})$ is an isomorphism in $\mathbf{DM}^{\mathrm{eff},\,\text{ét}}(k;\Lambda)$. It remains to show that $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K})$ is $\mathbb{A}^1$-local. But again, it follows formally form the construction that the homology presheaves of the complex $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K})$ are homotopy invariant (and admits transfers). Applying Theorem 4.8 to these and using a spectral sequence, one deduces that the maps $H^i_{\text{ét}}(U,\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K}))\to H^i_{\text{ét}}(\mathbb{A}^1\times U;\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K}))$ are isomorphisms. □

4.3. Application: morphisms between motivic sheaves.

A basic question about motivic sheaves is the following.

Question. *Given two motivic sheaves $\mathcal{M}$ and $\mathcal{N}$ over a base scheme S, how to compute the group* $\mathrm{hom}_{\mathbf{DA}^{\text{ét}}(S;\Lambda)}(\mathcal{M},\mathcal{N})$*?*

As said before, in theory, the formalism of the six operations reduces the above question to computing some groups of morphisms (usually many) in $\mathbf{DA}^{\text{ét}}(k;\Lambda)\simeq$

[15] In loc. cit., the result is established for the Nisnevich topology. However, it is an exercise to deduce the result for the étale topology using Suslin's rigidity theorem [24, Theorem 7.20] and the homotopy invariance of étale cohomology with values in $\Lambda/n\Lambda$ for n prime to the exponent-characteristic of k.

$\mathbf{DM}^{\text{ét}}(k;\Lambda)$ (for various fields k). Therefore, it is important to have a solution of this question when the base is a field.

Let k be a field and assume that the exponent-characteristic of k is invertible in Λ. We will explain the solution of the above question in the case where $\mathcal{M}$ and $\mathcal{N}$ are the motives of smooth k-varieties X and Y respectively. Hence, we concentrate on the groups

$$\hom_{\mathbf{DM}^{\text{ét}}(k;\Lambda)}(\mathrm{M}(X);\mathrm{M}(Y)[n]).$$

For simplicity, we assume that Y is proper of pure dimension d_Y. By Theorem 3.11, we know that $\mathrm{M}(Y)$ has a strong dual given by $\mathrm{M}(Y)^\vee = \mathrm{M}(Y)(-d_Y)[-2d_Y]$. Hence, we are left to compute the étale motivic cohomology groups

$$H^p_{\mathcal{L}}(Z;\Lambda(q)) := \hom_{\mathbf{DM}^{\text{ét}}(k;\Lambda)}(\mathrm{M}(Z);\Lambda(q)[p])$$

(for $Z = X \times_k Y$ and $q = d_Y$ and $p = n + 2d_Y$). The answer is as follows.

Theorem 4.12. *Let X be a smooth k-variety. Then there is a canonical isomorphism*

$$\hom_{\mathbf{DM}^{\text{ét}}(k;\Lambda)}(\mathrm{M}(X);\Lambda(q)[p]) \simeq H^{p-2q}_{\text{ét}}(X;\mathrm{Sing}^{\mathbb{A}^1}\Lambda_{\text{tr}}(\mathbb{P}^1_k,\infty_k)^{\wedge q}) \qquad (14)$$

where the right-hand side is the étale hypercohomology of X with values in the complex of étale sheaves $\mathrm{Sing}^{\mathbb{A}^1}\Lambda_{\text{tr}}(\mathbb{P}^1_k,\infty_k)^{\wedge q}$.

Remark 4.13. Theorem 4.12 is an immediate consequence of Theorem 4.8. Another theorem of Voevodsky asserts that the complex $\mathrm{Sing}^{\mathbb{A}^1}\Lambda_{\text{tr}}(\mathbb{P}^1_k,\infty_k)^{\wedge q}$ satisfies Nisnevich descent. Therefore, if Λ is a $\mathbb{Q}$-algebra (or when "étale" is replaced by "Nisnevich"), the right hand side in (14) is simply the cohomology of a concrete complex of cycles, namely $\mathrm{Cor}_k(\Delta^\bullet \times X, (\mathbb{P}^1_k,\infty_k)^{\wedge q}) \otimes \Lambda$.

5. Conjectures

There are many outstanding conjectures concerning motives and algebraic cycles. Some of these seem desperately out of reach such as the Hodge and Tate conjectures (that already made an appearance in Remark 2.27) or the Grothendieck and Kontsevich–Zagier conjectures on periods.

In this section we will discuss two other conjectures that, in comparison with the previous ones, seem more approachable. These two conjectures (as well as the previous ones) predict relations between algebro-geometric objects and transcendental objects, and each one of these conjectures fills some part of the gap between the two half-bridges discussed in the Introduction.

5.1. The conservativity conjecture. Let k be a field of characteristic zero and let $\sigma : k \hookrightarrow \mathbb{C}$ be a complex embedding. Given a finite type k-scheme X, denote by X_{an} the set $X(\mathbb{C})$ endowed with its analytic topology. One has a Betti realization functor [7]

$$\mathrm{B}^*_X : \mathbf{DA}^{\text{ét}}(X;\Lambda) \to \mathbf{D}(X_{\text{an}};\Lambda) \qquad (15)$$

where $\mathbf{D}(X_{\mathrm{an}};\Lambda)$ is the derived category of sheaves of Λ-modules on X_{an}. A central conjecture concerning motives states the following.

Conjecture 5.1 (Conservativity Conjecture). *The functor* B_X^**, restricted to the subcategory* $\mathbf{DA}^{\text{ét}}_{\mathrm{ct}}(X;\Lambda)$*, is conservative. Said differently, if* $\mathcal{M}$ *is a constructible motivic sheaf on* X *such that* $\mathrm{B}_X^*(\mathcal{M})\simeq 0$*, then necessarily* $\mathcal{M}\simeq 0$.

Lemma 5.2. *It suffices to prove Conjecture 5.1 for* $X=\mathrm{Spec}(k)$ *and* $\Lambda=\mathbb{Q}$.

Proof. The reduction to the case $\Lambda=\mathbb{Q}$ follows from Theorem 2.24. The reduction to the case $X=\mathrm{Spec}(k)$ is a consequence of the compatibility of the Betti realization with inverse images. □

Conjectures such as the Hodge and Tate Conjectures concern existence of algebraic cycles (and hence elements in motivic cohomology). On the contrary, Conjecture 5.1 concerns motives which makes it look more approachable. However, the next remark suggests that this hope might be too naive.

Remark 5.3. It is well-known that the category of Chow motives with rational coefficients embeds fully faithfully inside $\mathbf{DM}^{\text{ét}}(k;\mathbb{Q})$. Applying Conjecture 5.1 to Chow motives one obtains the following particular case. *Let* X *and* Y *be smooth and projective varieties over* k *of pure dimension* d*. Let* $\gamma\in\mathrm{CH}^d_{\mathbb{Q}}(X\times_k Y)$ *be an algebraic cycle inducing an isomorphism in cohomology* $\gamma: H^*(Y(\mathbb{C});\mathbb{Q})\xrightarrow{\sim} H^*(X(\mathbb{C});\mathbb{Q})$*. Then, there exists an algebraic cycle* $\delta\in\mathrm{CH}^d_{\mathbb{Q}}(Y\times_k X)$ *such that* $\delta\circ\gamma=[\Delta_X]$ *and* $\gamma\circ\delta=[\Delta_Y]$. This reveals a strong analogy/connexion between the Conservativity Conjecture and the Standard Conjecture of Lefschetz type [12].

Remark 5.4. On a more optimistic note, we mention that we formulated in [8, Conjecture B of §2.4] a concrete (although very complicated) conjecture that would implies Conjecture 5.1. We like to think that this is a non trivial step (although, probably, a very small one) towards a potential solution of the Conservativity Conjecture.

5.2. Existence of a motivic *t*-structure.

Keep the notation as in §5.1.

Conjecture 5.5 (t-Structure Conjecture). *The category* $\mathbf{DA}^{\text{ét}}_{\mathrm{ct}}(X;\Lambda)$ *carries a t-structure, called the* motivic t-structure*, making* B_X^* *exact. (Said differently, if* $\mathcal{M}$ *is a constructible X-motive which belongs to the heart of the motivic t-structure, then* $\mathrm{B}_X^*(\mathcal{M})$ *is concentrated in degree zero, i.e., is isomorphic to a constructible sheaf on* X_{an}*.) Moreover, this t-structure is independent of the choice of the complex embedding* σ*.*

Remark 5.6. Conjecture 5.5 can be reduced to the case where $X=\mathrm{Spec}(k)$ using gluing techniques. Moreover, these gluing techniques can also be used to define perverse motivic t-structures assuming the existence of the usual motivic t-structure.

Remark 5.7. It is important to note that we do not assume Λ to be a $\mathbb{Q}$-algebra in Conjecture 5.5. Indeed, the t-Structure Conjecture is expected to hold *integrally* for $\mathbf{DA}^{\text{ét}}_{\text{ct}}(X;\Lambda)$; in fact, assuming that $\mathbf{DA}^{\text{ét}}_{\text{ct}}(S;\mathbb{Q})$ admits a motivic t-structure, it is easy to construct a motivic t-structure on $\mathbf{DA}^{\text{ét}}_{\text{ct}}(X;\mathbb{Z})$ using Theorem 2.24.

This is particularly significant as it is well-known that $\mathbf{DM}_{\text{gm}}(k;\Lambda)$ (The "Nisnevich" variant of $\mathbf{DM}^{\text{ét}}_{\text{gm}}(k;\Lambda)$) cannot admit a motivic t-structure unless Λ is a $\mathbb{Q}$-algebra. (A simple explanation for this was given by Voevodsky [29, Remark on page 217].) This indicates that, in view of a future theory of *Abelian motivic sheaves*, it is more natural to impose étale descent.

Remark 5.8. In [8, Conjecture A of §2.4] we formulated a very concrete conjecture that, together with Conjecture B of loc. cit., should imply Conjecture 5.5 and more. (By "more", we have in mind the property that $\mathbf{DA}^{\text{ét}}_{\text{ct}}(S;\Lambda)$ is equivalent to the derived category of the heart of its motivic t-structure.)

Remark 5.9. As a measure of the deepness of Conjectures 5.1 and 5.5, we mention that they imply the Standard Conjectures in characteristic zero (as explained by Beilinson [10]). They imply many other well-established conjectures such as the Bloch Conjecture for surfaces and its generalizations, Kimura finiteness for Chow motives, the existence of the Bloch–Beilinson filtration on Chow groups, etc.

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