

# MOTIVES AND ALGEBRAIC CYCLES: A SELECTION OF CONJECTURES AND OPEN QUESTIONS

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*Für Steven mit Zuckerzusatz*

**ABSTRACT.** This article discusses a selection of conjectures and open questions in the study of motives and algebraic cycles. Special emphasis will be placed on the interplay between abstract properties of motives and concrete properties of Chow groups of algebraic cycles modulo rational equivalence.

*Keywords:* algebraic cycles; Bloch conjecture; Chow groups; conservativity conjecture; homotopy  $t$ -structure; motives; motivic  $t$ -structure; slice filtration; vanishing conjectures.

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**Introduction.** Since their invention by Grothendieck in the 60's, motives have served as a guide and a source of inspiration in Arithmetic Algebraic Geometry. However, up to the 90's, the theory of motives was lacking a solid foundation and the

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notion of motive was an elusive one. The situation changed drastically thanks to the work of Voevodsky and others, and we now have precise and natural mathematical entities that deserve to be called motives.<sup>1</sup> However, it is fair to say that the theory of motives is still consisting of more conjectures than theorems. Perhaps this is especially true only for “motives with rational coefficients” as our understanding of motivic cohomology with finite coefficients has been incredibly advanced with the solution of the Bloch–Kato conjecture by Voevodsky [52] with a crucial input of Rost [48].

We are mainly concerned with motives and motivic cohomology with rational coefficients. Unfortunately, in this context, breakthroughs are still to come, and, at present, we can only dream and fantasise about how beautiful and powerful the theory is ought to be. The goal of this article is to present a selection of conjectures and open problems about motives, and to explain their significance and impact on algebraic cycles. We hope to convey our enthusiasm and fascination about this beautiful subject. We have not tried to be exhaustive: there are many other fascinating conjectures that will not be mentioned at all. Also, we have refrained from extending too much the web of motivic conjectures: all the conjectures that we will discuss are folklore in one way or another.

In Section 1, we give a quick review of the theory of motives à la Voevodsky. The reader who is familiar with this theory can jump directly to Section 2 where we discuss the conservativity conjecture, one of the author’s favorite conjectures. In Section 3, we discuss the vanishing conjecture for the motivic Hopf algebra; this is different from the Beilinson–Soulé vanishing conjecture in motivic cohomology, although the two conjectures are undeniably related. In Section 4, we discuss two more conjectures around the slice filtration and the homotopy  $t$ -structure. These conjectures are less established than the previous ones and should be rather considered as open questions.

*Notations and conventions.* We fix a base-field  $k$  and a choice of an algebraic closure  $\bar{k}/k$  whose Galois group is denoted by  $\mathrm{Gal}(\bar{k}/k)$ . The symbol  $\Lambda$  always denotes the commutative ring of coefficients and  $\mathbf{D}(\Lambda)$  denotes the derived category of the abelian category of  $\Lambda$ -modules. We are mainly interested in the case where  $k$  has characteristic zero and where  $\Lambda = \mathbb{Q}$  (and, on one occasion, where  $\Lambda = \overline{\mathbb{Q}}$ ). Therefore, even though some of what we will say remains valid in some greater generality, we assume from now on that  $k$  has characteristic zero and that  $\Lambda$  is a  $\mathbb{Q}$ -algebra unless otherwise stated.

By  $k$ -variety we mean a finite type separated  $k$ -scheme. We denote by  $\mathrm{Sch}/k$  the category of  $k$ -varieties and by  $\mathrm{Sm}/k$  its full subcategory consisting of smooth  $k$ -varieties. Given a  $k$ -variety  $X$ , its Chow groups with coefficients in  $\Lambda$  are denoted by  $\mathrm{CH}^n(X; \Lambda)$ . Similarly, we denote by  $\mathrm{CH}^n(X, m; \Lambda)$  Bloch’s higher Chow groups of  $X$ . Given a complex embedding  $\sigma : k \hookrightarrow \mathbb{C}$ , and a  $k$ -variety  $X$ , we denote by

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<sup>1</sup>The reader might disagree with these statements. Indeed, it is possible to argue that Grothendieck’s construction of the category of Chow motives yields a solid foundation for pure motives just as Voevodsky’s construction of his triangulated category of motives yields a solid foundation for mixed motives. However, it is our personal opinion that the definition of Chow motives is ad hoc and not propitious to further structural developments in the theory of motives, whereas Voevodsky’s construction is natural — I would even say inevitable — and its true potential is yet to be unlocked. Let the future tell if this opinion is naive or not!

$X^{\text{an}}$  the associated complex analytic space. The singular cohomology groups of  $X^{\text{an}}$  with coefficients in  $\Lambda$  are denoted by  $H^n(X^{\text{an}}; \Lambda)$ .

By *mixed Hodge structure* we mean a rational graded-polarisable mixed Hodge structure, i.e., a triple  $(H, W, F)$  consisting of a finite dimensional  $\mathbb{Q}$ -vector space  $H$ , an increasing weight filtration  $W$  on  $H$  and a decreasing Hodge filtration  $F$  on  $H \otimes \mathbb{C}$  satisfying the usual conditions (see [24, Définition 2.3.1]) and such that the graded pieces for the weight filtration are polarisable pure Hodge structures (see [24, Définition 2.1.15]). The category of mixed Hodge structures is denoted by MHS. We will also consider the category of ind-objects in MHS which we denote by Ind-MHS.

An additive category is called *pseudo-abelian* if every projector has a kernel. In a monoidal category, the tensor product is denoted by  $- \otimes -$  and the internal Hom, when it exists, is denoted by  $\underline{\text{Hom}}(-, -)$ . The unit object of a general monoidal category is denoted by  $\mathbf{1}$ . The dual of an object  $M$  is given by  $M^\vee = \underline{\text{Hom}}(M, \mathbf{1})$ .

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## 1. REVIEW OF MOTIVES

As usual,  $\mathbf{DM}(k; \Lambda)$  denotes Voevodsky’s category of motives with coefficients in a commutative ring  $\Lambda$ .<sup>2</sup> Almost everywhere in this article, the category of motives can be used as a black box. Therefore, we will not recall its construction. Instead, we devote this section to a list of useful facts that should help the novice reader to grasp quickly the part of the theory that is needed later on. All these facts, except the last one, are due to Voevodsky and al., and their proofs can be found in [53, 36].

**1.1. Abstract categorical properties.** The category  $\mathbf{DM}(k; \Lambda)$  has, or is expected to have, formal properties analogous to those of the derived category of ind-mixed Hodge structures  $\mathbf{D}(\text{Ind-MHS})$ .<sup>3</sup> In this subsection, we list some of these properties. We will concentrate on what is known and leave the conjectures to later sections.

**FACT A** —  $\mathbf{DM}(k; \Lambda)$  has the structure of a closed monoidal triangulated category. Moreover, there is a natural functor  $M : \text{Sm}/k \rightarrow \mathbf{DM}(k; \Lambda)$  which takes a smooth  $k$ -variety  $X$  to its associated motive  $M(X)$ .<sup>4</sup>

Therefore, in  $\mathbf{DM}(k; \Lambda)$ , we can take the cone of a morphism, the tensor product of two objects and the internal Hom from an object with value in another one. Moreover, we have a supply of objects in  $\mathbf{DM}(k; \Lambda)$  coming from smooth  $k$ -varieties. In this paper, objects in  $\mathbf{DM}(k; \Lambda)$  will be referred to as *motives*. Elsewhere in the

<sup>2</sup>The knowledgeable reader will soon guess which of the variants of Voevodsky’s category of motives we have in mind. However, just in case, we spell it out:  $\mathbf{DM}(k; \Lambda)$  is the homotopy category with respect to the stable  $(\mathbb{A}^1, \text{Nis})$ -local model structure of the category of Tate spectra of complexes of presheaves with transfers on the category of smooth  $k$ -varieties with values in the category of  $\Lambda$ -modules. In particular, we do not impose any boundedness conditions.

<sup>3</sup>This is not totally true because of the failure of conservativity; see Remark 2.3 and Lemma 2.4.

<sup>4</sup>The functor  $M$  is naturally extendable to  $\text{Sch}/k$  but we do not need this extension in this article.

literature, the reader may also find the terms: mixed motives, triangulated motives, Voevodsky motives, etc.

The unit object of  $\mathbf{DM}(k; \Lambda)$  is usually denoted by  $\Lambda(0)$ . The motive  $M(X)$  serves as the universal homology object associated to the smooth  $k$ -variety  $X$ . For this reason, it is sometimes called the *homological motive* of  $X$  by contrast to its dual,  $M(X)^\vee$ , which is called the *cohomological motive* of  $X$ .

**Notation 1.1** — We set

$$\Lambda(1) := \text{Cone} \left\{ M(\text{Spec}(k)) \xrightarrow{-\infty} M(\mathbb{P}_k^1) \right\} [-2].$$

We think about  $\Lambda(1)$  as the reduced homology of the projective line.

The following property is imposed in the construction of  $\mathbf{DM}(k; \Lambda)$ .

**LEMMA 1.2** — *The objet  $\Lambda(1)$  is invertible for the tensor product. Its inverse, which is also its dual  $\underline{\text{Hom}}(\Lambda(1), \Lambda(0))$ , is denoted by  $\Lambda(-1)$ .*

**Notation 1.3** — As usual, we set for  $n \in \mathbb{Z}$ :

$$\Lambda(n) = \begin{cases} \Lambda(1)^{\otimes n} & \text{if } n \geq 0, \\ \Lambda(-1)^{\otimes -n} & \text{if } n < 0. \end{cases}$$

These are the *pure Tate motives*. Given an objet  $M \in \mathbf{DM}(k; \Lambda)$ , we set

$$M(n) = M \otimes \Lambda(n).$$

These are the *Tate twists* of  $M$ .

**FACT B** — *The motive  $M(X)$  of a smooth  $k$ -variety  $X$  is a compact object of  $\mathbf{DM}(k; \Lambda)$ . Moreover,  $\mathbf{DM}(k; \Lambda)$  is compactly generated by the Tate twists of motives of smooth  $k$ -varieties. In fact, it is enough to restrict to Tate twists of motives of smooth and projective  $k$ -varieties.*

We recall the meaning of compact objects and compactly generated triangulated categories.

**DEFINITION 1.4** — *Let  $\mathcal{T}$  be a triangulated category with infinite sums.*

- (a) *An object  $M \in \mathcal{T}$  is said to be compact if for every (possibly infinite) family  $(N_i)_{i \in I}$  of objects in  $\mathcal{T}$ , the obvious homomorphism*

$$\bigoplus_{i \in I} \text{hom}_{\mathcal{T}}(M, N_i) \longrightarrow \text{hom}_{\mathcal{T}}(M, \bigoplus_{i \in I} N_i)$$

*is a bijection.*

- (b) *Let  $\mathcal{B}$  be a set of objects in  $\mathcal{T}$ . We say that  $\mathcal{T}$  is compactly generated by  $\mathcal{B}$  if every objet in  $\mathcal{B}$  is compact and if the following implication holds for every object  $N \in \mathcal{T}$ :*

$$[\text{hom}_{\mathcal{T}}(M[i], N) = 0, \forall M \in \mathcal{B} \text{ and } \forall i \in \mathbb{Z}] \Rightarrow [N \simeq 0].$$

**Remark 1.5** — If  $\mathcal{T}$  is compactly generated by  $\mathcal{B}$ , then  $\mathcal{T}$  coincides with its smallest triangulated subcategory closed under infinite sums and containing the elements of  $\mathcal{B}$ . In other words, every objet of  $\mathcal{T}$  can be obtained by a (possibly infinite) iteration of cones and infinite sums from the objects belonging to  $\mathcal{B}$ . We leave it to the reader to contemplate the meaning of this in the case of  $\mathbf{DM}(k; \Lambda)$ .

The following lemma is contained in [41, Theorem 2.1].

**LEMMA 1.6** — *Let  $\mathcal{T}$  be a triangulated category with infinite sums and assume that  $\mathcal{T}$  is compactly generated by a set of objects  $\mathcal{B}$ . Let  $\mathcal{T}^{\text{co}} \subset \mathcal{T}$  be the full subcategory of compact objects of  $\mathcal{T}$ . Then  $\mathcal{T}^{\text{co}}$  is a triangulated subcategory of  $\mathcal{T}$  which coincides with the smallest triangulated subcategory of  $\mathcal{T}$  containing the elements of  $\mathcal{B}$  and which is pseudo-abelian.*

**DEFINITION 1.7** — *We denote by  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$  the full subcategory of compact objects in  $\mathbf{DM}(k; \Lambda)$ . The objects of  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$  are called geometric motives, aka., constructible motives.*

**Remark 1.8** — By Fact B, the functor  $M : \text{Sm}/k \rightarrow \mathbf{DM}(k; \Lambda)$  factors through  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$ . Moreover, by Lemma 1.6,  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$  is the smallest triangulated subcategory of  $\mathbf{DM}(k; \Lambda)$  which is pseudo-abelian, is closed under Tate twists and contains the motives of smooth projective  $k$ -varieties.

**Remark 1.9** — Under the analogy between  $\mathbf{DM}(k; \Lambda)$  and  $\mathbf{D}(\text{Ind-MHS})$ , the subcategory  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$  corresponds to  $\mathbf{D}^{\text{b}}(\text{MHS})$  which we identify with the triangulated subcategory of  $\mathbf{D}(\text{Ind-MHS})$  consisting of complexes of ind-mixed Hodge structures with finite dimensional total homology.

**FACT C** — *An object of  $\mathbf{DM}(k; \Lambda)$  is compact if and only if it is strongly dualisable. In particular, the tensor category  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$  is rigid. Moreover, if  $X$  is a smooth projective  $k$ -variety of pure dimension  $n$ , there is a motivic Poincaré duality isomorphism*

$$M(X)^{\vee} \simeq M(X)(-n)[-2n].$$

We recall the notion of strong duality.

**DEFINITION 1.10** — *Let  $\mathcal{M}$  be a tensor category and  $M$  an object of  $\mathcal{M}$ . We say that  $M$  admits a strong dual if there exist an object  $N \in \mathcal{M}$  and maps*

$$u : \mathbf{1} \rightarrow M \otimes N \quad \text{and} \quad v : N \otimes M \rightarrow \mathbf{1}$$

*making the following triangles commutative*

$$\begin{array}{ccc} N \otimes \mathbf{1} & \xrightarrow{N \otimes u} & N \otimes M \otimes N \\ & \searrow & \downarrow v \otimes N \\ & & \mathbf{1} \otimes N, \end{array} \quad \begin{array}{ccc} \mathbf{1} \otimes M & \xrightarrow{u \otimes M} & M \otimes N \otimes M \\ & \searrow & \downarrow M \otimes v \\ & & M \otimes \mathbf{1}. \end{array}$$

*Equivalently,  $N$  is a strong dual for  $M$  if there exists a natural isomorphism*

$$\underline{\text{Hom}}(M, -) \simeq N \otimes -.$$

*We say that  $\mathcal{M}$  is rigid if every object of  $\mathcal{M}$  is strongly dualisable.*

**1.2. Relation with algebraic cycles.** The category  $\mathbf{DM}(k; \Lambda)$  bears a strong relation with algebraic cycles. Let  $\mathbf{Chow}(k; \Lambda)$  be the category of Chow motives with coefficients in  $\Lambda$ .

**FACT D** — *There is a fully faithful embedding  $\mathbf{Chow}(k; \Lambda) \hookrightarrow \mathbf{DM}_{\text{gm}}(k; \Lambda)$  which is compatible with the tensor structures.*

In this paper, we use the homological convention for Chow motives. Objects in  $\mathbf{Chow}(k; \Lambda)$  are given by triples  $(X, \gamma, m)$  where  $X$  is a smooth projective variety,

$\gamma \in \mathrm{CH}_{\dim(X)}(X \times X; \Lambda)$  an idempotent and  $m \in \mathbb{Z}$ . Morphisms are given by<sup>5</sup>

$$\mathrm{hom}_{\mathbf{Chow}(k; \Lambda)}((X, \gamma, m), (Y, \delta, n)) = \delta \circ \mathrm{CH}_{\dim(X)+m-n}(X \times Y; \Lambda) \circ \gamma.$$

The embedding in Fact D takes a Chow motive  $(X, \gamma, m)$  to the image of the idempotent  $\gamma$  acting on the Voevodsky motive  $M(X)(m)[2m]$ .

**Remark 1.11** — By Fact B and Lemma 1.6, we know that the pseudo-abelian triangulated category  $\mathbf{DM}_{\mathrm{gm}}(k; \Lambda)$  is generated by the image of the fully faithful embedding  $\mathbf{Chow}(k; \Lambda) \hookrightarrow \mathbf{DM}_{\mathrm{gm}}(k; \Lambda)$ . Therefore, we can think about Voevodsky's category  $\mathbf{DM}_{\mathrm{gm}}(k; \Lambda)$  as being a “triangulated envelop” of Grothendieck's category of Chow motives.

**DEFINITION 1.12** — *Let  $X$  be a smooth  $k$ -variety. For  $(p, q) \in \mathbb{Z}^2$ , we set*

$$H^p(X; \Lambda(q)) := \mathrm{hom}_{\mathbf{DM}(k; \Lambda)}(M(X), \Lambda(q)[p]).$$

*These are the motivic cohomology groups of  $X$ . Similarly, we set*

$$H_p(X; \Lambda(q)) := \mathrm{hom}_{\mathbf{DM}(k; \Lambda)}(\Lambda(-q)[p], M(X)).$$

*These are the motivic homology groups of  $X$ . In the special case  $q = 0$ , these are also known as the Suslin homology groups of  $X$ .*

**FACT E** — *There is a natural isomorphism between motivic cohomology and Bloch's higher Chow groups [17] with coefficients in  $\Lambda$ :*

$$H^p(X; \Lambda(q)) \simeq \mathrm{CH}^q(X, 2q - p; \Lambda).$$

*In particular,  $H^{2n}(X; \Lambda(n))$  coincides with the usual Chow group  $\mathrm{CH}^n(X; \Lambda)$  of codimension  $n$  cycles.*

A proof of Fact E can be found in [50]. As a special case, and thanks to [42] and [49], one has the following relation between motivic cohomology and Milnor  $K$ -theory [37].

**COROLLARY 1.13** — *For  $n \geq 0$ , we have a natural isomorphism*

$$H^n(\mathrm{Spec}(k); \Lambda(n)) \simeq K_n^M(k) \otimes \Lambda$$

*where  $K_n^M(k)$  is the  $n$ -th Milnor  $K$ -theory group of the field  $k$ .*

There is also a cycle-theoretic description for the motivic homology groups. To explain this, we first recall the notion of finite correspondence which is basic in the theory of motives à la Voevodsky.

**DEFINITION 1.14** — *Let  $X$  and  $Y$  be smooth  $k$ -varieties. An elementary finite correspondence from  $X$  to  $Y$  is an integral closed subscheme  $Z \subset X \times Y$  such that the projection from  $Z$  to  $X$  is finite and surjective onto a connected component of  $X$ . A finite correspondence is a  $\Lambda$ -linear combination of elementary finite correspondences. We denote by  $\mathrm{Cor}(X, Y; \Lambda)$  the  $\Lambda$ -module of correspondences from  $X$  to  $Y$ . When  $\Lambda = \mathbb{Z}$ , we simply write  $\mathrm{Cor}(X, Y)$ . Clearly,*

$$\mathrm{Cor}(X, Y; \Lambda) \simeq \mathrm{Cor}(X, Y) \otimes \Lambda.$$

*There is a natural composition law on finite correspondences. Moreover, smooth  $k$ -varieties and finite correspondences form an additive category denoted by*

$$\mathrm{SmCor}(k; \Lambda).$$

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<sup>5</sup>This formula only makes sense when  $X$  has pure dimension. Otherwise, one has to interpret  $\dim(X)$  as a locally constant function on  $X \times Y$ .

There is a natural functor  $\mathrm{Sm}/k \rightarrow \mathrm{SmCor}(k; \Lambda)$  which is the identity on objects and which takes a morphism of  $k$ -varieties to its graph.

**Notation 1.15** — Given a pair  $(X, D)$  consisting of a smooth  $k$ -variety  $X$  and a normal crossing divisor  $D = \bigcup_{i \in I} D_i$ , with  $D_i$  irreducible, we denote by

$$\mathrm{Cor}((X, D), V; \Lambda)$$

the kernel of the homomorphism

$$\mathrm{Cor}(X, V; \Lambda) \rightarrow \bigoplus_{i \in I} \mathrm{Cor}(D_i, V; \Lambda).$$

Similarly, we denote by

$$\mathrm{Cor}(U, (X, D); \Lambda)$$

the cokernel of the homomorphism

$$\bigoplus_{i \in I} \mathrm{Cor}(U, D_i; \Lambda) \rightarrow \mathrm{Cor}(U, X; \Lambda).$$

**Notation 1.16** — We denote by  $\mathbb{E}^1 = \mathrm{Spec}(k[t, t^{-1}])$  the complement of the zero section in  $\mathbb{A}^1 = \mathrm{Spec}(k[t])$ . We consider the pair  $(\mathbb{E}^1, 1)$  and we denote by  $(\mathbb{E}^1, 1)^n$  the pair consisting of  $\mathbb{E}^n = \mathrm{Spec}(k[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$  and its normal crossing divisor defined by the equation  $(t_1 - 1) \cdots (t_n - 1) = 0$ .

For  $n \in \mathbb{N}$ , we consider the algebraic simplex

$$\Delta^n = \mathrm{Spec}(k[t_0, \dots, t_n]/(t_0 + \cdots + t_n - 1)).$$

Varying  $n$ , one gets a cosimplicial scheme  $\Delta^\bullet$ .

**FACT F** — *There are natural isomorphisms*

$$H_p(X; \Lambda(q)) \simeq \begin{cases} H_{p+q}(\mathrm{Cor}(\Delta^\bullet, X \times (\mathbb{E}^1, 1)^q; \Lambda)) & \text{if } q \geq 0, \\ H_{p+q}(\mathrm{Cor}((\mathbb{E}^1, 1)^{-q} \times \Delta^\bullet, X; \Lambda)) & \text{if } q \leq 0. \end{cases}$$

**1.3. Realisations.** Motives are rather abstract mathematical entities. Fortunately, we have at our disposal *realisation functors* that convert a motive into more familiar objects such as mixed Hodge structures or Galois representations. We only quote the following result (see [29] for the compact contravariant version).

**FACT G** — *Given a complex embedding  $\sigma : k \hookrightarrow \mathbb{C}$ , there exists a triangulated monoidal functor*

$$R_\sigma : \mathbf{DM}(k; \mathbb{Q}) \rightarrow \mathbf{D}(\mathrm{Ind}\text{-MHS})$$

*which takes the motive  $M(X)$  of a smooth  $k$ -variety  $X$  to a complex of ind-mixed Hodge structures computing the singular homology groups of  $X^{\mathrm{an}}$  together with their natural mixed Hodge structures. This functor is called the Hodge realisation.*

**Remark 1.17** — We will also consider a more basic realisation functor, namely the Betti realisation

$$B_\sigma : \mathbf{DM}(k; \Lambda) \rightarrow \mathbf{D}(\Lambda)$$

which takes the motive  $M(X)$  of a smooth  $k$ -variety  $X$  to the singular chain complex of  $X^{\mathrm{an}}$  with coefficients in  $\Lambda$ . When  $\Lambda = \mathbb{Q}$ , the functor  $B_\sigma$  is, up to a natural isomorphism, the composition of  $R_\sigma$  with the obvious forgetful functor.

**Remark 1.18** — Given a smooth  $k$ -variety  $X$ , we set  $C_{\text{Hdg}}(X) = R_\sigma(M(X))$ . It is known that

$$\text{hom}_{\mathbf{D}(\text{Ind-MHS})}(C_{\text{Hdg}}(X), \mathbb{Q}(q)[p])$$

coincides with the Deligne–Beilinson cohomology group  $H^p(X; \mathbb{Q}_{\mathcal{D}}(q))$ . Therefore, the functor  $R_\sigma$  induces a homomorphism

$$H^p(X, \mathbb{Q}(q)) \longrightarrow H^p(X; \mathbb{Q}_{\mathcal{D}}(q))$$

from motivic cohomology to Deligne–Beilinson cohomology. This is the so-called “regulator map” which extends the classical “cycle map” on Chow groups

$$\text{CH}^n(X; \mathbb{Q}) \longrightarrow H^{2n}(X; \mathbb{Q}_{\mathcal{D}}(n)).$$

This is to say that the functor  $R_\sigma$  encodes deep and complicated information which are not always easy to unravel.

## 2. THE CONSERVATIVITY CONJECTURE

There are many longstanding conjectures about motives but, arguably, the most basic one is the so-called *conservativity conjecture*.<sup>6</sup>

**2.1. Statement and general remarks.** Recall that a functor  $f : \mathcal{C} \longrightarrow \mathcal{D}$  is *conservative* if it detects isomorphisms, i.e., a morphism  $\alpha : A \longrightarrow B$  in  $\mathcal{C}$  is an isomorphism if and only if  $f(\alpha) : f(A) \longrightarrow f(B)$  is an isomorphism. In the triangulated setting,  $f$  is conservative if and only if it detects the zero objects.

Fix a complex embedding  $\sigma : k \hookrightarrow \mathbb{C}$ . The *conservativity conjecture* states the following.

**CONJECTURE 2.1** — *The Betti realisation functor*

$$B_\sigma : \mathbf{DM}_{\text{gm}}(k; \Lambda) \longrightarrow \mathbf{D}(\Lambda)$$

*is conservative.*

**Remark 2.2** — Our convention that  $\Lambda$  is a  $\mathbb{Q}$ -algebra is essential for Conjecture 2.1. Indeed, it is easy to construct nonzero geometric motives  $M \in \mathbf{DM}_{\text{gm}}(k; \mathbb{Z})$  with zero Betti realisation. The simplest example is obtained as follows. Assume that  $k$  contains the  $n$ -th roots of unity for some integer  $n \geq 2$ . Given that

$$\text{hom}_{\mathbf{DM}(k; \mathbb{Z})}(\mathbb{Z}(0), \mathbb{Z}(1)[r]) = \begin{cases} 0 & \text{if } r \neq 1, \\ k^\times & \text{if } r = 1, \end{cases}$$

we deduce that

$$\text{hom}_{\mathbf{DM}(k; \mathbb{Z})}(\mathbb{Z}/n\mathbb{Z}(0), \mathbb{Z}/n\mathbb{Z}(1)) = \mu_n(k).$$

Therefore, the choice of a generator  $\xi \in \mu_n(k)$  induces a morphism in  $\mathbf{DM}(k; \mathbb{Z})$

$$\xi : \mathbb{Z}/n\mathbb{Z}(0) \longrightarrow \mathbb{Z}/n\mathbb{Z}(1).$$

It is easy to see that  $B_\sigma(\xi)$  is an isomorphism whereas  $\xi$  is not.

**Remark 2.3** — Conjecture 2.1 has an obvious analogue in Hodge theory. Indeed, the counterpart in Hodge theory of the restriction of the Betti realisation to geometric motives is given by the forgetful functor

$$\mathbf{D}^b(\text{MHS}) \longrightarrow \mathbf{D}(\mathbb{Q})$$

<sup>6</sup>The author is aware that “conservativity” is not an English word as much as “conservativité” is not a French word. However, it seems that there is no better alternative for naming this conjecture.



which is obviously conservative. In fact, more generally, the forgetful functor

$$\mathbf{D}(\text{Ind-MHS}) \longrightarrow \mathbf{D}(\mathbb{Q})$$

is also conservative; this is particularly puzzling when contrasted with the next result which shows that the restriction to geometric motives is essential in Conjecture 2.1.

**LEMMA 2.4** — *If  $k$  has infinite transcendence degree over its prime field, then there are nonzero motives  $F \in \mathbf{DM}(k; \mathbb{Q})$  such that  $B_\sigma(F) \simeq 0$ .*

*Proof.* Recall by Corollary 1.13 that

$$H^n(\text{Spec}(k); \mathbb{Q}(n)) = K_n^M(k) \otimes \mathbb{Q}$$

where  $K_n^M(k)$  is the  $n$ -th Milnor  $K$ -theory group of the field  $k$ . Let  $(a_n)_{n \in \mathbb{N}}$  be a family of elements in  $k^\times$  which are algebraically independent. For each  $n \in \mathbb{N}$ , we consider the map

$$a_n : \mathbb{Q}(n)[n] \longrightarrow \mathbb{Q}(n+1)[n+1]$$

which corresponds to  $a_n \in k^\times$  modulo the chain of isomorphisms

$$\begin{aligned} \text{hom}_{\mathbf{DM}(k; \mathbb{Q})}(\mathbb{Q}(n)[n], \mathbb{Q}(n+1)[n+1]) &\simeq \text{hom}_{\mathbf{DM}(k; \mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(1)[1]) \\ &= H^1(\text{Spec}(k), \mathbb{Q}(1)) \\ &\simeq k^\times \otimes \mathbb{Q}. \end{aligned}$$

This gives an  $\mathbb{N}$ -inductive system  $\{\mathbb{Q}(n)[n]\}_{n \in \mathbb{N}}$  and we take  $F$  to be its homotopy colimit:

$$F = \text{hocolim}_{n \in \mathbb{N}} \mathbb{Q}(n)[n].$$

As  $B_\sigma$  commutes with direct sums, it also commutes with homotopy colimits of  $\mathbb{N}$ -inductive systems. Thus, we have

$$B_\sigma(F) \simeq \text{hocolim}_{n \in \mathbb{N}} B_\sigma(\mathbb{Q}(n)[n]) \simeq \text{hocolim}_{n \in \mathbb{N}} \mathbb{Q}[n] \simeq 0.$$

To finish, it remains to show that  $F$  is nonzero.

More precisely, we will show that the natural map

$$\alpha_\infty : \mathbb{Q}(0) \longrightarrow F$$

is nonzero. As  $\mathbb{Q}(0)$  is compact, we have an identification

$$\text{hom}_{\mathbf{DM}(k; \mathbb{Q})}(\mathbb{Q}(0), F) \simeq \text{colim}_{n \in \mathbb{N}} \text{hom}_{\mathbf{DM}(k; \mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)[n]).$$

Therefore,  $\alpha_\infty$  is zero if and only if there exists  $n \in \mathbb{N}$  such that the natural map

$$\alpha_n : \mathbb{Q}(0) \longrightarrow \mathbb{Q}(n)[n]$$

is zero. By the identification

$$\text{hom}_{\mathbf{DM}(k; \mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)[n]) \simeq K_n^M(k) \otimes \mathbb{Q},$$

$\alpha_n$  corresponds to the symbol  $\{a_0, \dots, a_n\} \in K_n^M(k) \otimes \mathbb{Q}$ . This symbol is nonzero under the hypothesis that the  $a_i$ 's are algebraically independent.  $\square$

**Remark 2.5** — Lemma 2.4 indicates that the analogy between  $\mathbf{DM}(k; \mathbb{Q})$  and  $\mathbf{D}(\text{Ind-MHS})$  breaks down when dealing with “big” motives. Said differently,  $\mathbf{DM}(k; \mathbb{Q})$  is not the correct candidate for the big derived category of mixed motives. Here is an obvious way to fix the problem.

**DEFINITION 2.6** — We call *fantom motive* an object of  $\mathbf{DM}(k; \Lambda)$  whose Betti realisation is zero.<sup>7</sup> *Fantom motives* form a localising triangulated subcategory of  $\mathbf{DM}(k; \Lambda)$  that we denote by  $\mathcal{F}ant$ . We define  $\mathbf{DM}^b(k; \Lambda)$  to be the Verdier quotient:

$$\mathbf{DM}^b(k; \Lambda) := \mathbf{DM}(k; \Lambda) / \mathcal{F}ant.$$

We call  $\mathbf{DM}^b(k; \Lambda)$  the category of *true motives*.

**Remark 2.7** — The Betti realisation on  $\mathbf{DM}(k; \Lambda)$  induces a Betti realisation functor

$$B_\sigma : \mathbf{DM}^b(k; \Lambda) \longrightarrow \mathbf{D}(\Lambda)$$

which is conservative by construction.

In the present state of knowledge, calling  $\mathbf{DM}^b(k; \Lambda)$  the category of true motives might sound a bit pretentious. However, here is a plausible conjecture that, if true, would justify such a terminology.

**CONJECTURE 2.8** — Let  $F \in \mathcal{F}ant$  be a *fantom motive* and let  $M \in \mathbf{DM}_{\text{gm}}(k; \Lambda)$  be a *geometric motive*. Then

$$\text{hom}_{\mathbf{DM}(k; \Lambda)}(F, M) = 0.$$

**Remark 2.9** — As a meagre evidence, we note that Conjecture 2.8 is satisfied for the motive  $F$  constructed in the proof of Lemma 2.4. This follows immediately from the following well-known property of geometric motives. Given  $M \in \mathbf{DM}_{\text{gm}}(k; \Lambda)$ , there exists an integer  $n_0$  (depending on  $M$ ) such that

$$\text{hom}_{\mathbf{DM}(k; \Lambda)}(\Lambda(n)[m], M) = 0$$

for all  $n \geq n_0$  and  $m \in \mathbb{Z}$ .

**PROPOSITION 2.10** — Assume Conjecture 2.8. Then the composite functor

$$\mathbf{DM}_{\text{gm}}(k; \Lambda) \hookrightarrow \mathbf{DM}(k; \Lambda) \twoheadrightarrow \mathbf{DM}^b(k; \Lambda)$$

is a fully faithful embedding.

*Proof.* This is a direct consequence of the construction of the Verdier localisation.  $\square$

**COROLLARY 2.11** — Conjecture 2.8 implies Conjecture 2.1.

Lemma 2.4 indicates that the failure of conservativity might be a phenomenon related to big base fields. Indeed, as far as we know, there is no reason to disbelieve the following conjecture which, if true, lies probably much deeper than Conjecture 2.1.

**CONJECTURE 2.12** — Assume that  $k$  has finite transcendence degree over its prime field. Then the Betti realisation functor

$$B_\sigma : \mathbf{DM}(k; \Lambda) \longrightarrow \mathbf{D}(\Lambda)$$

is conservative.

<sup>7</sup>The notion of fantom motive is independent of the choice of the complex embedding. This can be shown using the comparison theorem between the Betti and  $\ell$ -adic realisations. (Here, we have in mind a covariant triangulated functor  $\mathbf{DM}(k; \mathbb{Q}) \rightarrow \mathbf{D}(\mathbb{Q}_\ell)$ , commuting with arbitrary direct sums and sending the motive of a smooth  $k$ -variety  $X$  to a complex of  $\mathbb{Q}_\ell$ -vectorspaces computing the  $\ell$ -adic homology groups  $H_*(X \otimes_k \bar{k}; \mathbb{Q}_\ell)$ . The restriction of such a functor to geometric motives is constructed in [8] and there are standard ways to extend this to  $\mathbf{DM}(k; \mathbb{Q})$  in a continuous manner.)

**Remark 2.13** — In a way, the state of affair that we are describing is to be expected. Indeed, the motivic cohomological dimension of a field  $k$ , whatever this means, is expected to be  $\text{trdeg}(k/\mathbb{Q}) + 1$  and, in particular, should be infinite when the transcendence degree of  $k$  over its prime field is infinite. Moreover, it is well-known that “infinite cohomological dimension” in a Grothendieck abelian category can prevent its (unbounded) derived category from being compactly generated. Therefore, Fact B indicates that  $\mathbf{DM}(k; \Lambda)$  is not a reasonable candidate for the unbounded derived category of the abelian category of ind-mixed motives unless  $\text{degtr}(k/\mathbb{Q}) < \infty$ . On the contrary, when  $\text{degtr}(k/\mathbb{Q}) = \infty$ , the category  $\mathbf{DM}^b(k; \Lambda)$  is presumably not compactly generated as it was obtained by a Verdier localisation with respect to the triangulated subcategory  $\mathcal{F}ant$  which is not expected to be compactly generated (unless Conjecture 2.8 is totally wrong). Therefore, from this perspective, there is no obstacles for  $\mathbf{DM}^b(k; \Lambda)$  being the derived category of the abelian category of ind-mixed motives.

Another way of extending the conservativity conjecture to non-necessary geometric objects is given by the following conjecture (which, perhaps, deserves better the status of an open question).

**CONJECTURE 2.14** — Denote by  $\mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)$ <sup>8</sup> the smallest triangulated subcategory of  $\mathbf{DM}(k; \Lambda)$  closed under infinite sums and containing the motives  $M(X)$  for any smooth  $k$ -variety  $X$  of dimension  $\leq n$ . Then, the Betti realisation functor

$$B_\sigma : \mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda) \longrightarrow \mathbf{D}(\Lambda)$$

is conservative.

**2.2. Some consequences.** The conservativity conjecture has many concrete consequences on algebraic cycles. One of these consequences is a famous 40 years old conjecture of Bloch on 0-cycles on surfaces.<sup>9</sup>

**PROPOSITION 2.15** — Assume Conjecture 2.1. Let  $X$  be a smooth projective surface over an algebraically closed field  $k$  of characteristic zero. If  $p_g(X) = 0$ , then the Albanese homomorphism

$$\text{CH}_0(X) \longrightarrow \text{Alb}(X)(k)$$

is injective (where  $\text{Alb}(X)$  denotes the Albanese scheme of  $X$ ).<sup>10</sup>

*Proof.* By Roitman’s theorem [46], the kernel of the Albanese map is torsion free. Therefore, it is enough to prove that the map

$$\text{CH}_0(X; \mathbb{Q}) \longrightarrow \text{Alb}(X)(k) \otimes \mathbb{Q}$$

is injective. Recall that  $p_g(X) = \dim(\Omega^2(X))$  is the dimension of the space of global (holomorphic) differential 2-forms. Thus, the condition  $p_g(X) = 0$  implies that the Hodge structure on  $H_2(X^{\text{an}}; \mathbb{Q})$  is of type  $(-1, -1)$ . By the (dual of the) Lefschetz (1, 1) Theorem, this is also equivalent to the condition that the cycle class map

$$\text{NS}_1(X; \mathbb{Q}) \longrightarrow H_2(X^{\text{an}}; \mathbb{Q})$$

<sup>8</sup>The subcategories  $\mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)$  will be considered in more details in Subsection 4.2.

<sup>9</sup>See [22] for an interesting story about how Severi had stumbled upon the statement of the Bloch conjecture a few decades before the conjecture was stated.

<sup>10</sup>The converse of this statement is true (unconditionally) and is a well-known theorem of Mumford [38]: if the Albanese homomorphism is injective, then necessary  $p_g(X) = 0$ .

is an isomorphism. (Of course,  $\mathrm{NS}_1(X; \mathbb{Q})$  is the Néron–Severi group of 1-dimensional cycles, or equivalently divisors, on  $X$  with rational coefficients.)

The motive of the surface  $X$  has a Chow–Kunneth decomposition

$$M(X) = \bigoplus_{i=0}^4 M_i(X)[i]$$

such that  $B_\sigma(M_i(X))$  is isomorphic to  $H_i(X^{\mathrm{an}}; \mathbb{Q})$ . Furthermore, according to [33],  $M_2(X)$  decomposes into an algebraic and a transcendental part

$$M_2(X) = M_2^{\mathrm{alg}}(X) \oplus M_2^{\mathrm{tr}}(X).$$

The algebraic part  $M_2^{\mathrm{alg}}(X)$  is given by

$$M_2^{\mathrm{alg}}(X) = \mathrm{NS}_1(X; \mathbb{Q})(1)[2]$$

where the finite dimensional  $\mathbb{Q}$ -vector space  $\mathrm{NS}_1(X; \mathbb{Q})$  is considered as an Artin motive in the obvious way. The transcendental part  $M_2^{\mathrm{tr}}(X)$  determines the kernel of the Albanese map via the formula

$$\mathrm{hom}_{\mathbf{DM}(k; \mathbb{Q})}(\mathbb{Q}(0), M_2^{\mathrm{tr}}(X)) \simeq \ker \left\{ \mathrm{CH}_0(X; \mathbb{Q}) \rightarrow \mathrm{Alb}(X)(k) \otimes \mathbb{Q} \right\}.$$

Now, the isomorphisms

$$\mathrm{NS}_1(X; \mathbb{Q}) \simeq H_2(X^{\mathrm{an}}; \mathbb{Q}) \quad \text{and} \quad M_2^{\mathrm{alg}}(X) \simeq \mathrm{NS}_1(X; \mathbb{Q})(1)[2]$$

show that

$$B_\sigma(M_2^{\mathrm{alg}}(X)) \simeq B_\sigma(M_2(X)) \simeq H_2(X^{\mathrm{an}}; \mathbb{Q}).$$

It follows that  $B_\sigma(M_2^{\mathrm{tr}}(X)) = 0$ . Applying Conjecture 2.1, one deduces that  $M_2^{\mathrm{tr}}(X) \simeq 0$ . This finishes the proof thanks to the above description of the kernel of the Albanese map.  $\square$

**Remark 2.16** — Bloch’s conjecture for surfaces with  $p_g = 0$  has been checked for all surfaces which are not of general type in [18]. It is also known for some surfaces of general type admitting nice cyclic étale covers, see for example [31], [54], [55], [15], [16], etc. Perhaps, the most intriguing surfaces with  $p_g = 0$  for which Bloch’s conjecture seems intractable are the fake projective planes (aka., Mumford surfaces). Recall that a fake projective plane is a smooth proper surface with the same Betti numbers as  $\mathbb{P}^2$  but which is not isomorphic to  $\mathbb{P}^2$ . The first example of a fake projective plane was constructed by Mumford [39].

The conservativity conjecture applies much beyond the case of surfaces. For instance, one can prove the following statement using the same reasoning as in the proof of Proposition 2.15.

**PROPOSITION 2.17** — *Assume Conjecture 2.1. Let  $X$  be a smooth projective  $k$ -variety with algebraic cohomology, i.e., satisfying the following two conditions:*

- (a) for any  $m \in \mathbb{N}$ ,  $H^{2m+1}(X^{\mathrm{an}}; \mathbb{Q}) = 0$ ;
- (b) for any  $m \in \mathbb{N}$ , the cycle class map

$$\mathrm{CH}^m(X; \mathbb{Q}) \longrightarrow H^{2m}(X^{\mathrm{an}}; \mathbb{Q})$$

*is surjective.*

*Then,  $M(X)$  is a Tate motive. In particular, the cycle class maps in (b) are also injective.*

*Proof.* For  $m \in \mathbb{Z}$ , choose a family of elements  $(\alpha_{m,i})_{i \in I_m}$  in  $\mathrm{CH}^m(X; \mathbb{Q})$  whose images in singular cohomology form a basis of  $H^{2m}(X^{\mathrm{an}}; \mathbb{Q})$ . The  $\alpha_{m,i}$ 's induce a morphism in  $\mathbf{Chow}(k; \mathbb{Q})$ :

$$\beta : M(X) \longrightarrow \bigoplus_{m \geq 0} \bigoplus_{i \in I_m} \mathbb{Q}(m)[2m].$$

Moreover, the conditions (a) and (b) of the statement implies that  $B_\sigma(\beta)$  is an isomorphism. Applying Conjecture 2.1, we deduce that  $\beta$  was already an isomorphism.  $\square$

**Remark 2.18** — The conclusion of Proposition 2.17 is reminiscent to the following (unconditional) theorem of Jannsen (see [26] where this theorem is discussed and generalised): if the cycle class map from the Chow ring of a smooth projective  $k$ -variety to its singular cohomology is injective, then this cycle class map is in fact an isomorphism and the Chow motive of  $X$  is Tate. (Here, one needs to assume that  $k$  is algebraically closed with infinite transcendence degree.)

Here is another amusing consequence of the conservativity conjecture.

**PROPOSITION 2.19** — *Assume Conjecture 2.1. Assume also that the  $\mathbb{Q}$ -algebra  $\Lambda$  is connected (i.e., has no nontrivial idempotents). Let  $M \in \mathbf{DM}_{\mathrm{gm}}(k; \Lambda)$  be a geometric motive. Then the following two conditions are equivalent:*

- (i)  $M$  is invertible (for the tensor product),
- (ii) there exists a unique integer  $m$  such that  $H_i(B_\sigma(M))$  is zero unless  $i = m$  in which case it is an invertible  $\Lambda$ -module.

*Proof.* The implication (i)  $\Rightarrow$  (ii) is unconditional on Conjecture 2.1 and we leave it to the reader. (It is here where the connectedness of  $\Lambda$  is used.)

To prove that (ii)  $\Rightarrow$  (i), we remark that (ii) implies that the natural evaluation morphism in  $\mathbf{D}(\Lambda)$

$$B_\sigma(M) \otimes B_\sigma(M)^\vee \longrightarrow \Lambda$$

is an isomorphism. Now,  $B_\sigma$  is a monoidal functor and  $M$  is a strongly dualisable motive (because it is assumed to be geometric). This implies that  $B_\sigma(M)^\vee \simeq B_\sigma(M^\vee)$ . This shows that the evaluation map

$$M \otimes M^\vee \longrightarrow \Lambda(0)$$

becomes an isomorphism after applying  $B_\sigma$ . By Conjecture 2.1, we are done.  $\square$

**Remark 2.20** — In Proposition 2.19, it is useful to allow general coefficients rings. Indeed, it is expected that all invertible motives in  $\mathbf{DM}(k; \mathbb{Q})$  are Artin–Tate, i.e., of the form  $U(n)[m]$  where  $U$  is an Artin motive corresponding to a representation of  $\mathrm{Gal}(\bar{k}/k)$  on a 1-dimensional  $\mathbb{Q}$ -vector space. (Of course, such a representation factors through a character  $\mathrm{Gal}(\bar{k}/k) \rightarrow \{\pm 1\}$ .) On the contrary, it is easy to construct invertible motives in  $\mathbf{DM}(k; \overline{\mathbb{Q}})$  which are not Artin–Tate; here is an example. Let  $E$  be an elliptic curve with complex multiplication by an order  $\mathcal{O}$  in an imaginary quadratic extension  $L/\mathbb{Q}$ . The motive  $M_1(E)$  (corresponding to the first singular homology group of  $E^{\mathrm{an}}$ ) decomposes in  $\mathbf{DM}(k; L)$  in two parts

$$M_1(E) = M_1^+(E) \oplus M_1^-(E).$$

The part  $M_1^+(E)$  (resp.  $M_1^-(E)$ ) is characterized by the property that  $\mathcal{O}$  acts on the  $L$ -vector space  $B_\sigma(M_1^+(E))$  (resp.  $B_\sigma(M_1^-(E))$ ) via the embedding  $\mathcal{O} \hookrightarrow L$  (resp. the

composite embedding  $\mathcal{O} \hookrightarrow L \xrightarrow{c} L$  where  $c$  is the complex conjugation). It is easy to show that the motives  $M_1^\pm(E)$  are invertible.

To go further, we recall the following definitions from [34] (see also [3, §9] and [40, Chapter 4]).

**DEFINITION 2.21** — *Let  $\mathcal{C}$  be a  $\mathbb{Q}$ -linear pseudo-abelian tensor category. An object  $A \in \mathcal{C}$  is said to be even (resp. odd) if  $\bigwedge^m(A) = 0$  (resp.  $S^m(A) = 0$ ) for  $m$  large enough. An object  $A \in \mathcal{C}$  is said to be Kimura finite if it decomposes as the direct sum of an even object and an odd object.*

**PROPOSITION 2.22** — *Assume Conjecture 2.1. For simplicity, assume also that  $\Lambda$  is a field extension of  $\mathbb{Q}$ . Let  $M \in \mathbf{DM}_{\text{gm}}(k; \Lambda)$  be a geometric motive. Then the following conditions are equivalent.*

- (a)  $M$  is even (resp. odd);
- (b)  $H_i(B_\sigma(M)) = 0$  for  $i$  odd (resp.  $i$  even).

*In particular, a motive  $M$  is Kimura finite if and only if there exists a decomposition  $M = M_{\text{eve}} \oplus M_{\text{odd}}$  such that the complexes  $B_\sigma(M_{\text{eve}})$  and  $B_\sigma(M_{\text{odd}})$  have their homology in even and odd degrees respectively.*

*Proof.* This is clear. Indeed, a complex of  $\Lambda$ -vectorspaces with finite dimensional homology is even (resp. odd) if and only if its homology is in even (resp. odd) degrees.  $\square$

**Remark 2.23** — A conjecture of Kimura–O’Sullivan states that every Chow motive is Kimura finite. This conjecture does not follow from Conjecture 2.1. However, Proposition 2.22 shows that the conjecture of Kimura–O’Sullivan follows from the combination of Conjecture 2.1 and a weak version of the existence of a Chow–Kunneth decomposition (see [40, Definition 6.1.1]). Using this principle, one can deduce the following corollary.

**COROLLARY 2.24** — *Assume Conjecture 2.1.*

- (i) *The Chow motive of a smooth projective surface is Kimura finite.*
- (ii) *The Chow motive of a smooth hypersurface in a projective space is Kimura finite.*

The property of being Kimura finite is not closed under extensions in a triangulated category. As a consequence, it is easy to produce examples of geometric motives in  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$  which are not Kimura finite. A weaker property which behaves better in a triangulated context is the following one.

**DEFINITION 2.25** — *Let  $\mathcal{C}$  be a  $\mathbb{Q}$ -linear pseudo-abelian tensor category. An object  $A \in \mathcal{C}$  is said to be Schur finite if there exists an integer  $n \in \mathbb{N}$  and a nonzero projector  $\gamma \in \mathbb{Q}[\Sigma_n]$  in the group algebra of the symmetric group  $\Sigma_n$  which acts by zero on  $A^{\otimes n}$ .*

Clearly, a Kimura finite object is also Schur finite. If  $\mathcal{C}$  is a tensor triangulated category obtained as the homotopy category of a stable monoidal model category, we know by [28] that Schur finiteness has the “2 out 3” property in distinguished triangles. In particular, a perfect complex of  $\Lambda$ -modules is a Schur finite object of  $\mathbf{D}(\Lambda)$ . This immediately implies the following.

**PROPOSITION 2.26** — *Assume Conjecture 2.1. Then any object of  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$  is Schur finite.*

We close our list of consequences of the conservativity conjecture with the following statement.

**PROPOSITION 2.27** — *Assume Conjecture 2.1. Let  $K$  be a field endowed with a discrete valuation with residue field  $k$  of characteristic zero. Choose a uniformizer element  $\pi \in K$ . Then the “nearby motive” functor*

$$\Psi_\pi : \mathbf{DM}_{\text{gm}}(K; \mathbb{Q}) \longrightarrow \mathbf{DM}_{\text{gm}}(k; \mathbb{Q})$$

*is conservative.*

*Proof.* The functor  $\Psi_\pi$  is analogous to the formation of the limiting mixed Hodge structure in Hodge theory. We will not recall the construction of  $\Psi_\pi$  here. We direct the reader to [5, Chapitre 3], complemented by [11, Appendice 1.A], where the theory of nearby motivic sheaves is developed; see also [12, §2.3] for a quick recollection.

The idea of the proof of Proposition 2.27 is to use the compatibility of  $\Psi_\pi$  with the Betti realisation functors (see [6, §4]). This is possible when  $K$  is the fraction field of a curve endowed with the valuation associated to a close point. However, for more general  $K$ 's, one runs into some technical problems that are presumably solvable with some effort. Alternatively, one can use the compatibility of  $\Psi_\pi$  with the  $\ell$ -adic realisation functors [8, §10] which holds for general  $K$ 's. Indeed, thanks to the comparison theorem between Betti and  $\ell$ -adic cohomology, Conjecture 2.1 implies that the  $\ell$ -adic realisation functor is also conservative.  $\square$

**2.3. More about conservativity and Kimura finiteness.** In this subsection, we discuss further the relation between the conservativity conjecture and the Kimura finiteness for Chow motives. Our goal is to explain the following theorem.

**THEOREM 2.28** — *Assume that the Standard Conjecture  $D$  is true, i.e., that homological equivalence and numerical equivalence coincide (see [40, Chapter 3]). Assume also that the Kimura–O’Sullivan conjecture is true, i.e., that every Chow motive is Kimura finite. Then Conjecture 2.1 is also true.*

The main ingredient we need is a beautiful construction of Bondarko [19], based on his notion of *weight structure*, and generalising the construction of weight complexes of Gillet–Soulé [27]. Given an additive category  $\mathcal{C}$ , we denote by  $\mathbf{K}^b(\mathcal{C})$  the homotopy category of bounded complexes in  $\mathcal{C}$ .

**PROPOSITION 2.29** — *There exists a triangulated functor*

$$W : \mathbf{DM}_{\text{gm}}(k; \mathbb{Q}) \longrightarrow \mathbf{K}^b(\mathbf{Chow}(k; \mathbb{Q}))$$

*which is conservative and which makes the following triangle commutative (up to a canonical natural isomorphism)*

$$\begin{array}{ccc} \mathbf{Chow}(k; \mathbb{Q}) & \longrightarrow & \mathbf{DM}_{\text{gm}}(k; \mathbb{Q}) \\ & \searrow & \downarrow W \\ & & \mathbf{K}^b(\mathbf{Chow}(k; \mathbb{Q})). \end{array}$$

*Proof.* This is essentially [19, Proposition 6.3.1 and Remark 6.3.2].  $\square$

There is a similar construction for mixed Hodge structures. Let  $\text{CHS}$  denotes the full additive subcategory of  $\mathbf{D}^b(\text{MHS})$  whose objects are the direct sums  $\bigoplus_{n \in \mathbb{Z}} M_n[n]$  with  $M_n$  a polarisable pure Hodge structure of weight  $-n$ . By the decomposition theorem, the mixed Hodge complex associated to a smooth projective variety lies in  $\text{CHS}$ . Therefore, one has a commutative square

$$\begin{array}{ccc} \mathbf{Chow}(k; \mathbb{Q}) & \xrightarrow{R_\sigma} & \text{CHS} \\ \downarrow & & \downarrow \\ \mathbf{DM}_{\text{gm}}(k; \mathbb{Q}) & \xrightarrow{R_\sigma} & \mathbf{D}^b(\text{MHS}). \end{array}$$

From [20, 21], and especially [21, Proposition 2.1.1], one gets the following.

**PROPOSITION 2.30** — *There exists a triangulated functor*

$$W : \mathbf{D}^b(\text{MHS}) \longrightarrow \mathbf{K}^b(\text{CHS}),$$

*which is conservative and which makes the following diagrams commutative (up to canonical natural isomorphisms)*

$$\begin{array}{ccc} \text{CHS} & \longrightarrow & \mathbf{D}^b(\text{MHS}) \\ & \searrow & \downarrow W \\ & & \mathbf{K}^b(\text{CHS}), \end{array} \quad \begin{array}{ccc} \mathbf{DM}_{\text{gm}}(k; \mathbb{Q}) & \xrightarrow{W} & \mathbf{K}^b(\mathbf{Chow}(k; \mathbb{Q})) \\ \downarrow R_\sigma & & \downarrow R_\sigma \\ \mathbf{D}^b(\text{MHS}) & \xrightarrow{W} & \mathbf{K}^b(\text{CHS}). \end{array}$$

As a corollary, Theorem 2.28 follows from the following statement.

**LEMMA 2.31** — *Assume that the Standard Conjecture D is true. Assume also that the Kimura–O’Sullivan conjecture is true. Then the functor*

$$R_\sigma : \mathbf{K}^b(\mathbf{Chow}(k; \mathbb{Q})) \longrightarrow \mathbf{K}^b(\text{CHS})$$

*is conservative.*

*Proof.* Consider the category  $\mathbf{Num}(k; \mathbb{Q})$  of numerical motives (with rational coefficients). By [32],  $\mathbf{Num}(k; \mathbb{Q})$  is abelian semi-simple. The Kimura–O’Sullivan conjecture implies that the natural functor

$$\mathbf{K}^b(\mathbf{Chow}(k; \mathbb{Q})) \longrightarrow \mathbf{K}^b(\mathbf{Num}(k; \mathbb{Q})) = \mathbf{D}^b(\mathbf{Num}(k; \mathbb{Q}))$$

is conservative. Indeed, let  $M = M_\bullet$  be a bounded complex of Chow motives whose image in  $\mathbf{K}^b(\mathbf{Num}(k; \mathbb{Q}))$  is acyclic. We need to show that  $M$  is also acyclic. We argue by induction on the length of  $M$ . (If  $M$  is zero, there is nothing to prove.) Let  $n \in \mathbb{Z}$  be the smallest integer such that  $M_n \neq 0$ . We know that the differential  $\delta : M_{n+1} \rightarrow M_n$  induces a split surjective morphism in  $\mathbf{Num}(k; \mathbb{Q})$ . Therefore, we may find a morphism of Chow motives  $\eta : M_n \rightarrow M_{n+1}$  such that  $\delta \circ \eta - \text{id}_{M_n}$  is numerically equivalent to zero. By [34, Proposition 7.5], this implies that  $\delta \circ \eta - \text{id}_{M_n}$  is a nilpotent endomorphism of  $M_n$ , which in turn implies that  $\delta \circ \eta$  is an automorphism of  $M_n$ . Therefore, replacing  $\eta$  by  $\eta \circ (\delta \circ \eta)^{-1}$ , we may assume that  $\delta \circ \eta = \text{id}_{M_n}$ . This gives a decomposition  $M_{n+1} \simeq M'_{n+1} \oplus M_n$ . Let  $M' = M'_\bullet$  be the complex obtained from  $M$  by replacing the portion  $\{M_{n+1} \rightarrow M_n\}$  by  $\{M'_{n+1} \rightarrow 0\}$ . By construction, there is a map  $M' \rightarrow M$  which is a homotopy equivalence (i.e., an isomorphism in  $\mathbf{K}^b(\mathbf{Chow}(k; \mathbb{Q}))$ ). We now use induction to conclude.



To go further, let PHS be the full subcategory of MHS consisting of pure Hodge structures. There is a functor  $\text{CHS} \rightarrow \text{PHS}$  which takes  $\bigoplus_{n \in \mathbb{Z}} M_n[n]$  to  $\bigoplus_{n \in \mathbb{Z}} M_n$ . (Note that this is not an equivalence of categories!) By the Standard Conjecture D, there is a functor  $\mathbf{Num}(k; \mathbb{Q}) \rightarrow \text{PHS}$  and it is easy to see that the square

$$\begin{array}{ccc} \mathbf{K}^b(\mathbf{Chow}(k; \mathbb{Q})) & \longrightarrow & \mathbf{D}^b(\mathbf{Num}(k; \mathbb{Q})) \\ \downarrow & & \downarrow \\ \mathbf{K}^b(\text{CHS}) & \longrightarrow & \mathbf{D}^b(\text{PHS}) \end{array}$$

commutes up to a natural isomorphism. As  $\mathbf{Num}(k; \mathbb{Q}) \rightarrow \text{PHS}$  is clearly conservative, we are done.  $\square$

**Remark 2.32** — It is very unlikely that Theorem 2.28 is of any use for proving the conservativity conjecture in general. Indeed, proving new cases of the Kimura–O’Sullivan conjecture and/or the Standard Conjecture D by direct means seems quite hopeless and we expect things to go in the other way round. Nevertheless, one can use the known supply of motives for which these conjectures are known to give some evidence for the conservativity conjecture. This gives the following result of Wildeshaus [56, Theorem 1.12].

**COROLLARY 2.33** — *Let  $\mathcal{A} \subset \mathbf{DM}_{\text{gm}}(k; \mathbb{Q})$  be the smallest triangulated subcategory closed under direct summands, tensor product and duality, and containing the motives of smooth projective curves. Then the restriction of the Betti realisation to  $\mathcal{A}$  is conservative.*

*Proof.* Repeat the proof of Theorem 2.28 restricting to  $\mathcal{A}$  and using that the Standard Conjecture D and the Kimura–O’Sullivan conjecture are known for Chow motives which are direct summands of tensor products of motives of smooth projective curves.  $\square$

**Remark 2.34** — At the time of writing, the category  $\mathcal{A}$  is the largest subcategory of  $\mathbf{DM}_{\text{gm}}(k; \mathbb{Q})$  for which the conservativity conjecture is known to hold. Note that  $\mathcal{A}$  contains a lot of objects. For example, the motive of a finite type group-scheme, such as a semi-abelian variety, belongs to  $\mathcal{A}$ . Nevertheless,  $\mathcal{A}$  is still a very small portion of  $\mathbf{DM}_{\text{gm}}(k; \mathbb{Q})$ . To appreciate the gap to be filled, note that an effective polarisable pure Hodge structure of weight 2 with Hodge number  $h^{2,0} \geq 2$  and with generic Mumford–Tate group does not belong to the tannakian subcategory of MHS generated by those Hodge structures of type  $\{(1, 0), (0, 1)\}$ ; this follows from [25, Proposition 7.3]. Nevertheless, such weight 2 Hodge structures abound in algebraic geometry. Indeed, using [25, Proposition 7.5] (see also [2]), it can be shown that they appear as the middle primitive cohomology of general hypersurfaces  $S \subset \mathbb{P}^3$  of degree high enough. In particular, the motive of a general hypersurface  $S \subset \mathbb{P}^3$  of degree high enough is not in  $\mathcal{A}$ . For a more detailed analysis, we refer the reader to [25, §7].

### 3. THE VANISHING CONJECTURE FOR THE MOTIVIC PERIOD ALGEBRA

In this section we discuss a vanishing conjecture concerning the so-called motivic period algebra. This conjecture can be stated in a very elementary way, so we

decided to first give its statement and explain later its motivic origin and how it fits in a broader context.

**3.1. A concrete formulation.** We start by introducing some notation. For  $n \in \mathbb{N}$ , we denote by  $\overline{\mathbb{D}}^n = \overline{\mathbb{D}}^n(o, 1)$  the closed unit polydisc in  $\mathbb{C}^n$ :

$$\overline{\mathbb{D}}^n = \{(t_1, \dots, t_n) \in \mathbb{C}^n; |t_i| \leq 1, \forall 1 \leq i \leq n\}.$$

We will always consider  $\overline{\mathbb{D}}^n$  as a pro-analytic variety in the obvious way, i.e., as being the pro-system  $\{\mathbb{D}^n(o, r)\}_{r>1}$  of open polydiscs in  $\mathbb{C}^n$  of constant polyradius  $r > 1$ . In particular,  $\mathcal{O}(\overline{\mathbb{D}}^n)$  will denote the ring of overconvergent holomorphic functions on  $\overline{\mathbb{D}}^n$ , i.e., of those functions which are given by power series in  $n$  variables converging on  $\mathbb{D}^n(o, r)$  for some  $r > 1$ . Note that  $\mathcal{O}(\overline{\mathbb{D}}^n)$  is a noetherian regular ring.

As before, we fix a complex embedding  $\sigma : k \hookrightarrow \mathbb{C}$ . We consider a “semi-algebraic” version of  $\overline{\mathbb{D}}^n$  (see [9, §2.2.4]).

**DEFINITION 3.1** — *We denote by  $\overline{\mathbb{D}}_{\text{ét}}^n$  the pro- $k$ -variety of étale neighbourhoods of  $\overline{\mathbb{D}}^n$  in  $\mathbb{A}^n$ . More precisely, consider the category  $\mathcal{V}_{\text{ét}}(\overline{\mathbb{D}}^n/\mathbb{A}^n)$  whose objects are pairs  $(U, u)$  consisting of an étale  $\mathbb{A}^n$ -scheme  $U$  and a morphism of pro-analytic spaces  $u : \overline{\mathbb{D}}^n \rightarrow U^{\text{an}}$  making the following triangle commutative*

$$\begin{array}{ccc} \overline{\mathbb{D}}^n & \xrightarrow{u} & U^{\text{an}} \\ & \searrow & \downarrow \\ & & \mathbb{C}^n. \end{array}$$

Then  $\mathcal{V}_{\text{ét}}(\overline{\mathbb{D}}^n/\mathbb{A}^n)$  is a cofiltered category and the pro- $k$ -variety  $\overline{\mathbb{D}}_{\text{ét}}^n$  is given by the forgetful functor

$$\begin{aligned} \overline{\mathbb{D}}_{\text{ét}}^n : \mathcal{V}_{\text{ét}}(\overline{\mathbb{D}}^n/\mathbb{A}^n) &\longrightarrow \text{Sch}/k \\ (U, u) &\longmapsto U. \end{aligned}$$

By construction, we have a pro-étale morphism of pro- $k$ -varieties  $\overline{\mathbb{D}}_{\text{ét}}^n \rightarrow \mathbb{A}^n$ .

Note the following result.

**LEMMA 3.2** — *The pro- $k$ -variety  $\overline{\mathbb{D}}_{\text{ét}}^n$  is affine (i.e., isomorphic to a pro-object in the category of affine  $k$ -varieties) and  $\mathcal{O}(\overline{\mathbb{D}}_{\text{ét}}^n)$  is isomorphic to the sub- $k$ -algebra of  $\mathcal{O}(\overline{\mathbb{D}}^n)$  consisting of those overconvergent power series*

$$f = \sum_{\underline{e} \in \mathbb{N}^n} c_{\underline{e}} \cdot \underline{t}^{\underline{e}} \in \mathcal{O}(\overline{\mathbb{D}}^n)$$

which are algebraic over the field of rational functions  $k(t_1, \dots, t_n)$ .

*Proof.* This is the combination of [9, Propositions 2.58 and 2.102].  $\square$

We set

$$\mathcal{O}(\overline{\mathbb{D}}_{\text{ét}}^{\infty}) = \bigcup_{n \in \mathbb{N}} \mathcal{O}(\overline{\mathbb{D}}_{\text{ét}}^n).$$

(In the union above, we identify  $\mathcal{O}(\overline{\mathbb{D}}_{\text{ét}}^n)$  with a subset of  $\mathcal{O}(\overline{\mathbb{D}}_{\text{ét}}^{n+1})$  by considering a function of the variables  $(t_1, \dots, t_n)$  as a function of the variables  $(t_1, \dots, t_{n+1})$  which is constant with respect to  $t_{n+1}$ .)

Next, we consider differential forms of finite co-degree on  $\overline{\mathbb{D}}_{\text{ét}}^{\infty}$ .

**DEFINITION 3.3** — For  $d \in \mathbb{N}$ , we denote by  $\Omega^{\infty-d}(\overline{\mathbb{D}}_{\text{ét}}^{\infty})$  the  $\mathcal{O}(\overline{\mathbb{D}}_{\text{ét}}^{\infty})$ -module freely generated by symbols  $\widehat{dt}_I$ , one for each finite subset  $I \subset \mathbb{N} \setminus \{0\}$  of cardinality  $d$ :

$$\Omega^{\infty-d}(\overline{\mathbb{D}}_{\text{ét}}^{\infty}) = \bigoplus_{I \subset \mathbb{N} \setminus \{0\}; |I|=d} \mathcal{O}(\overline{\mathbb{D}}_{\text{ét}}^{\infty}) \cdot \widehat{dt}_I.$$

(If  $I = \{i_1 < \dots < i_d\}$ , we think about  $\widehat{dt}_I$  as the differential form of infinite degree

$$dt_1 \wedge \dots \wedge \widehat{dt}_{i_1} \wedge \dots \wedge \widehat{dt}_{i_2} \wedge \dots \wedge \widehat{dt}_{i_d} \wedge \dots$$

where the  $dt_{i_s}$  have been removed for  $1 \leq s \leq d$ .) Elements of  $\Omega^{\infty-d}(\overline{\mathbb{D}}_{\text{ét}}^{\infty})$  are called differential forms of co-degree  $d$  on  $\overline{\mathbb{D}}_{\text{ét}}^{\infty}$ .

**Remark 3.4** — Perhaps, a more natural way to define  $\Omega^{\infty-d}(\overline{\mathbb{D}}_{\text{ét}}^{\infty})$  is as follows. One has a inductive system  $\{\Omega^{n-d}(\overline{\mathbb{D}}_{\text{ét}}^n)\}_{n \geq d}$  with transition maps given by

$$- \wedge dt_{n+1} : \Omega^{n-d}(\overline{\mathbb{D}}_{\text{ét}}^n) \longrightarrow \Omega^{n+1-d}(\overline{\mathbb{D}}_{\text{ét}}^{n+1}).$$

Then  $\Omega^{\infty-d}(\overline{\mathbb{D}}_{\text{ét}}^{\infty})$  is the colimit of this inductive system.

**Notation 3.5** — For  $d \in \mathbb{N}$ , we denote by  $\widetilde{\Omega}^{\infty-d}(\overline{\mathbb{D}}_{\text{ét}}^{\infty})$  the subspace of  $\Omega^{\infty-d}(\overline{\mathbb{D}}_{\text{ét}}^{\infty})$  consisting of those differential forms that vanish at all the faces of  $\overline{\mathbb{D}}_{\text{ét}}^{\infty}$ . More precisely, a differential form of co-degree  $d$  on  $\overline{\mathbb{D}}_{\text{ét}}^{\infty}$

$$\omega = \sum_{I \subset \mathbb{N} \setminus \{0\}; |I|=d} f_I \cdot \widehat{dt}_I$$

belongs to  $\widetilde{\Omega}^{\infty-d}(\overline{\mathbb{D}}_{\text{ét}}^{\infty})$  if and only if  $f_I|_{t_i=c} = 0$  for all  $I \subset \mathbb{N} \setminus \{0\}$  of cardinality  $d$ ,  $i \in I$  and  $c \in \{0, 1\}$ . (For a function  $f$  depending on a complex variable  $t$ , the notation  $f|_{t=c}$  stands for the function obtained from  $f$  by substituting the variable  $t$  by the constant  $c \in \mathbb{C}$ .)

**DEFINITION 3.6** — There is an obvious de Rham differential

$$d : \Omega^{\infty-(d+1)}(\overline{\mathbb{D}}_{\text{ét}}^{\infty}) \longrightarrow \Omega^{\infty-d}(\overline{\mathbb{D}}_{\text{ét}}^{\infty})$$

which takes  $\widetilde{\Omega}^{\infty-(d+1)}(\overline{\mathbb{D}}_{\text{ét}}^{\infty})$  inside  $\widetilde{\Omega}^{\infty-d}(\overline{\mathbb{D}}_{\text{ét}}^{\infty})$ . We thus obtain a complex

$$\mathcal{P}_{\text{mot}}^{\text{eff}}(k, \sigma) := \widetilde{\Omega}^{\infty-\bullet}(\overline{\mathbb{D}}_{\text{ét}}^{\infty}),$$

concentrated in positive homological degrees, which is called the (effective) motivic period algebra (aka., algebra of motivic periods).

**Remark 3.7** — Contrary to what our terminology indicates, the complex  $\mathcal{P}_{\text{mot}}^{\text{eff}}(k, \sigma)$  is not an algebra on the nose. The algebra structure exists only in the derived category  $\mathbf{D}(k)$ . We refer the interested reader to [9, Proposition 2.108] for a description of the algebra product on  $H_0(\mathcal{P}_{\text{mot}}^{\text{eff}}(k, \sigma))$ .

We can now formulate the main conjecture of this section.

**CONJECTURE 3.8** — The motivic period algebra  $\mathcal{P}_{\text{mot}}^{\text{eff}}(k, \sigma)$  has no homology except in degree zero, i.e.,  $H_i(\mathcal{P}_{\text{mot}}^{\text{eff}}(k, \sigma)) = 0$  for  $i \in \mathbb{Z} \setminus \{0\}$ .

**Remark 3.9** — In degree zero, the homology of  $\mathcal{P}_{\text{mot}}^{\text{eff}}(k, \sigma)$  is expected to be very large. Indeed, there exists an “evaluation map”

$$\int_{[0,1]^{\infty}} : H_0(\mathcal{P}_{\text{mot}}^{\text{eff}}(k, \sigma)) \longrightarrow \mathbb{C}$$

which takes the class of a co-degree zero differential form

$$f \cdot \widehat{dt}_0 = f \cdot dt_1 \wedge \cdots,$$

with  $f \in \mathcal{O}(\overline{\mathbb{D}}_{\text{ét}}^n)$ , to the complex number  $\int_{[0,1]^n} f$ . It can be shown that the image of this evaluation map is the ring of effective periods obtained from the relative cohomology of pairs of  $k$ -varieties. When  $k$  is a number field (or if the complex embedding  $\sigma : k \hookrightarrow \mathbb{C}$  is generic), the evaluation map is expected to be injective; this is the famous period conjecture of Kontsevich–Zagier (see [35] and [10]).

**Remark 3.10** — There is a canonical lifting of  $2\pi\mathbf{i} \in \mathbb{C}$  to a class  $\underline{2\pi\mathbf{i}} \in H_0(\mathcal{P}_{\text{mot}}^{\text{eff}}(k; \sigma))$ . This class can be represented by the co-degree zero differential form

$$d \log f \wedge dt_2 \wedge \cdots$$

where  $f \in \mathcal{O}(\overline{\mathbb{D}}_{\text{ét}}^1)^\times$  is an invertible algebraic function in the variable  $t_1$  such that  $f(0) = f(1) = 1$  and  $f|_{[0,1]} : [0,1] \rightarrow \mathbb{C}^\times$  is counterclockwise simple loop around zero. (Note that the plane  $\mathbb{C}$  is oriented by the choice of  $\mathbf{i}$ .) We set

$$\mathcal{P}_{\text{mot}}(k, \sigma) := \mathcal{P}_{\text{mot}}^{\text{eff}}(k, \sigma)[(\underline{2\pi\mathbf{i}})^{-1}];$$

this is the (non-effective) *motivic period algebra* (aka., *algebra of motivic periods*).

For later references, we state the non-effective version of Conjecture 3.8.

**CONJECTURE 3.11** — *The non-effective motivic period algebra  $\mathcal{P}_{\text{mot}}(k, \sigma)$  has no homology except in degree zero, i.e.,  $H_i(\mathcal{P}_{\text{mot}}(k, \sigma)) = 0$  for  $i \in \mathbb{Z} \setminus \{0\}$ .*

The following statement is obvious.

**LEMMA 3.12** — *Conjecture 3.8 implies Conjecture 3.11.*

**3.2. The significance of Conjecture 3.8.** In this subsection, we explain the origin and significance of Conjecture 3.8. We start by recalling the weak tannakian formalism as developed in [9, §1].

**THEOREM 3.13** — *Let  $f : \mathcal{M} \rightarrow \mathcal{E}$  be a monoidal functor between two monoidal categories. (Everything is symmetric and unitary.) Assume the following conditions.*

- (i)  $f$  has a monoidal section  $e : \mathcal{E} \rightarrow \mathcal{M}$ , i.e.,  $f \circ e \simeq \text{id}_{\mathcal{E}}$ .
- (ii)  $f$  has a right adjoint  $g$  and  $e$  has a right adjoint  $u$ .
- (iii) The natural coprojection morphism

$$g(A') \otimes B \rightarrow g(A' \otimes f(B))$$

is an isomorphism for all  $A' \in \mathcal{E}$  and  $B \in \mathcal{M}$ .

Then, we have the following conclusions.

- (A)  $H = f(g(\mathbf{1}_{\mathcal{E}}))$  has a natural structure of a Hopf algebra in  $\mathcal{E}$ .
- (B) For every object  $M \in \mathcal{M}$ ,  $f(M)$  is naturally a left  $H$ -comodule. This gives a monoidal functor

$$\tilde{f} : \mathcal{M} \rightarrow \text{coMod}(H)$$

making the following triangle commutative

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\tilde{f}} & \text{coMod}(H) \\ & \searrow f & \downarrow \\ & & \mathcal{E}. \end{array}$$

- (C)  $H$  is the universal Hopf algebra satisfying (B).

As explained in [9, §2.1.3], Theorem 3.13 can be applied to the Betti realisation. Starting from now, we will denote by

$$B_\sigma^* : \mathbf{DM}(k; \Lambda) \longrightarrow \mathbf{D}(\Lambda),$$

instead of  $B_\sigma$ , the Betti realisation functor. As  $\mathbf{DM}(k; \Lambda)$  is compactly generated and  $B_\sigma^*$  commutes with infinite sums, the Brown representability theorem implies that  $B_\sigma^*$  has a right adjoint  $B_{\sigma,*}$  (see [41, Theorem 4.1]). On the other hand, the ‘‘Artin motive’’ functor

$$\mathbf{D}(\Lambda) \longrightarrow \mathbf{DM}(k; \Lambda)$$

is a monoidal section to  $B_\sigma^*$  and admits a right adjoint, again by the Brown representability theorem. Condition (iii) is also satisfied; it is an easy consequence of rigidity, in the monoidal sense (see [9, Lemme 2.8] and the proof of [9, Proposition 2.7]).

**DEFINITION 3.14** — *The Hopf algebra*

$$\mathcal{H}_{\text{mot}}(k, \sigma; \Lambda) := B_\sigma^* B_{\sigma,*} \Lambda$$

is called the motivic Hopf algebra of  $k$  (associated to  $\sigma : k \hookrightarrow \mathbb{C}$ ). Note that this is a Hopf algebra in  $\mathbf{D}(\Lambda)$ .

**Remark 3.15** — When  $\Lambda = \mathbb{Q}$ , we simply write  $\mathcal{H}_{\text{mot}}(k, \sigma)$  instead of  $\mathcal{H}_{\text{mot}}(k, \sigma; \mathbb{Q})$ . In fact, one gains nothing by allowing more general rings of coefficients as

$$\mathcal{H}_{\text{mot}}(k, \sigma; \Lambda) \simeq \mathcal{H}_{\text{mot}}(k, \sigma) \otimes \Lambda.$$

**Remark 3.16** — The motivic Hopf algebra co-acts on the singular homology of motives. More precisely, given a motive  $M \in \mathbf{DM}(k; \mathbb{Q})$ ,  $B_\sigma^*(M)$  is naturally a comodule over  $\mathcal{H}_{\text{mot}}(k, \sigma)$ . This co-action contains a lot of arithmetico-geometric information. Indeed, it determines the mixed Hodge structure on the Betti homology of  $M$  and the Galois action on the  $\ell$ -adic homology of  $M$ .

**CONJECTURE 3.17** — *The Hopf algebra  $\mathcal{H}_{\text{mot}}(k; \sigma)$  has no homology except in degree zero.*

**Remark 3.18** — According to Conjecture 3.17, the motivic Hopf algebra is the coordinate ring of an honest affine group-scheme. This group-scheme is the so-called *motivic Galois group* of  $k$  (associated to  $\sigma : k \hookrightarrow \mathbb{C}$ ).

There is a nice interpretation of Conjecture 3.17 in terms of operations on homology. We first make a definition.

**DEFINITION 3.19** — *Assume for simplicity that the  $\mathbb{Q}$ -algebra  $\Lambda$  is a field. An operation of degree  $d \in \mathbb{N}$  is a  $\Lambda$ -linear transformation*

$$H^0(B_\sigma^*(M)) \longrightarrow H^d(B_\sigma^*(M))$$

which is natural in  $M \in \mathbf{DM}(k; \Lambda)$ .

There are plenty of operations of degree zero. For instance, when  $\Lambda = \mathbb{Q}_\ell$ , every element of  $\text{Gal}(\bar{k}/k)$  induces an operation of degree zero. (Here  $\bar{k}$  is the algebraic closure of  $\sigma(k)$  in  $\mathbb{C}$ .) On the contrary, for operations of nonzero degrees, we conjecture the following.

**CONJECTURE 3.20** — *There are no nonzero operations of nonzero degree.*

**PROPOSITION 3.21** — *Conjecture 3.17 and Conjecture 3.20 are equivalent.*

*Proof.* Indeed, by [9, Proposition 1.32], an operation  $\gamma$  of degree  $d$  corresponds to a linear form on  $\mathcal{H}_{\text{mot}}(k, \sigma; \Lambda)$  of degree  $d$ , i.e., to a map

$$\ell : H_d(\mathcal{H}_{\text{mot}}(k, \sigma; \Lambda)) \longrightarrow \Lambda.$$

The correspondence works as follows. Given a linear form  $\ell$ , the corresponding operation is given by the composition of

$$\begin{array}{ccc} H^0(B_\sigma^*(M)) & \xrightarrow{\text{ca}} & H^0(\mathcal{H}_{\text{mot}}(k, \sigma; \Lambda) \otimes B_\sigma^*(M)) \\ & & \downarrow \\ & & H_d(\mathcal{H}_{\text{mot}}(k, \sigma; \Lambda)) \otimes H^d(B_\sigma^*(M)) \xrightarrow{\ell \otimes \text{id}} H^d(B_\sigma^*(M)). \end{array}$$

(In the above diagram, ca denotes the co-action map of  $\mathcal{H}_{\text{mot}}(k, \sigma; \Lambda)$  on  $B_\sigma^*(M)$ .) Conversely, given an operation  $\lambda$ , the corresponding linear form is given by the composition of

$$H^{-d}(B_\sigma^* B_{\sigma, *} \Lambda) \xrightarrow{\lambda} H^0(B_\sigma^* B_{\sigma, *} \Lambda) \xrightarrow{\text{cu}} \Lambda.$$

(In the above diagram, cu denotes the co-unit map of  $\mathcal{H}_{\text{mot}}(k, \sigma; \Lambda)$ .) The claimed result is now clear.  $\square$

The link with the vanishing conjectures of Subsection 3.1 follows from the following result.

**PROPOSITION 3.22** — *There is a natural quasi-isomorphism of complexes of  $\mathbb{C}$ -vectorspaces:*

$$\mathcal{H}_{\text{mot}}(k, \sigma) \otimes \mathbb{C} \simeq \mathcal{P}_{\text{mot}}(k, \sigma) \otimes_{k, \sigma} \mathbb{C}.$$

*Proof.* The Grothendieck comparison theorem between singular cohomology and algebraic de Rham cohomology can be stated in a compact form as an isomorphism in  $\mathbf{DM}(k; \mathbb{Q})$ :

$$(B_{\sigma, *} \mathbb{Q}) \otimes \mathbb{C} \simeq \mathbf{\Omega}_{/k} \otimes_{k, \sigma} \mathbb{C},$$

where  $\mathbf{\Omega}_{/k}$  is an object representing algebraic de Rham cohomology and constructed using the usual algebraic de Rham complexes of smooth  $k$ -varieties. This isomorphism induces an isomorphism in  $\mathbf{D}(\mathbb{Q})$

$$(B_\sigma^* B_{\sigma, *} \mathbb{Q}) \otimes \mathbb{C} \simeq (B_\sigma^* \mathbf{\Omega}_{/k}) \otimes_{k, \sigma} \mathbb{C}.$$

The left-hand side is  $\mathcal{H}_{\text{mot}}(k, \sigma) \otimes \mathbb{C}$  by construction and it remains to identify  $B_\sigma^* \mathbf{\Omega}_{/k}$  with  $\mathcal{P}_{\text{mot}}(k, \sigma)$ . This is a computation which can be found in [9, §2.3.1]; it relies on an approximation theorem for singular chains [9, Théorème 2.61] whose proof relies on Popescu's theorem [44, 45].  $\square$

**COROLLARY 3.23** — *Conjecture 3.11 and Conjecture 3.17 are equivalent.*

**Remark 3.24** — As  $H_d(\mathcal{P}_{\text{mot}}(k, \sigma)) = 0$  for  $d < 0$ , we deduce from Proposition 3.22 that  $H_d(\mathcal{H}_{\text{mot}}(k, \sigma)) = 0$  for  $d < 0$ . In particular, there are no nonzero operations of strictly negative degrees.

**Remark 3.25** — There exists an effective version of the motivic Hopf algebra which is denoted by  $\mathcal{H}_{\text{mot}}^{\text{eff}}(k, \sigma)$ . This is only a bialgebra, i.e., it does not have an antipode. It is defined by the same method while restricting to effective motives. More precisely, one considers the restriction  $B_\sigma^{\text{eff}, *}$  of the Betti realisation to the subcategory  $\mathbf{DM}^{\text{eff}}(k; \mathbb{Q})$  of effective motives (see Definition 4.1 below). This functor

has a right adjoint that we denote by  $B_{\sigma,*}^{\text{eff}}$ . The effective motivic bialgebra is given by

$$\mathcal{H}_{\text{mot}}^{\text{eff}}(k, \sigma) := B_{\sigma}^{\text{eff},*} B_{\sigma,*}^{\text{eff}} \mathbb{Q}.$$

It can be shown that  $\mathcal{H}_{\text{mot}}(k, \sigma) \simeq \mathcal{H}_{\text{mot}}^{\text{eff}}(k, \sigma)[\varsigma^{-1}]$  for a canonical element  $\varsigma \in H_0(\mathcal{H}_{\text{mot}}^{\text{eff}}(k, \sigma))$ . Moreover, as in Proposition 3.22, there exists a quasi-isomorphism of complexes of  $\mathbb{C}$ -vectorspaces

$$\mathcal{H}_{\text{mot}}^{\text{eff}}(k, \sigma) \otimes \mathbb{C} \simeq \mathcal{P}_{\text{mot}}^{\text{eff}}(k, \sigma) \otimes_{k, \sigma} \mathbb{C}.$$

(Under this isomorphism, the element  $\varsigma \otimes 1$  corresponds to  $\underline{2\pi i} \otimes (2\pi i)^{-1}$ .) In particular, this shows that Conjecture 3.8 is equivalent to the effective version of Conjecture 3.17, i.e., to the property that  $\mathcal{H}_{\text{mot}}^{\text{eff}}(k, \sigma)$  has no homology except in degree zero.

**3.3. Relation with the conjectural motivic  $t$ -structure.** Conjecture 3.17 is intimately related to the problem of constructing the so-called motivic  $t$ -structure. The existence of the motivic  $t$ -structure is one of the most central open problems in the theory of motives.

**CONJECTURE 3.26** — *Given a complex embedding  $\sigma : k \hookrightarrow \mathbb{C}$ , let  $\mathcal{T}_{\geq 0}^{\mathcal{M}}$  be the full subcategory of  $\mathbf{DM}(k; \Lambda)$  consisting of those motives  $M$  such that  $B_{\sigma}^*(M) \in \mathbf{D}_{\geq 0}(\Lambda)$ . Define  $\mathcal{T}_{< 0}^{\mathcal{M}}$  to be the right orthogonal to  $\mathcal{T}_{\geq 0}^{\mathcal{M}}$ , i.e., the full subcategory whose objects are the  $N \in \mathbf{DM}(k; \Lambda)$  such that*

$$\text{hom}_{\mathbf{DM}(k; \Lambda)}(M, N) = 0$$

for all  $M \in \mathcal{T}_{\geq 0}^{\mathcal{M}}$ . Then the following properties should hold.

- (i) *The pair  $(\mathcal{T}_{\geq 0}^{\mathcal{M}}, \mathcal{T}_{< 0}^{\mathcal{M}})$  is a  $t$ -structure which is independent from the choice of the complex embedding  $\sigma$ .*
- (ii) *The Betti realisation  $B_{\sigma}^* : \mathbf{DM}(k; \Lambda) \rightarrow \mathbf{D}(\Lambda)$  is  $t$ -exact. This is equivalent to saying that  $B_{\sigma}^*$  takes a motive in  $\mathcal{T}_{< 0}^{\mathcal{M}}$  to a complex in  $\mathbf{D}_{< 0}(\Lambda)$ .*
- (iii) *Assuming that  $\Lambda$  is a regular ring,<sup>11</sup> the  $t$ -structure  $(\mathcal{T}_{\geq 0}^{\mathcal{M}}, \mathcal{T}_{< 0}^{\mathcal{M}})$  restricts to  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$ , i.e., the pair*

$$(\mathcal{T}_{\geq 0}^{\mathcal{M}} \cap \mathbf{DM}_{\text{gm}}(k; \Lambda), \mathcal{T}_{< 0}^{\mathcal{M}} \cap \mathbf{DM}_{\text{gm}}(k; \Lambda))$$

*is a  $t$ -structure on  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$ . Said differently, the truncation functors associated to the  $t$ -structure  $(\mathcal{T}_{\geq 0}^{\mathcal{M}}, \mathcal{T}_{< 0}^{\mathcal{M}})$  preserve geometric motives.*

If it exists, the  $t$ -structure of Conjecture 3.26 is called the *motivic  $t$ -structure*.

**Remark 3.27** — It is not difficult to construct a  $t$ -structure on  $\mathbf{DM}(k; \Lambda)$  which, if Conjecture 3.26 was true, coincides with the motivic  $t$ -structure, at least after restricting to  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$  (under the assumption that  $\Lambda$  is regular). Indeed, let  $\mathcal{T}_{\geq 0}$  be the smallest subcategory of  $\mathbf{DM}(k; \Lambda)$  closed under infinite sums, extensions and suspensions, and containing the geometric motives whose Betti realisations belong to  $\mathbf{D}_{\geq 0}(\Lambda)$ . Also, let  $\mathcal{T}_{< 0}$  be the right orthogonal to  $\mathcal{T}_{\geq 0}$ . Then, by a general argument (see for example [4, Proposition 2.1.70]),  $(\mathcal{T}_{\geq 0}, \mathcal{T}_{< 0})$  is a  $t$ -structure, which is the alluded candidate for the motivic  $t$ -structure. Unfortunately, with such a definition, it is totally unclear how to establish properties (ii) and (iii) of Conjecture 3.26.

<sup>11</sup>This assumption is necessary. To see this, note that, at least when  $\Lambda$  is noetherian, the canonical  $t$ -structure on  $\mathbf{D}(\Lambda)$  restricts to a  $t$ -structure on the subcategory of perfect complexes if and only if  $\Lambda$  is regular.

**Remark 3.28** — If  $k$  has infinite transcendence degree over its prime field, the motivic  $t$ -structure on  $\mathbf{DM}(k; \Lambda)$  degenerates. Indeed, from the definition,  $\mathcal{T}_{\geq 0}^{\mathcal{M}}$  contains the triangulated subcategory  $\mathcal{F}ant$  of phantom motives (see Definition 2.6). However, under the assumption that  $\Lambda$  is regular, the conservativity conjecture (i.e., Conjecture 2.1) implies that the restriction of the motivic  $t$ -structure to  $\mathbf{DM}_{\text{gm}}(k; \Lambda)$  is non-degenerate. Indeed, if

$$M \in \mathbf{DM}_{\text{gm}}(k; \Lambda) \cap \left( \bigcap_{n \in \mathbb{N}} \mathcal{T}_{\geq n}^{\mathcal{M}} \right),$$

then necessarily  $B_{\sigma}^*(M) = 0$ , and thus  $M = 0$ .

**PROPOSITION 3.29** — *Conjecture 3.26 implies Conjecture 3.17.*

*Proof.* By Remark 3.24, we know that  $\mathcal{H}_{\text{mot}}(k, \sigma) \in \mathbf{D}_{\geq 0}(\mathbb{Q})$ . Therefore, we only need to show that  $\mathcal{H}_{\text{mot}}(k, \sigma) \in \mathbf{D}_{\leq 0}(\mathbb{Q})$ .

Now, assuming Conjecture 3.26, we have at our disposal a  $t$ -structure on  $\mathbf{DM}(k; \mathbb{Q})$  for which  $B_{\sigma}^*$  is  $t$ -exact. It follows, by a general argument, that the right adjoint  $B_{\sigma, *}$  is left  $t$ -exact, i.e., takes  $\mathbf{D}_{\leq 0}(\mathbb{Q})$  to  $\mathcal{T}_{\leq 0}^{\mathcal{M}}$ . In particular, we have

$$B_{\sigma, *} \mathbb{Q} \in \mathcal{T}_{\leq 0}^{\mathcal{M}}.$$

Using again that  $B_{\sigma}^*$  is  $t$ -exact, we deduce that

$$B_{\sigma}^* B_{\sigma, *} \mathbb{Q} \in \mathbf{D}_{\leq 0}(\mathbb{Q})$$

as needed. □

**Remark 3.30** — As far as we know, the vanishing conjecture for the motivic Hopf algebra (i.e., Conjecture 3.17) does not imply any longstanding conjecture on algebraic cycles, as was the case for the conservativity conjecture (i.e., Conjecture 2.1). Nonetheless, we believe that this is an important conjecture for at least two reasons. First, as explained above, the validity of Conjecture 3.17 is a necessary condition for the existence of the motivic  $t$ -structure. But more importantly, it is our strong believe that Conjecture 3.17 should constitute a crucial step in a future construction of the motivic  $t$ -structure. Indeed, in [9, §2.4], we stated a conjecture, [9, Conjecture B on page 127], which, if true, enables one to reconstruct a geometric motive from its Betti realisation using a descent procedure. We believe that Conjecture 3.26 follows, modulo some routine work, from the combination of [9, §2.4, Conjecture B] and Conjecture 3.17 (which is also [9, §2.4, Conjecture A]).

#### 4. HOMOTOPY $t$ -STRUCTURE, SLICE FILTRATION AND DIMENSION

In this section, we will formulate some plausible conjectures around the homotopy  $t$ -structure and the slice filtration, two standard tools in the theory of motives à la Voevodsky. Contrary to the ones stated previously, these conjectures are not part of the classical motivic paradise and rather originate from unexpected features of Voevodsky's construction of his category of motives. We will start by recalling the basic notion of effectiveness for motives.



**4.1. Effective motives, homotopy  $t$ -structure and slice filtration.** Recall that a mixed Hodge structure is said to be *effective* if its Hodge numbers  $h^{p,q}$  are zero unless both  $p$  and  $q$  are non-negative. Unfortunately, for motives, the notion of effectivity (see Definition 4.1 below) corresponds to the dual one: the dual of the Hodge realisation of a geometric effective motive belongs to the subcategory of  $\mathbf{D}^b(\text{MHS})$  generated by effective mixed Hodge structures.

**DEFINITION 4.1** — We denote by  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$  the smallest triangulated subcategory of  $\mathbf{DM}(k; \Lambda)$  closed under infinite sums and containing the homological motives  $M(X)$  for all smooth quasi-projective  $k$ -varieties  $X$ . (The crucial point here is that we do not allow negative Tate twists of the  $M(X)$ 's.) A motive is said to be effective if it lies in  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$ .

We also denote by  $\mathbf{DM}_{\text{gm}}^{\text{eff}}(k; \Lambda)$  the subcategory of compact objects of  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$ . This is the category of effective geometric motives.

**Remark 4.2** — Almost by definition, every geometric motive  $M \in \mathbf{DM}_{\text{gm}}(k; \Lambda)$  can be made effective after applying a sufficiently positive Tate twist. This is analogous (or rather dual) to the fact that every mixed Hodge structure becomes effective after a sufficiently negative Tate twist.

**Remark 4.3** — One interesting feature of the category  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$  is that it admits a more concrete description than the larger  $\mathbf{DM}(k; \Lambda)$ . Indeed, there is a fully faithful embedding

$$\mathbf{DM}^{\text{eff}}(k; \Lambda) \hookrightarrow \mathbf{D}(\mathbf{Str}_{\text{Nis}}(\text{Sm}/k; \Lambda))$$

into the derived category of the abelian category  $\mathbf{Str}_{\text{Nis}}(\text{Sm}/k; \Lambda)$  of Nisnevich sheaves with transfers on  $\text{Sm}/k$ . (Recall that a Nisnevich sheaf with transfers is a  $\Lambda$ -linear functor from  $\text{SmCor}(k; \Lambda)$  (see Definition 1.14) to the category of  $\Lambda$ -modules whose restriction to  $\text{Sm}/k$  is a sheaf for the Nisnevich topology.) Moreover, a complex of Nisnevich sheaves with transfers is in the image of this embedding if and only if its homology sheaves are homotopy invariant. (Recall that a Nisnevich sheaf  $F$  on  $\text{Sm}/k$  is homotopy invariant if the map  $F(X) \rightarrow F(\mathbb{A}^1 \times X)$ , induced by the natural projection to  $X$ , is an isomorphism for every  $X \in \text{Sm}/k$ .) As a consequence, one obtains the following result.

**LEMMA 4.4** — The canonical  $t$ -structure on  $\mathbf{D}(\mathbf{Str}_{\text{Nis}}(\text{Sm}/k; \Lambda))$  restricts to a  $t$ -structure on  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$  called the homotopy  $t$ -structure. The heart of the homotopy  $t$ -structure is equivalent to the category of homotopy invariant sheaves with transfers that we denote by  $\mathbf{HI}(k; \Lambda)$ . (Here and in the rest of the article, the word “sheaf” will always refer to the Nisnevich topology.)

**Remark 4.5** — It is possible to extend the homotopy  $t$ -structure to the full triangulated category of motives  $\mathbf{DM}(k; \Lambda)$ . The heart of this extended homotopy  $t$ -structure is more complicated to describe. Its objects are *Tate spectra*<sup>12</sup> with some properties. We direct the interested reader to [23] where the heart of the homotopy  $t$ -structure on  $\mathbf{DM}(k; \mathbb{Z})$  is described and shown to be equivalent to the category of Rost modules [47].

**Remark 4.6** — It should be noted that the homotopy  $t$ -structure is *not* the motivic  $t$ -structure of Conjecture 3.26. In fact, these two  $t$ -structures have nothing

<sup>12</sup>The notion of Tate spectra is analogous to the notion of spectra in stable homotopy theory, see for example [1].

in common. The homotopy  $t$ -structure does not correspond under realisations to any classical construction. Although, the Betti realisation

$$B_\sigma^{\text{eff},*} : \mathbf{DM}^{\text{eff}}(k; \Lambda) \longrightarrow \mathbf{D}(\Lambda)$$

is right  $t$ -exact when  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$  is endowed with the homotopy  $t$ -structure, it is very far from being left  $t$ -exact. Moreover, the homotopy  $t$ -structure does not restrict to  $\mathbf{DM}_{\text{gm}}^{\text{eff}}(k; \Lambda)$ . Indeed, the truncation functors with respect to the homotopy  $t$ -structure do not preserve geometric effective motives.

For later use, we introduce the following notation.

**Notation 4.7** — Given an effective motive  $M \in \mathbf{DM}^{\text{eff}}(k; \Lambda)$ , we denote by  $h_n(M)$  the homology objects associated to  $M$  with respect to the homotopy  $t$ -structure. These are homotopy invariant sheaves with transfers. As a Nisnevich sheaf,  $h_n(M)$  coincides with the sheafification of the following presheaf

$$U \in \text{Sm}/k \quad \mapsto \quad \text{hom}_{\mathbf{DM}^{\text{eff}}(k; \Lambda)}(M(U)[n], M).$$

Given a smooth  $k$ -variety  $X$ , we write  $h_n(X)$  instead of  $h_n(M(X))$ . Note that  $h_n(X) = 0$  for  $n < 0$  and that the group of global sections of  $h_n(X)$  coincides with the Suslin homology group  $H_n(X; \Lambda(0))$  (see Definition 1.12).

We will also need the slice filtration on motives (compare with [51]).

**DEFINITION 4.8** — *By the Brown representability theorem, the inclusion of  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$  into  $\mathbf{DM}(k; \Lambda)$  admits a right adjoint*

$$\nu^{\geq 0} : \mathbf{DM}(k; \Lambda) \longrightarrow \mathbf{DM}^{\text{eff}}(k; \Lambda).$$

For  $n \in \mathbb{Z}$ , we set

$$\nu^{\geq n}(M) := \nu^{\geq 0}(M(-n))(n).$$

(Note that  $\nu^{\geq n}$  takes values in the subcategory  $\mathbf{DM}^{\text{eff}}(k; \Lambda)(n)$  consisting of motives whose  $(-n)$ -th Tate twists are effective.) There are natural transformations

$$\dots \longrightarrow \nu^{\geq n} \longrightarrow \nu^{\geq n-1} \longrightarrow \dots$$

The system  $\{\nu^{\geq n}\}_{n \in \mathbb{N}}$  is called the slice filtration.

**Remark 4.9** — The slice filtration does not correspond via realisations to any classical construction. In fact, the functors  $\nu^{\geq n}$  are not well-behaved; see [30] for some conditional and unconditional negative properties. In particular, it is known that geometric motives are not preserved by these functors.

For later use, we record the following simple lemma.

**LEMMA 4.10** — *The triangulated category  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$  is monoidal and closed. Given two effective motives  $M$  and  $N$ , we denote by  $\underline{\text{Hom}}^{\text{eff}}(M, N)$  the internal hom from  $M$  to  $N$  computed in  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$ . We then have the formula*

$$\underline{\text{Hom}}^{\text{eff}}(M, N) = \nu^{\geq 0} \underline{\text{Hom}}(M, N).$$

*Proof.* The fact that the tensor product of two effective motives is also effective follows from the formula

$$M(X) \otimes M(Y) \simeq M(X \times Y)$$

for all  $X, Y \in \text{Sm}/k$ . To prove the formula for  $\underline{\text{Hom}}^{\text{eff}}(M, N)$ , it is enough to show that  $\nu^{\geq 0}\underline{\text{Hom}}(M, -)$  is right adjoint to  $M \otimes -$ . Given an effective motive  $L$ , we have the identifications

$$\begin{aligned} \text{hom}_{\mathbf{DM}^{\text{eff}}(k; \Lambda)}(L, \nu^{\geq 0}\underline{\text{Hom}}(M, N)) &\simeq \text{hom}_{\mathbf{DM}(k; \Lambda)}(L, \underline{\text{Hom}}(M, N)) \\ &\simeq \text{hom}_{\mathbf{DM}(k; \Lambda)}(M \otimes L, N) \\ &= \text{hom}_{\mathbf{DM}^{\text{eff}}(k; \Lambda)}(M \otimes L, N). \end{aligned}$$

This proves what we want.  $\square$

**PROPOSITION 4.11** — *Let  $n \geq 0$ . For  $M \in \mathbf{DM}^{\text{eff}}(k; \Lambda)$ , there is a natural isomorphism*

$$\nu^{\geq 0}(M(-n)) \simeq \underline{\text{Hom}}^{\text{eff}}(\Lambda(n), M).$$

*Proof.* Use that  $M(-n) = \underline{\text{Hom}}(\Lambda(n), M)$ .  $\square$

**Remark 4.12** — Using a similar reasoning, one sees that

$$\mathbf{B}_{\sigma, *}^{\text{eff}} \Lambda \simeq \nu^{\geq 0}(\mathbf{B}_{\sigma, *} \Lambda).$$

The Betti realisation of the motive  $\mathbf{B}_{\sigma, *}^{\text{eff}} \Lambda$  is the motivic bialgebra  $\mathcal{H}_{\text{mot}}^{\text{eff}}(k, \sigma; \Lambda)$ ; see Remark 3.25.

**4.2. Filtration by dimension.** In order to formulate our conjectures, we introduce the following subcategories.

**DEFINITION 4.13** — *For  $n \in \mathbb{N}$ , we denote by  $\mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)$  the smallest subcategory of  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$  closed under infinite sums and containing the motives  $M(X)$  for all smooth quasi-projective  $k$ -varieties  $X$  of dimension at most  $n$ .*

**Remark 4.14** — The categories  $\mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)$  are well understood for  $n \in \{0, 1\}$ . Indeed,  $\mathbf{DM}_{\leq 0}^{\text{eff}}(k; \Lambda)$  is equivalent to the derived category of continuous representations of  $\text{Gal}(\bar{k}/k)$  with coefficients in  $\Lambda$  (endowed with the discrete topology). Equivalently, there is an equivalence of categories

$$\mathbf{D}(\mathbf{Shv}_{\text{ét}}(\text{Et}/k; \Lambda)) \simeq \mathbf{DM}_{\leq 0}^{\text{eff}}(k; \Lambda)$$

where  $\text{Et}/k$  is the small étale site of  $\text{Spec}(k)$ . The case  $n = 1$  is also related to classical objects, namely Deligne 1-motives. Indeed, there is an equivalence of categories

$$\mathbf{D}(\text{Ind-}\mathcal{M}_1(k; \mathbb{Q})) \simeq \mathbf{DM}_{\leq 1}^{\text{eff}}(k; \mathbb{Q})$$

where  $\mathcal{M}_1(k; \mathbb{Q})$  is the abelian category of Deligne 1-motives up to isogeny. (In the geometric case, this equivalence is proven in [43].) It is possible to see in these statements some evidence, rather meagre admittedly, for the existence of the motivic  $t$ -structure (see Conjecture 3.26).

We note the following fact which shows that the situation is really very nice when  $n \in \{0, 1\}$ .

**PROPOSITION 4.15** — *For  $n \in \{0, 1\}$ , the inclusion of  $\mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)$  into  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$  admits a left adjoint. When  $n = 0$ , this left adjoint is denoted by*

$$\mathbf{L}\pi_0 : \mathbf{DM}^{\text{eff}}(k; \Lambda) \longrightarrow \mathbf{DM}_{\leq 0}^{\text{eff}}(k; \Lambda).$$

*When  $n = 1$ , this left adjoint is denoted by*

$$\mathbf{L}\text{Alb} : \mathbf{DM}^{\text{eff}}(k; \Lambda) \longrightarrow \mathbf{DM}_{\leq 1}^{\text{eff}}(k; \Lambda).$$

*Proof.* This is proven in [13]. See also [14] where the geometric case is treated using a different approach.  $\square$

**Remark 4.16** — For a geometric effective motive  $M \in \mathbf{DM}_{\text{gm}}^{\text{eff}}(k; \Lambda)$ , one has the following formula

$$\text{LAlb}(M) = \underline{\text{Hom}}^{\text{eff}}(\underline{\text{Hom}}^{\text{eff}}(M, \Lambda(1)), \Lambda(1)).$$

This formula was taken as a definition in [14, §5].

**Remark 4.17** — Contrary to right adjoints, the existence of left adjoints is often a nontrivial property. In fact, the functors  $L\pi_0$  and  $\text{LAlb}$  are very nice and they do correspond to classical constructions in Hodge theory. For instance, as the notation indicates,  $\text{LAlb}$  is somehow a “derived” version of the Albanese variety.

**Remark 4.18** — The situation becomes quickly extremely complicated starting from  $n = 2$ . Indeed, there is no known description of  $\mathbf{DM}_{\leq 2}^{\text{eff}}(k; \Lambda)$  using classical objects. Conjecturally,  $\mathbf{DM}_{\leq 2}^{\text{eff}}(k; \Lambda)$  should be the derived category of the abelian category of mixed 2-motives whose existence is out of reach (but see [7] for an attempt). There are also negative results. For instance, it is shown in [13, §2.5] that the inclusion of  $\mathbf{DM}_{\leq 2}^{\text{eff}}(k; \mathbb{Q})$  into  $\mathbf{DM}^{\text{eff}}(k; \mathbb{Q})$  cannot have a left adjoint (at least if  $k$  has infinite transcendence degree). Nevertheless, we will offer in Subsection 4.3 below two positive conjectures shedding light on the possibility of some beautiful structures in the theory of algebraic cycles.

We will also need the following related notion. (Recall Notation 4.7 for the significance of  $h_0(X)$ .)

**DEFINITION 4.19** — Let  $F$  be an object of  $\mathbf{HI}(k; \Lambda)$ , i.e., a homotopy invariant sheaf with transfers. We say that  $F$  is  $n$ -generated if there exists a surjection in  $\mathbf{HI}(k; \Lambda)$ :

$$\bigoplus_{i \in I} h_0(X_i) \twoheadrightarrow F$$

where the  $X_i$ 's have dimensions at most  $n$ . We say that  $F$  is  $n$ -presented if there exists an exact sequence in  $\mathbf{HI}(k; \Lambda)$ :

$$\bigoplus_{j \in J} h_0(Y_j) \longrightarrow \bigoplus_{i \in I} h_0(X_i) \longrightarrow F \longrightarrow 0$$

where the  $X_i$ 's and  $Y_j$ 's have dimensions at most  $n$ . (Above, we do not assume that  $I$  nor  $J$  is finite.) We denote by  $\mathbf{HI}_{\leq n}(k; \Lambda)$  the full subcategory of  $\mathbf{HI}(k; \Lambda)$  consisting of  $n$ -presented homotopy invariant sheaves with transfers.

**Remark 4.20** — In [13], an  $n$ -presented homotopy invariant sheaf with transfers was called an  $n$ -motivic sheaf. It is also conjectured in *loc. cit.* that the notions of  $n$ -generated and  $n$ -presented are equivalent (see [13, Conjecture 1.4.1 and Lemma 1.4.3]).

**Remark 4.21** — It is maybe useful to explain the idea of Definition 4.19 in a geometrically meaningful case. A basic example of an object in  $\mathbf{HI}(k; \mathbb{Q})$  is the Nisnevich sheaf  $\underline{\text{CH}}^d(X; \mathbb{Q})$  associated to the presheaf

$$\text{CH}^d(X \times -, \mathbb{Q}) : U \in \text{Sm}/k \quad \mapsto \quad \text{CH}^d(X \times U; \mathbb{Q}).$$

This makes sense for any  $k$ -variety  $X$  but, for simplicity, we will assume that  $X$  is smooth and proper. Saying that  $\underline{\mathrm{CH}}^d(X; \mathbb{Q})$  is  $n$ -generated amounts to saying the following. Every family of codimension  $d$  cycles on  $X$  parametrised by a quasi-projective smooth variety  $U$  is, up to rational equivalence and maybe after shrinking  $U$ , obtained from a family of codimension  $d$  cycles on  $X$  parametrised by a quasi-projective smooth variety  $V$  of dimension at most  $n$  by pulling-back along a finite correspondence from  $U$  to  $V$  (in the sense of Definition 1.14). In fact, knowing that  $\underline{\mathrm{CH}}^d(X; \mathbb{Q})$  is  $n$ -generated implies that there exists a denumerable family  $(S_\alpha)_\alpha$  of smooth and projective varieties of dimension at most  $n$ , and codimension  $d$ -cycles  $Z_\alpha \subset S_\alpha \times X$ , such that the induced map

$$\bigoplus_{\alpha} \mathrm{CH}_0(S_\alpha; \mathbb{Q}) \longrightarrow \mathrm{CH}^d(X; \mathbb{Q})$$

is universally surjective (i.e., remains surjective after extending the base field, say to  $\mathbb{C}$ ). Such properties are very nice and this is perhaps the next best thing to hope for, knowing the failure of representability of Chow groups in general [38].

**4.3. The conjectures.** In this last subsection, we will formulate two conjectures describing the behaviour of the homotopy  $t$ -structure and the slice filtration with respect to the filtration by dimension on the category of motives.

**CONJECTURE 4.22** — *The endofunctor  $\underline{\mathrm{Hom}}^{\mathrm{eff}}(\Lambda(1), -)$  of  $\mathrm{DM}^{\mathrm{eff}}(k; \Lambda)$  takes the subcategory  $\mathrm{DM}_{\leq n}^{\mathrm{eff}}(k; \Lambda)$  to the subcategory  $\mathrm{DM}_{\leq n-1}^{\mathrm{eff}}(k; \Lambda)$ .*

**Remark 4.23** — The endofunctor  $\underline{\mathrm{Hom}}^{\mathrm{eff}}(\Lambda(1)[1], -)$  is also known as Voevodsky's *contraction* and is usually denoted by  $(-)_-1$ . Voevodsky's contraction is indeed a remarkable operation. A notable property is that  $(-)_-1$  is exact with respect to the homotopy  $t$ -structure. The above conjecture is simply saying that Voevodsky's contraction sends  $\mathrm{DM}_{\leq n}^{\mathrm{eff}}(k; \Lambda)$  to  $\mathrm{DM}_{\leq n-1}^{\mathrm{eff}}(k; \Lambda)$ .

**Remark 4.24** — Using that

$$\underline{\mathrm{Hom}}^{\mathrm{eff}}(\Lambda(1), -)^{\circ m} \simeq \underline{\mathrm{Hom}}^{\mathrm{eff}}(\Lambda(m), -)$$

we see that, under Conjecture 4.22,  $\underline{\mathrm{Hom}}^{\mathrm{eff}}(\Lambda(m), -)$  induces a functor

$$\underline{\mathrm{Hom}}^{\mathrm{eff}}(\Lambda(m), -) : \mathrm{DM}_{\leq n}^{\mathrm{eff}}(k; \Lambda) \longrightarrow \mathrm{DM}_{\leq n-m}^{\mathrm{eff}}(k; \Lambda)$$

(with the convention that  $\mathrm{DM}_{\leq r}^{\mathrm{eff}}(k; \Lambda) = 0$  for  $r < 0$ ).

The following proposition gives some (meagre) evidence for Conjecture 4.22.

**PROPOSITION 4.25** — *Let  $n \in \mathbb{N}$  and  $M \in \mathrm{DM}_{\leq n}^{\mathrm{eff}}(k; \Lambda)$ . Then*

$$\underline{\mathrm{Hom}}^{\mathrm{eff}}(\Lambda(m), M) \in \mathrm{DM}_{\leq n-m}^{\mathrm{eff}}(k; \Lambda)$$

for all  $m \geq n - 1$ . (In particular,  $\underline{\mathrm{Hom}}^{\mathrm{eff}}(\Lambda(m), M) = 0$  if  $m \geq n + 1$ .)

*Proof.* We will assume that  $m \in \{n, n - 1\}$ . Indeed, the case  $m \geq n + 1$  follows from the case  $m = n$  and the first equivalence of categories in Remark 4.14 using the easy fact that the contraction of an étale-locally constant sheaf is zero.

It is enough to treat the case where  $M = M(X)$  with  $X$  a smooth projective  $k$ -variety of pure dimension  $n$ . Recall that the strong dual of  $M$  is given by  $M(-n)[-2n]$ . It follows that (Poincaré motivic duality)

$$M \simeq \underline{\mathrm{Hom}}^{\mathrm{eff}}(M, \Lambda(n)[2n]).$$

But we have natural isomorphisms

$$\begin{aligned} \underline{\mathbf{Hom}}^{\text{eff}}(\Lambda(m), \underline{\mathbf{Hom}}^{\text{eff}}(M, \Lambda(n))) &\simeq \underline{\mathbf{Hom}}^{\text{eff}}(M, \underline{\mathbf{Hom}}^{\text{eff}}(\Lambda(m), \Lambda(n))) \\ &\simeq \underline{\mathbf{Hom}}^{\text{eff}}(M, \Lambda(n-m)). \end{aligned}$$

Therefore, we are left to check that

$$\underline{\mathbf{Hom}}^{\text{eff}}(M, \Lambda(0)) \in \mathbf{DM}_{\leq 0}^{\text{eff}}(k; \Lambda) \quad \text{and} \quad \underline{\mathbf{Hom}}^{\text{eff}}(M, \Lambda(1)) \in \mathbf{DM}_{\leq 1}^{\text{eff}}(k; \Lambda).$$

We split the argument accordingly in two parts.

*Part 1:* The category  $\mathbf{DM}_{\leq 0}^{\text{eff}}(k; \Lambda)$  is monoidal closed; we denote by  $\underline{\mathbf{Hom}}_{\leq 0}^{\text{eff}}(-, -)$  its internal Hom. Also, the functor  $L\pi_0$  of Proposition 4.15 is monoidal. Given an effective motive  $L$ , we thus have the following isomorphisms

$$\begin{aligned} &\text{hom}_{\mathbf{DM}^{\text{eff}}(k; \Lambda)}(L, \underline{\mathbf{Hom}}^{\text{eff}}(M, \Lambda(0))) \\ &\simeq \text{hom}_{\mathbf{DM}^{\text{eff}}(k; \Lambda)}(L \otimes M, \Lambda(0)) \\ &\simeq \text{hom}_{\mathbf{DM}_{\leq 0}^{\text{eff}}(k; \Lambda)}(L\pi_0(L \otimes M), \Lambda(0)) \\ &\simeq \text{hom}_{\mathbf{DM}_{\leq 0}^{\text{eff}}(k; \Lambda)}(L\pi_0(L) \otimes L\pi_0(M), \Lambda(0)) \\ &\simeq \text{hom}_{\mathbf{DM}_{\leq 0}^{\text{eff}}(k; \Lambda)}(L\pi_0(L), \underline{\mathbf{Hom}}_{\leq 0}^{\text{eff}}(L\pi_0(M), \Lambda(0))) \\ &\simeq \text{hom}_{\mathbf{DM}^{\text{eff}}(k; \Lambda)}(L, \underline{\mathbf{Hom}}_{\leq 0}^{\text{eff}}(L\pi_0(M), \Lambda(0))). \end{aligned}$$

This shows that  $\underline{\mathbf{Hom}}^{\text{eff}}(M, \Lambda(0))$  is isomorphic to  $\underline{\mathbf{Hom}}_{\leq 0}^{\text{eff}}(L\pi_0(M), \Lambda(0))$ , and therefore belongs to  $\mathbf{DM}_{\leq 0}^{\text{eff}}(k; \Lambda)$  as wanted.

*Part 2:* The subcategory  $\mathbf{DM}_{\leq 1}^{\text{eff}}(k; \Lambda)$  is not closed under the tensor product. Nonetheless, it has a natural monoidal structure; its tensor product  $- \otimes_1 -$  is given by

$$M \otimes_1 N := \text{LAlb}(M \otimes N)$$

where  $\text{LAlb}$  is the left adjoint to the obvious inclusion (see Proposition 4.15). With this monoidal structure, the functor  $\text{LAlb} : \mathbf{DM}^{\text{eff}}(k; \Lambda) \rightarrow \mathbf{DM}_{\leq 1}^{\text{eff}}(k; \Lambda)$  is monoidal (see [14, Proposition 7.1.2]).

We denote by  $\underline{\mathbf{Hom}}_{\leq 1}^{\text{eff}}(-, -)$  the internal Hom relative to the monoidal structure on  $\mathbf{DM}_{\leq 1}^{\text{eff}}(k; \Lambda)$ . A similar computation as in part 1 shows that  $\underline{\mathbf{Hom}}^{\text{eff}}(M, \Lambda(1))$  is isomorphic to  $\underline{\mathbf{Hom}}_{\leq 1}^{\text{eff}}(\text{LAlb}(M), \Lambda(1))$ , and therefore belongs to  $\mathbf{DM}_{\leq 1}^{\text{eff}}(k; \Lambda)$  as wanted.  $\square$

**PROPOSITION 4.26** — *Assume Conjecture 4.22. Let  $M \in \mathbf{DM}_{\text{gm}}^{\text{eff}}(k; \Lambda)$  be an effective geometric motive. Then  $\underline{\mathbf{Hom}}^{\text{eff}}(M, -)$  preserves  $\mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)$ , i.e., if  $N$  belongs to  $\mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)$ , then so is  $\underline{\mathbf{Hom}}^{\text{eff}}(M, N)$ .*

*Proof.* One reduces to the case where  $M = M(X)$  with  $X$  smooth and projective of pure dimension  $d$ . Recall that  $M^\vee = M(X)(-d)[-2d]$  is the strong dual of  $M$ . We thus have the following isomorphisms

$$\underline{\mathbf{Hom}}(M, N) \simeq M^\vee \otimes N \simeq \underline{\mathbf{Hom}}(\Lambda(d), M \otimes N)[-2d].$$

Applying  $\nu^{\geq 0}$ , we deduce that

$$\underline{\mathbf{Hom}}^{\text{eff}}(M, N) \simeq \underline{\mathbf{Hom}}^{\text{eff}}(\Lambda(d), M \otimes N)[-2d].$$

Now, as  $M \in \mathbf{DM}_{\leq d}^{\text{eff}}(k; \Lambda)$  and  $N \in \mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)$ , we see that

$$M \otimes N \in \mathbf{DM}_{\leq n+d}^{\text{eff}}(k; \Lambda).$$

Applying Conjecture 4.22, the result follows.  $\square$

Our second conjecture concerns the homotopy  $t$ -structure.

**CONJECTURE 4.27** — *The homotopy  $t$ -structure on  $\mathbf{DM}^{\text{eff}}(k; \Lambda)$  restricts to a  $t$ -structure on  $\mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)$  for any  $n \in \mathbb{N}$ . Moreover, the heart of the homotopy  $t$ -structure on  $\mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)$  is the category  $\mathbf{HI}_{\leq n}(k; \Lambda)$  of  $n$ -presented homotopy invariant sheaves with transfers.*

**Remark 4.28** — As said before, an effective motive  $M \in \mathbf{DM}^{\text{eff}}(k; \Lambda)$  is a complex of sheaves with transfers on  $\text{Sm}/k$ . We denote by  $h_r(M)$  the homology sheaves of  $M$ ; these are objects of  $\mathbf{HI}(k; \Lambda)$ . Conversely, any object of  $\mathbf{HI}(k; \Lambda)$  is itself an effective motive in an obvious way. This is said, we can rephrase the first assertion in Conjecture 4.27 as the following implication:

$$[M \in \mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)] \quad \Rightarrow \quad [h_r(M) \in \mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda), \forall r \in \mathbb{Z}].$$

Also, the second assertion in Conjecture 4.27 gives the following equality

$$\mathbf{HI}_{\leq n}(k; \Lambda) = \mathbf{HI}(k; \Lambda) \cap \mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda).$$

Said differently, a homotopy invariant sheaf with transfers is  $n$ -presented if and only if it belongs to  $\mathbf{DM}_{\leq n}^{\text{eff}}(k; \Lambda)$ .

**LEMMA 4.29** — *Conjecture 4.27 holds when  $n \leq 1$ .*

*Proof.* This is proven in [13].  $\square$

**PROPOSITION 4.30** — *Assume Conjectures 4.22 and 4.27. Let  $X$  be a smooth  $k$ -variety. Then, for  $n \in \mathbb{N}$ ,  $\underline{\mathbf{CH}}^n(X; \mathbb{Q})$  is  $n$ -presented.*

*Proof.* Let  $\overline{X}$  be a smooth compactification of  $X$  whose complement is a normal crossing divisor with irreducible components  $D_i$ , for  $i \in I$ . There exists an exact sequence of homotopy invariant sheaves with transfers

$$\bigoplus_{i \in I} \underline{\mathbf{CH}}^{n-1}(D_i; \mathbb{Q}) \longrightarrow \underline{\mathbf{CH}}^n(\overline{X}; \mathbb{Q}) \longrightarrow \underline{\mathbf{CH}}^n(X; \mathbb{Q}) \longrightarrow 0.$$

Therefore, it is enough to prove the claim under the assumption that  $X$  is smooth and projective. Also, we may assume that  $X$  has pure dimension  $d$ . Of course, we only need to consider the case  $n \leq d$ .

Using duality, we have for  $U \in \text{Sm}/k$ :

$$\begin{aligned} \mathbf{CH}^n(X \times U; \mathbb{Q}) &\simeq \text{hom}_{\mathbf{DM}(k; \mathbb{Q})}(\mathbf{M}(X) \otimes \mathbf{M}(U), \mathbb{Q}(n)[2n]) \\ &\simeq \text{hom}_{\mathbf{DM}(k; \mathbb{Q})}(\mathbf{M}(U), \mathbf{M}(X) \otimes \mathbb{Q}(n-d)[2n-2d]) \\ &\simeq \text{hom}_{\mathbf{DM}(k; \mathbb{Q})}(\mathbf{M}(U)[2d-2n], \underline{\mathbf{Hom}}(\mathbb{Q}(d-n), \mathbf{M}(X))) \\ &\simeq \text{hom}_{\mathbf{DM}^{\text{eff}}(k; \mathbb{Q})}(\mathbf{M}(U)[2d-2n], \underline{\mathbf{Hom}}^{\text{eff}}(\mathbb{Q}(d-n), \mathbf{M}(X))). \end{aligned}$$

This shows that  $\underline{\mathbf{CH}}^n(X; \mathbb{Q})$  coincides with  $h_{2d-2n} \underline{\mathbf{Hom}}^{\text{eff}}(\mathbb{Q}(d-n), \mathbf{M}(X))$ . Using the combination of Conjecture 4.22 and Conjecture 4.27, we are done.  $\square$

**Remark 4.31** — Thanks to Lemma 4.29, the conclusion of Proposition 4.30 is known for  $n = 0, 1$ . (It is also known for  $n = \dim(X)$  for obvious reasons.) The case  $n = 2$  is already extremely interesting but also, unfortunately, completely out of reach.

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