ANABELIAN PRESENTATION OF THE MOTIVIC GALOIS GROUP IN CHARACTERISTIC ZERO

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Abstract. Let k be a field endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$, and let $\mathcal{G}_{\mathrm{mot}}$ be the associated motivic Galois group. The goal of this paper is to prove that $\mathcal{G}_{\mathrm{mot}}$ admits a natural manifestation in anabelian geometry. Roughly speaking, we prove that $\mathcal{G}_{\mathrm{mot}}$ coincides with the automorphism group of the functor $X \mapsto \pi_1^{\mathrm{geo}}(X)$ sending a k-variety X to the geometric completion of its topological fundamental group. This can be considered as a motivic version of the Ihara–Matsumoto–Oda Conjecture for Galois groups, greatly expanded and proven by Pop. In a sequel to this paper, we will develop the parallel story for the ℓ-adic realisation.

Contents

Key words and phrases. Anabelian geometry, motive, Betti realisation, motivic Galois group, (motivic) fundamental group.

The author is partially supported by the *Swiss National Science Foundation* (SNF), project 200020_178729.

Introduction

Our aim in this introduction is twofold. Firstly, we recall a few facts surrounding the classical Ihara–Matsumoto–Oda Conjecture, which is now a theorem of Pop [Pop19]. These facts will be of no use in the main body of the paper, but help putting our results into perspective. Secondly, we present a rough form of our main results.

Around the classical Ihara–Matsumoto–Oda conjecture.

Let k be a field and fix a separable closure $\overline{k}/k$ of k. Given a geometrically connected k-variety X and a geometric point $\overline{x} \to X$ over $\overline{k}$, one has the well-known short exact sequence of profinite groups

$$1 \to \overline{\pi}_1^{\text{ét}}(X, \overline{x}) \to \pi_1^{\text{ét}}(X, \overline{x}) \to \mathcal{G}(\overline{k}/k) \to 1, \tag{0.1}$$

where $\mathcal{G}(\overline{k}/k)$ is the Galois group of k, $\pi_1^{\text{ét}}(X, \overline{x})$ is the étale fundamental group of X and $\overline{\pi}_1^{\text{ét}}(X, \overline{x}) = \pi_1^{\text{ét}}(X \otimes_k \overline{k}, \overline{x})$ is the étale fundamental group of $X \otimes_k \overline{k}$; see [SGA71, Exposé IX, Théorème 6.1]. The short exact sequence (0.1) defines an action of $\mathcal{G}(\overline{k}/k)$ on $\overline{\pi}^{\text{ét}}(X, \overline{x})$ by outer automorphisms, i.e., a morphism of groups

$$\widetilde{\rho}_X : \mathcal{G}(\overline{k}/k) \to \mathrm{Out}(\overline{\pi}_1^{\text{ét}}(X, \overline{x})) = \frac{\mathrm{Aut}(\overline{\pi}_1^{\text{ét}}(X, \overline{x}))}{\mathrm{Inn}(\overline{\pi}_1^{\text{ét}}(X, \overline{x}))}. \tag{0.2}$$

In fact, the morphisms $\widetilde{\rho}_X$ are compatible with morphisms of k-varieties. To phrase this compatibility precisely, we introduce the category $\widetilde{\mathrm{Grp}}_{\mathrm{pf}}$ of profinite groups up to inner automorphisms. It is obtained from the category $\mathrm{Grp}_{\mathrm{pf}}$ of profinite groups by identifying the arrows that differ by inner automorphisms; explicitly, if H and G are profinite groups, then

$$\hom_{\widetilde{\mathrm{Grp}}_{\mathrm{pf}}}(H, G) = G \backslash \hom_{\mathrm{Grp}_{\mathrm{pf}}}(H, G),$$

where G is acting by conjugation on the set of morphisms from H to G. Using this category, the aforementioned compatibility can be summarised as follows.

(i) As an object of $\widetilde{\mathrm{Grp}}_{\mathrm{pf}}$, the profinite group $\overline{\pi}_1^{\text{ét}}(X, \overline{x})$ is independent of the choice of $\overline{x}$ up to a canonical isomorphism and we may denote it simply by $\overline{\pi}_1^{\text{ét}}(X)$.

(ii) Let $\mathrm{Sch}^0/k \subset \mathrm{Sch}/k$ be the category of geometrically connected k-varieties. The assignment $X \mapsto \overline{\pi}_1^{\text{ét}}(X)$ extends naturally into a functor

$$\overline{\pi}_1^{\text{ét}} : \mathrm{Sch}^0/k \to \widetilde{\mathrm{Grp}}_{\mathrm{pf}}. \tag{0.3}$$

(iii) The group $\mathcal{G}(\overline{k}/k)$ acts on the functor $\overline{\pi}_1^{\text{ét}}$ by invertible natural transformations; this action is given on $X \in \mathrm{Sch}^0/k$ by the morphism $\widetilde{\rho}_X$ in (0.2).

In particular, we obtain a morphism of groups

$$\widetilde{\rho} : \mathcal{G}(\overline{k}/k) \to \mathrm{Aut}(\overline{\pi}_1^{\text{ét}}). \tag{0.4}$$

More generally, given any subcategory $\mathcal{V} \subset \mathrm{Sch}^0/k$, we obtain a morphism of groups

$$\widetilde{\rho}_{\mathcal{V}} : \mathcal{G}(\overline{k}/k) \to \mathrm{Aut}(\overline{\pi}_1^{\text{ét}}|_{\mathcal{V}}). \tag{0.5}$$

We have the following remarkable theorem of Pop, see [Pop19, Theorem 2.7].

Theorem 0.1 (Pop). *Assume that k has characteristic zero and let $\mathcal{V} = \mathrm{Sm}^0/k$ be the subcategory of smooth k-varieties in Sch^0/k. Then $\widetilde{\rho}_{\mathcal{V}}$ is an isomorphism.*

Remark 0.2. In the 80's, Ihara asked whether the morphism $\widetilde{\rho}$, or some closely related variant, was an isomorphism for $k = \mathbb{Q}$ and, in the 90's, Matsumoto and Oda conjectured that this was indeed the case. In an unpublished manuscript, Pop gave a positive answer to Ihara's question. Soon after, he developed a new approach for proving much finer versions of the Ihara–Matsumoto–Oda Conjecture; the new approach was finally published in [Pop19]. Further refinements were obtained later by Topaz [Top18] and Silberstein [Sil13]. In the present paper, we will not be concerned with these finer versions. However, for the benefit of the interested reader, we mention the following.

- By [Pop19, Theorem 2.7], Theorem 0.1 holds true in positive characteristic if the étale fundamental groups $\overline{\pi}_1^{\text{ét}}(X)$ are replaced by their tame quotients.
- A version of Theorem 0.1 holds true if the étale fundamental groups $\overline{\pi}_1^{\text{ét}}(X)$ are replaced by their maximal pro-ℓ abelian-by-central quotient and Sm^0/k by much smaller subcategories. See [Pop19, Theorem 2.6] for a precise statement.
- A version of [Pop19, Theorem 2.6] holds true for the maximal mod-ℓ abelian-by-central quotients of the $\overline{\pi}_1^{\text{ét}}(X)$'s. See [Top18, Theorems A & B] for precise statements.
- When $k = \mathbb{Q}$ and X is a well-chosen geometrically integral algebraic surface with generic point η, the morphism
$$\widetilde{\rho}_{\eta} : \mathcal{G}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Out}(\overline{\pi}_1^{\text{ét}}(\eta))$$
is already an isomorphism, see [Sil13, Theorem 3].
- Finally, a p-adic analytic version of Theorem 0.1 was proven by André based on Pop's solution of the Ihara–Matsumoto–Oda conjecture; see [And03, Theorem 9.2.2].

In this paper, we will prove a motivic analogue of Theorem 0.1. In fact, strictly speaking, our main theorem is a motivic analogue of a variant of Theorem 0.1, and we start by explaining what this variant is about. To do so, we need a digression.

Given a noetherian scheme X, we denote by $\Pi^{\text{ét}}(X)$ the étale fundamental groupoid of X. Recall that $\Pi^{\text{ét}}(X)$ is a category enriched in profinite sets. The objects of $\Pi^{\text{ét}}(X)$ are the geometric points of X and, given two geometric points $\overline{x}_0 \to X$ and $\overline{x}_1 \to X$, the profinite set of maps from $\overline{x}_0$ to $\overline{x}_1$ in $\Pi^{\text{ét}}(X)$ is denoted by $\pi_1^{\text{ét}}(X, \overline{x}_0, \overline{x}_1)$. It is the set of natural isomorphisms between the fiber functors associated to $\overline{x}_0$ and $\overline{x}_1$ on the category of locally constant étale sheaves with finite stalks on X. For more details, see [SGA71, Exposé V, §5 & 7]. In fact, the fundamental groupoid has better functorial properties than the fundamental group. Indeed, let $\mathrm{Grpd}_{\mathrm{pf}}$ be the strict 2-category of groupoids enriched in profinite sets. Then, there is a strict 2-functor

$$\overline{\Pi}^{\text{ét}} : \mathrm{Sch}/k \to \mathrm{Grpd}_{\mathrm{pf}} \tag{0.6}$$

sending a k-variety X to the fundamental groupoid $\overline{\Pi}^{\text{ét}}(X) = \Pi^{\text{ét}}(X \otimes_k \overline{k})$ of its base change to $\overline{k}$. This 2-functor lifts and extends the functor $\overline{\pi}_1^{\text{ét}}$ in (0.3). More precisely, the following holds.

(i) Let $\mathrm{Grpd}^0_{\mathrm{pf}}$ be the full sub-2-category of $\mathrm{Grpd}_{\mathrm{pf}}$ whose objects are the connected groupoids. Then, there is an obvious functor

$$\mathrm{Grpd}^0_{\mathrm{pf}} \to \widetilde{\mathrm{Grp}}_{\mathrm{pf}}$$

exhibiting $\widetilde{\mathrm{Grp}}_{\mathrm{pf}}$ as the 1-categorical truncation of $\mathrm{Grpd}^0_{\mathrm{pf}}$.

(ii) There is a commutative triangle

$$\begin{tikzcd} \mathrm{Sch}^0/k \arrow[r, "\overline{\Pi}^{\text{ét}}"] \arrow[dr, "\overline{\pi}_1^{\text{ét}}"'] & \mathrm{Grpd}^0_{\mathrm{pf}} \arrow[d] \\ & \widetilde{\mathrm{Grp}}_{\mathrm{pf}}. \end{tikzcd}$$

There is also an action of the Galois group $\mathcal{G}(\overline{k}/k)$ on the 2-functor $\overline{\Pi}^{\text{ét}}$ in (0.6), lifting and extending the action $\widetilde{\rho}$ in (0.3). We now describe this action with some details. To fix ideas, we first recall what it means to give an automorphism of a 2-functor.

Recollection 0.3. We denote by $\mathrm{Aut}(\overline{\Pi}^{\text{ét}})$ the monoidal category of automorphisms of the strict 2-functor $\overline{\Pi}^{\text{ét}}$. An object of this category consists of a pair of families $((\xi_X)_X, (\alpha_f)_f)$ where:

- for an object X in Sch/k, ξ_X is an equivalence of the groupoid $\overline{\Pi}^{\text{ét}}(X)$ respecting the enrichment in profinite sets,
- for a morphism $f : Y \to X$ in Sch/k, α_f is a natural transformation

  $$\alpha_f : \xi_X \circ \overline{\Pi}^{\text{ét}}(f) \to \overline{\Pi}^{\text{ét}}(f) \circ \xi_Y$$

  which is necessarily invertible.

The α_f's are required to be compatible with composition in the obvious way: if $f : Y \to X$ and $g : Z \to Y$ are two composable morphisms in Sch/k, the following diagram commutes

$$\begin{tikzcd} \xi_X \circ \overline{\Pi}^{\text{ét}}(f \circ g) \arrow[rr, "\alpha_{f\circ g}"] \arrow[d, equal] & & \overline{\Pi}^{\text{ét}}(f \circ g) \circ \xi_Z \arrow[d, equal] \\ \xi_X \circ \overline{\Pi}^{\text{ét}}(f) \circ \overline{\Pi}^{\text{ét}}(g) \arrow[r, "\alpha_f"] & \overline{\Pi}^{\text{ét}}(f) \circ \xi_Y \circ \overline{\Pi}^{\text{ét}}(g) \arrow[r, "\alpha_g"] & \overline{\Pi}^{\text{ét}}(f) \circ \overline{\Pi}^{\text{ét}}(g) \circ \xi_Z. \end{tikzcd}$$

An arrow $\epsilon : ((\xi_X)_X, (\alpha_f)_f) \to ((\xi'_X)_X, (\alpha'_f)_f)$ in $\mathrm{Aut}(\overline{\Pi}^{\text{ét}})$ is a family of natural transformations $(\epsilon_X : \xi_X \to \xi'_X)_X$ such that for every $f : Y \to X$, the following square commutes

$$\begin{tikzcd} \xi_X \circ \overline{\Pi}^{\text{ét}}(f) \arrow[r, "\alpha_f"] \arrow[d, "\epsilon_X"] & \overline{\Pi}^{\text{ét}}(f) \circ \xi_Y \arrow[d, "\epsilon_Y"] \\ \xi'_X \circ \overline{\Pi}^{\text{ét}}(f) \arrow[r, "\alpha'_f"] & \overline{\Pi}^{\text{ét}}(f) \circ \xi'_Y. \end{tikzcd}$$

The monoidal structure on $\mathrm{Aut}(\overline{\Pi}^{\text{ét}})$ is given by composing functors and natural transformations:

$$((\xi_X)_X, (\alpha_f)_f) \circ ((\xi'_X)_X, (\alpha'_f)_f) = ((\xi_X \circ \xi'_X)_X, (\alpha_f \xi'_X \circ \xi_X \alpha'_f)_f).$$

Clearly, every object of $\mathrm{Aut}(\overline{\Pi}^{\text{ét}})$ is left and right invertible for this tensor product. Thus, $\mathrm{Aut}(\overline{\Pi}^{\text{ét}})$ is a (possibly noncommutative) Picard groupoid.

It is easy to check that there is a morphism of Picard groupoids

$$\rho : \mathcal{G}(\overline{k}/k) \to \mathrm{Aut}(\overline{\Pi}^{\text{ét}}) \tag{0.7}$$

sending an element $\sigma \in \mathcal{G}(\overline{k}/k)$ to the pair $((\xi_X)_X, (\alpha_f)_f)$ with:

- for X in Sch/k, ξ_X is the autoequivalence of $\Pi^{\text{ét}}(X \otimes_k \overline{k})$ induced by the automorphism $\mathrm{id}_X \otimes \mathrm{Spec}(\sigma^{-1})$ of the scheme $X \otimes_k \overline{k} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$,
- for $f : Y \to X$ in Sch/k, α_f is the identity natural transformation obtained by applying the strict 2-functor $\Pi^{\text{ét}}$ to the equality

$$(f \times \mathrm{id}_{\mathrm{Spec}(\overline{k})}) \circ (\mathrm{id}_Y \times \mathrm{Spec}(\sigma^{-1})) = (\mathrm{id}_X \times \mathrm{Spec}(\sigma^{-1})) \circ (f \times \mathrm{id}_{\mathrm{Spec}(\overline{k})}).$$

This said, we obtain a natural variant of the Ihara–Matsumoto–Oda question.

Question 0.4. *Is the functor ρ in* (0.7)*, or some closely related variant, an equivalence?*

We will prove in this paper that a closely related version of the functor ρ is indeed an equivalence of Picard groupoids. We refer the reader to Corollary 4.48 for a precise statement.

Remark 0.5. The truncation functor $\mathrm{Grpd}^{\circ}_{\mathrm{pf}} \to \widetilde{\mathrm{Grp}}_{\mathrm{pf}}$ induces a morphism of Picard groupoids

$$\mathrm{Aut}(\overline{\Pi}^{\text{ét}}|_{\mathcal{V}}) \to \mathrm{Aut}(\overline{\pi}_1^{\text{ét}}|_{\mathcal{V}})$$

for any subcategory $\mathcal{V} \subset \mathrm{Sch}^0/k$, and it is easy to check that the triangle

$$\begin{tikzcd} \mathcal{G}(\overline{k}/k) \arrow[r, "\rho_{\mathcal{V}}"] \arrow[dr, "\widetilde{\rho}_{\mathcal{V}}"'] & \mathrm{Aut}(\overline{\Pi}^{\text{ét}}|_{\mathcal{V}}) \arrow[d] \\ & \mathrm{Aut}(\overline{\pi}_1^{\text{ét}}|_{\mathcal{V}}) \end{tikzcd}$$

is commutative. Unfortunately, this morphism of Picard groupoids is not a priori essentially surjective on objects: given a compatible family of outer automorphisms of the $\overline{\pi}_1^{\text{ét}}(X)$'s, for $X \in \mathcal{V}$, there is an obstruction for lifting the family into an object of the Picard groupoid $\mathrm{Aut}(\overline{\Pi}^{\text{ét}}|_{\mathcal{V}})$. Thus, a positive answer to Question 0.4 does not formally imply a positive answer to the original Ihara–Matsumoto–Oda question. In particular, as far as we can see, the results obtained in this paper cannot be used to reprove some of Pop's results in [Pop19].

Before we start discussing our main results in the motivic setting, we need to move a small step further from the original Ihara–Matsumoto–Oda Conjecture. In fact, this extra step is merely cosmetical; it is based on the following observation.

Observation 0.6. The fundamental groupoid $\Pi^{\text{ét}}(X)$ of a scheme X carries exactly the same information as the multi-Galois category $\mathcal{E}(X)$ of locally constant étale sheaves on X with finite stalks. Indeed, $\mathcal{E}(X)$ is equivalent to the category of covariant functors from $\Pi^{\text{ét}}(X)$ to the category $\mathrm{Set}_{\mathrm{fin}}$ of finite sets (respecting the enrichment in profinite sets on $\Pi^{\text{ét}}(X)$) and, conversely, $\Pi_{\text{ét}}(X)$ is equivalent to the groupoid of fiber functors on $\mathcal{E}(X)$. More generally, we have a fully faithful 2-functor

$$\mathrm{Fun}(-, \mathrm{Set}_{\mathrm{fin}}) : (\mathrm{Grpd}_{\mathrm{pf}})^{\mathrm{op}} \to \mathrm{CAT}^{\mathrm{ex}}$$

to the 2-category of categories and exact functors.

Thus, instead of considering the 2-functor $\overline{\Pi}^{\text{ét}}$ in (0.6), one could consider the 2-functor

$$\overline{\mathcal{E}} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAT} \tag{0.8}$$

sending a k-variety X to the category $\overline{\mathcal{E}}(X) = \mathcal{E}(X \otimes_k \overline{k})$. It follows immediately from the above observation that we have a canonical equivalence of Picard groupoids

$$\mathrm{Aut}(\overline{\Pi}^{\text{ét}}) \simeq \mathrm{Aut}(\overline{\mathcal{E}}) \tag{0.9}$$

and hence also a morphism of Picard groupoids

$$\rho : \mathcal{G}(\overline{k}/k) \to \mathrm{Aut}(\overline{\mathcal{E}}). \tag{0.10}$$

The latter can be described as follows: it sends an element $\sigma \in \mathcal{G}(\overline{k}/k)$ to the family of pullback functors $((\mathrm{id}_X \times \mathrm{Spec}(\sigma))^*)_X$. This said, Question 0.4 is equivalent to the following.

Question 0.7. *Is the functor ρ in* (0.10)*, or some closely related variant, an equivalence?*

It is precisely Question 0.7 that we will generalise and answer in this paper. In fact, this formulation of Question 0.4 suggests immediately many variants where the categories of locally constant étale sheaves with finite stalks are replaced by similar ones such as:

- the entire étale topoi;
- categories of étale local systems with coefficients in a finite ring;
- categories of étale sheaves with coefficients in a finite ring.

At this stage, we may state a precise theorem about the 2-functors

$$\overline{\mathcal{E}}(-;\Lambda) : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAT} \qquad \text{and} \qquad \overline{\mathcal{F}}(-;\Lambda) : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAT}$$

sending a k-variety X to the categories $\overline{\mathcal{E}}(X;\Lambda)$ and $\overline{\mathcal{F}}(X;\Lambda)$ of étale local systems and étale sheaves on $X \otimes_k \overline{k}$ with coefficients in a finite ring Λ.

Theorem 0.8. *Assume that k has characteristic zero and that Λ is connected. Then, the natural functors of Picard groupoids*

$$\mathcal{G}(\overline{k}/k) \to \mathrm{Aut}(\overline{\mathcal{E}}(-;\Lambda)) \qquad \textit{and} \qquad \mathcal{G}(\overline{k}/k) \to \mathrm{Aut}(\overline{\mathcal{F}}(-;\Lambda))$$

are equivalences.

Remark 0.9. Theorem 0.8 will be obtained as a consequence of our motivic Theorems 2.10 and 4.37 combined with rigidity for torsion étale motives. See Corollaries 2.14 and 4.48. We also expect that there is a direct proof which is entirely parallel to the proofs of our motivic theorems; see Remark 2.15. Such a direct proof should also work for the 2-functor $\overline{\mathcal{T}} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAT}$ sending a a k-variety X to the étale topos of $X \otimes_k \overline{k}$.

Description of the main results.

We fix a field k endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$. In this paper, we are mainly concerned with variants of Questions 0.4 and 0.7 where the Galois group $\mathcal{G}(\overline{k}/k)$ is replaced with the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$. In fact, we first prove a variant of Theorem 0.8 for $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$, and then derive from it a statement which is more in the spirit of Question 0.7; this latter statement has a translation in terms of fundamental groupoids.

In the reminder of the introduction, we treat $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ as an affine group scheme defined over $\mathbb{Q}$ for the sake of simplicity. (In the main body of the paper, $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ will be considered also over deeper bases, such as the sphere spectrum, and, more importantly, we shall keep track of its natural derived structure, although the said structure is conjectured to be trivial.) The short exact sequence (0.1) admits a motivic version, at least generically and up to a caveat. Indeed, let K/k be a finitely generated extension in which k is algebraically closed, and let $\Sigma : K \hookrightarrow \mathbb{C}$ be a complex embedding extending σ. Let U be the pro-k-variety of open neighbourhoods of the generic point of a smooth model of K, and let U^{an} be the associated analytic pro-variety. Note that Σ determines a compatible system of base points in U^{an}. By [Ayo14c, Théorèmes 2.34 & 2.57], we have the following exact sequence of affine group schemes over $\mathbb{Q}$:

$$\pi_1^{\mathrm{alg}}(U^{\mathrm{an}}, \Sigma) \to \mathcal{G}_{\mathrm{mot}}(K, \Sigma) \to \mathcal{G}_{\mathrm{mot}}(k, \sigma) \to 1, \tag{0.11}$$

where $\pi_1^{\mathrm{alg}}(U^{\mathrm{an}}, \Sigma)$ is the pro-algebraic completion of the pro-discrete topological fundamental group of the analytic pro-variety U. The caveat here is that this sequence is not exact on the left (unless the extension K/k is trivial). Nevertheless, we obtain a short exact sequence

$$1 \to \pi_1^{\mathrm{geo}}(U, \Sigma) \to \mathcal{G}_{\mathrm{mot}}(K, \Sigma) \to \mathcal{G}_{\mathrm{mot}}(k, \sigma) \to 1 \tag{0.12}$$

if we define $\pi_1^{\mathrm{geo}}(U, \Sigma)$ to be the image of the morphism $\pi_1^{\mathrm{alg}}(U^{\mathrm{an}}, \Sigma) \to \mathcal{G}_{\mathrm{mot}}(K, \Sigma)$. This yields an action of $\mathcal{G}_{\mathrm{mot}}(k, \sigma)$ by outer automorphisms on the affine group scheme $\pi_1^{\mathrm{geo}}(U, \Sigma)$. Letting K vary, it is possible to formulate a generic analogue of the original Ihara–Matsumoto–Oda Conjecture for $\mathcal{G}_{\mathrm{mot}}(k, \sigma)$, but we will not do so here.

Instead, we move directly towards a motivic analogue of Question 0.7. For this, we need to understand precisely the Tannakian category which gives rise to $\pi_1^{\mathrm{geo}}(U, \Sigma)$. By construction, we have a surjection of affine group schemes

$$\pi_1^{\mathrm{alg}}(U^{\mathrm{an}}, \Sigma) \twoheadrightarrow \pi_1^{\mathrm{geo}}(U, \Sigma). \tag{0.13}$$

Since $\pi_1^{\mathrm{alg}}(U^{\mathrm{an}}, \Sigma)$ is the fundamental group of the Tannakian category $\mathbf{LS}(U^{\mathrm{an}}; \mathbb{Q})^{\heartsuit}$ of local systems of $\mathbb{Q}$-vector spaces on U^{an}, we may think of $\pi_1^{\mathrm{geo}}(U, \Sigma)$ as the fundamental group of a Tannakian subcategory $\mathbf{LS}_{\mathrm{geo}}(U; \mathbb{Q})^{\heartsuit} \subset \mathbf{LS}(U^{\mathrm{an}}; \mathbb{Q})^{\heartsuit}$ whose objects we call local systems of geometric origin. It turns out that, more generally, there is a good notion of sheaves of geometric origin over any k-variety, which specialises to the aforesaid one for the pro-k-variety U. Given a k-variety X, a constructible sheaf on X^{an} is of geometric origin if it belongs to the smallest abelian subcategory closed under extensions and containing the sheaves of the form $\mathrm{R}^n f_*^{\mathrm{an}}\mathbb{Q}$, for $n \in \mathbb{N}$ and $f : Y \to X$ a proper morphism. Deriving and closing up under colimits, one obtains the ∞-category $\mathbf{Sh}_{\mathrm{geo}}(X; \mathbb{Q})$. (See Definition 1.88, and Theorems 1.93 and 1.107.) By construction, these ∞-categories are stable by pullback, which yields a functor

$$\mathbf{Sh}_{\mathrm{geo}}(-; \mathbb{Q}) : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAT}_\infty. \tag{0.14}$$

In fact, we prove in Subection 1.6 that the ∞-categories $\mathbf{Sh}_{\mathrm{geo}}(X; \mathbb{Q})$, for $X \in \mathrm{Sch}/k$, are actually stable by the six operations on constructible sheaves. This is a nontrivial fact and a key ingredient in the proof of the next statement, which is our first main result.

Theorem 0.10. *There is a natural equivalence of H-groups*

$$\mathcal{G}_{\mathrm{mot}}(k, \sigma)(\mathbb{Q}) \xrightarrow{\sim} \mathrm{Auteq}(\mathbf{Sh}_{\mathrm{geo}}(-; \mathbb{Q})) \tag{0.15}$$

from the discrete group of $\mathbb{Q}$-rational points of $\mathcal{G}_{\mathrm{mot}}(k, \sigma)(\mathbb{Q})$ to the space of autoequivalences of the functor $\mathbf{Sh}_{\mathrm{geo}}(-; \mathbb{Q})$ considered as an object of the ∞-category of $\mathrm{CAlg}(\mathrm{CAT}_\infty)$-valued presheaves on Sch/k.

Remark 0.11. Of course, we will also give a similar interpretation of the Λ-points of $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ for any (classical) $\mathbb{Q}$-algebra Λ: $\mathcal{G}_{\mathrm{mot}}(k,\sigma)(\Lambda)$ is equivalent to the space of autoequivalences of the functor $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)$, sending $X \in \mathrm{Sch}/k$ to the ∞-category of sheaves of Λ-modules of geometric origin. However, it is worth mentioning that our main theorem is even more precise. Indeed, the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ carries a natural derived structure which we expect to be trivial, but this property has not been justified so far. Similarly, the right hand side of (0.15) is the global section of a derived group stack $\underline{\mathrm{Auteq}}(\mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q}))$. Our main theorem matches $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ and $\underline{\mathrm{Auteq}}(\mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q}))$ with their natural derived structures. We refer the reader to Theorem 2.10 for a precise statement.

Remark 0.12. An interesting byproduct of the proof of Theorem 0.10 is the following. The action of $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ on the functor $\mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q})$ can be used to construct a new six-functor formalism whose underlying pullback formalism is given by the functor

$$\mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q})^{\mathcal{G}_{\mathrm{mot}}} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAT}_{\infty}, \tag{0.16}$$

sending a k-variety X to the ∞-category $\mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Q})^{\mathcal{G}_{\mathrm{mot}}}$ of fixed objets in $\mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Q})$ for the action of $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$. The usual Betti realisation functor for motives, factors through $\mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q})^{\mathcal{G}_{\mathrm{mot}}}$ yielding a morphism of six-functor formalisms

$$\mathcal{R} : \mathbf{MSh}(-;\mathbb{Q}) \to \mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q})^{\mathcal{G}_{\mathrm{mot}}}. \tag{0.17}$$

(Here and below, $\mathbf{MSh}(X;\mathbb{Q})$ is the ∞-category of motivic sheaves on X.) It is expected that $\mathcal{R}$ yields an equivalence when restricted to the sub-∞-categories of constructible objects on both sides. In view of the main result of [CG17], objects in $\mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Q})^{\mathcal{G}_{\mathrm{mot}}}$ are entitled to be called Nori motivic sheaves. This gives a new construction of an abelian category of Nori motivic sheaves with the expected relation to the triangulated categories of motivic sheaves à la Voevodsky. See [Ara13], [Ara20], [Ivo17] and [IM19] for other approaches.

The proof of Theorem 0.10 is extraordinary simple! It relies on the following two ingredients.

(1) A motivic description of the functor $\mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q})$.
(2) The universality of the functor $\mathbf{MSh}(-;\mathbb{Q})$ as an object in the ∞-category of Voevodsky pullback formalisms.

The second ingredient is a recent result of Drew–Gallauer [DG20]. Roughly speaking, it is the property that the functor $\mathbf{MSh}(-;\mathbb{Q})$ is initial when viewed as an object of a certain ∞-category of functors satisfying enough of the six-functor formalism; see also Theorem 2.5. To describe the first ingredient, we note that the Betti realisation yields a natural transformation

$$\mathrm{B}^* : \mathbf{MSh}(-;\mathbb{Q}) \to \mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q}). \tag{0.18}$$

Let $\mathcal{B} \in \mathbf{MSh}(k;\mathbb{Q})$ be the algebra object representing singular cohomology of motives; this is the image of $\mathbb{Q}$ by the functor $\mathrm{B}_* : \mathrm{Mod}_{\mathbb{Q}} \to \mathbf{MSh}(k;\mathbb{Q})$, right adjoint to B^*. Applying a general ∞-categorical construction, we obtain a natural transformation

$$\widetilde{\mathrm{B}}^* : \mathbf{MSh}(-;\mathcal{B}) \to \mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q}) \tag{0.19}$$

factoring B^*. Here, for $X \in \mathrm{Sch}/k$, we denote by $\mathbf{MSh}(X;\mathcal{B})$ the ∞-category of $\mathcal{B}$-modules in $\mathbf{MSh}(X;\mathbb{Q})$. The motivic description of $\mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q})$ is the content of the following statement.

Theorem 0.13. *The morphism* $\widetilde{\mathrm{B}}^*$ *in* (0.19) *is an equivalence.*

A closely related variant of Theorem 0.13 was announced by Drew in [Dre18]; see the paragraph "Forthcoming and future work" in the introduction of loc. cit. The fully faithfulness part is an old observation due to Cisinski–Déglise, which is a direct consequence of the compatibility of the Betti realisation with the six operations; see Lemma 1.95 and Remark 1.97. The essential surjectivity of these functors relies on Deligne's semi-simplicity theorem [Del71, Théorème 4.2.6], but we offer another proof, avoiding Hodge theory and relying instead on results about the motivic Galois group, namely [Ayo14c, Théorèmes 2.34 & 2.57].

Using the above two ingredients, we get the following chain of equivalences of H-spaces:

$$\begin{aligned}
\text{Self-Map}(\mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q})) &\simeq \text{Self-Map}(\mathbf{MSh}(-;\mathcal{B})) \\
&\simeq \text{Self-Map}\begin{pmatrix}\mathbf{MSh}(-;\mathbb{Q})\\ \downarrow \\ \mathbf{MSh}(-;\mathcal{B})\end{pmatrix}_{\mathrm{id}} \\
&\simeq \lim_{f:Y\to X\in(\mathrm{Sch}/k)^{\mathrm{tw}}} \mathrm{Map}\begin{pmatrix}\mathbf{MSh}(X;\mathbb{Q}) & \mathbf{MSh}(Y;\mathbb{Q})\\ \downarrow \quad , & \downarrow \\ \mathbf{MSh}(X;\mathcal{B}) & \mathbf{MSh}(Y;\mathcal{B})\end{pmatrix}_{f^*}.
\end{aligned}$$

The subscripts "id" and "f^*" refer to taking the fibers at the points

$$\mathrm{id} \in \mathrm{End}(\mathbf{MSh}(-;\mathbb{Q})) \qquad \text{and} \qquad f^* \in \mathrm{Map}(\mathbf{MSh}(X;\mathbb{Q}), \mathbf{MSh}(Y;\mathbb{Q}))$$

respectively. Also, the self-mapping spaces are taken in the ∞-categories of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves on Sch/k, for the first two, and on $\Delta^1 \times \mathrm{Sch}/k$ for the third one. Finally, the mapping spaces after the limit are taken in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})^{\Delta^1}$. This said, we may use [Lur17, Theorem 4.8.5.21], to continue the above chain of equivalences as follows

$$\begin{aligned}
&\simeq \lim_{f:Y\to X\in(\mathrm{Sch}/k)^{\mathrm{tw}}} \mathrm{Map}_{\mathrm{Pr}^{\mathrm{CAlg}}}((\mathbf{MSh}(X;\mathbb{Q}),\mathcal{B}|_X),(\mathbf{MSh}(Y;\mathbb{Q}),\mathcal{B}|_Y))_{f^*} \\
&\simeq \lim_{f:Y\to X\in(\mathrm{Sch}/k)^{\mathrm{tw}}} \mathrm{Map}_{\mathrm{CAlg}(\mathbf{MSh}(Y;\mathbb{Q}))}(\mathcal{B}|_Y,\mathcal{B}|_Y) \\
&\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathbf{MSh}(k;\mathbb{Q}))}(\mathcal{B},\mathcal{B}) \\
&\simeq \mathcal{G}_{\mathrm{mot}}(k,\sigma)(\mathbb{Q}).
\end{aligned}$$

This finishes our sketch of proof of Theorem 0.10. For more details, we refer the reader to the proof of Theorem 2.10 where we actually use a slightly different argument.

In the remainder of the introduction, we briefly explain how to derive from Theorem 0.10 a similar statement about the action of the motivic Galois group on categories of local systems of geometric origin. For $X \in \mathrm{Sch}/k$, we denote by $\mathbf{LS}_{\mathrm{geo}}(X;\mathbb{Q})$ the full sub-∞-category of $\mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Q})$ spanned by dualizable objects.[1] Clearly, an autoequivalence of $\mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Q})$ restricts to an autoequivalence of $\mathbf{LS}_{\mathrm{geo}}(X;\mathbb{Q})$, which gives a map

$$\mathrm{Auteq}(\mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q})) \to \mathrm{Auteq}(\mathbf{LS}_{\mathrm{geo}}(-;\mathbb{Q})). \tag{0.20}$$

To construct a map in the opposite direction, it is enough to find a recipe for constructing the functor $\mathbf{Sh}_{\mathrm{geo}}(-;\mathbb{Q})$ from the functor $\mathbf{LS}_{\mathrm{geo}}(-;\mathbb{Q})$. This is not unreasonable since a constructible sheaf of geometric origin on X can be obtained by gluing local systems of geometric origin on the strata of some stratification of X. To produce such a recipe, we need to extend the functoriality of the categories $\mathbf{LS}_{\mathrm{geo}}(-;\mathbb{Q})$ in order to allow monodromic specialisation functors. For more details, we refer the reader to Section 3. Then, we need to show that these extra functorialities do not mess

[1] We use 'dualizable' to mean 'strongly dualizable'.

the autoequivalences of the functor $\mathbf{LS}_{\text{geo}}(-;\mathbb{Q})$. At the end, we are able to justify the following statement, which is our second main result; see Theorem 4.37.

Theorem 0.14. *There is a natural equivalence of H-groups*

$$\mathcal{G}_{\text{mot}}(k,\sigma)(\mathbb{Q}) \xrightarrow{\sim} \text{Auteq}(\mathbf{LS}_{\text{geo}}(-;\mathbb{Q})) \tag{0.21}$$

from the discrete group of $\mathbb{Q}$-rational points of $\mathcal{G}_{\text{mot}}(k,\sigma)(\mathbb{Q})$ to the space of autoequivalences of the functor $\mathbf{LS}_{\text{geo}}(-;\mathbb{Q})$ considered as an object of the ∞-category of $\text{CAlg}(\text{CAT}_\infty)$*-valued presheaves on* Sm/k.

Notation and conventions.

∞-Categories. We freely use the language of ∞-categories as developed in Lurie's books [Lur09a], [Lur17] and [Lur18]. The reader familiar with the content of these books will have no problem understanding our notation pertaining to higher category, higher algebra and higher algebraic geometry, which are often very close to those in loc. cit. Nevertheless, we list below some of the notations and conventions we frequently use.

As usual, we employ the device of Grothendieck universes, and we denote by Cat_∞ the ∞-category of small ∞-categories, and CAT_∞ the ∞-category of but possibly large ∞-categories. We denote by $\text{CAT}_\infty^{\text{L}}$ (resp. $\text{CAT}_\infty^{\text{R}}$) the wide sub-$\infty$-category of CAT_∞ spanned by functors which are left (resp. right) adjoints. Similarly, we denote by Pr^{L} (resp. Pr^{R}) the ∞-categories of presentable ∞-categories and left (resp. right) adjoint functors. We denote by $\text{Pr}^{\text{L}}_\omega \subset \text{Pr}^{\text{L}}$ (resp. $\text{Pr}^{\text{R}}_\omega \subset \text{Pr}^{\text{R}}$) the sub-$\infty$-category of compactly generated ∞-categories and compact-preserving functors (resp. functors commuting with filtered colimits).

We denote by $\mathcal{S}$ the ∞-category of spaces and by $\mathcal{S}p$ the ∞-category of spectra. The latter admits a natural t-structure $(\mathcal{S}p_{\geq 0}, \mathcal{S}p_{\leq 0})$ with $\mathcal{S}p_{\geq 0}$ the sub-∞-category of connective spectra.

Given an ∞-category $\mathcal{C}$, we denote by $\text{Map}_{\mathcal{C}}(x,y)$ the mapping space between two objects x and y in $\mathcal{C}$. We denote by $\text{Equi}_{\mathcal{C}}(x,y) \subset \text{Map}_{\mathcal{C}}(x,y)$ the subspace of equivalences between x and y. We also write $\text{Self-Map}_{\mathcal{C}}(x)$, instead of $\text{Map}_{\mathcal{C}}(x,x)$, for the self-mapping space of an object x of $\mathcal{C}$. Similarly, we write $\text{Auteq}_{\mathcal{C}}(x)$, instead of $\text{Equi}_{\mathcal{C}}(x,x)$, for the space of autoequivalences of x.

Given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\text{Fun}(\mathcal{C},\mathcal{D})$ the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$. If $\mathcal{C}$ is small, we denote by $\mathcal{P}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}},\mathcal{S})$ the ∞-category of presheaves on $\mathcal{C}$ and by $\text{y} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ the Yoneda embedding. More generally, we write $\text{Psh}(\mathcal{C},\mathcal{D})$ instead of $\text{Fun}(\mathcal{C}^{\text{op}},\mathcal{D})$ if we want to consider a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ as a $\mathcal{D}$-valued presheaf on $\mathcal{C}$. If $\mathcal{C}$ is endowed with a topology τ, we denote by $\mathcal{F}^{(\wedge)}_\tau(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ the full sub-∞-category of τ-(hyper)sheaves and by $\text{L}_\tau : \mathcal{P}(\mathcal{C}) \to \mathcal{F}^{(\wedge)}_\tau(\mathcal{C})$ the (hyper)sheafification functor. Similarly, we denote by $\text{Shv}^{(\wedge)}_\tau(\mathcal{C};\mathcal{D})$ the full sub-∞-category of $\text{Psh}(\mathcal{C};\mathcal{D})$ of $\mathcal{D}$-valued τ-(hyper)sheaves.

A symmetric monoidal ∞-category is a coCartesian fibration $\mathcal{C}^\otimes \to \text{Fin}_*$ such that the induced functor $(\rho^i_!)_i : \mathcal{C}_{\langle n\rangle} \to \prod_{1\leq i\leq n} \mathcal{C}_{\langle 1\rangle}$ is an equivalence for all $n \geq 0$. (Recall that Fin_* is the category of finite pointed sets, $\langle n\rangle = \{1,\ldots,n\} \cup \{*\}$ and $\rho^i : \langle n\rangle \to \langle 1\rangle$ is the unique map such that $(\rho^i)^{-1}(1) = \{i\}$.) The fiber $\mathcal{C}^\otimes_{\langle 1\rangle}$ over $\langle 1\rangle$ is called the underlying ∞-category, and is simply denoted by $\mathcal{C}$. The ∞-category of commutative algebras in $\mathcal{C}^\otimes$ is denoted by $\text{CAlg}(\mathcal{C})$. If $A \in \text{CAlg}(\mathcal{C})$, we denote by $\text{Mod}_A(\mathcal{C})$ the ∞-category of A-modules.

By Lurie's straightening construction, a symmetric monoidal category can be considered as a commutative algebra in $\text{CAT}_\infty^\times$. We use this to identify the ∞-category of symmetric monoidal ∞-categories with $\text{CAlg}(\text{CAT}_\infty)$. Similarly, the ∞-categories Pr^{L} and $\text{Pr}^{\text{L}}_\omega$ underly symmetric monoidal ∞-categories $\mathrm{Pr}^{\mathrm{L},\otimes}$ and $\mathrm{Pr}^{\mathrm{L},\otimes}_{\omega}$. A symmetric monoidal ∞-category is said to be presentable (resp. compactly generated) if it belongs to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ (resp. $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$).

Algebraic Geometry. If k is a field, we use "k-variety" as a shorthand for "k-scheme of finite type". If $\sigma : k \hookrightarrow \mathbb{C}$ is a complex embedding, we denote by X^{an} the complex analytic variety associated to a k-variety X. If we need to specify the role of σ, we write $X^{\sigma\text{-an}}$.

Unless otherwise stated, schemes will be assumed quasi-compact and quasi-separated. Given a scheme S, we denote by Sch/S the category of finite type S-schemes. We denote by $\mathrm{Sm}/S \subset \mathrm{Sch}/S$ and $\mathrm{\acute{E}t}/S \subset \mathrm{Sch}/S$ the full subcategories of smooth and étale S-schemes respectively. When S is the spectrum of a commutative ring R, we write Sch/R instead of $\mathrm{Sch}/\mathrm{Spec}(R)$, and similarly in the smooth and étale case. These categories are typically endowed with the étale topology, which we abbreviate by "ét". We usually denote by $- \times_S -$ (resp. $- \times_R -$) the direct product on Sch/S (resp. Sch/R). If k is a fixed base field, we often write $- \times -$ instead of $- \times_k -$.

Given a scheme S, we denote by $\mathbb{A}^n_S$ the n-dimensional relative affine space over S. If S is the spectrum of a commutative ring R, we write $\mathbb{A}^n_R$ instead of $\mathbb{A}^n_{\mathrm{Spec}(R)}$. If a base field k is fixed, we even write $\mathbb{A}^n$ instead of $\mathbb{A}^n_k$. Similarly, we denote by $\mathbb{P}^n_S$ the n-dimensional relative projective space over S, and use similar shorthands when S is the spectrum of a commutative ring or a fixed base field.

Motivic and ordinary sheaves. In this paper, we will depart from well-established notations in motivic homotopy theory: given a scheme X, we denote by $\mathbf{MSh}(X)$ the Morel–Voevodsky ∞-category of motivic sheaves on X in the étale topology. This is usually denoted by $\mathrm{SH}_{\mathrm{\acute{e}t}}(X)$, or similarly. Given $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$, we denote by $\mathbf{MSh}(X;\Lambda)$ the ∞-category of Λ-modules in $\mathbf{MSh}(X)$. Objects of $\mathbf{MSh}(X;\Lambda)$ will be called motivic sheaves with coefficients in Λ. More generally, if A is a commutative algebra $\mathbf{MSh}(S)$ and X is an S-scheme, we denote by $\mathbf{MSh}(X;A)$ the ∞-category of A-modules in $\mathbf{MSh}(X)$. (Here we use the symmetric monoidal functor given by pullback along the structural morphism $X \to S$.)

Similarly, given a complex analytic variety (or, more generally, any topological space) W, we denote by $\mathbf{Sh}(W)$ the ∞-category $\mathrm{Shv}(\mathrm{Op}(W);\mathcal{S}p)$ of $\mathcal{S}p$-valued sheaves on the site of opens in W. Given $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$, we denote by $\mathbf{Sh}(W;\Lambda)$ the ∞-category of Λ-modules in $\mathbf{Sh}(W)$ which we call sheaves on W with coefficients in Λ. The full sub-∞-category of $\mathbf{Sh}(W;\Lambda)$ of local systems is denoted by $\mathbf{LS}(W;\Lambda)$.

If X is a k-variety, with k a field endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$, we denote by $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$ the ind-completion of the ∞-category of sheaves on X^{an} which are constructible with respect to the analytification of a stratification of X. (We warn the reader that this is not a full sub-∞-category of $\mathbf{Sh}(X^{\mathrm{an}};\Lambda)$.) We denote by $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$ the full sub-∞-category of $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$ consisting of those sheaves which are of geometric origin. If we need to specify the role of σ, we write $\mathbf{Sh}_{\sigma\text{-ct}}(X;\Lambda)$ and $\mathbf{Sh}_{\sigma\text{-geo}}(X;\Lambda)$. If Λ is the sphere spectrum, we omit it from the notation, and write simply $\mathbf{Sh}_{\mathrm{ct}}(X)$ and $\mathbf{Sh}_{\mathrm{geo}}(X)$. The Betti realisation functor induces an exact functor

$$\mathrm{B}^*_X : \mathbf{MSh}(X;\Lambda) \to \mathbf{Sh}_{\mathrm{geo}}(X;\Lambda).$$

With X as above, we denote by $\widehat{\mathbf{LS}}(X;\Lambda)$ the full sub-∞-category of $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$ generated under colimits by $\mathbf{LS}(X^{\mathrm{an}};\Lambda)$. We denote by $\mathbf{LS}_{\mathrm{geo}}(X;\Lambda)$ the subcategory of $\mathbf{LS}(X^{\mathrm{an}};\Lambda)$ consisting of those local systems which are of geometric origin. Then, we denote by $\widehat{\mathbf{LS}}_{\mathrm{geo}}(X;\Lambda)$ the full sub-∞-category of $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$ generated under colimits by $\mathbf{LS}_{\mathrm{geo}}(X;\Lambda)$.

Galois and fundamental groups. Let k be a field. Given a separable closure $\overline{k}/k$ of k, we denote by $\mathcal{G}(\overline{k}/k)$ the absolute Galois group of k; this is a profinite group. If $\sigma : k \hookrightarrow \mathbb{C}$ is a complex embedding, we denote by $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ the motivic Galois group of k. This is naturally an affine derived group scheme. More precisely, it is the spectrum of a derived Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k,\sigma)$ which is concentrated in positive homological degrees.

If W is a connected paracompact complex analytic variety and $w \in W$ a point, we denote by $\pi_1(W,w)$ the fundamental group of W and $\pi_1^{\mathrm{alg}}(W,w)$ its pro-algebraic completion over $\mathbb{Q}$. Said differently, $\pi_1^{\mathrm{alg}}(W,w)$ is the fundamental group of the Tannakian category $\mathbf{LS}(W;\mathbb{Q})^{\heartsuit}$ neutralised by the fiber functor at w. If X is a connected k-variety and $x \in X^{\mathrm{an}}$ is a complex point of X, we denote by $\pi_1^{\mathrm{geo}}(X,x)$ the fundamental group of the Tannakian category $\mathbf{LS}_{\mathrm{geo}}(X;\mathbb{Q})^{\heartsuit}$ neutralised by the fiber functor at x. If we need to specify the role of σ, we write $\pi_1^{\sigma\text{-}\mathrm{geo}}(X,x)$.

1. Motives, realisation and the motivic Galois group

In this section, we recast the construction of the motivic Galois group, introduced in [Ayo14b] and revisited in [Ayo17a], using the language of ∞-categories. We also review some basic facts from [Ayo14b, Ayo14c] and relate them to the notion of local systems of geometric origin. All the results contained in this section are more or less known, but not always available in the generality we want to consider in this paper. The reader familiar with this material may skip this section and refer to it when needed.

1.1. **Motivic sheaves.**

In order to streamline the notation in the paper, we will depart from well-established notations in motivic homotopy theory and write $\mathbf{MSh}(X)$ for the Morel–Voevodsky ∞-category of motivic sheaves on X in the étale topology. This ∞-category is usually denoted by $\mathrm{SH}_{\text{ét}}(X)$, or similarly, in the literature. We will also write $\mathbf{MSh}(X;\Lambda)$ for the ∞-category of motivic sheaves with coefficients in a commutative ring spectrum $\Lambda \in \mathcal{S}p$. In this subsection, we recall the construction of these ∞-categories and review their basic properties. We also review the Betti realisation of motivic sheaves. We start by recalling some general facts.

Notation 1.1. Given a small ∞-category $\mathcal{C}$, we denote by $\mathcal{P}(\mathcal{C})$ the ∞-category of $\mathcal{S}$-valued presheaves on $\mathcal{C}$. (As usual, $\mathcal{S}$ is the ∞-category of Kan complexes.) If $\mathcal{C}$ is endowed with a Grothendieck topology τ, we denote by $\mathcal{F}_{\tau}^{(\wedge)}(\mathcal{C})$ the full sub-∞-category of $\mathcal{P}(\mathcal{C})$ spanned by the τ-(hyper)sheaves. More generally, if $\mathcal{D}$ is another ∞-category, we denote by $\mathrm{Psh}(\mathcal{C};\mathcal{D})$ the ∞-category of $\mathcal{D}$-valued presheaves on $\mathcal{C}$ and by $\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C};\mathcal{D})$ its full sub-∞-category spanned by τ-(hyper)sheaves. When $\mathcal{D} = \mathrm{Mod}_{\Lambda}$ is the ∞-category of Λ-modules for some commutative ring spectrum $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$, we write $\mathrm{Psh}(\mathcal{C};\Lambda)$ and $\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C};\Lambda)$ instead. There are obvious functors

$$\Lambda(-) : \mathcal{P}(\mathcal{C}) \to \mathrm{Psh}(\mathcal{C};\Lambda) \qquad \text{and} \qquad \Lambda_{\tau}(-) : \mathcal{F}_{\tau}^{(\wedge)}(\mathcal{C}) \to \mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C};\Lambda), \tag{1.1}$$

induced by the unique colimit-preserving functor $\mathcal{S} \to \mathrm{Mod}_{\Lambda}$ sending the one-point space to Λ. We also denote by $\Lambda(-)$ and $\Lambda_{\tau}(-)$ the composition of the functors in (1.1) with the Yoneda embedding and its (hyper)sheafified version (which fails to be an embedding in general).

Remark 1.2. Let $\mathcal{D}^{\otimes}$ be a symmetric monoidal ∞-category with underlying ∞-category $\mathcal{D}$. Applying [Lur09a, Proposition 3.1.2.1] to the coCartesian fibration $\mathcal{D}^{\otimes} \to \mathrm{Fin}_*$, we deduce that

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{D}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Fin}_*)} \mathrm{Fin}_* \to \mathrm{Fin}_*$$

defines a symmetric monoidal ∞-category $\mathrm{Psh}(\mathcal{C};\mathcal{D})^{\otimes}$ whose underlying ∞-category is $\mathrm{Psh}(\mathcal{C};\mathcal{D})$. By [Lur17, Proposition 2.2.1.9], if the symmetric monoidal ∞-category $\mathcal{D}^{\otimes}$ is presentable, then $\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C};\mathcal{D})$ underlies a unique symmetric monoidal ∞-category $\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C};\mathcal{D})^{\otimes}$ such that the (hyper)sheafification functor

$$\mathrm{L}_{\tau}:\mathrm{Psh}(\mathcal{C};\mathcal{D})\to\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C};\mathcal{D})$$

lifts to a symmetric monoidal functor. These considerations apply with $\mathcal{D}^{\otimes}=\mathrm{Mod}_{\Lambda}^{\otimes}$ the symmetric monoidal ∞-category of Λ-modules (with $\Lambda\in\mathrm{CAlg}(\mathcal{S}p)$ as above). The resulting symmetric monoidal ∞-categories are denoted by $\mathrm{Psh}(\mathcal{C};\Lambda)^{\otimes}$ and $\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C};\Lambda)^{\otimes}$ respectively. The functors $\Lambda(-)$ and $\Lambda_{\tau}(-)$ in (1.1) lift naturally to symmetric monoidal functors.

Remark 1.3. Let $\Lambda\in\mathrm{CAlg}(\mathcal{S}p)$ be a commutative ring spectrum. By [Lur09a, Proposition 5.5.3.6 & Remark 5.5.1.6], the ∞-categories $\mathrm{Psh}(\mathcal{C};\Lambda)$ and $\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C};\Lambda)$ are presentable. They are generated under colimits by their objects $\Lambda(X)$ and $\Lambda_{\tau}(X)$, for $X\in\mathcal{C}$. In fact, the objects $\Lambda(X)$ are compact, so that $\mathrm{Psh}(\mathcal{C};\Lambda)$ is compactly generated. More is true: the symmetric monoidal ∞-categories $\mathrm{Psh}(\mathcal{C};\Lambda)^{\otimes}$ and $\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C};\Lambda)^{\otimes}$ are presentable and, if $\mathcal{C}$ has finite products, $\mathrm{Psh}(\mathcal{C};\Lambda)^{\otimes}$ is even compactly generated.

Remark 1.4. If Λ is set to be the sphere spectrum $\mathbb{S}\in\mathcal{S}p$, we write $\mathrm{Psh}(\mathcal{C})$ and $\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C})$ instead of $\mathrm{Psh}(\mathcal{C};\mathbb{S})$ and $\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C};\mathbb{S})$. Said differently, $\mathrm{Psh}(\mathcal{C})$ and $\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C})$ are the ∞-categories of $\mathcal{S}p$-valued presheaves and τ-(hyper)sheaves respectively. In fact, this will be a general notational convention in the paper: given some ∞-categories depending on a commutative ring spectrum Λ, we simply remove "Λ" from the notation to indicate that Λ is set to be the sphere spectrum $\mathbb{S}\in\mathcal{S}p$. It is worth recalling that $\mathrm{Psh}(\mathcal{C})$ is tensored over $\mathcal{S}p^{\otimes}$ and that, by [Lur17, Proposition 4.8.1.17], the ∞-category $\mathrm{Psh}(\mathcal{C};\Lambda)$ is equivalent to the ∞-category of Λ-modules in $\mathrm{Psh}(\mathcal{C})$. Thus, we have an equivalence

$$\mathrm{Psh}(\mathcal{C})\otimes\mathrm{Mod}_{\Lambda}\simeq\mathrm{Psh}(\mathcal{C};\Lambda)$$

where the tensor product is taken in $\mathrm{Pr}^{\mathrm{L},\otimes}$. The same applies to the ∞-category $\mathrm{Shv}_{\tau}^{(\wedge)}(\mathcal{C};\Lambda)$.

As usual, for a scheme S, we denote by Sm/S the category of smooth S-schemes which we endow with the étale topology abbreviated by "ét". We also denote by $\mathbb{A}_S^n$ the n-dimensional relative affine space.

Definition 1.5. Let S be a scheme and $\Lambda\in\mathrm{CAlg}(\mathcal{S}p)$ a commutative ring spectrum. We denote by $\mathbf{MSh}^{\mathrm{eff}}(S;\Lambda)$ the full sub-∞-category of $\mathrm{Shv}_{\acute{e}t}^{\wedge}(\mathrm{Sm}/S;\Lambda)$ spanned by those étale hypersheaves which are $\mathbb{A}^1$-invariant. (Recall that a presheaf F is $\mathbb{A}^1$-invariant if, for every $X\in\mathrm{Sm}/S$, the projection map $p:\mathbb{A}_X^1\to X$ induces an equivalence $p^*:F(X)\simeq F(\mathbb{A}_X^1)$.)

Remark 1.6. The sub-∞-category $\mathbf{MSh}^{\mathrm{eff}}(S;\Lambda)$ is the localisation of $\mathrm{Shv}_{\acute{e}t}^{\wedge}(\mathrm{Sm}/S;\Lambda)$ with respect to the collection of maps of the form $\Lambda_{\acute{e}t}(\mathbb{A}_X^1)\to\Lambda_{\acute{e}t}(X)$, for $X\in\mathrm{Sm}/S$, and their desuspensions. In particular, the obvious inclusion admits a left adjoint

$$\mathrm{L}_{\mathbb{A}^1}:\mathrm{Shv}_{\acute{e}t}^{\wedge}(\mathrm{Sm}/S;\Lambda)\to\mathbf{MSh}^{\mathrm{eff}}(S;\Lambda).\tag{1.2}$$

The ∞-category $\mathbf{MSh}^{\mathrm{eff}}(S;\Lambda)$ is stable and, by [Lur17, Proposition 2.2.1.9], it underlies a unique symmetric monoidal ∞-category $\mathbf{MSh}^{\mathrm{eff}}(S;\Lambda)^{\otimes}$ such that $\mathrm{L}_{\mathbb{A}^1}$ lifts to a symmetric monoidal functor. Moreover, the symmetric monoidal ∞-category $\mathbf{MSh}^{\mathrm{eff}}(S;\Lambda)^{\otimes}$ is presentable.

Definition 1.7. Let S be a scheme and $\Lambda\in\mathrm{CAlg}(\mathcal{S}p)$ a commutative ring spectrum. We denote by T_S (or simply T is S is clear from the context) the image by the functor $\mathrm{L}_{\mathbb{A}^1}$ in (1.2) of the cofiber of the split inclusion $\Lambda_{\acute{e}t}(S) \to \Lambda_{\acute{e}t}(\mathbb{A}^1_S \setminus 0_S)$ induced by the unit section. With the notation of [Rob15, Definition 2.6], we set

$$\mathbf{MSh}(S;\Lambda)^{\otimes} = \mathbf{MSh}^{\mathrm{eff}}(S;\Lambda)^{\otimes}[\mathrm{T}_S^{-1}].$$

Thus, there is a morphism $\Sigma^{\infty}_{\mathrm{T}} : \mathbf{MSh}^{\mathrm{eff}}(S;\Lambda)^{\otimes} \to \mathbf{MSh}(S;\Lambda)^{\otimes}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, sending T_S to a $\otimes$-invertible object, and which is initial for this property. We denote by $\Omega^{\infty}_{\mathrm{T}}$ the right adjoint of $\Sigma^{\infty}_{\mathrm{T}}$.

Notation 1.8. Let X be a smooth S-scheme. We set

$$\mathrm{M}^{\mathrm{eff}}(X) = \mathrm{L}_{\mathbb{A}^1}(\Lambda_{\acute{e}t}(X)) \qquad \text{and} \qquad \mathrm{M}(X) = \Sigma^{\infty}_{\mathrm{T}}\mathrm{M}^{\mathrm{eff}}(X) = \Sigma^{\infty}_{\mathrm{T}}\mathrm{L}_{\mathbb{A}^1}(\Lambda_{\acute{e}t}(X)).$$

These are objects of $\mathbf{MSh}^{\mathrm{eff}}(S;\Lambda)$ and $\mathbf{MSh}(S;\Lambda)$. The object $\mathrm{M}(X)$ is called the homological motive associated to X.

Notation 1.9. We denote by Λ (or Λ_S) the monoidal unit of $\mathbf{MSh}(S;\Lambda)^{\otimes}$. For $n \in \mathbb{N}$, we denote by $\Lambda(n)$ the image of $\mathrm{T}_S^{\otimes n}[-n]$ by $\Sigma^{\infty}_{\mathrm{T}}$, and by $\Lambda(-n)$ the $\otimes$-inverse of $\Lambda(n)$. For $n \in \mathbb{Z}$, we denote by $M \mapsto M(n)$ the Tate twist given by tensoring with $\Lambda(n)$.

Lemma 1.10. *Let S be a scheme and $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$ a commutative ring spectrum. The symmetric monoidal ∞-category $\mathbf{MSh}^{\mathrm{eff}}(S;\Lambda)^{\otimes}$ is presentable and its underlying ∞-category is generated under colimits, and up to desuspension and negative Tate twists when applicable, by the motives $\mathrm{M}^{(\mathrm{eff})}(X)$ with $X \in \mathrm{Sm}/S$.*

Proof. See [AGV20, Lemma 2.1.20]. □

Under some mild hypotheses, Lemma 1.10 can be strengthened as in the next proposition. We refer the reader to [AGV20, Definition 2.4.8] for the notion of Λ-cohomological dimension when Λ is connective. Below, we extend this notion to nonconnective ring spectra by declaring that Λ-cohomological dimension is equal to the $\tau_{\geq 0}\Lambda$-cohomological dimension. Also, we will say that a scheme is Λ-good if it is $(\tau_{\geq 0}\Lambda, \acute{e}t)$-good in the sense of [AGV20, Definition 2.4.14].

Proposition 1.11. *Let k be a field of finite virtual Λ-cohomological dimension and S a k-variety. Then the symmetric monoidal ∞-category $\mathbf{MSh}^{(\mathrm{eff})}(S;\Lambda)^{\otimes}$ is compactly generated and its underlying ∞-category is generated under colimits, and up to desuspension and negative twists when applicable, by the objects $\mathrm{M}^{(\mathrm{eff})}(X)$ with $X \in \mathrm{Sm}/S$ assumed Λ-good.*

Proof. This is particular case of a more general result; see for example [AGV20, Proposition 2.4.22 & Remark 2.4.23]. □

Proposition 1.12. *Let S be a scheme and $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$ a commutative ring spectrum. The assignment $X \mapsto \mathbf{MSh}^{(\mathrm{eff})}(X;\Lambda)^{\otimes}$ extends naturally into a functor*

$$\mathbf{MSh}^{(\mathrm{eff})}(-;\Lambda)^{\otimes} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}). \tag{1.3}$$

Proof. We refer to [Rob15] for the construction of the functor $\mathbf{MSh}^{(\mathrm{eff})}(-;\Lambda)^{\otimes}$ in (1.3). □

Notation 1.13. Let $f : Y \to X$ be a morphism in Sch/S. The image of f by the functor $\mathbf{MSh}^{(\mathrm{eff})}(-;\Lambda)^{\otimes}$ in (1.3) is the symmetric monoidal functor

$$f^* : \mathbf{MSh}^{(\mathrm{eff})}(X;\Lambda)^{\otimes} \to \mathbf{MSh}^{(\mathrm{eff})}(Y;\Lambda)^{\otimes}$$

whose underlying functor, also denoted by f^*, is called the inverse image functor. The latter has a right adjoint f_*, called the direct image functor.

The functor $\mathbf{MSh}(-;\Lambda)^{\otimes}$ in (1.3) is an example of what we shall call a Voevodsky pullback formalism. (In [Ayo07a, §1.4.1], up to the monoidal structure, this was called a stable homotopy 2-functor. A closely related notion is that of a coefficient system; see [Dre18, Definition 5.3] and [DG20].)

Definition 1.14. Let S be a quasi-compact and quasi-separated scheme. A Voevodsky pullback formalism over S is a functor

$$\mathcal{H}^{\otimes} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{CAT}_{\infty}^{\mathrm{st}})$$

sending a finite type S-scheme X to a symmetric monoidal stable ∞-category $\mathcal{H}(X)^{\otimes}$ and a morphism $f : Y \to X$ of finite type S-schemes to a symmetric monoidal functor $f^* : \mathcal{H}(X)^{\otimes} \to \mathcal{H}(Y)^{\otimes}$ such that the following conditions are satisfied. (Below, we also write $f^* : \mathcal{H}(X) \to \mathcal{H}(Y)$ for the functor underlying f^*.)

(1) $\mathcal{H}(\emptyset)$ is equivalent to the final ∞-category with one object and one morphism.
(2) For every morphism $f : Y \to X$ in Sch/S, the functor f^* admits a right adjoint f_*. Moreover, given a Cartesian square

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

with g smooth, the exchange morphism $g^* f_* \to f'_* g'^*$ is an equivalence.
(3) If $f : Y \to X$ is a smooth morphism in Sch/S, the functor f^* admits a left adjoint $f_\sharp$. Moreover, for $A \in \mathcal{H}(X)$ and $B \in \mathcal{H}(Y)$, the obvious map $f_\sharp(f^*(A) \otimes B) \to A \otimes f_\sharp(B)$ is an equivalence.
(4) If $i : Z \hookrightarrow X$ is a closed immersion in Sch/S, the functor i_* is fully faithful. Moreover, if $j : U \hookrightarrow X$ is the complementary open immersion, then the pair (i^*, j^*) is conservative.
(5) If X is a k-variety and $p : \mathbb{A}^1_X \to X$ the obvious projection, then p^* is fully faithful.
(6) If X is a k-variety and $s : X \hookrightarrow \mathbb{A}^1_X$ a section to the morphism p in (5), then the endofunctor $p_\sharp \circ s_*$ is an equivalence.

We say that the Voevodsky pullback formalism $\mathcal{H}^{\otimes}$ is presentable if it factors through the ∞-category $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,st}})$.

Proposition 1.15. *The functor* $\mathbf{MSh}(-;\Lambda)$ *in* (1.3) *is a presentable Voevodsky pullback formalism.*

Proof. All the axioms in Definition 1.14 are direct consequences of the construction, except for axiom (4) which is a reformulation of the Morel–Voevodsky localisation theorem [MV99, §3.2, Theorem 2.21]. For a fully detailed exposition, see [Ayo07b, §4.5.2 & 4.5.3]. For a modern treatment of the Morel–Voevodsky localisation theorem, see [Hoy18, §1]. □

Remark 1.16. A Voevodsky pullback formalism gives rise to a full six-functor formalism as explained in [Ayo07a, Ayo07b]. More precisely, the axioms of Definition 1.14 imply the proper base change theorem, which can be used to define the exceptional adjunction

$$f_! : \mathcal{H}(Y) \rightleftarrows \mathcal{H}(X) : f^!$$

for every morphism $f : Y \to X$ in Sch/S.

1.2. **The Betti realisation.**

In this subsection, we recall the construction of the Betti realisation following [Ayo10]. Given a complex analytic variety V, we denote by AnSm/V the category of smooth complex analytic V-varieties which we endow with the classical topology, abbreviated by "cl". Unless otherwise stated, the notion of (hyper)sheaf on AnSm/V is always taken with respect to the classical topology. As usual, we denote by $\mathbb{D}^n_V$ the n-dimensional relative polydisc.

Definition 1.17. Let V be a complex analytic variety and $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$ a commutative ring spectrum. We denote by $\mathbf{AnSh}^{\mathrm{eff}}(V;\Lambda)$ the full sub-∞-category of $\mathrm{Shv}^{\wedge}_{\mathrm{cl}}(\mathrm{AnSm}/V;\Lambda)$ spanned by those hypersheaves which are $\mathbb{D}^1$-invariant. (Recall that a presheaf F is $\mathbb{D}^1$-invariant if, for every $W \in \mathrm{AnSm}/V$, the projection map $p : \mathbb{D}^1_W \to W$ induces an equivalence $p^* : F(W) \simeq F(\mathbb{D}^1_W)$.)

As explained in Remark 1.6, the ∞-category $\mathbf{AnSh}^{\mathrm{eff}}(V;\Lambda)$ underlies a presentable symmetric monoidal ∞-category $\mathbf{AnSh}^{\mathrm{eff}}(V;\Lambda)^{\otimes}$ and we have a symmetric monoidal functor

$$\mathrm{L}_{\mathbb{D}^1} : \mathrm{Shv}^{\wedge}_{\mathrm{cl}}(\mathrm{AnSm}/V;\Lambda)^{\otimes} \to \mathbf{AnSh}^{\mathrm{eff}}(V;\Lambda)^{\otimes} \tag{1.4}$$

whose underlying functor is left adjoint to the obvious inclusion.

Definition 1.18. Let V be a complex analytic variety and $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$ a commutative ring spectrum. We denote by T_V (or simply T is V is clear from the context) the image by the functor $\mathrm{L}_{\mathbb{D}^1}$ in (1.4) of the cofiber of the split inclusion $\Lambda_{\mathrm{cl}}(S) \to \Lambda_{\mathrm{cl}}(\mathbb{D}^1_S \smallsetminus 0_S)$ induced by the unit section. With the notation of [Rob15, Definition 2.6], we set

$$\mathbf{AnSh}(V;\Lambda)^{\otimes} = \mathbf{AnSh}^{\mathrm{eff}}(V;\Lambda)^{\otimes}[\mathrm{T}_V^{-1}].$$

Thus, there is a morphism $\Sigma^{\infty}_{\mathrm{T}} : \mathbf{AnSh}^{\mathrm{eff}}(V;\Lambda)^{\otimes} \to \mathbf{AnSh}(V;\Lambda)^{\otimes}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, sending T_V to a $\otimes$-invertible object, and which is initial for this property. We denote by $\Omega^{\infty}_{\mathrm{T}}$ the right adjoint of $\Sigma^{\infty}_{\mathrm{T}}$.

The ∞-categories introduced above are equivalent to much simpler ones. To state this, we introduce a notation.

Notation 1.19. Let W be a topological space and $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$ a commutative ring spectrum. We denote by $\mathbf{Sh}(W;\Lambda)^{\otimes}$ the symmetric monoidal ∞-category of sheaves on W with coefficients in Λ. In formulas, we set

$$\mathbf{Sh}(W;\Lambda) = \mathrm{Shv}^{\wedge}_{\mathrm{cl}}(\mathrm{Op}(W);\Lambda)$$

with $(\mathrm{Op}(W),\mathrm{cl})$ the site of opens in W with the classical topology, aka., the open cover topology.

Proposition 1.20. *Let V be a complex analytic space and $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$ a commutative ring spectrum. Then, the obvious functors*

$$\mathbf{Sh}(V;\Lambda) \xrightarrow{\iota_V^*} \mathbf{AnSh}^{\mathrm{eff}}(V;\Lambda) \xrightarrow{\Sigma^{\infty}_{\mathrm{T}}} \mathbf{AnSh}(V;\Lambda)$$

are equivalences of ∞-categories.

Proof. The first equivalence is the content of [Ayo10, Théorèmes 1.8]. The second equivalence follows from [Ayo10, Lemme 1.10] and the discussion right after this lemma in loc. cit. □

We now fix a field k endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$. Given a k-variety X, we denote by X^{an} the associated complex analytic variety.

Definition 1.21. Let X be a k-variety and $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$ a commutative ring spectrum. The Betti realisation for motivic sheaves on X is the symmetric monoidal functor

$$\mathrm{B}_X^* : \mathbf{MSh}(X;\Lambda)^{\otimes} \to \mathbf{Sh}(X^{\mathrm{an}};\Lambda)^{\otimes} \tag{1.5}$$

defined as the composition of

$$\mathbf{MSh}(X;\Lambda)^{\otimes} \xrightarrow{\mathrm{An}^*} \mathbf{AnSh}(X^{\mathrm{an}};\Lambda)^{\otimes} \simeq \mathbf{Sh}(X^{\mathrm{an}};\Lambda)^{\otimes}. \tag{1.6}$$

Here, An^* is induced from the functor $\mathrm{Sm}/X \to \mathrm{AnSm}/X^{\mathrm{an}}$, given by $Y \mapsto Y^{\mathrm{an}}$, by the functoriality of the constructions in Definitions 1.7 and 1.18. The functor B_X^* admits a right adjoint $\mathrm{B}_{X,*}$. If no confusion can arise, we sometimes write B^* and B_* instead of B_X^* and $\mathrm{B}_{X,*}$.

Proposition 1.22. *The functors* B_X^* *in* (1.5) *are part of a morphism of* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$*-valued presheaves*

$$\mathrm{B}^* : \mathbf{MSh}(-;\Lambda)^{\otimes} \to \mathbf{Sh}((-)^{\mathrm{an}};\Lambda)^{\otimes} \tag{1.7}$$

defined on Sch/k. *Moreover, if f is a smooth morphism in* Sch/k*, the natural transformation*

$$f_\sharp^{\mathrm{an}} \circ \mathrm{B}^* \to \mathrm{B}^* \circ f_\sharp$$

is an equivalence.

Proof. One argues as in [Rob15, §9.1] for the first assertion. The second assertion is clear. □

For later use, we need to adjust the Betti realisation in order to make its target a compactly generated ∞-category.

Definition 1.23. Let $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$ be a commutative ring spectrum.

(i) Let W be a complex analytic variety. We denote by $\mathbf{LS}(W;\Lambda)$ the full sub-∞-category of $\mathbf{Sh}(W;\Lambda)$ consisting of dualizable objects. Objects of $\mathbf{LS}(W;\Lambda)$ will be also called local systems on W; see Lemma 1.24 below. We also denote by $\widehat{\mathbf{LS}}(W;\Lambda)$ the full sub-∞-category of $\mathbf{Sh}(W;\Lambda)$ generated under colimits by the objects of $\mathbf{LS}(W;\Lambda)$. Objects of $\widehat{\mathbf{LS}}(W;\Lambda)$ will be called lisse sheaves on W.

(ii) Let X be a k-variety. A sheaf $F \in \mathbf{Sh}(X^{\mathrm{an}};\Lambda)$ is said to be constructible if for every point $x \in X$, there is a locally closed subvariety $Z \subset X$ containing x such that $F|_{Z^{\mathrm{an}}}$ is dualizable, i.e., belongs to $\mathbf{LS}(Z^{\mathrm{an}};\Lambda)$. We denote by $\mathbf{Ct}(X;\Lambda)$ the full sub-∞-category of $\mathbf{Sh}(X^{\mathrm{an}};\Lambda)$ consisting of constructible sheaves. We also set $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda) = \mathrm{Ind}(\mathbf{Ct}(X;\Lambda))$. There is an obvious colimit-preserving functor

$$\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda) \to \mathbf{Sh}(X^{\mathrm{an}};\Lambda) \tag{1.8}$$

which induces the identity functor on $\mathbf{Ct}(X;\Lambda)$, but which is not fully faithful unless X is zero-dimensional. Finally, we write $\mathbf{LS}(X;\Lambda)$ for $\mathbf{LS}(X^{\mathrm{an}};\Lambda)$ considered as a sub-∞-category of $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$ and we let $\widehat{\mathbf{LS}}(X;\Lambda)$ be the full sub-∞-category of $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$ generated under colimits by the objects of $\mathbf{LS}(X;\Lambda)$.

Below, we give two lemmas around the ∞-categories introduced in Definition 1.23. We start with the lemma that justifies our notion of local system.

Lemma 1.24. *Let W be a complex analytic variety, and $F \in \mathbf{Sh}(W;\Lambda)$ a sheaf on W. Then the following conditions are equivalent.*

(i) The sheaf F is dualizable.

(ii) There exists an open cover $(W_i)_{i\in I}$ of W such that the sheaves $F|_{W_i}$ are constant with value a perfect Λ-module.

(iii) For every contractible open subvariety $U \subset W$, the sheaf $F|_U$ is constant with value a perfect Λ-module.

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious. The implication (ii) $\Rightarrow$ (iii) is standard, but we include an argument for the reader's convenience. Assume that (ii) is satisfied. Replacing W by U, we may assume that W is contractible. In particular it is connected and all the fibers of F are equivalent to a fixed object $F_0 \in \mathrm{Mod}_\Lambda$. Let $Q = \mathrm{Equi}_{\mathrm{Mod}_\Lambda}(F_0, F)$ be the hypersheaf on W sending a connected open subset $V \subset W$ to the space of equivalences from F_0 to $F(V)$. Then Q admits an action of the H-group $G = \mathrm{Auteq}_{\mathrm{Mod}_\Lambda}(F_0)$. In fact, condition (ii) implies that Q is a G-torsor over W. Such a G-torsor is classified by a map $W \to \mathrm{B}(G)$, with $\mathrm{B}(G)$ the classifying space of G. Since W is contractible, every such map is null-homotopic, which implies that the G-torsor Q is necessarily trivial. A global section of Q yields an equivalence between F and the constant sheaf with value F_0 on W. This proves (iii).

It remains to show the implication (i) $\Rightarrow$ (ii). Assume that F is dualizable. We fix a point $x \in W$, and we show that F is constant in the neighbourhood of x. The functor $A \mapsto A_x$ is symmetric monoidal. This implies that the Λ-module F_x is perfect. Writing $F_x = \mathrm{colim}_{x\in U} F(U)$, where the colimit is over the open neighbourhoods of x, we deduce that the identity of F_x lifts to a morphism of Λ-modules $F_x \to F(U)$, for U small enough. Replacing W by U, we obtain a morphism $(F_x)_{\mathrm{cst}} \to F$ from the constant sheaf $(F_x)_{\mathrm{cst}}$ with value F_x, inducing the identity on the stalks at x. The cofiber G of this morphism is still dualizable, and it is enough to show that it is zero in the neighbourhood of x. Let $G^\vee$ be the dual of G, and consider the unit morphism $\eta : \Lambda_W \to G \otimes_\Lambda G^\vee$. Restricting to an open neighbourhood U of x and passing to global sections, we obtain a morphism of Λ-modules

$$\Lambda \to \Gamma(U; G \otimes_\Lambda G^\vee). \tag{1.9}$$

Taking the colimit over U, we obtain the map $\Lambda \to G_x \otimes_\Lambda G^\vee_x \simeq 0$. Since Λ is compact, we deduce that the morphism (1.9) is zero for U small enough. Said differently, the unit map $\Lambda_U \to G|_U \otimes_\Lambda G^\vee|_U$ is zero. This implies that $G|_U$ is zero as needed. □

We also record the following technical but reassuring result.

Lemma 1.25. *Let X be a k-variety. The functor in* (1.8) *restricts to an equivalence of ∞-categories*

$$\widehat{\mathbf{LS}}(X;\Lambda) \simeq \widehat{\mathbf{LS}}(X^{\mathrm{an}};\Lambda). \tag{1.10}$$

Proof. Recall that the functor in (1.8) induces the identity functor $\mathbf{LS}(X;\Lambda) = \mathbf{LS}(X^{\mathrm{an}};\Lambda)$, and that $\widehat{\mathbf{LS}}(X;\Lambda)$ and $\widehat{\mathbf{LS}}(X^{\mathrm{an}};\Lambda)$ are the sub-∞-categories of $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$ and $\mathbf{Sh}(X^{\mathrm{an}};\Lambda)$ generated under colimits by the local systems on X^{an}. Thus, $\widehat{\mathbf{LS}}(X;\Lambda)$ is compactly generated by the objects of $\mathbf{LS}(X;\Lambda)$ and it is enough to show the same for $\widehat{\mathbf{LS}}(X^{\mathrm{an}};\Lambda)$. More precisely, we need to show that a local system on X^{an} is a compact object of the ∞-category $\widehat{\mathbf{LS}}(X^{\mathrm{an}};\Lambda)$. We warn the reader that, on the contrary, a local system is not a compact object of $\mathbf{Sh}(X^{\mathrm{an}};\Lambda)$ in general.

Since local systems are dualizable, we are reduced to showing that Λ_{cst} is a compact object of $\widehat{\mathbf{LS}}(X^{\mathrm{an}};\Lambda)$ or, equivalently, that the functor $\Gamma(X^{\mathrm{an}};-)$ commutes with direct sums. Let $(M_\alpha)_\alpha$ be a family of lisse sheaves on X^{an} and $M = \bigoplus_\alpha M_\alpha$. We need to show that the natural map

$$\bigoplus_\alpha \Gamma(X^{\mathrm{an}}; M_\alpha) \to \Gamma(X^{\mathrm{an}}; M) \tag{1.11}$$

is an equivalence. Both sides of the map in (1.11) satisfy cdh excision. Using resolution of singularities in characteristic zero [Hir64], we reduce to the case where X is smooth, and admitting an open immersion $j : X \hookrightarrow \overline{X}$ with $\overline{X}$ proper and $\overline{X} \smallsetminus j(X)$ a strict normal crossing divisor. By [Lur09a, Corollary 7.3.4.12], the functor $\Gamma(\overline{X}^{\mathrm{an}}; -)$ commutes with direct sums. Thus, we are left to showing that the morphism

$$\bigoplus_{\alpha} j_* M_\alpha \to j_* M \tag{1.12}$$

is an equivalence in $\mathbf{Sh}(\overline{X}^{\mathrm{an}}; \Lambda)$. We check this on stalks, and fix a point $x \in \overline{X}^{\mathrm{an}}$. For U in a cofinal system of neighbourhoods of x in $\overline{X}^{\mathrm{an}}$, we can find isomorphisms

$$U \simeq \mathbb{D}^{m+n} \qquad \text{and} \qquad j^{-1}(U) \simeq \mathbb{D}^m \times (\mathbb{D}^1 \smallsetminus \{0\})^n,$$

for some integers $m, n \geq 0$. In particular, $j^{-1}(U)$ is then equivalent to the classifying space $\mathrm{B}(\mathbb{Z}^m)$ of $\mathbb{Z}^m$. It is enough to show that

$$\bigoplus_{\alpha} \Gamma(j^{-1}(U), M_\alpha|_{j^{-1}(U)}) \to \Gamma(j^{-1}(U), M|_{j^{-1}(U)}) \tag{1.13}$$

is an equivalence for such U's. The stalk of the lisse sheaf $M_\alpha|_{j^{-1}(U)}$ at a base point of $j^{-1}(U)$ is a Λ-module N_α with an action of $\mathbb{Z}^m$. Setting $N = \bigoplus_\alpha N_\alpha$, the morphism in (1.13) is equivalent to

$$\bigoplus_{\alpha} \Gamma(\mathbb{Z}^m; N_\alpha) \to \Gamma(\mathbb{Z}^m; N).$$

Thus, we are reduced to showing that $\Gamma(\mathbb{Z}^m; -)$ commutes with direct sums. By induction, we may assume that $m = 1$. The result follows then from the fact that $\Gamma(\mathbb{Z}; -)$ can be computed as the equalizer of the identity map and the action of $1 \in \mathbb{Z}$. $\square$

Proposition 1.26. *The six operations resulting from the Voevodsky pullback formalism*

$$\mathbf{Sh}((-)^{\mathrm{an}}; \Lambda)^{\otimes} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

preserve the sub-∞-categories of constructible sheaves.

Proof. This is well-known fact, at least when Λ is an ordinary ring. For the reader's convenience, we include a proof. To simplify notation, we shall write f^*, f_*, etc., instead of $f^{\mathrm{an},*}$, f^{an}_*, etc. We split the proof in several steps.

Step 1. The result is obvious for the ordinary pullback and the tensor product. In this step, we assume that the result is also known for the ordinary direct image functors, and we explain how to derive the result for the remaining operations.

Given a morphism f of k-varieties, to show that $f_!$ preserves constructible sheaves, we may assume that f is an open immersion or a proper morphism. The case of open immersions is clear and, in the case of proper morphisms, we use that $f_! \simeq f_*$ to conclude. Similarly, to prove that $f^!$ preserves constructible sheaves, we may assume that f is smooth or a closed immersion. If f is smooth, we conclude using that $f^!$ is equivalent to f^* up to twist and shift. If f is a closed immersion, we use the localisation triangle and the assumption that ordinary direct images along open immersions preserves constructible sheaves. Finally, to prove that $\underline{\mathrm{Hom}}(F, G)$ is constructible if $F, G \in \mathbf{Sh}(X^{\mathrm{an}}; \Lambda)$ are constructible, we may assume that F is of the form $u_! L$ where $u : Z \to X$ is a locally closed immersion and L is a local system on Z^{an}. In this case, we have equivalences

$$\underline{\mathrm{Hom}}(u_! L, G) \simeq u_* \underline{\mathrm{Hom}}(L, u^! G) \simeq u_*(L^\vee \otimes_\Lambda u^! G)$$

and the result follows from the previous considerations.

Step 2. By the previous step, it remains to see that f_* preserves constructible sheaves for every morphism $f : Y \to X$ of k-varieties.

We argue by induction on the dimension of Y. It is enough to show that f_*F is constructible for $F = h_!L$ with $h : T \hookrightarrow Y$ a locally closed immersion and L a local system on T^{an}. Replacing Y with the closure of $h(T)$, we may assume that h is the inclusion of a dense open subvariety $T \subset Y$. Let $V \subset T$ be the smooth locus of T, which is an open dense subvariety of Y. Let $v : V \hookrightarrow Y$ be the obvious inclusion and $s : Z \to Y$ the complementary closed immersion. We have an exact triangle

$$f_*F \to f_*v_*v^*F \to f_*s_*s^*C \to$$

with $C = \mathrm{cofib}(F \to v_*v^*F)$. Applying the induction hypothesis to $f \circ s : Z \to X$, we see that f_*F is constructible if $(f \circ v)_*F|_V$ and $v_*F|_V$ are constructible. This shows that we may replace f by $f \circ v$ and v. In particular, we may assume that Y is smooth and F is a local system on Y.

Step 3. Using resolution of singularities in characteristic zero [Hir64], we may find an open immersion $j : Y \to \overline{Y}$ over X, with $\overline{Y}$ smooth and $D = \overline{Y} \smallsetminus Y$ a normal crossing divisor. In this step, we prove that j_*F is constructible. (Recall that F is a local system on Y, by Step 2.)

Let $D_1, \ldots, D_m$ be the irreducible components of D and, for $I \subset \{1, \ldots, m\}$ nonempty, let

$$D_I = \bigcap_{a \in I} D_a \qquad \text{and} \qquad D_I^\circ = D_I \smallsetminus \bigcup_{b \notin I} D_b.$$

We claim that the restriction of j_*F to D_I° is a local system. Indeed, every point of $(D_I^\circ)^{\mathrm{an}}$ admits an open neighbourhood V in $\overline{Y}^{\mathrm{an}}$ such that

$$V \simeq \mathbb{D}^n \qquad \text{and} \qquad j^{-1}(V) \simeq \mathbb{D}^{n-r} \times (\mathbb{D}^1 \smallsetminus \{0\})^r.$$

(Of course, n is the local dimension of Y and r is the cardinality of I.) We form the commutative diagram with cartesian squares

$$\begin{tikzcd}
j^{-1}(V) \arrow[r] \arrow[d] & V \arrow[d] & V \cap (D_I^\circ)^{\mathrm{an}} \arrow[l] \arrow[d, "p"] \\
(\mathbb{D}^1 \smallsetminus \{0\})^r \arrow[r, "j_0"] & \mathbb{D}^r & 0^r. \arrow[l, "i_0"']
\end{tikzcd}$$

Clearly, the local system $F|_{j^{-1}(V)}$ is the pullback of a local system F_0 on $(\mathbb{D}^1 \smallsetminus \{0\})^r$. Using the (analytic) smooth base change theorem, it follows that the restriction of j_*F to $V \cap (D_I^\circ)^{\mathrm{an}}$ is isomorphic to $p^*i_0^*j_{0,*}F_0$. Moreover, arguing as in the proof of Lemma 1.25, we see that the Λ-module $i_0^*j_{0,*}F_0$ is perfect. This proves what we want.

Step 4. Let $g : \overline{Y} \to X$ be the structural morphism of the X-scheme $\overline{Y}$. Recall that our aim is to prove that $f_*F \simeq g_*j_*F$ is constructible.

Replacing $\overline{Y}$ with a connected component and X by the image of this component, we may assume that $\overline{Y}$ is integral and g dominant. Using the proper base change theorem, the constructibility of j_*F and induction on the dimension, we can replace X with any dense open subvariety of X. Thus, we may assume that g is smooth and that $D = \overline{Y} \smallsetminus Y$ is a relative normal crossing divisor, i.e., all the D_I's from Step 3 are smooth over X. We claim that in this situation, g_*j_*F is a local system on X. This is typically proven using Ehresmann's theorem. We will give below a different proof based on the six-functor formalism.

Step 5. In this last step, we finish the proof of the proposition. We work in the situation we have reached in Step 4. To show that f_*F is a local system, we may show that the obvious morphism

$$\underline{\mathrm{Hom}}(A,\Lambda)\otimes_\Lambda g_*j_*F \to \underline{\mathrm{Hom}}(A, g_*j_*F) \tag{1.14}$$

is an equivalence for every $A \in \mathbf{Sh}(X^{\mathrm{an}};\Lambda)$. We have obvious equivalences

$$\begin{aligned} \underline{\mathrm{Hom}}(A,\Lambda)\otimes_\Lambda g_*j_*F &\simeq g_*(g^*\underline{\mathrm{Hom}}(A,\Lambda)\otimes_\Lambda j_*F) \\ &\simeq g_*(\underline{\mathrm{Hom}}(g^*A,\Lambda)\otimes_\Lambda j_*F), \end{aligned}$$

$$\text{and}\quad \underline{\mathrm{Hom}}(A, g_*j_*F) \simeq g_*\underline{\mathrm{Hom}}(g^*A, j_*F).$$

The third equivalence is obvious, while the first two rely on the fact that g is proper and smooth. Therefore, it is enough to show that the natural morphism

$$\underline{\mathrm{Hom}}(g^*A,\Lambda)\otimes_\Lambda j_*F \to \underline{\mathrm{Hom}}(g^*A, j_*F) \tag{1.15}$$

is an equivalence. This is a local question over $\overline{Y}^{\mathrm{an}}$, and can be checked on stalks. We fix a point $y \in \overline{Y}^{\mathrm{an}}$. For V varying in a fundamental system of neighbourhoods of y, and U the image of V in X^{an}, we have:

(1) $U \simeq \mathbb{D}^m$, $V \simeq \mathbb{D}^n$ and, modulo these isomorphisms, $V \to U$ is given by the projection to the first m coordinates.
(2) $j^{-1}(V) \simeq \mathbb{D}^{n-r}\times(\mathbb{D}^1 \smallsetminus \{0\})^r$ with $r \leq n-m$.

It is enough to show that for these V's, the following morphism of Λ-modules

$$\Gamma(V;\underline{\mathrm{Hom}}(g^*A,\Lambda))\otimes_\Lambda \Gamma(V; j_*F) \to \Gamma(V;\underline{\mathrm{Hom}}(g^*A, j_*F)) \tag{1.16}$$

is an equivalence. We have a commutative diagram with Cartesian squares

$$\begin{tikzcd} j^{-1}(V) \arrow[r, "j'"] \arrow[d, "p''"] & V \arrow[r, "g'"] \arrow[d, "p'"] & U \arrow[d, "p"] \\ \mathbb{D}^{n-m-r}\times(\mathbb{D}^1 \smallsetminus \{0\})^r \arrow[r, "j_0"] & \mathbb{D}^{n-m} \arrow[r, "g_0"] & \mathrm{pt} \end{tikzcd}$$

and a local system F_0 on $\mathbb{D}^{n-m-r}\times(\mathbb{D}^1 \smallsetminus \{0\})^r$ such that $F|_{j^{-1}(V)} \simeq p''^*F_0$. Up to canonical equivalences, the morphism in (1.16) can be rewritten as

$$\Gamma(U;\underline{\mathrm{Hom}}(A|_U,\Lambda))\otimes_\Lambda \Gamma(\mathbb{D}^{n-m}; j_{0,*}F_0) \to \Gamma(V;\underline{\mathrm{Hom}}(g'^*A|_U; p'^*j_{0,*}F_0)). \tag{1.17}$$

We have a chain of natural equivalences

$$\begin{aligned} \Gamma(V;\underline{\mathrm{Hom}}(g'^*A|_U; p'^*j_{0,*}F_0)) &\overset{(1)}{\simeq} \Gamma(U;\underline{\mathrm{Hom}}(A|_U; g'_*p'^*j_{0,*}F_0)) \\ &\overset{(2)}{\simeq} \Gamma(U;\underline{\mathrm{Hom}}(A|_U; p^*g_{0,*}j_{0,*}F_0)) \\ &\overset{(3)}{\simeq} \Gamma(U;\underline{\mathrm{Hom}}(A|_U;\Lambda)\otimes_\Lambda p^*g_{0,*}j_{0,*}F_0) \\ &\overset{(4)}{\simeq} \Gamma(U;\underline{\mathrm{Hom}}(A|_U,\Lambda))\otimes_\Lambda \Gamma(\mathbb{D}^{n-m}; j_{0,*}F_0) \end{aligned}$$

where:

(1) is induced by the adjunction (g'^*, g'_*);
(2) follows from the (analytic) smooth base change theorem applied to the second square in the above commutative diagram;

(3, 4) follow from the fact that $p^* g_{0,*} j_{0,*} F_0$ is a (constant) local system with values $\Gamma(\mathbb{D}^{n-m}; j_{0,*}F_0)$ which is a perfect Λ-module.

This clearly proves that (1.17) is an equivalence, and finishes the proof of the proposition. □

Corollary 1.27. *There are two Voevodsky pullback formalisms*

$$\mathbf{Ct}(-;\Lambda)^{\otimes} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty) \qquad \textit{and} \qquad \mathbf{Sh}_{\mathrm{ct}}(-;\Lambda)^{\otimes} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$$

related by a morphism of six-functor formalisms $\mathbf{Ct}(-;\Lambda)^{\otimes} \to \mathbf{Sh}_{\mathrm{ct}}(-;\Lambda)^{\otimes}$*. Moreover, the four operations* f^**,* f_**,* $f_!$ *and* $f^!$ *associated to a morphism of k-varieties* f *by* $\mathbf{Sh}_{\mathrm{ct}}(-;\Lambda)$ *belong to* $\mathrm{Pr}^{\mathrm{L}}_{\omega}$*, i.e., are compact-preserving left adjoint functors.*

We summarise the main facts about the refined Betti realisation in the following statement.

Theorem 1.28. *Let* $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$ *be a commutative ring spectrum. There is a commutative triangle of Voevodsky pullback formalisms*

$$\begin{tikzcd} \mathbf{MSh}(-;\Lambda) \arrow[r, "\mathrm{B}^*_{\mathrm{ct}}"] \arrow[dr, "\mathrm{B}^*"'] & \mathbf{Sh}_{\mathrm{ct}}(-;\Lambda) \arrow[d] \\ & \mathbf{Sh}((-)^{\mathrm{an}};\Lambda) \end{tikzcd}$$

where the vertical arrow is given by the functors in (1.8)*. Moreover,* $\mathrm{B}^*_{\mathrm{ct}}$ *is actually a morphism of six-functor formalisms.*

Proof. To prove the theorem, we may replace k by an algebraic closure. In particular, we may assume that k has finite Λ-cohomological dimension. By Proposition 1.11, it follows that the ∞-categories $\mathbf{MSh}(X;\Lambda)$ are compactly generated for all $X \in \mathrm{Sch}/k$, i.e., we have equivalences

$$\mathrm{Ind}(\mathbf{MSh}(X;\Lambda)_{\omega}) \simeq \mathbf{MSh}(X;\Lambda). \tag{1.18}$$

By [Ayo07a, Théorème 2.2.37 & Corollaire 2.3.65], the six operations deduced from $\mathbf{MSh}(-;\Lambda)^{\otimes}$ respect compact objects. It is worth noting here that compactness for motivic sheaves (under our hypothesis on k) coincides with the notion of constructibility introduced in [Ayo07a, Définition 2.2.3] relatively to the set $\{\Lambda(n); n \in \mathbb{Z}\}$. (This is just Proposition 1.11.) This also implies immediately that the functors B^*_X in (1.5) take compact motivic sheaves to constructible sheaves, and thus induce a morphism of $\mathrm{CAlg}(\mathrm{Cat}_\infty)$-valued presheaves

$$\mathbf{MSh}(-;\Lambda)^{\otimes}_{\omega} \to \mathbf{Ct}(-;\Lambda)^{\otimes}.$$

Applying indization of ∞-categories and using (1.18), we obtain a morphism of Voevodsky pullback formalisms $\mathrm{B}^*_{\mathrm{ct}} : \mathbf{MSh}(-;\Lambda)^{\otimes} \to \mathbf{Sh}_{\mathrm{ct}}(-;\Lambda)^{\otimes}$. All the remaining assertions are clear. □

Remark 1.29. We will sometimes refer to the morphism of Voevodsky pull-back formalisms $\mathrm{B}^*_{\mathrm{ct}}$ as the refined Betti realisation, or simply as the Betti realisation if no confusion can arise. Also, we will often write B^* and B^*_X instead of $\mathrm{B}^*_{\mathrm{ct}}$ and $\mathrm{B}^*_{\mathrm{ct},X}$.

For later use, we note the following fact.

Proposition 1.30. *The functor*

$$\mathrm{B}_{\mathrm{ct},*} : \mathbf{Sh}_{\mathrm{ct}}(X;\Lambda) \to \mathbf{MSh}(X;\Lambda), \tag{1.19}$$

right adjoint to $\mathrm{B}^*_{\mathrm{ct}}$*, is colimit-preserving.*

Proof. Since $\mathrm{B}_{\mathrm{ct},*}$ is an exact functor between stable ∞-categories, it is enough to show that it preserves filtered colimits. If the base field k has finite virtual Λ-cohomological dimension, the functor $\mathrm{B}^*_{\mathrm{ct}}$ belongs to $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ and the claim follows from [Lur09a, Proposition 5.5.7.2(2)]. In general, we argue as follows. Let $\mathbf{MSh}_{\mathrm{nis}}(X;\Lambda)$ be the ∞-category of motivic sheaves on X in the Nisnevich topology, i.e., the ∞-category obtained by employing the Nisnevich topology instead of the étale topology in Definitions 1.5 and 1.7. There is an obvious functor

$$\mathrm{a}_{\text{ét}} : \mathbf{MSh}_{\mathrm{nis}}(X;\Lambda) \to \mathbf{MSh}(X;\Lambda)$$

which is a localisation functor, i.e., its right adjoint functor $\mathrm{o}_{\text{ét}}$ is fully faithful. The ∞-category $\mathbf{MSh}_{\mathrm{nis}}(X;\Lambda)$ is compactly generated and the composite functor $\mathrm{B}^*_{\mathrm{ct}}\circ \mathrm{a}_{\text{ét}}$ preserves compact objects. Using [Lur09a, Proposition 5.5.7.2(2)] as above, we conclude that $\mathrm{o}_{\text{ét}}\circ \mathrm{B}_{\mathrm{ct},*}$ is colimit-preserving. The result follows since $\mathrm{B}_{\mathrm{ct},*} \simeq \mathrm{a}_{\text{ét}}\circ \mathrm{o}_{\text{ét}}\circ \mathrm{B}_{\mathrm{ct},*}$ and $\mathrm{a}_{\text{ét}}$ is colimit-preserving. $\square$

1.3. Motivic Galois group, I. The general case.

We explain here the construction of the motivic Galois groupoid associated to a Weil spectrum taking advantage of the ∞-categorical edifice. In Subsection 1.4, we use this to recover the motivic Galois group associated to the Betti realisation as defined and studied in [Ayo14b, Ayo14c]. The motivic Galois groupoid arises naturally as a nonconnective affine spectral groupoid scheme in the sense of spectral algebraic geometry [Lur18]. We start by recalling the notions of group and groupoid objects in an ∞-category following [Lur09a, Definition 6.1.2.7].

Definition 1.31. Let $\mathcal{C}$ be an ∞-category. A groupoid object in $\mathcal{C}$ is a cosimplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that, for every integer $n \geq 0$ and every covering $\{0,\dots,n\} = I \cup J$ with $I \cap J = \{m\}$ a singleton, the square

$$\begin{tikzcd} X(\Delta^{\{0,\dots,n\}}) \arrow[r] \arrow[d] & X(\Delta^J) \arrow[d] \\ X(\Delta^I) \arrow[r] & X(\Delta^{\{m\}}) \end{tikzcd} \tag{1.20}$$

is Cartesian. We say that X is a group object if moreover $X(\Delta^0)$ is a final object.

The motivic Galois groupoid will be defined as a groupoid object in the ∞-category of nonconnective affine spectral schemes. We now recall this ∞-category.

Definition 1.32. A nonconnective spectral scheme $X = (|X|, \mathcal{O}_X)$ is a pair consisting of a topological space $|X|$ and a $\mathrm{CAlg}(\mathcal{S}p)$-valued hypersheaf $\mathcal{O}_X$, called the structural sheaf, such that the following properties are satisfied.

(i) The ringed space $X^{\mathrm{cl}} = (|X|, \pi_0\mathcal{O}_X)$ is a scheme in the classical sense;
(ii) For every $i \in \mathbb{Z}$, the $\pi_0\mathcal{O}_X$-module $\pi_i\mathcal{O}_X$ is quasi-coherent.

The scheme X^{cl} is called the underlying scheme of X. A spectral scheme X is a nonconnective spectral scheme X such that $\mathcal{O}_X$ is connective, i.e., the sheaves $\pi_i\mathcal{O}_X$ are trivial for $i < 0$. A (nonconnective) spectral scheme X is said to be affine if X^{cl} is affine.

Notation 1.33. We denote by $\mathrm{SpSCH}^{\mathrm{nc}}$ the ∞-category of nonconnective spectral schemes and morphisms of locally spectrally ringed spaces in the sense of [Lur18, Definition 1.1.5.1]. We denote by SpSCH the full sub-∞-category of $\mathrm{SpSCH}^{\mathrm{nc}}$ consisting of spectral schemes. We also denote by $\mathrm{SpAFF}^{(\mathrm{nc})}$ the full sub-∞-category of $\mathrm{SpSCH}^{(\mathrm{nc})}$ consisting of affine (nonconnective) spectral schemes.

Remark 1.34. The point of view of spectrally ringed spaces will be of little use in this paper. In fact, we will rather think about spectral schemes via their functors of points. More precisely, to a commutative ring spectrum A, one can associate an affine nonconnective spectral scheme $\mathrm{Spec}(A)$, whose underlying scheme is $\mathrm{Spec}(\pi_0 A)$. By [Lur18], this construction yields equivalences of ∞-categories

$$\mathrm{Spec} : \mathrm{CAlg}(\mathcal{S}p)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{SpAFF}^{\mathrm{nc}} \qquad \text{and} \qquad \mathrm{Spec} : \mathrm{CAlg}(\mathcal{S}p_{\geq 0})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{SpAFF}.$$

Now, using the Yoneda embedding, one obtains a fully faithful functor

$$\mathrm{SpSCH}^{(\mathrm{nc})} \to \mathscr{P}(\mathrm{SpAFF}^{(\mathrm{nc})}).$$

The image of a (nonconnective) spectral scheme X by this functor will be also denoted by X. The latter is really the "functor of points" associated to X. More generally, an object of $\mathscr{P}(\mathrm{SpAFF}^{(\mathrm{nc})})$ will be called a (nonconnective) spectral prestack.

Notation 1.35. Given a (nonconnective) commutative ring spectrum $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$, we denote by $\mathrm{SpSCH}^{(\mathrm{nc})}_{/\Lambda}$ and $\mathrm{SpAFF}^{(\mathrm{nc})}_{/\Lambda}$ the ∞-categories of objects over $\mathrm{Spec}(\Lambda)$ in $\mathrm{SpSCH}^{(\mathrm{nc})}$ and $\mathrm{SpAFF}^{(\mathrm{nc})}$.

Definition 1.36. Let Λ be a (nonconnective) commutative ring spectrum. A (nonconnective) spectral group(oid) Λ-scheme G is a group(oid) object in the ∞-category $\mathrm{SpSCH}^{(\mathrm{nc})}_{/\Lambda}$. We say that G is affine if it is a group(oid) object in the ∞-category $\mathrm{SpAFF}^{(\mathrm{nc})}_{/\Lambda}$.

Remark 1.37. The ∞-category of affine spectral group(oid) schemes over Λ is equivalent to the ∞-category of group(oid) objects in $\mathrm{CAlg}(\mathcal{S}p_{\geq 0})^{\mathrm{op}}$ and hence to the opposite of the ∞-category of commutative Hopf algebra (resp. algebroid) objects in $\mathcal{S}p_{\geq 0}$. Said differently, every affine spectral group(oid) scheme G arises in an essentially unique way as the spectrum of a commutative Hopf algebra (resp. algebroid) object in $\mathcal{S}p_{\geq 0}$. The same applies in the nonconnective case. Recall that a commutative Hopf algebra (algebroid) object in a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is a cosimplicial object $H : \mathbf{\Delta} \to \mathrm{CAlg}(\mathcal{C})$ satisfying the dual conditions for a group(oid) object in Definition 1.31.

To go further we need the notion of a Weil spectrum.

Definition 1.38. Let k be a field and Λ a commutative ring spectrum. A Weil Λ-spectrum is a commutative algebra $\mathscr{A} \in \mathrm{CAlg}(\mathbf{MSh}(k;\Lambda))$ satisfying the following two conditions.

(i) For every $\Gamma(k;\mathscr{A})$-module L, the obvious morphism
$$L \to \Gamma(k; \mathscr{A} \otimes_{\Gamma(k;\mathscr{A})} L) \tag{1.21}$$
is an equivalence.

(ii) For every motive $M \in \mathbf{MSh}(k;\Lambda)$, the composition of
$$\mathscr{A} \otimes_{\Gamma(k;\mathscr{A})} \Gamma(k; \mathscr{A} \otimes_\Lambda M) \to \mathscr{A} \otimes_{\Gamma(k;\mathscr{A})} \mathscr{A} \otimes_\Lambda M \to \mathscr{A} \otimes_\Lambda M \tag{1.22}$$
is an equivalence. (The first morphism is deduced from the unit of the adjunction between Γ and its left adjoint given by tensoring Λ-modules with the unit motive.)

We will say that $\mathscr{A}$ is neutral over Λ if the obvious map $\Lambda \to \Gamma(k;\mathscr{A})$ is an equivalence.

Remark 1.39. Condition (i) in Definition 1.38 is automatically satisfied when k has finite Λ-cohomological dimension. Indeed, in this case, the functor $\Gamma(k;-)$ is colimit-preserving by Proposition 1.11. (Note that $\mathrm{Spec}(k)$ is Λ-good.) Since the desuspensions of $\Gamma(k;\mathscr{A})$ form a set of compact generators of the ∞-category $\mathrm{Mod}_{\Gamma(k;\mathscr{A})}$, it is enough to check that the morphism in (1.21) is an equivalence for $L = \Gamma(k;\mathscr{A})$, which is obvious.

Remark 1.40. The notion of a Weil spectrum is closely related to the notion of a Weil cohomology as formalised in [CD12, Definition 2.1.4] and [Ayo20, Définition 1.1]. Indeed, a Weil cohomology Γ_W in the sense of [Ayo20, Définition 1.1] determines a commutative algebra $\mathbf{\Gamma}_W$ in $\mathbf{MSh}(k)$ which is a Weil spectrum by [Ayo20, Proposition 2.8]. Conversely, if one modifies [Ayo20, Définition 1.1] in order to allow commutative ring spectra instead of only allowing commutative dg $\mathbb{Q}$-algebras, then a Weil spectrum $\mathscr{A}$ defines a Weil cohomology given by $X \mapsto \Gamma(k; \underline{\mathrm{Hom}}(\mathrm{M}(X), \mathscr{A}))$. One can easily set up an equivalence between ∞-categories of Weil spectra and Weil cohomologies. We leave this to the interested reader, since this will be irrelevant for us.

The next lemma is a generalisation of [CD12, Theorem 2.6.2] with an essentially identical proof.

Lemma 1.41. *Let $\mathscr{A}$ be a Weil Λ-spectrum. Then, the obvious functor*

$$\mathscr{A} \otimes_{\Gamma(k;\mathscr{A})} - : \mathrm{Mod}_{\Gamma(k;\mathscr{A})} \to \mathbf{MSh}(k; \mathscr{A}) \tag{1.23}$$

is an equivalence of ∞-categories.

Proof. The functor in (1.23) has a right adjoint given by $\mathcal{M} \mapsto \Gamma(k; \mathcal{M})$. Condition (i) in Definition 1.38 says precisely that the unit of this adjunction is an equivalence. Therefore, the functor in (1.23) is fully faithful. Since $\mathbf{MSh}(k; \mathscr{A})$ is generated under colimits by objects of the form $\mathscr{A} \otimes_\Lambda M$, with $M \in \mathbf{MSh}(k; \Lambda)$, we deduce that the functor in (1.23) is also essentially surjective. □

The following statement generalises property (ii) in Definition 1.38.

Corollary 1.42. *Let $\mathscr{A}$ be a Weil Λ-spectrum. The obvious morphism*

$$\mathscr{A} \otimes_{\Gamma(k;\mathscr{A})} \Gamma(k; \mathcal{M}) \to \mathcal{M} \tag{1.24}$$

is an equivalence for every $\mathscr{A}$-module $\mathcal{M}$ in $\mathbf{MSh}(k; \Lambda)$.

Proof. This follows from Lemma 1.41. Indeed, the morphism in (1.24) is the counit of the adjunction between the functor in (1.23) and its right adjoint. □

Notation 1.43. Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category and let $A \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra in $\mathcal{C}^\otimes$. The cobar construction on A is the cosimplicial commutative algebra

$$\mathrm{cobar}^\bullet(A) : \mathbf{\Delta} \to \mathrm{CAlg}(\mathcal{C})$$

which is the left Kan extension along the inclusion $\mathbf{\Delta}^{\leq 0} \subset \mathbf{\Delta}$ of the functor $[0] \mapsto A$. Informally, for an integer $n \geq 0$, we have $\mathrm{cobar}^n(A) \simeq A^{\otimes n+1}$, and the face maps are induced by the unit morphism $\mathbf{1}_\mathcal{C} \to A$ while the degeneracy maps are induced by the multiplication morphism $A \otimes A \to A$. The cobar construction on A has a natural augmentation given by $\mathrm{cobar}^{-1}(A) = \mathbf{1}_\mathcal{C}$.

Theorem 1.44. *Let $\mathscr{A} \in \mathrm{CAlg}(\mathbf{MSh}(k; \Lambda))$ be a Weil Λ-spectrum.*

- *(i) The cosimplicial commutative algebra $\Gamma(k; \mathrm{cobar}(\mathscr{A}))$ is a Hopf algebroid in $\mathrm{Mod}^\otimes_\Lambda$.*
- *(ii) For $M \in \mathbf{MSh}(k; \Lambda)$, the $\Gamma(k; \mathrm{cobar}(\mathscr{A}))$-module $\Gamma(k; \mathrm{cobar}(\mathscr{A}) \otimes_\Lambda M)$ is naturally a comodule over the Hopf algebroid $\Gamma(k; \mathrm{cobar}(\mathscr{A}))$.*

Thus, one obtains a symmetric monoidal functor

$$\Gamma(k; \mathrm{cobar}(\mathscr{A}) \otimes_\Lambda -) : \mathbf{MSh}(k; \Lambda)^\otimes \to \mathrm{coMod}^\otimes_{\Gamma(k;\mathrm{cobar}(\mathscr{A}))}. \tag{1.25}$$

Proof. As usual, we extend the assignment $[n] \mapsto \mathrm{cobar}^n(A)$ to nonempty finite linearly ordered sets. To prove (i), we need to show that for every partition $\{0, \ldots, n\} = I \cup J$ with $I \cap J = \{m\}$ a singleton, the natural map

$$\Gamma(k; \mathrm{cobar}^I(\mathscr{A})) \otimes_{\Gamma(k;\mathrm{cobar}^{\{m\}}(\mathscr{A}))} \Gamma(k; \mathrm{cobar}^J(\mathscr{A})) \to \Gamma(k, \mathrm{cobar}^{\{0,\ldots,n\}}(\mathscr{A})) \tag{1.26}$$

is an equivalence in Mod_Λ. To do so, we start by noting that we have an equivalence

$$\mathrm{cobar}^I(\mathcal{A}) \otimes_{\mathrm{cobar}^{\{m\}}(\mathcal{A})} \mathrm{cobar}^J(\mathcal{A}) \xrightarrow{\sim} \mathrm{cobar}^{\{0,\ldots,n\}}(\mathcal{A}) \tag{1.27}$$

in $\mathbf{MSh}(k;\Lambda)$. On the other hand, by Corollary 1.42, we also have natural equivalences

$$\mathrm{cobar}^{\{m\}}(\mathcal{A}) \otimes_{\Gamma(k;\mathrm{cobar}^{\{m\}}(\mathcal{A}))} \Gamma(k;\mathrm{cobar}^L(\mathcal{A})) \xrightarrow{\sim} \mathrm{cobar}^L(\mathcal{A})$$

for all subsets $L \subset \{0,\ldots,n\}$ containing m. This shows that the domain of the equivalence in (1.27) is naturally equivalent to

$$\mathrm{cobar}^{\{m\}}(\mathcal{A}) \otimes_{\Gamma(k;\mathrm{cobar}^{\{m\}}(\mathcal{A}))} (\Gamma(k;\mathrm{cobar}^I(\mathcal{A})) \otimes_{\Gamma(k;\mathrm{cobar}^{\{m\}}(\mathcal{A}))} \Gamma(k;\mathrm{cobar}^J(\mathcal{A})))$$

whereas its codomain is naturally equivalent to

$$\mathrm{cobar}^{\{m\}}(\mathcal{A}) \otimes_{\Gamma(k;\mathrm{cobar}^{\{m\}}(\mathcal{A}))} \mathrm{cobar}^{\{0,\ldots,n\}}(\mathcal{A}).$$

Thus, up to natural equivalences, we see that the equivalence in (1.27) can be obtained from the morphism in (1.26) by applying $\mathcal{A} \otimes_{\Gamma(k;\mathcal{A})} -$. Since this functor is an equivalence by Lemma 1.41, the result follows.

The proof of (ii) is very similar to that of (i), but we include it for the reader's convenience. Here, for every integers $0 \leq m \leq n$, we need to check that the obvious morphism

$$\Gamma(k;\mathrm{cobar}^{\{0,\ldots,n\}}(\mathcal{A})) \otimes_{\Gamma(k;\mathrm{cobar}^{\{m\}}(\mathcal{A}))} \Gamma(k;\mathrm{cobar}^{\{m\}}(\mathcal{A}) \otimes_\Lambda M) \to \Gamma(k;\mathrm{cobar}^{\{0,\ldots,n\}}(\mathcal{A}) \otimes_\Lambda M) \tag{1.28}$$

is an equivalence in Mod_Λ. As above, we start by noting that we have an equivalence

$$\mathrm{cobar}^{\{0,\ldots,n\}}(\mathcal{A}) \otimes_{\mathrm{cobar}^{\{m\}}(\mathcal{A})} (\mathrm{cobar}^{\{m\}}(\mathcal{A}) \otimes_\Lambda M) \xrightarrow{\sim} \mathrm{cobar}^{\{0,\ldots,n\}}(\mathcal{A}) \otimes_\Lambda M \tag{1.29}$$

in $\mathbf{MSh}(k;\Lambda)$. By Corollary 1.42, we have natural equivalences

$$\mathrm{cobar}^{\{m\}}(\mathcal{A}) \otimes_{\Gamma(k;\mathrm{cobar}^{\{m\}}(\mathcal{A}))} \Gamma(k;\mathrm{cobar}^L(\mathcal{A}) \otimes_\Lambda N) \xrightarrow{\sim} \mathrm{cobar}^{\{0,\ldots,n\}}(\mathcal{A}) \otimes_\Lambda N$$

for all subsets $L \subset \{0,\ldots,n\}$ containing m and all objects $N \in \mathbf{MSh}(k;\Lambda)$. Using this for $L = \{m\}$ or $L = \{0,\ldots,n\}$ and $N = \Lambda$ or $N = M$, we see that, up to natural equivalences, the equivalence in (1.29) can be obtained from the morphism in (1.28) by applying $\mathcal{A} \otimes_{\Gamma(k;\mathcal{A})} -$. We then conclude using Lemma 1.41. ☐

Definition 1.45. Keep the assumptions of Theorem 1.44. The Hopf algebroid $\Gamma(k;\mathrm{cobar}(\mathcal{A}))$ is called the motivic Hopf algebroid associated to $\mathcal{A}$ and is denoted by $\mathcal{H}_{\mathrm{mot}}(k,\mathcal{A})$. We set

$$\mathcal{G}_{\mathrm{mot}}(k,\mathcal{A}) = \mathrm{Spec}(\mathcal{H}_{\mathrm{mot}}(k,\mathcal{A}));$$

this is a nonconnective spectral affine groupoid Λ-scheme, which we call the motivic Galois groupoid associated to $\mathcal{A}$. When the Weil spectrum $\mathcal{A}$ is neutral over Λ, $\mathcal{H}_{\mathrm{mot}}(k,\mathcal{A})$ is a Hopf algebra and $\mathcal{G}_{\mathrm{mot}}(k,\mathcal{A})$ is a group.

We end this subsection with a description of the points of the groupoid $\mathcal{G}_{\mathrm{mot}}(k,\mathcal{A})$.

Theorem 1.46. *Let $\mathcal{A} \in \mathrm{CAlg}(\mathbf{MSh}(k;\Lambda))$ be a Weil Λ-spectrum.*

(i) The commutative algebra $\mathcal{A}$ admits a natural structure of a comodule over $\mathcal{H}_{\mathrm{mot}}(k,\mathcal{A})$. More precisely, there is a natural lift of $\mathcal{A}$ to a commutative algebra object in the symmetric monoidal ∞-category $\mathrm{coMod}^{\otimes}_{\mathcal{H}_{\mathrm{mot}}(k,\mathcal{A})}(\mathbf{MSh}(k;\Lambda))$.

(ii) Let $s,t \in \mathcal{G}_{\mathrm{mot}}(k,\mathcal{A})_0(\Lambda)$ be two Λ-points corresponding to two Λ-algebra morphisms from $\Gamma(k;\mathcal{A})$ to Λ. We set $\mathcal{A}_s = \mathcal{A} \otimes_{\Gamma(k;\Lambda),s} \Lambda$ and $\mathcal{A}_t = \mathcal{A} \otimes_{\Gamma(k;\Lambda),t} \Lambda$. Then, the coaction in (i) induces an equivalence of Kan complexes

$$\mathcal{G}_{\mathrm{mot}}(k,\mathcal{A})_1 \times_{\mathcal{G}_{\mathrm{mot}}(k,\mathcal{A})_0 \times \mathcal{G}_{\mathrm{mot}}(k,\mathcal{A})_0} (s,t) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathbf{MSh}(k;\Lambda))}(\mathcal{A}_s,\mathcal{A}_t).$$

Proof. The lift of $\mathcal{A}$ to a commutative algebra in $\mathrm{coMod}^{\otimes}_{\mathcal{H}_{\mathrm{mot}}(k,\mathcal{A})}(\mathbf{MSh}(k;\Lambda))$ is simply given by the cosimplicial commutative algebra $\mathrm{cobar}(\mathcal{A})$ considered as an algebra over the Hopf algebroid $\Gamma(k;\mathrm{cobar}(\mathcal{A}))$. The key property to be checked here is that the obvious morphism

$$\mathrm{cobar}^{\{m\}}(A)\otimes_{\Gamma(k,\mathrm{cobar}^{\{m\}}(A))}\Gamma(k,\mathrm{cobar}^{\{0,\dots,n\}}(A))\to\mathrm{cobar}^{\{0,\dots,n\}}(A)$$

is an equivalence for all integers $0\leq m\leq n$, which follows from Corollary 1.42. This proves (i).

For (ii), we need to show that the coaction of $\mathcal{H}_{\mathrm{mot}}(k,\mathcal{A})$ on $\mathcal{A}$ induces an equivalence of Kan complexes

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_{\Lambda})}(\Gamma(k;\mathcal{A}_s\otimes_{\Lambda}\mathcal{A}_t),\Lambda)\simeq\mathrm{Map}_{\mathrm{CAlg}(\mathbf{MSh}(k;\Lambda))}(\mathcal{A}_s,\mathcal{A}_t). \tag{1.30}$$

The recipe for the above map is as follows. To a morphism $\alpha:\Gamma(k;\mathcal{A}_s\otimes_{\Lambda}\mathcal{A}_t)\to\Lambda$, we associate a morphism $\widetilde{\alpha}:\mathcal{A}_s\to\mathcal{A}_t$ by taking the composition of

$$\mathcal{A}_s\to\mathcal{A}_s\otimes_{\Lambda}\mathcal{A}_t\simeq\Gamma(k;\mathcal{A}_s\otimes_{\Lambda}\mathcal{A}_t)\otimes_{\Lambda}\mathcal{A}_t\xrightarrow{\alpha\otimes\mathrm{id}}\mathcal{A}_t.$$

Conversely, given a morphism $\beta:\mathcal{A}_s\to\mathcal{A}_t$, we associate a morphism $\widehat{\beta}:\Gamma(k;\mathcal{A}_s\otimes_{\Lambda}\mathcal{A}_t)\to\Lambda$ by applying $\Gamma(k;-)$ to

$$\mathcal{A}_s\otimes_{\Lambda}\mathcal{A}_t\xrightarrow{\beta\otimes\mathrm{id}}\mathcal{A}_t\otimes_{\Lambda}\mathcal{A}_t\xrightarrow{\mu}\mathcal{A}_t$$

and using the equivalence $\Gamma(k;\mathcal{A}_t)\simeq\Lambda$. It is easy to construct maps of spaces from the recipes $\alpha\mapsto\widetilde{\alpha}$ and $\beta\mapsto\widehat{\beta}$. It is also easy to see that these maps are compatible with base change along morphisms of commutative algebras $\Lambda\to\Lambda'$. Thus, to finish the proof, it remains to see that the recipes $\alpha\mapsto\widetilde{\alpha}$ and $\beta\mapsto\widehat{\beta}$ are inverses of each other up to homotopy, i.e., as maps on the π_0's of the spaces in (1.30). This is again an easy but tedious exercise that we leave to the reader. □

In the case where the Weil Λ-spectrum is neutralised over Λ, we can be more precise. To do so, we need some preliminaries on group actions in the ∞-categorical setting.

Notation 1.47. Given an ∞-category $\mathcal{C}$, we denote by $\mathcal{C}^{\simeq}$ the largest wide sub-∞-category of $\mathcal{C}$ which is an ∞-groupoid.

Construction 1.48. Let $\mathcal{C}$ be an ∞-category and let $X\in\mathcal{C}$. We define a (small) bisimplicial set

$$\mathrm{Auteq}_{\mathcal{C}}(X):\mathbf{\Delta}^{\mathrm{op}}\to\mathrm{Set}_{\Delta}$$

by the following Cartesian square of (possibly large) bisimplicial sets

$$\begin{tikzcd}
\mathrm{Auteq}_{\mathcal{C}}(X)_{\bullet} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^{\bullet},\mathcal{C}^{\simeq}) \arrow[d] \\
\mathrm{pt} \arrow[r, "(X,\dots,X)"] & \mathrm{bar}_{\bullet}(\mathcal{C}^{\simeq})
\end{tikzcd}$$

where $\mathrm{bar}_{\bullet}(-)$ denotes the bar construction. (Recall that the bar construction on an object is the right Kan extension along $(\mathbf{\Delta}^{\leq 0})^{\mathrm{op}}\hookrightarrow\mathbf{\Delta}^{\mathrm{op}}$ of the unique functor from $(\mathbf{\Delta}^{\leq 0})^{\mathrm{op}}$ pointing to that object.) Since $\mathcal{C}^{\simeq}$ is a Kan complex, the right vertical map in the above square is a Kan fibration in each simplicial degree. It follows readily from the construction that $\mathrm{Auteq}_{\mathcal{C}}(X)$ defines a group object in $\mathcal{S}$. This is the group of autoequivalences of $X\in\mathcal{C}$.

Remark 1.49. If one does not care about explicit models, the H-group $\mathrm{Auteq}_{\mathcal{C}}(X)$ can also be defined by the following Cartesian square of simplicial objects in $\widehat{\mathcal{S}}$

$$\begin{tikzcd} \mathrm{Auteq}_{\mathcal{C}}(X)_\bullet \arrow[r] \arrow[d] & \mathcal{C}^{\simeq} \arrow[d, "\mathrm{diag}"] \\ \mathrm{pt} \arrow[r, "{(X,\dots,X)}"] & \mathrm{bar}_\bullet(\mathcal{C}^{\simeq}) \end{tikzcd}$$

where pt and $\mathcal{C}^{\simeq}$ are considered as constant simplicial objects in $\widehat{\mathcal{S}}$. Indeed, for $n \geq 0$, the Kan complexes $\mathrm{Fun}(\Delta^n, \mathcal{C}^{\simeq})$ are all equivalent to $\mathcal{C}^{\simeq}$. In fact, we will be using below this simpler description. However, the description given in Construction 1.48 is perhaps more intuitive.

Remark 1.50. Given a groupoid object $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$ in spaces, we set $\mathrm{B}(G) = \mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} G$ which we call the classifying space of G. By construction, $\mathrm{B}(G)$ is the value at $\Delta^{\emptyset} \in \mathbf{\Delta}_+$ of the augmented simplicial object G^+ extending G to a colimit diagram. Moreover, the groupoid G is effective in the sense of [Lur09a, Definition 6.1.2.14], i.e., G^+ is the Čech nerve associated to the map $G_0 \to \mathrm{B}(G)$; see [Lur09a, Corollary 6.1.3.20] for a proof of this classical fact.

We now return to Construction 1.48, and consider the space $\mathrm{B}(\mathrm{Auteq}_{\mathcal{C}}(X))$ as an ∞-groupoid (i.e., an ∞-category where all maps are equivalences). Since $\mathrm{Auteq}_{\mathcal{C}}(X)_\bullet$ is the Čech nerve associated to $X : \mathrm{pt} \to \mathcal{C}^{\simeq}$, we have a map of spaces

$$\mathrm{B}(\mathrm{Auteq}_{\mathcal{C}}(X)) \to \mathcal{C}^{\simeq}$$

which identify $\mathrm{B}(\mathrm{Auteq}_{\mathcal{C}}(X))$ to the full sub-∞-groupoid of $\mathcal{C}^{\simeq}$ spanned by the object X.

We will need a parametrised version of Construction 1.48.

Construction 1.51. Let K be a small simplicial set and let $\mathcal{C} : K \to \mathrm{CAT}_\infty$ be a diagram in ∞-categories. Let $X : \mathrm{pt}_K \to \mathcal{C}$ be a natural transformation from the constant functor $\mathrm{pt}_K : K \to \mathrm{CAT}_\infty$ pointing the final ∞-category pt. We define a diagram

$$\mathrm{Auteq}_{\mathcal{C}}(X) : \mathbf{\Delta}^{\mathrm{op}} \times K \to \mathcal{S}$$

by the following Cartesian square of simplicial objects in $\mathrm{Fun}(K, \widehat{\mathcal{S}})$

$$\begin{tikzcd} \mathrm{Auteq}_{\mathcal{C}}(X)_\bullet \arrow[r] \arrow[d] & \mathcal{C}^{\simeq} \arrow[d, "\mathrm{diag}"] \\ \mathrm{pt}_K \arrow[r, "{(X,\dots,X)}"] & \mathrm{bar}_\bullet(\mathcal{C}^{\simeq}). \end{tikzcd}$$

Here $\mathcal{C}^{\simeq} : K \to \widehat{\mathcal{S}}$ is the functor obtained from $\mathcal{C}$ by composing with $(-)^{\simeq} : \mathrm{CAT}_\infty \to \widehat{\mathcal{S}}$. We may view $\mathrm{Auteq}_{\mathcal{C}}(X)$ as a diagram

$$\mathrm{Auteq}_{\mathcal{C}}(X) : K \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}),$$

taking values in the full sub-∞-category of group objects in $\mathcal{S}$, and sending a vertex $v \in K$ to the group object $\mathrm{Auteq}_{\mathcal{C}(v)}(X(v))$.

We need to spell out in which sense the group object $\mathrm{Auteq}_{\mathcal{C}}(X)$ acts on X. We do this directly in the parametrised case.

Lemma 1.52. *Keep the notations of Construction 1.51, and assume that K is an ∞-category and that $\mathcal{C}$ takes values in Cat_∞. Let $p : \int_K \mathcal{C} \to K$ be the coCartesian fibration classified by the functor $\mathcal{C}$. There is a functor*

$$\mathrm{Act}_{\mathcal{C}}(X) : \mathbf{\Delta}^{\mathrm{op}} \times \textstyle\int_K \mathcal{C} \to \mathcal{S}$$

and a natural transformation $\mathrm{Act}_{\mathcal{C}}(X) \to \mathrm{Auteq}_{\mathcal{C}}(X) \circ (\mathrm{id}_{\mathbf{\Delta}^{\mathrm{op}}} \times p)$ such that the following conditions are satisfied.

(i) The functor $\mathrm{Act}_{\mathcal{C}}(X)_0 : \int_K \mathcal{C} \to \mathcal{S}$ is given informally by $(v, A) \mapsto \mathrm{Map}_{\mathcal{C}(v)}(X(v), A)$.

(ii) For all integers $0 \leq m \leq n$ the square

$$\begin{tikzcd}
\mathrm{Act}_{\mathcal{C}}(X)_{\{0,\dots,n\}} \arrow[r] \arrow[d] & \mathrm{Auteq}_{\mathcal{C}}(X)_{\{0,\dots,n\}} \circ p \arrow[d] \\
\mathrm{Act}_{\mathcal{C}}(X)_{\{m\}} \arrow[r] & \mathrm{Auteq}_{\mathcal{C}}(X)_{\{m\}} \circ p
\end{tikzcd}$$

is Cartesian.

(iii) For every object (v, A) in $\int_K \mathcal{C}$, the induced action of $\mathrm{Auteq}_{\mathcal{C}(v)}(X(v))$ on $\mathrm{Map}_{\mathcal{C}(v)}(X(v), A)$ in $\mathrm{h}\mathcal{S}$ is the one given by composition.

Proof. Consider the functor

$$\mathcal{F} : \textstyle\int_K \mathcal{C} \to \mathrm{CAT}_\infty$$

sending an object (v, A) to the over ∞-category $\mathcal{C}(v)_{/A}$. We have a natural transformation $\mathcal{F} \to \mathcal{C} \circ p$ given informally at (v, A) by the forgetful functor $\mathcal{C}(v)_{/A} \to \mathcal{C}(v)$. This induces an natural transformation $\mathcal{F}^{\simeq} \to \mathcal{C}^{\simeq} \circ p$. This said, we define $\mathrm{Act}_{\mathcal{C}}(X)$ by the Cartesian square of simplicial objects in $\mathrm{Fun}(\int_K \mathcal{C}, \mathcal{S})$

$$\begin{tikzcd}
\mathrm{Act}_{\mathcal{C}}(X)_\bullet \arrow[r] \arrow[d] & \mathcal{F}^{\simeq} \arrow[d] \\
\mathrm{pt}_K \circ p \arrow[r, "{(X,\dots,X)}"] & \mathrm{bar}_\bullet(\mathcal{C}^{\simeq} \circ p).
\end{tikzcd}$$

The right vertical arrow is the composition of $\mathcal{F}^{\simeq} \to \mathcal{C}^{\simeq} \circ p$ followed by the diagonal embedding. The properties (i)–(iii) follow readily from the construction. $\square$

We now go back to the motivic Galois group associated to a Weil Λ-spectrum.

Notation 1.53. Let $\mathcal{R}$ be a commutative ring spectrum in $\mathbf{MSh}(k; \Lambda)$. (We are interested in the case of a Weil spectrum, but this is not relevant for this notation.) We apply Construction 1.51 with the functor

$$\mathrm{CAlg}(\mathbf{MSh}(k; -)) : (\mathrm{SpAFF}^{\mathrm{nc}}_{/\Lambda})^{\mathrm{op}} \to \mathrm{CAT}_\infty,$$

sending $\mathrm{Spec}(\Lambda')$ to $\mathrm{CAlg}(\mathbf{MSh}(k; \Lambda'))$, and the natural transformation $\mathrm{pt} \to \mathrm{CAlg}(\mathbf{MSh}(k; -))$, given at $\mathrm{Spec}(\Lambda')$ by the functor pointing at $\mathcal{R} \otimes_\Lambda \Lambda'$. This yields a nonconnective spectral group Λ-prestack of automorphisms of $\mathcal{R}$ which we denote by $\underline{\mathrm{Auteq}}(\mathcal{R})$.

Theorem 1.54. *Let $\mathcal{A} \in \mathrm{CAlg}(\mathbf{MSh}(k; \Lambda))$ be a Weil Λ-spectrum which is neutral over Λ. Then there is a canonical equivalence of nonconnective spectral group Λ-prestacks*

$$\mathcal{G}_{\mathrm{mot}}(k, \mathcal{A}) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathcal{A}).$$

Proof. By Theorem 1.46, we know that $\underline{\mathrm{Auteq}}(\mathcal{A})$ is affine and has the same underlying nonconnective spectral scheme as $\mathcal{G}_{\mathrm{mot}}(k, \mathcal{A})$. We need to show that this identification of the underlying spectral prestacks extends to the group structure. To simplify notation, we write $X_\bullet = \mathrm{Spec}(E^\bullet)$ for the simplicial object in $\mathrm{SpAFF}^{\mathrm{nc}}$ representing $\underline{\mathrm{Auteq}}(\mathcal{A})$. By Lemma 1.52, we have a functor

$$\underline{\mathrm{Act}}(\mathcal{A}) : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Fun}\left(\int_{\mathrm{CAlg}(\mathrm{Mod}_\Lambda)} \mathbf{MSh}(k; -), \mathcal{S}\right)$$

and, for every integer $n \geq 0$, the functor

$$\underline{\mathrm{Act}}(\mathcal{A})_n : \int_{\mathrm{CAlg}(\mathrm{Mod}_\Lambda)} \mathbf{MSh}(k; -) \to \mathcal{S}$$

is corepresentable by the pair $(E^n, \mathcal{A} \otimes_\Lambda E^n)$. (For the last assertion, we use properties (i) and (ii) in Lemma 1.52.) Thus, forgetting the structure of E^n-modules, we obtain a cosimplicial commutative algebra $\widehat{\mathcal{A}}^\bullet$ in $\mathbf{MSh}(k; \Lambda)$ with the following properties:

- (i) $\widehat{\mathcal{A}}^0 \simeq \mathcal{A}$;
- (ii) $\Gamma(k; \widehat{\mathcal{A}}^\bullet) \simeq E^\bullet$;
- (iii) the induced action of $\mathrm{Spec}(E^1)$ on $\mathcal{A}$ in the homotopy category coincides with the tautological action of $\underline{\mathrm{Auteq}}(\mathcal{A})_1$.

It is now easy to conclude. Indeed, being a left Kan extension, there is a morphism of cosimplicial commutative algebras $\mathrm{cobar}(\mathcal{A}) \to \widehat{\mathcal{A}}$ and, as recalled at the beginning of the proof, the induced morphism $\Gamma(k; \mathrm{cobar}^1 \mathcal{A}) \to \Gamma(k; \widehat{\mathcal{A}}^1)$. This readily implies that the morphism of cosimplicial commutative algebras $\Gamma(k; \mathrm{cobar}^\bullet \mathcal{A}) \to \Gamma(k; \widehat{\mathcal{A}}^\bullet)$ is an equivalence degreewise. □

Remark 1.55. Given a Weil Λ-spectrum $\mathcal{A}$ neutral over Λ, we have the associated realisation functor given by

$$\mathfrak{R}_{\mathcal{A}} : \mathbf{MSh}(k; \Lambda)^\otimes \xrightarrow{\mathcal{A} \otimes_\Lambda -} \mathbf{MSh}(k; \mathcal{A})^\otimes \simeq \mathrm{Mod}_\Lambda^\otimes,$$

where the last equivalence is the one described in Lemma 1.41. It follows from [Lur17, Corollary 4.8.5.21] that we have an equivalence of groups

$$\mathrm{Auteq}(\mathcal{A}) \simeq \mathrm{Auteq}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathbf{MSh}(k;\Lambda)^\otimes /}}(\mathfrak{R}_{\mathcal{A}}).$$

On the other hand, the obvious map

$$\mathrm{Auteq}_{\mathrm{Fun}(\mathbf{MSh}(k;\Lambda)^\otimes, \mathrm{Mod}_\Lambda^\otimes)}(\mathfrak{R}_{\mathcal{A}}) \to \mathrm{Auteq}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathbf{MSh}(k;\Lambda)^\otimes /}}(\mathfrak{R}_{\mathcal{A}})$$

is an equivalence. Indeed, every autoequivalence of $\mathfrak{R}_{\mathcal{A}}$ which is the identity on $\mathbf{MSh}(k; \Lambda)^\otimes$ is also the identity on $\mathrm{Mod}_\Lambda^\otimes$ since $\mathfrak{R}_{\mathcal{A}}$ admits a section. Thus, we have an equivalence of groups

$$\mathrm{Auteq}(\mathcal{A}) \simeq \mathrm{Auteq}_{\mathrm{Fun}(\mathbf{MSh}(k;\Lambda)^\otimes, \mathrm{Mod}_\Lambda^\otimes)}(\mathfrak{R}_{\mathcal{A}}).$$

This holds equally with $\mathcal{A}$ replaced by $\mathcal{A} \otimes_\Lambda \Lambda'$ for any commutative Λ-algebra Λ'. Using this and Theorem 1.54, it is easy to show that the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(k, \mathcal{A})$ is equivalent to the one defined by Iwanari [Iwa14, Definition 5.13]. Since this is not needed later on, we leave the details to the interested reader.

1.4. **Motivic Galois group, II. The Betti case.**

We fix a base field k and a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$. In this subsection, we compare the constructions of Subsection 1.3 in the Betti case with the construction in [Ayo17a]; this is actually quite straightforward. Then, we recall a few known properties of the motivic Galois group in the Betti case. The construction in [Ayo17a] relies on the following observation.

Proposition 1.56. *For $M \in \mathbf{MSh}(k; \Lambda)$ and $N \in \mathrm{Mod}_\Lambda$, the obvious morphism*

$$M \otimes_\Lambda \mathrm{B}_*(N) \to \mathrm{B}_*(\mathrm{B}^*(M) \otimes_\Lambda N) \tag{1.31}$$

is an equivalence.

Proof. By Proposition 1.30, the domain and codomain of the morphism in (1.31) are colimit-preserving in the variable M. Thus, we may assume that M is dualizable and use [Ayo14b, Lemme 2.8] to conclude. □

Notation 1.57. For a commutative ring spectrum Λ, we set

$$\mathcal{B}_\Lambda = \mathrm{B}_*(\Lambda)$$

which we view as an object of $\mathrm{CAlg}(\mathbf{MSh}(k; \Lambda))$. We refere to $\mathcal{B}_\Lambda$ as the Betti spectrum. When $\Lambda = \mathbb{S}$ is the sphere spectrum, we simply write $\mathcal{B}$.

Corollary 1.58.

(i) There is a canonical equivalence of commutative algebras $\mathcal{B}_\Lambda \simeq \mathcal{B} \otimes \Lambda$.
(ii) The commutative algebra $\mathcal{B}_\Lambda$ in $\mathbf{MSh}(k; \Lambda)$ is a Weil Λ-spectrum neutral over Λ.
(iii) The functor $\Gamma(k; -) : \mathbf{MSh}(k; \mathcal{B}_\Lambda) \to \mathrm{Mod}_\Lambda$ is an equivalence of ∞-categories.
(iv) The functor $\mathrm{B}^* : \mathbf{MSh}(k; \Lambda) \to \mathrm{Mod}_\Lambda$ is equivalent to $\Gamma(k; \mathcal{B}_\Lambda \otimes_\Lambda -)$.

Proof. We start by noting that there is an equivalence

$$\mathrm{B}_*(\Lambda) \otimes_\Lambda L \xrightarrow{\sim} \mathrm{B}_*(L) \tag{1.32}$$

for every Λ-module L. Indeed, by Proposition 1.30, the domain and codomain of the morphism in (1.32) are colimit-preserving in the variable L. Thus, we may assume that $L = \Lambda$, and the claim becomes obvious. This readily implies assertion (i) by taking for Λ and L the sphere spectrum and Λ respectively. The equivalence in (1.32) can also be used to verify condition (i) of Definition 1.38 for $\mathcal{B}_\Lambda$. Indeed, we are then reduced to showing that $L \to \Gamma(k; \mathrm{B}_*(L))$ is an equivalence, which is obvious since $\Gamma(k; \mathrm{B}_*(-))$ is equivalent to the identity functor of Mod_Λ. (Its left adjoint is the composite functor $\mathrm{B}^* \circ (-)_{\mathrm{cst}}$ where $(-)_{\mathrm{cst}} : \mathrm{Mod}_\Lambda \to \mathbf{MSh}(k; \Lambda)$ is the "constant" motive functor.)

Next, we prove property (iii) of the statement. The proof is very similar to that of Lemma 1.41. We actually show that the left adjoint

$$\mathcal{B}_\Lambda \otimes_\Lambda - : \mathrm{Mod}_\Lambda \to \mathbf{MSh}(k; \mathcal{B}_\Lambda)$$

is an equivalence. Fully faithfulness follows immediately from condition (i) of Definition 1.38 which we just verified for $\mathcal{B}_\Lambda$. Essential surjectivity follows from Proposition 1.56 and the equivalences in (1.32). Indeed, for every $M \in \mathbf{MSh}(k; \Lambda)$, we have a chain of equivalences:

$$\mathcal{B}_\Lambda \otimes_\Lambda M \simeq \mathrm{B}_*\mathrm{B}^*(M) \simeq \mathcal{B}_\Lambda \otimes_\Lambda \mathrm{B}^*(M).$$

We now finish the proof of property (ii) of the statement by verifying that condition (ii) of Definition 1.38 holds true for $\mathcal{B}_\Lambda$. (Condition (i) the that definition was verified above.) We need to show that the composition of

$$\mathrm{B}_*(\Lambda)\otimes_\Lambda \Gamma(k;\mathrm{B}_*(\Lambda)\otimes_\Lambda M)\to \mathrm{B}_*(\Lambda)\otimes_\Lambda \mathrm{B}_*(\Lambda)\otimes_\Lambda M\to \mathrm{B}_*(\Lambda)\otimes_\Lambda M$$

is an equivalence. By Proposition 1.56, we may rewrite this composition as follows:

$$\mathrm{B}_*(\Lambda)\otimes_\Lambda \Gamma(k;\mathrm{B}_*\mathrm{B}^*(M))\to \mathrm{B}_*(\Lambda)\otimes_\Lambda \mathrm{B}_*\mathrm{B}^*(M)\to \mathrm{B}_*\mathrm{B}^*(M).$$

This is a morphism of $\mathcal{B}_\Lambda$-modules. By the property (iii) just proved, it is enough to show that it induces an equivalence after applying $\Gamma(k;-)$ which is clear.

It remains to show property (iv) of the statement. For this we use again the natural equivalence $\mathcal{B}_\Lambda\otimes M\simeq \mathrm{B}^*\mathrm{B}_*(M)$ provided by Proposition 1.56 and the fact, mentioned previously in the proof, that $\Gamma(k;\mathrm{B}_*(-))$ is equivalent to the identity functor. □

Construction 1.59. Let $\Lambda\in\mathrm{CAlg}(\mathcal{S}p)$ be a commutative ring spectrum. We simply write

$$\mathrm{B}^*:\mathbf{MSh}(k;\Lambda)^\otimes\to\mathbf{Sh}(\mathrm{pt};\Lambda)^\otimes\simeq\mathrm{Mod}_\Lambda^\otimes \tag{1.33}$$

for the Betti realisation functor over $\mathrm{Spec}(k)$. (Note that in this case, there is no distinction between the Betti realisation functor and its refined version given by Theorem 1.28.) The functor B^* in (1.33) admits a right adjoint which we denote by B_*. By [Lur17, Proposition 4.7.3.3] applied to the right adjoint functor $(\mathrm{B}^*)^{\mathrm{op}}$, the composite functor $\mathrm{B}_*\mathrm{B}^*$ underlies a coalgebra structure in the ∞-category $\mathrm{EndFun}(\mathbf{MSh}(k;\Lambda)^\otimes)$ of right-lax symmetric monoidal endofunctors of $\mathbf{MSh}(k;\Lambda)^\otimes$. Said differently, there exists a cosimplicial object $\widehat{\widehat{\mathrm{B}}}:\mathbf{\Delta}\to\mathrm{EndFun}(\mathbf{MSh}(k;\Lambda)^\otimes)$ which is informally given as follows.

(1) For every integer $n\geq 0$, there is an equivalence $\widehat{\widehat{\mathrm{B}}}{}^n\simeq(\mathrm{B}_*\mathrm{B}^*)^{\circ n+1}$.
(2) For $0\leq i\leq n+1$, the i-th face map $\widehat{\widehat{\mathrm{B}}}{}^n\to\widehat{\widehat{\mathrm{B}}}{}^{n+1}$ is given by

$$(\mathrm{B}_*\mathrm{B}^*)^{\circ i}\circ\mathrm{id}\circ(\mathrm{B}_*\mathrm{B}^*)^{\circ n+1-i}\xrightarrow{\eta}(\mathrm{B}_*\mathrm{B}^*)^{\circ i}\circ(\mathrm{B}_*\mathrm{B}^*)\circ(\mathrm{B}_*\mathrm{B}^*)^{\circ n+1-i},$$

where η is the unit of the adjunction $(\mathrm{B}^*,\mathrm{B}_*)$.
(3) For $0\leq i\leq n-1$, the i-th codegeneracy map $\widehat{\widehat{\mathrm{B}}}{}^n\to\widehat{\widehat{\mathrm{B}}}{}^{n-1}$ is given by

$$\mathrm{B}_*\circ(\mathrm{B}^*\mathrm{B}_*)^{\circ i}\circ(\mathrm{B}^*\mathrm{B}_*)\circ(\mathrm{B}^*\mathrm{B}_*)^{\circ n-i-1}\circ\mathrm{B}^*\xrightarrow{\delta}\mathrm{B}_*\circ(\mathrm{B}^*\mathrm{B}_*)^{\circ i}\circ\mathrm{id}\circ(\mathrm{B}^*\mathrm{B}_*)^{\circ n-i-1}\circ\mathrm{B}^*,$$

where δ is the counit of the adjunction $(\mathrm{B}^*,\mathrm{B}_*)$.

We may think of $\widehat{\widehat{\mathrm{B}}}$ as a right-lax symmetric monoidal functor

$$\widehat{\widehat{\mathrm{B}}}:\mathbf{MSh}(k;\Lambda)^\otimes\to\mathrm{Fun}(\mathbf{\Delta},\mathbf{MSh}(k;\Lambda))^\otimes \tag{1.34}$$

and, as usual, we also write $\widehat{\widehat{\mathrm{B}}}$ for the underlying functor. In particular, $\widehat{\widehat{\mathrm{B}}}(\Lambda)$ is a cosimplicial commutative algebra in $\mathbf{MSh}(k;\Lambda)^\otimes$.

Definition 1.60. By [Ayo17a, Theorem 8.3], the cosimplicial commutative algebra $\Gamma(k;\widehat{\widehat{\mathrm{B}}}(\Lambda))$ is a commutative Hopf algebra, which we call the motivic Hopf algebra and denote by $\mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda)$. (This also follows from Theorem 1.44, Corollary 1.58 and 1.62 below.) Its spectrum is denoted by $\mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda)$ and is called the motivic Galois group. By definition, this is a nonconnective spectral group Λ-scheme. When Λ is the sphere spectrum, we simply write $\mathcal{H}_{\mathrm{mot}}(k,\sigma)$ and $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$.

Remark 1.61. It follows from Corollary 1.58 and Lemma 1.62 below that there is an equivalence of commutative Hopf algebras $\mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda)\simeq\mathcal{H}_{\mathrm{mot}}(k,\sigma)\otimes\Lambda$. Thus, $\mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda)$ is simply the base change of $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ along $\mathrm{Spec}(\Lambda)\to\mathrm{Spec}(\mathbb{S})$.

Lemma 1.62. *The cosimplicial commutative algebra*

$$\widehat{\mathrm{B}}(\Lambda) : \mathbf{\Delta} \to \mathrm{CAlg}(\mathbf{MSh}(k; \Lambda))$$

is the left Kan extension of its restriction to $\mathbf{\Delta}^{\leq 0}$. Said differently, there is an equivalence of cosimplicial commutative algebras $\mathrm{cobar}^\bullet(\mathcal{B}_\Lambda) \xrightarrow{\sim} \widehat{\mathrm{B}}^\bullet(\Lambda)$.

Proof. We need to show that the morphism

$$\overbrace{\mathrm{B}_*(\Lambda) \otimes_\Lambda \ldots \otimes_\Lambda \mathrm{B}_*(\Lambda)}^{n \text{ times}} \to \overbrace{\mathrm{B}_*\mathrm{B}^* \circ \ldots \circ \mathrm{B}_*\mathrm{B}^*}^{n \text{ times}}(\Lambda)$$

is an equivalence. This follows by induction from Proposition 1.56. □

Corollary 1.63. *The motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k, \sigma; \Lambda)$ is canonically equivalent to the motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k, \mathcal{B}_\Lambda)$ associated to the Betti spectrum $\mathcal{B}_\Lambda$.*

Corollary 1.64. *There is a canonical equivalence of nonconnective spectral group Λ-prestacks*

$$\mathcal{G}_{\mathrm{mot}}(k, \sigma; \Lambda) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathcal{B}_\Lambda).$$

Proof. This is the combination of Theorem 1.54 and Corollary 1.63. □

The reminder of this subsection is devoted to reviewing some of the more concrete properties of the motivic Galois group obtained in [Ayo14b, Ayo14c]. We start with the following useful fact.

Proposition 1.65. *Assume that k is a filtered union of a family of subfields $(k_\alpha)_\alpha$ and let $\sigma_\alpha = \sigma|_{k_\alpha}$ be the complex embedding of k_α obtained by restriction from σ. Then, there is an equivalence of commutative Hopf algebras*

$$\operatorname*{colim}_\alpha \mathcal{H}_{\mathrm{mot}}(k_\alpha, \sigma_\alpha; \Lambda) \xrightarrow{\sim} \mathcal{H}_{\mathrm{mot}}(k, \sigma; \Lambda).$$

Proof. For this proof, we need to work with motives in the Nisnevich topology. This is indeed possible arguing as in the proof of Proposition 1.30. In fact, all the results of Subsection 1.3 are valid equally in the Nisnevich topology. Below, we use freely that the ∞-categories of motives in the Nisnevich topology are compactly generated.

The morphism $e_\alpha : \mathrm{Spec}(k) \to \mathrm{Spec}(k_\alpha)$ induces a pair of adjoint functors

$$e_\alpha^* : \mathbf{MSh}_{\mathrm{nis}}(k_\alpha; \Lambda) \rightleftarrows \mathbf{MSh}_{\mathrm{nis}}(k; \Lambda) : e_{\alpha,*}$$

and, by [AGV20, Proposition 2.5.11], we have an equivalence in $\mathrm{Pr}^{\mathrm{L}}_\omega$:

$$\mathbf{MSh}_{\mathrm{nis}}(k; \Lambda) \simeq \operatorname*{colim}_\alpha \mathbf{MSh}_{\mathrm{nis}}(k_\alpha; \Lambda).$$

Concretely, this means that for any object $M \in \mathbf{MSh}_{\mathrm{nis}}(k; \Lambda)$, we have an equivalence

$$M \simeq \operatorname*{colim}_\alpha e_\alpha^* e_{\alpha,*} M. \tag{1.35}$$

(It is standard but not completely obvious to set up an equivalence as above; we refer the reader to [AGV20, Lemma 3.5.7 & Remark 3.5.8] where a similar construction is discussed in a different situation.) Denote by $\mathrm{B}_\alpha^* : \mathbf{MSh}(k_\alpha; \Lambda) \to \mathrm{Mod}_\Lambda$ the Betti realisation functor associated to σ_α and let $\mathrm{B}_{\alpha,*}$ be its right adjoint. We have an equivalence $\mathrm{B}_\alpha^* \simeq \mathrm{B}^* \circ e_\alpha^*$ inducing an equivalence of right-lax monoidal functors $\mathrm{B}_{\alpha,*} \simeq e_{\alpha,*} \circ \mathrm{B}_*$. In particular, letting $\mathcal{B}_{\alpha,\Lambda} = \mathrm{B}_{\alpha,*}(\Lambda)$, we have an equivalence of commutative algebras $\mathcal{B}_{\alpha,\Lambda} \simeq e_{\alpha,*}\mathcal{B}_{\Lambda}$. Applying the equivalence in (1.35), we obtain an equivalence of commutative algebras

$$\operatorname*{colim}_{\alpha} e_{\alpha}^{*}\mathcal{B}_{\alpha,\Lambda} \simeq \mathcal{B}_{\Lambda}. \tag{1.36}$$

Since the above colimit is filtered and the functors e_{α}^{*} symmetric monoidal, we also deduce an equivalence of cosimplicial commutative algebras

$$\operatorname*{colim}_{\alpha} e_{\alpha}^{*}\mathrm{cobar}(\mathcal{B}_{\alpha,\Lambda}) \simeq \mathrm{cobar}(\mathcal{B}_{\Lambda}). \tag{1.37}$$

We now apply $\Gamma(k;-)$ and use the fact that the obvious morphism

$$\operatorname*{colim}_{\alpha} \Gamma(k_{\alpha};\mathrm{cobar}(\mathcal{B}_{\alpha,\Lambda})) \to \operatorname*{colim}_{\alpha} \Gamma(k; e_{\alpha}^{*}\mathrm{cobar}(\mathcal{B}_{\alpha,\Lambda}))$$

is an equivalence to conclude. □

To go further, we need a short digression.

Construction 1.66. Given a profinite group G, we denote by B_G the category of finite continuous G-sets, which we endow with the topology τ_G generated by jointly surjective families of morphisms of G-sets. For a commutative ring spectrum Λ, we define the ∞-category $\mathrm{Rep}(G;\Lambda)$ of G-representations with coefficients in Λ as the ∞-category of τ_G-hypersheaves on B_G with coefficients in Λ, i.e., we set

$$\mathrm{Rep}(G;\Lambda) = \mathrm{Shv}^{\wedge}_{\tau_G}(\mathrm{B}_G;\Lambda).$$

When G is finite, we recover the usual definition of the ∞-category of G-representations with coefficients in Λ. The site (B_G, τ_G) has a canonical point given by G considered as a profinite G-set via the regular left action. Taking stalks at this point gives a forgetful functor

$$\mathrm{f}_G^{*} : \mathrm{Rep}(G;\Lambda) \to \mathrm{Mod}_{\Lambda}.$$

We denote by $\mathrm{f}_{G,*}$ its right adjoint.

Notation 1.67. Let X be a profinite set. Given a Λ-module M, we let $\mathcal{C}^0(X;M)$ be the Λ-module

$$\mathcal{C}^0(X;M) = \operatorname*{colim}_{X \twoheadrightarrow F} M^F$$

where the colimit is over the filtered set of surjections from X to finite sets and $M^F = \prod_F M$ is the direct product of copies or M indexed by F. When $M = \Lambda$, the resulting Λ-module $\mathcal{C}^0(X;\Lambda)$ is naturally a commutative Λ-algebra. In fact, we have equivalences $\mathcal{C}^0(X;M) \simeq \mathcal{C}^0(X;\Lambda) \otimes_{\Lambda} M$. As usual, when Λ is the sphere spectrum, we simply write $\mathcal{C}^0(X)$.

Lemma 1.68. *Let G be a profinite group.*

- *(i) The functor* $\mathrm{f}_{G,*} : \mathrm{Mod}_{\Lambda} \to \mathrm{Rep}(G;\Lambda)$ *is colimit-preserving.*
- *(ii) The cosimplicial algebra* $\Gamma(G;\mathrm{cobar}^{\bullet}(\mathrm{f}_{G,*}(\Lambda)))$ *is equivalent to the commutative Hopf algebra* $\mathcal{C}^0(\mathrm{B}_{\bullet}(G);\Lambda)$ *where* $\mathrm{B}_{\bullet}(G)$ *is the classifying simplicial profinite set of G.*

Proof. This is an easy exercise. We give some details for the reader's convenience. The functor $\mathrm{f}_{G,*}$ sends a Λ-module M to the τ_G-sheaf $F \in \mathrm{B}_G \mapsto M^F$. This proves (i) since finite direct products in Mod_{Λ} are colimit-preserving.

For (ii), we remark that, for $m \geq 0$, the G-representation $\mathrm{f}_{G,*}(\Lambda)^{\otimes m}$ is the τ_G-hypersheaf associated to the presheaf $F \in \mathrm{B}_G \mapsto \Lambda^{F^m}$. Thus, $\mathrm{f}_{G,*}(\Lambda)^{\otimes m}$ is given by $F \in \mathrm{B}_G \mapsto \mathcal{C}^0(F \times^G G^m;\Lambda)$ with G acting diagonally on G^m. From this, we deduce easily that $\Gamma(G;\mathrm{cobar}^{\bullet}(\mathrm{f}_{G,*}(\Lambda)))$ is equivalent to $\mathcal{C}^0(G\backslash\mathrm{bar}_{\bullet}(G);\Lambda)$. This is precisely the commutative Hopf algebra $\mathcal{C}^0(\mathrm{B}_{\bullet}(G);\Lambda)$. □

Remark 1.69. To ease notation, we will often use the same symbol to denote a commutative Hopf algebra, its underlying commutative algebra, its underlying spectrum, etc. For example, the expression "the commutative Hopf algebra $\mathcal{C}^0(G;\Lambda)$" should be read "the commutative Hopf algebra $\mathcal{C}^0(\mathrm{B}(G);\Lambda)$". Similarly, the expression "the Λ-module $\mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda)$" should be read "the Λ-module $\mathrm{B}^*\mathrm{B}_*(\Lambda)$".

Theorem 1.70. *Let $\overline{k}/k$ be an algebraic closure of k and $\overline{\sigma} : \overline{k} \hookrightarrow \mathbb{C}$ a complex embedding extending σ. We have the following properties.*

(i) There is a coCartesian square of commutative Hopf Λ-algebras

$$\begin{tikzcd}
\mathcal{C}^0(\mathcal{G}_{\overline{k}/k};\Lambda) \arrow[r] \arrow[d] & \mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda) \arrow[d] \\
\Lambda \arrow[r] & \mathcal{H}_{\mathrm{mot}}(\overline{k},\overline{\sigma};\Lambda).
\end{tikzcd}$$

(ii) The morphism of commutative Hopf Λ-algebras

$$\mathcal{C}^0(\mathcal{G}_{\overline{k}/k};\Lambda) \to \mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda) \tag{1.38}$$

becomes an equivalence after ℓ-adic completion, for every prime ℓ. Said differently, the cofiber of the morphism of Λ-modules in (1.38) *belongs to* $\mathrm{Mod}_{\Lambda_{\mathbb{Q}}}$.

Proof. We have a commutative triangle in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$

$$\begin{tikzcd}
\mathrm{Rep}(\mathcal{G}_{\overline{k}/k};\Lambda) \arrow[r, "\mathrm{c}^*"] \arrow[dr, "\mathrm{f}^*"'] & \mathbf{MSh}(k;\Lambda) \arrow[d, "\mathrm{B}^*"] \\
& \mathrm{Mod}_{\Lambda}
\end{tikzcd}$$

where c^* is the "Artin motive" functor and where we write "f^*" for "$\mathrm{f}^*_{\mathcal{G}_{\overline{k}/k}}$". This induces a morphism of commutative algebras $\mathrm{c}^*\mathrm{f}_*(\Lambda) \to \mathrm{B}_*(\Lambda)$ and hence a morphism of cosimplicial algebras

$$\mathrm{c}^*(\mathrm{cobar}(\mathrm{f}_*(\Lambda))) \to \mathrm{cobar}(\mathrm{B}_*(\Lambda)).$$

Applying $\Gamma(k;-)$ and using the natural transformation $\Gamma(\mathcal{G}_{\overline{k}/k};-) \to \Gamma(k;\mathrm{c}^*(-))$, we obtain, by Lemma 1.68, a morphism of commutative Hopf Λ-algebras

$$\mathcal{C}^0(\mathcal{G}_{\overline{k}/k};\Lambda) \to \mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda).$$

Since this construction is natural in the pair (k,σ), we also obtain a commutative square as in part (i) of the statement. To prove that this square is coCartesian, we may replace the commutative Hopf algebras by their underlying commutative algebras. The result follows then from Proposition 1.65 and Lemma 1.71 below.

We now turn to the proof of (ii). Using Proposition 1.65, it is enough to check the assertion that the cofiber of the morphism in (1.38) belongs to $\mathrm{Mod}_{\Lambda_{\mathbb{Q}}}$ when k is a finitely generated extension of $\mathbb{Q}$. In particular, we may assume that k has finite virtual Λ-cohomological dimension.

Given a stable presentable ∞-category $\mathcal{C}$, we let $\mathcal{C}_{\ell\text{-cpl}}$ be the full sub-∞-category of ℓ-complete objects in $\mathcal{C}$. Recall that the obvious inclusion admits a left adjoint

$$(-)_{\ell\text{-cpl}} : \mathcal{C} \to \mathcal{C}_{\ell\text{-cpl}}$$

exhibiting $\mathcal{C}_{\ell\text{-cpl}}$ as the localisation of $\mathcal{C}$ with respect to ℓ-divisible objects (i.e., those objects for which multiplication by ℓ is an equivalence). The Betti realisation functor B^* and its right adjoint

B_* are colimit-preserving, and hence are left adjoint functors. Moreover, they preserve ℓ-divisible objects. Thus, by the universal properties of localisation, we have commutative squares

$$\begin{tikzcd}
\mathbf{MSh}(k;\Lambda) \arrow[r, "B^*"] \arrow[d, "(-)_{\ell\text{-cpl}}"] & \mathrm{Mod}_\Lambda \arrow[d, "(-)_{\ell\text{-cpl}}"] \\
\mathbf{MSh}(k;\Lambda)_{\ell\text{-cpl}} \arrow[r, "B_\ell^*"] & (\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{Mod}_\Lambda \arrow[r, "B_*"] \arrow[d, "(-)_{\ell\text{-cpl}}"] & \mathbf{MSh}(k;\Lambda) \arrow[d, "(-)_{\ell\text{-cpl}}"] \\
(\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}} \arrow[r, "B_{\ell,*}"] & \mathbf{MSh}(k;\Lambda)_{\ell\text{-cpl}},
\end{tikzcd}$$

and $B_{\ell,*}$ is right adjoint to B_ℓ^*. Since the Λ-module $\mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda)$ is given $B^*B_*\Lambda$, we deduce that its ℓ-completion $\mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda)_{\ell\text{-cpl}}$ is given by $B_\ell^* B_{\ell,*}\Lambda_\ell$. A similar reasoning shows that the ℓ-completion of $\mathcal{C}^0(\mathcal{G}_{\bar{k}/k};\Lambda) \simeq f^*f_*\Lambda$ is given by $f_\ell^* f_{\ell,*}\Lambda_\ell$ where f_ℓ^* and $f_{\ell,*}$ are defined similarly as B_ℓ^* and $B_{\ell,*}$. (Here, we implicitly use Lemma 1.68(i).) Thus, to finish the proof, it would suffices to show that the obvious functor

$$c_\ell^* : \mathrm{Rep}(\mathcal{G}_{\bar{k}/k})_{\ell\text{-cpl}} \to \mathbf{MSh}(k;\Lambda)_{\ell\text{-cpl}}$$

is an equivalence. This is precisely the content of the rigidity theorem, proven by Bachmann [Bac18, Theorem 6.6]. See also [AGV20, Theorem 2.10.4]. □

Lemma 1.71. *Let k'/k be a finite extension and $\sigma' : k' \hookrightarrow \mathbb{C}$ a complex embedding extending σ. Then, we have a coCartesian square of commutative algebras*

$$\begin{tikzcd}
\Lambda^{\mathrm{Hom}_\sigma(k',\mathbb{C})} \arrow[r] \arrow[d, "\sigma'^*"] & \mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda) \arrow[d] \\
\Lambda \arrow[r] & \mathcal{H}_{\mathrm{mot}}(k',\sigma';\Lambda)
\end{tikzcd}$$

where $\mathrm{Hom}_\sigma(k',\mathbb{C})$ *is the set of complex embeddings extending σ and σ'^* is "evaluation at σ'".*

Proof. Let $e : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ be the obvious morphism. For $M' \in \mathbf{MSh}(k';\Lambda)$, the object $e_*(M')$ is an $e_*(\Lambda)$-module and the morphism

$$e^*e_*(M') \otimes_{e^*e_*(\Lambda)} \Lambda \to M' \tag{1.39}$$

is an equivalence. To prove this, we remark that the domain and codomain of the morphism in (1.39) are colimit-preserving in the variable M'. (Here, we use that $e_* \simeq e_!$ admits a right adjoint.) On the other hand, $\mathbf{MSh}(k';\Lambda)$ is generated under colimits by the image of e^*. Indeed, for X' a smooth k'-variety, the motive $\mathrm{M}(X') \in \mathbf{MSh}(k';\Lambda)$ is a direct summand of $\mathrm{M}(X' \otimes_k k')$. Thus, it is enough to show that the morphism in (1.39) is an equivalence when $M' = e^*M$. In this case, by the projection formula, we have $e_*M' \simeq e_*(\Lambda) \otimes M$, and the result is then obvious.

Let $B'^* : \mathbf{MSh}(k',\Lambda) \to \mathrm{Mod}_\Lambda$ the Betti realisation functor associated to σ', B'_* its right adjoint and $\mathcal{B}'_\Lambda = B'_*(\Lambda)$. Since $B^* \simeq B'^* \circ e^*$, we have $\mathcal{B}_\Lambda \simeq e_*(\mathcal{B}'_\Lambda)$. Using the equivalence in (1.39), we obtain an equivalence of commutative algebras

$$e^*\mathcal{B}_\Lambda \otimes_{e^*e_*\Lambda} \Lambda \xrightarrow{\sim} \mathcal{B}'_\Lambda.$$

Applying B'^* to this equivalence, the result follows. □

Theorem 1.72. *Assume that Λ is connective. Then, the commutative Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda)$ is also connective, i.e., it is a commutative Hopf algebra in the symmetric monoidal ∞-category $\mathrm{Mod}^{\otimes}_{\Lambda,\geq 0}$. In other words, $\mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda)$ is a spectral affine group scheme over Λ.*

Proof. We only need to show that the underlying Λ-module of $\mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda)$ is connective. Since $\mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda)\simeq\mathcal{H}_{\mathrm{mot}}(k,\sigma)\otimes\Lambda$, it is enough to show that $\mathcal{H}_{\mathrm{mot}}(k,\sigma)$ is connective. Since $C^0(\mathcal{G}_{\bar{k}/k})$ is connective, it is enough to show that the cofiber of

$$C^0(\mathcal{G}_{\bar{k}/k})\to\mathcal{H}_{\mathrm{mot}}(k,\sigma)$$

is a connective spectrum. By Theorem 1.70(ii), this cofiber belongs to $\mathrm{Mod}_{\mathbb{Q}}$. Since tensoring with $\mathbb{Q}$ is exact, we are left to show that the cofiber of

$$C^0(\mathcal{G}_{\bar{k}/k};\mathbb{Q})\to\mathcal{H}_{\mathrm{mot}}(k,\sigma;\mathbb{Q})$$

is connective. This follows immediately from [Ayo14b, Corollaire 2.105]. □

Remark 1.73. Keep assuming that Λ is connective. It follows from Theorem 1.72 that the ordinary $\pi_0\Lambda$-algebra

$$\mathcal{H}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma;\Lambda)=\pi_0\mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda)$$

is an ordinary commutative Hopf algebra. Its spectrum, denoted $\mathcal{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma;\Lambda)$, is the classical affine group scheme underlying $\mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda)$. This is an affine group scheme over $\pi_0\Lambda$. Taking $\Lambda=\mathbb{Q}$, we obtain an affine pro-algebraic group $\mathcal{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma;\mathbb{Q})$ which is isomorphic to Nori's motivic Galois group by [CG17, Theorem 9.1].

1.5. **The fundamental sequence.**

In this subsection, we discuss some of the results obtained in [Ayo14c, §2] relating the motivic Galois groups to the topological fundamental groups. The main facts are summarised in Theorem 1.86 below. As usual, $\Lambda\in\mathrm{CAlg}(\mathcal{S}p)$ is a commutative ring spectrum.

Construction 1.74. Fix a base field k and a complex embedding $\sigma:k\hookrightarrow\mathbb{C}$. Let $X=(X_\alpha)_\alpha$ be a pro-k-variety and $x\in\lim_\alpha X_\alpha(\mathbb{C})$ a compatible system of base points. We can associate to the pair (X,x) a commutative Hopf algebra object $\mathcal{F}_\sigma(X,x;\Lambda)$ in $\mathrm{Mod}^{\otimes}_{\Lambda}$ by adapting Construction 1.59. Indeed, consider the symmetric monoidal functor

$$\phi_x^*:\widehat{\mathbf{LS}}(X;\Lambda)^{\otimes}\to\mathrm{Mod}^{\otimes}_{\Lambda}\tag{1.40}$$

obtained by restricting the inverse image functor along the inclusion $x:\mathrm{Spec}(k)\to X$. Here and below, we define $\widehat{\mathbf{LS}}(X;\Lambda)^{\otimes}$ to be the filtered colimit in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$ of the symmetric monoidal ∞-categories $\widehat{\mathbf{LS}}(X_\alpha;\Lambda)^{\otimes}$ introduced in Definition 1.23. The functor ϕ_x^* in (1.40) is a morphism in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$. In particular, ϕ_x^* admits a right adjoint $\phi_{x,*}$ which right-lax monoidal and commutes with colimits. By [Lur17, Proposition 4.7.3.3], the composite functor $\phi_{x,*}\phi_x^*$ underlies a coalgebra structure in the ∞-category $\mathrm{EndFun}(\widehat{\mathbf{LS}}(X;\Lambda)^{\otimes})$ of right-lax symmetric monoidal endofunctors of $\widehat{\mathbf{LS}}(X;\Lambda)^{\otimes}$. Said differently, there exists a cosimplicial object

$$\widehat{\phi}_x:\mathbf{\Delta}\to\mathrm{EndFun}(\widehat{\mathbf{LS}}(X;\Lambda)^{\otimes})$$

such that $\widehat{\phi}^n_x=(\phi_{x,*}\phi_x^*)^{\circ n+1}$, and the faces and codegeneracies are induced by the unit and counit of the adjunction $(\phi_x^*,\phi_{x,*})$. Arguing as in the proof of Lemma 1.62, one can show that the cosimplicial commutative algebra $\widehat{\phi}_x\Lambda$ is the left Kan extension of its restriction to $\mathbf{\Delta}^{\leq 0}$.[2] Said differently, there is an equivalence of cosimplicial commutative algebras

$$\mathrm{cobar}(\phi_{x,*}(\Lambda))\xrightarrow{\sim}\widehat{\phi}_x(\Lambda).$$

[2] It is maybe worth developing a more general formalism that we can apply here.

We define a cosimplicial commutative algebra in $\mathrm{Mod}_\Lambda^\otimes$ by setting

$$\mathcal{F}(X, x; \Lambda) = \pi_{X,*}(\widehat{\phi}_x(\Lambda))$$

where $\pi_X : X \to \mathrm{Spec}(k)$ is the structural projection and $\pi_{X,*}$ is the usual direct image functor.

Notation 1.75. With the notations as in Construction 1.74, we set

$$\pi_1^{\mathrm{alg}}(X, x; \Lambda) = \mathrm{Spec}(\mathcal{F}(X, x; \Lambda)).$$

This is a nonconnective spectral affine group Λ-scheme.

Remark 1.76. If Λ' is a commutative Λ-algebgra, we have a commutative square

$$\begin{tikzcd}
\widehat{\mathbf{LS}}(X; \Lambda)^\otimes \arrow[r, "\phi_x^*"] \arrow[d, "-\otimes_\Lambda \Lambda'"] & \mathrm{Mod}_\Lambda \arrow[d, "-\otimes_\Lambda \Lambda'"] \\
\widehat{\mathbf{LS}}(X; \Lambda')^\otimes \arrow[r, "\phi_x^*"] & \mathrm{Mod}_{\Lambda'}
\end{tikzcd}$$

inducing a morphism of commutative Hopf Λ'-algebras

$$\mathcal{F}(X, x; \Lambda) \otimes_\Lambda \Lambda' \to \mathcal{F}(X, x; \Lambda'). \tag{1.41}$$

In contrast with Remark 1.61, this morphism is not an equivalence in general.

Lemma 1.77. *Keep the assumptions and notations of Construction 1.74. Assume that Λ is an ordinary regular ring. Then the Λ-module underlying $\mathcal{F}(X, x; \Lambda)$ is coconnective.*

Proof. The condition that Λ is an ordinary regular ring implies that the canonical t-structure on $\mathbf{Sh}(X^{\mathrm{an}}; \Lambda)$ induces a t-structure on $\mathbf{LS}(X; \Lambda)$ which, by indization, induces a t-structure on $\widehat{\mathbf{LS}}(X; \Lambda)$. Moreover, the functor ϕ_x^* is t-exact. This implies that its right adjoint $\phi_{x,*}$ is a left exact. The result follows since the Λ-module underlying $\mathcal{F}(X, x; \Lambda)$ is given by $\phi_x^* \phi_{x,*}(\Lambda)$. □

We recall the following standard definition.

Definition 1.78.

(i) An elementary fibration is an affine morphism of schemes $f : X \to S$ which is part of a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & Z \arrow[l, "i"'] \arrow[dl, "e"] \\
& S &
\end{tikzcd}$$

such that j is a fiberwise dense open immersion, $\overline{f}$ is smooth, proper, geometrically connected and of relative dimension 1, and e is étale.

(ii) A k-variety X is said to be an Artin neighbourhood if its structural morphism $X \to \mathrm{Spec}(k)$ can be factored as

$$X = X_d \xrightarrow{f_d} X_{d-1} \to \cdots \to X_1 \xrightarrow{f_1} X_0 = \mathrm{Spec}(k)$$

where, for $1 \leq r \leq d$, the morphism $f_r : X_r \to X_{r-1}$ is an elementary fibration. (In particular, an Artin neighbourhood is smooth and geometrically connected over k.)

Lemma 1.79. *Assume that the field k is infinite. Then, every smooth and geometrically connected k-variety admits an open covering by Artin neighbourhoods.*

Proof. This is proven in [SGA73, Exposé XI, Proposition 3.3] under the assumption that k is algebraically closed. However, the argument in loc. cit. can be easily adapted to the case where k is only assumed to be infinite. Indeed, in [SS16, Lemma 6.3], the argument was adapted to cover the case where k is perfect infinite, and where one is working in the neighbourhood of a rational point. Both assumptions are only used in the last paragraph of the proof of [SS16, Lemma 6.3]; they can be removed using the following simple fact: for a geometrically irreducible k-variety S, and nonempty open subsets $U_1, \dots, U_n$ of $S \otimes_k \overline{k}$, with $\overline{k}/k$ an algebraic closure, there is an open subset $V \subset S$ such that $V \otimes_k \overline{k}$ is contained in all the U_i's. □

The following proposition is a "pro-algebraic" version of the well-known property that Artin neighbourhoods are of type $K(\pi, 1)$; see [SS16, §2.3 & Proposition 2.8] for a discussion in the context of pro-finite étale homotopy theory. (In the statement below, we implicitly use the fact that pushforwards along elementary fibrations preserve local systems, which follows from Ehresmann's theorem; see also Step 5 of the proof of Proposition 1.26.)

Proposition 1.80 (Beĭlinson). *Assume that Λ is a field or a Dedekind domain with finite residue fields at all maximal ideals. Let X be an Artin neighbourhood. Then there is an equivalence of ∞-categories*

$$\theta_X : \mathrm{D}(\widehat{\mathbf{LS}}(X;\Lambda)^{\heartsuit}) \xrightarrow{\sim} \widehat{\mathbf{LS}}(X;\Lambda).$$

More generally, assume that for every k-variety S we are given a full sub-∞-category $\widehat{\mathcal{L}}(S) \subset \widehat{\mathbf{LS}}(S;\Lambda)$ such that the following conditions are satisfied:

- *each $\widehat{\mathcal{L}}(S)$ is stable under colimits, desuspension, tensor product, and truncations with respect to the obvious t-structure;*
- *the $\widehat{\mathcal{L}}(S)$'s are stable by pullbacks and by pushforwards along finite étale morphisms and elementary fibrations;*
- *each $\mathcal{L}(S) = \widehat{\mathcal{L}}(S) \cap \mathbf{LS}(S;\Lambda)$ is stable under duality and generates $\widehat{\mathcal{L}}(S)$ by colimits.*

With $\widehat{\mathcal{L}}(X)^{\heartsuit} = \widehat{\mathcal{L}}(X) \cap \widehat{\mathbf{LS}}(X;\Lambda)^{\heartsuit}$, there is an equivalence of ∞-categories $\mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit}) \xrightarrow{\sim} \widehat{\mathcal{L}}(X)$.

Proof. The essential part of the statement can be obtained by adapting Beĭlinson's proof of [Beĭ87, Lemma 2.1.1]; some extra work is needed for dealing with unbounded complexes. For the reader's convenience, we give a complete proof which we split into three steps.

Step 1. We argue by induction on the dimension of X. When X is zero-dimensional, there is nothing to prove. Thus, we may assume that the dimension of X is ≥ 1, and we fix an elementary fibration $f : X \to S$ with S an Artin neighbourhood. By the induction hypothesis, the functor

$$\theta_S : \mathrm{D}(\widehat{\mathcal{L}}(S)^{\heartsuit}) \to \widehat{\mathcal{L}}(S)$$

is an equivalence of ∞-categories. Consider the commutative square

$$\begin{tikzcd} \mathrm{D}(\widehat{\mathcal{L}}(S)^{\heartsuit}) \arrow[r, "f^*"] \arrow[d, "\theta_S", "\sim"'] & \mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit}) \arrow[d, "\theta_X"] \\ \widehat{\mathcal{L}}(S) \arrow[r, "f^*"] & \widehat{\mathcal{L}}(X), \end{tikzcd} \tag{1.42}$$

where we denote by f^* the two functors given by pullback along f. These two functors admit right adjoints that we denote by

$$\mathrm{R}f_* : \mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit}) \to \mathrm{D}(\widehat{\mathcal{L}}(S)^{\heartsuit}) \qquad \text{and} \qquad f_* : \widehat{\mathcal{L}}(X) \to \widehat{\mathcal{L}}(S).$$

We will prove in Steps 2 and 3 below that the square (1.42) is right adjointable, i.e., that the natural transformation

$$\theta_S \circ \mathrm{R}f_* \to f_* \circ \theta_X \tag{1.43}$$

is an equivalence. This would suffices to conclude. Indeed, θ_X is colimit-preserving and its image generates $\widehat{\mathcal{L}}(X)$ under colimits. Thus, it is enough to show that θ_X is fully faithful. Let M and N be two objects of $\mathrm{D}(\widehat{\mathcal{L}}(X)^\heartsuit)$, and consider the map

$$\mathrm{Map}_{\mathrm{D}(\widehat{\mathcal{L}}(X)^\heartsuit)}(M, N) \to \mathrm{Map}_{\widehat{\mathcal{L}}(X)}(M, N). \tag{1.44}$$

(Since θ_X is the identity on objects, we simply write M and N for $\theta_X(M)$ and $\theta_X(N)$.) The domain and codomain of the map in (1.44) transform colimits in the variable M into limits in $\mathcal{S}$. The ∞-category $\mathrm{D}(\widehat{\mathcal{L}}(X)^\heartsuit)$ is generated under colimits by the objects of $\mathcal{L}(X)^\heartsuit$, and these objects are clearly dualizable with respect to the symmetric monoidal structure on $\mathrm{D}(\widehat{\mathcal{L}}(X)^\heartsuit)$ given by Lemma 1.81 below. Thus, we may assume that M is dualizable with dual $M^\vee$. Replacing N by $N \otimes_\Lambda M^\vee$, we may assume that $M = \Lambda_X$ is the unit object. In this case, the map (1.44) can be identified with the map

$$\mathrm{Map}_{\mathrm{D}(\widehat{\mathcal{L}}(S)^\heartsuit)}(\Lambda_S, \mathrm{R}f_*N) \to \mathrm{Map}_{\widehat{\mathcal{L}}(S)}(\Lambda_S, f_*N) \tag{1.45}$$

induced from the equivalence θ_S and the natural transformation (1.43). This proves our claimed reduction.

Step 2. It remains to see that the natural transformation (1.43) is an equivalence. Note that the functor f_* has cohomological amplitude in $[0, 1]$, i.e., takes an object of $M \in \widehat{\mathcal{L}}(X)^\heartsuit$ to an object $f_*M \in \widehat{\mathcal{L}}(S)$ concentrated in cohomological degrees zero and one.

We start by showing that (1.43) is an equivalence when evaluated at an injective object J of the Grothendieck abelian category $\widehat{\mathcal{L}}(X)^\heartsuit$. The condition that J is injective insures that the complex $\mathrm{R}f_*J$ is concentrated in degree zero. Since $\mathrm{R}f_*J \to f_*J$ induces an equivalence in degree zero, it remains to see that $\mathrm{H}^1(f_*J)$ vanishes. Since the functor $f^* : \widehat{\mathcal{L}}(S)^\heartsuit \to \widehat{\mathcal{L}}(X)^\heartsuit$ is exact, it follows that $\mathrm{H}^0(f_*J)$ is an injective object of $\widehat{\mathcal{L}}(S)^\heartsuit$. Using that θ_S is an equivalence, we deduce that f_*J is isomorphic to $\mathrm{H}^0(f_*J) \oplus \mathrm{H}^1(f_*J)[-1]$. To conclude, it is thus enough to show that the morphism $f^*f_*J \to J$ is zero on the factor $f^*\mathrm{H}^1(f_*J)[-1]$. But $\mathrm{cofib}(f^*\mathrm{H}^1(f_*J)[-1] \to J)$ belongs to $\widehat{\mathcal{L}}(X)^\heartsuit$ and is an extension of $f^*\mathrm{H}^1(f_*J)$ by J. This extension must split because J is injective.

Next, we show that (1.43) is an equivalence when evaluated at any object M of $\widehat{\mathcal{L}}(X)^\heartsuit$. Choose an exact sequence

$$M \to I^0 \to I^1 \to \ldots \to I^m \to N \to 0$$

where the I^i's are injective objects of $\widehat{\mathcal{L}}(X)^\heartsuit$. Letting $I = [I^0 \to \ldots \to I^m]$, with I^0 placed in degree zero, we obtain an exact triangle

$$M \to I \to N[-m] \to$$

in $\mathrm{D}(\widehat{\mathcal{L}}(X)^\heartsuit)$. The morphism $\mathrm{R}f_*I \to f_*I$ is an equivalence by the previous discussion and induction. We deduce an equivalence

$$\mathrm{cofib}(\mathrm{R}f_*M \to f_*M) \simeq \mathrm{cofib}(\mathrm{R}f_*N \to f_*N)[-m-1].$$

The right hand side in concentrated in cohomological degrees $\geq m$. This shows that the morphism $\mathrm{H}^i(\mathrm{R}f_*M) \to \mathrm{H}^i(f_*M)$ is an isomorphism for $i \leq m-1$. Since m can be taken arbitrary large, this proves that $\mathrm{R}f_*M \to f_*M$ is an equivalence.

For use in Step 3 below, we note the following consequence. We have two "global sections" functors

$$\mathrm{R}\Gamma(X;-):\mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit})\to \mathrm{D}(\mathrm{Mod}_{\Lambda}^{\heartsuit})\simeq \mathrm{Mod}_{\Lambda}\qquad \text{and}\qquad \Gamma(X;-):\widehat{\mathcal{L}}(X)\to \mathrm{Mod}_{\Lambda},$$

right adjoint to the obvious functors sending a Λ-module to the associated constant sheaf. Using the induction hypothesis, and the discussion above, we immediately see that the obvious natural transformation $\mathrm{R}\Gamma(X;-)\to\Gamma(X,-)\circ\theta_X$ is an equivalence when evaluated at any object of $\mathrm{D}^{\mathrm{b}}(\widehat{\mathcal{L}}(X)^{\heartsuit})$. In particular, using Artin's vanishing theorem for the cohomology of affine varieties in the Betti setting (see the beginning of [Nor02, §1] for a proof), we deduce that the functor $\mathrm{R}\Gamma(X;-)$ has cohomological amplitude in $[0,\dim(X)]$.

Step 3. At this stage, we know that (1.43) is an equivalence when evaluated at any object of $\mathrm{D}^{\mathrm{b}}(\widehat{\mathcal{L}}(X)^{\heartsuit})$. This sub-$\infty$-category generates $\mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit})$ under colimits. On the other hand, the functors f_* and θ_X are colimit-preserving. (In the case of f_*, we use that $f^*:\widehat{\mathcal{L}}(S)\to\widehat{\mathcal{L}}(X)$ belongs to $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ which follows immediately from Definition 1.23; see also Lemma 1.25.) Thus, to conclude, it remains to see that $\mathrm{R}f_*$ is also colimit-preserving. For this, we will show that $f^*:\mathrm{D}(\widehat{\mathcal{L}}(S)^{\heartsuit})\to\mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit})$ belongs to $\mathrm{Pr}^{\mathrm{L}}_{\omega}$. That $\mathrm{D}(\widehat{\mathcal{L}}(S)^{\heartsuit})$ is compactly generated follows from the induction hypothesis since $\widehat{\mathcal{L}}(S)$ has this property. Using the symmetric monoidal structures provided by Lemma 1.81 below, we see that $f^*:\mathrm{D}(\widehat{\mathcal{L}}(S)^{\heartsuit})\to\mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit})$ preserves dualizable objects. Since dualizable objects generate $\mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit})$ under colimits, we can conclude if dualizability in $\mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit})$ implies compactness. This is the case if and only if Λ_X is compact in $\mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit})$. Said differently, we are left to show that the "global sections" functor $\mathrm{R}\Gamma(X;-):\mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit})\to\mathrm{Mod}_{\Lambda}$ is colimit-preserving.

In order to do so, we first prove that every object of $\mathrm{D}(\widehat{\mathcal{L}}(X)^{\heartsuit})$ is Postnikov complete in the sense of [CM19, Definition 2.4] (see also [CM19, Example 2.6]). By [CM19, Proposition 2.10], it is enough to show that every torsion-free object M of $\mathcal{L}(X)^{\heartsuit}=\mathcal{L}(X)\cap\widehat{\mathcal{L}}(X)^{\heartsuit}$ has cohomological dimension $\leq\dim(X)$. (Here we use the fact that every object of $\mathcal{L}(X)^{\heartsuit}$ is a quotient of a torsion-free object; see the proof of Lemma 1.81 below.) This follows immediately from the following properties: M is dualizable, the functor $M^{\vee}\otimes-$ is t-exact, and, for $N\in\widehat{\mathcal{L}}(X)^{\heartsuit}$, the complex of Λ-modules $\mathrm{R}\Gamma(X;N)$ has cohomological amplitude in $[0,\dim(X)]$. (The last property was proven in Step 2.) This said, it is easy to see that $\mathrm{R}\Gamma(X;-)$ is colimit-preserving. Indeed, let $L:K^{\triangleright}\to\mathrm{D}(\widehat{\mathcal{L}}(X))$ be a colimit diagram with K a filtered ordinary category, and denote by ∞ the cone point of $K^{\triangleright}$. For every $\alpha\in K^{\triangleright}$, we have an equivalence in Mod_{Λ}:

$$\mathrm{R}\Gamma(X;L(\alpha))\simeq\lim_{m\in\mathbb{N}}\mathrm{R}\Gamma(X;\tau_{\leq m}L(\alpha)),$$

just using that $L(\alpha)$ is Postnikov complete. Moreover, for a fixed cohomological index n, the tower of ordinary Λ-modules $(\mathrm{R}^n\Gamma(X;\tau_{\leq m}L(\alpha)))_m$ is constant starting from $m\geq\dim(X)-n$. (This follows from the fact proven in Step 2 that $\mathrm{R}\Gamma(X;-)$ has cohomological amplitude in $[0,\dim(X)]$.) Using Milnor exact sequence for $\lim^1$ (see for example [Wei94, Theorem 3.5.8]), we deduce that

$$\mathrm{R}^n\Gamma(X;L(\alpha))\to\mathrm{R}^n\Gamma(X;\tau_{\leq m}L(\alpha))$$

is an isomorphism for $m\geq\dim(X)-n$. Thus, we are reduced to showing that $\mathrm{R}^n\Gamma(X;\tau_{\leq m}L(\infty))$ is the colimit of the $\mathrm{R}^n\Gamma(X;\tau_{\leq m}L(\alpha))$, for $\alpha\in K$. Said differently, we may replace L with $\tau_{\geq -n}\tau_{\leq m}L$ and assume that the $L(\alpha)$'s, for $\alpha\in K^{\triangleright}$, are bounded. In this case, by Step 2, we may replace $\mathrm{R}\Gamma(X;-)$ with $\Gamma(X;-)\circ\theta_X$ and conclude using that $\Gamma(X;-)$ is colimit-preserving. □

Lemma 1.81. *Assume that Λ is a field or a Dedekind domain with finite residue fields at all maximal ideals, and assume we are given full sub-∞-categories $\widehat{\mathcal{L}}(S) \subset \widehat{\mathbf{LS}}(S;\Lambda)$ as in the statement of Proposition 1.80. Then the ∞-categories $\mathrm{D}(\widehat{\mathcal{L}}(S)^{\heartsuit})$ admit natural symmetric monoidal structures such that the pullback functors and the functors θ_S lift to symmetric monoidal functors.*

Proof. The case where Λ is a field is clear, so we assume that Λ is a Dedekind domain. It is enough to show that every local system M in $\widehat{\mathcal{L}}(S)^{\heartsuit}$ admits a resolution by a torsion-free local system in $\widehat{\mathcal{L}}(S)^{\heartsuit}$. Let M_{tor} be the subsheaf of M consisting of torsion sections. By our assumption on the residue fields of the Dedekind domain Λ, the local system M_{tor} has finite monodromy, i.e., there exists a finite étale cover $e : S' \to S$ such that e^*M_{tor} is constant. Since $e_!e^*M_{\mathrm{tor}} \to M_{\mathrm{tor}}$ is surjective, we may find a surjection $N \to M_{\mathrm{tor}}$ with N torsion-free. (More precisely, we may take for N the image by $e_!$ of the constant sheaf associated to a torsion-free ordinary Λ-module surjecting onto the ordinary global sections of e^*M_{tor}.) We then obtain a surjection $M \times_{M_{\mathrm{tor}}} N \to M$ from a torsion free local system in $\widehat{\mathcal{L}}(S)^{\heartsuit}$, whose kernel is also torsion-free. □

Lemma 1.82. *Assume that Λ is a field or a Dedekind domain with finite residue fields at all maximal ideals. Let X be a pro-k-variety and $x \in \lim X(\mathbb{C})$. Denote by $\overline{\pi}_1^{\text{ét}}(X,x)$ the profinite étale fundamental group of the pair (X^{an}, x). The morphism of commutative Hopf Λ-algebras*

$$C^0(\overline{\pi}_1^{\text{ét}}(X,x);\Lambda) \to \mathcal{F}(X,x;\Lambda) \tag{1.46}$$

becomes an equivalence after tensoring with $\Lambda/\mathfrak{p}$ for every maxmial ideal $\mathfrak{p} \subset \Lambda$. Said differently, the cofiber of the morphism of Λ-modules in (1.46) *belongs to* $\mathrm{Mod}_{\mathrm{Frac}(\Lambda)}$.

Proof. The argument is identical to the one used in the proof of Theorem 1.70(ii); instead of the rigidity theorem for motivic sheaves, one uses that torsion local systems over X are étale locally constant. We leave the details to the reader. □

We can now prove the following theorem.

Theorem 1.83. *Assume that Λ is a field or a Dedekind domain with finite residue fields at all maximal ideals. Let X be an Artin neighbourhood and let $x \in X(\mathbb{C})$. Then $\mathcal{F}(X,x;\Lambda)$ is concentrated in degree zero and $\pi_1^{\mathrm{alg}}(X,x;\Lambda)$ is a classical affine group Λ-scheme.*

Proof. By Lemma 1.77, we already know that $\mathcal{F}(X,x;\Lambda)$ is coconnective. Thus, it remains to see that it is connective. Using Lemma 1.82, it is enough to prove this after tensoring with $\mathrm{Frac}(\Lambda)$. Since the functor $\phi_{x,*} : \mathrm{Mod}_{\Lambda} \to \widehat{\mathbf{LS}}(X;\Lambda)$ is colimit-preserving, we have

$$\mathcal{F}(X,x;\Lambda) \otimes_{\Lambda} \mathrm{Frac}(\Lambda) \simeq \phi_x^*\phi_{x,*}(\mathrm{Frac}(\Lambda)).$$

Now, by Proposition 1.80, the functor $\phi_{x,*}$ can be identified with the right derived functor $\mathrm{R}\phi_{x,*} : \mathrm{D}(\mathrm{Mod}_{\Lambda}^{\heartsuit}) \to \mathrm{D}(\widehat{\mathbf{LS}}(X;\Lambda)^{\heartsuit})$ associated to the right adjoint of $\phi_x^* : \widehat{\mathbf{LS}}(X;\Lambda)^{\heartsuit} \to \mathrm{Mod}_{\Lambda}^{\heartsuit}$. Since $\mathrm{Frac}(\Lambda)$ is an injective object of $\mathrm{Mod}_{\Lambda}^{\heartsuit}$, we deduce that $\mathrm{R}\phi_{x,*}(\mathrm{Frac}(\Lambda))$ is concentrated in degree zero. The same is thus true for $\phi_x^*\mathrm{R}\phi_{x,*}(\mathrm{Frac}(\Lambda))$ as needed. □

Corollary 1.84. *Assume that Λ is a field or a Dedekind domain with finite residue fields at all maximal ideals. Let K/k be a field extension and $\Sigma : K \hookrightarrow \mathbb{C}$ a complex embedding extending σ. Consider* $\mathrm{Spec}(K)$ *as a pro-k-variety in the obvious way and Σ as a point of the limit of the analytic pro-variety* $\mathrm{Spec}(K)^{\mathrm{an}}$*. Then, $\mathcal{F}(K,\Sigma;\Lambda)$ is concentrated in degree zero and $\pi_1^{\mathrm{alg}}(K,\Sigma;\Lambda)$ is a classical affine group Λ-scheme.*

Proof. This is an immediate consequence of Lemma 1.79 and Theorem 1.83. □

Remark 1.85. When Λ is a field, the affine group scheme $\pi_1^{\mathrm{alg}}(X,x;\Lambda)$ in Theorem 1.83 is the pro-algebraic completion of the topological fundamental group $\pi_1(X^{\mathrm{an}},x)$. This can be easily obtained from the fact that $\mathbf{LS}(X;\Lambda)^{\heartsuit}$ is equivalent to the ordinary category of finite-dimensional representations of $\pi_1(X^{\mathrm{an}},x)$ with coefficients in Λ.

In the next statement, we summarise the relation between motivic Galois groups and algebraic completions of fundamental groups. The main part of the statement was essentially proven in [Ayo14c, §2]. (See also [Ayo14d] for some corrections.)

Theorem 1.86. *Let K/k be a field extension, $\Sigma : K \hookrightarrow \mathbb{C}$ a complex embedding of K extending $\sigma : k \to \mathbb{C}$. Consider the induced morphism of spectral affine group Λ-schemes*

$$\rho_{K/k} : \mathcal{G}_{\mathrm{mot}}(K,\Sigma;\Lambda) \to \mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda), \tag{1.47}$$

and let $\mathcal{G}_{\mathrm{rel}}(K/k,\Sigma;\Lambda)$ be its kernel. We have the following properties.

- *(i) The morphism $\rho_{K/k}$ in (1.47) is flat. Moreover, it is faithfully flat if and only if k is algebraically closed in K.*
- *(ii) If k is algebraically closed, the morphism $\rho_{K/k}$ in (1.47) admits a splitting exhibiting $\mathcal{G}_{\mathrm{mot}}(K,\Sigma;\Lambda)$ as a semi-direct product of $\mathcal{G}_{\mathrm{rel}}(K/k,\Sigma;\Lambda)$ by $\mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda)$.*
- *(iii) The spectral scheme $\mathcal{G}_{\mathrm{rel}}(K/k,\Sigma;\Lambda)$ is flat over Λ. In particular, if Λ is an ordinary ring, $\mathcal{G}_{\mathrm{rel}}(K/k,\Sigma;\Lambda)$ is classical.*
- *(iv) Assume that Λ is a field or a Dedekind domain with finite residue fields at all maximal ideals. Then, the obvious morphism of classical affine group Λ-schemes*

$$\pi_1^{\mathrm{alg}}(K/k,\Sigma;\Lambda) \to \mathcal{G}_{\mathrm{rel}}(K/k,\Sigma;\Lambda) \tag{1.48}$$

is faithfully flat.

In particular, if Λ is a field or a Dedekind domain with finite residue fields at all maximal ideals, and if k is algebraically closed in K, we have an exact sequence of classical affine group Λ-schemes

$$\pi_1^{\mathrm{alg}}(K/k;\Sigma;\Lambda) \to \mathcal{G}^{\mathrm{cl}}_{\mathrm{mot}}(K,\Sigma;\Lambda) \to \mathcal{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma;\Lambda) \to \{1\} \tag{1.49}$$

Proof. Let $k' \subset K$ be the algebraic closure of k in K and $\sigma' = \Sigma|_{k'} : k' \hookrightarrow \mathbb{C}$. We have a factorisation $\rho_{K/k} = \rho_{k'/k} \circ \rho_{K/k'}$. By Lemma 1.71, the morphism $\rho_{k'/k} : \mathcal{G}_{\mathrm{mot}}(k',\sigma';\Lambda) \to \mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda)$ is the base change of $\mathrm{Spec}(\sigma'^*) : \mathrm{Spec}(\Lambda) \to \mathrm{Spec}(\Lambda^{\mathrm{Hom}_\sigma(k',\mathbb{C})})$ which is a pro-open immersion. Thus, to prove that (1.47) is flat, we may replace k by k'. Said differently, it is enough to prove the second assertion in (i). Next, we want to reduce to the case where k is algebraically closed. Fix an algebraic closure $\overline{k}/k$ of k and a complex embedding $\overline{\sigma} : \overline{k} \to \mathbb{C}$ extending σ, and let $\overline{K} = K \otimes_k \overline{k}$ and $\overline{\Sigma} : \overline{K} \to \mathbb{C}$ be the complex embedding that restricts to $\overline{\sigma}$ and Σ. We have a commutative diagram of spectral affine group Λ-schemes with Cartesian squares

$$\begin{tikzcd}
\mathcal{G}_{\mathrm{mot}}(\overline{K},\overline{\Sigma};\Lambda) \arrow[r, "\rho_{\overline{K}/\overline{k}}"] \arrow[d] & \mathcal{G}_{\mathrm{mot}}(\overline{k},\overline{\sigma};\Lambda) \arrow[r] \arrow[d] & \{1\}_\Lambda \arrow[d] \\
\mathcal{G}_{\mathrm{mot}}(K,\Sigma;\Lambda) \arrow[r, "\rho_{K/k}"] & \mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda) \arrow[r] & \mathcal{G}(\overline{k}/k)_\Lambda,
\end{tikzcd}$$

where we wrote $\mathcal{G}(\overline{k}/k)_\Lambda$ for the Galois group of $\overline{k}/k$ considered as a constant spectral group scheme over Λ. To show that $\rho_{K/k}$ is faithfully flat, we may argue locally around each point $\gamma \in \mathcal{G}(\overline{k}/k)$. Since $\mathcal{G}(\overline{k}/k)$ is profinite, localising around the inverse image of γ is the same as taking the fiber at γ. Thus, we are left to prove that $\mathcal{G}_{\mathrm{mot}}(K,\Sigma;\Lambda)_\gamma \to \mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda)_\gamma$ is faitfully flat (where "γ" in subscript refers to taking the fiber at γ). But $\mathcal{G}_{\mathrm{mot}}(K,\Sigma;\Lambda)_\gamma$ and $\mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda)_\gamma$ are torsors over $\mathcal{G}_{\mathrm{mot}}(\overline{K},\overline{\Sigma};\Lambda)$ and $\mathcal{G}_{\mathrm{mot}}(\overline{k},\overline{\sigma};\Lambda)$. So, it is enough to treat the case $\gamma=1$ as needed.

Now that we have reduced property (i) to the case where k is algebraically closed, we see that it follows from properties (ii) and (iii). We next prove (ii) and (iii), starting with (iii) which requires less work. By Remark 1.61, it is enough to show that $\mathcal{G}_{\mathrm{rel}}(K/k,\Sigma)$ is flat over the sphere spectrum. Using Theorem 1.70(ii), we have equivalences

$$\begin{array}{rl} & \mathrm{cofib}(C^0(\overline{\pi}_1^{\text{ét}}(K/k,\Sigma))\to\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma)) \\ \simeq & \mathrm{cofib}(C^0(\overline{\pi}_1^{\text{ét}}(K/k,\Sigma);\mathbb{Q})\to\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma;\mathbb{Q})) \\ \simeq & \mathrm{cofib}(C^0(\overline{\pi}_1^{\text{ét}}(K/k,\Sigma);\mathbb{Z})\to\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma;\mathbb{Z})). \end{array}\tag{1.50}$$

By [Ayo14c, Theorem 2.55], $\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma;\mathbb{Z})$ is concentrated in degree zero, which implies that the $\mathbb{Q}$-modules in (1.50) are also concentrated in degree zero. Thus $\pi_0\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma)$ is isomorphic to the ordinary commutative ring $\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma;\mathbb{Z})$ and that the latter is torsion-free, and hence flat over $\mathbb{Z}$. To finish the proof of (iii), it remains to show that, for $i\geq 1$, the obvious morphism

$$\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma;\mathbb{Z})\otimes\pi_i\mathbb{S}\to\pi_i\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma)\tag{1.51}$$

is an equivalence. Since $\pi_i\mathbb{S}$ is a torsion group, Theorem 1.70(ii) implies that the left hand side is isomorphic to $C^0(\overline{\pi}_1^{\text{ét}}(K/k,\Sigma);\mathbb{Z})\otimes\pi_i\mathbb{S}$. Using again the equivalences (1.50) and [Ayo14c, Theorem 2.55], we see that the morphism

$$C^0(\overline{\pi}_1^{\text{ét}}(K/k,\Sigma))\to\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma)$$

induces an isomorphism on the i-th homotopy groups. Thus, we are left to show that

$$C^0(\overline{\pi}_1^{\text{ét}}(K/k,\Sigma);\mathbb{Z})\otimes\pi_i\mathbb{S}\to\pi_iC^0(\overline{\pi}_1^{\text{ét}}(K/k,\Sigma))$$

is an equivalence. This is obvious since $C^0(\overline{\pi}_1^{\text{ét}}(K/k,\Sigma))$ is flat over the sphere spectrum.

We now prove (ii). Without loss of generality, we may assume that K is also algebraically closed. Applying Zorn's lemma to the ordered set of pairs (L,s) consisting of an algebraically closed subfield $L\subset K$ containing k and a section s of $\rho_{L/k}$, we are reduced to showing (ii) in the case where K/k has transcendence degree 1. Let t be an indeterminate and, for $n\in\mathbb{N}^\times$, fix an n-th root $t^{1/n}$ of t. Let A_n be the henselisation of $k[t^{1/n}]$ at the ideal generated by $t^{1/n}$, and $K_n=A_n[t^{-1}]$. Then $\bigcup_{n\in\mathbb{N}^\times}K_n$ is an algebraically closed extension of k of transcendence degree 1. Thus, without loss of generality, we may assume that $K=\bigcup_{n\in\mathbb{N}^\times}K_n$. For $n\in\mathbb{N}^\times$, we denote by

$$\Psi_n:\mathbf{MSh}(K_n;\Lambda)^\otimes\to\mathbf{MSh}(k;\Lambda)^\otimes$$

the "nearby motive" functor associated to the uniformizer $t^{1/n}\in A_n$. For the construction and the basic properties of this functor, we refer the reader to [Ayo07b, §3.5]; see also [AIS17, §4.3] for a shorter account of the construction. (The constructions of loc. cit. are done using the language of derivators but are easily translated into the language of ∞-categories.) By [Ayo07b, Proposition 3.5.9], for $m,n\in\mathbb{N}^\times$, we have an equivalence of symmetric monoidal functors $\Psi_{mn}\circ(e_{n,mn})_\eta^*\simeq\Psi_n$, where $e_m:\mathrm{Spec}(A_{mn})\to\mathrm{Spec}(A_n)$ is the obvious morphism. Passing to the colimit in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, we obtain a symmetric monoidal functor

$$\Psi_\infty:\mathbf{MSh}(K;\Lambda)^\otimes\to\mathbf{MSh}(k;\Lambda)^\otimes.$$

Composing with the Betti realisation functor associated to $\sigma : k \hookrightarrow \mathbb{C}$, we obtain the tangential Betti realisation functor

$$\mathrm{TgB}^* : \mathbf{MSh}(K;\Lambda)^{\otimes} \xrightarrow{\Psi_\infty} \mathbf{MSh}(k;\Lambda)^{\otimes} \xrightarrow{\mathrm{B}^*} \mathrm{Mod}_\Lambda^{\otimes} \tag{1.52}$$

considered in [Ayo15, §2.5]. We claim that this functor is non canonically equivalent to $\mathrm{B}^*_\Sigma : \mathbf{MSh}(K;\Lambda)^{\otimes} \to \mathrm{Mod}^{\otimes}_\Lambda$. Before saying anything about this claim, we explain why it suffices for proving (ii). Using the claimed equivalence, we see that the $\mathcal{H}_{\mathrm{mot}}(K,\Sigma;\Lambda)$ is equivalent to the commutative Hopf algebra associated to the Weil spectrum $\mathrm{TgB}_*(\Lambda)$, where TgB_* is the right adjoint of TgB^*. By the very construction of TgB^*, we have a morphism of commutative algebras $\Psi_\infty(\mathrm{TgB}_*(\Lambda)) \to \mathrm{B}_*(\Lambda)$ yielding a morphism of commutative Hopf algebras

$$\mathcal{H}_{\mathrm{mot}}(K, \mathrm{TgB}_*(\Lambda)) \to \mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda).$$

This gives the required splitting.

We now say a few words about why the functor TgB^* in (1.52) is equivalent to B^*_Σ. The proof is very similar to that of [Ayo14c, Proposition 2.20] and [Ayo15, Théorème 2.18], and we will not repeat the details here. One needs a variant of [Ayo14c, Lemme 2.21] insuring the existence of a family of paths $(\gamma_n : [0,1] \to \mathbb{C})_{n\in\mathbb{N}^\times}$ with the following properties:

- $\gamma_n(0) = 0$, $\gamma_n'(0) = 1$ and $\gamma_n(1)$ is the image of $t^{1/n}$ by $\Sigma : K \hookrightarrow \mathbb{C}$,
- for $m, n \in \mathbb{N}^\times$, we have $(\gamma_{mn})^m = \gamma_n$,
- γ_n admits a lift to the analytic pro-variety $\mathrm{Spec}(A_n)^{\mathrm{an}}$ sending 0 to the origin.

We leave the construction of such a family of paths to the reader.

To finish the proof, it remains to prove (iv). Without loss of generality, we may assume that k is algebraically closed. The morphism in (1.48) is over $\mathcal{G}(\overline{K}/K)_\Lambda$, where $\overline{K}$ is the algebraic closure of K in $\mathbb{C}$. Arguing as in the beginning of the proof, we reduce to showing that the morphism in (1.48) is faithfully flat after taking the fiber along $\{1\}_\Lambda \subset \mathcal{G}(\overline{K}/K)_\Lambda$. Said differently, we may assume that K is algebraically closed. In this case, we have a Cartesian square of ordinary rings

$$\begin{tikzcd}
\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma;\Lambda) \arrow[r] \arrow[d] & \Lambda \arrow[d] \\
\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma;\Lambda) \otimes_\Lambda \mathrm{Frac}(\Lambda) \arrow[r] & \mathrm{Frac}(\Lambda)
\end{tikzcd}$$

where the horizontal arrows are the counit morphisms. This follows immediately from Theorem 1.70(ii), even when Λ has positive characteristic. Similarly, by Lemma 1.82, we have a Cartesian square of ordinary rings

$$\begin{tikzcd}
\mathcal{F}(K/k,\Sigma;\Lambda) \arrow[r] \arrow[d] & \Lambda \arrow[d] \\
\mathcal{F}(K/k,\Sigma;\Lambda) \otimes_\Lambda \mathrm{Frac}(\Lambda) \arrow[r] & \mathrm{Frac}(\Lambda).
\end{tikzcd}$$

Applying Lemma 1.87 below, we are reduced to showing that the morphism of $\mathrm{Frac}(\Lambda)$-algebras

$$\mathcal{H}_{\mathrm{rel}}(K/k,\Sigma;\Lambda) \otimes_\Lambda \mathrm{Frac}(\Lambda) \to \mathcal{F}(K/k,\Sigma;\Lambda) \otimes_\Lambda \mathrm{Frac}(\Lambda)$$

is flat. Since these are commutative Hopf algebras over a field, it is enough to show that the induced morphism of classical affine schemes is surjective. Thus, it is enough to show that

$$\pi_1^{\mathrm{alg}}(K/k,\Sigma;\mathrm{Frac}(\Lambda)) \to \mathcal{G}_{\mathrm{rel}}(K/k,\Sigma;\mathrm{Frac}(\Lambda))$$

is surjective, and we may assume that the field $\mathrm{Frac}(\Lambda)$ has characteristic zero. The result follows then from [Ayo14c, Théorème 2.57]. $\square$

Lemma 1.87. *Let R be an integral domain, and consider a commutative triangle of ordinary rings*

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[dr, "a"'] & B' \arrow[d, "b"] \\ & \mathrm{Frac}(R) \end{tikzcd}$$

where a and b are surjective. Define subrings $A \subset A'$ and $B \subset B'$ by $A = a^{-1}(R)$ and $B = b^{-1}(R)$, and let $f : A \to B$ be the induced morphism. Then f is flat if and only if f' is flat.

Proof. It is clear that A' and B' are localisations of A and B. Thus, if f is flat, then so is f'. The converse follows from [Fer03, Théorème 2.2(iv)]. Indeed, assume that f' is flat. Let $C' = B' \otimes_{A'} \mathrm{Frac}(R)$ and $C = c^{-1}(R)$, where $c : C' \to \mathrm{Frac}(A)$ is the obvious morphism. Then B is also the inverse image of $C \subset C'$ along the obvious map $B' \to C'$. Said differently, the A-module B is the image by the functor

$$S : \mathrm{Mod}_R^\heartsuit \times_{\mathrm{Mod}^\heartsuit_{\mathrm{Frac}(R)}} \mathrm{Mod}^\heartsuit_{A'} \to \mathrm{Mod}^\heartsuit_A,$$

as in [Fer03, page 559], of the triple (C, s, B'), where $s : C \otimes_R \mathrm{Frac}(R) \simeq C'$ is the obvious identification. Thus, it remains to see that C' is flat over R, which is clear. $\square$

1.6. **Constructible sheaves of geometric origin.**

In this subsection, we give a precise definition of what we mean by sheaves of geometric origin. (Similar notions exist already in the literature, see for example [BBD82, §6.2.4], but we will not attempt to make precise comparisons here.) We then prove that this class of sheaves has good stability properties. Along the way, we prove a Betti version of [Dre18, Desiderata 1.1(7)]. As usual, we fix a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$.

Definition 1.88. Let $\Lambda \in \mathrm{CAlg}(\mathcal{S}p_{\geq 0})$ be a connective commutative ring spectrum, and let X be a k-variety.[3]

(i) We denote by $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^\heartsuit$ the full subcategory of $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)^\heartsuit$ generated under kernels, cokernels, extensions and filtered colimits by the sheaves of the form $\mathrm{H}^p(f_*\Lambda)$ with $f : Y \to X$ a proper k-morphism and $p \in \mathbb{Z}$.

(ii) We denote by $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$ the full sub-∞-category of $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$ consisting of those objects with homology sheaves in $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^\heartsuit$. A sheaf in $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$ is said to be of geometric origin.

(iii) We denote by $\mathbf{Ct}_{\mathrm{geo}}(X;\Lambda)$ the full sub-∞-category of $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$ spanned by constructible sheaves of geometric origin, i.e., the intersection $\mathbf{Ct}(X;\Lambda) \cap \mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$. Similarly, we denote by $\mathbf{LS}_{\mathrm{geo}}(X;\Lambda)$ and $\widehat{\mathbf{LS}}_{\mathrm{geo}}(X;\Lambda)$ the intersection of $\mathbf{LS}(X;\Lambda)$ and $\widehat{\mathbf{LS}}(X;\Lambda)$ with $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$. Objects in $\mathbf{LS}_{\mathrm{geo}}(X;\Lambda)$ are called local systems of geometric origin.

Remark 1.89. By construction, the natural t-structure on $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$ restricts to a t-structure on $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$ whose heart is $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^\heartsuit$. Also, for $f : Y \to X$ a proper morphism, the sheaf $f_*\Lambda$ belongs to $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$. In fact, we can characterise the full sub-∞-category $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda) \subset \mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$ as being the one generated under colimits, desuspension and truncation by sheaves of the

[3] This definition needs to be modified: as defined, it is not at all clear that the ∞-category $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$ admits a t-structure, compatible with the one on $\mathbf{Sh}(X^{\mathrm{an}};\Lambda)$.

form $f_*\Lambda$, with f proper. In other words, this is the full sub-∞-category generated under colimits and desuspension by the smallest abelian subcategory of $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)^\heartsuit$ containing the sheaves of the form $\mathrm{H}^p(f_*\pi_0\Lambda)$ for $f : Y \to X$ a proper k-morphism and $p \in \mathbb{N}$.

Remark 1.90. Using the proper base change theorem in the Betti context, we see that sheaves of geometric origin are stable under pullbacks. Thus, we have a CAT_∞-valued presheaf

$$\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda) : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAT}_\infty. \tag{1.53}$$

Given a k-variety X, [Ayo07a, Lemme 2.2.23] implies that the ∞-category $\mathbf{MSh}(X;\Lambda)$ is generated under colimits, desuspension and negative Tate twists by motives of the form $f_*\Lambda$, with $f : Y \to X$ proper. This shows that the refined Betti realisation of Theorem 1.28 factors through a morphism

$$\mathrm{B}^*_{\mathrm{geo}} : \mathbf{MSh}(-;\Lambda) \to \mathbf{Sh}_{\mathrm{geo}}(-;\Lambda) \tag{1.54}$$

of CAT_∞-values presheaves. If no confusion can arise, we simply write B^* for this morphism. Using Remark 1.89, one obtains yet another characterisation of the full sub-∞-category $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda) \subset \mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$: it is the one generated under colimits, desuspension and truncation by the image of the refined Betti realisation functor $\mathrm{B}^* : \mathbf{MSh}(X;\Lambda) \to \mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$.

Before stating our main results concerning sheaves of geometric origin, we give the following construction.

Construction 1.91. Recall that we denote by $\mathcal{B}_\Lambda$ the commutative algebra object of $\mathbf{MSh}(k;\Lambda)$ given by $\mathrm{B}_*(\Lambda)$, where $\mathrm{B}_* : \mathrm{Mod}_\Lambda \to \mathbf{MSh}(k;\Lambda)$ is the right adjoint of the Betti realisation functor. There is a Voevodsky pullback formalism

$$\mathbf{MSh}(-;\mathcal{B}_\Lambda)^\otimes : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \tag{1.55}$$

sending a k-variety X to the symmetric monoidal ∞-category $\mathbf{MSh}(X;\mathcal{B}_\Lambda)^\otimes$ of $\mathcal{B}_\Lambda$-modules in $\mathbf{MSh}(X;\Lambda)^\otimes$. Moreover, we have a factorisation of the refined Betti realisation

$$\mathrm{B}^* : \mathbf{MSh}(-;\Lambda)^\otimes \xrightarrow{\mathcal{B}_\Lambda\otimes-} \mathbf{MSh}(-;\mathcal{B}_\Lambda)^\otimes \xrightarrow{\widetilde{\mathrm{B}}^*} \mathbf{Sh}_{\mathrm{ct}}(-;\Lambda)^\otimes, \tag{1.56}$$

where $\widetilde{\mathrm{B}}^*$ is informally given by the formula $\widetilde{\mathrm{B}}^*(-) = \mathrm{B}^*(-) \otimes_{\mathrm{B}^*\mathrm{B}_*(\Lambda)} \Lambda$. We now sketch the construction of the functor (1.55) and the factorisation (1.56). For this, we adapt the method used in [AGV20, §3.4]. Recall that we have a functor $\mathrm{CAlg} : \mathrm{CAlg}(\mathrm{CAT}_\infty) \to \mathrm{CAT}_\infty$ sending a symmetric monoidal ∞-category $\mathcal{C}^\otimes$ to the ∞-category $\mathrm{CAlg}(\mathcal{C})$ of commutative algebra objects in $\mathcal{C}^\otimes$. By [AGV20, Construction 3.4.4 & Remark 3.4.5], we have a functor $\mathrm{Mod}(-)^\otimes : \mathrm{CAlg}(\mathrm{CAT}_\infty) \to \mathrm{CAT}_\infty$ endowed with a natural transformation $\mathrm{Mod}(-)^\otimes \to \mathrm{Fin}_* \times \mathrm{CAlg}(-)$. This gives us a commutative square of CAT_∞-valued presheaves on Sch/k

$$\begin{tikzcd}
\mathrm{Mod}(\mathbf{MSh}(-;\Lambda))^\otimes \arrow[r, "\mathrm{Mod}(\mathrm{B}^*)^\otimes"] \arrow[d, "f_0"] & \mathrm{Mod}(\mathbf{Sh}_{\mathrm{ct}}(-;\Lambda))^\otimes \arrow[d, "f_1"] \\
\mathrm{Fin}_* \times \mathrm{CAlg}(\mathbf{MSh}(-;\Lambda)) \arrow[r, "\mathrm{CAlg}(\mathrm{B}^*)"] & \mathrm{Fin}_* \times \mathrm{CAlg}(\mathbf{Sh}_{\mathrm{ct}}(-;\Lambda)).
\end{tikzcd}$$

Applying Lurie's unstraightening construction [Lur09a, §3.2], we get a commutative diagram

$$\begin{tikzcd}
\mathfrak{M}_0^{\otimes} \arrow[rr, "H^{\otimes}"] \arrow[d, "q_0"] & & \mathfrak{M}_1^{\otimes} \arrow[d, "q_1"] \\
\mathrm{Fin}_* \times \Xi_0 \arrow[rr, "F"] \arrow[dr, "p_0"'] & & \mathrm{Fin}_* \times \Xi_1 \arrow[dl, "p_1"] \\
& \mathrm{Fin}_* \times (\mathrm{Sch}/k)^{\mathrm{op}}. &
\end{tikzcd}$$

It follows from [AGV20, Lemma 3.4.6] and [Lur09a, Proposition 2.4.2.3(3)] that the functors p_i, q_i and $p_i \circ q_i$ are coCartesian fibrations for $i \in \{0,1\}$. To conclude, consider the coCartesian sections

$$s_{\mathcal{B}_\Lambda} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \Xi_0, \quad s_{0,\Lambda} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \Xi_0 \quad \text{and} \quad s_{1,\Lambda} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \Xi_1$$

sending $\mathrm{Spec}(k)$ to $\mathrm{B}_*(\Lambda) \in \mathrm{CAlg}(\mathbf{MSh}(k;\Lambda))$, $\Lambda \in \mathrm{CAlg}(\mathbf{MSh}(k;\Lambda))$ and $\Lambda \in \mathrm{CAlg}(\mathbf{Sh}_{\mathrm{ct}}(k;\Lambda))$ respectively. We have morphisms $s_{0,\Lambda} \to s_{\mathcal{B}_\Lambda}$ and $F s_{\mathcal{B}_\Lambda} \to s_{1,\Lambda}$, which we use to obtain coCartesian fibrations

$$\begin{aligned}
\Phi_0 &= \mathfrak{M}_0^{\otimes} \times_{\Xi_0,\, s_{0,\Lambda} \to s_{\mathcal{B}_\Lambda}} (\Delta^1 \times (\mathrm{Sch}/k)^{\mathrm{op}}) \to \Delta^1 \times \mathrm{Fin}_* \times (\mathrm{Sch}/k)^{\mathrm{op}}, \\
\Phi_1 &= \mathfrak{M}_1^{\otimes} \times_{\Xi_1,\, s_{1,\Lambda} \to F s_{\mathcal{B}_\Lambda} \to s_{1,\Lambda}} (\Delta^2 \times (\mathrm{Sch}/k)^{\mathrm{op}}) \to \Delta^2 \times \mathrm{Fin}_* \times (\mathrm{Sch}/k)^{\mathrm{op}}.
\end{aligned}$$

Applying Lurie's straightening construction [Lur09a, §3.2] to these coCartesian fibrations, we get the following commutative diagram of $\mathrm{CAlg}(\mathrm{CAT}_\infty)$-valued presheaves on Sch/k:

$$\begin{tikzcd}
\mathbf{MSh}(-;\Lambda)^{\otimes} \arrow[r, "-\otimes_\Lambda \mathcal{B}_\Lambda"] \arrow[d, "\mathrm{B}^*"] & \mathbf{MSh}(-;\mathcal{B}_\Lambda)^{\otimes} \arrow[d, "\mathrm{B}^*"] & \\
\mathbf{Sh}_{\mathrm{ct}}(-;\Lambda)^{\otimes} \arrow[r, "-\otimes_\Lambda \mathrm{B}^*\mathcal{B}_\Lambda"] \arrow[rr, bend right, equal] & \mathbf{Sh}_{\mathrm{ct}}(-;\mathrm{B}^*\mathcal{B}_\Lambda)^{\otimes} \arrow[r, "-\otimes_{\mathrm{B}^*\mathcal{B}_\Lambda} \Lambda"] & \mathbf{Sh}_{\mathrm{ct}}(-;\Lambda)^{\otimes}.
\end{tikzcd}$$

This finishes the construction.

Remark 1.92. Combining Remark 1.53 and the morphism $\widetilde{\mathrm{B}}^*$ in (1.56), we obtain a morphism of $\mathrm{CAlg}(\mathrm{CAT}_\infty)$-valued presheaves

$$\widetilde{\mathrm{B}}^*_{\mathrm{geo}} : \mathbf{MSh}(-;\mathcal{B}_\Lambda)^{\otimes} \to \mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}. \tag{1.57}$$

If no confusion can arise, we simply write $\widetilde{\mathrm{B}}^*$ for this morphism.

Below, we summarise some essential properties of sheaves of geometric origin.

Theorem 1.93.

(i) Given a k-variety X, the functor

$$\widetilde{\mathrm{B}}^* : \mathbf{MSh}(X;\mathcal{B}_\Lambda) \to \mathbf{Sh}_{\mathrm{ct}}(X;\Lambda) \tag{1.58}$$

is fully faithful with essential image $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$. *Said differently, the morphism* $\widetilde{\mathrm{B}}^*_{\mathrm{geo}}$ *in* (1.57) *is an equivalence.*

(ii) The sub-∞-categories $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda) \subset \mathbf{Sh}_{\mathrm{ct}}(-;\Lambda)$ *are closed under the four operations* f^*, f_*, $f_!$ *and* $f^!$, *associated to a morphism* f *of k-varieties, as well as tensor product and internal homomorphism from a constructible geometric sheaf.*

(iii) The abelian subcategory $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit} \subset \mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)^{\heartsuit}$ *is stable under subquotients.*

Remark 1.94. The fully faithfulness of the functor $\widetilde{\mathrm{B}}^*$ in (1.58) was first observed by Cisinski–Déglise in [CD19, Example 17.1.7]. Since this can be obtained by a simple application of the six-functor formalism and its compatibility with the Betti realisation, we include a proof in Lemma 1.95 below. As far as we know, the determination of the essential image of the functor $\widetilde{\mathrm{B}}^*$ in (1.58) is not stated explicitly in the literature, but see Remark 1.101.

The reminder of this section is mostly devoted to proving Theorem 1.93.

Lemma 1.95. *The functor* $\widetilde{\mathrm{B}}^*$ *in* (1.58) *is fully faithful.*

Proof. For the sake of clarity, we shall write $\widetilde{\mathrm{B}}^*_X$ for the functor in (1.58) and denote by $\widetilde{\mathrm{B}}_{X,*}$: $\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda) \to \mathbf{MSh}(X;\mathcal{B}_\Lambda)$ its right adjoint. Informally, the latter sends an object $F \in \mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)$ to $\mathrm{B}_{X,*}(F)$ endowed with its natural structure of $\mathcal{B}_\Lambda$-module. Combining Proposition 1.30 with [Lur17, Corollary 3.4.4.6], we deduce that $\widetilde{\mathrm{B}}_{X,*}$ is colimit-preserving. To prove that the functor $\widetilde{\mathrm{B}}^*_X$ is fully faithful, we need to show that the unit natural transformation $\mathrm{id} \to \widetilde{\mathrm{B}}_{X,*}\widetilde{\mathrm{B}}^*_X$ is an equivalence. Since its domain and codomain are colimit-preserving, it is enough to prove that this natural transformation is an equivalence when evaluated on a set of objects generating $\mathbf{MSh}(X;\mathcal{B}_\Lambda)$ under colimits. Using [Ayo07a, Proposition 2.2.27], we are reduced to showing that

$$g_*(\Lambda) \otimes_\Lambda \mathcal{B}_\Lambda|_X \to \widetilde{\mathrm{B}}_{X,*}\widetilde{\mathrm{B}}^*_X(g_*(\Lambda) \otimes_\Lambda \mathcal{B}_\Lambda|_X) = \mathrm{B}_{X,*}\mathrm{B}^*_X(g_*(\Lambda))$$

is an equivalence when $g : Y \to X$ is a proper morphism from a smooth k-variety Y. By the projection formula (see, for example, [AGV20, Proposition 4.1.7]), we have $g_*(\Lambda) \otimes_\Lambda \mathcal{B}_\Lambda|_X \simeq g_*(\mathcal{B}_\Lambda|_Y)$. By Theorem 1.28, we have an equivalence $\mathrm{B}_{X,*}\mathrm{B}^*_X(g_*(\Lambda)) \simeq g_*\mathrm{B}_{Y,*}(\Lambda)$. Thus, to conclude, we need to show that $\mathcal{B}_\Lambda|_Y \to \mathrm{B}_{Y,*}(\Lambda)$ is an equivalence. Using that the refined Betti realisation commute with extension by zero, we deduce that the functors $\mathrm{B}_{-,*}$ commute with the inverse image along open immersions. Thus, we may replace Y by a smooth compactification and assume that Y is also proper. Using [Ayo07a, Proposition 2.2.27] for a second time, we see it is enough to show that

$$\mathrm{Map}_{\mathbf{MSh}(Y;\Lambda)}(h_!\Lambda(m)[n], \mathcal{B}_\Lambda|_Y) \to \mathrm{Map}_{\mathbf{MSh}(Y;\Lambda)}(h_!\Lambda(m)[n], \mathrm{B}_{Y,*}(\Lambda))$$

is an equivalence for $h : Z \to Y$ a proper morphism from a smooth k-variety Z, and $m, n \in \mathbb{Z}$. Using adjunction and Theorem 1.28, we reduce to showing that

$$\mathrm{Map}_{\mathbf{MSh}(Z;\Lambda)}(\Lambda(m)[n], E \otimes_\Lambda \mathcal{B}_\Lambda|_Z) \to \mathrm{Map}_{\mathbf{MSh}(Z;\Lambda)}(\Lambda(m)[n], \mathrm{B}_{Z,*}\mathrm{B}^*_Z E)$$

where $E = h^!(\Lambda)$ is the relative Thom space associated to the virtual normal bundle of h. Finally, by adjunction, we are left to show that

$$p_*(E \otimes_\Lambda p^*\mathcal{B}_\Lambda) \to p_*\mathrm{B}_{Z,*}\mathrm{B}^*_Z(E)$$

is an equivalence, where $p : Z \to \mathrm{Spec}(k)$ is the structural morphism which is smooth and proper. Using the projection formula [AGV20, Proposition 4.1.7] and Theorem 1.28 as we did previousely, we can rewrite this morphism as $p_*(E) \otimes_\Lambda \mathcal{B}_\Lambda \to \mathrm{B}_*\mathrm{B}^*p_*(E)$. That this is an equivalence is a particular case of Proposition 1.56. ◻

Remark 1.96. The proof of Lemma 1.95 shows that the commutative algebra $\mathcal{B}_\Lambda|_X$, obtained by pulling back $\mathcal{B}_\Lambda$ to X, coincides with $\mathrm{B}_{X,*}(\Lambda)$. This is in fact a formal consequence of the fully faithfulness of the functor $\widetilde{\mathrm{B}}^*$ in (1.58) which implies, more generally, that

$$M \otimes_\Lambda \mathcal{B}_\Lambda|_X \to \mathrm{B}_{X,*}\mathrm{B}^*_X(M)$$

is an equivalence for every $M \in \mathbf{MSh}(X;\Lambda)$.

Remark 1.97. One could give a shorter proof of Lemma 1.95 under the hypothesis that the field k has finite virtual Λ-cohomological dimension. Indeed, in this case, we easily reduce to showing that

$$\mathrm{Map}_{\mathbf{MSh}(X;\Lambda)}(M, N \otimes_\Lambda \mathcal{B}_\Lambda|_X) \to \mathrm{Map}_{\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)}(\mathrm{B}^*(M), \mathrm{B}^*(N)) \tag{1.59}$$

is an equivalence when M is compact. We then have an equivalence

$$\underline{\mathrm{Hom}}(M, -) \otimes_\Lambda \mathcal{B}_\Lambda|_X \simeq \underline{\mathrm{Hom}}(M, - \otimes_\Lambda \mathcal{B}_\Lambda|_X)$$

since $\mathcal{B}_\Lambda|_X$ can be written as a filtered colimit of dualizable objects. By Theorem 1.28, we have an equivalence $\mathrm{B}^* \circ \underline{\mathrm{Hom}}(M, -) \simeq \underline{\mathrm{Hom}}(\mathrm{B}^*(M), \mathrm{B}^*(-))$. Putting these facts together, we may rewrite the map in (1.59) as follows:

$$\mathrm{Map}_{\mathbf{MSh}(X;\Lambda)}(\Lambda, \underline{\mathrm{Hom}}(M, N) \otimes_\Lambda \mathcal{B}_\Lambda|_X) \to \mathrm{Map}_{\mathbf{Sh}_{\mathrm{ct}}(X;\Lambda)}(\Lambda, \mathrm{B}^*(\underline{\mathrm{Hom}}(M, N))). \tag{1.60}$$

Let $p : X \to \mathrm{Spec}(k)$ be the structural projection. Bor the same reasons as before, we have equivalences $p_*(- \otimes_\Lambda \mathcal{B}_\Lambda|_X) \simeq p_*(-) \otimes_\Lambda \mathcal{B}_\Lambda$ and $\mathrm{B}^* \circ p_* \simeq p_* \circ \mathrm{B}^*$. Thus, it is enough to show that

$$(p_*\underline{\mathrm{Hom}}(M, N)) \otimes_\Lambda \mathcal{B}_\Lambda \to \mathrm{B}_*\mathrm{B}^*(p_*\underline{\mathrm{Hom}}(M, N))$$

is an equivalence, which is the case by Proposition 1.56.

The crucial step in proving Theorem 1.93 is to show that the functor $\widetilde{\mathrm{B}}^*$ in (1.58) is essentially surjective. We start by proving a reduction.

Lemma 1.98. *To prove that the functor* $\widetilde{\mathrm{B}}^*_{\mathrm{geo}}$ *in* (1.57) *is an equivalence, it is enough to treat the case where* $\Lambda = \mathbb{Q}$.

Proof. Using Lemma 1.95, it remains to see that $\mathbf{Sh}_{\mathrm{geo}}(X; \Lambda)$ is generated under colimits by the image of the functor $\mathrm{B}^* : \mathbf{MSh}(X; \Lambda) \to \mathbf{Sh}_{\mathrm{geo}}(X; \Lambda)$. We assume that this is known when $\Lambda = \mathbb{Q}$ and we explain how to deduce the general case. We split the argument in two small steps.

Step 1. Let Λ' be a commutative Λ-algebra. The forgetful functor $\mathbf{Sh}_{\mathrm{ct}}(X; \Lambda') \to \mathbf{Sh}_{\mathrm{ct}}(X; \Lambda)$ is t-exact, colimit-preserving and conservative. Using Remark 1.90, we deduce that this functor takes $\mathbf{Sh}_{\mathrm{geo}}(X; \Lambda')$ into $\mathbf{Sh}_{\mathrm{geo}}(X; \Lambda)$, inducing a functor

$$g : \mathbf{Sh}_{\mathrm{geo}}(X; \Lambda') \to \mathbf{Sh}_{\mathrm{geo}}(X; \Lambda)$$

which is also t-exact, colimit-preserving and conservative. Moreover, the functor g admits a left adjoint $f : \mathbf{Sh}_{\mathrm{geo}}(X; \Lambda) \to \mathbf{Sh}_{\mathrm{geo}}(X; \Lambda')$ given by $M \mapsto M \otimes_\Lambda \Lambda'$.

We claim that the image of f generates $\mathbf{Sh}_{\mathrm{geo}}(X; \Lambda')$ under colimits. This follows for example from [AGV20, Proposition 3.1.14 & Lemma 3.1.15]. Alternatively, we may argue more concretely as follows. Fix $M' \in \mathbf{Sh}_{\mathrm{geo}}(X; \Lambda')$. Using [Lur17, Proposition 4.7.3.3], we have an augmented simplicial object $R'_\bullet$ in $\mathbf{Sh}_{\mathrm{geo}}(X; \Lambda')$ given informally by $R'_n = (f \circ g)^{\circ n+1} M'$, for $n \geq -1$. It is enough to show that $R'_\bullet$ is a colimit diagram. Since g is colimit-preserving and conservative, it is enough to show that $g(R'_\bullet)$ is a colimit diagram, which follows from the fact that the simplicial object $g(R'_\bullet)$ is split, see [Lur17, Example 4.7.2.7].

This said, we see that the image of $\mathrm{B}^* : \mathbf{MSh}(X; \Lambda') \to \mathbf{Sh}_{\mathrm{geo}}(X; \Lambda')$ generates $\mathbf{Sh}_{\mathrm{geo}}(X; \Lambda')$ under colimits if this is the case for the image of $\mathrm{B}^* : \mathbf{MSh}(X; \Lambda) \to \mathbf{Sh}_{\mathrm{geo}}(X; \Lambda)$. This will be used several times in Step 2 below.

Step 2. By Step 1, we only need to treat the case where Λ is the sphere spectrum. Given an object $M \in \mathbf{Sh}_{\text{geo}}(X)$, the object $M \otimes \mathbb{Q}$ belongs to $\mathbf{Sh}_{\text{geo}}(X; \mathbb{Q})$ and hence, by our assumption, to the image of the fully faithful embedding $\widetilde{\mathrm{B}}^* : \mathbf{MSh}(X; \mathcal{B}) \to \mathbf{Sh}_{\text{geo}}(X)$. Thus, we may replace M with the cofiber of $M \to M \otimes \mathbb{Q}$ and assume that M is ℓ-nilpotent for some prime ℓ. We may also assume that M belongs to the heart of the t-structure. Since $M \simeq \pi_0(M \otimes_{\mathbb{S}} \mathbb{Z})$ belongs to $\mathbf{Sh}_{\text{geo}}(X; \mathbb{Z})$ by construction, it is enough to show that M belongs to the image of the functor $\widetilde{\mathrm{B}}^* : \mathbf{MSh}(X; \mathcal{B}_{\mathbb{Z}}) \to \mathbf{Sh}_{\text{geo}}(X; \mathbb{Z})$. Since M is ℓ-nilpotent, we are finally reduced to showing that the image of the functor

$$\mathrm{B}^* : \mathbf{MSh}(X; \mathbb{Z}/\ell^\nu) \to \mathbf{Sh}_{\text{geo}}(X; \mathbb{Z}/\ell^\nu)$$

generates $\mathbf{Sh}_{\text{geo}}(X; \mathbb{Z}/\ell^\nu)$ under colimits. (Here $\nu \geq 1$ is an integer.) By construction, $\mathbf{Sh}_{\text{geo}}(X; \mathbb{Z}/\ell^\nu)$ is generated under colimits and desuspensions by constructible ordinary étale sheaves of $\mathbb{Z}/\ell^\nu$-modules on X^{an}. Those are clearly in the image of the Betti realisation. □

We now establish the essential surjectivity of the functor $\widetilde{\mathrm{B}}^*$ in (1.58) at the generic point, assuming that $\Lambda = \mathbb{Q}$. We will give two proofs, one relying on Deligne's semi-simplicity theorem [Del71, Théorème 4.2.6] and one avoiding semi-simplicity, relying instead on Theorem 1.86. We believe that the argument using Deligne's semi-simplicity theorem was also known to Drew; see Remark 1.101 below. For the next statement, we remark that, for $\Lambda = \mathbb{Q}$, the morphism in (1.57) clearly belongs to the ∞-category of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$-valued presheaves. We also implicitly use the continuity property of ∞-categories of motives, as stated for example in [AGV20, Proposition 2.5.11].

Lemma 1.99. *Let K/k be a field extension. Then, the functor*

$$\widetilde{\mathrm{B}}^* : \mathbf{MSh}(K; \mathcal{B}_{\mathbb{Q}}) \to \mathbf{Sh}_{\text{geo}}(K; \mathbb{Q}), \tag{1.61}$$

obtained from (1.57) *via colimit in* $\mathrm{Pr}^{\mathrm{L}}_{\omega}$*, is an equivalence. Moreover, the abelian subcategory* $\mathbf{Sh}_{\text{geo}}(K; \mathbb{Q})^{\heartsuit} \subset \mathbf{Sh}_{\text{ct}}(K; \mathbb{Q})^{\heartsuit}$ *is stable under subquotients.*

First proof of Lemma 1.99. By [Ayo07a, Proposition 2.2.27] and Remark 1.90, the ∞-category $\mathbf{Sh}_{\text{geo}}(K; \mathbb{Q})$ is generated under colimits, desuspension and truncations by objects of the form $f_*\mathbb{Q}$, with $f : Y \to \mathrm{Spec}(K)$ proper and smooth. We need to show that the word "truncation" is superfluous here. To do so, remark that one can reformulate the previous sentence as follows: the ∞-category $\mathbf{Sh}_{\text{geo}}(K; \mathbb{Q})$ is generated under colimits and desuspension by the smallest abelian subcategory of $\mathbf{Sh}_{\text{ct}}(K; \mathbb{Q})^{\heartsuit}$ stable by extension and containing the objects of the form $\mathrm{H}^p(f_*\mathbb{Q})$, for $f : Y \to \mathrm{Spec}(K)$ proper and smooth, and $p \in \mathbb{N}$. Thus, by Lemma 1.100 below, we see that it suffices to show that every subquotient of $\mathrm{H}^p(f_*\mathbb{Q})$ is a direct summand of $f_*\mathbb{Q}$. This follows from Deligne's semi-simplicity theorem [Del71, Théorème 4.2.6] combined with [Del68, Proposition 2.1]. Alternatively, one could also use the decomposition theorem [BBD82, Théorème 6.2.5]. □

Lemma 1.100. *Let $\mathcal{A}$ be an abelian category, and S a set of objects in $\mathcal{A}$. Let $\mathcal{B}$ be the smallest abelian subcategory of $\mathcal{A}$ stable by extensions and containing the objects of S. Then, every object of $\mathcal{B}$ admits a finite, separated and exhaustive filtration whose graded pieces are subquotients of objects in S.*

Remark 1.101. The above argument using the decomposition theorem [BBD82, Théorème 6.2.5] shows that the essential image of the fully faithful embedding

$$\widetilde{\mathrm{B}}^* : \mathbf{MSh}(X; \mathcal{B}_{\mathbb{Q}}) \to \mathbf{Sh}_{\text{ct}}(X; \mathbb{Q})$$

is closed under truncation with respect to the perverse t-structure on $\mathbf{Sh}_{\text{ct}}(X; \mathbb{Q})$. A closely related result was announced by Drew in [Dre18]. Most probably, this was based on the same argument.

We will give another proof of Lemma 1.99, avoiding Deligne's semi-simplicity theorem. This relies on the following result which is of independent interest.

Proposition 1.102. *Let K/k be a field extension and $\Sigma : K \hookrightarrow \mathbb{C}$ a complex embedding extending σ. Denote by $\mathrm{B}^*_\Sigma : \mathbf{MSh}(K;\Lambda) \to \mathrm{Mod}_\Lambda$ the associated Betti realisation functor and set $\mathcal{B}_{\Sigma,\Lambda} = \mathrm{B}_{\Sigma,*}(\Lambda)$ which we endow with its natural structure of a commutative $\mathcal{B}_\Lambda$-algebra. Said differently, we view $\mathcal{B}_{\Sigma,\Lambda}$ as an object of $\mathrm{CAlg}(\mathbf{MSh}(K;\mathcal{B}_\Lambda))$. Denote by $\Sigma^* : \mathbf{Sh}_{\mathrm{ct}}(K;\Lambda) \to \mathrm{Mod}_\Lambda$ the fiber functor associated to point Σ of $\mathrm{Spec}(K)^{\mathrm{an}}$. Then, the following conditions are satisfied.*

(i) There is an equivalence of commutative algebras $\Sigma^*\widetilde{\mathrm{B}}^*\mathcal{B}_{\Sigma,\Lambda} \simeq \mathcal{H}_{\mathrm{rel}}(K/k,\Sigma;\Lambda)$ which is compatible with the coaction of $\mathcal{F}(K/k;\Sigma;\Lambda)$.

(ii) Assume that Λ is an ordinary commutative ring. Then $\widetilde{\mathrm{B}}^*\mathcal{B}_{\Sigma,\Lambda}$ belongs to $\mathbf{Sh}_{\mathrm{geo}}(K;\Lambda)^\heartsuit$.

Proof. By construction, we have equivalences

$$\begin{aligned}
\Sigma^*\widetilde{\mathrm{B}}^*\mathcal{B}_{\Sigma,\Lambda} &\simeq \Sigma^*(\mathrm{B}^*\mathcal{B}_{\Sigma,\Lambda} \otimes_{\mathrm{B}^*\mathcal{B}_\Lambda} \Lambda)\\
&\simeq (\Sigma^*\mathrm{B}^*\mathcal{B}_{\Sigma,\Lambda}) \otimes_{\mathrm{B}^*\mathcal{B}_\Lambda} \Lambda\\
&\simeq (\mathrm{B}^*_\Sigma\mathcal{B}_{\Sigma,\Lambda}) \otimes_{\mathrm{B}^*\mathcal{B}_\Lambda} \Lambda\\
&\simeq \mathcal{H}_{\mathrm{mot}}(K,\Sigma;\Lambda) \otimes_{\mathcal{H}_{\mathrm{mot}}(k,\sigma;\Lambda)} \Lambda
\end{aligned}$$

showing the first assertion. For the second assertion, we may assume that $\Lambda = \mathbb{Z}$. Then, $\widetilde{\mathrm{B}}^*\mathcal{B}_{\Sigma,\mathbb{Z}}$ belongs to $\mathbf{Sh}_{\mathrm{geo}}(K;\mathbb{Z})^\heartsuit$ if and only if $\Sigma^*\widetilde{\mathrm{B}}^*\mathcal{B}_{\Sigma,\Lambda}$ belongs to $\mathrm{Mod}_{\mathbb{Z}}^\heartsuit$, which is indeed the case by Theorem 1.86(iii). □

Second proof of Lemma 1.99. As in the first proof, the essential point is to show that every subquotient Q of $\mathrm{H}^pf_*\mathbb{Q}$ belongs to the essential image of the functor $\widetilde{\mathrm{B}}^*$ in (1.61). We use the notation in the statement of Proposition 1.102. The action of $\pi_1^{\mathrm{alg}}(K/k,\Sigma;\mathbb{Q})$ on $\Sigma^*\mathrm{H}^pf_*\mathbb{Q}$ factors through the quotient $\mathcal{G}_{\mathrm{rel}}(K/k,\Sigma;\mathbb{Q})$, and the same is true for Σ^*Q. Thus, we may find a $\pi_1^{\mathrm{alg}}(K/k,\Sigma;\mathbb{Q})$-equivariant resolution

$$\Sigma^*(Q) \to J^0 \otimes \mathcal{H}_{\mathrm{rel}}(K/k,\Sigma;\mathbb{Q}) \to \cdots \to J^n \otimes \mathcal{H}_{\mathrm{rel}}(K/k,\Sigma;\mathbb{Q}) \to \cdots$$

where the J^n's are $\mathbb{Q}$-vector spaces endowed with a trivial action of $\pi_1^{\mathrm{alg}}(K/k,\Sigma;\mathbb{Q})$. By Proposition 1.102, this gives a resolution in $\mathbf{Sh}_{\mathrm{ct}}(K/k;\mathbb{Q})^\heartsuit$ of the form

$$Q \to J^0 \otimes \widetilde{\mathrm{B}}^*\mathcal{B}_{\Sigma,\mathbb{Q}} \to \cdots \to J^n \otimes \widetilde{\mathrm{B}}^*\mathcal{B}_{\Sigma,\mathbb{Q}} \to \cdots .$$

Truncating stupidly and using that (1.61) is fully faithful (by Lemma 1.95), we obtain a tower $(P_n)_{n\geq 0}$ in $\mathbf{MSh}(K;\mathcal{B}_{\mathbb{Q}})$ such that $\widetilde{\mathrm{B}}^*(P_n) = [J^0 \otimes \widetilde{\mathrm{B}}^*\mathcal{B}_{\Sigma,\mathbb{Q}} \to \cdots \to J^n \otimes \widetilde{\mathrm{B}}^*\mathcal{B}_{\Sigma,\mathbb{Q}}]$. In particular, we see that

$$\mathrm{H}^i\widetilde{\mathrm{B}}^*(P_n) \simeq \begin{cases} Q & \text{si} \quad i = 0,\\ 0 & \text{si} \quad i \notin \{0,n\}.\end{cases}$$

Since the cohomological dimension of $\mathbf{Sh}_{\mathrm{ct}}(K;\mathbb{Q})^\heartsuit$ is bounded by the transcendence degree of the extension K/k, we have

$$\widetilde{\mathrm{B}}^*(P_n) \simeq Q \oplus \mathrm{H}^n\widetilde{\mathrm{B}}^*(P_n)[-n]$$

for n big enough. Using that $\widetilde{\mathrm{B}}^*$ is fully faithful, it follows that Q is the image of a direct summand of P_n as needed. □

Notation 1.103. Let X be a pro-k-variety and $x \in \lim X(\mathbb{C})$. Repeating Construction 1.74 with $\widehat{\mathbf{LS}}_{\mathrm{geo}}(X;\Lambda)^{\otimes}$ instead of $\widehat{\mathbf{LS}}(X;\Lambda)^{\otimes}$ we obtain a commutative Hopf algebra $\mathcal{F}^{\mathrm{geo}}(X,x;\Lambda)$. We set

$$\pi_1^{\mathrm{geo}}(X,x;\Lambda) = \mathrm{Spec}(\mathcal{F}^{\mathrm{geo}}(X,x;\Lambda)).$$

This is a nonconnective spectral affine group Λ-scheme.

Remark 1.104. The statement of Theorem 1.83 holds true for $\pi_1^{\mathrm{geo}}(X,x;\Lambda)$ with the same proof.

Corollary 1.105. *Let K/k be a field extension and $\Sigma : K \hookrightarrow \mathbb{C}$ a complex embedding extending σ. Then, there is an equivalence of spectral affine group Λ-schemes*

$$\pi_1^{\mathrm{geo}}(K,\Sigma;\Lambda) \xrightarrow{\sim} \mathcal{G}_{\mathrm{rel}}(K/k,\Sigma;\Lambda).$$

In particular, $\pi_1^{\mathrm{geo}}(K,\Sigma;\Lambda)$ is flat over Λ. Moreover, assuming that k is algebraically closed in K, we have a short exact sequence

$$\{1\}_\Lambda \to \pi_1^{\mathrm{geo}}(K,\Sigma;\Lambda) \to \mathcal{G}_{\mathrm{mot}}(K,\Sigma;\Lambda) \to \mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda) \to \{1\}_\Lambda.$$

Finally, when Λ is a field or a Dedekind domain with finite residue fields at all maximal ideals, we have a faithfully flat morphism of classical affine group Λ-schemes

$$\pi_1^{\mathrm{alg}}(K/k,\Sigma;\Lambda) \to \pi_1^{\mathrm{geo}}(K/k,\Sigma;\Lambda).$$

Proof. This follows immediately from Theorem 1.86 and Proposition 1.102. □

We are now ready to finish the proof of Theorem 1.93.

Proof of Theorem 1.93, (i) and (ii). We first note that part (ii) follows from (i) combined with Theorem 1.28. Concerning (i), the fully faithfulness of the functor $\widetilde{\mathrm{B}}^*$ in (1.58) was established in Lemma 1.95. Moreover, by Lemma 1.98, we may assume that $\Lambda = \mathbb{Q}$. Furthermore, by Lemma 1.99, we know the result at the generic points. Thus, it remains to explain how to deduce essential surjectivity of $\widetilde{\mathrm{B}}^*$ in the general case from knowing it in the generic case (under the assumption that $\Lambda = \mathbb{Q}$).

We fix a compact object $F \in \mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Q})$ and we prove that it belongs to the image of $\widetilde{\mathrm{B}}^*$. We argue by noetherian induction on X. Let η be a generic point of X and $K = \kappa(\eta)$. By Lemma 1.99, we my find a compact object $M_\eta \in \mathbf{MSh}(K;\mathcal{B}_{\mathbb{Q}})$ such that $\widetilde{\mathrm{B}}^* M_\eta \simeq \eta^* F$. By the the continuity property of ∞-categories of motives (see for example [AGV20, Proposition 2.5.11]), there exists $M \in \mathbf{MSh}(X;\mathcal{B}_{\mathbb{Q}})$ compact such that $M_\eta \simeq \eta^* M$. Similarly, the equivalence $\widetilde{\mathrm{B}}^* M_\eta \simeq \eta^* F$ spreads into an equivalence $\widetilde{\mathrm{B}}^* j^* M \simeq j^* F$, with $j : U \hookrightarrow X$ the inclusion of an open neighbourhood of η. Let $i : Z = X \smallsetminus U \hookrightarrow X$ be the complementary closed immersion. We have an exact triangle

$$\widetilde{\mathrm{B}}^* j_! j^* M \to F \to i_* i^* F \to .$$

By the induction hypothesis, there exists $L \in \mathbf{MSh}(Z;\mathcal{B}_{\mathbb{Q}})$ such that $\widetilde{\mathrm{B}}^* L \simeq i^* F$, and the result follows. □

Before proving assertion (iii) of Theorem 1.93, we note the following corollary of assertion (i).

Corollary 1.106. *Let Λ' be a commutative Λ-algebra.*

(i) We have a t-exact equivalence of ∞-categories

$$\mathrm{Mod}_{\Lambda'}(\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)) \simeq \mathbf{Sh}_{\mathrm{geo}}(X;\Lambda').$$

In particular, if Λ and Λ' are ordinary rings, then we have an equivalence of ordinary categories $\mathrm{Mod}_{\Lambda'}(\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit}) \simeq \mathbf{Sh}_{\mathrm{geo}}(X;\Lambda')^{\heartsuit}$.

(ii) If Λ and Λ' are ordinary rings,[4] *we have an equivalence of ∞-categories*

$$\mathrm{Mod}_{\Lambda'}(\widehat{\mathbf{LS}}_{\mathrm{geo}}(X;\Lambda)) \simeq \widehat{\mathbf{LS}}_{\mathrm{geo}}(X;\Lambda').$$

Proof. Part (i) follows immediately from Theorem 1.93 using the analogous property for the ∞-categories of motivic sheaves. For (ii), it is enough to treat the case of an ordinary $\mathbb{Z}$-algebra Λ, i.e., to prove that $\mathrm{Mod}_{\Lambda}(\widehat{\mathbf{LS}}_{\mathrm{geo}}(X;\mathbb{Z})) \simeq \widehat{\mathbf{LS}}_{\mathrm{geo}}(X;\Lambda)$. We claim that the forgetful functor

$$\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda) \to \mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Z})$$

takes $\widehat{\mathbf{LS}}_{\mathrm{geo}}(X;\Lambda)$ into $\widehat{\mathbf{LS}}_{\mathrm{geo}}(X;\mathbb{Z})$. Indeed, given a local system L of Λ-modules on X of geometric origin, the sheaves $\mathrm{H}_i(L)$ can be written as a filtered colimit of constructible subsheaves of $\mathbb{Z}$-modules of geometric origin. But, every constructible subsheaf of $\mathbb{Z}$-modules of $\mathrm{H}_i(L)$ is contained in a sub-local system of $\mathbb{Z}$-modules, which must be also of geometric origin. This proves our claim. It follows that the functor

$$-\otimes_{\mathbb{Z}}\Lambda : \widehat{\mathbf{LS}}_{\mathrm{geo}}(X;\mathbb{Z}) \to \widehat{\mathbf{LS}}_{\mathrm{geo}}(X;\Lambda)$$

admits a right adjoint, given by the forgetful functor, which is thus conservative and colimit-preserving. We may apply [Lur17, Theorem 4.7.3.5] to conclude. □

Proof of Theorem 1.93, (iii). By Corollary 1.106(i), we may assume that $\Lambda = \mathbb{Z}$. First, we note that the result is true generically, i.e., if K/k is a field extension, then the abelian subcategory $\mathbf{Sh}_{\mathrm{geo}}(K;\mathbb{Z})^{\heartsuit} \subset \mathbf{Sh}_{\mathrm{ct}}(K;\mathbb{Z})^{\heartsuit}$ is stable under subquotients. This follows from Lemma 1.99 using the fact that any torsion sheaf of geometric origin is the realisation of a torsion étale sheaf.

Next, we consider the general case of a k-variety X. Let $F \in \mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Z})^{\heartsuit}$ and $G \subset F$ a subobject of F in $\mathbf{Sh}_{\mathrm{ct}}(X;\mathbb{Z})$. We must show that G is of geometric origin. We may assume that F and G are constructible. We argue by noetherian induction on X. Let η be a generic point of X. By the above discussion, we know that η^*G is of geometric origin. Thus, we may find a constructible sheaf G' of geometric origin on X such that $\eta^*G \simeq \eta^*G'$. This isomorphism extends to an open neighbourhood U of η on X. In particular, j^*G is of geometric origin, with $j : U \hookrightarrow X$ the obvious inclusion. To conclude, it remains to see that i^*G is of geometric origin, with $i : Z = X \smallsetminus U \to X$ the inclusion of the complement of U. But i^*Z is a subsheaf of i^*F, and we may conclude by induction. □

We end the subsection with the following result which is essentially due to Nori.

Theorem 1.107 (Nori)**.** *Assume that Λ is an ordinary commutative ring. Then, the functor*

$$\mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit}) \to \mathbf{Sh}_{\mathrm{geo}}(X;\Lambda) \tag{1.62}$$

is an equivalence of ∞-categories. If Λ is a field or a Dedekind domain with finite residue fields at all maximal ideals, then the same is true with "$\mathbf{Sh}_{\mathrm{ct}}$" instead of "$\mathbf{Sh}_{\mathrm{geo}}$".

We will not give a self-contained proof of Theorem 1.107. Instead, we will explain how to deduce it from results in [Nor02]. We first prove some reductions.

Lemma 1.108. *Assume that Λ is an ordinary ring. We have an equivalence of ∞-categories*

$$\mathrm{Mod}_{\Lambda}(\mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Z})^{\heartsuit})) \simeq \mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit}).$$

In particular, to show that the functor in (1.62) *is an equivalence, it is enough to consider the case $\Lambda = \mathbb{Z}$.*

[4] Is this really necessary?

Proof. The functor $-\otimes\Lambda : \mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Z})^{\heartsuit} \to \mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit}$ is right exact. By Corollary 1.106(i), it admits a right adjoint given by the forgetful functor. Arguing as in the proof of Lemma 1.81, this adjunction can be lifted to the derived setting:

$$\mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Z})^{\heartsuit}) \rightleftarrows \mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit}).$$

It is immediate to see that [Lur17, Theorem 4.7.3.5] applies to the above adjunction yielding the equivalence in the statement. This prove the first statement. Combining this with Corollary 1.106(i), we obtain also the second statement. □

Lemma 1.109. *Assume that Λ is a field or a Dedekind domain with finite residue fields at all maximal ideals. To show that the functor in* (1.62) *is an equivalence, it is enough to show that the functor*

$$\mathrm{D}^{\mathrm{b}}(\mathbf{Ct}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit}) \to \mathbf{Ct}_{\mathrm{geo}}(X;\Lambda) \tag{1.63}$$

is an equivalence. The same is true for "$\mathbf{Sh}_{\mathrm{ct}}$" and "$\mathbf{Ct}$" instead of "$\mathbf{Sh}_{\mathrm{geo}}$" and "$\mathbf{Ct}_{\mathrm{geo}}$".

Proof. Since $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$ is the indization of $\mathbf{Ct}_{\mathrm{geo}}(X;\Lambda)$, it is enough to show that $\mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit})$ is the indization of $\mathrm{D}^{\mathrm{b}}(\mathbf{Ct}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit})$. The argument is very similar to the one used in Step 3 of the proof of Proposition 1.80. Indeed, using that the functor in (1.63) is an equivalence, we deduce that every object of $\mathbf{Ct}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit}$ has cohomological dimension bounded by $2\dim(X)+1$. Using [CM19, Proposition 2.10], we deduce that every object of $\mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit})$ is Postnikov complete. From this, it follows easily that every object in $\mathrm{D}^{\mathrm{b}}(\mathbf{Ct}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit})$ determines a compact object of $\mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit})$. The result follows since $\mathrm{D}^{\mathrm{b}}(\mathbf{Ct}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit})$ generates $\mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit})$ under filtered colimits. The case of "$\mathbf{Sh}_{\mathrm{ct}}$" and "$\mathbf{Ct}$" is treated similarly. □

Proof of Theorem 1.107. We only treat the case of sheaves of geometric origin. By Lemmas 1.108 and 1.109, it is enough to show that the functor in (1.63) is an equivalence. Only fully faithfulness is needed. We will prove more generally that

$$\mathrm{Map}_{\mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit})}(F,G) \to \mathrm{Map}_{\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)}(F,G) \tag{1.64}$$

is an equivalence for $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{Ct}_{\mathrm{geo}}(X;\Lambda))$ and $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda))$.

Step 1. We first assume that G is torsion. Since F is constructible, the domain and codomain of (1.64) commute with colimits of uniformly bounded above inductive systems in the variable G. Thus, we may assume that G is a $\mathbb{Z}/\ell^{\nu}$-module. By adjunction, we may then replace Λ with Λ/ℓ^{ν} and F with $F\otimes\mathbb{Z}/\ell^{\nu}$. In this case, the result follows from the fact that derived ∞-category of étale sheaves of $\mathbb{Z}/\ell^{\nu}$-modules on $X\otimes_k\mathbb{C}$ is equivalent to the ∞-category $\mathbf{Sh}_{\mathrm{geo}}(X;\mathbb{Z}/\ell^{\nu})$. This ultimately relies on Artin's comparison theorem [SGA73, Exposé XI, Théorème 4.4 & Exposé XVI, Théorème 4.1].

Step 2. By Step 1, we may replace G with $G\otimes_{\Lambda}\mathrm{Frac}(\Lambda)$ since $\mathrm{cofib}(G\to G\otimes_{\Lambda}\mathrm{Frac}(\Lambda))$ is torsion. By adjunction, we can further replace Λ and F with $\mathrm{Frac}(\Lambda)$ and $F\otimes_{\Lambda}\mathrm{Frac}(\Lambda)$. In this case, the result follows from [Nor02, Theorem 3]. Strictly speaking, Nori's theorem is stated for constructible sheaves, but a quick look at his proof shows that the result is also valid for constructible sheaves of geometric origin (and constructible sheaves of $\mathrm{Frac}(\Lambda)$-modules definable over Λ). □

2. The main theorem for constructible sheaves

This section contains the main results of this paper. In particular, we show that the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(k, \sigma)$ arises naturally as the group of autoequivalences of the functor

$$\mathbf{Sh}_{\mathrm{geo}}(-)^{\otimes} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{CAT}_{\infty}).$$

This is Theorem 2.10 whose proof is given in Subsection 2.2. A key ingredient for the proof is a result of Drew–Gallauer [DG20] that we review in Subsection 2.1.

2.1. Universal Voevodsky pullback formalisms.

In this subsection, we review the main result of [DG20] which, roughly speaking, asserts that $\mathbf{MSh}_{\mathrm{nis}}(-)^{\otimes}$ is initial among all Voevodsky pullback formalisms. The following definition agrees with [DG20, Definition 2.10] except for the condition that $\mathcal{H}(\emptyset)$ is final.

Definition 2.1. Let S be a quasi-compact and quasi-separated scheme. A pullback formalism is a functor

$$\mathcal{H}^{\otimes} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{CAT}_{\infty})$$

sending a finite type S-scheme X to a symmetric monoidal ∞-category $\mathcal{H}(X)^{\otimes}$ and a morphism $f : Y \to X$ of finite type S-schemes to a symmetric monoidal functor $f^* : \mathcal{H}(X)^{\otimes} \to \mathcal{H}(Y)^{\otimes}$ such that the following conditions are satisfied. (Below, we also write $f^* : \mathcal{H}(X) \to \mathcal{H}(Y)$ for the functor underlying f^*.)

(i) $\mathcal{H}(\emptyset)$ is equivalent to the final ∞-category with one object and one morphism.
(ii) If $f : Y \to X$ is a smooth morphism, the functor f^* admits a left adjoint $f_\sharp$ satisfying the projection formula: the morphism $f_\sharp(f^*(A) \otimes B) \to A \otimes f_\sharp(B)$ is an equivalence for every $A \in \mathcal{H}(X)$ and $B \in \mathcal{H}(Y)$.
(iii) Given a Cartesian square of finite type S-schemes

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

with g smooth, the exchange morphism $g'_\sharp f'^* \to f^* g_\sharp$ is an equivalence.

A morphisms of pullback formalisms is a natural transformation $\theta : \mathcal{H}^{\otimes} \to \mathcal{H}'^{\otimes}$ such that the natural morphisms $f_\sharp \circ \theta_Y \to \theta_X \circ f_\sharp$ are equivalences for all smooth morphisms $f : Y \to X$ in Sch/S. We denote by $\mathrm{PB}(S)$ the sub-∞-category of $\mathrm{Psh}(\mathrm{Sch}/S; \mathrm{CAlg}(\mathrm{CAT}_{\infty}))$ spanned by the pullback formalisms and their morphisms.

The following is a key technical result in [DG20]. We will give a sketch of proof for the reader's convenience. For a much more systematic treatment, we refer the reader to [DG20, Theorem 3.25].

Proposition 2.2. *The ∞-category $\mathrm{PB}(S)$ admits an initial object given by the functor*

$$(\mathrm{Sm}/-)^{\times} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{CAT}_{\infty}) \tag{2.1}$$

sending $X \in \mathrm{Sch}/S$ to the ordinary category Sm/X endowed with its Cartesian symmetric monoidal structure.

Proof. We will only explain how to construct a morphism of pullback formalisms

$$\theta : (\mathrm{Sm}/-)^{\times} \to \mathcal{H}^{\otimes}$$

for any $\mathcal{H}^{\otimes}$ in $\mathrm{PB}(S)$. The construction is functorial enough and, with some effort, can be made into a section of the left fibration $\mathrm{PB}(S)_{(\mathrm{Sm}/-)^{\times}/} \to \mathrm{PB}(S)$ sending $(\mathrm{Sm}/-)^{\times}$ to its identity functor. This would be enough to conclude, but we will not carry out the details here. Informally, the functor $\theta_X : \mathrm{Sm}/X \to \mathcal{H}(X)$ sends a smooth X-scheme Y with structural morphism f to the object $f_\sharp \mathbf{1}_Y$, where $\mathbf{1}_Y$ is the monoidal unit of $\mathcal{H}(Y)^{\otimes}$.

The construction of the natural transformation $\theta : (\mathrm{Sm}/-) \to \mathcal{H}$, without its compatibility with the symmetric monoidal structures, is quite straightforward. (For instance, one can adapt the proof of [AGV20, Lemma 2.6.12].) The question of how to incorporate the symmetric monoidal structures was resolved in [DG20]. In retrospective, one needs to exploit the fact that the functors $f_\sharp$ are left-lax monoidal. The problem is that the theory of symmetric monoidal ∞-categories is built in a way that allows to speak easily about right-lax monoidal functors, but not about the left-lax monoidal ones. Thus, one is lead to work with the induced symmetric monoidal structures on the ∞-categories $\mathcal{H}^{\mathrm{op}}(X)$'s. (Here we write "$\mathcal{H}^{\mathrm{op}}(X)$" instead of "$\mathcal{H}(X)^{\mathrm{op}}$".)

Recall that $(\mathrm{Sch}/S)^{\mathrm{op},\amalg}$ is the ordinary category whose objects are given by pairs $(\langle n\rangle, (X_i)_{1\leq i\leq n})$, where $n \geq 0$ is an integer and the X_i's are S-schemes of finite type. An arrow

$$(\langle n\rangle, (X_i)_{1\leq i\leq n}) \to (\langle n'\rangle, (X'_j)_{1\leq j\leq n'})$$

between two such pairs is a morphism $r : \langle n\rangle \to \langle n'\rangle$ and, for every $0 \leq j \leq n'$, a morphism of S-schemes $X'_j \to \prod_{i\in r^{-1}(j)}(X_i/S)$. We have an obvious functor $(\mathrm{Sch}/S)^{\mathrm{op},\amalg} \to \mathrm{Fin}_*$ which defines the coCartesian monoidal structure on $(\mathrm{Sch}/S)^{\mathrm{op}}$. We also have an obvious diagonal functor $d : \mathrm{Fin}_* \times (\mathrm{Sch}/S)^{\mathrm{op}} \to (\mathrm{Sch}/S)^{\mathrm{op},\amalg}$ sending a pair $(\langle n\rangle, X)$ to the pair $(\langle n\rangle, (X)_{1\leq i\leq n})$.

Similarly, we consider the ordinary category D whose objects are pairs $(\langle n\rangle, (f_i : Y_i \to X_i)_{1\leq i\leq n})$, where $n \geq 0$ is an integer and the f_i's are smooth morphisms in Sch/S. An arrow

$$(\langle n\rangle, (f_i : Y_i \to X_i)_{1\leq i\leq n}) \to (\langle n'\rangle, (f'_j : Y'_j \to X'_j)_{1\leq j\leq n'})$$

between two such pairs is a morphism $r : \langle n\rangle \to \langle n'\rangle$ and, for every $0 \leq j \leq n'$, a commutative square of S-schemes

$$\begin{tikzcd} Y'_j \arrow[r] \arrow[d, "f'_j"] & \prod_{i\in r^{-1}(j)}(Y_i/S) \arrow[d] \\ X'_j \arrow[r] & \prod_{i\in r^{-1}(j)}(X_i/S). \end{tikzcd}$$

We have obvious functors

$$s : D \to (\mathrm{Sch}/S)^{\mathrm{op},\amalg} \qquad \text{and} \qquad t : D \to (\mathrm{Sch}/S)^{\mathrm{op},\amalg}$$

sending the object $(\langle n\rangle, (f_i : Y_i \to X_i)_{1\leq i\leq n})$ to $(\langle n\rangle, (Y_i)_{1\leq i\leq n})$ and $(\langle n\rangle, (X_i)_{1\leq i\leq n})$ respectively. We also have a natural transformation $\phi : t \to s$ given at the previously considered object by $\mathrm{id}_{\langle n\rangle}$ and the f_i's.

Next, we consider the coCartesian fibration

$$p : \Xi^{\otimes} \to (\mathrm{Sch}/S)^{\mathrm{op},\amalg}, \tag{2.2}$$

whose fiber at $(\langle n\rangle, (X_i)_{1\leq i\leq n})$ is the Cartesian product of the ∞-categories $\mathcal{H}^{\mathrm{op}}(X_i)$'s. The existence of such a coCartesian fibration is insured by [DG20, Corollary A.12] as explained in [DG20, Remark A.13]. Note that the base change of p by $d|_{\mathrm{Fin}_*\times\{X\}}$, for $X \in \mathrm{Sch}/S$, is the coCartesian fibration $\mathcal{H}^{\mathrm{op}}(X)^{\otimes} \to \mathrm{Fin}_*$ defining the symmetric monoidal structure on the opposite of $\mathcal{H}(X)$. Pulling back along s and t, we obtain a commutative triangle

$$\begin{tikzcd} \Xi_t^{\otimes} \arrow[rr, "\phi^*"] \arrow[dr, "p_t"'] & & \Xi_s^{\otimes} \arrow[dl, "p_s"] \\ & D, & \end{tikzcd}$$

where p_s and p_t are coCartesian fibrations, and ϕ^* preserves coCartesian edges. Informally, over the previously considered object $(\langle n\rangle, (f_i : Y_i \to X_i)_{1\leq i\leq n})$, ϕ^* is given by the Cartesian product of the inverse image functors f_i^* and thus admits a right adjoint by assumption. By [Lur17, Proposition 7.3.2.6], the functor ϕ^* admits a right adjoint $\phi_\sharp$ relative to D in the sense of [Lur17, Definition 7.3.2.2]. Writing $\mathbf{1}$ for the coCartesian section of p_s given by the monoidal units, we obtain a section $\phi_\sharp\mathbf{1} : D \to \Xi_t^{\otimes}$ of p_t. Equivalently, we have constructed a commutative triangle

$$\begin{tikzcd} D \arrow[rr, "h"] \arrow[dr, "t"'] & & \Xi^{\otimes} \arrow[dl, "p"] \\ & (\mathrm{Sch}/S)^{\mathrm{op},\amalg} & \end{tikzcd}$$

with the following properties. The base change of t by $d|_{\mathrm{Fin}_*\times\{X\}}$, for $X \in \mathrm{Sch}/S$, is the coCartesian fibration

$$t_X : D_X = (\mathrm{Sm}/X)^{\mathrm{op},\amalg} \to \mathrm{Fin}_*$$

defining the coCartesian symmetric monoidal structure on $(\mathrm{Sm}/X)^{\mathrm{op}}$. Similarly, the base change of h by $d|_{\mathrm{Fin}_*\times\{X\}}$, for $X \in \mathrm{Sch}/S$, is a symmetric right-lax monoidal functor

$$h_X : (\mathrm{Sm}/X)^{\mathrm{op},\amalg} \to \mathcal{H}^{\mathrm{op}}(X)^{\otimes}$$

sending a smooth X-scheme Y with structural morphism f to $f_\sharp\mathbf{1}_Y$. If follows from the projection formula that h_X is actually symmetric monoidal. By straightening the base changes of t and p by d, we thus obtain a morphism $(\mathrm{Sm}/-)^{\mathrm{op},\amalg} \to \mathcal{H}^{\mathrm{op},\otimes}$ in $\mathrm{Psh}(\mathrm{Sch}/S; \mathrm{CAlg}(\mathrm{CAT}_\infty))$. Applying the involution $(-)^{\mathrm{op}}$ of $\mathrm{CAlg}(\mathrm{CAT}_\infty)$ to this morphism, yields the desired result. □

Definition 2.3.

(i) A pullback formalism $\mathcal{H}^{\otimes}$ is called presentable if it factors through $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. Similarly, a morphism of presentable pullback formalisms is a morphism of pullback formalisms that belongs to the ∞-category $\mathrm{Psh}(\mathrm{Sch}/S; \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))$. We denote by $\mathrm{PrPB}(S)$ the sub-∞-category of $\mathrm{PB}(S)$ spanned by the presentable pullback formalisms and their morphisms.

(ii) A pullback formalism $\mathcal{H}^{\otimes}$ is called stable if it factors $\mathrm{CAlg}(\mathrm{CAT}_\infty^{\mathrm{st}})$. Similarly, a morphism of stable pullback formalisms is a morphism of pullback formalisms that belongs to the ∞-category $\mathrm{Psh}(\mathrm{Sch}/S; \mathrm{CAlg}(\mathrm{CAT}_\infty^{\mathrm{st}}))$. Stable presentable pullback formalisms form a full sub-∞-category of $\mathrm{PrPB}(S)$ that we denote by $\mathrm{PrPB}^{\mathrm{st}}(S)$.

Definition 2.4. We define $\mathrm{VPB}(S)$ to be the full sub-∞-category of $\mathrm{PB}^{\mathrm{st}}(S)$ consisting of Voevodsky pullback formalisms in the sense of Definition 1.14. Similarly, we define $\mathrm{PrVPB}(S)$ to be the full sub-∞-category of $\mathrm{PrPB}^{\mathrm{st}}(S)$ consisting of Voevodsky pullback formalisms. Objects of $\mathrm{PrVPB}(S)$ are the presentable Voevodsky pullback formalisms.

We can now state the main result of [DG20, Theorem 7.14]. Since this result is crucial for us, we give a sketch of proof relying on Robalo's universality result [Rob15, Corollary 2.39]. For a more detailed and self-contained proof, we refer the reader to [DG20].

Theorem 2.5 (Drew–Gallauer). *Let S be a quasi-compact and quasi-separated scheme, locally of finite Krull dimension. Then the Voevodsky pullback formalism*

$$\mathbf{MSh}_{\mathrm{nis}}(-)^{\otimes} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

is an initial object of $\mathrm{PrVPB}(S)$*.*

Proof. In fact, one proves that $\mathbf{MSh}_{\mathrm{nis}}(-)$ is initial in the ∞-category $\mathcal{W}$ defined below, which is much larger than $\mathrm{PrVPB}(S)$. Consider the functor $\mathrm{Sch}/S \to \mathrm{CAT}_\infty$ sending $X \in \mathrm{Sch}/S$ to the ∞-category

$$\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}})_{(\mathrm{Sm}/X)^{\times}/} = \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}}) \times_{\mathrm{CAlg}(\mathrm{CAT}_\infty)} \mathrm{CAlg}(\mathrm{CAT}_\infty)_{(\mathrm{Sm}/X)^{\times}/},$$

and form the associated Cartesian fibration

$$p : \oint_{(\mathrm{Sch}/S)^{\mathrm{op}}} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}})_{(\mathrm{Sm}/-)^{\times}/} \to (\mathrm{Sch}/S)^{\mathrm{op}}.$$

The ∞-category $\mathrm{Sect}(p)$ of sections of p is equivalent to the fiber product

$$\mathrm{Psh}(\mathrm{Sch}/S, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}})) \times_{\mathrm{Psh}(\mathrm{Sch}/S, \mathrm{CAlg}(\mathrm{CAT}_\infty))} \mathrm{Psh}(\mathrm{Sch}/S, \mathrm{CAlg}(\mathrm{CAT}_\infty))_{(\mathrm{Sm}/-)^{\times}/}.$$

Thus, an object of $\mathrm{Sect}(p)$ is given by a morphism of $\mathrm{CAlg}(\mathrm{CAT}_\infty)$-valued presheaves on Sch/S from $(\mathrm{Sm}/-)^{\times}$ to a $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}})$-valued presheaf. Let $\mathcal{W} \subset \mathrm{Sect}(p)$ be the full sub-∞-category spanned by those presheaves morphisms $\theta : (\mathrm{Sm}/-)^{\times} \to \mathcal{H}(-)^{\otimes}$ satisfying the following conditions for every $X \in \mathrm{Sch}/S$.

(i) The functor θ_X takes a Nisnevich square of smooth X-schemes to a coCartesian square in the ∞-category $\mathcal{H}(X)$.

(ii) For every $U \in \mathrm{Sm}/X$, the functor θ_X takes the projection $\mathbb{A}^1_U \to U$ to an equivalence in the ∞-category $\mathcal{H}(X)$.

(iii) The object $\mathrm{cofib}(\theta_X(\infty_X) \to \theta_X(\mathbb{P}^1_X))$ of the symmetric monoidal ∞-category $\mathcal{H}(X)^{\otimes}$ is $\otimes$-invertible.

Alternatively, we can define $\mathcal{W}$ as the ∞-category $\mathrm{Sect}(q)$ of sections of the Cartesian fibration $q : \mathfrak{Q} \to (\mathrm{Sch}/S)^{\mathrm{op}}$ where

$$\mathfrak{Q} \subset \oint_{(\mathrm{Sch}/S)^{\mathrm{op}}} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}})_{(\mathrm{Sm}/-)^{\times}/}$$

is the full sub-∞-category whose fiber $\mathfrak{Q}_X$ at $X \in \mathrm{Sch}/S$ is spanned by those symmetric monoidal functors $\varsigma : (\mathrm{Sm}/X)^{\times} \to \mathcal{K}^{\otimes}$ satisfying the analogs of conditions (i)–(iii) above.

By Proposition 2.2, we have a faithful functor

$$\mathrm{PrPB}^{\mathrm{st}}(S) \to \mathrm{Sect}(p)$$

whose restriction to $\mathrm{PrVPB}(S)$ factors through $\mathcal{W} = \mathrm{Sect}(q)$. (Recall that a functor is said to be faithful if it induces monomorphisms on mapping spaces; a monomorphism in $\mathcal{S}$ is a (-1)-truncated map in the sense of [Lur09a, Definition 5.5.6.8].) We claim that it is enough to show that the natural transformation

$$(\mathrm{Sm}/-)^{\times} \to \mathbf{MSh}_{\mathrm{nis}}(-)^{\otimes}$$

is an initial object of $\mathscr{W}$. Indeed, given a presentable Voevodsky pullback formalism $\mathscr{H}^{\otimes}$ and a triangle of natural transformations

$$\begin{tikzcd} (\mathrm{Sm}/-)^{\times} \arrow[r] \arrow[dr] & \mathbf{MSh}_{\mathrm{nis}}(-)^{\otimes} \arrow[d, "\Phi"] \\ & \mathscr{H}^{\otimes} \end{tikzcd}$$

with Φ in the ∞-category $\mathrm{Psh}(\mathrm{Sch}/S, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}}))$ and the two others in $\mathrm{PB}(S)$, it is easy to deduce that Φ is necessarily a morphism of presentable Voevodsky pullback formalisms. This said, the result follows now from [Rob15, Corollary 2.39] combined with [Lur09a, Proposition 2.4.4.9] applied to the Cartesian fibration q. □

In fact, for later use, we will need a Λ-linear version of Theorem 2.5 which follows immediately from the $\mathbb{S}$-linear version.

Definition 2.6. Let $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$ be a commutative ring spectrum. A Λ-linear presentable Voevodsky pullback formalism is a functor

$$\mathscr{H}^{\otimes} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}})_{\mathrm{Mod}_{\Lambda}^{\otimes}/}$$

whose composition with the forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}})_{\mathrm{Mod}_{\Lambda}^{\otimes}/} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}})$ is a Voevodsky pullback formalism. We denote by $\mathrm{PrVPB}(S)_{\Lambda}$ the ∞-category of Λ-linear presentable Voevodsky pullback formalisms. A morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}})_{\mathrm{Mod}_{\Lambda}^{\otimes}/}$-valued presheaves on Sch/S belongs to $\mathrm{PrVPB}(S)_{\Lambda}$ if its underlying morphism belongs to $\mathrm{PrVPB}(S)$.

Lemma 2.7. *The obvious forgetful functor* $\mathrm{PrVPB}(S)_{\Lambda} \to \mathrm{PrVPB}(S)$ *admits a left adjoint sending a presentable Voevodsky pullback formalism* $\mathscr{H}(-)^{\otimes}$ *to the functor* $\mathrm{Mod}_{\Lambda}(\mathscr{H}(-))^{\otimes}$.

Using Lemma 2.7, we deduce that Theorem 2.5 admits the following Λ-linear version.

Theorem 2.8 (Drew–Gallauer)**.** *Let S be a quasi-compact and quasi-separated scheme, locally of finite Krull dimension. Then the Voevodsky pullback formalism*

$$\mathbf{MSh}_{\mathrm{nis}}(-;\Lambda)^{\otimes} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}_{\Lambda}^{\otimes}/}$$

is an initial object of $\mathrm{PrVPB}(S)_{\Lambda}$. *Said differently, the obvious forgetful functor*

$$\mathrm{PrVPB}(S)_{\mathbf{MSh}_{\mathrm{nis}}(-;\Lambda)^{\otimes}/} \to \mathrm{PrVPB}(S)_{\Lambda}$$

is an equivalence of ∞-categories.

2.2. **The first main theorem.**

In this subsection, we prove our main theorem for constructible sheaves of geometric origin and derive a few complements. We start by introducing the prestack of autoequivalences of the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaf $\mathbf{Sh}_{\mathrm{geo}}^{\otimes}$.

Definition 2.9. Let k be a field and $\sigma : k \hookrightarrow \mathbb{C}$ a complex embedding. We define the nonconnective spectral group $\mathbb{S}$-prestack $\underline{\mathrm{Auteq}}(\mathbf{Sh}_{\mathrm{geo}}^{\otimes})$ by applying Construction 1.51 to

- the functor $\mathcal{C} : (\mathrm{SpAFF}^{\mathrm{nc}})^{\mathrm{op}} \to \mathrm{CAT}_{\infty}$ sending $\mathrm{Spec}(\Lambda)$ to the ∞-category

$$\mathrm{Psh}(\mathrm{Sch}/k; \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}_{\Lambda}^{\otimes}/})$$

  of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}_{\Lambda}^{\otimes}/}$-valued presheaves on Sch/S, and

- the natural transformation $\mathrm{pt} \to \mathcal{C}$ sending $\mathrm{Spec}(\Lambda)$ to the functor

$$\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/}.$$

Thus, informally, the group of Λ-points of $\underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo}})$ is the group of autoequivalences of the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/}$-valued presheaf $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}$. If we want to stress that $\underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo}})$ depends on the complex embedding σ, we will write $\underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\sigma\text{-geo}})$.

The following is our main theorem for constructible sheaves.

Theorem 2.10 (Main theorem for constructible sheaves)**.** *Let k be a field and $\sigma : k \hookrightarrow \mathbb{C}$ a complex embedding. There is an equivalence of nonconnective spectral group $\mathbb{S}$-prestacks*

$$\mathcal{G}_{\mathrm{mot}}(k,\sigma) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\sigma\text{-geo}}). \tag{2.3}$$

In particular, the right hand side is a spectral affine group scheme.

Proof. We split the proof in two steps.

Step 1. Using Theorem 1.54, it is enough to construct an equivalence of nonconnective spectral group $\mathbb{S}$-prestacks

$$\underline{\mathrm{Auteq}}(\mathcal{B}) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo}})$$

where $\mathcal{B} \in \mathrm{CAlg}(\mathbf{MSh}(k)^{\otimes})$ is the Betti spectrum introduced in Notation 1.57. By Theorem 1.93(i), we have an equivalence of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/}$-valued presheaves

$$\widetilde{\mathrm{B}}^{*} : \mathbf{MSh}(-;\mathcal{B}_{\Lambda})^{\otimes} \to \mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}.$$

Thus, in Definition 2.9, we may as well take the natural transformation $\mathrm{pt} \to \mathcal{C}$ sending $\mathrm{Spec}(\Lambda)$ to the functor

$$\mathbf{MSh}(-;\mathcal{B}_{\Lambda})^{\otimes} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/}.$$

Now, recall that the nonconnective spectral group $\mathbb{S}$-prestack $\underline{\mathrm{Auteq}}(\mathcal{B})$ is obtained by applying Construction 1.51 to

- the functor $\mathcal{D} : (\mathrm{SpAFF}^{\mathrm{nc}})^{\mathrm{op}} \to \mathrm{CAT}_{\infty}$ sending $\mathrm{Spec}(\Lambda)$ to the ∞-category

$$\mathrm{CAlg}(\mathbf{MSh}(k;\Lambda))$$

  of commutative algebra objects in $\mathbf{MSh}(k;\Lambda)^{\otimes}$, and
- the natural transformation $\mathrm{pt} \to \mathcal{D}$ sending $\mathrm{Spec}(\Lambda)$ to $\mathcal{B}_{\Lambda}$.

There is a natural transformation $\mathcal{D} \to \mathcal{C}$ sending $\mathrm{Spec}(\Lambda)$ to the functor

$$\mathrm{CAlg}(\mathbf{MSh}(k;\Lambda)) \to \mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/})$$

taking a commutative algebra object $\mathcal{A}$ of $\mathbf{MSh}(k;\Lambda)^{\otimes}$ to the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/}$-valued presheaf

$$\mathbf{MSh}(-;\mathcal{A})^{\otimes} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/}.$$

Moreover, the following triangle of CAT_{∞}-valued presheaves on $\mathrm{SpAFF}^{\mathrm{nc}}$

$$\begin{tikzcd} \mathrm{pt} \arrow[r] \arrow[dr] & \mathcal{D} \arrow[d] \\ & \mathcal{C} \end{tikzcd}$$

is commutative. Applying Construction 1.51 with the CAT_∞-valued presheaf on $\Delta^1 \times \mathrm{SpAFF}^{\mathrm{nc}}$ corresponding to $\mathcal{D} \to \mathcal{C}$, we obtain a morphism

$$\underline{\mathrm{Auteq}}(\mathcal{B}) \to \underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo}}), \tag{2.4}$$

and it remains to see that this morphism is an equivalence. This will be proven in the next steps.

Step 2. Evaluating the morphism (2.4) at $\mathrm{Spec}(\Lambda)$, for $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$, yields a morphism of groups objects in $\mathcal{S}$:

$$\mathrm{Auteq}_{\mathrm{CAlg}(\mathbf{MSh}(k;\Lambda))}(\mathcal{B}_\Lambda) \to \mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/})}(\mathbf{MSh}(-;\mathcal{B}_\Lambda)^{\otimes}). \tag{2.5}$$

We only need to show that (2.5) induces an equivalence on the underlying spaces.

First, we note that any autoequivalence of the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/}$-valued presheaf $\mathbf{MSh}(-;\mathcal{B}_\Lambda)^{\otimes}$ is automatically an autoequivalence of Λ-linear presentable Voevodsky pullback formalisms. Thus, the codomain of the map in (2.5) can be rewritten as follows:

$$\mathrm{Auteq}_{\mathrm{PrVPB}(k)_\Lambda}(\mathbf{MSh}(-;\mathcal{B}_\Lambda)^{\otimes}) \tag{2.6}$$

where $\mathrm{PrVPB}(k)_\Lambda$ is the ∞-category of introduced in Definition 2.6. Applying Theorem 2.8, we see that the space in (2.6) is equivalent to

$$\mathrm{Auteq}_{\mathrm{PrVPB}(k)_{\mathbf{MSh}(-;\Lambda)^{\otimes}/}}(\mathbf{MSh}(-;\Lambda)^{\otimes} \to \mathbf{MSh}(-;\mathcal{B}_\Lambda)^{\otimes}). \tag{2.7}$$

Using again that the autoequivalences of the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaf $\mathbf{MSh}(-;\mathcal{B}_\Lambda)^{\otimes}$ belong to $\mathrm{PrVPB}(k)$, we may rewrite the space in (2.7) as

$$\mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))_{\mathbf{MSh}(-;\Lambda)^{\otimes}/}}(\mathbf{MSh}(-;\Lambda)^{\otimes} \to \mathbf{MSh}(-;\mathcal{B}_\Lambda)^{\otimes}). \tag{2.8}$$

Now, remark that we have a commutative triangle

$$\begin{tikzcd}
\mathrm{Auteq}_{\mathrm{CAlg}(\mathbf{MSh}(k;\Lambda))}(\mathcal{B}_\Lambda) \arrow[r, "(a)"] \arrow[dr, "(c)"'] & \mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))_{\mathbf{MSh}(-;\Lambda)^{\otimes}/}}(\mathbf{MSh}(-;\Lambda)^{\otimes} \to \mathbf{MSh}(-;\mathcal{B}_\Lambda)^{\otimes}) \arrow[d, "(b)"] \\
& \mathrm{Auteq}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathbf{MSh}(k;\Lambda)^{\otimes}/}}(\mathbf{MSh}(k;\Lambda)^{\otimes} \to \mathbf{MSh}(k;\mathcal{B}_\Lambda)^{\otimes})
\end{tikzcd}$$

where (a) is the map in (2.5) modulo the above identifications, and (b) is the map induced by evaluating at the final object of Sch/k. It is easy to see that the map (c) is induced by the functor

$$\mathrm{CAlg}(\mathbf{MSh}(k;\Lambda)) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathbf{MSh}(k;\Lambda)^{\otimes}/}$$

sending a commutative algebra object $\mathcal{A}$ in $\mathbf{MSh}(k;\Lambda)^{\otimes}$ to the object $\mathbf{MSh}(k;\Lambda)^{\otimes} \to \mathbf{MSh}(k;\mathcal{A})^{\otimes}$. By [Lur17, Corollary 4.8.5.21], this functor is fully faithful, which implies that the map (c) is an equivalence. To end the proof, it remains to see that the map (b) is an equivalence.

Step 3. To prove that the map (b) is an equivalence we remark that for every finite type k-scheme X, the following square

$$\begin{tikzcd}
\mathbf{MSh}(k;\Lambda)^{\otimes} \arrow[r] \arrow[d] & \mathbf{MSh}(k;\mathcal{B}_\Lambda)^{\otimes} \arrow[d] \\
\mathbf{MSh}(X;\Lambda)^{\otimes} \arrow[r] & \mathbf{MSh}(X;\mathcal{B}_\Lambda)^{\otimes}
\end{tikzcd}$$

is coCartesian in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. Indeed, by the usual formula for the pushout of commutative algebras (which follows by combining [Lur17, Proposition 3.2.4.7] with the proof of [Lur17, Lemma 1.3.3.10]), it is enough to show that the base change functor

$$\mathrm{Mod}_{\mathbf{MSh}(k;\Lambda)^{\otimes}}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Mod}_{\mathbf{MSh}(X;\Lambda)^{\otimes}}(\mathrm{Pr}^{\mathrm{L}})$$

takes the $\mathbf{MSh}(k;\Lambda)^{\otimes}$-module $\mathbf{MSh}(k;\mathcal{B}_\Lambda)$ to the $\mathbf{MSh}(X;\Lambda)^{\otimes}$-module $\mathbf{MSh}(X;\mathcal{B}_\Lambda)$. Thus, we are reduced to showing that the obvious functor

$$\mathrm{Mod}_{\mathcal{B}_\Lambda}(\mathbf{MSh}(k;\Lambda)) \otimes_{\mathbf{MSh}(k;\Lambda)^{\otimes}} \mathbf{MSh}(X;\Lambda) \to \mathrm{Mod}_{\mathcal{B}_\Lambda}(\mathbf{MSh}(X;\Lambda))$$

is an equivalence. (Note that here, we are free to forget the symmetric monoidal structure on $\mathbf{MSh}(X;\Lambda)$, remembering only its left $\mathbf{MSh}(k;\Lambda)^{\otimes}$-module structure.) The claimed result is then a particular case of [BZFN10, Proposition 4.1].

This said, it is now easy to prove that the map (b) is an equivalence. Indeed, the object

$$\mathbf{MSh}(-;\Lambda)^{\otimes} \to \mathbf{MSh}(-;\Lambda)^{\otimes}$$

of $\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))_{\mathbf{MSh}(-;\mathcal{B}_\Lambda)^{\otimes}/}$ appears now as the image of the object

$$(\mathbf{MSh}(k;\Lambda)^{\otimes})_{\mathrm{cst}} \to (\mathbf{MSh}(k;\mathcal{B}_\Lambda)^{\otimes})_{\mathrm{cst}}$$

by the cobase change functor

$$\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))_{(\mathbf{MSh}(k;\Lambda)^{\otimes})_{\mathrm{cst}}/} \to \mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))_{\mathbf{MSh}(-;\Lambda)^{\otimes}/}.$$

(As usual, the subscript "cst" refers to "constant presheaf".) Thus, the result follows readily from Lemma 2.11 below. □

Lemma 2.11. *Let $\mathcal{C}$ be a small ∞-category admitting a final object* pt*, and let $\mathcal{D}$ be an ∞-category admitting pushouts. Let $F \in \mathrm{Psh}(\mathcal{C};\mathcal{D})$ be a $\mathcal{D}$-valued presheaf on $\mathcal{C}$. Let $f : A \to B$ be a map in $\mathcal{D}$, and suppose we have a pushout square in* $\mathrm{Psh}(\mathcal{C};\mathcal{D})$*:*

$$\begin{tikzcd} A_{\mathrm{cst}} \arrow[r] \arrow[d, "f_{\mathrm{cst}}"] & F \arrow[d] \\ B_{\mathrm{cst}} \arrow[r] & G. \end{tikzcd}$$

Then, the functor $\mathrm{Psh}(\mathcal{C};\mathcal{D}) \to \mathcal{D}$*, given by evaluating at* pt*, induces an equivalence of group objects in $\mathcal{S}$:*

$$\mathrm{Auteq}_{\mathrm{Psh}(\mathcal{C};\mathcal{D})_{F/}}(F \to G) \simeq \mathrm{Auteq}_{\mathcal{D}_{F(\mathrm{pt})/}}(F(\mathrm{pt}) \to G(\mathrm{pt})). \tag{2.9}$$

Proof. Without loss of generality, we may assume that $A = F(\mathrm{pt})$ which implies that $B = G(\mathrm{pt})$. Evaluation at pt gives a map

$$\mathrm{Auteq}_{\mathrm{Psh}(\mathcal{C};\mathcal{D})_{F/}}(F \to G) \to \mathrm{Auteq}_{\mathcal{D}_{A/}}(A \to B), \tag{2.10}$$

and we want to show that this map is an equivalence. By construction, the object $F \to G$ is the image of the object $A_{\mathrm{cst}} \to B_{\mathrm{cst}}$ by the cobase change functor $\mathrm{Psh}(\mathcal{C};\mathcal{D})_{A_{\mathrm{cst}}/} \to \mathrm{Psh}(\mathcal{C};\mathcal{D})_{F/}$ which admits a right adjoint given composition with the morphism $A_{\mathrm{cst}} \to F$. Combining this with the fact that $(-)_{\mathrm{cst}}$ is left adjoint to evaluating at pt, we obtain the equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{Psh}(\mathcal{C};\mathcal{D})_{F/}}(F \to G, F \to G) &\simeq \mathrm{Map}_{\mathrm{Psh}(\mathcal{C};\mathcal{D})_{A_{\mathrm{cst}}/}}(A_{\mathrm{cst}} \to B_{\mathrm{cst}}, A_{\mathrm{cst}} \to G) \\ &\simeq \mathrm{Map}_{\mathcal{D}_{A/}}(A \to B, A \to B). \end{aligned}$$

It is easy to see that the composite equivalence sends the subspace $\mathrm{Auteq}_{\mathrm{Psh}(\mathcal{C};\mathcal{D})_{F/}}(F \to G)$ to the subspace $\mathrm{Auteq}_{\mathcal{D}_{A/}}(A \to B)$ yielding the map (2.10). It remains to see that this map is surjective on π_0, which follows from the fact that it admits a section. $\square$

Our next task is to derive a version of Theorem 2.10 for the classical affine group scheme underlying $\mathcal{G}_{\mathrm{mot}}(k, \sigma)$. For that, we need a "classical" version of Definition 2.9.

Definition 2.12. Let k be a field and $\sigma : k \hookrightarrow \mathbb{C}$ a complex embedding. We define the (noncommutative) Picard $\mathbb{Z}$-prestack $\underline{\mathrm{Auteq}}(\mathbf{Sh}_{\mathrm{geo}}^{\heartsuit,\otimes})$ to be the presheaf of Picard groupoids on AFF sending $\mathrm{Spec}(\Lambda)$, with Λ an ordinary commutative ring, to the Picard groupoid of autoequivalences of the functor $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\heartsuit,\otimes}$ from $(\mathrm{Sch}/k)^{\mathrm{op}}$ to the 2-category of ordinary Λ-linear symmetric monoidal categories. If we want to stress that this depends on the complex embedding σ, we will write $\underline{\mathrm{Auteq}}(\mathbf{Sh}_{\sigma\text{-geo}}^{\heartsuit,\otimes})$ instead.

Corollary 2.13. *Let k be a field and $\sigma : k \hookrightarrow \mathbb{C}$ a complex embedding. There is an equivalence of classical Picard $\mathbb{Z}$-prestacks*

$$\mathcal{G}_{\mathrm{mot}}^{\mathrm{cl}}(k, \sigma) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathbf{Sh}_{\sigma\text{-geo}}^{\heartsuit,\otimes}).$$

In particular, the right hand side is an affine group scheme.

Proof. Recall that, for an ordinary commutative ring Λ, we have $\mathcal{G}_{\mathrm{mot}}^{\mathrm{cl}}(k, \sigma)(\Lambda) = \mathcal{G}_{\mathrm{mot}}(k, \sigma)(\Lambda)$. Thus, by Theorem 2.10, it remains to construct natural equivalences of group objects in $\mathcal{S}$:

$$\begin{aligned}&\mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{CAT}_\infty)_{\mathrm{Mod}_\Lambda^{\otimes}/})}(\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}) \\ &\qquad\xrightarrow{\sim} \mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{CAT}_{\mathrm{ord}})_{\mathrm{Mod}_\Lambda^{\heartsuit,\otimes}/})}(\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\heartsuit,\otimes}).\end{aligned} \tag{2.11}$$

(Here, we write $\mathrm{CAT}_{\mathrm{ord}}$ for the full sub-∞-category of CAT_∞ spanned by ordinary categories. In particular, we only retain invertible natural transformations between functors.) We split the proof in three steps.

Step 1. Note that any autoequivalence Θ of the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,\,st}})$-valued presheaf $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}$ has to be t-exact, i.e., should respect the natural t-structures on the stable ∞-categories $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)$, for $X \in \mathrm{Sch}/k$. This follows from the following two observations:

- for any finite extension l/k, Θ induces a t-exact autoequivalence on the stable ∞-category $\mathbf{Sh}_{\mathrm{geo}}(l;\Lambda) \simeq (\mathrm{Mod}_\Lambda)^{\hom_k(l,\mathbb{C})}$ since it preserves colimits and the $\otimes$-unit,
- Θ commutes with the t-exact functors $x^* : \mathbf{Sh}_{\mathrm{geo}}(X;\Lambda) \to \mathbf{Sh}_{\mathrm{geo}}(x;\Lambda)$, for all closed points $x \in X$, and these functors can be used to detect connective and coconnective objects.

In particular, we deduce an equivalence of group objects in $\mathcal{S}$:

$$\begin{aligned}&\mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{CAT}_\infty)_{\mathrm{Mod}_\Lambda^{\otimes}/})}(\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}) \\ &\qquad\xrightarrow{\sim} \mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{CAT}_\infty)_{\mathrm{Mod}_{\Lambda,\geq 0}^{\otimes}/})}(\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)_{\geq 0}^{\otimes}).\end{aligned} \tag{2.12}$$

Moreover, for every $X \in \mathrm{Sch}/k$, the ordinary category $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit}$ can be identified with the full sub-∞-category of discrete objects in $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)_{\geq 0}$ or, equivalently, of those objects which are local for all maps between 1-connective objects. This implies the existence of a natural map of group objects in $\mathcal{S}$:

$$\begin{aligned}\mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{CAT}_\infty)_{\mathrm{Mod}^{\otimes}_{\Lambda,\geq 0}/})}(\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}_{\geq 0}) \\ \to \mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{CAT}_{\mathrm{ord}})_{\mathrm{Mod}^{\heartsuit,\otimes}_{\Lambda}/})}(\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\heartsuit,\otimes}).\end{aligned} \tag{2.13}$$

Clearly, the maps in (2.12) and (2.13) can be made functorial in the ordinary ring Λ. Thus, to finish the proof, it suffices to show that the map in (2.13) is an equivalence.

Step 2. We first note that the map in (2.13) admits a section. Indeed, a symmetric monoidal autoequivalence of the functor $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\heartsuit,\otimes}$ induces a symmetric monoidal autoequivalence of the functor $\mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\heartsuit})^{\otimes}_{\geq 0}$ which, by Theorem 1.107, is equivalent to $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}_{\geq 0}$. (This actually requires the construction of a natural transformation $\mathrm{D}(\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\heartsuit})^{\otimes} \to \mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}$ extending the functors considered in Theorem 1.107. This can be easily achieved using, for example, a model-theoretic construction of the symmetric monoidal ∞-categories of sheaves on analytic spaces.) The domain of the map in (2.13) is discrete, being equivalent to the set of Λ-points of the affine group scheme $\mathcal{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma)$ by Theorem 2.10 combined with the equivalence in (2.12). It follows that the codomain of the map in (2.13) is also discrete. Even more, we see that the ordinary presheaf on AFF sending $\mathrm{Spec}(\Lambda)$ to the group

$$\mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{CAT}_{\mathrm{ord}})_{\mathrm{Mod}^{\heartsuit,\otimes}_{\Lambda}/})}(\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\heartsuit,\otimes})$$

is representable by an affine group scheme which is a split closed subgroup of $\mathcal{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma)$. To end the proof, we need to show that this closed subscheme is dense, and for that it is enough to show that this closed subscheme and $\mathcal{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma)$ have the same Λ-points for every field Λ of characteristic zero. In this way, we are reduced to showing that the map in (2.13) is an equivalence when Λ is a field of characteristic zero. Under this assumption, the tensor products on the abelian categories $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit}$ are exact in both variables.

Step 3. The remainder of the argument consists in quoting some results from [Lur18]. Following loc. cit., we denote by Groth_∞ the ∞-category of Grothendieck prestable ∞-categories. This is the full sub-∞-category of Pr^{L} whose objects are the prestable ∞-categories in which filtered colimits are exact; see [Lur18, Proposition C.1.4.1, Definitions C.1.4.2 & C.3.0.5]. We also need the wide sub-∞-category $\mathrm{Groth}^{\mathrm{lex}}_\infty \subset \mathrm{Groth}_\infty$ spanned by those functors which are left exact (in addition to being colimit-preserving); see [Lur18, Notation C.3.2.3]. We need as well the full sub-∞-categories $\mathrm{Groth}^{\mathrm{lex,sep}}_\infty \subset \mathrm{Groth}^{\mathrm{lex}}_\infty$ and $\mathrm{Groth}^{\mathrm{lex}}_{\mathrm{ab}} \subset \mathrm{Groth}^{\mathrm{lex}}_\infty$ spanned by the separated Grothendieck prestable ∞-categories and by the Grothendieck abelian categories respectively; see [Lur18, Proposition C.3.6.1 & Definition C.5.4.1]. The ∞-categories Groth_∞, $\mathrm{Groth}^{\mathrm{lex}}_\infty$ and $\mathrm{Groth}^{\mathrm{lex}}_{\mathrm{ab}}$ have natural symmetric monoidal structures, and the inclusion functors to $\mathrm{Pr}^{\mathrm{Add}}$ are compatible with these structures; see [Lur18, Theorem C.4.2.1 & Corollary C.4.4.2]. There is a functor $(-)_{\mathrm{sep}} : \mathrm{Groth}^{\mathrm{lex}}_\infty \to \mathrm{Groth}^{\mathrm{lex,sep}}_\infty$ which is left adjoint to the obvious inclusion. By [Lur18, Corollary C.4.6.2], there is a unique symmetric monoidal structure on $\mathrm{Groth}^{\mathrm{lex,sep}}_\infty$ such that $(-)_{\mathrm{sep}}$ is symmetric monoidal. Now, by [Lur18, Theorem C.5.4.9 & Remark C.5.4.10], the construction $\mathcal{A} \to \mathrm{D}(\mathcal{A})_{\geq 0}$ defines a fully faithful functor

$$\mathrm{D}(-)_{\geq 0} : \mathrm{Groth}^{\mathrm{lex}}_{\mathrm{ab}} \to \mathrm{Groth}^{\mathrm{lex,sep}}_\infty \tag{2.14}$$

from the 2-category $\mathrm{Groth}^{\mathrm{lex}}_{\mathrm{ab}}$ of Grothendieck abelian categories and colimit-preserving exact functors. The induced functor

$$\mathrm{D}(-)_{\geq 0} : \mathrm{Mod}_{\mathrm{Mod}^{\heartsuit}_{\mathbb{Q}}}(\mathrm{Groth}^{\mathrm{lex}}_{\mathrm{ab}}) \to \mathrm{Mod}_{\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Q}}}(\mathrm{Groth}^{\mathrm{lex,\,sep}}_{\infty}) \tag{2.15}$$

is symmetric monoidal, i.e., given two $\mathbb{Q}$-linear Grothendieck abelian categories $\mathcal{A}$ and $\mathcal{A}'$, the natural functor

$$\mathrm{D}(\mathcal{A})_{\geq 0} \otimes \mathrm{D}(\mathcal{A}')_{\geq 0} \to \mathrm{D}(\mathcal{A} \otimes \mathcal{A}')_{\geq 0}$$

is an equivalence. To see this, we use [Lur18, Corollary C.2.1.8] to view $\mathcal{A}$ and $\mathcal{A}'$ as exact localisations of $\mathrm{Mod}^{\heartsuit}_{R}$ and $\mathrm{Mod}^{\heartsuit}_{R'}$, where R and R' are ordinary $\mathbb{Q}$-algebras. In this case, $\mathrm{D}(\mathcal{A})_{\geq 0}$ and $\mathrm{D}(\mathcal{A}')_{\geq 0}$ are exact localisations of $\mathrm{Mod}^{\mathrm{cn}}_{R}$ and $\mathrm{Mod}^{\mathrm{cn}}_{R'}$. The result then follows from the equivalence $\mathrm{Mod}^{\mathrm{cn}}_{R} \otimes \mathrm{Mod}^{\mathrm{cn}}_{R'} \simeq \mathrm{Mod}^{\mathrm{cn}}_{R\otimes R'}$, noting that $R \otimes R'$ is an ordinary $\mathbb{Q}$-algebra (since R and R' are flat over $\mathbb{Q}$). Having said all this, it is now easy to conclude.

From the fully faithful symmetric monoidal embedding in (2.15), we obtain a fully faithful embedding

$$\mathrm{CAlg}(\mathrm{Groth}^{\mathrm{lex}}_{\mathrm{ab}})_{\mathrm{Mod}^{\heartsuit}_{\mathbb{Q}}/} \to \mathrm{CAlg}(\mathrm{Groth}^{\mathrm{lex,\,sep}}_{\infty})_{\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Q}}/}. \tag{2.16}$$

The functor $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\heartsuit,\otimes}$ can be considered as a $\mathrm{CAlg}(\mathrm{Groth}^{\mathrm{lex}}_{\mathrm{ab}})_{\mathrm{Mod}^{\heartsuit}_{\mathbb{Q}}/}$-valued presheaf. Similarly, the functor $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}_{\geq 0}$ can be considered as a $\mathrm{CAlg}(\mathrm{Groth}^{\mathrm{lex,\,sep}}_{\infty})_{\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Q}}/}$-valued presheaf. Moreover, the latter is obtained from the former by composing with the fully faithful embedding in (2.16). Thus, we obtain an equivalence of group objects in $\mathcal{S}$:

$$\begin{aligned} &\mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{Groth}^{\mathrm{lex,\,sep}}_{\infty})_{\mathrm{Mod}^{\mathrm{cn}}_{\Lambda}/})}(\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}_{\geq 0}) \\ &\qquad \to \mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{Groth}^{\mathrm{lex}}_{\mathrm{ab}})_{\mathrm{Mod}^{\heartsuit}_{\Lambda}/})}(\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\heartsuit,\otimes}). \end{aligned} \tag{2.17}$$

Since any autoequivalence of $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)_{\geq 0}$ (resp. $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit}$) is colimit-preserving and left exact, we see that the map in (2.13) coincides with the one in (2.17). □

By base change to positive characteristic rings, one obtains the following particular case of Corollary 2.13.

Corollary 2.14. *Let k be a field of charactersitic zero, $\overline{k}/k$ an algebraic closure of k and Λ a torsion connected ring. Consider the functor $\overline{\mathcal{F}}(-;\Lambda) : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAT}_{\mathrm{ord}}$ sending a k-variety X to the ordinary category of étale sheaves on $X \otimes_k \overline{k}$ with coefficients in Λ. Then, there is an equivalence of Picard groupoids*

$$\mathcal{G}(\overline{k}/k) \simeq \mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{CAT}_{\mathrm{ord}}))}(\overline{\mathcal{F}}(-;\Lambda)^{\otimes}).$$

In particular, the right hand side is discrete.

Proof. This follows immediately from Corollary 2.13. Indeed, fix a complex embedding $\overline{\sigma} : \overline{k} \to \mathbb{C}$ and set $\sigma = \overline{\sigma}|_k$. By Theorem 1.70, $\mathcal{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma;\Lambda) \simeq \mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda)$ is isomorphic to the constant group Λ-scheme associated to the profinite group $\mathcal{G}(\overline{k}/k)$. On the other hand, for every $X \in \mathrm{Sch}/k$, the category $\overline{\mathcal{F}}(X;\Lambda)$ is equivalent $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\heartsuit}$. (To prove this, one reduces to the case where Λ is finite and use [SGA73, Exposé XI, Théorème 4.4(i)].) □

Remark 2.15. The proof we gave of Corollary 2.14 is not satisfactory. Indeed, one can obtain an elementary and more direct argument, which is moreover valid for an arbitrary field k and an arbitrary commutative ring Λ, by following the same path used in proving Theorem 2.10. More precisely, one replaces the Drew–Gallauer universality theorem for $\mathbf{MSh}(-;\Lambda)$ (i.e., Theorem 2.8) by a similar one for the functor $\mathcal{F}(-;\Lambda)$ sending a k-variety X to the category of étale sheaves of Λ-modules on X. Then, one interprets $\overline{\mathcal{F}}(X;\Lambda)$ as the category of modules in $\mathcal{F}(X;\Lambda)$ over $\mathcal{C}^0(\mathcal{G}(\bar{k}/k);\Lambda)$ seen as an algebra object of $\mathcal{F}(k;\Lambda)$. This reduces the computation of the autoequivalence groupoids of $\overline{\mathcal{F}}(-;\Lambda)$ to the automorphism group of an algebra object of $\mathcal{F}(k;\Lambda)$ which is more manageable. We leave the details to the interested reader.

2.3. **A complement to the first main theorem.**

In this subsection, we derive an interesting complement to our main theorem for constructible sheaves, that is Theorem 2.10. We first start by reformulating this theorem.

Notation 2.16. Let k be a field and $\sigma : k \hookrightarrow \mathbb{C}$ a complex embedding. We denote by $\mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ the spectral $\mathbb{S}$-prestack sending $\mathrm{Spec}(\Lambda) \in \mathrm{SpAFF}$ to the space

$$\mathrm{B}(\mathcal{G}_{\mathrm{mot}}(k,\sigma)(\Lambda)) = \operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathcal{G}_{\mathrm{mot}}(k,\sigma)_n(\Lambda).$$

Below, we will consider $\mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ as a presheaf on SpAFF with values in ∞-groupoids and, in particular, as a Cat_∞-valued presheaf. Given a connective commutative ring spectrum Λ, we also denote by $\mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma;\Lambda)$ the restriction of $\mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ to $\mathrm{SpAFF}_{/\Lambda}$.

Notation 2.17. We denote by $\mathrm{LinPr}^\otimes$ the $\mathrm{CAlg}(\mathrm{CAT}_\infty)$-valued presheaf on SpAFF sending $\mathrm{Spec}(\Lambda)$ to the symmetric monoidal ∞-category $\mathrm{LinPr}(\Lambda)^\otimes = \mathrm{Mod}_{\mathrm{Mod}^\otimes_\Lambda}(\mathrm{Pr}^{\mathrm{L}})^\otimes$ of presentable ∞-categories tensored over the symmetric monoidal ∞-category $\mathrm{Mod}^\otimes_\Lambda$. We deduce from this a CAT_∞-valued presheaf $\mathrm{CAlg}(\mathrm{LinPr})$ on SpAFF sending $\mathrm{Spec}(\Lambda)$ to the ∞-category $\mathrm{CAlg}(\mathrm{LinPr}(\Lambda))$ which we identify with the ∞-category $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^\otimes_\Lambda/}$ of presentable symmetric monoidal ∞-categories endowed with a symmetric monoidal functor from $\mathrm{Mod}^\otimes_\Lambda$. (This identification follows from [Lur17, Corollary 3.4.1.7].) Below, we simply write

$$\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{LinPr}))$$

for the CAT_∞-valued presheaf on SpAFF sending $\mathrm{Spec}(\Lambda)$ to the ∞-category

$$\mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{LinPr}(\Lambda))).$$

Note that we have a natural transformation

$$\mathrm{pt} \to \mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{LinPr})) \tag{2.18}$$

pointing at $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^\otimes$ for every $\mathrm{Spec}(\Lambda) \in \mathrm{SpAFF}$.

Theorem 2.18. *Let k be a field and $\sigma : k \hookrightarrow \mathbb{C}$ a complex embedding. There is a morphism of* CAT_∞*-valued presheaves on* SpAFF*:*

$$\mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma) \to \mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{LinPr})) \tag{2.19}$$

such that, for every $\mathrm{Spec}(\Lambda) \in \mathrm{SpAFF}$*, the functor*

$$\mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma)(\Lambda) \to \mathrm{Psh}(\mathrm{Sch}/k;\mathrm{CAlg}(\mathrm{LinPr}(\Lambda)))$$

induces an equivalence between its domain and the full sub-∞-groupoid of its codomain spanned by the object $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^\otimes$*. In particular, the morphism in* (2.18) *factors through the morphism in* (2.19).

Proof. It follows immediately from Construction 1.51 and Remark 1.50 that there is a morphism of CAT_∞-valued presheaves on SpAFF:

$$\mathrm{B}(\underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo}})) \to \mathrm{Psh}(\mathrm{Sch}/k; \mathrm{CAlg}(\mathrm{LinPr}))$$

with the required property. Thus, the result follows from Theorem 2.10. $\square$

Construction 2.19. The morphism in (2.19) determines a morphism of CAT_∞-valued presheaves on SpAFF:

$$(\mathrm{Sch}/k)^{\mathrm{op}} \times \mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma) \to \mathrm{CAlg}(\mathrm{LinPr}). \tag{2.20}$$

Roughly speaking, at $\mathrm{Spec}(\Lambda) \in \mathrm{SpAFF}$, this morphism is given by the functor sending a pair $(X, \star)$ consisting of $X \in \mathrm{Sch}/k$ and the base point $\star$ of $\mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma)(\Lambda)$ to $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\otimes}$. Applying Lurie's unstraightening [Lur09a, §3.2] to the functor $\mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma)$, we obtain the coCartesian fibration

$$p : \Phi = \int_{(\mathrm{SpAFF})^{\mathrm{op}}} \mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma) \to (\mathrm{SpAFF})^{\mathrm{op}},$$

which is in fact a left fibration. Similarly, applying Lurie's unstraightening [Lur09a, §3.2] to the functor CAlg(LinPr), we obtain a coCartesian fibration

$$q : \Psi = \int_{(\mathrm{SpAFF})^{\mathrm{op}}} \mathrm{CAlg}(\mathrm{LinPr}) \to (\mathrm{SpAFF})^{\mathrm{op}}.$$

The morphism in (2.20) induces a commutative triangle

$$\begin{tikzcd} (\mathrm{Sch}/k)^{\mathrm{op}} \times \Phi \arrow[rr, "h"] \arrow[dr, "p\circ r"'] & & \Psi \arrow[dl, "q"] \\ & (\mathrm{SpAFF})^{\mathrm{op}} & \end{tikzcd}$$

with $r : (\mathrm{Sch}/k)^{\mathrm{op}} \times \Phi \to \Phi$ the projection to the second factor and h a functor preserving coCartesian edges. There is a functor $l : \Psi \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ whose restriction to the fiber at $\mathrm{Spec}(\Lambda)$ is the obvious forgetful functor $\mathrm{CAlg}(\mathrm{LinPr}(\Lambda)) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. (We leave it to the reader to construct the functor l.) Consider the composite functor

$$l \circ h : (\mathrm{Sch}/k)^{\mathrm{op}} \times \Phi \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}). \tag{2.21}$$

Roughly speaking, this functor sends a triple $(X, \Lambda, \star)$, with $X \in \mathrm{Sch}/k$, $\mathrm{Spec}(\Lambda) \in \mathrm{SpAFF}$ and $\star$ the base point of $\mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma)(\Lambda)$, to the symmetric monoidal ∞-category $\mathbf{Sh}_{\mathrm{geo}}(X;\Lambda)^{\otimes}$. By adjunction, we obtain a functor

$$(\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Fun}(\Phi, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})). \tag{2.22}$$

We denote by

$$\mathbf{Sh}^{\mathcal{G}_{\mathrm{mot}}}_{\mathrm{geo}}(-)^{\otimes} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \tag{2.23}$$

the functor obtained from the one in (2.22) by composition with the limit functor

$$\lim_{\Phi} : \mathrm{Fun}(\Phi, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

Informally, for $X \in \mathrm{Sch}/k$, an object of the ∞-category $\mathbf{Sh}^{\mathcal{G}_{\mathrm{mot}}}_{\mathrm{geo}}(X)$ is an object $M \in \mathbf{Sh}_{\mathrm{geo}}(X)$ endowed with compatible equivalences $M \otimes \Lambda \simeq \gamma^*(M \otimes \Lambda)$ for every $\mathrm{Spec}(\Lambda) \in \mathrm{SpAFF}$ and $\gamma \in \mathcal{G}_{\mathrm{mot}}(k,\sigma)(\Lambda)$. Said differently, $\mathbf{Sh}^{\mathcal{G}_{\mathrm{mot}}}_{\mathrm{geo}}(X)$ is the ∞-category of geometric sheaves fixed by the action of $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ on the ∞-category $\mathbf{Sh}_{\mathrm{geo}}(X)$.

Proposition 2.20. *The functor* $\mathbf{Sh}_{\mathrm{geo}}^{\mathcal{G}_{\mathrm{mot}}}(-)^{\otimes}$ *in* (2.23) *is a presentable Voevodsky pullback formalism. Moreover, the forgetful functors yield a natural transformation*

$$\mathrm{ff} : \mathbf{Sh}_{\mathrm{geo}}^{\mathcal{G}_{\mathrm{mot}}}(-)^{\otimes} \to \mathbf{Sh}_{\mathrm{geo}}(-)^{\otimes} \tag{2.24}$$

which is a strong morphism of presentable Voevodsky pullback formalisms.

Proof. This is a consequence of the fact that limits of ∞-categories have excellent formal properties. Inspecting Definition 1.14, we see that it suffices to show that the squares

$$\begin{tikzcd} \mathbf{Sh}_{\mathrm{geo}}^{\mathcal{G}_{\mathrm{mot}}}(X) \arrow[r, "f^*"] \arrow[d, "\mathrm{ff}_X"] & \mathbf{Sh}_{\mathrm{geo}}^{\mathcal{G}_{\mathrm{mot}}}(Y) \arrow[d, "\mathrm{ff}_Y"] \\ \mathbf{Sh}_{\mathrm{geo}}(X) \arrow[r, "f^*"] & \mathbf{Sh}_{\mathrm{geo}}(Y) \end{tikzcd} \tag{2.25}$$

are right adjointable for all morphisms $f : Y \to X$ in Sch/k and left adjointable for all smooth ones. This follows easily from [Lur17, Corollary 4.7.4.18]. Indeed, the functor

$$\Phi \to \mathrm{Fun}(\Delta^1, \mathrm{CAT}_\infty), \tag{2.26}$$

deduced from (2.21) by restricting along $(f, \mathrm{id}) : \Delta^1 \times \Phi \to (\mathrm{Sch}/k)^{\mathrm{op}} \times \Phi$ and using adjunction, factors through $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{CAT}_\infty)$; see [Lur17, Definition 4.7.4.16]. (This follows form the fact that the $\mathrm{B}\mathcal{G}_{\mathrm{mot}}(k,\sigma)(\Lambda)$'s are ∞-groupoids and that the operation f_* on sheaves of geometric origin commutes with extension of scalars.) Consider a limit diagram

$$\Phi^{\triangleleft} \to \mathrm{Fun}(\Delta^1, \mathrm{CAT}_\infty) \tag{2.27}$$

extending the diagram in (2.26). By construction, the cone point of $\Phi^{\triangleleft}$ is mapped to the functor $f^* : \mathbf{Sh}_{\mathrm{geo}}^{\mathcal{G}_{\mathrm{mot}}}(X) \to \mathbf{Sh}_{\mathrm{geo}}^{\mathcal{G}_{\mathrm{mot}}}(Y)$. By [Lur17, Corollary 4.7.4.18], the diagram in (2.27) factors through $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{CAT}_\infty)$. Since the square in (2.25) is the image of the edge relating the cone point of $\Phi^{\triangleleft}$ to the object $(\mathrm{Spec}(\mathbb{S}), \star)$ of Φ, the result follows. When f is smooth, the functor in (2.26) factors through $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{CAT}_\infty)$, and we may conclude similarly. □

Corollary 2.21. *There is a strong morphism of presentable Voevodsky pullback formalisms*

$$\mathrm{B}^{\mathcal{G}_{\mathrm{mot}}} : \mathbf{MSh}(-)^{\otimes} \to \mathbf{Sh}_{\mathrm{geo}}^{\mathcal{G}_{\mathrm{mot}}}(-)^{\otimes}.$$

Proof. This follows from Proposition 2.20 and the Drew–Gallauer universality theorem (i.e., Theorem 2.5). □

Remark 2.22. It is expected that the functor $\mathrm{B}^{\mathcal{G}_{\mathrm{mot}}} : \mathbf{MSh}(X) \to \mathbf{Sh}_{\mathrm{geo}}^{\mathcal{G}_{\mathrm{mot}}}(X)$, for $X \in \mathrm{Sch}/k$, is very close to being an equivalence of ∞-categories. More precisely, it is expected to induce an equivalence between the ∞-category of constructible motivic sheaves $\mathbf{MCt}(X)$ and the ∞-category $\mathbf{Ct}_{\mathrm{geo}}^{\mathcal{G}_{\mathrm{mot}}}(X)$ of constructible sheaves of geometric origin fixed by $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ acting on $\mathbf{Ct}_{\mathrm{geo}}(X)$. This property certainly implies the conservativity conjecture (see for example [Ayo17b, §2.1]), and it is plausible that both statements are actually equivalent. We do not know how to prove "formally" the equivalence of these two statements, but see [Pri20, §2.5].

Since $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ is affine, it is possible to describe the ∞-categories $\mathbf{Sh}_{\mathrm{geo}}^{\mathcal{G}_{\mathrm{mot}}}(-)^{\otimes}$ more simply. This is explained in the following remark.

Remark 2.23. We write $\mathscr{G}_{\mathrm{mot}}(k,\sigma)_\bullet$ for the simplicial object in SpAFF defining the spectral group $\mathbb{S}$-scheme $\mathscr{G}_{\mathrm{mot}}(k,\sigma)$. By construction, $\mathrm{B}\mathscr{G}_{\mathrm{mot}}(k,\sigma)$ is the colimit of $\mathscr{G}_{\mathrm{mot}}(k,\sigma)_\bullet$ taken in the ∞-category of $\mathbb{S}$-prestacks, i.e., of presheaves on SpAFF. Thus, with the notation of Construction 2.19, the cosimplicial diagram $\mathscr{G}_{\mathrm{mot}}(k,\sigma)_\bullet : \mathbf{\Delta} \to (\mathrm{SpAFF})^{\mathrm{op}}$ admits a canonical lift to Φ, i.e., there is a commutative triangle

$$\begin{tikzcd} & \Phi \arrow[d, "p"] \\ \mathbf{\Delta} \arrow[ur, "\rho"] \arrow[r, "\mathscr{G}_{\mathrm{mot}}(k,\sigma)_\bullet"'] & (\mathrm{SpAFF})^{\mathrm{op}}. \end{tikzcd}$$

Moreover, ρ induces an equivalence between the colimits of the diagrams p and $\mathscr{G}_{\mathrm{mot}}(k,\sigma)_\bullet$ taken in the ∞-category of $\mathbb{S}$-prestacks. Thus, composing ρ with the functor in (2.21), one obtains a functor

$$\mathbf{Sh}_{\mathrm{geo}}(-;\mathscr{O}(\mathscr{G}_{\mathrm{mot}}(k,\sigma)_\bullet))^{\otimes} : (\mathrm{Sch}/k)^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \tag{2.28}$$

and an equivalence

$$\mathbf{Sh}_{\mathrm{geo}}^{\mathscr{G}_{\mathrm{mot}}}(-)^{\otimes} \simeq \lim_{[n]\in\mathbf{\Delta}} \mathbf{Sh}_{\mathrm{geo}}(-;\mathscr{O}(\mathscr{G}_{\mathrm{mot}}(k,\sigma)_n))^{\otimes} \tag{2.29}$$

of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves on Sch/k. It is important to note here that the functor in (2.28) is by no mean the obvious one obtained by applying $\mathbf{Sh}_{\mathrm{geo}}(-;-)^{\otimes}$ to the cosimplicial ring spectrum $\mathscr{O}(\mathscr{G}_{\mathrm{mot}}(k,\sigma)_\bullet)$. In fact, this functor encodes the action of $\mathscr{G}_{\mathrm{mot}}(k,\sigma)$ on $\mathbf{Sh}_{\mathrm{geo}}(-)^{\otimes}$.

Remark 2.24. There is also an "ordinary" version of the previous results, where $\mathscr{G}_{\mathrm{mot}}(k,\sigma)$ is replaced with its underlying ordinary group scheme $\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma)$. More precisely, replacing $\mathscr{G}_{\mathrm{mot}}(k,\sigma)$ by $\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma)$ in Construction 2.19, we obtain a functor

$$\mathbf{Sh}_{\mathrm{geo}}^{\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}}(-)^{\otimes} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}). \tag{2.30}$$

Arguing as in Proposition 2.20, we see that this functor is a presentable Voevodsky pullback formalism, and the obvious natural transformation

$$\mathbf{Sh}_{\mathrm{geo}}^{\mathscr{G}_{\mathrm{mot}}}(-)^{\otimes} \to \mathbf{Sh}_{\mathrm{geo}}^{\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}}(-)^{\otimes} \tag{2.31}$$

is a strong morphism of presentable Voevodsky pullback formalisms. Arguing as in Remark 2.23, we also obtain the following simpler description

$$\mathbf{Sh}_{\mathrm{geo}}^{\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}}(-)^{\otimes} \simeq \lim_{[n]\in\mathbf{\Delta}} \mathbf{Sh}_{\mathrm{geo}}(-;\mathscr{O}(\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma)_n))^{\otimes}. \tag{2.32}$$

Let $\mathbf{\Delta}' \subset \mathbf{\Delta}$ be the wide subcategory of strictly increasing maps. The obvious inclusion is coinitial by [Lur09a, Lemma 6.5.3.7], and thus we also have an equivalence

$$\mathbf{Sh}_{\mathrm{geo}}^{\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}}(-;\Lambda)^{\otimes} \simeq \lim_{[n]\in\mathbf{\Delta}'} \mathbf{Sh}_{\mathrm{geo}}(-;\mathscr{O}(\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma;\Lambda)_n))^{\otimes}. \tag{2.33}$$

(Here, the left hand side is defined as in Construction 2.19 but working Λ-linearly.) Since $\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma;\Lambda)$ is a flat affine group Λ-scheme, we see that the semi-cosimplicial diagram

$$\mathbf{Sh}_{\mathrm{geo}}(-;\mathscr{O}(\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma;\Lambda)_\bullet))^{\otimes} : \mathbf{\Delta}' \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

lifts to a diagram of stable ∞-categories with t-structures and t-exact functors. It follows that $\mathbf{Sh}_{\mathrm{geo}}^{\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}}(X;\Lambda)$ admits a t-structure such that

$$\mathbf{Sh}_{\mathrm{geo}}^{\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}}(X;\Lambda)^{\heartsuit} = \lim_{[n]\in\mathbf{\Delta}'} \mathbf{Sh}_{\mathrm{geo}}(-;\mathscr{O}(\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma;\Lambda)_n))^{\heartsuit}.$$

By the main result of [CG17], the abelian category $\mathbf{Sh}_{\mathrm{geo}}^{\mathcal{G}^{\mathrm{cl}}_{\mathrm{mot}}}(k;\mathbb{Q})^{\heartsuit}$ is equivalent to the indization of the abelian category of Nori motives. Thus, for $X \in \mathrm{Sch}/k$, an object of $\mathbf{Sh}_{\mathrm{geo}}^{\mathcal{G}^{\mathrm{cl}}_{\mathrm{mot}}}(X;\mathbb{Q})^{\heartsuit}$ whose underlying sheaf is constructible, is entitled to be called a Nori motivic sheaf on X. See [Ara13], [Ara20], [Ivo17] and [IM19] for other approaches.

3. Monodromic specialisation, stratification and exit-path

In this section, we develop a machinery which we use in Section 4 to extract from Theorem 2.10 a description of $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ as a group of autoequivalences of a functor $\mathbf{LS}^{\circ}_{\mathrm{geo}}(-)^{\otimes}$ taking values in ∞-categories of local systems of geometric origin. (For a precise statement, see Theorem 4.37.) In fact, this machinery allows for a description of the ∞-categories $\mathbf{Sh}_{\mathrm{geo}}(X)$, for $X \in \mathrm{Sch}/k$, in terms of ∞-categories of local systems of geometric origin in a highly structured and "coordinate free" manner. We expect this machinery to be also useful in other contexts.

3.1. Regularly stratified varieties and deformations to normal cones.

In this subsection, we gather some geometric constructions needed in the remainder of this section. To fix ideas, we start by recalling the notion of a stratification.

Definition 3.1. Let X be a noetherian spectral space (see for example [Sta18, Tag 08YF]). A stratification $\mathcal{P}$ of X is a set of connected and locally closed subspaces of X, called $\mathcal{P}$-strata, such that the following condition is satisfied.

(i) The $\mathcal{P}$-strata form a partition of X, i.e., we have a set-theoretic decomposition $X = \coprod_{S\in\mathcal{P}} S$.
(ii) The closure of a $\mathcal{P}$-stratum is a union of $\mathcal{P}$-strata.

A subset $C \subset X$ is called $\mathcal{P}$-constructible if it is a union of $\mathcal{P}$-strata.

Remark 3.2. Let X be a noetherian spectral space.

(i) Let $\mathcal{P}$ be a stratification of X. The set $\mathcal{P}$ of $\mathcal{P}$-strata is finite. We define a partial order $\leq$ on $\mathcal{P}$ by setting $T \leq S$ if $T \subset \overline{S}$. A $\mathcal{P}$-stratum is maximal for this order if and only if it is open. Moreover, the union of open $\mathcal{P}$-strata is dense in X.
(ii) Let $\mathcal{P}$ and $\mathcal{Q}$ be two stratifications on X. We say that $\mathcal{Q}$ is finer than $\mathcal{P}$ is every $\mathcal{P}$-constructible subset is $\mathcal{Q}$-constructible. More generally, a continuous map $f : Y \to X$ of noetherian spectral spaces is said to be compatible with stratifications $\mathcal{P}$ and $\mathcal{Q}$ on X and Y if the inverse image of a $\mathcal{P}$-constructible subset is $\mathcal{Q}$-constructible. In this case, there is an induced map $f_* : \mathcal{Q} \to \mathcal{P}$ sending a $\mathcal{Q}$-stratum D to the unique $\mathcal{P}$-stratum C such that $f(D) \subset C$.
(iii) Given a finite family $(Z_i)_{i\in I}$ of closed subsets of X, there is a coarsest stratification on X for which the Z_i's are constructible. The strata of this stratification are the connected components of the subsets $(\bigcap_{j\in J} Z_j) \smallsetminus (\bigcup_{i\in I\smallsetminus J} Z_i)$ for $J \subset I$.

Definition 3.3. A stratified scheme is a pair $(X, \mathcal{P}_X)$ consisting of a noetherian scheme X and a stratification $\mathcal{P}_X$ of the topological space underlying X, which we call the structural stratification. Whenever possible, we shall omit the mention of the structural stratification $\mathcal{P}_X$. A morphism of stratified schemes $f : Y \to X$ is a morphism of schemes which is compatible with the structural stratifications (as in Remark 3.2(ii)).

Remark 3.4.

(i) If no confusion can arise, the $\mathscr{P}_X$-strata of a stratified scheme X are simply called the strata of X. Similarly, the $\mathscr{P}_X$-constructible subsets of X are simply called the constructible subsets of X.
(ii) Let S be a base scheme. By the expression "stratified S-scheme", we mean a stratified scheme whose underlying scheme is endowed with a morphism to S. Similarly for the expression "morphism of stratified S-schemes", etc.

Notation 3.5. Let X be a stratified scheme. We denote by X° the union of the open strata of X. As said in Remark 3.2(i), this is a dense open subscheme of X.

Definition 3.6.
(i) Let X be a regular noetherian scheme. A stratification $\mathscr{P}$ on X is said to be regular if there exists a strict normal crossing divisor D on X whose irreducible components are $\mathscr{P}$-constructible and such that $\mathscr{P}$ is the coarsest stratification on X with this property. (Said differently, if $(D_i)_{\in I}$ are the irreducible components of D, then the $\mathscr{P}$-strata are the connected components of the subsets $(\bigcap_{j\in J} D_j) \smallsetminus (\bigcup_{i\in I\smallsetminus J} D_i)$, for $J \subset I$, as in Remark 3.2(iii).)
(i′) A regularly stratified scheme X is a stratified scheme X whose underlying scheme is regular and whose structural stratification $\mathscr{P}_X$ is also regular.
(ii) Let S be a noetherian scheme and X a smooth S-scheme. A stratification $\mathscr{P}$ on X is said to be smooth (over S) if there exists a relative strict normal crossing divisor D on X which is a union of $\mathscr{P}$-constructible smooth divisors and such that $\mathscr{P}$ is the coarsest stratification on X with this property.
(ii′) Let S be a noetherian scheme. A smoothly stratified S-scheme is a stratified S-scheme X whose underlying S-scheme is smooth and whose structural stratification $\mathscr{P}_X$ is also smooth.

Notation 3.7. We denote by $\mathrm{SCH}\text{-}\Sigma$ the category of stratified schemes and $\mathrm{REG}\text{-}\Sigma$ its full subcategory of regularly stratified schemes. Let S be a noetherian scheme. We denote by $\mathrm{Sch}\text{-}\Sigma/S$ the category of finite type stratified S-schemes. We denote by $\mathrm{Reg}\text{-}\Sigma/S$ (resp. $\mathrm{Sm}\text{-}\Sigma/S$) the full subcategory of $\mathrm{Sch}\text{-}\Sigma/S$ spanned by the regularly stratified S-schemes (resp. the smoothly stratified S-schemes). If A is a ring and $S = \mathrm{Spec}(A)$, we write $\mathrm{Sch}\text{-}\Sigma/A$, $\mathrm{Reg}\text{-}\Sigma/A$ and $\mathrm{Sm}\text{-}\Sigma/A$ instead. Note that for a perfect field k, we have $\mathrm{Sm}\text{-}\Sigma/k = \mathrm{Reg}\text{-}\Sigma/k$.

Our next task is to introduce a version of the classical deformation to the normal cone which plays an important role in the whole section. This is the subject of Construction 3.10 below; the relation with the classical deformation to the normal cone is explained in Remark 3.12. We start by introducing some useful notations.

Notation 3.8. Let X be a regularly stratified scheme and let C be a stratum of X. We denote by $\mathrm{R}^\circ_X(C)$ the group of Cartier divisors of X freely generated by the constructible irreducible divisors of X containing C. (Said differently, $\mathrm{R}^\circ_X(C)$ is a lattice having a basis indexed by the 1-codimensional strata D in X such that $C \leq D$.) We denote by $\mathrm{R}_X(C) \subset \mathrm{R}^\circ_X(C)$ the submonoid of effective Cartier divisors. We set

$$\mathrm{T}^\circ_X(C) = \mathrm{Spec}(\mathbb{Z}[\mathbf{t}^\nu;\ \nu \in \mathrm{R}^\circ_X(C)]) \qquad \text{and} \qquad \mathrm{T}_X(C) = \mathrm{Spec}(\mathbb{Z}[\mathbf{t}^\nu;\ \nu \in \mathrm{R}_X(C)]).$$

(Here, the $\mathbf{t}^\nu$'s are monomials such that $\mathbf{t}^\nu \cdot \mathbf{t}^{\nu'} = \mathbf{t}^{\nu+\nu'}$.) Thus, $\mathrm{T}^\circ_X(C)$ is the split torus dual to the lattice $\mathrm{R}^\circ_X(C)$ and we have an equivariant embedding $\mathrm{T}^\circ_X(C) \hookrightarrow \mathrm{T}_X(C)$. Such an embedding will be called a split torus-embedding; see Definition 3.104 below. In fact, $\mathrm{T}_X(C)$ is isomorphic to $\mathbb{A}^c$, with c the codimension of C in X, and $\mathrm{T}^\circ_X(C)$ corresponds to the complement of the union of the coordinate hyperplanes. We use this to consider $\mathrm{T}_X(C)$ as a regularly stratified scheme, with open stratum $\mathrm{T}^\circ_X(C)$. We denote by o_C he unique closed stratum of $\mathrm{T}_X(C)$; it is isomorphic to $\mathrm{Spec}(\mathbb{Z})$. Often, when working over a base scheme S, we continue writing $\mathrm{T}^\circ_X(C)$, $\mathrm{T}_X(C)$ and o_C for the base change of these schemes to S.

Notation 3.9. Let X be a regular scheme. Given a reduced divisor D on X, we denote as usual by $\mathcal{O}_X(D)$ the fractional ideal associated to D, i.e., the inverse of the ideal of $\mathcal{O}_X$ defining D. More generally, if $v = e_1 \cdot D_1 + \ldots + e_n \cdot D_n$ is a Cartier divisor on X, such that the D_i's are reduced, we set $\mathcal{O}_X(v) = \mathcal{O}_X(D_1)^{e_1} \cdots \mathcal{O}_X(D_n)^{e_n}$. (Note that $\mathcal{O}_X(v)$ is an ideal of $\mathcal{O}_X$ if and only if $-v$ is effective.)

Construction 3.10. Let X be a regularly stratified scheme and let C be a stratum of X. Assuming that X is connected, we set

$$\mathrm{Df}^\circ_X(C) = X^\circ \times \mathrm{T}^\circ_X(C) \qquad \text{and} \qquad \mathrm{Df}_X(C) = \mathrm{Spec}\left(\bigoplus_{v \in \mathrm{R}^\circ_X(C)} (\mathcal{O}_X(v) \cap \mathcal{O}_X) \cdot \mathbf{t}^v\right). \tag{3.1}$$

By construction, we have an evident morphism $\mathrm{Df}_X(C) \to X \times \mathrm{T}_X(C)$, and Cartesian squares

$$\begin{tikzcd} \mathrm{Df}^\circ_X(C) \arrow[r] \arrow[d] & \mathrm{Df}_X(C) \arrow[d] \\ X^\circ \arrow[r] & X \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X \times \mathrm{T}^\circ_X(C) \arrow[r] \arrow[d] & \mathrm{Df}_X(C) \arrow[d] \\ \mathrm{T}^\circ_X(C) \arrow[r] & \mathrm{T}_X(C). \end{tikzcd}$$

The scheme $\mathrm{Df}_X(C)$ is regular and the open subscheme $\mathrm{Df}^\circ_X(C) \subset \mathrm{Df}_X(C)$ is the complement of a strict normal crossing divisor. We use this to make $\mathrm{Df}_X(C)$ into a regularly stratified scheme so that $\mathrm{Df}^\circ_X(C)$ is its open stratum. Note also that the obvious action of $\mathrm{T}^\circ_X(C)$ on $\mathrm{Df}^\circ_X(C)$ extends to an action on $\mathrm{Df}_X(C)$. (Indeed, $\mathrm{Df}_X(C)$ is the spectrum of a $\mathrm{R}^\circ_X(C)$-graded $\mathcal{O}_X$-algebra.)

If X is no longer assumed to be connected, we set $\mathrm{Df}_X(C) = \mathrm{Df}_{X'}(C)$ where X' is the connected component of X containing C.

We also introduce a constructible open subscheme of $\mathrm{Df}(X)$, which we will need later on.

Construction 3.11. Keep the assumptions and notations as in Construction 3.10. Let $D \subset X$ be an irreducible constructible divisor containing C, corresponding to an element $v \in \mathrm{R}_X(C)$. The ideal of $\mathcal{O}_{\mathrm{Df}_X(C)}$ generated by the sections of $\mathcal{O}_X(-v) \cdot \mathbf{t}^{-v}$ is the ideal of an irreducible smooth divisor $D^\flat \subset \mathrm{Df}_X(C)$, namely the closure of $D \times \mathrm{T}^\circ_X(C)$ in $\mathrm{Df}_X(C)$. We define the constructible open subset $\mathrm{Df}^\flat_X(C) \subset \mathrm{Df}_X(C)$ as the complement of the divisors $D^\flat$, for $D \subset X$ irreducible constructible and containing C. Using the second equality in (3.1), we readily obtain the following description

$$\mathrm{Df}^\flat_X(C) = \mathrm{Spec}\left(\bigoplus_{v \in \mathrm{R}^\circ_X(C)} \mathcal{O}_X(v) \cdot \mathbf{t}^v\right). \tag{3.2}$$

Thus, the X-scheme $\mathrm{Df}^\flat_X(C)$ is a torsor over T. In particular, the morphism $\mathrm{Df}^\flat_X(C) \to X$ is smooth.

Remark 3.12. Keep the notation as in Construction 3.10 and assume that X is connected. Let $D_1, \ldots, D_c$ be the irreductible constructible divisors containing C. For $1 \leq i \leq c$, we have the classical deformation to the normal cone of D_i, which we denote by W_i. Recall that W_i is the complement of the strict transform of $\mathrm{o} \times X$ in the blowup of $\mathbb{A}^1 \times X$ along $\mathrm{o} \times D_i$, i.e.,

$$W_i = \operatorname{Spec}\left(\mathcal{O}_X[t] \oplus \bigoplus_{n \geq 1} \mathcal{O}_X(-nD_i) \cdot t^{-n}\right).$$

It follows immediately that $\mathrm{Df}_X(C)$ is the fiber product of the W_i's over X. Also, letting $\widetilde{D}_i$ be the strict transform of $\mathbb{A}^1 \times D_i$ in W_i, the open subscheme $\mathrm{Df}_X^\flat(C)$ corresponds to the fiber product of the $W_i' \smallsetminus \widetilde{D}_i$'s. The main reason for having introduced the scheme $\mathrm{Df}_X(C)$ the way we did in Construction 3.10 is to render its naturality in X and C more transparent; see Theorem 3.22 below.

Notation 3.13. Let X be a regularly stratified scheme and let C be a stratum of X. We define $\mathrm{N}_X(C)$ by the Cartesian square

$$\begin{tikzcd} \mathrm{N}_X(C) \arrow[r] \arrow[d] & \mathrm{Df}_X(C) \arrow[d] \\ \mathrm{o}_C \arrow[r] & \mathrm{T}_X(C). \end{tikzcd} \tag{3.3}$$

The closed subscheme $\mathrm{N}_X(C) \subset \mathrm{Df}_X(C)$ is constructible, and hence inherits a stratification from the one of $\mathrm{Df}_X(C)$. This makes $\mathrm{N}_X(C)$ into a connected regularly stratified scheme (see Lemma 3.14 below). We write $\mathrm{N}_X^\circ(C)$ for the open stratum of $\mathrm{N}_X(C)$. Note also that $\mathrm{N}_X(C)$ inherits an action of the torus $\mathrm{T}_X^\circ(C)$ from the one on $\mathrm{Df}_X(C)$.

Lemma 3.14. *Let X be a regularly stratified scheme and let C be a stratum of X. Let $D_1, \ldots, D_c$ be the irreducible constructible divisors containing C. For $1 \leq i \leq c$, let $N_i \to D_i$ be the normal bundle of the closed immersion $D_i \to X$. Then $\mathrm{N}_X(C)$ is isomorphic to*

$$(N_1 \times_{D_1} \overline{C}) \times_{\overline{C}} \ldots \times_{\overline{C}} (N_1 \times_{D_c} \overline{C})$$

endowed with the coarsest stratification $\mathcal{P}$ for which the inverse images of the zero sections of the N_i's and the inverse images of the irreducible components of $\overline{C} \smallsetminus C$ are $\mathcal{P}$-constructible. In particular, the following properties hold:

(i) $\mathrm{N}_X(C)$ *is regularly stratified and its underlying scheme is isomorphic to the normal cone of the closed immersion $\overline{C} \to X$;*
(ii) $\mathrm{N}_X^\circ(C)$ *is naturally a torsor over* $\mathrm{T}_X^\circ(C)$ *defined over C.*

Proof. If C is open, $\mathrm{N}_X(C) = \overline{C}$ by construction, and there is nothing to prove. Thus, we may assume that C has codimension ≥ 1. A direct computation shows that $\mathrm{N}_X(C)$ is isomorphic to the spectrum of the $\mathrm{R}_X^\circ(C)$-graded $\mathcal{O}_{\overline{C}}$-algebra

$$\bigoplus_{\nu \in \mathrm{R}_X^\circ(C),\, \nu < 0} \overline{\mathcal{O}}_X(\nu) \cdot \mathbf{t}^\nu,$$

where $\overline{\mathcal{O}}_X(\nu)$ is the quotient of $\mathcal{O}_X(\nu)$ by the sub-$\mathcal{O}_X$-module $\sum_{\nu' < \nu} \mathcal{O}_X(\nu')$. (Of course the inequality sign "<" refers to the additive order on the group $\mathrm{R}_X^\circ(C)$ for which $\mathrm{R}_X(C)$ is the monoid of positive elements.) The statement follows readily from this. □

We now describe with some details the stratification of $\mathrm{Df}_X(C)$.

Remark 3.15. Let X be a regularly stratified scheme and let C be a stratum of X. Given a second stratum $D \geq C$, we have direct sum decompositions

$$\mathrm{R}_X^\circ(C) = \mathrm{R}_X^\circ(D) \oplus \mathrm{R}_{X|D}^\circ(C) \qquad \text{and} \qquad \mathrm{R}_X(C) = \mathrm{R}_X(D) \oplus \mathrm{R}_{X|D}(C) \tag{3.4}$$

where $\mathrm{R}^{\circ}_{X|D}(C)$ is the group of Cartier divisors of X generated by the constructible irreducible divisors of X containing C but not D, and $\mathrm{R}_{X|D}(C) \subset \mathrm{R}^{\circ}_{X|D}(C)$ is the submonoid of effective Cartier divisors. Dually, we obtain the following direct product decompositions

$$\mathrm{T}^{\circ}_{X}(C) = \mathrm{T}^{\circ}_{X}(D) \times \mathrm{T}^{\circ}_{X|D}(C) \qquad \text{and} \qquad \mathrm{T}_{X}(C) = \mathrm{T}_{X}(D) \times \mathrm{T}_{X|D}(C). \tag{3.5}$$

In particular, we have a Cartesian square

$$\begin{tikzcd}
\mathrm{T}_{X|D}(C) \arrow[r] \arrow[d] & \mathrm{T}_{X}(C) \arrow[d] \\
\mathrm{o}_{D} \arrow[r] & \mathrm{T}_{X}(D)
\end{tikzcd}$$

which we use to identify $\mathrm{T}_{X|D}(C)$ with the constructible closed subscheme of $\mathrm{T}_{X}(C)$ whose ideal is generated by the $\mathbf{t}^{\nu}$ for $\nu \in \mathrm{R}_{X}(D)$. Using this identification, $\mathrm{T}^{\circ}_{X|D}(C)$ is then a stratum of $\mathrm{T}_{X}(C)$. It is easy to check that the map $D \geq C \mapsto \mathrm{T}^{\circ}_{X|D}(C)$ is a bijection between the strata of $\mathrm{T}_{X}(C)$ and those strata of X containing C in their closure.

Notation 3.16. Let X be a regularly stratified scheme and let $C \leq D$ be strata of X. We set

$$\mathrm{Df}_{X|D}(C) = \mathrm{Df}_{X}(C) \times_{\mathrm{T}_{X}(C)} \mathrm{T}_{X|D}(C) \simeq \mathrm{Df}_{X}(C) \times_{\mathrm{T}_{X}(D)} \mathrm{o}_{D}.$$

This is a constructible closed subscheme of $\mathrm{Df}_{X}(C)$, and hence inherits a stratification from the one of $\mathrm{Df}_{X}(C)$. This makes $\mathrm{Df}_{X|D}(C)$ into a connected regularly stratified scheme (see Lemma 3.17 below). As usual, we denote by $\mathrm{Df}^{\circ}_{X|D}(C)$ the open stratum of $\mathrm{Df}_{X|D}(C)$ and set $\mathrm{Df}^{\flat}_{X|D} = \mathrm{Df}^{\flat}_{X} \cap \mathrm{Df}_{X|D}$.

Lemma 3.17. *Let X be a regularly stratified scheme and let $C \leq D$ be strata of X. Then, there is a commutative square of stratified schemes*

$$\begin{tikzcd}
\mathrm{Df}_{X|D}(C) \arrow[r, "\sim"] \arrow[d] & \mathrm{Df}_{\overline{D}}(C) \times_{\overline{D}} \mathrm{N}_{X}(D) \arrow[d] \\
\mathrm{T}_{X|D}(C) \arrow[r, "\sim"] & \mathrm{T}_{\overline{D}}(C)
\end{tikzcd} \tag{3.6}$$

where the horizontal arrows are isomorphisms. In particular, the following properties hold:

(i) $\mathrm{Df}_{X|D}(C)$ *is regularly stratified and there is a morphism* $\mathrm{Df}_{X|D}(C) \to \mathrm{Df}_{\overline{D}}(C)$ *making* $\mathrm{Df}_{X|D}(C)$ *into a vector bundle over* $\mathrm{Df}_{\overline{D}}(C)$*;*
(ii) *there is an isomorphism* $\mathrm{Df}^{\circ}_{X|D}(C) \simeq \mathrm{T}^{\circ}_{\overline{D}}(C) \times \mathrm{N}^{\circ}_{X}(D)$*.*

Proof. If D is open, we have by construction $\mathrm{Df}_{X|D}(C) \simeq \mathrm{Df}_{X}(C) = \mathrm{Df}_{\overline{D}}(C)$, and there is nothing to prove. Thus, we may assume that D has codimension ≥ 1. Using the decompositions in (3.4), we have an isomorphism of $\mathcal{O}_X$-algebras

$$\mathcal{O}(\mathrm{Df}_{X}(C)) \simeq \left(\bigoplus_{\nu \in \mathrm{R}^{\circ}_{X|D}(C)} (\mathcal{O}_{X}(\nu) \cap \mathcal{O}_{X}) \cdot \mathbf{t}^{\nu} \right) \otimes_{\mathcal{O}_X} \mathcal{O}(\mathrm{Df}_{X}(D)).$$

It follows by construction that

$$\begin{aligned}
\mathcal{O}(\mathrm{Df}_{X|D}(C)) &\simeq \left(\bigoplus_{v\in \mathrm{R}^\circ_{X|D}(C)} (\mathcal{O}_X(v)\cap \mathcal{O}_X)\cdot \mathbf{t}^v\right)\otimes_{\mathcal{O}_X} \mathcal{O}(\mathrm{N}_X(D)) \\
&\simeq \left(\left(\bigoplus_{v\in \mathrm{R}^\circ_{X|D}(C)} (\mathcal{O}_X(v)\cap \mathcal{O}_X)\cdot \mathbf{t}^v\right)\otimes_{\mathcal{O}_X} \mathcal{O}_{\overline{D}}\right)\otimes_{\mathcal{O}_{\overline{D}}} \mathcal{O}(\mathrm{N}_X(D)) \\
&\simeq \mathcal{O}(\mathrm{Df}_{\overline{D}}(C))\otimes_{\mathcal{O}_{\overline{D}}} \mathcal{O}(\mathrm{N}_X(D)).
\end{aligned}$$

For the last isomorphism in the above chain, we use the fact that the lattice $\mathrm{R}^\circ_{\overline{D}}(C)$ can be identified with $\mathrm{R}^\circ_{X|D}(C)$. This proves the lemma. ◻

In the remainder of this subsection, we will describe the functoriality of the previous constructions in X and C. Given a stratified scheme Y and a sequence of strata $(D_j)_{1\leq j\leq n}$ in Y, we will call "relevant" the open strata of Y containing at least one of the D_j's in their closure.

Lemma 3.18. *Let $f : Y \to X$ be a morphism of regularly stratified schemes taking the relevant open strata of Y to open strata of X.*

(i) Let D be a stratum of Y and $C = f_*(D)$. Pulling back Cartier divisors from the smallest constructible neighbourhood of C to the smallest constructible neighbourhood of D induces a homomorphism $f^* : \mathrm{R}^\circ_X(C) \to \mathrm{R}^\circ_Y(D)$ respecting the submonoids of effective elements.

(ii) Let $D_0 \geq D_1$ be strata of Y and $C_0 \geq C_1$ their images by f_*. Then, the following square

$$\begin{tikzcd}
\mathrm{R}^\circ_X(C_0) \arrow[r] \arrow[d, "f^*"] & \mathrm{R}^\circ_X(C_1) \arrow[d, "f^*"] \\
\mathrm{R}^\circ_Y(D_0) \arrow[r] & \mathrm{R}^\circ_X(D_1),
\end{tikzcd}$$

where the horizontal arrows are the obvious inclusions, is commutative. Moreover, the homomorphism $f^* : \mathrm{R}^\circ_X(C_1) \to \mathrm{R}^\circ_Y(D_1)$ sends $\mathrm{R}^\circ_{X|C_0}(C_1)$ into $\mathrm{R}^\circ_{Y|D_0}(D_1)$ inducing a homomorphism $f^* : \mathrm{R}^\circ_{X|C_0}(C_1) \to \mathrm{R}^\circ_{Y|D_0}(D_1)$ rendering the square

$$\begin{tikzcd}
\mathrm{R}^\circ_{X|C_0}(C_1) \arrow[r, "\sim"] \arrow[d, "f^*"] & \mathrm{R}^\circ_{\overline{C}_0}(C_1) \arrow[d, "f_0^*"] \\
\mathrm{R}^\circ_{Y|D_0}(D_1) \arrow[r, "\sim"] & \mathrm{R}^\circ_{\overline{D}_0}(D_1)
\end{tikzcd}$$

commutative (with $f_0 : \overline{D}_0 \to \overline{C}_0$ the morphism induced by f).

Proof. Everything is clear except the second assertion in part (ii). For this one needs to remark that if $E \subset X$ is an irreducible constructible divisor containing C_1 but not C_0, then every component of $f^{-1}(E)$ that contains D_1 does not contain D_0. ◻

Proposition 3.19. *Let $f : Y \to X$ be a morphism of regularly stratified schemes.*

(i) Let D be a stratum of Y and $C = f_*(D)$. There is an induced commutative diagram of regularly stratified schemes

$$\begin{tikzcd}
Y \arrow[d] & \mathrm{Df}_Y(D) \arrow[l] \arrow[r] \arrow[d] & \mathrm{T}_Y(D) \arrow[d] \\
X & \mathrm{Df}_X(C) \arrow[l] \arrow[r] & \mathrm{T}_X(C)
\end{tikzcd}$$

(ii) Let $D_0 \geq D_1$ be strata of Y and $C_0 \geq C_1$ their images by f_*. There is an induced commutative diagram of regularly stratified schemes

$$\begin{tikzcd}[column sep=small]
& \mathrm{T}_{\overline{D}_0}(D_1) \arrow[dd] & & \mathrm{T}_{Y|D_0}(D_1) \arrow[ll, "\sim"'] \arrow[rr] \arrow[dd] & & \mathrm{T}_Y(D_1) \arrow[rr] \arrow[dd] & & \mathrm{T}_Y(D_0) \arrow[dd] \\
\mathrm{Df}_{\overline{D}_0}(D_1) \arrow[ur] \arrow[dd] & & \mathrm{Df}_{Y|D_0}(D_1) \arrow[ll] \arrow[rr] \arrow[ur] \arrow[dd] & & \mathrm{Df}_Y(D_1) \arrow[rr] \arrow[ur] \arrow[dd] & & \mathrm{Df}_Y(D_0) \arrow[ur] \arrow[dd] & \\
& \mathrm{T}_{\overline{C}_0}(C_1) & & \mathrm{T}_{X|C_0}(C_1) \arrow[ll, "\sim"'] \arrow[rr] & & \mathrm{T}_X(C_1) \arrow[rr] & & \mathrm{T}_X(C_0) \\
\mathrm{Df}_{\overline{C}_0}(C_1) \arrow[ur] & & \mathrm{Df}_{X|C_0}(C_1) \arrow[ll] \arrow[rr] \arrow[ur] & & \mathrm{Df}_X(C_1) \arrow[rr] \arrow[ur] & & \mathrm{Df}_X(C_0). \arrow[ur] &
\end{tikzcd}$$

Proof. We first assume that f takes the relevant open stratum to an open stratum. In this case, part (i) follows from Construction 3.10 and Lemma 3.18(i). Similarly, part (ii) follows from Lemma 3.18(ii) and the construction of the morphisms $\mathrm{Df}_{X|C_0}(C_1) \to \mathrm{Df}_{\overline{C}_0}(C_1)$ and $\mathrm{Df}_{Y|D_0}(D_1) \to \mathrm{Df}_{\overline{D}_0}(D_1)$ which can be extracted from the proof of Lemma 3.17.

Next, we assume that $Y \to X$ is the inclusion of an irreducible constructible closed subscheme. In this case, Y° is a stratum of X and we have $C = D$, $C_0 = D_0$ and $C_1 = D_1$. We define the vertical morphisms of the right square in the commutative diagram of (i) by the compositions

$$\mathrm{Df}_Y(D) = \mathrm{Df}_Y(C) \xrightarrow{s_0} \mathrm{Df}_Y(C) \times_Y \mathrm{N}_X(Y^\circ) \overset{(\star)}{\simeq} \mathrm{Df}_{X|Y^\circ}(C) \to \mathrm{Df}_X(C)$$

$$\text{and} \qquad \mathrm{T}_Y(D) = \mathrm{T}_Y(C) \simeq \mathrm{T}_{X|Y^\circ}(C) \to \mathrm{T}_X(C).$$

(Here, s_0 denotes the zero section of $\mathrm{N}_X(Y^\circ)$ and $(\star)$ is the inverse of the isomorphism given in Lemma 3.17.) The commutativity of the diagram in (i) is then clear by construction. We leave the commutativity of the diagram in (ii) to the reader. □

Theorem 3.20. *There are functors*

$$\mathrm{Df},\ \mathrm{T} : \int_{\text{REG-}\Sigma} \mathscr{P} \to \text{REG-}\Sigma$$

sending a pair (X, C), consisting of a regularly stratified scheme X and a stratum $C \subset X$ to the regularly stratified schemes $\mathrm{Df}_X(C)$ and $\mathrm{T}_X(C)$ respectively. These functors are characterised by the following properties.

(i) For a fixed X and strata $C_0 \geq C_1$ in X, the associated morphisms $\mathrm{Df}_X(C_1) \to \mathrm{Df}_X(C_0)$ and $\mathrm{T}_X(C_1) \to \mathrm{T}_X(C_0)$ are the obvious ones.

(ii) A morphism $f : (Y, D) \to (X, C)$ with $C = f_*(D)$ is sent to the obvious morphisms $\mathrm{Df}_Y(D) \to \mathrm{Df}_X(C)$ and $\mathrm{T}_Y(D) \to \mathrm{T}_X(C)$. (See Proposition 3.19(i).)

Proof. This follows from Proposition 3.19 by direct verification. The details are omitted. □

To state the next result, we need to introduce some notations.

Notation 3.21. Given a stratified scheme X, we let $\mathscr{P}'_X$ be the sub-poset of $(\mathscr{P}_X, \geq) \times (\mathscr{P}_X, \leq)$ whose elements are pairs (C_-, C_+) of strata in X such that $C_- \geq C_+$. Thus, an arrow $(C'_-, C'_+) \to (C_-, C_+)$ witnesses a chain of inequalities $C'_- \geq C_- \geq C_+ \geq C'_+$. Moreover, we let $\mathscr{P}''_X$ be the sub-poset of $(\mathscr{P}_X, \geq) \times (\mathscr{P}_X, \leq) \times (\mathscr{P}_X, \leq)$ whose elements are triples (C_-, C_0, C_+) of strata in X such that $C_- \geq C_0 \geq C_+$. We have an obvious functor $\mathscr{P}''_X \to \mathscr{P}'_X$ whose fiber at (C_-, C_+) is the set of strata between C_+ and C_- partially ordered by $\leq$.

Theorem 3.22. *There are functors*

$$\mathrm{Df},\ \mathrm{T} : \int_{\text{REG-}\Sigma} \mathscr{P}'' \to \text{REG-}\Sigma$$

sending a pair $(X, (C_-, C_0, C_+))$, consisting of a regularly stratified scheme X and an object of $\mathscr{P}''_X$, to the regularly stratified schemes $\mathrm{Df}_{\overline{C}_-|C_0}(C_+)$ and $\mathrm{T}_{\overline{C}_-|C_0}(C_+)$ respectively. These functors are characterised by the following properties.

(i) For a fixed X and an arrow in $\mathscr{P}''_X$ of the form $(C_-, C_0, C_+) \to (C_-, C_-, C_+)$ the associated morphisms $\mathrm{Df}_{\overline{C}_-|C_0}(C_+) \to \mathrm{Df}_{\overline{C}_-}(C_+)$ and $\mathrm{T}_{\overline{C}_-|C_0}(C_+) \to \mathrm{T}_{\overline{C}_-}(C_+)$ are the obvious inclusions.

(ii) For a fixed X and an arrow in $\mathscr{P}''_X$ of the form $(C_-, C_-, C'_+) \to (C_-, C_-, C_+)$ the associated morphisms $\mathrm{Df}_{\overline{C}_-}(C'_+) \to \mathrm{Df}_{\overline{C}_-}(C_+)$ and $\mathrm{T}_{\overline{C}_-}(C'_+) \to \mathrm{T}_{\overline{C}_-}(C_+)$ are the obvious ones.

(iii) For a fixed X and an arrow in $\mathscr{P}''_X$ of the form $(C_-, C_0, C_+) \to (C_0, C_0, C_+)$ the associated morphism $\mathrm{Df}_{\overline{C}_-|C_0}(C_+) \to \mathrm{Df}_{\overline{C}_0}(C_+)$ is the one described in Lemma 3.17, and the associated morphism $\mathrm{T}_{\overline{C}_-|C_0}(C_+) \to \mathrm{T}_{\overline{C}_0}(C_+)$ is the obvious isomorphism.

(iv) A coCartesian morphism of the form $f : (Y, (D_-, D_-, D_+)) \to (X, (C_-, C_-, C_+))$ is sent to the obvious morphisms $\mathrm{Df}_{\overline{D}_-}(D_+) \to \mathrm{Df}_{\overline{C}_-}(C_+)$ and $\mathrm{T}_{\overline{D}_-}(D_+) \to \mathrm{T}_{\overline{C}_-}(C_+)$. (See Proposition 3.19(i).)

Furthermore, we have a natural transformation $\mathrm{Df} \to \mathrm{T}$ *given by the obvious morphisms and the functor* $\mathrm{N} = \mathrm{Df} \times_{\mathrm{T}} o$ *factors through* $\int_{\text{REG-}\Sigma} \mathscr{P}'$ *yielding a functor*

$$\mathrm{N} : \int_{\text{REG-}\Sigma} \mathscr{P}' \to \text{REG-}\Sigma$$

sending a pair $(X, (C_-, C_+))$ to the regularly stratified scheme $\mathrm{N}_{\overline{C}_-}(C_+)$.

Proof. This follows from Proposition 3.19 by direct verification. The details are omitted. □

3.2. **Monodromic specialisation, I. Definition and basic properties.**

In this subsection, we construct the monodromic specialisation functors. Classically, monodromic specialisations were introduced by Verdier in [Ver83, §8], and they are closely related to the nearby cycles functors. Roughly speaking, monodromic specialisation along a closed subvariety $Z \subset X$ is the nearby cycles functor associated to the deformation to the normal cone of Z. When Z is a principal divisor, this allows one to encode the monodromy action on the sheaf of nearby cycles via variations over the fibers of a relative 1-dimensional torus. In the motivic setting, a similar but more restrictive formalism was recently developed by Ivorra–Sebag in [IS18]. (Indeed, the map $f^{\mathbf{G}_\mathrm{m}}$ used in [IS18, §4.1] is the projection to $\mathbb{A}^1$ of an open subvariety of the deformation to the normal cone of the central fiber of f.) Our monodromic specialisation formalism is closely related to the aforementioned constructions, but differs in some aspects related to functoriality. We start by generalising slightly some constructions from [Ayo07b, §3.4 & 3.5]. Throughout this subsection, we fix a base scheme S and a presentable Voevodsky pullback formalism

$$\mathcal{H}^\otimes : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

(See Definitions 1.14 and 2.4.)

Construction 3.23. Let T be a split torus over S (or any other base scheme). We define a diagram of T-schemes $\mathcal{Y}^T$ as follows. The indexing category of $\mathcal{Y}^T$ is $\mathbf{\Delta} \times \mathbb{N}^\times$ where $\mathbb{N}^\times = \mathbb{N} \smallsetminus \{0\}$ is partially ordered with the opposite of the divisibility relation. We set $\mathcal{Y}^T([n], r) = T^{n+1}$, with structural morphism

$$T^{n+1} \to T, \quad (x_0, \ldots, x_n) \mapsto x_0^r.$$

For $r, d \in \mathbb{N}^\times$, the morphism $\mathcal{Y}^T([n], rd) \to \mathcal{Y}^T([n], r)$ is evaluation to the power d. The cosimplicial scheme $\mathcal{Y}^T(-, r)$ is independent of r once we forget the structural morphism to T. Its coface morphism $d^i : \mathcal{Y}^T([n], r) \to \mathcal{Y}^T([n+1], r)$ is induced by the diagonal immersion $T \to T \times T$ of the i-th factor in T^{n+1}, if $0 \leq i \leq n$, and is given by inserting 1 at the $n+1$-th factor of T^{n+2}, if $i = n+1$. (Note that we are numbering the factors of T^{n+1} and T^{n+2} starting from 0 to n and $n+1$ respectively.) Its codegeneracy morphism $s^j : \mathcal{Y}^T([n], r) \to \mathcal{Y}^T([n-1], r)$ is given by the projection parallel to the $j+1$-th factor, for $0 \leq j \leq n-1$. Said differently, $\mathcal{Y}^T(-, r)$ is the cosimplicial split torus $T \mathbin{\widetilde{\times}}_T 1$, obtained by applying [Ayo07b, Lemme 3.4.1], and considered as a cosimplicial T-scheme using the composition of

$$T \mathbin{\widetilde{\times}}_T 1 \to T \xrightarrow{(-)^r} T.$$

In fact, we even have $\mathcal{Y}^T = \mathcal{E}^T \mathbin{\widetilde{\times}}_{\mathcal{E}^T} 1$, where we apply [Ayo07b, Lemme 3.4.1] in the category of $\mathbb{N}^\times$-diagrams of tori with $\mathcal{E}^T$ the object sending $r \in \mathbb{N}^\times$ to T and an arrow $rd \to r$ in $\mathbb{N}^\times$ to the elevation to the power d. (See also [Ayo07b, Définitions 3.5.1 & 3.5.3].)

Construction 3.24. Keep the notations as in Construction 3.23. Let $\theta : \mathcal{Y}^T \to (T, \mathbf{\Delta} \times \mathbb{N}^\times)$ be the natural transformation from $\mathcal{Y}^T$ to the constant $\mathbf{\Delta} \times \mathbb{N}^\times$-diagram given by the structural projections to T. We have a morphism of coCartesian fibrations

$$\begin{tikzcd}
(\mathbf{\Delta} \times \mathbb{N}^\times)^{\mathrm{op}} \times \mathcal{H}(T)^\otimes \arrow[rr, "\theta^*"] \arrow[dr, "p"'] & & \int_{(\mathbf{\Delta} \times \mathbb{N}^\times)^{\mathrm{op}}} \mathcal{H}(\mathcal{Y}^T)^\otimes \arrow[dl] \\
& (\mathbf{\Delta} \times \mathbb{N}^\times)^{\mathrm{op}} \times \mathrm{Fin}_*. &
\end{tikzcd}$$

By [Lur17, Proposition 7.3.2.6], the functor θ^* admits a relative right adjoint θ_*. By applying $\theta_*\theta^*$ to the section of p given by the $\otimes$-unit object of $\mathcal{H}(T)^\otimes$, we obtain the section $\theta_*\mathbf{1}$ of p which we may view as a diagram $\theta_*\mathbf{1} : (\mathbf{\Delta} \times \mathbb{N}^\times)^{\mathrm{op}} \to \mathrm{CAlg}(\mathcal{H}(T))$. We set

$$\mathcal{U}_T = \operatorname*{colim}_{(\mathbf{\Delta} \times \mathbb{N}^\times)^{\mathrm{op}}} \theta_*\mathbf{1}. \tag{3.7}$$

We note that the above colimit is sifted, and thus can be computed on the underlying objects in $\mathcal{H}(T)$. We will also need a variant of the above construction where we use the subdiagram $\mathcal{Y}_1^T = \mathcal{Y}^T(-, 1)$ instead of $\mathcal{Y}^T$. This yields the commutative algebra $\mathcal{L}_T$ given by

$$\mathcal{L}_T = \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}} \times \{1\}} \theta_*\mathbf{1}. \tag{3.8}$$

By construction, we have a morphism of commutative algebras $\mathcal{L}_T \to \mathcal{U}_T$. We refer to $\mathcal{L}_T$ as the logarithmic algebra over T.

Notation 3.25. Let T° be a split torus and $j : T^\circ \hookrightarrow T$ a split torus-embedding (as in Definition 3.104 below). We set

$$\mathcal{L}_T = j_*\mathcal{L}_{T^\circ} \quad \text{and} \quad \mathcal{U}_T = j_*\mathcal{U}_{T^\circ}. \tag{3.9}$$

Since j_* is a lax symmetric monoidal functor, these are also commutative algebras in $\mathcal{H}(T)^{\otimes}$.

Remark 3.26. The commutative algebras $\mathcal{L}_T$ and $\mathcal{U}_T$ introduced in Construction 3.24 and Notation 3.25 are motivic in the following sense: the initial morphism of Voevodsky pullback formalisms

$$\mathbf{MSh}_{\mathrm{nis}}(-)^{\otimes} \to \mathcal{H}(-)^{\otimes},$$

given by Theorem 2.8, takes the commutative algebras $\mathcal{L}_T$ and $\mathcal{U}_T$ in $\mathbf{MSh}_{\mathrm{nis}}(T)^{\otimes}$ to the commutative algebras $\mathcal{L}_T$ and $\mathcal{U}_T$ in $\mathcal{H}(T)^{\otimes}$.

We now discuss a few properties of the commutative algebras $\mathcal{L}_T$ and $\mathcal{U}_T$.

Lemma 3.27.

(i) *Let T' and T'' be two split torus-embeddings, and let $T = T' \times T''$. There are canonical equivalences of commutative algebras $\mathcal{L}_T \simeq \mathcal{L}_{T'} \boxtimes \mathcal{L}_{T''}$ and $\mathcal{U}_T \simeq \mathcal{U}_{T'} \boxtimes \mathcal{U}_{T''}$.*

(ii) *Let T be a split torus-embedding over S, and let $p : T \to S$ be the structural projection. The unit morphisms $\mathbf{1} \to p_*\mathcal{L}_T$ and $\mathbf{1} \to p_*\mathcal{U}_T$ are equivalences.*

(iii) *Let T be a split torus-embedding over S, and let $E^{\circ} \subset T$ be a stratum with closure E. There are equivalences of commutative algebras $\mathcal{L}_T|_E \simeq \mathcal{L}_E$ and $\mathcal{U}_T|_E \simeq \mathcal{U}_E$.*

Proof. Let $j : T^{\circ} \hookrightarrow T$, $j' : T'^{\circ} \hookrightarrow T'$ and $j'' : T''^{\circ} \hookrightarrow T''$ be the split torus-embeddings considered in (i). There is an isomorphism of $\mathbf{\Delta} \times \mathbb{N}^{\times}$-diagrams of T-schemes

$$\mathcal{Y}^{T^{\circ}} \simeq \mathcal{Y}^{T'^{\circ}} \times_S \mathcal{Y}^{T''^{\circ}}.$$

Since the tensor product commutes with sifted colimits, we are reduced to showing that the obvious morphism $j'_*(A') \boxtimes j''_*(A'') \to j_*(A' \boxtimes A'')$ is an equivalence, when A' and A'' are motives in $\mathcal{H}(T')$ and $\mathcal{H}(T'')$ obtained via pullback from $\mathcal{H}(S)$. This is easy, and left to the reader.

To prove (ii), we may assume that $T = T^{\circ}$ is a torus. Also, it suffices to treat the case of the logarithmic algebra. Using (i), we may reduce to the case of $T = \mathbf{G}_{\mathrm{m}}$ which is treated in [Ayo07b, page 78]. In fact, the argument in loc. cit. is valid for a general T. The point is that $p_*\mathcal{L}_T$ is the geometric realisation of $(p \circ \theta_1)_*\mathbf{1}$. We then use [Ayo07b, Lemme 3.4.1(B) & Corollaire 3.4.12] to conclude.

For (iii), we may assume that $T = E \times \mathbb{A}^c_S$ so that E is identified with the subscheme $E \times 0$. In this case, we may use (i) to reduce to the case where $T = \mathbb{A}^c$ and $E = 0_S$. Using (i) again, we may further reduce to the case where $T = \mathbb{A}^1$. The result follows then from [Ayo07b, Proposition 3.4.9(1) & Lemme 3.5.10]. We may also deduce it from (ii), but we leave this to the reader. $\square$

To state our next result, we need to introduce some notations.

Notation 3.28. Let T be a split torus over S. We denote by $q : T \to S$ the structural projection and by $e_r : T \to T$, for $r \in \mathbb{N}^{\times}$, the elevation to the power r on T.

(i) We denote by $\mathcal{H}(T)_{\mathrm{un}/S}$, or simply $\mathcal{H}(T)_{\mathrm{un}}$ if S is understood, the stable localising sub-∞-category of $\mathcal{H}(T)$ generated by the image of q^*. We denote by $\phi^*_{\mathrm{un}} : \mathcal{H}(T)_{\mathrm{un}} \to \mathcal{H}(S)$ the inverse image functor along the unit section of T, and by ϕ^{un}_* its right adjoint.

(ii) Similarly, we denote by $\mathcal{H}(T)_{\mathrm{qun}/S}$, or simply $\mathcal{H}(T)_{\mathrm{qun}}$ if S is understood, the stable localising sub-∞-category of $\mathcal{H}(T)$ generated by the image of the functors $e_{r,*} \circ q^*$, for $r \in \mathbb{N}^{\times}$. We denote by $\phi^*_{\mathrm{qun}} : \mathcal{H}(T)_{\mathrm{qun}} \to \mathcal{H}(S)$ the inverse image functor along the unit section of T, and by ϕ^{qun}_* its right adjoint.

Objects of $\mathcal{H}(T)_{\mathrm{un}}$ are said to be unipotent, and those of $\mathcal{H}(T)_{\mathrm{qun}}$ are said to be quasi-unipotent.

Proposition 3.29. *Let T be a split torus over S. There are equivalences*

$$\phi_*^{\mathrm{un}}(-) \simeq \mathcal{L}_T \otimes q^*(-) \quad \textit{and} \quad \phi_*^{\mathrm{qun}}(-) \simeq \mathcal{U}_T \otimes q^*(-). \tag{3.10}$$

In particular, we have $\mathcal{L}_T \simeq \phi_*^{\mathrm{un}}(\mathbf{1})$ and $\mathcal{U}_T \simeq \phi_*^{\mathrm{qun}}(\mathbf{1})$.

Proof. This is a generalisation of [Ayo14c, Proposition 2.10] and the proof given in loc. cit. still works in the generality we are considering. For the reader's convenience, we sketch the proof.

We only treat the quasi-unipotent case. Using the unit sections of the tori $\mathcal{Y}^T([n], r)$, for $[n] \in \mathbf{\Delta}$ and $r \in \mathbb{N}^\times$, we see that there is a commutative algebra morphism $\epsilon : \phi_{\mathrm{qun}}^*(\mathcal{U}_T) \to \mathbf{1}$. We may use this to define a natural transformation $\mathcal{U}_T \otimes q^*(-) \to \phi_*^{\mathrm{qun}}(-)$. We take the one corresponding by adjunction to the composition of

$$\phi_{\mathrm{qun}}^*(\mathcal{U}_T \otimes q^*(-)) \simeq \phi_{\mathrm{qun}}^*(\mathcal{U}_T) \otimes \phi_{\mathrm{qun}}^*(q^*(-)) \xrightarrow{\epsilon} \phi_{\mathrm{qun}}^*(q^*(-)) \simeq \mathrm{id}.$$

We will show that for every $M \in \mathcal{H}(T)_{\mathrm{qun}}$ and $N \in \mathcal{H}(S)$, the induced map

$$\mathrm{Map}_{\mathcal{H}(T)}(M, \mathcal{U}_T \otimes q^*(N)) \to \mathrm{Map}_{\mathcal{H}(T)}(M, \phi_*^{\mathrm{qun}}(N)) \tag{3.11}$$

is an equivalence. By the definition of $\mathcal{H}(T)_{\mathrm{qun}}$, we may assume that M belongs to the image of $e_{r,*}q^*$. Note that we have equivalences $e_{r,*}\mathcal{U}_T \simeq \mathcal{U}_T$ and $e_{r,*} \circ \phi_*^{\mathrm{qun}} \simeq \phi_*^{\mathrm{qun}}$ fitting in a commutative diagram of natural transformations

$$\begin{tikzcd}
\mathcal{U}_T \otimes q^*(-) \arrow[rrr] \arrow[d, "\sim"] & & & \phi_*^{\mathrm{qun}}(-) \arrow[d, "\sim"] \\
e_{r,*}\mathcal{U}_T \otimes q^*(-) \arrow[r, "\sim"] & e_{r,*}(\mathcal{U}_T \otimes e_r^* q^*(-)) \arrow[r, "\sim"] & e_{r,*}(\mathcal{U}_T \otimes q^*(-)) \arrow[r] & e_{r,*}\phi_*^{\mathrm{qun}}(-).
\end{tikzcd}$$

Thus, the morphism $\mathcal{U}_T \otimes q^*(N) \to \phi_*^{\mathrm{qun}}(N)$, inducing the map in (3.11), is equivalent to its image by $e_{r,*}$. Using adjunction and the fact that the image of $e_r^* e_{r,*} q^*$ is contained in the image of q^*, we are left to show that (3.11) is an equivalence with $M = q^* M_0$, for $M_0 \in \mathcal{H}(S)$. By adjunction, we are thus left to show that

$$q_*(\mathcal{U}_T \otimes q^*(N)) \to q_*(\phi_*^{\mathrm{qun}}(N))$$

is an equivalence. Since $\phi_{\mathrm{qun}}^* \circ q^* \simeq \mathrm{id}$, we deduce that the codomain of this morphism is equivalent to N. On the other hand, the domain is easily seen to be equivalent to $q_*(\mathcal{U}_T) \otimes N$. We finally conclude using Lemma 3.27(ii). ◻

Corollary 3.30. *Let $T' \to T$ be a morphism of split tori over S.*

(i) The right adjoint to the inverse image functor $\mathcal{H}(T)_{\mathrm{un}} \to \mathcal{H}(T')_{\mathrm{un}}$ takes $\mathcal{L}_{T'}$ to $\mathcal{L}_T$.
(ii) The right adjoint to the inverse image functor $\mathcal{H}(T)_{\mathrm{qun}} \to \mathcal{H}(T')_{\mathrm{qun}}$ takes $\mathcal{U}_{T'}$ to $\mathcal{U}_T$.

Proof. This follows immediately from Proposition 3.29. ◻

We now come to the definition of the nearby cycles functors.

Definition 3.31. Let T be a split torus-embedding over S and $E^\circ \subset T$ a stratum with closure E. Let $f : X \to T$ be a finite type morphism and form a commutative diagram with Cartesian squares

$$\begin{tikzcd}
X_\eta \arrow[r, "j"] \arrow[d, "f_\eta"] & X \arrow[d, "f"] & X_\sigma \arrow[l, "i"'] \arrow[d, "f_\sigma"] \\
T^\circ \arrow[r, "j"] & T & E. \arrow[l, "i"']
\end{tikzcd}$$

We define functors $\Upsilon_{f,E},\ \Psi_{f,E} : \mathcal{H}(X_\eta) \to \mathcal{H}(X_\sigma)$ by the formulae:

$$\Upsilon_{f,E}(-) = i^* j_*(\mathcal{L}_{T^\circ} \otimes -) \qquad \text{and} \qquad \Psi_{f,E}(-) = i^* j_*(\mathcal{U}_{T^\circ} \otimes -).$$

These functors are called the unipotent and the quasi-unipotent nearby functors.

Remark 3.32. The functors $\Upsilon_{f,E}$ and $\Psi_{f,E}$ are lax monoidal. Moreover, when the T-scheme X varies, these functors form a specialisation system over (T, j, i), in the sense of [Ayo07b, Définition 3.1.1]. The proof of this is a straightforward application of the six-functor formalism; see for instance the proof of [Ayo07b, Proposition 3.2.9]. Moreover, we have obvious natural transformations $\Upsilon_{f,E} \to \Psi_{f,E}$ defining a morphism of specialisation systems.

For later use, we record the following lemma.

Lemma 3.33. *If f is smooth, there are equivalences*

$$f_\sigma^*(\mathcal{L}_E) \xrightarrow{\sim} \Upsilon_{f,E}(\mathbf{1}) \qquad \textit{and} \qquad f_\sigma^*(\mathcal{U}_E) \xrightarrow{\sim} \Psi_{f,E}(\mathbf{1}).$$

In particular, if the stratum E° has relative dimension 0*, then we have equivalences*

$$\mathbf{1} \simeq \Upsilon_{f,E}(\mathbf{1}) \simeq \Psi_{f,E}(\mathbf{1}).$$

Proof. This follows from Lemma 3.27(iii) using the smooth base change theorem. □

We now come to the main players of this section, namely the monodromic nearby functors. From now on, we assume that S is noetherian.

Definition 3.34. Let X be a regularly stratified S-scheme, and let C be a stratum of X. Consider the following commutative diagram with Cartesian squares

$$\begin{tikzcd} X & X \times \mathrm{T}^\circ_X(C) \arrow[l, "p"'] \arrow[r, "j"] \arrow[d, "\rho_\eta"] & \mathrm{Df}_X(C) \arrow[d, "\rho"] & \mathrm{N}_X(C) \arrow[l, "i"'] \arrow[d, "\rho_\sigma"] \\ & \mathrm{T}^\circ_X(C) \arrow[r, "j"] & \mathrm{T}_X(C) & \mathrm{o}_C. \arrow[l, "i"'] \end{tikzcd}$$

(See Construction 3.10.) We define the functors $\widetilde{\Upsilon}_C,\ \widetilde{\Psi}_C : \mathcal{H}(X) \to \mathcal{H}(\mathrm{N}_X(C))$ by the formulae

$$\widetilde{\Upsilon}_C = \Upsilon_\rho \circ p^* \qquad \text{and} \qquad \widetilde{\Psi}_C = \Psi_\rho \circ p^*.$$

The functors $\widetilde{\Upsilon}_C$ and $\widetilde{\Psi}_C$ are called the monodromic specialisation functors. The first one is said to be unipotent and the second one is said to be quasi-unipotent.

Remark 3.35. We will also need a variant of Definition 3.34 where we employ the open deformation space $\mathrm{Df}^\flat_X(C)$ instead of $\mathrm{Df}_X(C)$. More precisely, we consider the commutative diagram with Cartesian squares

$$\begin{tikzcd} X^\circ \arrow[r, "p'"] & X^\circ \times \mathrm{T}^\circ_X(C) \arrow[r, "j"] \arrow[d, "\rho'_\eta"] & \mathrm{Df}^\flat_X(C) \arrow[d, "\rho'"] & \mathrm{N}^\circ_X(C) \arrow[l, "i"'] \arrow[d, "\rho'_\sigma"] \\ & \mathrm{T}^\circ_X(C) \arrow[r, "j"] & \mathrm{T}_X(C) & \mathrm{o}_C \arrow[l, "i"'] \end{tikzcd}$$

and define the functors $\widetilde{\Upsilon}_C,\ \widetilde{\Psi}_C : \mathcal{H}(X^\circ) \to \mathcal{H}(\mathrm{N}^\circ_X(C))$ by the formulae

$$\widetilde{\Upsilon}_C = \Upsilon_{\rho'} \circ p'^* \qquad \text{and} \qquad \widetilde{\Psi}_C = \Psi_{\rho'} \circ p'^*.$$

These functors will be also called monodromic specialisation functors. They are related to the previous ones by the following commutative squares

$$\begin{tikzcd}
\mathcal{H}(X) \arrow[r, "\widetilde{\Upsilon}_C\ (\text{resp. } \widetilde{\Psi}_C)"] \arrow[d] & \mathcal{H}(\mathrm{N}_X(C)) \arrow[d] \\
\mathcal{H}(X^\circ) \arrow[r, "\widetilde{\Upsilon}_C\ (\text{resp. } \widetilde{\Psi}_C)"] & \mathcal{H}(\mathrm{N}^\circ_X(C))
\end{tikzcd}$$

where the vertical arrows are the obvious restriction functors.

Corollary 3.36. *Let X be a regularly stratified S-scheme and C a stratum of X. There are equivalences $\mathbf{1} \simeq \widetilde{\Upsilon}_C(\mathbf{1}) \simeq \widetilde{\Psi}_C(\mathbf{1})$ in $\mathcal{H}(\mathrm{N}_X(C))$.*

Proof. This actually requires some hypotheses, such as S being regular and the unit object $\mathbf{1} \in \mathcal{H}(S)$ satisfying absolute purity as in [Ayo14a, Définition 7.1]. In case S is the spectrum of a field, this follows immediately from Lemma 3.33 since $\mathrm{Df}_X(C)$ is smooth over $\mathrm{T}_X(C)$. □

We now give two results describing the rough functoriality of the monodromic specialisations functors. More structured results will be discussed in Subsection 3.3.

Proposition 3.37. *Let $f : Y \to X$ be a morphism of regularly stratified S-schemes. Let $D \subset Y$ be a stratum of Y and let $C = f_*(D)$. We assume that f takes the relevant open stratum of Y to an open stratum of X. Then, we have natural transformations*

$$f^*_\sigma \circ \widetilde{\Upsilon}_C \to \widetilde{\Upsilon}_C \circ f^*_\eta \qquad \textit{and} \qquad f^*_\sigma \circ \widetilde{\Psi}_C \to \widetilde{\Psi}_C \circ f^*_\eta,$$

which are equivalences if f is smooth and D is open in $f^{-1}(C)$.

Proof. The construction of the natural transformations is easy and left to the reader. For the last assertion, we remark that the hypotheses on f and D imply that $\mathrm{T}_Y(D) \to \mathrm{T}_X(C)$ is an isomorphism. This said, the result follows immediately form the smooth base change theorem. □

Proposition 3.38. *Let X be a regularly stratified S-scheme, and let $C_0 \geq C_1$ be strata of X. Let $E \subset \mathrm{N}_X(C_0)$ be the largest stratum over $C_1 \subset \overline{C}_0$ relative to the projection $\mathrm{N}_X(C_0) \to \overline{C}_0$. Modulo the identifications $\mathrm{N}_X(C_1) \simeq \mathrm{N}_X(C_0) \times_{\overline{C}_0} \mathrm{N}_{\overline{C}_0}(C_1) \simeq \mathrm{N}_{\mathrm{N}_X(C_0)}(E)$, there are natural transformations*

$$\widetilde{\Upsilon}_{C_1} \to \widetilde{\Upsilon}_E \circ \widetilde{\Upsilon}_{C_0} \quad \textit{and} \quad \widetilde{\Psi}_{C_1} \to \widetilde{\Psi}_E \circ \widetilde{\Psi}_{C_0}.$$

Proof. Consider the commutative diagram with Cartesian squares

$$\begin{tikzcd}
 & & & \mathrm{N}_X(C_1) \arrow[dl, bend right, "i'"'] \arrow[d, "i_0"] \\
 & & \mathrm{Df}_X(C_1) & \mathrm{Df}_{X|C_0}(C_1) \arrow[l, "i'_1"'] \\
 & X \times \mathrm{T}^\circ_X(C_1) \arrow[ur, bend left, "j'"] \arrow[r, "j_1"] \arrow[d, "q''"] \arrow[dl, bend right, "p'"'] & \mathrm{Df}_X(C_1) \times_{\mathrm{T}_{X|C_0}(C_1)} \mathrm{T}^\circ_{X|C_0}(C_1) \arrow[u, "j'_0"] \arrow[d, "q'"] & \mathrm{N}_X(C_0) \times \mathrm{T}^\circ_{X|C_0}(C_1) \arrow[l, "i_1"'] \arrow[u, "j_0"] \arrow[d, "q"] \\
X & X \times \mathrm{T}^\circ_X(C_0) \arrow[l, "p"'] \arrow[r, "j"] & \mathrm{Df}_X(C_0) & \mathrm{N}_X(C_0). \arrow[l, "i"']
\end{tikzcd}$$

The composite functor $\widetilde{\Psi}_E \circ \widetilde{\Psi}_{C_0}$ can be identified with

$$i_0^* j_{0,*}(\mathcal{U}_{\mathrm{T}^\circ_{X|C_0}(C_1)} \otimes q^* i^* j_*(\mathcal{U}_{\mathrm{T}^\circ_X(C_0)} \otimes p^*(-))). \tag{3.12}$$

The morphism q' being smooth, we have an equivalence $q^* i^* j_* \simeq i_1^* j_{1,*} q''^*$ by the smooth base change theorem. Using the definition of $\mathcal{U}_{\mathrm{T}^\circ_{X|C_0}(C_1)}$, we see easily that the natural transformation

$$\mathcal{U}_{\mathrm{T}^\circ_{X|C_0}(C_1)} \otimes i_1^* j_{1,*}(-) \to j_{1,*} i_1^*(\mathcal{U}_{\mathrm{T}^\circ_{X|C_0}(C_1)} \otimes -)$$

is an equivalence. It follows that the functor in (3.12) is equivalent to

$$i_0^* j_{0,*} i_1^* j_{1,*}(\mathcal{U}_{\mathrm{T}^\circ_{X|C_0}(C_1)} \otimes q''^*(\mathcal{U}_{\mathrm{T}^\circ_X(C_0)} \otimes p^*(-))). \tag{3.13}$$

Since $\mathcal{U}_{\mathrm{T}^\circ_{X|C_0}(C_1)} \otimes q''^* \mathcal{U}_{\mathrm{T}^\circ_X(C_0)} \simeq \mathcal{U}_{\mathrm{T}^\circ_X(C_1)}$, we obtain an equivalence

$$\widetilde{\Psi}_E \circ \widetilde{\Psi}_{C_0} \simeq i_0^* j_{0,*} i_1^* j_{1,*}(\mathcal{U}_{\mathrm{T}^\circ_X(C_1)} \otimes p'^*(-)). \tag{3.14}$$

On the other hand, we have a natural transformation $i'^* j'^* \to i_0^* j_{0,*} i_1^* j_{1,*}$ induced by the exchange morphism $i_1'^* j'_{0,*} \to j_{0,*} i_1^*$. This finishes the proof since $\widetilde{\Psi}_{C_1} = i'^* j'_*(\mathcal{U}_{\mathrm{T}^\circ_X(C_1)} \otimes p'^*(-))$. □

A version of the following result was proven in [IS18, Theorems 4.1.1 & 4.2.1] in the case of $\mathbf{MSh}_{\mathrm{nis}}(-)$. We give below a considerably simpler proof which requires however étale descent (and thus cannot be used in the case of $\mathbf{MSh}_{\mathrm{nis}}(-)$).

Proposition 3.39. *Assume that $\mathcal{H}$ satisfies étale descent.*

(i) Let X be a regularly stratified S-scheme and $C \subset X$ a stratum. Then, for any $M \in \mathcal{H}(X^\circ)$, the object $\widetilde{\Psi}_C(M) \in \mathcal{H}(\mathrm{N}^\circ_X(C))$ is quasi-unipotent (see Notation 3.28).[5] Thus $\widetilde{\Psi}_C$ induced a functor $\widetilde{\Psi}_C : \mathcal{H}(X^\circ) \to \mathcal{H}(\mathrm{N}^\circ_X(C))_{\mathrm{qun}}$.

(ii) Let T be a split torus-embedding admitting a stratum o_T of relative dimension 0. Let X be a regularly stratified S-scheme, $f : X \to T$ a morphism of stratified S-schemes and $C \subset X$ a stratum above o_T. Assume that the induced morphism $\mathrm{T}_X(C) \to T$ is an isomorphism. Then f determines a section $s_f : C \to \mathrm{N}^\circ_X(C)$, and there is an equivalence

$$s_f^* \circ \widetilde{\Psi}_C \xrightarrow{\sim} \Psi_{f, o_T}$$

between functors from $\mathcal{H}(X^\circ)$ to $\mathcal{H}(C)$.

Proof. In this statement, the functor $\widetilde{\Psi}_C$ is the one discussed in Remark 3.35. There is a commutative diagram with Cartesian squares

$$\begin{tikzcd} X^\circ \times \mathrm{T}^\circ_X(C) \arrow[r, "j"] \arrow[d, "\alpha'"] & \mathrm{Df}^\flat_X(C) \arrow[d, "\alpha"] & \mathrm{N}^\circ_X(C) \arrow[l, "i"'] \arrow[d, "\alpha''"] \\ X^\circ \arrow[r, "j"] & X & C, \arrow[l, "i"'] \end{tikzcd}$$

and the morphism α is smooth. Going back to the construction of $\widetilde{\Psi}_C$, we see that (i) follows from the following assertion. The object $i^* j_*(\mathrm{id}_{X^\circ} \times e_r)_* \alpha'^* M$ is quasi-unipotent for every $r \geq 1$ and $M \in \mathcal{H}(X^\circ)$. (As usual, $e_r : \mathrm{T}^\circ_X(C) \to \mathrm{T}^\circ_X(C)$ is elevation to the r-th power.) Given a Kummer étale cover $X' \to X$, with X considered as a log scheme in the obvious way, the induced morphism $\mathrm{N}^\circ_{X'}(C') \to \mathrm{N}^\circ_X(C)$, with C' a connected component of $X' \times_X C$, is étale. Moreover, letting C' vary, we obtain an étale cover $(\mathrm{N}^\circ_{X'}(C') \to \mathrm{N}^\circ_X(C))_{C'}$ of $\mathrm{N}^\circ_X(C)$. Using étale descent, we may thus replace

[5] Extended to torsors over torus in the obvious way.

X with X'. Taking X' sufficiently ramified around C, we reduce to the case $r = 1$. Said differently, we are left to check that $i^* j_* \alpha'^* M$ is quasi-unipotent. This is obvious since $i^* j_* \alpha'^* M \simeq \alpha'' i^* j_* M$ by the smooth base change theorem.

We now prove part (ii). Fix an identification $T = \mathrm{Spec}(\mathcal{O}_S[t_1, \dots, t_n])$. There is a diagonal embedding $T \to \mathrm{Df}_T(o_T)$ given by the $\mathcal{O}_S$-algebra homomorphism

$$\mathcal{O}_S\left[\frac{t_1}{t_1'}, \dots, \frac{t_n}{t_n'}, t_1', \dots, t_n'\right] \to \mathcal{O}_S[t_1, \dots, t_n]$$

sending t_i and t_i' to t_i. One immediately sees that the composition

$$\mathrm{Df}_X(C) \times_{\mathrm{Df}_T(o_T)} T \to \mathrm{Df}_X(C) \to X$$

is an isomorphism. Moreover, the induced map $X \to \mathrm{Df}_X(C)$ factors through $\mathrm{Df}_X^\flat(C)$. This given a commutative diagram with Cartesian squares

$$\begin{tikzcd}
X^\circ \arrow[r, "j"] \arrow[d, "\beta'"] & X \arrow[d, "\beta"] & C \arrow[l, "i"'] \arrow[d, "\beta''"] \\
X^\circ \times \mathrm{T}_X^\circ(C) \arrow[r, "j"] & \mathrm{Df}_X^\flat(C) & \mathrm{N}_X^\circ(C). \arrow[l, "i"']
\end{tikzcd}$$

The section s_f in the statement is the morphism β''. We have an obvious natural transformation

$$\beta''^* i^* j^* \to i^* j^* \beta'^*$$

and we need to show that it induces equivalences on objects of the form $(\mathrm{id}_{X^\circ} \times e_r)_* \alpha'^* M$ with $r \geq 1$ and $M \in \mathcal{H}(X^\circ)$. Arguing as in the proof of (i), we reduce to the case $r = 1$, which follows from the smooth base change theorem. □

Notation 3.40. Let X be a regularly stratified S-scheme and $C \subset X$ a stratum. We define a functor

$$\chi_C : \mathcal{H}(X^\circ) \to \mathcal{H}(C)$$

by $\chi_C = i^* j_*$, with $j : X^\circ \to X$ and $i : C \to X$.

Proposition 3.41. *Assume that $\mathcal{H}$ satisfies étale descent. Let X be a regularly stratified S-scheme and $C \subset X$ a stratum. We denote by $q : \mathrm{N}_X^\circ(C) \to C$ the obvious projection. Then, there are equivalences*

$$\chi_C \simeq q_* \circ \widetilde{\Upsilon}_C \simeq q_* \circ \widetilde{\Psi}_C.$$

Proof. Consider the commutative diagram with Cartesian squares

$$\begin{tikzcd}
X^\circ \times \mathrm{T}_X^\circ(C) \arrow[r, "j"] \arrow[d, "q''"] & \mathrm{Df}_X^\flat(C) \arrow[d, "q'"] & \mathrm{N}_X^\circ(C) \arrow[l, "i"'] \arrow[d, "q"] \\
X^\circ \arrow[r, "j"] & X & C. \arrow[l, "i"']
\end{tikzcd}$$

There is a natural transformation

$$q_* i^* j_* \to i^* j_* q''_* = \chi_C \circ q''_*. \tag{3.15}$$

We claim that the natural transformation in (3.15) is an equivalence when evaluated at objects of the form $(\mathrm{id}_{X^\circ} \times e_r)_* q''^*(M)$ for $r \geq 1$ and $M \in \mathcal{H}(X^\circ)$. The question being local for the Kummer log étale topology, we may assume that $r = 1$. In this case, the result follows from the fact that

q_*q^* is locally on X a direct sum of Tate twists. This said, we are left to show that the natural transformation

$$\mathrm{id} \to q''_*(\mathcal{U}_{\mathrm{T}^\circ_X(C)} \otimes q''^*(-))$$

is an equivalence. This follows from the fact that $q''_*(\mathcal{U}_{\mathrm{T}^\circ_X(C)}) \simeq \mathbf{1}$; see Lemma 3.27(ii). □

Lemma 3.42. *Let $p : Q \to S$ be the projection of a torsor on S over a split torus. Consider the functor $\mathcal{H}(Q) \to \mathcal{H}(S; p_*\mathbf{1})$ sending $M \in \mathcal{H}(Q)$ to p_*M considered as a module over the commutative algebra $p_*\mathbf{1}$. This functor restricts to an equivalence of ∞-categories*

$$\mathcal{H}(Q)_{\mathrm{un}} \simeq \mathcal{H}(S; p_*\mathbf{1}).$$

Notation 3.43. Let X be a regularly stratified S-scheme and $C \subset X$ a stratum. We denote by

$$\widetilde{\chi}_C : \mathcal{H}(X^\circ) \to \mathcal{H}(C; \chi\mathbf{1})$$

the functor sending $M \in \mathcal{H}(X^\circ)$ to $\chi_C(M)$ considered as a module over $\chi_C(\mathbf{1})$.

Corollary 3.44. *Let X be a regularly stratified S-scheme and $C \subset X$ a stratum. There a commutative triangle*

$$\begin{tikzcd} \mathcal{H}(X^\circ) \arrow[r, "\widetilde{\Upsilon}_C"] \arrow[dr, "\widetilde{\chi}_C"'] & \mathcal{H}(\mathrm{N}^\circ_X(C)) \arrow[d, "\sim"] \\ & \mathcal{H}(C; \chi\mathbf{1}) \end{tikzcd}$$

where the vertical arrow is an equivalence of ∞-categories.

Proof. Let $q : \mathrm{N}^\circ_X(C) \to C$ be the obvious projection, and denote by $\widetilde{q}_* : \mathcal{H}(\mathrm{N}^\circ_X(C)) \simeq \mathcal{H}(C; q_*\mathbf{1})$ the equivalence provided by Lemma 3.42. Using Corollary 3.36 and Proposition 3.41, we obtain an equivalence of commutative algebras $q_*\mathbf{1} \simeq \chi_C(\mathbf{1})$. Modulo this identification, Proposition 3.41 gives an equivalence $\widetilde{q}_* \circ \widetilde{\Upsilon} \simeq \widetilde{\chi}_C$ as needed. □

3.3. **Logarithmicity and tameness at the boundary.**

In this section, we introduce and study the notions of logarithmicity and tameness at the boundary of regularly stratified schemes. We fix a Voevodsky pullback formalism

$$\mathcal{H}(-)^{\otimes} : \mathrm{Sch}/S \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}),$$

defined over a noetherian base scheme S, and assumed to satisfy absolute purity, étale descent and compact generation.

Lemma 3.45. *Let $j : U \to X$ be an open immersion.*

- *(i) The functor $\widetilde{j}^* : \mathcal{H}(X; j_*\mathbf{1}) \to \mathcal{H}(U)$, restricted the full sub-∞-category of $\mathcal{H}(X; j_*\mathbf{1})$ spanned by the dualizable objects, is fully faithful.*
- *(ii) Let $\widetilde{j}_* : \mathcal{H}(U) \to \mathcal{H}(X; j_*\mathbf{1})$ be the right adjoint to $\widetilde{j}^*$. If $N \in \mathcal{H}(X; j_*\mathbf{1})$ is dualizable, then the obvious morphism $N \to \widetilde{j}_*\widetilde{j}^*N$ is an equivalence.*

Proof. Properties (i) and (ii) follow from the following assertion. Given two $j_*\mathbf{1}$-modules M and N, with N dualizable, the map

$$\mathrm{Map}_{\mathcal{H}(X; j_*\mathbf{1})}(M, N) \to \mathrm{Map}_{\mathcal{H}(U)}(\widetilde{j}^*(M), \widetilde{j}^*(N))$$

is an equivalence. Since the functor $\widetilde{j}^*$ is monoidal, we may replace M with $M \otimes_{j_*\mathbf{1}} N^\vee$, and reduce to the case where $N = j_*\mathbf{1}$. Said differently, it is enough to show that the map

$$\mathrm{Map}_{\mathcal{H}(X;j_*\mathbf{1})}(M, j_*\mathbf{1}) \to \mathrm{Map}_{\mathcal{H}(U)}(\widetilde{j}^*(M), \mathbf{1})$$

is an equivalence. This is clear since the right adjoint to $\widetilde{j}^*$ takes $\mathbf{1}$ to $j_*\mathbf{1}$ considered as a $j_*\mathbf{1}$-module in the obvious way. □

Notation 3.46. Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category and $M \in \mathcal{C}$ a dualizable object. We denote by $\langle M\rangle^\otimes$ the full monoidal sub-∞-category of $\mathcal{C}^\otimes$ generated by M and its dual.

Proposition 3.47. *Let X be a regularly stratified scheme, and let $M \in \mathcal{H}(X^\circ)$ be a dualizable object. We denote by $j : X^\circ \to X$ the obvious inclusion. The following conditions are equivalent.*

(i) The restriction of the right-lax monoidal functor $\widetilde{\Upsilon}_C : \mathcal{H}(X^\circ)^\otimes \to \mathcal{H}(\mathrm{N}^\circ_X(C))^\otimes$ to $\langle M\rangle^\otimes$ is monoidal for every stratum $C \subset X$.

(ii) The restriction of the right-lax monoidal functor $\widetilde{\chi}_C : \mathcal{H}(X^\circ)^\otimes \to \mathcal{H}(C;\chi\mathbf{1})^\otimes$ to $\langle M\rangle^\otimes$ is monoidal for every stratum $C \subset X$.

(iii) The $j_*\mathbf{1}$-module j_*M is dualizable as an object of $\mathcal{H}(X; j_*\mathbf{1})^\otimes$.

(iv) There is a dualizable object N of $\mathcal{H}(X; j_*\mathbf{1})^\otimes$ such that $M \simeq j^*(N)$.

Proof. The equivalence between (i) and (ii) is immediate from Corollary 3.44 and the equivalence between (iii) and (iv) follows from Lemma 3.45. We will prove the proposition by showing the implications (ii) ⇒ (iii) and (iv) ⇒ (ii). We split the proof in two parts.

Part 1. Here we assume that M satisfies property (ii) and show that the $j_*\mathbf{1}$-module j_*M is dualizable. For this, it is enough to check that the obvious morphism

$$j_*M \otimes_{j_*\mathbf{1}} \underline{\mathrm{Hom}}_{j_*\mathbf{1}}(j_*M, j_*\mathbf{1}) \to \underline{\mathrm{Hom}}_{j_*\mathbf{1}}(j_*M, j_*M)$$

is an equivalence. By adjunction, we may rewrite this morphism as

$$j_*M \otimes_{j_*\mathbf{1}} j_*\underline{\mathrm{Hom}}(M, \mathbf{1}) \to j_*\underline{\mathrm{Hom}}(M, M).$$

Since M is dualizable, we may rewrite again this morphism as

$$j_*M \otimes_{j_*\mathbf{1}} j_*M^\vee \to j_*(M \otimes M^\vee)$$

where $M^\vee$ is the dual of M. By the localisation property, it is enough to show that the above morphism becomes an equivalence after restriction to each stratum $C \subset X$. This restriction is the obvious morphism

$$\widetilde{\chi}_C(M) \otimes_{\chi_C\mathbf{1}} \widetilde{\chi}_C(M^\vee) \to \widetilde{\chi}_C(M \otimes M^\vee)$$

which is indeed an equivalence by (ii).

Part 2. Assume that property (iv) is satisfied. By Lemma 3.45, the functor j^* induces an equivalence of symmetric monoidal ∞-categories $\langle N\rangle^\otimes \to \langle M\rangle^\otimes$ with inverse given by $E \mapsto \widetilde{j}_*(E)$. It follows that $\widetilde{\chi}_C|_{\langle M\rangle^\otimes}$ is the composition of

$$\langle M\rangle^\otimes \simeq \langle N\rangle^\otimes \hookrightarrow \mathcal{H}(X; j_*\mathbf{1})^\otimes \xrightarrow{\iota_C^*} \mathcal{H}(C;\chi\mathbf{1})^\otimes$$

where $\iota_C : C \to X$ is the obvious inclusion. Since ι_C^* is symmetric monoidal, the same is true for the functor $\widetilde{\chi}_C|_{\langle M\rangle^\otimes}$. This finishes the proof of the proposition. □

Definition 3.48. Let X be a regularly stratified S-scheme and $M \in \mathcal{H}(X^\circ)$ dualizable. We say that M is logarithmic at the boundary of X if it satisfies the equivalent conditions of Proposition 3.47.

Corollary 3.49. *Let X and Y be regularly stratified S-schemes. Let $f : Y \to X$ be a morphism of stratified S-schemes sending an open stratum to an open stratum, and let $f^\circ : Y^\circ \to X^\circ$ be the induced morphism. The functor $f^{\circ,*} : \mathcal{H}(X^\circ) \to \mathcal{H}(Y^\circ)$ preserves logarithmicity at the boundary.*

Proof. It is easiest to see that $f^{\circ,*}$ preserves property (iv) of Proposition 3.47. Indeed, let $j_X : X^\circ \hookrightarrow X$ and $j_Y : Y^\circ \hookrightarrow Y$ be the obvious inclusions. If $M \in \mathcal{H}(X^\circ)$ is equivalent to $j_X^* N$, with N a dualizable $j_{X,*}\mathbf{1}$-module, then $f^{\circ,*}M$ is equivalent to $j_Y^*(f^*N \otimes_{f^*j_{X,*}\mathbf{1}} j_{Y,*}\mathbf{1})$ and the $j_{Y,*}\mathbf{1}$-module $f^*N \otimes_{f^*j_{X,*}\mathbf{1}} j_{Y,*}\mathbf{1}$ is dualizable. □

Lemma 3.50. *Let X be a regularly stratified S-scheme and $Y \subset X$ a regular constructible locally closed subscheme. Denote by $i : Y \to X$, $j_X : X^\circ \to X$ and $j_Y : Y^\circ \to Y$ the obvious inclusions. Let N be a dualizable $j_{X,*}\mathbf{1}$-module. Then $i^*(N)$ is a dualizable $j_{Y,*}\mathbf{1}$-module.*

Proof. Since i^* is monoidal, we see that $i^*(N)$ is dualizable as a $i^*j_{X,*}\mathbf{1}$-module. By [Lur17, Proposition 4.6.4.4], it is enough to prove that $i^*j_{X,*}\mathbf{1}$ is a dualizable $j_{Y,*}\mathbf{1}$-module. By [Ayo07b, Théorème 3.3.10] applied to the canonical specialisation system (in the sense of [Ayo07b, Exemple 3.1.4]), we have an equivalence $i^*j_{X,*}\mathbf{1} \simeq j_{Y,*}j_Y^*(i^*j_{X,*}\mathbf{1})$ and $j_Y^*(i^*j_{X,*}\mathbf{1})$ is, locally for the Zariski topology, a finite direct sum of Tate twists. □

Proposition 3.51. *Let X be a regularly stratified S-scheme, and let $M \in \mathcal{H}(X^\circ)$ be dualizable and logarithmic at the boundary of X.*

- *(i) Let C be a stratum of X. Then $\chi_C(M) \in \mathcal{H}(C)$ is dualizable and logarithmic at the boundary of $\overline{C}$.*
- *(ii) Let $C_0 \geq C_1$ be strata of X. The natural morphism $\chi_{C_1}(M) \to \chi_{C_1 \subset \overline{C}_0} \circ \chi_{C_0}(M)$ is an equivalence. (We write '$\chi_{C_1 \subset \overline{C}_0}$' to indicate that C_1 is considered as a stratum of $\overline{C}_0$.)*

Proof. Property (i) follows from Lemma 3.50. In the situation considered in (ii), we denote by $j : X^\circ \to X$, $j_0 : C_0 \to \overline{C}_0$ and $i : C_1 \to \overline{C}_0$ the obvious inclusions, and we set $N = j_*M$. We need to show that $N|_{C_1} \to i^*j_{0,*}(N|_{C_0})$ is an equivalence. By Lemma 3.50, the $j_{0,*}\mathbf{1}$-module $N|_{\overline{C}_0}$ is dualizable. By Lemma 3.45(ii), this implies that $N|_{\overline{C}_0} \to j_{0,*}(N|_{C_0})$ is an equivalence. The result follows since $N|_{C_1} \simeq (N|_{\overline{C}_0})|_{C_1}$. □

Proposition 3.51 admits a variant for the functors $\widetilde{\Upsilon}_C$.

Proposition 3.52. *Let X be a regularly stratified S-scheme, and let $M \in \mathcal{H}(X^\circ)$ be dualizable and logarithmic at the boundary.*

- *(i) Let C be a stratum of X. Then $\widetilde{\Upsilon}_C(M) \in \mathcal{H}(\mathrm{N}_X^\circ(C))$ is dualizable and logarithmic at the boundary of $\mathrm{N}_X(C)$.*
- *(ii) Let $C_0 \geq C_1$ be strata of X. Let $E \subset \mathrm{N}_X(C_0)$ be the largest stratum over $C_1 \subset \overline{C}_0$ relative to the projection $\mathrm{N}_X(C_0) \to \overline{C}_0$. The natural morphism $\widetilde{\Upsilon}_{C_1}(M) \to \widetilde{\Upsilon}_E \circ \widetilde{\Upsilon}_{C_0}(M)$ is an equivalence. (See Proposition 3.38.)*

Proof. To prove (i), we apply Proposition 3.51(i) to the regularly stratified scheme $\mathrm{Df}_X(C)$ and the object $\mathcal{L}_{\mathrm{T}_X^\circ(C)} \otimes p'^*M$ in $\mathcal{H}(X^\circ \times \mathrm{T}_X^\circ(C))$. (Here, we use the notations of Definition 3.34 and Remark 3.35.) It follows that $\widetilde{\Upsilon}_C(M)$ is a filtered colimit of dualizable objects which are logarithmic at the boundary. Thus, it remains to see that $\widetilde{\Upsilon}_C(M)$ is dualizable. This follows immediately from the characterisation (i) in Proposition 3.47. A proof of (ii) can be obtained similarly by applying Proposition 3.51(ii) to $\mathrm{Df}_X(C)$. □

Our next task is to prove a variant of Proposition 3.52 for the functors $\widetilde{\Psi}_C$ and for a larger class of dualizable objects in $\mathcal{H}(X^\circ)$, namely those which we call tame at the boundary (see Definition 3.54 below). We will need the following result.

Proposition 3.53. *Let X be a regularly stratified S-scheme, C a stratum of X, and $M \in \mathcal{H}(X^\circ)$ dualizable and logarithmic at the boundary. Then, the morphism $\widetilde{\Upsilon}_C(M) \to \widetilde{\Psi}_C(M)$ is an equivalence.*

Proof. As usual, we consider the commutative diagram with Cartesian squares

$$\begin{tikzcd}
X^\circ \times \mathrm{T}^\circ_X(C) \arrow[r, "j"] \arrow[d, "q''"] & \mathrm{Df}^\flat_X(C) \arrow[d, "q'"] & \mathrm{N}^\circ_X(C) \arrow[l, "i"'] \arrow[d, "q"] \\
X^\circ \arrow[r, "j"] & X & C. \arrow[l, "i"']
\end{tikzcd}$$

By Corollary 3.49, q''^*M is logarithmic at the boundary of $\mathrm{Df}^\flat_X(C)$. It follows that the obvious morphism

$$j_*(\mathcal{U}_{\mathrm{T}^\circ_X(C)}) \otimes_{j_*\mathbf{1}} j_*(q''^*(M)) \to j_*(\mathcal{U}_{\mathrm{T}^\circ_X(C)} \otimes q''^*(M)) \tag{3.16}$$

is an equivalence, since it can be identified with the equivalence

$$\underline{\mathrm{Hom}}_{j_*\mathbf{1}}(j_*(q''^*(M^\vee)), j_*(\mathcal{U}_{\mathrm{T}^\circ_X(C)})) \simeq j_*\underline{\mathrm{Hom}}(q''^*(M^\vee), \mathcal{U}_{\mathrm{T}^\circ_X(C)}).$$

Applying i^* to the morphism in (3.16), we deduce an equivalence

$$\widetilde{\Psi}_C(M) \simeq \widetilde{\Psi}_C(\mathbf{1}) \otimes_{i^*j_*\mathbf{1}} i^*j_*(q''^*(M)). \tag{3.17}$$

Arguing similarly with $\mathcal{L}_{\mathrm{T}^\circ_X(C)}$ instead of $\mathcal{U}_{\mathrm{T}^\circ_X(C)}$, we also obtain an equivalence

$$\widetilde{\Upsilon}_C(M) \simeq \widetilde{\Upsilon}_C(\mathbf{1}) \otimes_{i^*j_*\mathbf{1}} i^*j_*(q''^*(M)). \tag{3.18}$$

The result follows now from Corollary 3.36. □

Definition 3.54. Let X be a regularly stratified S-scheme and $M \in \mathcal{H}(X^\circ)$ dualizable. We say that M is tame at the boundary of X if, locally for the Kummer log étale topology on X, M is logarithmic at the boundary of X.

Theorem 3.55. *Let X be a regularly stratified S-scheme, and let $M \in \mathcal{H}(X^\circ)$ be dualizable and tame at the boundary of X.*

- *(i) Let C be a stratum of X. Then $\widetilde{\Psi}_C(M) \in \mathcal{H}(\mathrm{N}^\circ_X(C))$ is dualizable and tame at the boundary of $\mathrm{N}_X(C)$. Moreover, the restriction of $\widetilde{\Psi}_C$ to $\langle M\rangle^\otimes$ is monoidal.*
- *(ii) Let $C_0 \geq C_1$ be strata of X. Let $E \subset \mathrm{N}_X(C_0)$ be the largest stratum over $C_1 \subset \overline{C}_0$ relative to the projection $\mathrm{N}_X(C_0) \to \overline{C}_0$. The natural morphism $\widetilde{\Psi}_{C_1}(M) \to \widetilde{\Psi}_E \circ \widetilde{\Psi}_{C_0}(M)$ is an equivalence. (See Proposition 3.38.)*

Proof. Since the problem is local for the Kummer log étale topology on X, we may reduce to the case where M is logarithmic at the boundary of X. In this case, the theorem follows by combining Propositions 3.52 and 3.53. □

Definition 3.56. Let X be a stratified S-scheme. An object $M \in \mathcal{H}(X)$ is said to be constructible (resp. ind-constructible) if $M|_C \in \mathcal{H}(C)$ is dualizable (resp. ind-dualizable) for every stratum $C \subset X$. We denote by $\mathcal{H}_{\mathrm{ct}}(X)$ (resp. $\mathcal{H}_{\mathrm{ict}}(X)$) the full sub-$\infty$-category of $\mathcal{H}(X)$ spanned by the constructible (resp. ind-constructible) objects. We note that the ∞-category $\mathcal{H}_{\mathrm{ict}}(X)$ is equivalent to the indization of $\mathcal{H}_{\mathrm{ct}}(X)$.

Definition 3.57. Let X be a regularly stratified S-scheme, and $U \subset X$ an open stratum of X. We denote by $\mathcal{H}_{\mathrm{tame}}(U/X)$ (resp. $\mathcal{H}_{\mathrm{log}}(U/X)$) the full sub-$\infty$-category of $\mathcal{H}(U)$ spanned by dualizable objects which are tame (resp. logarithmic) at the boundary of X. We also denote by $\mathcal{H}_{\mathrm{itame}}(U/X)$ (resp. $\mathcal{H}_{\mathrm{ilog}}(U/X)$) for the full sub-$\infty$-category of $\mathcal{H}(U)$ generated under colimits by $\mathcal{H}_{\mathrm{tame}}(U/X)$ (resp. $\mathcal{H}_{\mathrm{log}}(U/X)$). If X is understood and there is no risk of confusion, we simply write $\mathcal{H}_{\mathrm{tame}}(U)$ and $\mathcal{H}_{\mathrm{itame}}(U)$ (resp. $\mathcal{H}_{\mathrm{log}}(X)$ and $\mathcal{H}_{\mathrm{ilog}}(X)$).

Definition 3.58. Let X be a regularly stratified S-scheme, and $U \subset X$ a constructible open subscheme.

(i) An object $M \in \mathcal{H}(U)$ is said to be tamely constructible (resp. logarithmically constructible) with respect to X if $M|_C \in \mathcal{H}(C)$ is dualizable and tame (resp. dualizable and logarithmic) at the boundary of $\overline{C}$ for every stratum $C \subset U$. (Here the closure $\overline{C}$ is taken in X.) We denote by $\mathcal{H}_{\text{ct-tm}}(U/X)$ (resp. $\mathcal{H}_{\text{ct-log}}(U/X)$) the full sub-$\infty$-category of $\mathcal{H}(U)$ spanned by the tamely (resp. logarithmically) constructible objects. When X is understood and there is no risk of confusion, we sometimes write simply $\mathcal{H}_{\text{ct-tm}}(U)$ (resp. $\mathcal{H}_{\text{ct-log}}(U)$); this is for instance systematically used when $U = X$.

(ii) We denote by $\mathcal{H}_{\text{ict-tm}}(U/X)$ (resp. $\mathcal{H}_{\text{ict-log}}(U/X)$) the full sub-$\infty$-category of $\mathcal{H}(U)$ generated under colimits by the objects of $\mathcal{H}_{\text{ct-tm}}(U/X)$ (resp. $\mathcal{H}_{\text{ct-log}}(U/X)$). Objects in $\mathcal{H}_{\text{ict-tm}}(U/X)$ (resp. $\mathcal{H}_{\text{ict-log}}(U/X)$) are said to be tamely (resp. logarithmically) ind-constructible. When X is understood and there is no risk of confusion, we sometimes write $\mathcal{H}_{\text{ict-tm}}(U)$ (resp. $\mathcal{H}_{\text{ict-log}}(U)$); this is for instance systematically used when $U = X$.

Remark 3.59. The sub-∞-categories $\mathcal{H}_{\mathrm{ict}}(-) \subset \mathcal{H}(-)$ are stable under tensor product and pullback along morphisms of stratified S-schemes. In particular, we obtain a $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaf

$$\mathcal{H}_{\mathrm{ict}}(-)^{\otimes} : (\text{Sch-}\Sigma/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

Similarly, letting $(\text{Sch-}\Sigma/S)^{\mathrm{op}}_{\mathrm{open}}$ be the category of constructible open immersions of regularly stratified S-schemes, we obtain two $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves

$$\mathcal{H}_{\text{ict-log}}(-/-)^{\otimes} \text{ and } \mathcal{H}_{\text{ict-tm}}(-/-)^{\otimes} : (\text{Reg-}\Sigma/S)^{\mathrm{op}}_{\mathrm{open}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

(This relies on Corollary 3.49.)

Proposition 3.60. *Let X be a regularly stratified S-scheme, $U \subset X$ a constructible open subscheme and $Y \subset X$ a regular constructible locally closed subscheme. We form the Cartesian square*

$$\begin{array}{ccc} V & \xrightarrow{j'} & Y \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ U & \xrightarrow{j} & X. \end{array}$$

Assume that V is dense in Y. Then, for any $M \in \mathcal{H}_{\text{ict-tm}}(U/X)$, the natural morphism

$$i^* j_* M \to j'_* i'^* M$$

is an equivalence.

Proof. Without loss of generality, we may assume that X and Y are connected. The question is local on X for the étale topology. Thus, we may assume that $M = e'_* M'$, where $e : X' \to X$ is a finite Kummer log étale morphism, $e' : U' \to U$ its base change and $M' \in \mathcal{H}_{\text{ct-log}}(U'/X')$. Using the finite base change theorem, we are reduced to treat the case where M is itself logarithmically constructible. We may even assume that $M = \iota_{C,*}N$, where $C \subset X$ is a stratum contained in U, $\iota_C : C \to U$ its inclusion and $N \in \mathcal{H}(C)$ dualizable and logarithmic at the boundary of $\overline{C}$. If $\overline{C}$ is disjoint from Y or U, there is nothing to prove. Thus, we may assume that $C \subset U$ and that $\overline{C} \cap Y \neq \emptyset$. Since X is regularly stratified, this actually implies that $Y \subset \overline{C}$. Thus, we may replace X with $\overline{C}$ and assume that $C = X^\circ$. We now write $u : X^\circ \to X$ and $v : Y^\circ \to Y$ for the obvious inclusions. We see that it is enough to show that the morphism $i^*u_*N \to v_*v^*i^*u_*N$ is an equivalence. Now, recall that the $u_*\mathbf{1}$-module u_*N is dualisable. It follows that the $i^*u_*\mathbf{1}$-module i^*u_*N is also dualisable. Using [Ayo07b, Théorème 3.3.10] in the case of the canonical specialisation system (in the sense of [Ayo07b, Exemple 3.1.4]), we obtain an equivalence of commutative algebras $i^*u_*\mathbf{1} \simeq v_*v^*i^*u_*\mathbf{1}$. Since $v_*v^*i^*u_*\mathbf{1}$ is dualizable as a $v_*\mathbf{1}$-module, we deduce that i^*u_*N is a dualizable $v_*\mathbf{1}$-module. We conclude using Lemma 3.45. □

Corollary 3.61. *Let X be a regularly stratified S-scheme and $C \subset X$ a stratum. We denote by $j : X^\circ \to X$ and $j' : \mathrm{N}^\circ_X(C) \to \mathrm{N}_X(C)$ the obvious inclusions. Then, we have commutative squares of ∞-categories*

$$\begin{array}{ccc} \mathcal{H}_{\text{ict-log}}(X^\circ) & \xrightarrow{\widetilde{\Upsilon}_C} & \mathcal{H}_{\text{ict-log}}(\mathrm{N}^\circ_X(C)) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j'_*} \\ \mathcal{H}_{\text{ict-log}}(X) & \xrightarrow{\widetilde{\Upsilon}_C} & \mathcal{H}_{\text{ict-log}}(\mathrm{N}_X(C)), \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathcal{H}_{\text{ict-tm}}(X^\circ) & \xrightarrow{\widetilde{\Psi}_C} & \mathcal{H}_{\text{ict-tm}}(\mathrm{N}^\circ_X(C)) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j'_*j'^*} \\ \mathcal{H}_{\text{ict-tm}}(X) & \xrightarrow{\widetilde{\Psi}_C} & \mathcal{H}_{\text{ict-tm}}(\mathrm{N}_X(C)). \end{array}$$

The same holds true if we replace j_* and j'_* with $j_!$ and $j'_!$ respectively. (In the squares above, logarithmic and tame constructibility is taken relatively to X and $\mathrm{N}_X(C)$.)

Proof. This follows immediately from Proposition 3.60 applied to $\mathrm{Df}_X(C)$ and well-chosen open and locally closed subschemes. □

Corollary 3.62. *Let X be a regularly stratified S-scheme, $Y \subset X$ a regular locally closed constructible subscheme, C a stratum of X and D a stratum of Y. We assume that D is open in $Y \cap \overline{C}$. Let $i : Y \to X$ and $i' : \mathrm{N}_Y(D) \to \mathrm{N}_X(C)$ be the obvious inclusions. Then, we have commutative squares of ∞-categories*

$$\begin{array}{ccc} \mathcal{H}_{\text{ict-log}}(X) & \xrightarrow{i^*} & \mathcal{H}_{\text{ict-log}}(Y) \\ \downarrow{\scriptstyle \widetilde{\Upsilon}_C} & & \downarrow{\scriptstyle \widetilde{\Upsilon}_D} \\ \mathcal{H}_{\text{ict-log}}(\mathrm{N}_X(C)) & \xrightarrow{i'^*} & \mathcal{H}_{\text{ict-log}}(\mathrm{N}_Y(D)), \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathcal{H}_{\text{ict-tm}}(X) & \xrightarrow{i^*} & \mathcal{H}_{\text{ict-tm}}(Y) \\ \downarrow{\scriptstyle \widetilde{\Psi}_C} & & \downarrow{\scriptstyle \widetilde{\Psi}_D} \\ \mathcal{H}_{\text{ict-tm}}(\mathrm{N}_X(C)) & \xrightarrow{i'^*} & \mathcal{H}_{\text{ict-tm}}(\mathrm{N}_Y(D)). \end{array}$$

Moreover, if $Y \subset X$ is closed, the above squares are right adjointable.

Proof. This follows immediately from Proposition 3.60 applied to $\mathrm{Df}_X(C)$ and well-chosen open and locally closed subschemes. □

Theorem 3.63. *Let X be a regularly stratified S-scheme.*

(i) Let C be a stratum of X. The functors $\widetilde{\Upsilon}_C$ and $\widetilde{\Psi}_C$ restrict to symmetric monoidal functors

$$\mathcal{H}_{\text{ict-log}}(X)^{\otimes} \xrightarrow{\widetilde{\Upsilon}_C} \mathcal{H}_{\text{ict-log}}(\mathrm{N}_X(C))^{\otimes} \quad \textit{and} \quad \mathcal{H}_{\text{ict-tm}}(X)^{\otimes} \xrightarrow{\widetilde{\Psi}_C} \mathcal{H}_{\text{ict-tm}}(\mathrm{N}_X(C))^{\otimes}. \tag{3.19}$$

(ii) Let $C_0 \geq C_1$ be strata of X. Let $E \subset \mathrm{N}_X(C_0)$ be the largest stratum over $C_1 \subset \overline{C}_0$ relative to the projection $\mathrm{N}_X(C_0) \to \overline{C}_0$. The natural transformations in Proposition 3.38 induce commutative triangles of symmetric monoidal functors

$$\begin{tikzcd} \mathcal{H}_{\text{ict-log}}(X)^{\otimes} \arrow[r, "\widetilde{\Upsilon}_{C_0}"] \arrow[dr, "\widetilde{\Upsilon}_{C_1}"'] & \mathcal{H}_{\text{ict-log}}(\mathrm{N}_X(C_0))^{\otimes} \arrow[d, "\widetilde{\Upsilon}_E"] \\ & \mathcal{H}_{\text{ict-log}}(\mathrm{N}_X(C_1))^{\otimes}, \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H}_{\text{ict-tm}}(X)^{\otimes} \arrow[r, "\widetilde{\Psi}_{C_0}"] \arrow[dr, "\widetilde{\Psi}_{C_1}"'] & \mathcal{H}_{\text{ict-tm}}(\mathrm{N}_X(C_0))^{\otimes} \arrow[d, "\widetilde{\Psi}_E"] \\ & \mathcal{H}_{\text{ict-tm}}(\mathrm{N}_X(C_1))^{\otimes}. \end{tikzcd}$$

Proof. The ∞-category $\mathcal{H}_{\text{ict-tm}}(X)$ is generated under colimits by objects of the form $\iota_{E,!}M$ where $E \subset X$ is a stratum, and $M \in \mathcal{H}(E)$ is dualizable and tame at the boundary of $\overline{E}$. Thus, using Corollaries 3.61 and 3.62, we are reduced to prove the Theorem with $\mathcal{H}_{\text{ict-tm}}(X)$ replaced with its sub-∞-category spanned by objects of the form $u_!M$, with $u : X^\circ \to X$ the obvious inclusion and $M \in \mathcal{H}(X^\circ)$ dualizable and tame at the boundary of X. Using Corollarie 3.61 again, the result follows from Theorem 3.55. (The case of $\widetilde{\Upsilon}$ is similar; one uses Proposition 3.52 instead.) □

In the remainder of this subsection, we consider the notions of logarithmicity and tameness at the boundary in the Betti realisation, and more generally for sheaves in the classical topology on complex varieties. We fix a commutative ring spectrum $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$.

Lemma 3.64. *Let D be a 1-dimension complex open disc, $o \in D$ its center and $D^* = D \smallsetminus \{o\}$. Let $n \geq 0$ be an integer and denote by $q : (D^*)^n \to \mathrm{pt}$ the obvious projection. The functor*

$$\widetilde{q}^* : \mathrm{Perf}_{q_*\Lambda} \to \mathbf{LS}((D^*)^n; \Lambda), \tag{3.20}$$

sending a perfect $q_*\Lambda$-module M to $q^*M \otimes_{q^*q_*\Lambda} \Lambda$, is fully faithful with image the stable idempotent complete full sub-∞-category of $\mathbf{LS}((D^*)^n; \Lambda)$ generated by the constant local system Λ_{cst}.

Proof. This is immediate. □

Definition 3.65. We denote by $\mathbf{LS}((D^*)^n; \Lambda)_{\mathrm{un}}$ the essential image of the functor $\widetilde{q}^*$ in (3.20). A local system over $(D^*)^n$ which belongs to $\mathbf{LS}((D^*)^n; \Lambda)_{\mathrm{un}}$ is said to be unipotent.

Proposition 3.66. *Let D be a 1-dimension complex open disc, $o \in D$ its center and $D^* = D \smallsetminus \{o\}$. Let $n \geq 0$ be an integer and denote by $j : (D^*)^n \to D^n$ the obvious inclusion. Let L be a local system of Λ-modules on $(D^*)^n$. The following conditions are equivalent.*

(i) The $j_*\Lambda$-module j_*L is dualizable.

(ii) The local system L is unipotent.

Proof. The implication (ii) ⇒ (i) is obvious, so we only need to show that (i) implies (ii). We split the proof in two parts. In the first part, we treat the case $n = 1$. In the second part, we treat the general case by induction on n.

Part 1. Let L be a local system on D^* such that the $j_*\Lambda$-module j_*L is dualizable. Let $i : o \to D$ be the complement of j, and let $q : D^* \to \mathrm{pt}$ be the obvious projection. Since L is a local system, we have an equivalence $q_*L \simeq i^*j_*L$. (Indeed, there is a cofinal system of deleted neighbourhoods of o whose cohomology with values in L is precisely q_*L.) Since i^* is monoidal, we deduce that q_*L is a dualizable $q_*\Lambda$-module. We define a new local system L' on D^* by

$$L' = \mathrm{cofib}(q^*q_*L \otimes_{q^*q_*\Lambda} \Lambda \to L).$$

The local system $L_0 = q^*q_*L \otimes_{q^*q_*\Lambda} \Lambda$ is unipotent by definition, so that we are left to show that L' is unipotent. Since $\widetilde{q}^*$ is fully faithful (by Lemma 3.64), the unit morphism $\mathrm{id} \to \widetilde{q}_*\widetilde{q}^*$ is an equivalence. This implies that the morphism $L_0 \to L$ induces an equivalence $q_*L_0 \simeq q_*L$. Thus, we may replace L with L' and assume that $q_*L \simeq 0$. As explained above, this is equivalent to the condition that $i^*j_*L \simeq 0$. Arguing as in the proof of Proposition 3.47, we see that this property holds true for any object in $\langle L\rangle^{\otimes}$. In particular, we see that $q_*(L \otimes_\Lambda L^\vee) \simeq 0$. Thus, the coevaluation morphism $\Lambda \to L \otimes_\Lambda L^\vee$ is necessarily zero, and this implies that $L \simeq 0$.

Part 2. Here we assume that $n \geq 2$ and that the implication (i) $\Rightarrow$ (ii) is known for $n-1$. Consider the commutative diagram

$$\begin{tikzcd}
 & D^{n-1} \times D^* \arrow[rr] \arrow[ddrr, "q"] & & D^{n-1} \times D \arrow[dd, "p"] \\
(D^*)^{n-1} \times D^* \arrow[ur, "j'''"] \arrow[r, "j'"] \arrow[dr, "q'"'] & (D^*)^{n-1} \times D \arrow[urr] \arrow[d, "p'"] & & \\
 & (D^*)^{n-1} \arrow[rr, "j''"] & & D^{n-1}.
\end{tikzcd}$$

By assumption, the $j'_*\Lambda$-module j'_*L is dualizable. Thus, by Step 1, we have an equivalence

$$q'^*q'_*L \otimes_{q'^*q'_*\Lambda} \Lambda \xrightarrow{\sim} L. \tag{3.21}$$

Set $L' = q'_*L$ considered as an object of $\mathbf{LS}((D^*)^{n-1}; q'_*\Lambda)$. We claim that L' satisfies the property (i) of the statement, i.e., that j''_*L' is a dualizable $j''_*q'_*\Lambda$-module. (Note that $q'_*\Lambda \simeq \Lambda \oplus \Lambda[-1]$ is a constant sheaf of commutative ring spectra on $(D^*)^{n-1}$.) To prove this, we note that we have an equivalence $j''_*q'_*\Lambda \simeq q_*j'''_*\Lambda$. Moreover, by a relative version of Lemma 3.64, we have a fully faithful symmetric monoidal functor

$$\widetilde{q}^* : \mathbf{Sh}(D^{n-1}; q_*j'''_*\Lambda) \to \mathbf{Sh}(D^{n-1} \times D^*; j'''_*\Lambda).$$

Thus, it is enough to show that $\widetilde{q}^*j''_*L'$ is dualizable. We have a chain of equivalences

$$\begin{aligned}
\widetilde{q}^*j''_*L' &\overset{(1)}{\simeq} q^*j''_*L' \otimes_{q^*q_*j'''_*\Lambda} j'''_*\Lambda \\
&\overset{(2)}{\simeq} j'''_*q'^*L' \otimes_{j'''_*q'^*q'_*\Lambda} j'''_*\Lambda \\
&\overset{(3)}{\simeq} j'''_*q'^*L' \otimes_{q_*q^*\Lambda} \Lambda \\
&\overset{(4)}{\simeq} j'''_*(q'^*q'_*L \otimes_{q'^*q'_*\Lambda} \Lambda) \\
&\overset{(5)}{\simeq} j'''_*L,
\end{aligned}$$

where (1) is by definition, (2) by the smooth base change theorem, (3) follows from the obvious equivalence $j'''_*q'^*q'_*\Lambda \simeq j'''_*\Lambda \otimes_\Lambda q^*q_*\Lambda$, (4) is given by the projection formula and (5) follows from the equivalence in (3.21). Our claim follows now since j'''_*L is dualizable over $j'''_*\Lambda$ by assumption. This said, we may apply the induction hypothesis to L' to deduce that it is unipotent as a local system of $q'_*\Lambda$-modules on $(D^*)^{n-1}$. We conclude using again the equivalence in (3.21). □

Lemma 3.67. *Let D be a 1-dimension complex open disc, $o \in D$ its center and $D^* = D \smallsetminus \{o\}$. Fix an integer $n \geq 0$, and let $\gamma_1, \ldots, \gamma_n$ be the generators of the fundamental group $\pi_1((D^*)^n, x) \simeq \mathbb{Z}^n$ at some base point x. Let L be a local system on $(D^*)^n$.*

(i) If L is unipotent, then the γ_i's act unipotently on L_x.

(ii) When Λ is an ordinary regular ring, the converse is also true: L is unipotent if and only if the γ_i's act unipotently on L_x.

Proof. Part (i) follows immediately from the fact that L_x, as a Λ-module with an action by $\mathbb{Z}^n$, is a successive extension of Λ endowed with the trivial action of $\mathbb{Z}^n$. If Λ is regular, the ordinary sheaves $\mathrm{H}_i(L)$ are also local systems. Thus, to prove the converse when Λ is regular, we may assume that L is an ordinary local system. Since a finite type ordinary Λ-module with a unipotent action by $\mathbb{Z}^n$ is a successive extension of finite type ordinary Λ-modules with a trivial action, we reduce to the case of a constant sheaf, which is clear. □

Definition 3.68. A local system of Λ-modules L over $(D^*)^n$ is said to be quasi-unipotent if there is a finite étale cover of the form $e : (D'^*)^n \to (D^*)^n$ such that e^*L is unipotent. We denote by $\mathbf{LS}((D^*)^n; \Lambda)_{\mathrm{qun}}$ the full sub-∞-category of $\mathbf{LS}((D^*)^n; \Lambda)$ spanned by the quasi-unipotent local systems.

More generally, we make the following definition.

Definition 3.69. Let W be a smooth complex variety and $W^\circ \subset W$ an open subset. We assume that every point of W admits a neighbourhood U such that the pair $(U, U \cap W^\circ)$ is isomorphic to a pair of the form $(D^n, (D^*)^m \times D^{n-m})$ for some integers $0 \leq m \leq n$. (Said differently, $W \smallsetminus W^\circ$ is locally a normal crossing divisor on W.) A local system L on W° is said to be quasi-unipotent (resp. unipotent) near the boundary of W if for every U as above the local system $L|_{U\cap W^\circ}$ is quasi-unipotent (resp. unipotent).

We now fix a base field k endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$.

Proposition 3.70. *Let X be a smoothly stratified k-variety, and let $L \in \mathbf{LS}(X^\circ; \Lambda)$ be a local system. Then, L is tame (resp. logarithmic) at the boundary of X if and only if it is quasi-unipotent (resp. unipotent) near the boundary of X^{an}.*

Proof. The equivalence between logarithmicity at the boundary of X and unipotence near the boundary of X^{an} is clear using the characterisation (iii) in Proposition 3.47 and Proposition 3.66. Indeed, with $j : X^\circ \to X$ the obvious inclusion, the property that the $j_*\Lambda$-module j_*L is dualizable is local on X^{an} for the analytic topology.

It follows readily from the respective case that if L is tame at the boundary of X then it is also quasi-unipotent near the boundary of X^{an}. For the converse, assume that L is quasi-unipotent near the boundary of X^{an}. Let $(U_i)_{i\in I}$ be an analytic cover of X^{an} such that the pairs $(U_i, U_i \cap X^{\circ,\mathrm{an}})$ are isomorphic to $(D^n, (D^*)^m \times D^{n-m})$ for some integers $0 \leq m \leq n$. For every $i \in I$, we fix a finite cover $e_i : U_i' \to U_i$, unramified over $U_i^\circ = U_i \cap X^{\circ,\mathrm{an}}$ and such that $e_i^{\circ,*}L|_{U^\circ}$ is unipotent. (Here, we write $e_i^\circ : U_i'^\circ \to U_i^\circ$ for the finite étale cover obtained from e_i by base change.) We let d_i be the degree of e_i, which we assume to be minimal. Let C be a stratum of X. For the U_i's intersecting C^{an}, it is easy to see that the numbers d_i's are the same. Thus, we may find a Kummer log étale cover $e : X' \to X$ such that $X'^{\mathrm{an}} \times_{X^{\mathrm{an}}} U_i$ dominates U_i' whenever U_i intersects C^{an}. Since X admits finitely many strata, we can actually assume that $X'^{\mathrm{an}} \times_{X^{\mathrm{an}}} U_i$ dominates U_i' for every $i \in I$. Letting $e^\circ : X'^\circ \to X^\circ$ be the base change of e, it follows that $e'^{\circ,*}L$ is unipotent near the boundary of X'^{an}, and thus logarithmic at the boundary of X' as needed. □

Theorem 3.71. *Assume that Λ is connective. Let X be a smoothly stratified k-variety, and let $L \in \mathbf{LS}_{\mathrm{geo}}(X^\circ; \Lambda)$ be a local system of geometric origin. Then L is tame at the boundary of X.*

Proof. We split the proof in two steps. In the first step, we treat the case $\Lambda = \mathbb{Z}$ which is essentially well-known. In the second step, we explain how to treat the general case.

Step 1. Here we assume that $\Lambda = \mathbb{Z}$. In this case, it is enough to treat the case where L is an ordinary local system. Using Lemma 3.67(ii) and Proposition 3.70, we only need to check that the action of loops around 1-codimensional strata in X act quasi-unipotently on the fibers of L. For this, it is enough to show that for every morphism $i : E \to X$ from a smooth curve E transversal to $X \smallsetminus X^\circ$, the local system $i^{\circ,*}L$ is tame at the boundary of E. (As usual, $i^\circ : E^\circ \to X^\circ$ is the base change of i.) Said differently, we are reduced to the case where X itself is a smooth curve. We may even assume that X is affine and $X^\circ = X \smallsetminus o$ for some rational point $o \in X$. We may also replace X with the pro-system of open neighbourhoods of o. In this case, the result follows immediately from the Grothendieck local monodromy theorem (see [SGA72a, Exposé I, Corollaire 3.4] or [Ill94, Théorème 2.1.2]) and the definition of constructible sheaves of geometric origin (see Definition 1.88(i)).

Step 2. Here we explain how to deduce the general case from the case $\Lambda = \mathbb{Z}$. We start by treating the case where Λ is a general ordinary commutative ring. In this case, by Corollary 1.106(ii), we have an equivalence of ∞-categories

$$\mathrm{Mod}_\Lambda(\widehat{\mathbf{LS}}_{\mathrm{geo}}(X^\circ;\mathbb{Z})) \simeq \widehat{\mathbf{LS}}_{\mathrm{geo}}(X^\circ;\Lambda).$$

It follows that the stable idempotent complete ∞-category $\mathbf{LS}_{\mathrm{geo}}(X^\circ;\Lambda)$ is generated by the image of the functor $-\otimes_{\mathbb{Z}}\Lambda : \mathbf{LS}_{\mathrm{geo}}(X^\circ;\mathbb{Z}) \to \mathbf{LS}_{\mathrm{geo}}(X^\circ;\Lambda)$. Thus, the property that every object of $\mathbf{LS}_{\mathrm{geo}}(X^\circ;\Lambda)$ is tame follows from the first step.

We now assume that Λ is any connective commutative ring spectrum. Let $L \in \mathbf{LS}_{\mathrm{geo}}(X^\circ;\Lambda)$ and let $L' = L \otimes_\Lambda \pi_0\Lambda$ considered as an object of $\mathbf{LS}_{\mathrm{geo}}(X^\circ;\pi_0\Lambda)$. By the previous discussion, we know that L' is tame at the boundary of X. Thus, replacing X by a Kummer log étale cover, we may assume that L' is logarithmic at the boundary of X. In this case, we will show that L is also logarithmic at the boundary of X using the characterisation (i) in Proposition 3.47. Let $C \subset X$ be a stratum. We have a commutative square of right-lax symmetric monoidal functors

$$\begin{tikzcd}
\langle L\rangle^\otimes \arrow[r, "\widetilde{\Upsilon}_C"] \arrow[d, "-\otimes_\Lambda\pi_0\Lambda"'] & \mathbf{Sh}_{\mathrm{geo}}(\mathrm{N}_X^\circ(C);\Lambda)^\otimes \arrow[d, "-\otimes_\Lambda\pi_0\Lambda"] \\
\langle L'\rangle^\otimes \arrow[r, "\widetilde{\Upsilon}_C"] & \mathbf{Sh}_{\mathrm{geo}}(\mathrm{N}_X^\circ(C);\pi_0\Lambda)^\otimes
\end{tikzcd}$$

where the lower horizontal arrow is monoidal and the vertical arrows are monoidal and conservative. This implies that the upper horizontal arrow is monoidal as needed. □

Corollary 3.72. *Assume that Λ is connective. Let X be a smoothly stratified k-variety.*

(i) Let C be a stratum of X. The functor $\widetilde{\Psi}_C$ restricts to a symmetric monoidal functor

$$\widetilde{\Psi}_C : \widehat{\mathbf{LS}}_{\mathrm{geo}}(X^\circ;\Lambda)^\otimes \to \widehat{\mathbf{LS}}_{\mathrm{geo}}(\mathrm{N}_X^\circ(C);\Lambda)^\otimes. \tag{3.22}$$

(ii) Let $C_0 \geq C_1$ be strata of X. Let $E \subset \mathrm{N}_X(C_0)$ be the largest stratum over $C_1 \subset \overline{C}_0$ relative to the projection $\mathrm{N}_X(C_0) \to \overline{C}_0$. The natural transformation in Proposition 3.38 induces a

commutative triangle of symmetric monoidal functors

$$\begin{tikzcd} \widehat{\mathbf{LS}}_{\mathrm{geo}}(X^\circ;\Lambda)^\otimes \arrow[r, "\widetilde{\Psi}_{C_0}"] \arrow[dr, "\widetilde{\Psi}_{C_1}"'] & \widehat{\mathbf{LS}}_{\mathrm{geo}}(\mathrm{N}^\circ_X(C_0);\Lambda)^\otimes \arrow[d, "\widetilde{\Psi}_E"] \\ & \widehat{\mathbf{LS}}_{\mathrm{geo}}(\mathrm{N}^\circ_X(C_1);\Lambda)^\otimes. \end{tikzcd}$$

Proof. This follows immediately from Theorems 3.55 and 3.71, or directly from Theorem 3.63. □

3.4. Some ∞-categorical constructions.

Here we gather some general ∞-categorical constructions needed in Subsection 3.5 for building a highly structured formalism of monodromic specialisations for tamely constructible sheaves. These constructions are rather tedious and technical. We recommend the reader who is willing to assume the existence of a lax 2-functor as described in Remark 3.98 below (or to supply his own construction of such a lax 2-functor), to skip this subsection and go directly to Subsection 3.5.

Construction 3.73. Let $\mathcal{C}$ be an ∞-category admitting pushouts, and let $p : \Xi \to \mathcal{C}$ be a coCartesian fibration. For $u : A \to B$ in $\mathcal{C}$, we denote by $u_! : \Xi_A \to \Xi_B$ the induced functor on the fibers of p. We assume that p is also a Cartesian fibration, i.e., that $u_!$ admits a right adjoint $u^!$ for every morphism u in $\mathcal{C}$. Consider the coCartesian fibrations

$$p' : \Xi' = \mathcal{C}^{\Delta^1} \times_{\mathrm{ev}_0,\mathcal{C}} \Xi \to \mathcal{C}^{\Delta^1} \quad \text{and} \quad p'' : \Xi'' = \mathcal{C}^{\Delta^1} \times_{\mathrm{ev}_1,\mathcal{C}} \Xi \to \mathcal{C}^{\Delta^1} \tag{3.23}$$

obtained from p by base change. (Here, for $i \in \{0,1\}$, $\mathrm{ev}_i : \mathcal{C}^{\Delta^1} \to \mathcal{C}$ denotes the evaluation functor at the point $i \in \Delta^1$.) The obvious natural transformation $\theta : \mathrm{ev}_0 \to \mathrm{ev}_1$ induces a morphism of coCartesian fibrations

$$\begin{tikzcd} \Xi' \arrow[rr, "\theta_!"] \arrow[dr, "p'"'] & & \Xi'' \arrow[dl, "p''"] \\ & \mathcal{C}^{\Delta^1}. & \end{tikzcd} \tag{3.24}$$

The fiber of $\theta_!$ at a morphism $u : A \to B$, considered as an object of $\mathcal{C}^{\Delta^1}$, is the functor $u_!$. Thus, by [Lur17, Proposition 7.3.2.6], $\theta_!$ admits a relative right adjoint $\theta^!$. The endofunctor $\theta^!\theta_!$ of Ξ' gives rise to a commutative triangle of ∞-categories

$$\begin{tikzcd} \mathcal{C}^{\Delta^1} \times_{\mathrm{ev}_0,\mathcal{C}} \Xi \arrow[rr, "\Theta_0"] \arrow[dr, "\mathrm{ev}_0\circ p'"'] & & \Xi \arrow[dl, "p"] \\ & \mathcal{C}. & \end{tikzcd} \tag{3.25}$$

We set $\mathcal{E}_0 = \mathcal{C}^{\Delta^1}$ and write $e_0 : \mathcal{E}_0 \to \mathcal{C}$ for the functor ev_0. Thus, the diagram in (3.25) can be considered as a morphism

$$\Theta_0 : \mathcal{E}_0 \times_{\mathcal{C}} \Xi \to \Xi \tag{3.26}$$

in the ∞-category $\mathrm{CAT}_{\infty/\mathcal{C}}$. Since $\mathcal{C}$ admits pushouts, we see that e_0 is a coCartesian fibration classified by the functor $\mathcal{C}_{-/} : \mathcal{C} \to \mathrm{CAT}_\infty$. The projection e_0 admits a section $s_0 : \mathcal{C} \to \mathcal{E}_0$ sending an object A of $\mathcal{C}$ to the initial object of $\mathcal{C}_{A/}$. It follows immediately from [Lur17, Proposition 7.3.2.5] and the construction that there is a commutative triangle

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{C}} \Xi \arrow[r, "(s_0,\mathrm{id}_\Xi)"] \arrow[dr, "\sim"'] & \mathcal{E}_0 \times_{\mathcal{C}} \Xi \arrow[d, "\Theta_0"] \\ & \Xi \end{tikzcd} \tag{3.27}$$

in the ∞-category $\mathrm{CAT}_{\infty/\mathcal{C}}$.

Remark 3.74. The functor Θ_0 in (3.26) admits the following informal description. An object of $\mathcal{E}_0 \times_{\mathcal{C}} \Xi$ is a pair $(f : A \to B, M)$, where f is a morphism in $\mathcal{C}$ and M is an object of Ξ_A. To such a pair, the functor Θ_0 associates the object $f^! f_! M$ of Ξ. A morphism

$$(f : A \to B, M) \to (g : C \to D, N)$$

in $\mathcal{E}_0 \times_{\mathcal{C}} \Xi$, corresponding to a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[d, "v"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

and a morphism $u_! M \to N$ in Ξ_C, is sent by Θ_0 to the composition of

$$u_! f^! f_! M \to g^! v_! f_! M \simeq g^! g_! u_! M \to g^! g_! N.$$

Construction 3.75. Keep the notations and assumptions from Construction 3.73. Recall that the endofunctor of $\mathrm{CAT}_{\infty/\mathcal{C}}$ given by fiber product with a coCartesian fibration over $\mathcal{C}$ admits a right adjoint. In particular, we have an ∞-category $\underline{\mathrm{EndFun}}_{\mathcal{C}}(\Xi)$ in $\mathrm{CAT}_{\infty/\mathcal{C}}$ obtained by taking the internal hom of Ξ with itself. An n-simplex of this ∞-category is a pair consisting of a functor $\Delta^n \to \mathcal{C}$ and an endofunctor of $\Delta^n \times_{\mathcal{C}} \Xi$ respecting the projection to Δ^n. In fact, by [Lur17, Corollary 4.7.1.40], $\underline{\mathrm{EndFun}}_{\mathcal{C}}(\Xi)$ underlies a monoid object $\underline{\mathrm{EndFun}}_{\mathcal{C}}(\Xi)^{©}$ in $\mathrm{CAT}_{\infty/\mathcal{C}}$, acting on Ξ on the left, and where multiplication is given by composition. The functor Θ_0 in (3.26) gives rise to a functor $\Theta'_0 : \mathcal{E}_0 \to \underline{\mathrm{EndFun}}_{\mathcal{C}}(\Xi)$ over $\mathcal{C}$. From the commutative triangle in (3.27), we deduce that the composition of

$$\mathcal{C} \xrightarrow{s_0} \mathcal{E}_0 \xrightarrow{\Theta'_0} \underline{\mathrm{EndFun}}_{\mathcal{C}}(\Xi) \tag{3.28}$$

is the identity morphism of the monoid object $\underline{\mathrm{EndFun}}_{\mathcal{C}}(\Xi)^{©}$. Said differently, we may view Θ'_0 as a morphism of $\mathbb{E}_0$-algebras in the symmetric momoidal ∞-category $\mathrm{CAT}^{\times}_{\infty/\mathcal{C}}$. By [Lur17, Propositions 3.1.3.2 & 3.1.3.3], Θ'_0 extends uniquely to a morphism of monoids

$$\Theta' : \mathcal{E}^{\otimes} \to \underline{\mathrm{EndFun}}_{\mathcal{C}}(\Xi)^{©}, \tag{3.29}$$

where $\mathcal{E}^{\otimes}$ is the free $\mathbb{E}_1$-algebra object associated to the $\mathbb{E}_0$-algebra object $\mathcal{E}_0$. Moreover, the underlying ∞-category $\mathcal{E}$ of $\mathcal{E}^{\otimes}$ admits the following description. Let $\check{\mathcal{E}}^{\bullet}$ be the semi-cosimplicial object in $\mathrm{CAT}_{\infty/\mathcal{C}}$ given in degree n by

$$\check{\mathcal{E}}^n = \overbrace{\mathcal{E}_0 \times_{\mathcal{C}} \ldots \times_{\mathcal{C}} \mathcal{E}_0}^{n+1 \text{ times}}$$

and where the i-th coface map $\check{\mathcal{E}}^n \to \check{\mathcal{E}}^{n+1}$, for $0 \le i \le n+1$, is given by inserting the section s_0 at the i-th place. Then

$$\mathcal{E} = \operatorname*{colim}_{[n]\in\mathbf{\Delta}'} \check{\mathcal{E}}^n \simeq \left(\int_{[n]\in\mathbf{\Delta}'} \check{\mathcal{E}}^n\right)[W_{\mathrm{cocart}}^{-1}]$$

is the localisation of the domain of the coCartesian fibration classified by $\check{\mathcal{E}}^\bullet$ with respect to the coCartesian edges. In particular, we see immediately that the projection $e : \mathcal{E} \to \mathcal{C}$ is a coCartesian fibration whose fiber at $A \in \mathcal{C}$ is the free monoid associated to the $\mathbb{E}_0$-algebra object $\mathcal{C}_{A/}$ in CAT_∞. By adjunction, the functor Θ' in (3.29) gives rise to a commutative triangle

$$\begin{tikzcd} \mathcal{E} \times_{\mathcal{C}} \Xi \arrow[rr, "\Theta"] \arrow[dr] & & \Xi \arrow[dl, "p"] \\ & \mathcal{C}. & \end{tikzcd} \tag{3.30}$$

The functor Θ underlies a left $\mathcal{E}^\otimes$-module structure on Ξ in $\mathrm{CAT}^\times_{\infty/\mathcal{C}}$.

Remark 3.76. Objects of the ∞-category $\check{\mathcal{E}}^m$ are families $(f_i : A \to C_i)_{0\le i\le m}$ of morphisms in $\mathcal{C}$ emanating from a single object. It follows that $\int_{\mathbf{\Delta}'} \check{\mathcal{E}}$ has the following informal description:

- objects are pairs $([m], (f_i : A \to C_i)_{0\le i\le m})$ where the f_i's are morphisms in $\mathcal{C}$;
- a morphism
$$(r, u, (v_i)_{0\le i\le m}) : ([m], (f_i : A \to C_i)_{0\le i\le m}) \to ([n], (g_j : B \to D_i)_{0\le j\le n})$$
between two such pairs consists of a strictly increasing map $r : [m] \to [n]$, a morphism $u : A \to B$, a family of morphisms $(v_i : C_i \to D_{r(i)})_{0\le i\le m}$ and commutative squares
$$\begin{tikzcd} A \arrow[r, "f_i"] \arrow[d, "u"] & C_i \arrow[d, "v_i"] \\ B \arrow[r, "g_{r(i)}"] & D_{r(i)}. \end{tikzcd}$$

The category $\mathcal{E}$ is obtained from the previous one by localising with respect to those morphisms $(r, u, (v_i)_{0\le i\le m})$ as above, such that u, the v_i's, for $0 \le i \le m$, and the g_j's, for $j \notin r([m])$, are all equivalences. The tensor product

$$([m], (f_i : A \to C_i)_{0\le i\le m}) \otimes ([n], (g_i : A \to D_j)_{0\le j\le n})$$

in $\mathcal{E}^\otimes_A$ is given by $([m+n+1], (f_i)_{0\le i\le m} \sqcup (g_{i-m-1})_{m+1\le i\le m+n+1})$. Finally, the left action of a pair $([m], (f_i : A \to C_i)_{0\le i\le m})$ on Ξ_A takes an object M to the object $f_0^! f_{0,!} \circ \ldots \circ f_m^! f_{m,!} M$.

Remark 3.77. In order to fix ideas, we explicitly state our conventions concerning monoids and their actions. Recall that a monoid object in an ∞-category with finite products is a simplicial object $M_\bullet$ such that M_0 is a final object and, for every $m \ge 1$, the morphism $M_m \to \prod_{i=1}^m M_1$, induced by the inclusions $\{i-1, i\} \subset \{0, \ldots, m\}$, is an equivalence. The object M_1 is called the underlying object; it is endowed with the multiplication given by the composition of

$$M_1 \times M_1 \xrightarrow{(d_0, d_2)^{-1}} M_2 \xrightarrow{d_1} M_1.$$

Note that we use the equivalence (d_0, d_2) instead of (d_2, d_0) for compatibility with the nerve construction. (In this way, if M is an ordinary monoid, the nerve of the ordinary category associated to M gives back the multiplication of M using the above composition.) Similarly, a left action of the monoid $M_\bullet$ is a morphism of simplicial objects $a_\bullet : X_\bullet \to M_\bullet$ such that, for $m \geq 1$, the morphism $(i_0, a_m) : X_m \to X_0 \times M_m$, where $i_0 : \{0\} \hookrightarrow \{0, \ldots, m\}$ is the obvious inclusion, is an equivalence. The underlying object of $X_\bullet$ is X_0. The left action of M_1 on X_0 in the homotopy category is the composition of

$$M_1 \times X_0 \xrightarrow{(a_1,d_1)^{-1}} X_1 \xrightarrow{d_0} X_0.$$

A right action of $M_\bullet$ is defined similarly: instead if i_0 one uses the inclusions $i_m : \{0\} \hookrightarrow \{0, \ldots, m\}$ with image m.

We need to express the outcome of Construction 3.75 using the language of $(\infty, 2)$-category. There are several ways to convey the idea of what an $(\infty, 2)$-category is ought to be; see [Lur09b, Theorem 0.0.3] for a comparison between some of the different approaches. Here, we choose to work with the following definition (as in [GR17, Chapter 10, §2.1]).

Definition 3.78.

(i) A category object in CAT_∞ is a simplicial object $\mathcal{D}_\bullet$ in CAT_∞ such that, for every $m \geq 2$, the obvious functor

$$\mathcal{D}_m \to \overbrace{\mathcal{D}_1 \times_{\mathcal{D}_0} \ldots \times_{\mathcal{D}_0} \mathcal{D}_1}^{m \text{ times}},$$

induced by the inclusions $\{i-1, i\} \subset \{0, \ldots, m\}$, for $1 \leq i \leq m$, is an equivalence.

(ii) A Segal ∞-category is a category object $\mathcal{D}_\bullet$ in CAT_∞ such that $\mathcal{D}_0$ is an ∞-groupoid.

(iii) A Segal ∞-category $\mathcal{D}_\bullet$ is said to be complete if it is local with respect to the morphism of simplicial ordinary discrete categories $\mathrm{N}_\bullet(\{0\}) \hookrightarrow \mathrm{N}_\bullet(\{0 \rightleftarrows 1\})$, where $\mathrm{N}_\bullet$ denotes the usual nerve.

A complete Segal ∞-category is also called an $(\infty, 2)$-category. We denote by SGL the ∞-category of Segal ∞-categories and by cSGL its full sub-∞-category spanned by the complete ones.

Remark 3.79. The condition for a Segal ∞-category $\mathcal{D}_\bullet$ to be complete depends only on the underlying Segal space $\mathcal{D}_\bullet^\simeq$. In fact, a Segal ∞-category $\mathcal{D}_\bullet$ is complete if and only if its underlying Segal space $\mathcal{D}_\bullet^\simeq$ is complete in the sense of Rezk [Rez01, §6]. Since the ∞-category of Segal ∞-categories is presentable, the obvious inclusion admits a left adjoint $\mathrm{c} : \mathrm{SGL} \to \mathrm{cSGL}$.

Example 3.80.

(i) Let $\mathcal{C}$ be an ∞-category. We define a Segal space $\mathrm{Seg}_\bullet(\mathcal{C})$ by the formula

$$\mathrm{Seg}_\bullet(\mathcal{C}) = \mathrm{Map}_{\mathrm{CAT}_\infty}([\bullet], \mathcal{C}).$$

It is easy to see that $\mathrm{Seg}_\bullet(\mathcal{C})$ is complete and thus defines an $(\infty, 2)$-category where every 2-morphism is invertible.

(ii) Let $\mathcal{E}^\otimes$ be a monoidal ∞-category. Being a monoid object in CAT_∞, $\mathcal{E}^\otimes$ is a category object in CAT_∞. Since $\mathcal{E}^\otimes_{[0]} = \star$, we see that $\mathcal{E}^\otimes$ is a Segal ∞-category which is however incomplete in general. We think of the associated complete Segal ∞-category $\mathrm{c}(\mathcal{E}^\otimes)$ as the $(\infty, 2)$-category with one object having $\mathcal{E}^\otimes$ as its ∞-category of endomorphisms.

The following construction generalises both (i) and (ii) in Example 3.80.

Construction 3.81. Let $\mathcal{C}$ be an ∞-category and $\mathcal{E}^\otimes \to \mathcal{C} \times \mathbf{\Delta}^{\mathrm{op}}$ be a coCartesian fibration defining a monoid object in $\mathrm{CAT}_{\infty/\mathcal{C}}$. (For example, $\mathcal{C}$ and $\mathcal{E}^\otimes$ can be as in Construction 3.75.) Our goal is to associate to $\mathcal{E}^\otimes$ an $(\infty, 2)$-category endowed with a functor to $\mathcal{C}$ whose fiber at $A \in \mathcal{C}$ is equivalent to the $(\infty,2)$-category associated to the monoidal ∞-category $\mathcal{E}_A^{\otimes}$ as in Example 3.80(ii). For $m \geq 0$, we denote by $\widetilde{[m]}$ the poset

$$\widetilde{[m]} = \{(i,j); 0 \leq i \leq j \leq m\}$$

ordered by $(i,j) \leq (i',j')$ if $i' \leq i \leq j \leq j'$. We have a functor $\widetilde{[m]} \to [m]$, given by $(i,j) \mapsto j$, which is a coCartesian fibration. For $n \geq 0$, we set

$$\mathrm{G}^n = \left(\int_{[m]\to[n]\in\mathbf{\Delta}_{/[n]}} \widetilde{[m]}^{\mathrm{op}}\right)^{\mathrm{op}}.$$

Explicitly, an object of G^n is a pair $([m] \to [n], (i,j))$ with $(i,j) \in \widetilde{[m]}$. A morphism

$$r : ([m] \to [n], (i,j)) \to ([m'] \to [n], (i',j')) \tag{3.31}$$

in G^n is a morphism $r : [m'] \to [m]$ in $\mathbf{\Delta}_{/[n]}$ such that $(i,j) \leq (r(i'), r(j'))$ in the poset $\widetilde{[m]}$. In particular, we have an obvious functor $\mathrm{G}^n \to [n]$ sending $([m] \to [n], (i,j))$ to the image of j in $[n]$. This defines a cosimplicial category $\mathrm{G}^\bullet$ endowed with a cosimplicial functor $\mathrm{G}^\bullet \to [\bullet]$. Consider the simplicial ∞-category $\mathrm{R}_\bullet(\mathcal{E}/\mathcal{C})$ given by

$$\mathrm{R}_\bullet(\mathcal{E}/\mathcal{C}) = \mathrm{Fun}(\mathrm{G}^\bullet, \mathcal{E}^{\otimes}) \times_{\mathrm{Fun}(\mathrm{G}^\bullet,\mathcal{C})} \mathrm{Fun}([\bullet], \mathcal{C}).$$

Thus, the objects of $\mathrm{R}_n(\mathcal{E}/\mathcal{C})$ are the commutative squares of ∞-categories

$$\begin{tikzcd} \mathrm{G}^n \arrow[r, "B"] \arrow[d] & \mathcal{E}^{\otimes} \arrow[d] \\ {[n]} \arrow[r, "A"] & \mathcal{C}. \end{tikzcd} \tag{3.32}$$

We define a simplicial ∞-category $\mathrm{Seg}_\bullet(\mathcal{E}/\mathcal{C})$ by specifying sub-∞-categories of the $\mathrm{R}_n(\mathcal{E}/\mathcal{C})$'s. To do so, we note that there is a functor $\mathrm{g}^n : \mathrm{G}^n \to \mathbf{\Delta}^{\mathrm{op}}$ sending a pair $([m] \to [n], (i,j))$ to $[i]$ and a morphism r as in (3.31) to the map $r|_{[i']} : [i'] \to [i]$ considered as an arrow in $\mathbf{\Delta}^{\mathrm{op}}$. A commutative square as in (3.32) belongs to $\mathrm{Seg}_n(\mathcal{E}/\mathcal{C})$ if the following condition is satisfied.

($\star$) The following triangle is commutative

$$\begin{tikzcd} \mathrm{G}^n \arrow[r, "B"] \arrow[dr, "\mathrm{g}^n"'] & \mathcal{E}^{\otimes} \arrow[d] \\ & \mathbf{\Delta}^{\mathrm{op}}. \end{tikzcd}$$

Moreover, the functor B sends every morphism in G^n to a coCartesian edge of $\mathcal{E}^{\otimes}$ with respect to the coCartesian fibration $\mathcal{E}^{\otimes} \to \mathcal{C} \times \mathbf{\Delta}^{\mathrm{op}}$.

A morphism $(A,B) \to (A',B')$ in $\mathrm{R}_n(\mathcal{E}/\mathcal{C})$ between two squares as in (3.32) satisfying the condition ($\star$) belongs to $\mathrm{Seg}_n(\mathcal{E}/\mathcal{C})$ if for every $0 \leq j \leq n$, the induced map $A(j) \to A'(j)$ in $\mathcal{C}$ is an equivalence. By Lemma 3.82 below, $\mathrm{Seg}_\bullet(\mathcal{E}/\mathcal{C})$ is a Segal ∞-category. Its completion $\mathrm{cSeg}_\bullet(\mathcal{E}/\mathcal{C})$ is the $(\infty,2)$-category we set to construct.

Lemma 3.82. *The ∞-category $\mathrm{Seg}_0(\mathcal{E}/\mathcal{C})$ is equivalent to the groupoid $\mathcal{C}^{\simeq}$. Moreover, for $n \geq 1$ and a functor $A : [n] \to \mathcal{C}$, there is an equivalence of ∞-categories*

$$\mathrm{Seg}_n(\mathcal{E}/\mathcal{C}) \times_{\mathrm{Seg}_n(\mathcal{C})} \{A\} \simeq \prod_{s=1}^{n} \mathcal{E}_{A(s)}. \tag{3.33}$$

In particular, $\mathrm{Seg}_\bullet(\mathcal{E}/\mathcal{C})$ is a Segal ∞-category.

Proof. Only the equivalence in (3.33) requires a proof. Let (A, B) be an object of $\mathrm{Seg}_n(\mathcal{E}/\mathcal{C})$ given by a square as in (3.32). From property ($\star$), it follows that B is uniquely determined by its restriction to $\widetilde{[n]}$. Thus the left hand side in (3.33) is equivalent to the ∞-category of coCartesian sections of

$$\mathcal{E}^{\otimes} \times_{\mathbf{\Delta}^{\mathrm{op}}\times\mathcal{C}} \widetilde{[n]} \to \widetilde{[n]}, \tag{3.34}$$

where the base change is with respect to the functor $\widetilde{[n]} \to \mathbf{\Delta}^{\mathrm{op}} \times \mathcal{C}$ given by $(i, j) \mapsto ([i], A(j))$. Since the functor $\widetilde{[n]} \to \mathbf{\Delta}^{\mathrm{op}}$, given by $(i, j) \mapsto [i]$, factors through the wide subcategory of inert morphisms in $\mathbf{\Delta}^{\mathrm{op}}$, we see that the functor in (3.34) decomposes as a direct product of coCartesian fibrations $\mathcal{E}_A^{(s)} \to \widetilde{[n]}$, for $1 \leq s \leq n$, admitting the following description. Let $\widetilde{[n]}_s$ be the full subcategory of $\widetilde{[n]}$ spanned by the objects (i, j) with $i \geq s$. Then, $\mathcal{E}_A^{(s)} \times_{\widetilde{[n]}} \widetilde{[n]}_s$ coincides with $\mathcal{E}_A \times_{[n]} \widetilde{[n]}_s$, where $\widetilde{[n]}_s \to [n]$ is the functor given by $(i, j) \mapsto j$. On the other hand, for $(i, j) \in \widetilde{[n]}$ with $0 \leq i < s$, the fiber of $\mathcal{E}_A^{(s)}$ at (i, j) is the final category. It is immediate to see that a coCartesian section of $\mathcal{E}_A^{(s)} \to \widetilde{[n]}$ is uniquely determined by its value at (s, s), which is an object of $\mathcal{E}_{A(s)}$. This finishes the proof. □

Remark 3.83. The $(\infty, 2)$-category $\mathrm{cSeg}_\bullet(\mathcal{E}/\mathcal{C})$ obtained in Construction 3.81 admits the following informal description.

(i) Its objects are the objects of $\mathcal{C}$.
(ii) Given two objects A and B, a 1-morphism between A and B in is a pair (E, u) where $u : A \to B$ is a morphism in $\mathcal{C}$ and E is an object of $\mathcal{E}_B$.
(iii) A 2-morphism $(E, u) \to (E', u')$ between two 1-morphisms as above is a pair consisting of an equivalence between u and u' and a morphism $E \to E'$ in $\mathcal{E}_B$.
(iv) Composition of two 1-morphisms $(E, u) : A \to B$ and $(F, v) : B \to C$ is given by the pair $(F \otimes v_!(E), v \circ u)$.

In particular, we see that the hypothesis that $\mathcal{E}^{\otimes} \to \mathcal{C} \times \mathbf{\Delta}^{\mathrm{op}}$ is a Cartesian fibration is directly related to the associativity of composition of 1-morphisms.

To go further, we introduce the $(\infty, 2)$-category of ∞-categories. We start with some notations which will be also useful later on.

Notation 3.84. Given a simplicial set S, we denote by $\mathrm{CAT}_{\infty/S}$ the ∞-category of inner fibrations with codomain S. (When S is an ∞-category, this equivalent to the over ∞-category of ∞-categories with functors to S.) We denote by $\mathrm{CAT}_{\infty/S}^{\mathrm{cart}}$ (resp. $\mathrm{CAT}_{\infty/S}^{\mathrm{cocart}}$) the full sub-$\infty$-categories of $\mathrm{CAT}_{\infty/S}$ spanned by the Cartesian (resp. coCartesian) fibrations. We write $\mathrm{CAT}_{\infty/S}^{\text{st-cart}}$ (resp. $\mathrm{CAT}_{\infty/S}^{\text{st-cocart}}$) for the wide sub-$\infty$-category of $\mathrm{CAT}_{\infty/S}^{\mathrm{cart}}$ (resp. $\mathrm{CAT}_{\infty/S}^{\mathrm{cocart}}$) where the functors are required to respect Cartesian (resp. coCartesian) edges. On the other hand, we denote by $\mathrm{CAT}_{\infty/S}^{\text{loc-cart}}$ (resp. $\mathrm{CAT}_{\infty/S}^{\text{loc-cocart}}$) the full sub-$\infty$-categories of $\mathrm{CAT}_{\infty/S}$ spanned by the locally Cartesian (resp. locally coCartesian) fibrations.

Definition 3.85. The $(\infty, 2)$-category $\underline{\mathrm{CAT}}_\infty$ of ∞-categories is the complete Segal ∞-category given in degree n by the wide sub-∞-category of $\mathrm{CAT}_{\infty/[n]^{\mathrm{op}}}^{\mathrm{cart}}$ defined as follows. Given two Cartesian fibrations $\mathcal{C} \to [n]^{\mathrm{op}}$ and $\mathcal{D} \to [n]^{\mathrm{op}}$, a commutative triangle

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr] & & \mathcal{D} \arrow[dl] \\ & {[n]^{\mathrm{op}}} & \end{tikzcd}$$

is a morphism in $\underline{\mathrm{CAT}}_\infty([n])$ if F induces equivalences $\mathcal{C}_i \simeq \mathcal{D}_i$ on the fibers, for every $0 \le i \le n$.

In the remainder of this subsection, we will be concerned with the following problem. Assume we are given a coCartesian fibration $\Xi \to \mathcal{C}$ underlying a left $\mathcal{E}^\otimes$-module in $\mathrm{CAT}_{\infty/\mathcal{C}}$; construct a lax 2-functor from $\mathrm{cSeg}_\bullet(\mathcal{E}/\mathcal{C})$ to $\underline{\mathrm{CAT}}_\infty$ extending the functor $\mathcal{C} \to \mathrm{CAT}_\infty$ associated to Ξ. We start by explicitly describing the situation we will consider.

Situation 3.86. Let $\mathcal{C}$ be an ∞-category, and let $\mathcal{E}^\otimes \to \mathcal{C} \times \mathbf{\Delta}^{\mathrm{op}}$ be a coCartesian fibration defining a monoid object in $\mathrm{CAT}_{\infty/\mathcal{C}}$. Let $\Xi \to \mathcal{C}$ be a coCartesian fibration endowed with a left module structure over $\mathcal{E}^\otimes$ in the ∞-category $\mathrm{CAT}_{\infty/\mathcal{C}}$. Thus, we have a commutative diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[dr] & \mathcal{E}^\otimes \arrow[r] \arrow[d] & \mathcal{C} \times \mathbf{\Delta}^{\mathrm{op}} \arrow[dl] \\ & \mathbf{\Delta}^{\mathrm{op}}, & \end{tikzcd} \tag{3.35}$$

with the following properties:

(i) the functor $\mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}$ is a coCartesian fibration and the functor $\mathcal{M} \to \mathcal{E}^\otimes$ preserves coCartesian edges over $\mathbf{\Delta}^{\mathrm{op}}$;

(ii) $\mathcal{M}_{[0]}$ is equivalent to Ξ compatibly with the projection to $\mathcal{C}$;

(iii) for all $n \ge 0$, the induced functors

$$\mathcal{M}_{[n]} \to \mathcal{M}_{[0]} \times_{\mathcal{C}} \mathcal{E}^\otimes_{[n]} = \Xi \times_{\mathcal{C}} \mathcal{E}^\otimes_{[n]} \tag{3.36}$$

are equivalences.

(In (3.36), we use the functors $i_{0,!} : \mathcal{M}_{[n]} \to \mathcal{M}_{[0]}$ corresponding to the obvious inclusions $i_0 : [0] \hookrightarrow [n]$; see Remark 3.77 for our convention on left actions.)

Remark 3.87. By [Lur09a, Proposition 2.4.2.11], the properties (i) and (iii) in Situation 3.86 imply that $\mathcal{M} \to \mathcal{E}^\otimes$ is a locally coCartesian fibration. For later use, we describe some of the locally coCartesian edges of $\mathcal{M} \to \mathcal{E}^\otimes$. An object of $\mathcal{E}^\otimes_{[n]}$ is a tuple $(A, E_1, \dots, E_n)$ where $A \in \mathcal{C}$ and $E_i \in \mathcal{E}_A$. Let $u : A \to B$ be a morphism in $\mathcal{C}$, and consider the unique morphism $\widetilde{u} : (A, E_1, \dots, E_n) \to (B, \emptyset)$ over u and the map $i_n : [0] \to [n]$ given by $i_n(0) = n$. Furthermore, consider an object in $\mathcal{M}$ over $(A, E_1, \dots, E_n)$ given by a tuple $(A, X_0, E_1, \dots, E_n)$ where X_0 is an object of Ξ_A. Then, a locally coCartesian edge over $\widetilde{u}$ with domain $(A, X_0, E_1, \dots, E_n)$ is of the form

$$(A, X_0, E_1, \dots, E_n) \to (B, u_!(E_n \otimes \dots \otimes E_1 \otimes X_0)). \tag{3.37}$$

We note also that the commutative triangle

$$\begin{tikzcd} (A, E_1, \dots, E_n) \arrow[d] \arrow[dr] & \\ (B, u_!(E_1) \otimes \dots \otimes u_!(E_n)) \arrow[r] & (B, \emptyset) \end{tikzcd}$$

induces the morphism

$$(B, u_!(E_n \otimes \dots \otimes E_1 \otimes X_0)) \to (B, u_!(E_n) \otimes \dots \otimes u_!(E_1) \otimes u_!(X_0)) \tag{3.38}$$

which is not an equivalence in general.

Notation 3.88. Consider the twisted arrow category $\mathbf{\Delta}^{\mathrm{tw}}$ whose objects are maps $r : [m] \to [n]$ in $\mathbf{\Delta}$ and where a morphism $(a, b) : r \to r'$ from $r : [m] \to [n]$ to $r' : [m'] \to [n']$ is given by a commutative square

$$\begin{tikzcd} {[m]} \arrow[r, "r"] & {[n]} \arrow[d, "b"] \\ {[m']} \arrow[u, "a"] \arrow[r, "r'"] & {[n'].} \end{tikzcd} \tag{3.39}$$

Thus, sending $[m] \to [n]$ to $[m]$ (resp. $[n]$) yields a functor $\mathbf{\Delta}^{\mathrm{tw}} \to \mathbf{\Delta}^{\mathrm{op}}$ (resp. $\mathbf{\Delta}^{\mathrm{tw}} \to \mathbf{\Delta}$) which is a coCartesian fibration. We will write $\mathbf{\Delta}^{\mathrm{twop}}$ instead of $(\mathbf{\Delta}^{\mathrm{tw}})^{\mathrm{op}}$. Let

$$\mathrm{H} = \left(\int_{[m]\to[n]\in\mathbf{\Delta}^{\mathrm{twop}}} [m] \right)^{\mathrm{op}} \tag{3.40}$$

be the codomain of the Cartesian fibration $\mathrm{H} \to \mathbf{\Delta}^{\mathrm{tw}}$ corresponding to the functor $\mathbf{\Delta}^{\mathrm{twop}} \to \mathrm{Cat}$ sending $r : [m] \to [n]$ to $[m]^{\mathrm{op}}$. Explicitly, an object of H is a pair $(r : [m] \to [n], j)$, with $j \in [m]$, and a morphism

$$(a, b) : (r : [m] \to [n], j) \to (r' : [m'] \to [n'], j') \tag{3.41}$$

is a pair of maps $a : [m'] \to [m]$ and $b : [n] \to [n']$ making the square in (3.39) commutative, and such that $a(j') \leq j$. We denote by

$$\mathrm{h} : \mathrm{H} \to \mathbf{\Delta} \qquad \text{and} \qquad \mathrm{k} : \mathrm{H} \to \mathbf{\Delta} \tag{3.42}$$

the functors sending an object $(r : [m] \to [n], j)$ to $[\![r(j), n]\!]$ and $[n]$ respectively. We have an obvious natural transformation $\mathrm{h} \to \mathrm{k}$ given by the obvious inclusions $[\![r(j), n]\!] \hookrightarrow [n]$.

Construction 3.89. We work in Situation 3.86. We denote by $\mathcal{M}' \to \mathbf{\Delta}$ and $\mathcal{E}'^{\otimes} \to \mathbf{\Delta}$ the Cartesian fibrations which are dual to the coCartesian fibrations $\mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}$ and $\mathcal{E}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$, i.e., which are classified by the same functors. (For an explicit construction of the dual Cartesian fibration, see [BGN18].) We have a commutative diagram

$$\begin{tikzcd} \mathcal{M}' \arrow[r] \arrow[dr] & \mathcal{E}'^{\otimes} \arrow[r] \arrow[d] & \mathcal{C} \times \mathbf{\Delta} \arrow[dl] \\ & \mathbf{\Delta}, & \end{tikzcd} \tag{3.43}$$

where the slanted arrows are Cartesian fibrations, and both triangles are morphisms of Cartesian fibrations. By base change along $\mathrm{h} : \mathrm{H} \to \mathbf{\Delta}$, we obtain a morphism of Cartesian fibrations

$$\begin{tikzcd} \mathcal{M}' \times_{\mathbf{\Delta},\mathrm{h}} \mathrm{H} \arrow[rr] \arrow[dr] & & \mathcal{E}'^{\otimes} \times_{\mathbf{\Delta},\mathrm{h}} \mathrm{H} \arrow[dl] \\ & \mathrm{H}. & \end{tikzcd} \tag{3.44}$$

Moreover, the natural transformation $\mathrm{h} \to \mathrm{k}$ gives rise to a morphism of Cartesian fibrations

$$\begin{tikzcd} \mathcal{E}'^{\otimes} \times_{\mathbf{\Delta},\mathrm{k}} \mathrm{H} \arrow[rr] \arrow[dr] & & \mathcal{E}'^{\otimes} \times_{\mathbf{\Delta},\mathrm{h}} \mathrm{H} \arrow[dl] \\ & \mathrm{H}. & \end{tikzcd} \tag{3.45}$$

We define the ∞-category $\mathcal{N}'$ as the base change of $\mathcal{M}'$ along the composite functor

$$\mathcal{E}'^{\otimes} \times_{\mathbf{\Delta},\mathrm{k}} \mathrm{H} \to \mathcal{E}'^{\otimes} \times_{\mathbf{\Delta},\mathrm{h}} \mathrm{H} \to \mathcal{E}'^{\otimes}, \tag{3.46}$$

so to have a Cartesian square of ∞-categories

$$\begin{tikzcd} \mathcal{N}' \arrow[r] \arrow[d] & \mathcal{E}'^{\otimes} \times_{\mathbf{\Delta},\mathrm{k}} \mathrm{H} \arrow[d] \\ \mathcal{M}' \times_{\mathbf{\Delta},\mathrm{h}} \mathrm{H} \arrow[r] & \mathcal{E}'^{\otimes} \times_{\mathbf{\Delta},\mathrm{h}} \mathrm{H}. \end{tikzcd} \tag{3.47}$$

In particular, we see that the projection $\mathcal{N}' \to \mathrm{H}$ is a Cartesian fibration. Composing with the obvious Cartesian fibration $\mathrm{H} \to \mathbf{\Delta}^{\mathrm{tw}}$, we obtain the Cartesian fibration $\mathcal{N}' \to \mathbf{\Delta}^{\mathrm{tw}}$.

Lemma 3.90. *Consider the commutative triangle of ∞-categories*

$$\begin{tikzcd} \mathcal{N}' \arrow[rr, "\rho'"] \arrow[dr] & & \mathcal{E}'^{\otimes} \times_{\mathbf{\Delta}} \mathbf{\Delta}^{\mathrm{tw}} \arrow[dl] \\ & \mathbf{\Delta}^{\mathrm{tw}}. & \end{tikzcd} \tag{3.48}$$

The slanted arrows are Cartesian fibrations and the functor ρ' preserves Cartesian edges over $\mathbf{\Delta}^{\mathrm{tw}}$. (Said differently, the triangle in (3.48) *is a morphism of Cartesian fibrations.) Given an object $r : [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{tw}}$, we denote by $\rho_r : \mathcal{N}_r \to \mathcal{E}^{\otimes}_{[n]}$ the functor induced on fibers and, if $r = \mathrm{id}_{[n]}$, we write $\rho_{[n]} : \mathcal{N}_{[n]} \to \mathcal{E}^{\otimes}_{[n]}$ instead. Then, the following properties are satisfied.*

- *(i) For every $r : [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{tw}}$, we have a functor $\mathcal{N}_r \to [m]^{\mathrm{op}}$ and, for every $a : [m'] \to [m]$, we have an equivalence $\mathcal{N}_{r\circ a} \simeq \mathcal{N}_r \times_{[m]^{\mathrm{op}},a} [m']^{\mathrm{op}}$.*
- *(ii) For every $r : [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{tw}}$, the functor $\rho_r : \mathcal{N}_r \to \mathcal{E}^{\otimes}_{[n]}$ is a coCartesian fibration.*
- *(iii) For every morphism $(a, b) : r \to r'$ in $\mathbf{\Delta}^{\mathrm{tw}}$ as in* (3.39)*, the functor $(a, b)^* : \mathcal{N}_{r'} \to \mathcal{N}_r$ takes a $\rho_{r'}$-coCartesian edge to a ρ_r-coCartesian edge.*
- *(iv) For $(A, E_1, \dots, E_n)$ an object of $\mathcal{E}^{\otimes}_{[n]}$, with $A \in \mathcal{C}$ and $E_1, \dots, E_n \in \mathcal{E}_A$, the induced functor $(\mathcal{N}_{[n]})_{A,E_1,\dots,E_n} \to [n]^{\mathrm{op}}$ is the Cartesian fibration classified by the sequence of functors*

$$\Xi_A \xrightarrow{E_1\otimes-} \Xi_A \xrightarrow{E_2\otimes-} \dots \xrightarrow{E_n\otimes-} \Xi_A.$$

Proof. We only sketch the proof of (ii) and (iii). For (ii), we simply describe the functor induced by a morphism

$$(u, e_1, \dots, e_n) : (A, E_1, \dots, E_n) \to (B, F_1, \dots, F_n)$$

where $u : A \to B$ is a morphism in $\mathcal{C}$ and $e_i : u_!(E_i) \to F_i$ is a morphism in $\mathcal{E}_B$. Recall that we need to specify a functor between the ∞-categories

$$(\mathcal{N}_{[n]})_{A,E_1,\dots,E_n} \to (\mathcal{N}_{[n]})_{B,F_1,\dots,F_n}.$$

Informally, an object of the domain is a pair (i, X) where $0 \le i \le n$ and $X \in \Xi_A$. Our functor takes (i, X) to $(i, u_!(X))$. A morphism $(j, X') \to (i, X)$ in the domain exists if $i \le j$ and corresponds to a morphism $X' \to E_j \otimes \dots \otimes E_i \otimes X$. Our functor takes such a morphism to a morphism

$(j, u_!(X')) \to (i, u_!(X))$ corresponding to the composition of

$$\begin{tikzcd} u_!(X') \arrow[r] & u_!(E_j \otimes \ldots \otimes E_i \otimes X) \arrow[r] & u_!(E_j) \otimes \ldots \otimes u_!(E_i) \otimes u_!(X) \arrow[d] \\ & & u_!(F_j) \otimes \ldots \otimes u_!(F_i) \otimes u_!(X). \end{tikzcd}$$

We now check (iii). Assume for simplicity that (a, b) is of the form

$$(r, \mathrm{id}_{[n]}) : \mathrm{id}_{[n]} \to (r : [m] \to [n]).$$

In this case, we need to check that the square

$$\begin{tikzcd} (\mathcal{N}_r)_{A, E_{r(1)}, \ldots, E_{r(m)}} \arrow[r] \arrow[d] & (\mathcal{N}_r)_{B, F_{r(1)}, \ldots, F_{r(m)}} \arrow[d] \\ (\mathcal{N}_{[n]})_{A, E_1, \ldots, E_n} \arrow[r] & (\mathcal{N}_{[n]})_{B, F_1, \ldots, F_n} \end{tikzcd}$$

is commutative. This follows immediately form the fact that the vertical arrows are the obvious functors sending an object (i, X), with $1 \leq i \leq m$ and $X \in \Xi_A$, to $(r(i), X)$. $\square$

Corollary 3.91. *Keep the notations as in Construction 3.89. Let $\mathcal{N} \to \mathbf{\Delta}^{\mathrm{twop}}$ be the coCartesian fibration which is dual to the Cartesian fibration $\mathcal{N}' \to \mathbf{\Delta}^{\mathrm{tw}}$. Then, we have a morphism of coCartesian fibrations*

$$\begin{tikzcd} \mathcal{N} \arrow[rr, "\rho"] \arrow[dr] & & \mathcal{E}^{\otimes} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{twop}} \arrow[dl] \\ & \mathbf{\Delta}^{\mathrm{twop}}. & \end{tikzcd} \tag{3.49}$$

Moreover, the functor ρ is a coCartesian fibration.

Construction 3.92. We continue working in Situation 3.86 and keep the notations above. The coCartesian fibration ρ is classified by a functor

$$\mathcal{F} : \mathcal{E}^{\otimes} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{twop}} \to \mathrm{CAT}_\infty$$

endowed with a natural transformation to the composite functor

$$\mathcal{E}^{\otimes} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{twop}} \to \mathbf{\Delta}^{\mathrm{twop}} \xrightarrow{([m] \to [n]) \mapsto [m]^{\mathrm{op}}} \mathrm{CAT}_\infty.$$

Consider the functor $p : \mathcal{E}^{\otimes} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{twop}} \to \mathcal{E}^{\otimes}$, which is a Cartesian fibration. Let

$$\widetilde{\mathcal{F}} : \mathcal{E}^{\otimes} \to \mathrm{CAT}_\infty$$

be the right Kan extension of $\mathcal{F}$ along p. Using [Lur09a, Proposition 4.3.3.10], we have an equivalences of ∞-categories

$$\widetilde{\mathcal{F}}([n], E_1, \ldots, E_n) \simeq \lim_{r:[m] \to [n]} (\mathcal{N}_r)_{E_1, \ldots, E_n} \simeq (\mathcal{N}_{[n]})_{E_1, \ldots, E_n}.$$

Moreover, there is a natural transformation from $\widetilde{\mathcal{F}}$ to the composite functor

$$\mathcal{E}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{[n] \mapsto [n]^{\mathrm{op}}} \mathrm{CAT}_\infty.$$

This gives a simplicial functor $\mathcal{E}^{\otimes}_\bullet \to \mathrm{CAT}_{\infty/[\bullet]^{\mathrm{op}}}$ which obviously lands in $\mathrm{CAT}^{\mathrm{cart}}_{\infty/[\bullet]^{\mathrm{op}}}$.

Remark 3.93. If in Situation 3.86 we have $\mathcal{C} = \mathrm{pt}$, the functor $\mathcal{E}^{\otimes}_{\bullet} \to \mathrm{CAT}_{\infty/[\bullet]^{\mathrm{op}}}$ obtained in Construction 3.92 already defines a 2-functor $\mathrm{c}(\mathcal{E}^{\otimes}) \to \underline{\mathrm{CAT}}_{\infty}$. In the general case, we still need to work a little bit more.

Construction 3.94. We continue working in Situation 3.86 and keep the notations above. We will also use some notation from Construction 3.81. We have a morphism of coCartesian fibrations

$$\begin{tikzcd} \widetilde{\mathcal{n}} \arrow[rr, "\widetilde{\rho}"] \arrow[dr] & & \mathcal{E}^{\otimes} \arrow[dl] \\ & \mathbf{\Delta}^{\mathrm{op}} & \end{tikzcd} \tag{3.50}$$

where $\widetilde{\rho}$ is the coCartesian fibration classified by the functor $\widetilde{\mathcal{F}}$ obtained in Construction 3.92. Consider the obvious evaluation functor $\mathrm{Seg}_n(\mathcal{E}/\mathcal{C}) \times \mathrm{G}^n \to \mathcal{E}^{\otimes}$ and form the Cartesian square

$$\begin{tikzcd} \mathcal{K}_n \arrow[r, "\mu"] \arrow[d] & \mathrm{Seg}_n(\mathcal{E}/\mathcal{C}) \times \mathrm{G}^n \arrow[d] \\ \widetilde{\mathcal{n}} \arrow[r, "\widetilde{\rho}"] & \mathcal{E}^{\otimes} \end{tikzcd}$$

The functor μ being a coCartesian fibration, we obtain a simplicial functor

$$\mathrm{Seg}_{\bullet}(\mathcal{E}/\mathcal{C}) \to \mathrm{CAT}^{\text{st-cocart}}_{\infty/\mathrm{G}^{\bullet}}.$$

(See Notation 3.84.) The obvious functor $\mathrm{G}^n \to [n]$ admits a left adjoint $\delta_n : [n] \to \mathrm{G}^n$ sending $j \in [n]$ to the object $(\mathrm{id}_{[n]}, (0, j))$. Thus, for each $n \geq 0$, the functor $\mathrm{CAT}^{\text{st-cocart}}_{\infty/[n]} \to \mathrm{CAT}^{\text{st-cocart}}_{\infty/\mathrm{G}^n}$ admits a right adjoint given by pullback along δ_n. Using [Lur17, Proposition 7.3.2.6], we obtain a relative right adjoint functor

$$\int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Seg}_n(\mathcal{E}/\mathcal{C}) \to \int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{CAT}^{\text{st-cocart}}_{\infty/[n]}. \tag{3.51}$$

It is easy to see that this functor sends an object

$$O = (A_0 \xrightarrow{u_1} A_1 \xrightarrow{u_2} \dots \xrightarrow{u_n} A_n, E_1, \dots, E_n) \tag{3.52}$$

to the domain of the coCartesian fibration classified by the functor $[n] \to \mathrm{CAT}_{\infty}$ sending $i \in [n]$ to the domain of the Cartesian fibration classified by the sequence

$$\Xi_{A_i} \xrightarrow{u_{i,!}\dots u_{2,!}(E_1)\otimes -} \Xi_{A_i} \xrightarrow{u_{i,!}\dots u_{3,!}(E_2)\otimes -} \dots \xrightarrow{E_i\otimes -} \Xi_{A_i}. \tag{3.53}$$

It follows from this that the functor in (3.51) preserve coCartesian edges over $\mathbf{\Delta}^{\mathrm{op}}$. Thus, it is induced by a simplicial functor $\mathrm{Seg}_{\bullet}(\mathcal{E}/\mathcal{C}) \to \mathrm{CAT}^{\text{st-cocart}}_{\infty/[\bullet]}$. On the other hand, we have a simplicial equivalence $\mathrm{CAT}^{\text{st-cocart}}_{\infty/[\bullet]} \simeq \mathrm{CAT}^{\text{st-cart}}_{\infty/[\bullet]^{\mathrm{op}}}$. In this way, we obtain a simplicial functor

$$\mathrm{Seg}_{\bullet}(\mathcal{E}/\mathcal{C}) \to \mathrm{CAT}^{\text{loc-cart}}_{\infty/\widehat{[\bullet]}^{\mathrm{op}}} \tag{3.54}$$

sending the object O in (3.52) to a locally Cartesian fibration with codomain $\widehat{[n]}^{\mathrm{op}}$. Restricting further along the diagonal embedding $[\bullet] \hookrightarrow \widehat{[\bullet]}$, given by $i \mapsto (i, i)$, we obtain a simplicial functor

$$\mathrm{Seg}_{\bullet}(\mathcal{E}/\mathcal{C}) \to \mathrm{CAT}^{\text{loc-cart}}_{\infty/[\bullet]^{\mathrm{op}}}. \tag{3.55}$$

Applying [Lur17, Proposition 7.3.2.6] and Lemma 3.95 below, we obtain a relative right adjoint

$$\int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Seg}_n(\mathcal{E}/\mathcal{C}) \to \int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{CAT}^{\text{cart}}_{\infty/[n]^{\mathrm{op}}}. \tag{3.56}$$

This functor factors through

$$\int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{cSeg}_n(\mathcal{E}/\mathcal{C}) \to \int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \underline{\mathrm{CAT}}_\infty([n]) \tag{3.57}$$

which is the lax 2-functor we set to construct.

Lemma 3.95. *For every $n \in \mathbb{N}$, the inclusion $\mathrm{CAT}^{\mathrm{cart}}_{\infty/[n]} \subset \mathrm{CAT}^{\mathrm{loc\text{-}cart}}_{\infty/[n]}$ admits a right adjoint. Moreover, for every $r : [m] \to [n]$ in $\mathbf{\Delta}$, the commutative square*

$$\begin{tikzcd}
\mathrm{CAT}^{\mathrm{cart}}_{\infty/[n]} \arrow[r] \arrow[d, "-\times_{[n]}[m]"] & \mathrm{CAT}^{\mathrm{loc\text{-}cart}}_{\infty/[n]} \arrow[d, "-\times_{[n]}[m]"] \\
\mathrm{CAT}^{\mathrm{cart}}_{\infty/[m]} \arrow[r] & \mathrm{CAT}^{\mathrm{loc\text{-}cart}}_{\infty/[m]}
\end{tikzcd}$$

is right adjointable.

Proof. Let $\mathrm{i}_n : \mathrm{CAT}^{\mathrm{cart}}_{\infty/[n]} \to \mathrm{CAT}^{\mathrm{loc\text{-}cart}}_{\infty/[n]}$ be the obvious inclusion. We define a functor

$$\mathrm{f}_n : \mathrm{CAT}^{\mathrm{loc\text{-}cart}}_{\infty/[n]} \to \mathrm{CAT}^{\mathrm{cart}}_{\infty/[n]}$$

as follows. Consider the simplicial subset $\mathrm{A}^n \subset \Delta^n$ given by the union of the edges $\Delta^{\{i-1,i\}}$, for $1 \leq i \leq n$. If $\mathcal{M} \to \Delta^n$ is a locally Cartesian fibration, then $\mathcal{M} \times_{\Delta^n} \mathrm{A}^n \to \mathrm{A}^n$ is a Cartesian fibration. Since the inclusion $\mathrm{A}^n \to \Delta^n$ is anodyne, there is an equivalence of ∞-categories $\mathrm{CAT}^{\mathrm{cart}}_{\infty/\Delta^n} \simeq \mathrm{CAT}^{\mathrm{cart}}_{\infty/\mathrm{A}^n}$, and we define f_n by the composition of

$$\mathrm{CAT}^{\mathrm{loc\text{-}cart}}_{\infty/[n]} \xrightarrow{-\times_{\Delta^n}\mathrm{A}^n} \mathrm{CAT}^{\mathrm{cart}}_{\infty/\mathrm{A}^n} \simeq \mathrm{CAT}^{\mathrm{cart}}_{\infty/[n]}.$$

Explicitly, given a locally Cartesian fibration $\mathcal{F} \to [n]$, the Cartesian fibration $\mathrm{f}_n(\mathcal{F})$ is given by $\mathrm{N}\mathfrak{C}(\mathcal{F} \times_{\Delta^n} \mathrm{A}^n)_{\mathrm{fib}}$ where N is the simplicial nerve, $\mathfrak{C}$ its left adjoint and $(-)_{\mathrm{fib}}$ is a fibrant replacement in the model category of simplicially enriched categories. In particular, we see that the obvious inclusion $\mathcal{F} \times_{\Delta^n} \mathrm{A}^n \to \mathcal{F}$ extends uniquely into a functor $\mathrm{i}_n \circ \mathrm{f}_n(\mathcal{F}) \to \mathcal{F}$. This gives a natural transformation $\mathrm{i}_n \circ \mathrm{f}_n \to \mathrm{id}$ which is easily seen to be a counit of an adjunction. □

Remark 3.96. Using the informal description of the $(\infty, 2)$-category $\mathrm{cSeg}_\bullet(\mathcal{E}/\mathcal{C})$ given in Remark 3.83, we can informally describe the lax 2-functor in (3.57) as follows. It takes an object $A \in \mathcal{C}$ to the ∞-category Ξ_A. It takes a pair $(E, u) : A \to B$ to the functor $(E\otimes-)\circ u_!$. Given two composable 1-morphisms $(E, u) : A \to B$ and $(F, v) : B \to C$, it associates the natural transformation

$$(F \otimes -) \circ v_! \circ (E \otimes -) \circ u_! \to (F \otimes g_!(E) \otimes -) \circ (v \circ u)_!$$

which is not invertible in general.

3.5. **Monodromic specialisation, II. Functoriality.**

It is of the utmost importance for the proof of our second main theorem to keep track of the coherence properties of the monodromic specialisation functors introduced in Subsection 3.2, at the ∞-categorical level. This will be achieved in this subsection. We fix a base scheme S and a presentable Voevodsky pullback formalism

$$\mathcal{H}^\otimes : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

(See Definitions 1.14 and 2.4.) We will gradually impose several conditions on S and $\mathcal{H}^\otimes$.

Construction 3.97. Let
$$\mathcal{C}_{\mathcal{H}} = \int_{X \in (\mathrm{Sch}/S)^{\mathrm{op}}} \mathrm{CAlg}(\mathcal{H}(X)) \tag{3.58}$$
to be the codomain of the coCartesian fibration classified by $\mathrm{CAlg}(\mathcal{H})^{\mathrm{op}} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAT}_\infty$. An object of $\mathcal{C}_{\mathcal{H}}$ is a pair $(X, \mathcal{A}_X)$ where $X \in \mathrm{Sch}/S$ is a finite type S-scheme and $\mathcal{A}_X \in \mathrm{CAlg}(\mathcal{H}(X))$ is a commutative algebra in $\mathcal{H}(X)^{\otimes}$. A morphism $(f, \theta) : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ between two such pairs consists of a morphism $f : Y \to X$ of S-schemes and a morphism of commutative algebras $\theta : f^*\mathcal{A}_X \to \mathcal{A}_Y$ in $\mathcal{H}(Y)^{\otimes}$. There is a functor
$$\mathcal{H}(-;-)^{\otimes} : \mathcal{C}_{\mathcal{H}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \tag{3.59}$$
sending a pair $(X, \mathcal{A}_X)$ to the symmetric monoidal ∞-category $\mathcal{H}(X; \mathcal{A}_X)^{\otimes} = \mathrm{Mod}_{\mathcal{A}_X}(\mathcal{H}(X))^{\otimes}$. (See for example [AGV20, §3.4] for the construction of a similar functor.) Note also that $\mathcal{C}_{\mathcal{H}}$ admits pushouts. We apply Construction 3.75 to the coCartesian fibration
$$\Xi^{\otimes}_{\mathcal{H}} = \int_{\mathcal{C}_{\mathcal{H}}} \mathcal{H}(-;-)^{\otimes} \to \mathcal{C}_{\mathcal{H}} \tag{3.60}$$
classified by $\mathcal{H}(-;-)^{\otimes}$. This yields a monoid object $\mathcal{E}^{\otimes}_{\mathcal{H}}$ in $\mathrm{CAT}_{\infty/\mathcal{C}_{\mathcal{H}}}$ acting on $\Xi^{\otimes}_{\mathcal{H}}$ by lax symmetric monoidal functors. With the notations of Construction 3.81, we set
$$\mathbb{D}_{\mathcal{H}} = \mathrm{cSeg}(\mathcal{E}_{\mathcal{H}}/\mathcal{C}_{\mathcal{H}})^{1\text{-op},\,2\text{-op}}. \tag{3.61}$$
Let $\underline{\mathrm{CAT}}_{\infty/\mathrm{Fin}_*}$ be the $(\infty,2)$-category of ∞-categories endowed with a functor to Fin_*, defined by replacing the ∞-categories $\mathrm{CAT}^{\mathrm{cart}}_{\infty/[n]^{\mathrm{op}}}$ in Definition 3.85 with the ∞-categories $(\mathrm{CAT}^{\mathrm{cart}}_{\infty/[n]^{\mathrm{op}}})_{/\mathrm{Fin}_* \times [n]^{\mathrm{op}}}$. By Construction 3.94 applied to the coCartesian fibration $\Xi^{\otimes}_{\mathcal{H}} \to \mathcal{C}_{\mathcal{H}}$ with its left $\mathcal{E}^{\otimes}_{\mathcal{H}}$-module structure, we have a lax 2-functor
$$\mathrm{cSeg}(\mathcal{E}_{\mathcal{H}}/\mathcal{C}_{\mathcal{H}}) \rightsquigarrow \underline{\mathrm{CAT}}_{\infty/\mathrm{Fin}_*}, \tag{3.62}$$
and it follows from Remark 3.96 that it factors through the sub-$(\infty,2)$-category $\underline{\mathrm{SMCAT}}_\infty \subset \underline{\mathrm{CAT}}_{\infty/\mathrm{Fin}_*}$ spanned by the symmetric monoidal ∞-categories and right-lax monoidal functors between them. This gives a lax 2-functor
$$\mathcal{H}(-;-)^{\otimes} : \mathbb{D}^{1\text{-op},\,2\text{-op}}_{\mathcal{H}} \rightsquigarrow \underline{\mathrm{SMCAT}}_\infty. \tag{3.63}$$
This is essentially the only take we need from Subsection 3.4.

Combining Remarks 3.76, 3.83 and 3.96, we obtain the following.

Remark 3.98. To ease notations, we denote an object of $\mathcal{C}^{\mathrm{op}}_{\mathcal{H}}$ by X, Y, etc., instead of $(X, \mathcal{A}_X)$, $(Y, \mathcal{A}_Y)$, etc. Similarly, a morphism in $\mathcal{C}^{\mathrm{op}}_{\mathcal{H}}$ is simply denoted by f, g, etc., instead of (f, θ), (g, θ), etc. Thus, a morphism $f : Y \to X$ in $\mathcal{C}^{\mathrm{op}}_{\mathcal{H}}$ consists of a morphism of finite type S-schemes $f : Y \to X$ together with a morphism of commutative algebras $f^*\mathcal{A}_X \to \mathcal{A}_Y$ in $\mathcal{H}(Y)^{\otimes}$. This said, the $(\infty,2)$-category $\mathbb{D}_{\mathcal{H}}$ admits the following informal description.

(i) The objects of $\mathbb{D}_{\mathcal{H}}$ are precisely the objects of $\mathcal{C}^{\mathrm{op}}_{\mathcal{H}}$.

(ii) A 1-morphism $(f, ([m], (u_i)_{0 \leq i \leq m})) : Y \to X$ in $\mathbb{D}_{\mathcal{H}}$ consists of a morphism $f : Y \to X$ in $\mathcal{C}^{\mathrm{op}}_{\mathcal{H}}$, an object $[m]$ in $\mathbf{\Delta}'$ and a sequence $(u_i : Y_i \to Y)_{0 \leq i \leq m}$ of morphisms in $\mathcal{C}^{\mathrm{op}}_{\mathcal{H}}$ with codomain Y. Given a second 1-morphism $(g, ([n], (v_j)_{0 \leq j \leq n})) : Z \to Y$ the composite 1-morphism $Z \to X$ is given by $(f \circ g, ([m+n+1], (\widetilde{u}_i)_{0 \leq i \leq m} \sqcup (v_{j-m-1})_{m+1 \leq j \leq m+n+1}))$ where the $\widetilde{u}_i$'s are the base change of the u_i's along g.

(iii) A 2-morphism

$$(f', ([m'], (u'_j)_{0\leq j\leq m'})) \Rightarrow (f, ([m], (u_i)_{0\leq i\leq m})) \tag{3.64}$$

between two 1-morphisms from Y to X as in (ii) can exist only when f is equivalent to f'. Assuming that $f = f'$, an effective 2-morphism $(r, (e_i)_{0\leq i\leq m})$ as in (3.64) is given by a strictly increasing map $r : [m] \hookrightarrow [m']$ and a sequence of commutative triangles

$$\begin{tikzcd} \bullet \arrow[r, "e_i"] \arrow[dr, "u'_{r(i)}"'] & \bullet \arrow[d, "u_i"] \\ & Y \end{tikzcd}$$

in $C^{\mathrm{op}}_{\mathcal{H}}$. A general 2-morphism is obtained from the effective ones by localisation with respect to those effective 2-morphisms $(r, (e_i)_{0\leq i\leq m})$ as above such that the e_i's, for $i \in [m]$, and the u_j's, for $j \notin r([m])$, are equivalences.

Moreover, the lax 2-functor in (3.63) admits the following informal description.

(iv) It takes an object $X = (X, \mathcal{A}_X)$ to the symmetric monoidal ∞-category $\mathcal{H}(X; \mathcal{A}_X)^{\otimes}$.

(v) It takes a 1-morphism $(f, ([m], (u_i)_{0\leq i\leq m})) : Y \to X$ to the composite functor

$$u_{m,*} \circ u_m^* \circ \ldots \circ u_{0,*} \circ u_0^* \circ f^* : \mathcal{H}(X; \mathcal{A}_X)^{\otimes} \to \mathcal{H}(Y; \mathcal{A}_Y)^{\otimes}.$$

Given a second 1-morphism $(g, ([n], (v_j)_{0\leq j\leq n})) : Z \to Y$, it associates the obvious natural transfromation

$$\begin{aligned}(v_{n,*} \circ v_n^* \circ \ldots \circ v_{0,*} \circ v_0^* \circ g^*) \circ (u_{m,*} \circ u_m^* \circ \ldots \circ u_{0,*} \circ u_0^* \circ f^*) \\ \to v_{n,*} \circ v_n^* \circ \ldots \circ v_{0,*} \circ v_0^* \circ \widetilde{u}_{m,*} \circ \widetilde{u}_m^* \circ \ldots \circ \widetilde{u}_{0,*} \circ \widetilde{u}_0^* \circ (f \circ g)^*.\end{aligned}$$

(vi) It takes a 2-morphism $(r, (e_i)_{0\leq i\leq m})$ to the natural transformation induced by the unit morphisms $\mathrm{id} \to e_{i,*} \circ e_i^*$, for $0 \leq i \leq m$, and $\mathrm{id} \to u'_{j,*}u'^*_j$, for $j \in [m'] \smallsetminus r([m])$.

For later use, we introduce a simpler version of the $(\infty, 2)$-category $\mathbb{D}_{\mathcal{H}}$.

Construction 3.99. Let $\mathbb{D}_S$ the ordinary bicategory admitting the same description as $\mathbb{D}_{\mathcal{H}}$ but with $C^{\mathrm{op}}_{\mathcal{H}}$ replaced by Sch/S. The objects of $\mathbb{D}_S$ are the finite type S-schemes. The 1-morphisms and the 2-morphisms are as in (ii) and (iii) of Remark 3.98 with the difference that the f, u_i's, u'_j's, etc., are just morphisms of S-schemes. The informal description in Remark 3.98, (ii) and (iii), gives a rigorous definition of $\mathbb{D}_S$ since Sch/S is an ordinary category. This said, we have a 2-functor

$$\mathbb{D}_S \times_{\mathrm{Sch}/S} C^{\mathrm{op}}_{\mathcal{H}} \to \mathbb{D}_{\mathcal{H}} \tag{3.65}$$

identifying the domain with the wide 2-full sub-$(\infty, 2)$-category of $\mathbb{D}_{\mathcal{H}}$ spanned by the 1-morphisms of the form $(f, ([m], (u_i : Y_i \to Y)_{0\leq i\leq m}))$ such that the u_i's induce equivalences $u_i^*\mathcal{A}_Y \xrightarrow{\sim} \mathcal{A}_{Y_i}$.

Notation 3.100. In order to simplify notations when speaking of 1-morphisms in $\mathbb{D}_S$, we employ the following conventions.

(i) We identify $\mathrm{Reg}\text{-}\Sigma/S$ with the image of the functor $\mathrm{Reg}\text{-}\Sigma/S \to \mathbb{D}_S$, which is the identity on objects and sends a morphism $f : Y \to X$ of regularly stratified S-schemes to the 1-morphism $(f, ([0], \mathrm{id}_Y))$. In particular, we write f to indicate the 1-morphism $(f, ([0], \mathrm{id}_Y))$.

(ii) Given a morphism $h : Z \to X$ of regularly stratified S-schemes, we denote by $[Z]$ or $[h]$ the 1-endomorphism of X given by $(\mathrm{id}_X, ([0], h))$.

With these conventions, a general 1-morphism $(f, ([m], (u_i : Y_i \to Y)_{0\leq i\leq m}))$ can be written as the composite $f \circ [Y_0] \circ \ldots \circ [Y_m]$ or $f \circ [u_0] \circ \ldots \circ [u_m]$.

Definition 3.101.

(i) Recall the functor $\mathscr{P} : \text{SCH-}\Sigma \to \text{Cat}$ sending a stratified scheme X to the poset $(\mathscr{P}_X, \leq)$ of strata of X. We denote by $\mathscr{P}_+ : \text{SCH-}\Sigma \to \text{Cat}$ the functor sending a stratified scheme X to the poset $(\mathscr{P}_{X,+}, \leq)$ obtained by adding a greatest element to $\mathscr{P}_X$ which is preserved by the functors $f_* : \mathscr{P}_{Y,+} \to \mathscr{P}_{X,+}$ for all morphisms $f : Y \to X$. We set

$$\text{SCH-}\Sigma^{\mathrm{m}} = \int_{\text{SCH-}\Sigma} \mathscr{P} \qquad \text{and} \qquad \text{SCH-}\Sigma^{\mathrm{m}}_+ = \int_{\text{SCH-}\Sigma} \mathscr{P}_+.$$

An object of $\text{SCH-}\Sigma^{\mathrm{m}}$ is called a marked stratified scheme; it is a pair (X, C) consisting of a stratified scheme X and a stratum $C \subset X$. A morphism of marked stratified schemes $f : (Y, D) \to (X, C)$ is a morphism of stratified schemes $f : Y \to X$ such that $f_*(D) \leq C$. We have an obvious fully faithful functor $\text{SCH-}\Sigma \to \text{SCH-}\Sigma^{\mathrm{m}}_+$ which we use to identify the complement of $\text{SCH-}\Sigma^{\mathrm{m}}$ in $\text{SCH-}\Sigma^{\mathrm{m}}_+$ to $\text{SCH-}\Sigma$. We define similarly the categories $\text{REG-}\Sigma^{\mathrm{m}}_{(+)}$, $\text{Sch-}\Sigma^{\mathrm{m}}_{(+)}/S$, $\text{Reg-}\Sigma^{\mathrm{m}}_{(+)}/S$ and $\text{Sm-}\Sigma^{\mathrm{m}}_{(+)}/S$.

(ii) Recall the functor $\mathscr{P}' : \text{SCH-}\Sigma \to \text{Cat}$ sending a stratified scheme X to the poset $\mathscr{P}'_X$; see Notations 3.21. We denote by $\mathscr{P}'_+ : \text{SCH-}\Sigma \to \text{Cat}$ the functor sending a stratified scheme X to the poset $\mathscr{P}'_{X,+}$ obtained by adding a greatest element to $\mathscr{P}'_X$ which is preserved by the functors f_* for all morphisms $f : Y \to X$. We set

$$\text{SCH-}\Sigma^{\mathrm{dm}} = \int_{\text{SCH-}\Sigma} \mathscr{P}' \qquad \text{and} \qquad \text{SCH-}\Sigma^{\mathrm{dm}}_+ = \int_{\text{SCH-}\Sigma} \mathscr{P}'_+.$$

An object of $\text{SCH-}\Sigma^{\mathrm{dm}}$ is called a demarcated stratified scheme; it is a triple (X, C_-, C_+) consisting of a stratified scheme X and strata $C_- \geq C_+$ of X. A morphism of demarcated stratified schemes $f : (Y, D_-, D_+) \to (X, C_-, C_+)$ is a morphism of stratified schemes $f : Y \to X$ such that

$$f_*(D_-) \geq C_- \geq C_+ \geq f_*(D_+).$$

We have an obvious fully faithful functor $\text{SCH-}\Sigma \to \text{SCH-}\Sigma^{\mathrm{dm}}_+$ which we use to identify the complement of $\text{SCH-}\Sigma^{\mathrm{dm}}$ in $\text{SCH-}\Sigma^{\mathrm{dm}}_+$ to $\text{SCH-}\Sigma$. We define similarly the categories $\text{REG-}\Sigma^{\mathrm{dm}}_{(+)}$, $\text{Sch-}\Sigma^{\mathrm{dm}}_{(+)}/S$, $\text{Reg-}\Sigma^{\mathrm{dm}}_{(+)}/S$ and $\text{Sm-}\Sigma^{\mathrm{dm}}_{(+)}/S$.

Notation 3.102.

(i) Let (X, C_-, C_+) be a demarcated regularly stratified scheme. We denote by $\mathrm{N}^{\diamond}_X(C_-, C_+)$ the constructible open subscheme of $\mathrm{N}_X(C_+)$ making the following square Cartesian

$$\begin{array}{ccc} \mathrm{N}^{\diamond}_X(C_-, C_+) & \longrightarrow & \mathrm{N}_X(C_+) \\ \big\downarrow & & \big\downarrow \\ \mathrm{N}^{\circ}_{\overline{C}_-}(C_+) & \longrightarrow & \mathrm{N}_{\overline{C}_-}(C_+). \end{array}$$

We denote by $\mathrm{i}_{C_-,C_+} : \mathrm{N}^{\diamond}_X(C_-, C_+) \to \mathrm{Df}_X(C_+)$ the obvious inclusion.

(ii) More generally, let X be a regularly stratified scheme and (C_-, C_0, C_+) in $\mathscr{P}''_X$. (See Notation 3.21.) We denote by $\mathrm{Df}^{\diamond}_{X|C_0}(C_-, C_+)$ the constructible open subscheme of $\mathrm{Df}_{X|C_0}(C_+)$

making the following square Cartesian

$$\begin{tikzcd}
\mathrm{Df}^{\diamond}_{X|C_0}(C_-, C_+) \arrow[r] \arrow[d] & \mathrm{Df}_{X|C_0}(C_+) \arrow[d] \\
\mathrm{N}^{\diamond}_X(C_-, C_0) \arrow[r] & \mathrm{N}_X(C_0).
\end{tikzcd}$$

We denote by $\mathrm{i}_{C_-,C_0,C_+} : \mathrm{Df}^{\diamond}_{X|C_0}(C_-, C_+) \to \mathrm{Df}_X(C_+)$ the obvious inclusion. Note that when $C_0 = C_+$, we get back the morphism i_{C_-,C_+} of (i).

By Theorem 3.20, we have a functor

$$\mathrm{Df} : \mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}_+/S \to \mathrm{Sch}/S \tag{3.66}$$

sending a demarcated regularly stratified S-scheme (X, C_-, C_+) to the S-scheme $\mathrm{Df}_X(C_+)$. The following proposition provides a lift of this functor into an oplax 2-functor with values in $\mathbb{D}_S$.

Proposition 3.103. *There is an oplax 2-functor*

$$\underline{\mathrm{Df}} : \mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}_+/S \rightsquigarrow \mathbb{D}_S \tag{3.67}$$

admitting the following description.

(i) It extends the obvious functor $\mathrm{Reg}\text{-}\Sigma/S \to \mathbb{D}_S$ and sends a demarcated regularly stratified S-scheme (X, C_-, C_+) to the S-scheme $\mathrm{Df}_X(C_+)$.

(ii) Let (Y, D_-, D_+) be a demarcated regularly stratified S-scheme and let $f : Y \to X$ be a morphism of regularly stratified S-schemes. Then $\underline{\mathrm{Df}}$ sends $f : (Y, D_-, D_+) \to X$ to the 1-morphism $f \circ q \circ [\mathrm{i}_{D_-,D_+}]$, where $q : \mathrm{Df}_Y(D_+) \to Y$ is the obvious morphism.

(ii′) Let $f : (Y, D_-, D_+) \to (X, C_-, C_+)$ be a morphism of demarcated regularly stratified S-schemes. Then $\underline{\mathrm{Df}}$ sends f to the 1-morphism

$$[\mathrm{i}_{C_-,C_+}] \circ \mathrm{Df}(f) \circ [\mathrm{i}_{D_-,D_+}]$$

where $\mathrm{Df}(f) : \mathrm{Df}_Y(D_+) \to \mathrm{Df}_X(C_+)$ is the morphism induced by f.

(iii) The image by $\underline{\mathrm{Df}}$ of the identity of a demarcated regularly stratified S-scheme (X, C_-, C_+) is related to the identity 1-morphism by the obvious 2-morphism

$$[\mathrm{i}_{C_-,C_+}]^{\circ 2} \to \mathrm{id}_{\mathrm{Df}_X(C_+)}.$$

(iv) Let $f : Y \to X$ be a morphism of regularly stratified S-schemes and let $g : (Z, E_-, E_+) \to (Y, D_-, D_+)$ be a morphism of demarcated regularly stratified S-schemes. Then, the associated 2-morphism $\underline{\mathrm{Df}}(f) \circ \underline{\mathrm{Df}}(g) \to \underline{\mathrm{Df}}(f \circ g)$ is the composition of

$$\begin{array}{rl}
& f \circ q \circ [\mathrm{i}_{D_-,D_+}] \circ [\mathrm{i}_{D_-,D_+}] \circ \mathrm{Df}(g) \circ [\mathrm{i}_{E_-,E_+}] \\
\to & f \circ q \circ \mathrm{id}_{\mathrm{Df}_Y(D_+)} \circ \mathrm{Df}(g) \circ [\mathrm{i}_{E_-,E_+}] \\
\simeq & f \circ g \circ q' \circ [\mathrm{i}_{E_-,E_+}]
\end{array}$$

where $q' : \mathrm{D}_Z(E_+) \to Z$ is the obvious morphism.

(iv′) Let $f : (Y, D_-, D_+) \to (X, C_-, C_+)$ and $g : (Z, E_-, E_+) \to (Y, D_-, D_+)$ be two composable morphisms between demarcated regularly stratified S-schemes. Then, the associated 2-morphism $\underline{\mathrm{Df}}(f) \circ \underline{\mathrm{Df}}(g) \to \underline{\mathrm{Df}}(f \circ g)$ is the composition of

$$\begin{aligned} & [\mathrm{i}_{C_-,C_+}] \circ \mathrm{Df}(f) \circ [\mathrm{i}_{D_-,D_+}] \circ [\mathrm{i}_{D_-,D_+}] \circ \mathrm{Df}(g) \circ [\mathrm{i}_{E_-,E_+}] \\ \to \quad & [\mathrm{i}_{C_-,C_+}] \circ \mathrm{Df}(f) \circ \mathrm{id}_{\mathrm{Df}_Y(D_+)} \circ \mathrm{Df}(g) \circ [\mathrm{i}_{E_-,E_+}] \\ \simeq \quad & [\mathrm{i}_{C_-,C_+}] \circ \mathrm{Df}(f \circ g) \circ [\mathrm{i}_{E_-,E_+}]. \end{aligned}$$

Proof. This is proven by an easy direct verification. The details are omitted. □

Our next task is to define a functor from $\mathrm{Reg}\text{-}\Sigma_+^{\mathrm{dm}}/S$ to $\mathcal{C}_{\mathcal{H}}^{\mathrm{op}}$.

Definition 3.104. We define the category TEmb of split torus-embeddings as follows. An object of TEmb is a triple (T, T°, j_T) where T is a smooth affine $\mathbb{Z}$-scheme, T° is a split torus over $\mathbb{Z}$ acting on T and $j_T : T^\circ \hookrightarrow T$ is an equivariant dense open immersion. Such an object will be simply denoted by T; it is isomorphic to a triple of the form

$$((\mathbb{A}^1 \smallsetminus 0)^m \times \mathbb{A}^n, (\mathbf{G}_\mathrm{m})^{m+n}, j),$$

with $m, n \in \mathbb{N}$ and j the obvious inclusion. An object T of TEmb is regularly stratified by the orbits of the action of T°. The kernel of the action of T° on an orbit E° of T can be identified with the split torus $\mathrm{T}_T^\circ(E)$. (See Notation 3.8.) The closure of the $\mathrm{T}_T^\circ(E)$-orbit of $1 \in T^\circ$ intersect E° in a $\mathbb{Z}$-point. This induces an equivariant isomorphism $E^\circ \simeq T^\circ/\mathrm{T}_T^\circ(E)$ and, in particular, gives E° the structure of a split torus. The closure E of the strata E° in T is then an object of TEmb. We can now complete the description of the category TEmb: a morphism $T' \to T$ in TEmb is a morphism of stratified schemes, inducing a morphism of tori from T'° to a stratum E°, and such that the morphism $T' \to E$ is T'°-equivariant. When $E^\circ = T^\circ$ we call such a morphism strict. Strict morphisms form a wide subcategory of TEmb which we denote by TEmb′. We also write STor for the full subcategory of TEmb (and TEmb′) spanned by split tori.

Given a split torus-embedding T, we will also write T for its base change to S. Our next task is to construct, for every $T \in \mathrm{TEmb}$, a commutative algebra $\mathcal{U}_T$ in $\mathcal{H}^\otimes(T)$, which is functorial in morphisms in TEmb. We first construct these algebras functorially for strict morphisms.

Construction 3.105. Since the diagram $\mathcal{Y}^T$ described in Construction 3.23 is functorial in the split torus T, we have sections

$$\mathcal{L}_{\mathrm{STor}}, \mathcal{U}_{\mathrm{STor}} : \mathrm{STor}^{\mathrm{op}} \to \int_{\mathrm{STor}^{\mathrm{op}}} \mathrm{CAlg}(\mathcal{H}) \tag{3.68}$$

sending a split torus T to the commutative algebras $\mathcal{L}_T$ and $\mathcal{U}_T$. Alternatively, one can construct these sections using Proposition 3.29 and [Lur17, Proposition 7.3.2.6]. Using the obvious functor $\mathrm{TEmb}' \to \mathrm{STor}$, given by $T \mapsto T^\circ$, and its left adjoint given by the obvious inclusion, one may extend $\mathcal{L}_{\mathrm{STor}}$ and $\mathcal{U}_{\mathrm{STor}}$ to sections

$$\mathcal{L}', \mathcal{U}' : \mathrm{TEmb}'^{\mathrm{op}} \to \int_{\mathrm{TEmb}'^{\mathrm{op}}} \mathrm{CAlg}(\mathcal{H}) \tag{3.69}$$

sending a split torus-embedding T to the commutative algebras $\mathcal{L}_T = j_{T,*}\mathcal{L}_{T^\circ}$ and $\mathcal{U}_T = j_{T,*}\mathcal{U}_{T^\circ}$ where $j_T : T^\circ \to T$ is the structural embedding.

Construction 3.106. We form the diagram of functors and natural transformations

$$\begin{tikzcd} & & \int_{\mathrm{TEmb}^{\mathrm{op}}} \mathrm{CAlg}(\mathcal{H}) \arrow[d, "p"] \\ \mathrm{TEmb}'^{\mathrm{op}} \arrow[r, "\iota"'] \arrow[urr, bend left, "\mathcal{L}', \mathcal{U}'"] & \mathrm{TEmb}^{\mathrm{op}} \arrow[r, equal] \arrow[ur, "\mathcal{L}, \mathcal{U}"'] \arrow[u, phantom, "\Rightarrow"] & \mathrm{TEmb}^{\mathrm{op}} \end{tikzcd} \tag{3.70}$$

where ι is the obvious inclusion, $\mathcal{L}'$ and $\mathcal{U}'$ are the functors deduced from the sections in (3.69), and

$$\mathcal{L},\, \mathcal{U} : \mathrm{TEmb}^{\mathrm{op}} \to \int_{\mathrm{TEmb}^{\mathrm{op}}} \mathrm{CAlg}(\mathcal{H}) \tag{3.71}$$

are the left Kan extensions of $\mathcal{L}'$ and $\mathcal{U}'$ relative to the coCartesian fibration p. Said differently, for a split torus-embedding T, we have

$$\mathcal{L}(T) = \operatorname*{colim}_{e:T\to T'} e^*\mathcal{L}'(T') \qquad \text{and} \qquad \mathcal{U}(T) = \operatorname*{colim}_{e:T\to T'} e^*\mathcal{U}'(T') \tag{3.72}$$

where the colimit is over $\mathrm{TEmb}' \times_{\mathrm{TEmb}} \mathrm{TEmb}_{T/}$. By Lemma 3.107 below, $\mathcal{L}$ and $\mathcal{U}$ are extensions of $\mathcal{L}'$ and $\mathcal{U}'$ in the usual sense. Thus, the functors $\mathcal{L}$ and $\mathcal{U}$ take a split torus-embedding T to the commutative algebras $\mathcal{L}_T$ and $\mathcal{U}_T$ defined in Construction 3.105.

Lemma 3.107. *The natural transformations $\mathcal{L}' \to \mathcal{L} \circ \iota$ and $\mathcal{U}' \to \mathcal{U} \circ \iota$ depicted in the diagram* (3.70) *are equivalences.*

Proof. We fix $T \in \mathrm{TEmb}$ and use the formulae in (3.72). We have an adjunction

$$\beta_T : \mathrm{TEmb}' \times_{\mathrm{TEmb}} \mathrm{TEmb}_{T/} \rightleftarrows \mathrm{TEmb}'_{T/} : \iota_T$$

where ι_T is the obvious inclusion. The functor β_T sends an object $e : T \to T'$ to the object $\widetilde{e} : T \to \widetilde{T}'$, where $\widetilde{T}' \subset T'$ is the closure of the stratum of T' containing $e(T^\circ)$. The unit map $\mathrm{id} \to \iota_T \circ \beta_T$ is given, at $e : T \to T'$, by the commutative triangle

$$\begin{tikzcd} T \arrow[r, "e"] \arrow[dr, "\widetilde{e}"'] & T' \arrow[d, "p"] \\ & \widetilde{T}' \end{tikzcd}$$

where $p : T' \to \widetilde{T}'$ is the quotient map identifying $\widetilde{T}'$ with the quotient of T' by the kernel of the action of T'° on the stratum of T' containing $e(T^\circ)$. It follows from Lemma 3.27 that the obvious morphisms $e^*p^*\mathcal{L}_{\widetilde{T}'} \to e^*\mathcal{L}_{T'}$ and $e^*p^*\mathcal{U}_{\widetilde{T}'} \to e^*\mathcal{U}_{T'}$ are equivalences. Thus, we are left to compute $\operatorname{colim} F \circ \beta_T^{\mathrm{op}}$ for a functor F with domain $(\mathrm{TEmb}'_{T/})^{\mathrm{op}}$. (We are interested in the case where F is given by $(e : T \to T') \mapsto e^*\mathcal{L}_{T'}$ or by $(e : T \to T') \mapsto e^*\mathcal{U}_{T'}$.) The functor $F \mapsto F \circ \beta_T^{\mathrm{op}}$ being left adjoint to the functor $G \mapsto G \circ \iota_T^{\mathrm{op}}$, we deduce that $F \circ \beta_T^{\mathrm{op}}$ is the left Kan extension of F along the inclusion ι_T^{op}. It follows that $\operatorname{colim} F \circ \beta_T^{\mathrm{op}} \simeq \operatorname{colim} F \simeq F(\mathrm{id}_T)$. This finishes the proof. □

Construction 3.108. The sections $\mathcal{L}$ and $\mathcal{U}$ in (3.71) give rise to functors $\mathcal{L},\, \mathcal{U} : \mathrm{TEmb} \to \mathcal{C}_{\mathcal{H}}^{\mathrm{op}}$ sending a split torus-embedding T to the pairs $(T, \mathcal{L}_T)$ and $(T, \mathcal{U}_T)$ respectively. Consider the functor

$$\mathrm{T} : \mathrm{Reg}\text{-}\Sigma_+^{\mathrm{dm}}/S \to \mathrm{TEmb} \tag{3.73}$$

sending a regularly stratified S-scheme to the trivial torus and a demarcated regularly stratified S-scheme (X, C_-, C_+) to the split torus-embedding $\mathrm{T}_{X|C_-}(C_+)$. We have a natural transformation

$\mathrm{Df} \to \mathrm{T}$ from the functor (3.66) to the composition of (3.73) with $\mathrm{TEmb} \to \mathrm{Sch}/S$. Base changing along this natural transformation, we deduce functors

$$\mathcal{L},\ \mathcal{U} : \text{Reg-}\Sigma_+^{\mathrm{dm}}/S \to \mathcal{C}_{\mathcal{H}}^{\mathrm{op}} \tag{3.74}$$

admitting the following description. They send a regularly stratified S-scheme X to the pair $(X, \mathbf{1})$, and a demarcated regularly stratified S-scheme (X, C_-, C_+) to the pairs $(\mathrm{Df}_X(C_+), \mathcal{L}_{C_-,C_+})$ and $(\mathrm{Df}_X(C_+), \mathcal{U}_{C_-,C_+})$ respectively, where $\mathcal{L}_{C_-,C_+}$ and $\mathcal{U}_{C_-,C_+}$ are the pullbacks of $\mathcal{L}_{\mathrm{T}_{X|C_-}(C_+)}$ and $\mathcal{U}_{\mathrm{T}_{X|C_-}(C_+)}$ along the natural map $\mathrm{Df}_X(C_+) \to \mathrm{T}_{X|C_-}(C_+)$. Combining the oplax 2-functor in (3.67) with the 2-functors in (3.74) we deduce two oplax 2-functors

$$(\underline{\mathrm{Df}}, \mathcal{L}),\ (\underline{\mathrm{Df}}, \mathcal{U}) : \text{Reg-}\Sigma_+^{\mathrm{dm}}/S \to \mathbb{D}_{\mathcal{H}}. \tag{3.75}$$

Combining the oplax 2-functors in (3.75) with the lax 2-functor in (3.63), we obtain two lax 2-functors

$$\mathcal{H}(\underline{\mathrm{Df}}(-); \mathcal{L})^{\otimes},\ \mathcal{H}(\underline{\mathrm{Df}}(-); \mathcal{U})^{\otimes} : (\text{Reg-}\Sigma_+^{\mathrm{dm}}/S)^{\mathrm{op}} \rightsquigarrow \underline{\mathrm{SMCAT}}_{\infty}. \tag{3.76}$$

By construction, the restriction of these lax 2-functors to $(\text{Reg-}\Sigma/S)^{\mathrm{op}}$ yield the obvious functor sending a regularly stratified S-scheme X to $\mathcal{H}(X)^{\otimes}$.

Remark 3.109. We now give an informal description of the lax 2-functors in (3.76). We only discuss the case of $\mathcal{H}(\underline{\mathrm{Df}}(-), \mathcal{U})^{\otimes}$.

(i) It takes a regularly stratified S-scheme X to $\mathcal{H}(X)^{\otimes}$. It takes a demarcated regularly stratified S-scheme (X, C_-, C_+) to $\mathcal{H}(\mathrm{Df}_X(C_+); \mathcal{U}_{C_-,C_+})^{\otimes}$.

(ii) Let (Y, D_-, D_+) be a demarcated regularly stratified S-scheme and let $f : Y \to X$ be a morphism of regularly stratified S-schemes. It sends the morphism $f : (Y, D_-, D_+) \to X$ to the composition of

$$\begin{tikzcd} \mathcal{H}(X) \arrow[r, "f^*"] & \mathcal{H}(Y) \arrow[r, "q^*"] & \mathcal{H}(\mathrm{Df}_Y(D_+); \mathcal{U}_{D_-,D_+}) \arrow[d, "(\mathrm{i}_{D_-,D_+})_*(\mathrm{i}_{D_-,D_+})^*"] \\ & & \mathcal{H}(\mathrm{Df}_Y(D_+); \mathcal{U}_{D_-,D_+}). \end{tikzcd}$$

(ii′) It sends a morphism $f : (Y, D_-, D_+) \to (X, C_-, C_+)$ of demarcated regularly stratified S-schemes to the composition of

$$\begin{tikzcd} \mathcal{H}(\mathrm{Df}_X(C_+); \mathcal{U}_{C_-,C_+}) \arrow[r, "\mathrm{Df}(f)^*"] & \mathcal{H}(\mathrm{Df}_Y(D_+); \mathcal{U}_{D_-,D_+}) \arrow[d, "(\mathrm{i}_{D_-,D_+})_*(\mathrm{i}_{D_-,D_+})^*(\mathrm{i}'_{C_-,C_+})_*(\mathrm{i}'_{C_-,C_+})^*"] \\ & \mathcal{H}(\mathrm{Df}_Y(D_+); \mathcal{U}_{D_-,D_+}) \end{tikzcd}$$

where i'_{C_-,C_+} is the base change of i_{C_-,C_+} along $\mathrm{Df}(f)$.

The lax compatibility with composition in $\text{Reg-}\Sigma_+^{\mathrm{dm}}/S$ is witnessed by obvious natural transformations given by composing units of inverse-direct image adjunctions. We will abstain from writing these explicitly now; instead, we will describe them below when needed.

The next step in our construction consists in defining sub-lax 2-functors of the ones in (3.76) which are particularly well-behaved. We will need the following lemma.

Lemma 3.110. *Let (X, C_-, C_+) be a demarcated regularly stratified S-scheme. Then, the functors*

$$\mathcal{H}(\mathrm{N}^\circ_{\overline{C}_-}(C_+)) \to \mathcal{H}(\mathrm{N}^\diamond_X(C_-, C_+)), \quad \mathcal{H}(\mathrm{N}^\diamond_X(C_-, C_+)) \xrightarrow{(\mathrm{i}_{C_-,C_+})_*} \mathcal{H}(\mathrm{Df}_X(C_+); \mathcal{L}_{C_-,C_+})$$

$$\textit{and} \quad \mathcal{H}(\mathrm{N}^\diamond_X(C_-, C_+)) \xrightarrow{(\mathrm{i}_{C_-,C_+})_*} \mathcal{H}(\mathrm{Df}_X(C_+); \mathcal{U}_{C_-,C_+})$$

are fully faithful.

Proof. For the first functor, this follows from $\mathbb{A}^1$-invariance and the fact that $\mathrm{N}^\diamond_X(C_-, C_+)$ is a vector bundle over $\mathrm{N}^\circ_{\overline{C}_-}(C_+)$. For the second and third functors, this follows from the fact that i_{C_-,C_+} is a locally closed immersion and that $\mathbf{1} \simeq (\mathrm{i}_{C_-,C_+})^*\mathcal{L}_{C_-,C_+} \simeq (\mathrm{i}_{C_-,C_+})^*\mathcal{U}_{C_-,C_+}$. □

Definition 3.111. We define sub-lax 2-functors

$$\mathcal{H}^\Upsilon_+(-)^\otimes \subset \mathcal{H}(\underline{\mathrm{Df}}(-); \mathcal{L})^\otimes \qquad \text{and} \qquad \mathcal{H}^\Psi_+(-)^\otimes \subset \mathcal{H}(\underline{\mathrm{Df}}(-); \mathcal{U})^\otimes$$

as follows. If X is a regularly stratified S-scheme, we set

$$\mathcal{H}^\Upsilon_+(X) = \mathcal{H}_{\text{ict-log}}(X) \qquad \text{and} \qquad \mathcal{H}^\Psi_+(X) = \mathcal{H}_{\text{ict-tm}}(X).$$

If (X, C_-, C_+) is a demarcated regularly stratified S-scheme, we define

$$\mathcal{H}^\Upsilon_+(X, C_-, C_+) \subset \mathcal{H}(\mathrm{Df}_X(C_+); \mathcal{L}_{C_-,C_+})$$

to be the essential image of the composite functor

$$\mathcal{H}_{\mathrm{ilog}}(\mathrm{N}^\circ_{\overline{C}_-}(C_+)) \to \mathcal{H}(\mathrm{N}^\diamond_X(C_-, C_+)) \xrightarrow{(\mathrm{i}_{C_-,C_+})_*} \mathcal{H}(\mathrm{Df}_X(C_+); \mathcal{L}_{C_-,C_+})$$

which is fully faithful by Lemma 3.110. Similarly, we define

$$\mathcal{H}^\Psi_+(X, C_-, C_+) \subset \mathcal{H}(\mathrm{Df}_X(C_+); \mathcal{U}_{C_-,C_+})$$

to be the essential image of the fully faithful composite embedding

$$\mathcal{H}_{\mathrm{itame}}(\mathrm{N}^\circ_{\overline{C}_-}(C_+)) \to \mathcal{H}(\mathrm{N}^\diamond_X(C_-, C_+)) \xrightarrow{(\mathrm{i}_{C_-,C_+})_*} \mathcal{H}(\mathrm{Df}_X(C_+); \mathcal{U}_{C_-,C_+}).$$

It follows from Remark 3.109 that this indeed defines sub-lax 2-functors

$$\mathcal{H}^{\Upsilon,\otimes}_{(+)}, \mathcal{H}^{\Psi,\otimes}_{(+)} : (\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}_{(+)}/S)^{\mathrm{op}} \rightsquigarrow \underline{\mathrm{SMCAT}}_\infty \tag{3.77}$$

of the ones in (3.76). See also the proof of part (ii) of Theorem 3.112 below, from which an argument can be easily extracted.

Theorem 3.112. *For a demarcated regularly stratified S-scheme (X, C_-, C_+), there is an equivalence of symmetric monoidal ∞-categories*

$$\mathcal{H}^\Psi_+(X, C_-, C_+)^\otimes \simeq \mathcal{H}_{\mathrm{itame}}(\mathrm{N}^\circ_{\overline{C}_-}(C_+))^\otimes \tag{3.78}$$

where ind-tameness is with respect to the boundary of $\mathrm{N}_{\overline{C}_-}(C_+)$. Modulo these equivalences, the lax 2-functor $\mathcal{H}^\Psi$ admits the following description.

(i) A morphism in $\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}_+/S$ of the form $f : (Y, D_-, D_+) \to X$ is sent to the inverse image functor

$$\mathrm{N}^\circ(f)^* : \mathcal{H}_{\text{ict-tm}}(X) \to \mathcal{H}_{\mathrm{itame}}(\mathrm{N}^\circ_{\overline{D}_-}(D_+))$$

along the morphism $\mathrm{N}^\circ(f) : \mathrm{N}^\circ_{\overline{D}_-}(D_+) \to X$ induced by f.

(ii) *A morphism in* $\mathrm{Reg}\text{-}\Sigma_+^{\mathrm{dm}}/S$ *of the form* $f:(Y,D_-,D_+)\to(X,C_-,C_+)$ *is sent to the composition of*

$$\mathcal{H}_{\mathrm{itame}}(\mathrm{N}^{\circ}_{\overline{C}_-}(C_+)) \xrightarrow{\widetilde{\Psi}_{C'_+}} \mathcal{H}_{\mathrm{itame}}(\mathrm{N}^{\circ}_{\overline{C}_-}(C'_+)) \xrightarrow{\mathrm{N}^{\circ}(f)^*} \mathcal{H}_{\mathrm{itame}}(\mathrm{N}^{\circ}_{\overline{D}_-}(D_+))$$

where $C'_+=f_*(D_+)$ *and* $\mathrm{N}^{\circ}(f):\mathrm{N}^{\circ}_{\overline{D}_-}(D_+)\to\mathrm{N}^{\circ}_{\overline{C}_-}(C'_+)$ *is the morphism induced by* f.

Moreover, the restriction of $\mathcal{H}_+^{\Psi,\otimes}$ *to the subcategory* $\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S\subset\mathrm{Reg}\text{-}\Sigma_+^{\mathrm{dm}}/S$ *is strict, i.e., is a functor*

$$\mathcal{H}^{\Psi,\otimes}:(\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}}\to\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}). \tag{3.79}$$

Finally, the previous properties hold also for the lax 2-functor $\mathcal{H}^{\Upsilon,\otimes}$ *when replacing "tameness" with "logarithmicity" at the boundary.*

Proof. The equivalence in (3.78) is clear by construction. Part (i) is also clear. We now prove part (ii). By definition, the functor $\mathcal{H}^{\Psi}(f)$ is given by the composition of

$$\begin{tikzcd} \mathcal{H}(\mathrm{N}^{\diamond}_X(C_-,C_+)) \arrow[r] & \mathcal{H}(Z;\mathcal{U}_{D_-,D_+}) \arrow[d, "(\mathrm{i}'_{C_-,C_+})_*"] & \\ & \mathcal{H}(\mathrm{Df}_Y(D_+);\mathcal{U}_{D_-,D_+}) \arrow[r, "(\mathrm{i}_{D_-,D_+})^*"] & \mathcal{H}(\mathrm{N}^{\diamond}_Y(D_-,D_+)), \end{tikzcd} \tag{3.80}$$

where $Z=\mathrm{N}^{\diamond}_X(C_-,C_+)\times_{\mathrm{Df}_X(C_+)}\mathrm{Df}_Y(D_+)$. Set $C'_-=f_*(D_-)$ and $Z'=\mathrm{N}^{\diamond}_X(C'_-,C_+)\times_{\mathrm{Df}_X(C_+)}\mathrm{Df}_Y(D_+)$. It is easy to see that there is a constructible neighbourhood of $\mathrm{N}^{\diamond}_Y(D_-,D_+)$ which intersects Z with Z'. Thus, the composition of (3.80) is equivalent to the composition of

$$\begin{tikzcd} \mathcal{H}(\mathrm{N}^{\diamond}_X(C_-,C_+)) \arrow[r] & \mathcal{H}(\mathrm{N}^{\diamond}_X(C'_-,C_+)) \arrow[r] & \mathcal{H}(Z';\mathcal{U}_{D_-,D_+}) \arrow[d, "(\mathrm{i}'_{C'_-,C_+})_*"] & \\ & & \mathcal{H}(\mathrm{Df}_Y(D_+);\mathcal{U}_{D_-,D_+}) \arrow[r, "(\mathrm{i}_{D_-,D_+})^*"] & \mathcal{H}(\mathrm{N}^{\diamond}_Y(D_-,D_+)). \end{tikzcd} \tag{3.81}$$

Thus, we may replace C_- with C'_- and assume that $f_*(D_-)=C_-$. In this case, we may replace X and Y by $\overline{C}_-$ and $\overline{D}_-$, and assume that X and Y are connected, f takes Y° to X°, and that $C_-=X^{\circ}$ and $D_-=Y^{\circ}$. The functor $\mathcal{H}^{\Psi}(f)$ is then given by the composition of

$$\begin{tikzcd} \mathcal{H}(\mathrm{N}^{\circ}_X(C_+)) \arrow[r] & \mathcal{H}(Z;\mathcal{U}_{D_+}) \arrow[d, "(\mathrm{i}'_{C_+})_*"] & \\ & \mathcal{H}(\mathrm{Df}_Y(D_+);\mathcal{U}_{D_+}) \arrow[r, "(\mathrm{i}_{D_+})^*"] & \mathcal{H}(\mathrm{N}^{\circ}_Y(D_+)) \end{tikzcd} \tag{3.82}$$

where $Z=\mathrm{N}^{\circ}_X(C_+)\times_{\mathrm{Df}_X(C_+)}\mathrm{Df}_Y(D_+)$, and i'_{C_+} and i_{D_+} are the obvious inclusions. Let $\overline{Z}$ be the closure of Z in $\mathrm{Df}_Y(D_+)$, and consider the Cartesian square

$$\begin{tikzcd} Z \arrow[r, "g'"] \arrow[d, "j'"] & \mathrm{N}^{\circ}_X(C_+) \arrow[d, "j"] \\ \overline{Z} \arrow[r, "g"] & \mathrm{N}_X(C_+). \end{tikzcd}$$

It follows from Lemma 3.45 that we have an induced commutative square

$$\begin{tikzcd}
\mathcal{H}_{\mathrm{itame}}(\mathrm{N}^{\circ}_{X}(C_+)) \arrow[r, "g'^*"] \arrow[d, "j_*"] & \mathcal{H}(Z;\mathcal{U}_{D_+}) \arrow[d, "j'_*"] \\
\mathcal{H}_{\mathrm{ict\text{-}tm}}(\mathrm{N}_{X}(C_+)) \arrow[r, "g^*\otimes_{g^*j_*\mathbf{1}}\mathcal{U}_{D_+}|_{\overline{Z}}"] & \mathcal{H}(\overline{Z};\mathcal{U}_{D_+}).
\end{tikzcd}$$

From this, we deduce that, after restriction to ind-tame objects, the composition of (3.82) is equivalent to the composition of

$$\begin{tikzcd}
\mathcal{H}(\mathrm{N}^{\circ}_{X}(C_+)) \arrow[r] & \mathcal{H}(\mathrm{Df}^{\circ}_{X|C_+}(C'_+);\mathcal{U}_{C'_+}) \arrow[d, "u_*"] & & \\
& \mathcal{H}(\mathrm{Df}_{X|C_+}(C'_+);\mathcal{U}_{C'_+}) \arrow[r, "(\mathrm{i}_{C'_+})^*"] & \mathcal{H}(\mathrm{N}^{\circ}_{X}(C'_+)) \arrow[r] & \mathcal{H}(\mathrm{N}^{\circ}_{Y}(D_+))
\end{tikzcd} \tag{3.83}$$

where $u : \mathrm{Df}^{\circ}_{X|C_+}(C'_+) \to \mathrm{Df}_{X|C_+}(C'_+)$ is the obvious inclusion. The composition of the first three functors in (3.83) is equivalent to $\widetilde{\Psi}_{C'_+}$ as needed.

It remains to show that the restriction of $\mathcal{H}^{\Psi,\otimes}_{+}$ to $\text{Reg-}\Sigma^{\mathrm{dm}}/S$ is strict. We start by noting that the above argument shows that $\mathcal{H}^{\Psi,\otimes}$ takes a commutative triangle of the form

$$\begin{tikzcd}
(Y,D_-,D_+) \arrow[r] \arrow[dr] & (X,C'_-,C'_+) \arrow[d] \\
& (X,C_-,C_+)
\end{tikzcd}$$

to a commutative diagram in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. Thus, we are left to check that $\mathcal{H}^{\Psi,\otimes}$ takes a commutative triangle of the form

$$\begin{tikzcd}
(X,C''_-,C''_+) \arrow[r] \arrow[dr] & (X,C'_-,C'_+) \arrow[d] \\
& (X,C_-,C_+)
\end{tikzcd}$$

to a commutative diagram in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. This follows from Theorem 3.63 combined with Corollary 3.61. We leave the details to the reader. □

3.6. **An exit-path theorem.**

We keep the notations and assumptions from Subsection 3.5. Here, we use Theorem 3.112 to show how to reconstruct, under some hypothesis, the presentable Voevodsky pullback formalism $\mathcal{H}^{\otimes}$ from the functor $\mathcal{H}^{\Psi,\otimes} : (\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. This reconstruction is reminiscent to an exit-path type phenomenon, which we will not pursue here.

We start with some general ∞-categorical preliminaries. We need a result about Cartesian fibrations which is probably well-known, but for which we do not know a reference. To state it, we recall a few facts in the following remark.

Remark 3.113. For a simplicial set S, we have the category $(\mathrm{Set}^{+}_{\Delta})_{/S}$ of marked simplicial sets over S endowed with the Cartesian model structure (see [Lur09a, Proposition 3.1.3.7]). Given a marked simplicial set (T,A) and a map $p : T \to S$, we have an adjunction

$$p^*_A : (\mathrm{Set}^{+}_{\Delta})_{/S} \rightleftarrows (\mathrm{Set}^{+}_{\Delta})_{/T} : p^A_*$$

where p_A^* takes a marked simplicial set (X, E) over S to $(X \times_S T, E \times_{S_1} A)$. The right adjoint p_*^A takes a marked simplicial set (Y, F) over T to the marked simplicial set $(p_*(Y), F^A)$ admitting the following description.

- An n-simplex of $p_*(Y)$ consists of morphisms $\Delta^n \to S$ and $s : \Delta^n \times_S T \to Y$ in $(\mathrm{Set}_\Delta)_{/T}$.
- An edge of $p_*(Y)$ consisting of morphisms $\Delta^1 \to S$ and $s : \Delta^1 \times_S T \to Y$ is marked if s sends $(\Delta^1)_1 \times_{S_1} A$ into F.

In general, the adjunction (p_A^*, p_*^A) is not Quillen with respect to the Cartesian model structures. Nevertheless, we have the following result.

Proposition 3.114. *Keep the notations as in Remark 3.113, and assume that $p : T \to S$ is a coCartesian fibration and that A consists of the p-coCartesian edges. Then the adjunction (p_A^*, p_*^A) is Quillen with respect to the Cartesian model structures.*

Proof. We will adapt the proof of [Lur09a, Proposition 4.1.2.15]. Arguing as in the proof of [Lur09a, Proposition 4.1.2.8], we reduce to showing that p_A^* takes a marked right anodyne morphism in $(\mathrm{Set}_\Delta^+)_{/S}$ to a trivial cofibration in $(\mathrm{Set}_\Delta^+)_{/T}$ for the Cartesian model structure. The class of marked anodyne morphisms is the weakly saturated class of morphisms (in the sense of [Lur09a, Definition A.1.2.2]) generated by the those listed in [Lur09a, Definition 3.1.1.1]. Thus, it is enough to check the required property for the morphisms listed in [Lur09a, Definition 3.1.1.1]. We will only treat the case of the inclusions

$$(\Lambda_n^n, E') \subset (\Delta^n, E), \tag{3.84}$$

for $n \geq 1$, where E is the set of all degenerate edges together with the final edge $\Delta^{\{n-1,n\}}$, and $E' = E \cap (\Lambda_n^n)_1$; the remaining ones can be treated similarly.

By base change along the map $\Delta^n \to S$, we reduce to the case where $S = \Delta^n$. In this case, we use [Lur09a, Proposition 3.2.2.7(1)], to find a composable sequence of simplicial sets

$$\phi : C^0 \to \ldots \to C^n$$

and a quasi-equivalence $\mathrm{M}^{\mathrm{op}}(\phi) \to T$ in the sense of [Lur09a, Definition 3.2.2.6]. Let A' be the set of edges in $\mathrm{M}^{\mathrm{op}}(\phi)$ of the form

$$([1] \xrightarrow{r} [n], [1] \to [0] \to C_{r(0)}).$$

Given a marked simplicial set (Q, J) over Δ^n, we claim that

$$(\mathrm{M}^{\mathrm{op}}(\phi), A') \times_{\Delta^n} (Q, J) \to (T, A) \times_{\Delta^n} (Q, J)$$

is a Cartesian equivalence in $(\mathrm{Set}_\Delta^+)_{/Q}$. This follows from [Lur09a, Proposition 3.2.2.7(2)], which implies that the underlying morphism $\mathrm{M}^{\mathrm{op}}(\phi) \times_{\Delta^n} Q \to T \times_{\Delta^n} Q$ is a categorical equivalence, and the observation that every edge in A can be obtained from an edge in A' by composing with an equivalence, and that this property is preserved by fiber product with (Q, J). Applying this with the domain and codomain of the inclusion in (3.84), we are reduced to showing that

$$(\mathrm{M}^{\mathrm{op}}(\phi), A') \times_{\Delta^n} (\Lambda_n^n, E') \to (\mathrm{M}^{\mathrm{op}}(\phi), A') \times_{\Delta^n} (\Delta^n, E)$$

is right anodyne. But this map is a pushout of $C^0 \times (\Lambda_n^n, E') \to C^0 \times (\Delta^n, E)$. □

Corollary 3.115. *Let $p : \mathcal{C} \to \mathcal{B}$ be a coCartesian fibration of ∞-categories. We denote by $p^* : \mathrm{CAT}_{\infty/\mathcal{B}} \to \mathrm{CAT}_{\infty/\mathcal{C}}$ the base change functor, and by p_* its right adjoint. Then p_* preserves Cartesian fibrations. More precisely, if $\mathcal{D} \to \mathcal{C}$ is a Cartesian fibration, so is $p_*(\mathcal{D}) \to \mathcal{B}$, and the latter is classified by a functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{CAT}_\infty$ admitting the following informal description.*

(i) It sends an object $A \in \mathcal{B}$ to the ∞-category $\mathrm{Sect}(\mathcal{D}_A/\mathcal{C}_A)$ of sections of the Cartesian fibration $\mathcal{D}_A \to \mathcal{C}_A$.

(ii) It sends a morphism $u : B \to A$ in $\mathcal{B}$ to a functor $u^* : \mathrm{Sect}(\mathcal{D}_A/\mathcal{C}_A) \to \mathrm{Sect}(\mathcal{D}_B/\mathcal{C}_B)$ which can be informally described as follows. Fix a section $s : \mathcal{C}_A \to \mathcal{D}_A$. For $Y \in \mathcal{C}_B$, we consider the coCartesian edge $v : Y \to u_!(Y)$ in $\mathcal{C}$ over u, and then the Cartesian edge $v^* s(u_!(Y)) \to s(u_!(Y))$ in $\mathcal{D}$ over v. Then the section $u^*(s) : \mathcal{C}_B \to \mathcal{D}_B$ takes Y to $v^* s(u_!(Y))$.

Proof. This follows readily from Proposition 3.114 since a fibrant object for the Cartesian model structure is precisely a Cartesian fibration marked by its Cartesian edges. □

Construction 3.116. Let $\mathcal{C}$ be an ∞-category, and let

$$F : \Delta^1 \times \mathcal{C} \rightsquigarrow \underline{\mathrm{CAT}}_\infty \tag{3.85}$$

be a lax 2-functor whose restrictions to $\{0\} \times \mathcal{C}$ and $\{1\} \times \mathcal{C}$ are strict, i.e., define functors

$$F_0 : \mathcal{C} \to \mathrm{CAT}_\infty \qquad \text{and} \qquad F_1 : \mathcal{C} \to \mathrm{CAT}_\infty. \tag{3.86}$$

The lax 2-functor F gives rise to a commutative triangle of the form

$$\begin{tikzcd} \Delta^1 \times \int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Map}([n], \mathcal{C}) \arrow[rr, "\xi"] \arrow[dr] & & \int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{CAT}^{\mathrm{cart}}_{\infty/[n]^{\mathrm{op}}} \arrow[dl] \\ & \mathbf{\Delta}^{\mathrm{op}}, & \end{tikzcd} \tag{3.87}$$

and hence to a natural transformation $\xi_0 \to \xi_1$ over $\mathbf{\Delta}^{\mathrm{op}}$ between the functors

$$\xi_0, \xi_1 : \int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Map}([n], \mathcal{C}) \to \int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{CAT}^{\mathrm{cart}}_{\infty/[n]^{\mathrm{op}}} \tag{3.88}$$

incarnating the functors F_0 and F_1. We denote by $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}$ the domain of the functors in (3.88) which is also the domain of the coCartesian fibration classified by the simplicial ∞-groupoid $\mathrm{Map}([\bullet], \mathcal{C})$. We may view ξ_0 and ξ_1 as defining sections ξ_0' and ξ_1' of the coCartesian fibration

$$\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}} \times_{\mathbf{\Delta}^{\mathrm{op}}} \int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{CAT}^{\mathrm{cart}}_{\infty/[n]^{\mathrm{op}}} \to \mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}. \tag{3.89}$$

Since F_0 and F_1 are strict functors, the sections ξ_0' and ξ_1' are coCartesian. Thus, the induced natural transformation $\xi_0' \to \xi_1'$ defines an edge in the ∞-category

$$\lim_{[n]\to\mathcal{C}\in\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}} \mathrm{CAT}^{\mathrm{cart}}_{\infty/[n]^{\mathrm{op}}} \simeq \mathrm{CAT}^{\mathrm{cart}}_{\infty/\mathcal{C}^{\mathrm{op}}}. \tag{3.90}$$

We obtain in this way a commutative triangle

$$\begin{tikzcd} \mathcal{D}_0 \arrow[rr, "\theta"] \arrow[dr, "p_0"'] & & \mathcal{D}_1 \arrow[dl, "p_1"] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd} \tag{3.91}$$

where p_0 and p_1 are the Cartesian fibrations classified by the functors F_0 and F_1.

Construction 3.117. Consider the forgetful functor

$$p : \mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S \to \mathrm{Reg}\text{-}\Sigma/S \tag{3.92}$$

given by $p(X, C_-, C_+) = X$. Also, denote by

$$i_0 : \text{Reg-}\Sigma/S \hookrightarrow \text{Reg-}\Sigma_+^{\mathrm{dm}}/S \qquad \text{and} \qquad i_1 : \text{Reg-}\Sigma^{\mathrm{dm}}/S \hookrightarrow \text{Reg-}\Sigma_+^{\mathrm{dm}}/S \tag{3.93}$$

the obvious fully faithful inclusions. We have a natural transformation $i_1 \to i_0 \circ p$ sending an object (X, C_-, C_+) to the morphism $(X, C_-, C_+) \to X$ given by the identity of X. We denote by

$$\phi : (\Delta^1)^{\mathrm{op}} \times \text{Reg-}\Sigma^{\mathrm{dm}}/S \to \text{Reg-}\Sigma_+^{\mathrm{dm}}/S \tag{3.94}$$

the functor classified by this natural transformation. Precomposing with ϕ^{op}, yields a lax 2-functor

$$\mathcal{H}_+^{\Psi,\otimes} \circ \phi : \Delta^1 \times (\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}} \to \underline{\mathrm{SMCAT}}_\infty \tag{3.95}$$

whose restrictions to $\{\epsilon\} \times (\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}}$, for $\epsilon \in \{0, 1\}$, are strict by Theorem 3.112. Applying Construction 3.116 to $\mathcal{H}_+^{\Psi,\otimes} \circ \phi$ yields a commutative triangle

$$\begin{tikzcd} \displaystyle\oint_{\text{Reg-}\Sigma^{\mathrm{dm}}/S} \mathcal{H}_+^{\Psi,\otimes} \circ i_0 \circ p \arrow[rr, "\theta"] \arrow[dr] & & \displaystyle\oint_{\text{Reg-}\Sigma^{\mathrm{dm}}/S} \mathcal{H}_+^{\Psi,\otimes} \circ i_1 \arrow[dl] \\ & \text{Reg-}\Sigma^{\mathrm{dm}}/S & \end{tikzcd} \tag{3.96}$$

where the slanted arrows are Cartesian fibrations. We denote by p^* the base change along the functor p, and p_* its right adjoint. The domain of θ can be rewritten as

$$p^*\left(\oint_{\text{Reg-}\Sigma/S} \mathcal{H}^{\otimes}_{\text{ict-tm}}\right).$$

Thus, by adjunction, the triangle in (3.96) gives rise to the following commutative triangle

$$\begin{tikzcd} \displaystyle\oint_{\text{Reg-}\Sigma/S} \mathcal{H}^{\otimes}_{\text{ict-tm}} \arrow[rr, "\theta'"] \arrow[dr] & & p_*\left(\displaystyle\oint_{\text{Reg-}\Sigma^{\mathrm{dm}}/S} \mathcal{H}^{\Psi,\otimes}\right) \arrow[dl] \\ & \text{Reg-}\Sigma/S & \end{tikzcd} \tag{3.97}$$

where the slanted arrows are Cartesian fibrations: for the left one, this is by construction and, for the right one, this follows from Corollary 3.115 since the functor p is a coCartesian fibration.

The following result describes the fibers of functor θ'.

Theorem 3.118. *The triangle in* (3.97) *is a morphism of Cartesian fibrations, i.e., the functor θ' preserves Cartesian edges. For a regularly stratified S-scheme X, the functor*

$$\theta'_X : \mathcal{H}_{\text{ict-tm}}(X) \to \mathrm{Sect}\left(\oint_{\mathcal{P}'_X} \mathcal{H}^{\Psi} \middle/ \mathcal{P}'_X\right) \tag{3.98}$$

induced by θ' on the fibers at X admits the following informal description. Fix an object $M \in \mathcal{H}_{\text{ict-tm}}(X)$. The section $\theta'_X(M)$ takes $(C_-, C_+) \in \mathcal{P}'_X$ to $M|_{\mathrm{N}^\circ_{\overline{C}_-}(C_+)} \in \mathcal{H}_{\text{itame}}(\mathrm{N}^\circ_{\overline{C}_-}(C_+))$. It takes an arrow $(C'_-, C'_+) \to (C_-, C_+)$ in $\mathcal{P}'_X$ to the morphism corresponding to the composition of

$$M|_{\mathrm{N}^\circ_{\overline{C}'_-}(C'_+)} \simeq (M|_{\mathrm{N}^\circ_{\overline{C}_-}(C'_+)})|_{\mathrm{N}^\circ_{\overline{C}'_-}(C'_+)} \to \widetilde{\Psi}_{C'_+}(M|_{\mathrm{N}^\circ_{\overline{C}_-}(C_+)})|_{\mathrm{N}^\circ_{\overline{C}'_-}(C'_+)}. \tag{3.99}$$

Moreover, the functor θ'_X is fully faithful with essential image the sub-∞-category spanned by those sections sending an arrow of the form $(C'_-, C_+) \to (C_-, C_+)$ to a Cartesian edge.

Proof. The description of the functor θ'_X follows readily from the constructions. The assertion that θ' preserves Cartesian edges reduces to the following property: given a morphism of regularly stratified S-schemes $f : Y \to X$, an object $(D_-, D_+) \in \mathscr{P}'_Y$ with image $(C_-, C_+) \in \mathscr{P}'_X$, and an object $M \in \mathcal{H}_{\text{ict-tm}}(X)$, the natural morphism

$$\mathrm{N}^\circ(f)^*\theta'_X(M)(C_-, C_+) \to \theta'_Y(f^*M)(D_-, D_+),$$

with $\mathrm{N}^\circ(f) : \mathrm{N}^\circ_{\overline{D}_-}(D_+) \to \mathrm{N}^\circ_{\overline{C}_-}(C_+)$ the morphism induced by f, is an equivalence. This is immediate using the description of the sections $\theta'_X(M)$ and $\theta'_Y(f^*M)$ on the objects of $\mathscr{P}'_X$ and $\mathscr{P}'_Y$. It remains to prove the last assertion concerning the fully faithfulness of θ'_X and its essential image. We divide the proof of this in three steps.

Step 1. For a regularly stratified S-scheme X, we denote by Sect_X the codomain of the functor θ'_X in (3.98), and Sect'_X the full sub-∞-category of Sect_X spanned by those sections sending an arrow of the form $(C'_-, C_+) \to (C_-, C_+)$ to a Cartesian edge. Clearly, θ'_X factors through Sect'_X, and we need to show that it induces an equivalence of ∞-categories $\mathcal{H}_{\text{ict-tm}}(X) \simeq \mathrm{Sect}'_X$. We denote by $\alpha_X : \mathrm{Sect}'_X \to \mathrm{Sect}_X$ the obvious inclusion, and by $\beta_X : \mathrm{Sect}_X \to \mathrm{Sect}'_X$ its right adjoint.

Let $i_Z : Z \subset X$ be the inclusion of a regular constructible locally closed subscheme of X. We have a pair of adjoint functors

$$i_Z^* : \mathrm{Sect}_X \rightleftarrows \mathrm{Sect}_Z : i_{Z,*}, \tag{3.100}$$

where i_Z^* is the restriction along the inclusion $\mathscr{P}'_Z \hookrightarrow \mathscr{P}'_X$ and $i_{Z,*}$ is the relative right Kan extension along $\mathscr{P}'_Z \hookrightarrow \mathscr{P}'_X$. Since

$$\oint_{\mathscr{P}'_X} \mathcal{H}^\Psi \to \mathscr{P}'_X$$

is a Cartesian fibration, the functor $i_{Z,*}$ admits a very simple description: for a section s defined over $\mathscr{P}'_Z$, the section $i_{Z,*}(s)$ evaluated at (C_-, C_+) is given by

$$i_{Z,*}(s)(C_-, C_+) = \lim_{(C_-,C_+)\to(E_-,E_+),\ (E_-,E_+)\in\mathscr{P}'_Z} \widetilde{\Psi}_{C_+} s(E_-, E_+)|_{\mathrm{N}^\circ_{\overline{C}_-}(C_+)}. \tag{3.101}$$

The adjunction $(i_Z^*, i_{Z,*})$ gives rise to an adjunction

$$i'^*_Z : \mathrm{Sect}'_X \rightleftarrows \mathrm{Sect}'_Z : i'_{Z,*}, \tag{3.102}$$

where $i'^*_Z \simeq \beta_Z \circ i^*_Z \circ \alpha_X$ and $i'_{Z,*} \simeq \beta_X \circ i'_{Z,*} \circ \alpha_Z$. We claim that the following commutative square

$$\begin{tikzcd} \mathcal{H}_{\text{ict-tm}}(X) \arrow[r, "i_Z^*"] \arrow[d, "\theta'_X"] & \mathcal{H}_{\text{ict-tm}}(Z) \arrow[d, "\theta'_Z"] \\ \mathrm{Sect}'(X) \arrow[r, "i'^*_Z"] & \mathrm{Sect}'(Z) \end{tikzcd} \tag{3.103}$$

is right adjointable. This will be proven in the second step. In the third step, we use this to conclude the proof.

Step 2. Here we prove that the square (3.103). Note that this is evident when i_Z is a closed immersion. Indeed, in this case, the formula in (3.102) gives

$$i_{Z,*}(s)(C_-, C_+) = \begin{cases} s(E_-, C_+)|_{\mathrm{N}^\circ_{\overline{C}_-}(C_+)} & \text{if} \quad C_+ \subset Z, \\ 0 & \text{if} \quad C_+ \not\subset Z, \end{cases}$$

where E_- is the open stratum of $Z \cap \overline{C}_-$. It follows that $i_{Z,*}$ takes Sect'_Z to Sect'_X, so that its is enough to show that the natural transformation $\theta'_X \circ i_{Z,*} \to i_{Z,*} \circ \theta'_Z$ is an equivalence, which is clear. Thus, it is enough to treat the case of an open immersion $i_U : U \to X$ and we may assume that U is the complement of a closed stratum $C_0 \subset X$.

Fix a section $s \in \mathrm{Sect}'_U$. We will give a formula for $i'_{U,*}(s)$. First, note that $i_{U,*}(s)(C_-, C_+) \simeq s(C_-, C_+)$ when $C_+ \neq C_0$. On the other hand, we have:

$$i_{U,*}(s)(C_-, C_0) = \lim_{C_- \leq C_+ < C_0} \widetilde{\Psi}_{C_0} s(C_-, C_+). \tag{3.104}$$

Let $\iota : \mathcal{Q} = (\mathcal{P}_X^{\mathrm{op}})_{C_0/} \hookrightarrow \mathcal{P}'_X$ be the inclusion of the sub-poset of $\mathcal{P}'_X$ consisting of those elements of the form (C_-, C_0). We also have two pairs of adjoint functors

$$\iota^* : \mathrm{Sect}_X \rightleftarrows \mathrm{Sect}_{\mathcal{Q}} : \iota_* \qquad \text{and} \qquad \alpha_{\mathcal{Q}} : \mathrm{Sect}'_{\mathcal{Q}} \rightleftarrows \mathrm{Sect}'_{\mathcal{Q}} : \beta_{\mathcal{Q}},$$

where

$$\mathrm{Sect}_{\mathcal{Q}} = \mathrm{Sect}\left(\left. \int_{\mathcal{Q}} \mathcal{H}^{\Psi} \right/ \mathcal{Q} \right)$$

and $\mathrm{Sect}'_{\mathcal{Q}}$ is the full sub-∞-category of $\mathrm{Sect}_{\mathcal{Q}}$ spanned by the Cartesian sections. We claim that $i'_{U,*}(s)$ is the section rendering the following square of Sect_X Cartesian

$$\begin{tikzcd} i'_{U,*}(s) \arrow[r] \arrow[d] & i_{U,*}(s) \arrow[d] \\ \iota_* \beta_{\mathcal{Q}} \iota^* i_{U,*}(s) \arrow[r] & \iota_* \iota^* i_{U,*}(s). \end{tikzcd}$$

Indeed, the section rendering this square Cartesian belongs clearly to Sect'_X since it coincides with $i_{U,*}(s)$ on (C_-, C_+), when $C_+ \neq C_0$, and with $\beta_{\mathcal{Q}} \iota^* i_{U,*}(s)$ on (C_-, C_0), for $C_- \leq C_0$. On the other hand, given $t \in \mathrm{Sect}'_X$, we have equivalences

$$\begin{aligned} \mathrm{Map}(t, \iota_* \beta_{\mathcal{Q}} \iota^* i_{U,*}(s)) &\simeq \mathrm{Map}(\iota^*(t), \beta_{\mathcal{Q}} \iota^* i_{U,*}(s)) \\ &\simeq \mathrm{Map}(\iota^*(t), \iota^* i_{U,*}(s)) \\ &\simeq \mathrm{Map}(t, \iota_* \iota^* i_{U,*}(s)), \end{aligned}$$

which implies the required property $\mathrm{Map}(t, i'_{U,*}(s)) \simeq \mathrm{Map}(t, i_{U,*}(s))$. Thus, we only need to describe $\beta_{\mathcal{Q}}$. Since $\mathcal{Q}$ admits a final object, we have an equivalence of ∞-categories $\mathrm{Sect}'_{\mathcal{Q}} \simeq \mathcal{H}_{\mathrm{itame}}(C_0)$. We also denote by $\beta_{\mathcal{Q}} : \mathrm{Sect}_{\mathcal{Q}} \to \mathcal{H}_{\mathrm{itame}}(C_0)$ the functor obtained from $\beta_{\mathcal{Q}}$ using this identification. For strata $E \leq D \leq C$, let $p_{E,D} : \mathrm{N}_{\overline{E}}(C_0) \to \mathrm{N}_{\overline{D}}(C_0)$ be the obvious morphism. Given a section $r \in \mathrm{Sect}_{\mathcal{Q}}$, we have

$$\beta_{\mathcal{Q}}(r) = \lim_{C_- \leq C_+ \leq C_0} (p_{C_-, C_0})_* (p_{C_-, C_+})^* r(C_+, C_0).$$

Note that for $C'_- \leq C_- \leq C_+ \leq C'_+ \leq C_0$, the map $r(C_+, C_0) \to (p_{C_+, C'_+})^* r(C'_+, C_0)$ gives rise to a morphism in $\mathcal{H}_{\mathrm{itame}}(C_0)$:

$$(p_{C_-, C_0})_* (p_{C_-, C_+})^* r(C_+, C_0) \to (p_{C'_-, C_0})_* (p_{C'_-, C'_+})^* r(C'_+, C_0).$$

We are now ready to finish the proof of the right adjointability of the square in (3.103).

Fix an object $M \in \mathcal{H}_{\text{ict-tm}}(U)$, and assume that $s = \theta'_U(M)$. We need to prove that $\beta_{\mathcal{Q}}(\iota^* i_{U,*}(s))$ is canonically equivalent to $i_{U,*}(M)|_{C_0}$. Said differently, we need to show that the map

$$i_{U,*}(M)|_{C_0} \to \lim_{C_- \leq C_+ \leq C_0} (p_{C_-, C_0})_* (p_{C_-, C_+})^* i_{U,*}(\theta'_U(M))(C_+, C_0)$$

is an equivalence. To do so, we may assume that $M = j_!M'$ where j is the inclusion of a stratum of U. We easily reduce to the case where $j : X^\circ \hookrightarrow U$ is the inclusion of the open stratum of X. In this case, $M|_{\mathrm{N}^\circ_{\overline{C_-}}(C_+)}$ is zero unless $C_- = C_+ = X^\circ$. Thus, the formula in (3.104) gives us

$$i_{U,*}(\theta'_U(M))(C_+, C_0) \simeq \begin{cases} \widetilde{\Psi}_{C_0}(M')[1-c_0] & \text{if} \quad C_+ = X^\circ \\ 0 & \text{if} \quad C_+ \neq X^\circ \end{cases}$$

where c_0 is the codimension of C_0 in X. Hence, we need to show that the morphism

$$(i_{U,*}j_!M')|_{C_0} \to (p_{X^\circ,C_0})_*\widetilde{\Psi}_{C_0}(M')[1-c_0]$$

is an equivalence. Since p_{X°,C_0} is the obvious projection $\mathrm{N}^\circ_X(C_0) \to C_0$, we have an equivalence $(p_{X^\circ,C_0})_*\widetilde{\Psi}_{C_0}(M') \simeq (i_{U,*}j_*M')|_{C_0}$. Finally, we are left to show that

$$(i_{U,*}j_!M')|_{C_0} \to (i_{U,*}j_*M')|_{C_0}[1-c_0]$$

is an equivalence, which is an easy exercise left to the reader.

Step 3. We now use the right adjointability of the square in (3.103) to prove that the functor $\theta'_X : \mathcal{H}_{\text{ict-tm}}(X) \to \mathrm{Sect}'(X)$ is an equivalence. To prove that θ'_X is fully faithful, it is enough to show that the map

$$\mathrm{Map}_{\mathcal{H}_{\text{ict-tm}}(X)}(M, i_{C,*}N) \to \mathrm{Map}_{\mathrm{Sect}'_X}(\theta'_X(M), \theta'_X(i_{C,*}N))$$

is an equivalence for every stratum C of X, and every objects $M \in \mathcal{H}_{\text{ict-tm}}(X)$ and $N \in \mathcal{H}_{\text{itame}}(C)$. By the the right adjointability of the square in (3.103), we have a natural equivalence $\theta'_X \circ i_{C,*} \simeq i'_{C,*} \circ \theta'_C$. Using the adjunction $(i'^*_C, i'_{C,*})$ and the commutation $i'^*_C \circ \theta'_X \simeq \theta'_C \circ i^*_C$, we reduce to showing that

$$\mathrm{Map}_{\mathcal{H}_{\text{itame}}(C)}(i^*_C M, N) \to \mathrm{Map}_{\mathrm{Sect}'_C}(\theta'_C(i^*_C M), \theta'_C N)$$

is an equivalence, which is obvious since θ'_C is an equivalence of ∞-categories.

To prove essential surjectivity for θ'_X, we argue by induction on the number of strata of X. Thus, if $C_0 \subset X$ is a closed stratum and $U = X \smallsetminus C_0$ its complement, we may assume that the functor $\theta'_U : \mathcal{H}_{\text{ict-tm}}(U) \to \mathrm{Sect}'_U$ is an equivalence. Using the right adjointability of the square in (3.103), we deduce that every object in the image of $i'_{U,*} : \mathrm{Sect}'_U \to \mathrm{Sect}'_X$ belongs to the essential image of θ'_X. Now, if s is a general section in Sect'_X, we may consider a fiber sequence

$$F \to s \to i'_{U,*}i'^*_U(s)$$

in Sect'_X. Since θ'_X is a fully faithful exact functor between stable ∞-categories, we are left to show that F belongs to its image. But the section F has the property that $F(C_-, C_+) = 0$ unless $C_+ = C_0$. Since it belongs to Sect'_X, we see that it is isomorphic to $i'_{C_0,*}F(C_0, C_0) \simeq \theta'_X(i_{C_0,*}F(C_0, C_0))$ as needed. This finishes the proof. □

Corollary 3.119 (Exit-path Theorem). *Denote by $p : \text{Reg-}\Sigma^{\mathrm{dm}}/S \to \text{Reg-}\Sigma/S$ the functor forgetting the demarcation. Then, the functor*

$$\theta' : \oint_{\text{Reg-}\Sigma/S} \mathcal{H}^{\otimes}_{\text{ict-tm}} \to p_*\left(\oint_{\text{Reg-}\Sigma^{\mathrm{dm}}/S} \mathcal{H}^{\Psi,\otimes}\right) \tag{3.105}$$

is fully faithful and its essential image consists of those pairs (X, s), with X a regularly stratified S-scheme and

$$s : \mathcal{P}'_X \to \oint_{\mathcal{P}'_X} \mathcal{H}^{\Psi}$$

a section taking the edges $(C'_-, C_+) \to (C_-, C_+)$, with $C'_- \geq C_- \geq C_+$, to Cartesian edges.

Proof. This follows immediately form Theorem 3.118. □

4. The main theorem for local systems

This section is devoted to extracting from Theorem 2.10 a description of $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ as the group of autoequivalences of the functor

$$\mathbf{LS}_{\mathrm{geo}}(-)^{\otimes} : (\mathrm{Sm}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{CAT}_{\infty}).$$

As explained in the introduction, this can be viewed as a motivic non truncated version of the Ihara–Matsumoto–Oda Conjecture. Our method relies on the machinery developed in Section 3 and, in particular, on our exit-path theorem (see Corollary 3.119).

4.1. Stratification, constructibility and cdh descent.

We fix a field k endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$. In this subsection, we prove a version of Theorem 2.10 where the ∞-categories of sheaves are replaced with ∞-categories of sheaves that are locally constant over the strata of a given stratification; see Corollary 4.7 below. We need to introduce the cdh topology on the category of stratified schemes.

Definition 4.1. A family $(f_i : X_i \to X)_i$ in SCH-Σ is said to be a cdh-cover if it is so after forgetting the stratifications. We denote by cdh the topology generated by cdh-covers on SCH-Σ or similar categories such as Sch-Σ/S, for a noetherian scheme S, and Reg-Σ/S, for an excellent scheme S of characteristic zero.

We will use the following simple fact.

Lemma 4.2. *Let S be a noetherian scheme.*

(i) The forgetful functor β_S : Sch-$\Sigma/S \to$ Sch$/S$ induces an equivalence of ∞-topoi

$$\beta_{S,*} : \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}/S) \xrightarrow{\sim} \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}\text{-}\Sigma/S). \tag{4.1}$$

(ii) Assume that S is excellent of characteristic zero. Then, the obvious inclusions ι_S and forgetful functors β_S induce a commutative square of equivalences of ∞-topoi

$$\begin{tikzcd}
\mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}/S) \arrow[r, "\beta_{S,*}", "\sim"'] \arrow[d, "\iota_{S,*}", "\sim"'] & \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}\text{-}\Sigma/S) \arrow[d, "\iota_{S,*}", "\sim"'] \\
\mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Reg}/S) \arrow[r, "\beta_{S,*}", "\sim"'] & \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Reg}\text{-}\Sigma/S).
\end{tikzcd} \tag{4.2}$$

Proof. To prove (i), we note that the functor β_S admits a right adjoint α_S : Sch$/S \to$ Sch-Σ/S sending an S-scheme X to itself endowed with the trivial stratification (i.e., the strata are the connected components). Moreover, the counit morphism $\beta_S \circ \alpha_S \to \mathrm{id}$ is invertible. We deduce from this a pair of adjoint functors

$$\beta_S^* : \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}\text{-}\Sigma/S) \rightleftarrows \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}/S) : \alpha_S^*$$

such that $\beta_S^* \circ \alpha_S^* \simeq \mathrm{id}$. We will prove that the unit morphism $\mathrm{id} \to \alpha_S^* \circ \beta_S^*$ is an equivalence. Since α_S^* and β_S^* commute with arbitrary colimits, it is enough to show that $\mathrm{id} \to \alpha_S^* \circ \beta_S^*$ is an equivalence when applied to objects of the form $\mathrm{L}_{\mathrm{cdh}}\mathrm{y}(X)$, for $X \in$ Sch-Σ/S. Thus, we are left to show that the morphism $X \to \alpha_S\beta_S(X)$ induces an equivalence after cdh sheafification. This is clear since $X \to \alpha_S\beta_S(X)$ is both a monomorphism and a cdh-cover.

To prove (ii), it is enough to show that the inclusions $\iota_S : \mathrm{Reg}(\text{-}\Sigma)/S \to \mathrm{Sch}(\text{-}\Sigma)/S$ induce equivalences of topoi. By resolution of singularities for excellent schemes [Tem08], every (stratified) S-scheme X admits a cdh-cover by (regularly stratified) regular S-schemes. Thus, the result follows from [SGA72b, Exposé III, Théorème 4.1] □

Proposition 4.3. *Let $\mathcal{H}^{\otimes} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a presentable Voevodsky pullback formalism. Then $\mathcal{H}^{\otimes}$ is a* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$*-valued* cdh*-sheaf.*

Proof. The proof of [Hoy17, Proposition 6.24], which deals with the case of $\mathbf{MSh}_{\mathrm{nis}}(-)$, is also valid for a general presentable Voevodsky pullback formalism. For the reader's convenience, we include a proof. It follows from [Voe10, Theorem 4.5] that the ∞-category $\mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}/S)$ is the localisation of $\mathcal{P}(\mathrm{Sch}/S)$ with respect to the following morphisms of presheaves.

(i) The inclusion of the empty presheaf $\emptyset \hookrightarrow \mathrm{y}(\emptyset)$.

(ii) The morphism $\mathrm{y}(U)\coprod_{\mathrm{y}(U')}\mathrm{y}(X') \to \mathrm{y}(X)$ associated to a Nisnevich square

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "e'"] & X' \arrow[d, "e"] \\ U \arrow[r, "j"] & X. \end{tikzcd}$$

Recall that the square is Cartesian, e is étale, j is an open immersion and the induced morphism $X' \smallsetminus U' \to X \smallsetminus U$ is an isomorphism.

(iii) The morphism $\mathrm{y}(Z)\coprod_{\mathrm{y}(Z')}\mathrm{y}(X') \to \mathrm{y}(X)$ associated to an abstract blowup square

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "e'"] & X' \arrow[d, "e"] \\ Z \arrow[r, "i"] & X. \end{tikzcd}$$

Recall that the square is Cartesian, e is proper, i is a closed immersion and the induced morphism $X' \smallsetminus Z' \to X \smallsetminus Z$ is an isomorphism.

Thus, it is enough to check that the left Kan extension of $\mathcal{H}$ along the Yoneda embedding $\mathrm{y} : \mathrm{Sch}/S \to \mathcal{P}(\mathrm{Sch}/S)$ tranforms the above morphisms into equivalences. This is clear for (i) since $\mathcal{H}(\emptyset)$ is the final ∞-category. We only treat the case of (iii) since (ii) is entirely similar. We need to show that the obvious functor

$$\theta^* : \mathcal{H}(X) \xrightarrow{(i^*,e^*)} \mathcal{H}(Z) \times_{\mathcal{H}(Z')} \mathcal{H}(X')$$

is an equivalence. This functor admits a right adjoint θ_* sending an object $(A, B, u : e'^*(A) \simeq i'^*(B))$ to the limit of the diagram

$$\begin{tikzcd} & & e_*(B) \arrow[d] \\ i_*(A) \arrow[r] & i_*e'_*e'^*(A) \arrow[r, "u", "\sim"'] & e_*i'_*i'^*(B). \end{tikzcd}$$

We claim that the unit morphism $\mathrm{id} \to \theta_*\theta^*$ is an equivalence. Writing $s : Z' \to X$ for the composite morphisms $e \circ i' = i \circ e'$, we need to show that the square

$$\begin{tikzcd} M \arrow[r] \arrow[d] & i_*i^*(M) \arrow[d] \\ e_*e^*(M) \arrow[r] & s_*s^*(M) \end{tikzcd}$$

is Cartesian for all $M \in \mathcal{H}(X)$. Let $j : X \smallsetminus Z \to X$ and $j' : X' \smallsetminus Z' \to X'$ be the obvious inclusions. Since the pair (j^*, i^*) is conservative, it is enough to show that the above square becomes Cartesian after applying j^* and i^*. This is clear for j^* and follows from the proper base change theorem for i^*. It remains to see that θ^* is essentially surjective. But a general object (A, B, u) as above is part of a triangle where the first and third terms are

$$(0, j'_! j'^! B, 0 \simeq i'^* j'_! j'^! B) \qquad \text{and} \qquad (A, i'_* e'^* A, e'^* A \simeq i'^* i'_* e'^* A).$$

Both these objects are clearly in the image of θ. This finishes the proof. □

Proposition 4.4. *Let $\beta : \mathrm{Sch}\text{-}\Sigma/k \to \mathrm{Sch}/k$ be the functor sending a smoothly stratified k-variety to its underlying k-variety. Then, the morphism of* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$*-valued presheaves*

$$\mathbf{Sh}_{\mathrm{geo,\,ict}}(-; \Lambda)^{\otimes} \to \mathbf{Sh}_{\mathrm{geo}}(-; \Lambda)^{\otimes} \circ \beta \tag{4.3}$$

exhibits $\mathbf{Sh}_{\mathrm{geo}}(-; \Lambda)^{\otimes} \circ \beta$ as the cdh*-sheafification of $\mathbf{Sh}_{\mathrm{geo,\,ict}}(-; \Lambda)^{\otimes}$.*

Proof. In fact, more is true: the morphism in (4.3) exhibits $\mathbf{Sh}_{\mathrm{geo}}(-; \Lambda) \circ \beta$ as the τ-sheafification of $\mathbf{Sh}_{\mathrm{geo,\,ict}}(-; \Lambda)$, where τ is the topology on $\mathrm{Sch}\text{-}\Sigma/k$ generated by the morphisms of stratified k-varieties of the form $\mathrm{id}_X : (X, \mathscr{P}') \to (X, \mathscr{P})$. Indeed, for a k-variety X, we have an equivalence

$$\operatorname*{colim}_{\mathscr{P}} \mathbf{Sh}_{\mathrm{geo}}((X, \mathscr{P}); \Lambda) \xrightarrow{\sim} \mathbf{Sh}_{\mathrm{geo}}(X; \Lambda),$$

where the colimit is taken in Pr^{L} and is indexed by the stratifications of X. □

Proposition 4.5. *Let S be an excellent scheme of characteristic* 0*, and let*

$$\mathcal{H}^{\otimes} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

be a presentable Voevodsky pullback formalism. Assume that the conclusion of Proposition 4.4 is satisfied for $\mathcal{H}^{\otimes}$, i.e., that the natural transformation $\mathcal{H}^{\otimes}_{\mathrm{ict}} \to \mathcal{H}^{\otimes} \circ \beta$ exhibits $\mathcal{H}^{\otimes} \circ \beta$ as the cdh*-sheafification of $\mathcal{H}^{\otimes}_{\mathrm{ict}}$. There is a commutative square of equivalences of group objects in $\mathcal{S}$:*

$$\begin{tikzcd} \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\mathrm{ict}}) \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\mathrm{ict}}|_{\mathrm{Reg}\text{-}\Sigma/S}) \arrow[d, "\sim"] \\ \mathrm{Auteq}(\mathcal{H}^{\otimes}) \arrow[r, "\sim"] & \mathrm{Auteq}(\mathcal{H}^{\otimes}|_{\mathrm{Reg}/S}). \end{tikzcd} \tag{4.4}$$

Proof. By Lemma 4.2, we have an equivalence of ∞-categories

$$\mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}/S; \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})) \xrightarrow{\iota_*} \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Reg}/S; \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))$$

sending the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued sheaf $\mathcal{H}^{\otimes}$ to $\mathcal{H}^{\otimes}|_{\mathrm{Reg}/S}$ and inducing the bottom horizontal equivalence of the square in (4.4). We will prove the proposition by showing that the vertical morphisms in this square are equivalences. Using Lemma 4.2, we may replace the commutative square in (4.4) by the following one:

$$\begin{tikzcd} \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\mathrm{ict}}) \arrow[r] \arrow[d] & \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\mathrm{ict}}|_{\mathrm{Reg}\text{-}\Sigma/S}) \arrow[d] \\ \mathrm{Auteq}(\mathcal{H}^{\otimes} \circ \beta) \arrow[r] & \mathrm{Auteq}(\mathcal{H}^{\otimes} \circ \beta|_{\mathrm{Reg}\text{-}\Sigma/S}). \end{tikzcd} \tag{4.5}$$

The vertical morphisms in this square are induced by the cdh-sheafification functors

$$\mathrm{L}_{\mathrm{cdh}} \;:\; \mathrm{Psh}(\mathrm{Sch}\text{-}\Sigma/S\,;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})) \;\to\; \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}\text{-}\Sigma/S\,;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))$$
$$\mathrm{L}_{\mathrm{cdh}} \;:\; \mathrm{Psh}(\mathrm{Reg}\text{-}\Sigma/S\,;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})) \;\to\; \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Reg}\text{-}\Sigma/S\,;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))$$

which, by Proposition 4.4 (and Lemma 4.2), send $\mathcal{H}^{\otimes}_{\mathrm{ict}}$ and $\mathcal{H}^{\otimes}_{\mathrm{ict}}|_{\mathrm{Reg}\text{-}\Sigma/S}$ to $\mathcal{H}^{\otimes} \circ \beta$ and $\mathcal{H}^{\otimes} \circ \beta|_{\mathrm{Reg}\text{-}\Sigma/S}$ respectively. We denote by $t : \mathcal{H}^{\otimes}_{\mathrm{ict}} \to \mathcal{H}^{\otimes} \circ \beta$ the obvious natural transformation. By Lemma 4.6 below, the obvious maps

$$\mathrm{Auteq}(t) \to \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\mathrm{ict}}) \qquad \text{and} \qquad \mathrm{Auteq}(t|_{\mathrm{Reg}\text{-}\Sigma/S}) \to \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\mathrm{ict}}|_{\mathrm{Reg}\text{-}\Sigma/S})$$

are equivalences. Thus, it remains to see that the the obvious maps

$$\mathrm{Auteq}(t) \to \mathrm{Auteq}(\mathcal{H}^{\otimes} \circ \beta) \qquad \text{and} \qquad \mathrm{Auteq}(t|_{\mathrm{Reg}\text{-}\Sigma/S}) \to \mathrm{Auteq}(\mathcal{H}^{\otimes} \circ \beta|_{\mathrm{Reg}\text{-}\Sigma/S})$$

are also equivalences. We will only deal with the first map; the case of the second map is entirely similar. We first note that the map $\mathrm{Auteq}(t) \to \mathrm{Auteq}(\mathcal{H}^{\otimes} \circ \beta)$ is an epimorphism. Indeed, let θ be an autoequivalence of $\mathcal{H}^{\otimes} \circ \beta$. For $X \in \mathrm{Sch}\text{-}\Sigma/S$, the endofunctor θ_X of $\mathcal{H}(X)$ preserves the full sub-∞-category $\mathcal{H}_{\mathrm{ict}}(X)$ since its compact generators are characterised by the property of being dualizable over every stratum of X. (Here we use that $\theta_X(-)|_C \simeq \theta_C(-|_C)$ for a stratum C of X.) Thus, θ extends to an autoequivalence of the inclusion $t : \mathcal{H}^{\otimes}_{\mathrm{ict}} \hookrightarrow \mathcal{H}^{\otimes} \circ \beta$ as needed. It remains to show that the map $\mathrm{Auteq}(t) \to \mathrm{Auteq}(\mathcal{H}^{\otimes} \circ \beta)$ is a monomorphism. We have a Cartesian square

$$\begin{tikzcd} \mathrm{Map}(t,t) \arrow[r] \arrow[d] & \mathrm{Map}(\mathcal{H}^{\otimes} \circ \beta, \mathcal{H}^{\otimes} \circ \beta) \arrow[d] \\ \mathrm{Map}(\mathcal{H}^{\otimes}_{\mathrm{ict}}, \mathcal{H}^{\otimes}_{\mathrm{ict}}) \arrow[r] & \mathrm{Map}(\mathcal{H}^{\otimes}_{\mathrm{ict}}, \mathcal{H}^{\otimes} \circ \beta). \end{tikzcd}$$

Since the bottom horizontal map is a monomorphism, the result follows. $\square$

Lemma 4.6. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subset \mathcal{C}$ be a reflexive sub-∞-category (in the sense of* [Lur09a, Remark 5.2.7.9]*). Let $\mathcal{D} \subset \mathcal{C}^{\Delta^1}$ be the full sub-∞-category spanned by the edges $C_0 \to C_1$ in $\mathcal{C}$ exhibiting C_1 as the localisation of C_0 relative de $\mathcal{C}'$. Then, evaluating at* 0 *yields an equivalence of ∞-categories $\mathcal{D} \to \mathcal{C}$.*

Proof. The functor is essentially surjective since every object C_0 admits a localisation relative to $\mathcal{C}'$. So, it remains to show that the functor is fully faithful. Given two objects $c : C_0 \to C_1$ and $d : D_0 \to D_1$ in $\mathcal{D}$, we have a Cartesian square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{D}}(c,d) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(C_1, D_1) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}(C_0, D_0) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(C_0, D_1). \end{tikzcd}$$

Since D_1 belongs to $\mathcal{C}'$, and C_1 is the localisation of C_0 relatively to $\mathcal{C}'$, the right vertical map of this square is an equivalence. Thus, its left vertical map is also an equivalence as needed. □

Corollary 4.7. *We have a commutative square of equivalences of derived group $\mathbb{S}$-prestacks*

$$\begin{tikzcd} \underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo,\,ict}}) \arrow[r, "\sim"] \arrow[d, "\sim"] & \underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo,\,ict}}|_{\mathrm{Sm}\text{-}\Sigma/k}) \arrow[d, "\sim"] \\ \underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo}}) \arrow[r, "\sim"] & \underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo}}|_{\mathrm{Sm}/k}). \end{tikzcd} \tag{4.6}$$

In particular, we have an equivalence of derived group $\mathbb{S}$-prestacks

$$\mathcal{G}_{\mathrm{mot}}(k,\sigma) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo,\,ict}}|_{\mathrm{Sm}\text{-}\Sigma/k}). \tag{4.7}$$

Proof. This follows from Proposition 4.5. For the last assertion, we use Theorem 2.10. □

4.2. **An application of the exit-path theorem.**

We fix a field k endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$. In this subsection, we use our exit-path theorem, i.e., Corollary 3.119, to prove a version of Theorem 2.10 where the ∞-categories of sheaves are replaced with ∞-categories of local systems.

Remark 4.8. For a commutative connective ring spectrum Λ, the functor $\mathbf{Sh}^{\Psi}_{\mathrm{geo}}(-;\Lambda)^{\otimes}$, obtained by applying Theorem 3.112 with $\mathcal{H}(-)^{\otimes} = \mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}$, takes a demarcated smoothly stratified k-variety (X, C_-, C_+) to the symmetric monoidal ∞-category $\widehat{\mathbf{LS}}_{\mathrm{geo}}(\mathrm{N}^{\circ}_{\overline{C}_-}(C_+);\Lambda)^{\otimes}$. Indeed, by Theorem 3.71, every local system of geometric origin on $\mathrm{N}^{\circ}_{\overline{C}_-}(C_+)$ is tame on the boundary of $\mathrm{N}_{\overline{C}_-}(C_+)$. For this reason, the functor $\mathbf{Sh}^{\Psi}_{\mathrm{geo}}(-;\Lambda)^{\otimes}$ will be denoted instead by

$$\widehat{\mathbf{LS}}^{\Psi}_{\mathrm{geo}}(-;\Lambda)^{\otimes} : (\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}). \tag{4.8}$$

Taking the sub-∞-categories of dualizable objects, we also obtain a functor

$$\mathbf{LS}^{\Psi}_{\mathrm{geo}}(-;\Lambda)^{\otimes} : (\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{CAT}_{\infty}). \tag{4.9}$$

In fact, the functor $\widehat{\mathbf{LS}}^{\Psi}_{\mathrm{geo}}(-;\Lambda)^{\otimes}$ can be obtained from the functor $\mathbf{LS}^{\Psi}_{\mathrm{geo}}(-;\Lambda)^{\otimes}$ by indization. In particular, an autoequivalence of the former induces an autoequivalence of the latter and vice versa.

Definition 4.9. We define the spectral group $\mathbb{S}$-prestack $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\Psi,\otimes}_{\mathrm{geo}})$ as in Definition 2.9. Namely, we apply Construction 1.51 to

- the functor $\mathcal{C} : (\mathrm{SpAFF}^{\mathrm{nc}})^{\mathrm{op}} \to \mathrm{CAT}_{\infty}$ sending $\mathrm{Spec}(\Lambda)$ to the ∞-category

  $$\mathrm{Psh}(\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k; \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/})$$

  of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/}$-valued presheaves on $\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k$, and
- the natural transformation $\mathrm{pt} \to \mathcal{C}$ sending $\mathrm{Spec}(\Lambda)$ to the functor

  $$\widehat{\mathbf{LS}}^{\Psi}_{\mathrm{geo}}(-;\Lambda)^{\otimes} : (\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/}.$$

If we want to stress that $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\Psi,\otimes}_{\mathrm{geo}})$ depends on σ, we will write $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\Psi,\otimes}_{\sigma\text{-geo}})$.

In this subsection, we will prove the following version of Theorem 2.10.

Theorem 4.10. *Let k be a field and $\sigma : k \hookrightarrow \mathbb{C}$ a complex embedding. There is an equivalence of spectral group $\mathbb{S}$-prestacks*

$$\mathcal{G}_{\mathrm{mot}}(k,\sigma) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathbf{LS}^{\Psi,\otimes}_{\sigma\text{-geo}}). \tag{4.10}$$

In particular, the right hand side is a spectral affine group scheme.

By Corollary 4.7, we only need to show that there is an equivalence

$$\underline{\mathrm{Auteq}}(\mathbf{LS}^{\Psi,\otimes}_{\mathrm{geo}}) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo,\,ict}}). \tag{4.11}$$

This relies on the exit-path theorem, i.e., Corollary 3.119. For later use, we will prove this in a greater generality; see Theorem 4.15 below. We recall the Kummer étale topology.

Definition 4.11. Let $f : Y \to X$ be a morphism in Reg-Σ/S. We say that f is Kummer étale if it satisfies the following conditions.

(i) Locally for the étale topology on X and Y, the morphism f is of the form

$$\mathrm{Spec}(\mathcal{O}_X[t_1,\dots,t_m]/(t_1^{e_1}-u_1,\dots,t_1^{e_m}-u_m)) \to X$$

where $t_1,\dots,t_m$ are indeterminates, $e_1,\dots,e_m$ are positive integers invertible on X, and $u_1,\dots,u_m \in \mathcal{O}(X)$ define distinct reduced irreducible constructible divisors of X.

(ii) The stratification of Y is the pullback of the stratification of X.

The Kummer étale topology on Reg-Σ/S is the topology generated by the jointly surjective families of Kummer étale morphisms. We also define a Kummer étale topology on Reg-Σ^{dm}/S as follows: it is the one generated by the families $(e_i : (X_i, C_{i,-}, C_{i,+}) \to (X, C_-, C_+))_i$ such that the e_i's are Kummer étale and the induced family $(\overline{C}_{i,+}, \overline{C}_+)_i$ is jointly surjective (and hence a Kummer étale cover). Similarly, we have a Kummer étale topology on Reg-Σ^{dm}_+/S. We denote all these topologies by két.

Proposition 4.12. *Let Λ be a commutative ring spectrum. The functor $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)^{\otimes}$, restricted to* Sm-$\Sigma/S$*, satisfies Kummer étale hyperdescent.*

Proof. Given a Kummer étale hypercover $X_\bullet \to X_{-1}$ in Sm-Σ/k, we need to show that

$$\mathbf{Sh}_{\mathrm{geo}}(X_{-1};\Lambda) \to \lim_{[n]\in\mathbf{\Delta}} \mathbf{Sh}_{\mathrm{geo}}(X_n;\Lambda) \tag{4.12}$$

is an equivalence. We argue by induction on the number of strata in X_{-1}. We fix a closed stratum $Z_{-1} \subset X_{-1}$ and set $U_{-1} = X_{-1} \smallsetminus Z_{-1}$. We also set $U_\bullet = X_\bullet \times_{X_{-1}} U_{-1}$ and $Z_\bullet = (X_\bullet \times_{X_{-1}} Z_{-1})_{\mathrm{red}}$, and denote by $j_\bullet : U_\bullet \to X_\bullet$ and $i_\bullet : Z_\bullet \to X_\bullet$ the obvious inclusions. Then $U_\bullet \to U_{-1}$ is a Kummer étale hypercover and $Z_\bullet \to Z_{-1}$ is an étale hypercover. Every object $(F_n)_n$ in the codomain of the functor in (4.12) is part of a cofiber sequence

$$(j_{n,!}j_n^! F_n)_n \to (F_n)_n \to (i_{n,*}i_n^* F_n)_n.$$

Using the induction hypothesis and étale hyperdescent, we deduce that $(j_{n,!}j_n^! F_n)_n$ and $(i_{n,*}i_n^* F_n)_n$ belong to the essential image of the functor in (4.12). (Note that étale hyperdescent for $\mathbf{Sh}_{\mathrm{geo}}(-;\Lambda)$ can be deduced from Theorem 1.93 and the corresponding property for $\mathbf{MSh}(X;\Lambda)$; see for example [AGV20, Proposition 3.2.1].) Thus, it is enough to show that the functor in (4.12) is fully faithful. Said differently, given $F \in \mathbf{Sh}_{\mathrm{geo}}(X_{-1};\Lambda)$, we need to show that

$$F \to \lim_{[n]\in\mathbf{\Delta}} e_{n,*}e_n^* F \tag{4.13}$$

is an equivalence. To do so, we may assume that $\Lambda = \mathbb{S}$. We claim that F is Postnikov complete in the sense of [CM19, Definition 2.4]. To prove this, we may assume that F is uniquely divisible, i.e., is a sheaf of $\mathbb{Q}$-vector spaces, or that F is ℓ-torsion for some prime ℓ. In the first case, Postnikov completeness was discussed in the proof of Lemma 1.109. In the second case, we use the equivalences of ∞-categories

$$\mathbf{Sh}_{\mathrm{geo}}(-)_{\ell\text{-nil}} \simeq \mathrm{Shv}_{\acute{\mathrm{e}}\mathrm{t}}(\acute{\mathrm{E}}\mathrm{t}/(- \times_k \mathbb{C}))_{\ell\text{-nil}}$$

to reduce to the Postnikov completeness of étale sheaves on small étale sites of varieties of bounded cohomological dimension. (See for example [AGV20, Lemma 2.4.5].) This said, we can write $F = \lim_{m\in\mathbb{N}} \tau_{\leq m}F$. Since e_n^* is exact, we also have $e_n^*F = \lim_{m\in\mathbb{N}} e_n^*\tau_{\leq m}F$. Thus, we are reduced to showing that the morphism in (4.13) is an equivalence when F is truncated. In this case, we may assume that $X_\bullet \to X_{-1}$ is a Čech nerve. Using étale hyperdescent, we reduce to the case where $X_\bullet \to X_{-1}$ is the Čech nerve of a finite Kummer étale cover $X_0 \to X_{-1}$. In this case, all the e_n's are finite. Given a stratum C of X_{-1}, we denote by $\iota_C : C \to X_{-1}$ its inclusion. It is enough to show that the morphism in (4.13) is an equivalence after applying ι_C^*, for all the strata $C \subset X_{-1}$. Using that F is truncated and that the e_n's are finite, we have equivalences

$$\iota_C^* \lim_{[n]\in\mathbf{\Delta}} e_{n,*}\, e_n^*\, F \simeq \lim_{[n]\in\mathbf{\Delta}} \iota_C^*\, e_{n,*}\, e_n^*\, F \simeq \lim_{[n]\in\mathbf{\Delta}} e_{C,n,*}\, e_{C,n}^*\, \iota_C^*\, F,$$

where $e_{C,\bullet} : (X_\bullet \times_X C)_{\mathrm{red}} \to C$ is the obvious morphism. Thus, we are reduced to showing that

$$\iota_C^*\, F \to \lim_{[n]\in\mathbf{\Delta}} e_{C,n,*}\, e_{C,n}^*\, \iota_C^*\, F$$

is an equivalence, which is true by étale descent since $e_{C,\bullet}$ is the Čech nerve of a finite étale cover of C. □

In the reminder of this subsection, we fix an excellent scheme S of characteristic 0 and a presentable Voevodsky pullback formalism $\mathcal{H}^{\otimes} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

Proposition 4.13. *Assume that $\mathcal{H}^{\otimes}$ satisfies Kummer étale hyperdescent. Then, the lax 2-functor $\mathcal{H}_+^{\Psi,\otimes}$ admits Kummer étale hyperdescent, and the morphism $\mathcal{H}_+^{\Upsilon,\otimes} \to \mathcal{H}_+^{\Psi,\otimes}$ exhibits $\mathcal{H}_+^{\Psi,\otimes}$ as the Kummer étale hypersheafification of $\mathcal{H}_+^{\Upsilon,\otimes}$ computed in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

Proof. We may prove the statement after restriction to $\mathrm{Reg}\text{-}\Sigma/S$ and $\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S$, in which cases the lax 2-functors become strict functors. We split the proof in three small steps.

Step 1. Let $e_\bullet : X_\bullet \to X_{-1}$ be a hypercover in $\mathrm{Reg}\text{-}\Sigma/S$. By assumption, we have an equivalence of ∞-categories

$$\mathcal{H}(X_{-1}) \xrightarrow{\sim} \lim_{[n]\in\mathbf{\Delta}} \mathcal{H}(X_n).$$

Since $\mathcal{H}_{\mathrm{ict\text{-}tm}}(X_n) \subset \mathcal{H}(X_n)$ is a full sub-∞-category, for every $n \geq -1$, we see that it is enough to prove the following property: an object $M \in \mathcal{H}(X_{-1})$ belongs to $\mathcal{H}_{\mathrm{ict\text{-}tm}}(X_{-1})$ if and only if $M|_{X_0}$ belongs to $\mathcal{H}_{\mathrm{ict\text{-}tm}}(X_0)$. It is enough to prove this for the objects $\iota_{C,!}\iota_C^*M$, where $\iota_C : C \to X$ is the inclusion of a stratum of X. In this case, it is enough to show that $M|_C$ belongs to $\mathcal{H}_{\mathrm{itame}}(C/\overline{C})$ assuming that $M|_D$ belongs to $\mathcal{H}_{\mathrm{itame}}(D/\overline{D})$, for every stratum $D \subset X_0$ over C. Consider the obvious morphism $e_{C,\bullet} : (X_\bullet \times_{X_{-1}} X_\bullet)_{\mathrm{red}} \to C$. This is an étale hypercover. By étale hyperdescent, we deduce an equivalence

$$M|_C \simeq \operatorname*{colim}_{[n]\in\mathbf{\Delta}} e_{C,n,\sharp}\, e_{C,n}^*\,(M|_C).$$

For $n \geq 0$, $e_{C,n}^*(M|_C)$ is a pullback of $e_{C,0}^*(M|_C)$, and hence is ind-tame at the boundary. This implies tha $M|_C$ is ind-tame as needed.

Step 2. Now, let $e_\bullet : (X_\bullet, C_{\bullet,-}, C_{\bullet,+}) \to (X_{-1}, C_{-1,-}, C_{-1,+})$ be a Kummer étale hypercover in $\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S$. We need to show that we have an equivalence of ∞-categories

$$\mathcal{H}^{\Psi}(X_{-1}, C_{-1,-}, C_{-1,+}) \xrightarrow{\sim} \lim_{[n]\in\mathbf{\Delta}} \mathcal{H}^{\Psi}(X_n, C_{n,-}, C_{n,+}).$$

This follows from the first step since $\mathrm{N}^{\circ}_{\overline{C}_{\bullet,-}}(C_{\bullet,+}) \to \mathrm{N}^{\circ}_{\overline{C}_{-1,-}}(C_{-1,+})$ is a Kummer étale hypercover and, for every $n \geq -1$, $\mathcal{H}^{\Psi}(X_n, C_{n,-}, C_{n,+})$ is equivalent to $\mathcal{H}_{\mathrm{itame}}(\mathrm{N}^{\circ}_{\overline{C}_{n,-}}(C_{n,+}))$ and hence, to the full sub-∞-category of $\mathcal{H}_{\text{ict-tm}}(\mathrm{N}_{\overline{C}_{n,-}}(C_{n,+}))$ spanned by those objects whose restriction to the boundary is zero.

Step 3. It remains to prove the assertion about the hypersheafification of $\mathcal{H}^{\Upsilon}_{+}$. We only treat the case of $\mathcal{H}_{\text{ict-log}}$; the case of $\mathcal{H}^{\Upsilon}$ follows then easily using the reasoning in the second step. We fix $X \in \mathrm{Reg}\text{-}\Sigma/S$. For an integer $d \geq 1$, we denote by $H_d \subset \mathrm{y}(X)$ the sieve consisting of those Kummer étale morphisms $X' \to X$ whose ramification index divides d on each constructible irreducible divisor of X. Clearly, H_d is a covering sieve for the Kummer étale topology. Let $\mathcal{H}_{\text{ict-log}}(H_d)$ be the value at H_d of the right Kan extension of $\mathcal{H}_{\text{ict-log}}(-)$ along the Yoneda embedding of $\mathrm{Reg}\text{-}\Sigma/S$. Arguing as in the first step, we see that $\mathcal{H}_{\text{ict-log}}(H_d)$ is the full sub-∞-category of $\mathcal{H}(X)$ generated under colimit by the constructible sheaves which are logarithmic after locally extracting d-th roots of the equations defining the irreducible constructible divisors. In particular, $\mathcal{H}_{\text{ict-log}}(H_d)$ is compactly generated and we have an equivalence in Pr^{L}

$$\operatorname*{colim}_{d\in\mathbb{N}^{\times}} \mathcal{H}_{\text{ict-log}}(H_d) \simeq \mathcal{H}_{\text{ict-tm}}(X).$$

This finishes the proof. □

Corollary 4.14. *Assume that $\mathcal{H}^{\otimes}$ satisfies Kummer étale hyperdescent. Then, we have natural equivalences of group objects in $\mathcal{S}$:*

$$\mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}}) \xrightarrow{\sim} \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-tm}}) \qquad \textit{and} \qquad \mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes}) \xrightarrow{\sim} \mathrm{Auteq}(\mathcal{H}^{\Psi,\otimes}). \tag{4.14}$$

Proof. The strategy is similar to the one used in the proof of Proposition 4.5. We consider the natural transformations $t : \mathcal{H}^{\otimes}_{\text{ict-log}} \to \mathcal{H}^{\otimes}_{\text{ict-tm}}$ and $t' : \mathcal{H}^{\Upsilon,\otimes} \to \mathcal{H}^{\Psi,\otimes}$. By Proposition 4.13 and Lemma 4.6 we have equivalences of group objects in $\mathcal{S}$:

$$\mathrm{Auteq}(t) \xrightarrow{\sim} \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}}) \qquad \text{and} \qquad \mathrm{Auteq}(t') \xrightarrow{\sim} \mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes}).$$

Thus, it remains to show that the obvious maps

$$\mathrm{Auteq}(t) \to \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-tm}}) \qquad \text{and} \qquad \mathrm{Auteq}(t') \to \mathrm{Auteq}(\mathcal{H}^{\Psi,\otimes})$$

are equivalences. Using that the natural transformations t and t' are given by fully faithful embeddings, and arguing as in the proof of Proposition 4.5, we only need to show that these maps are epimorphisms. Said differently, given autoequivalences $(\theta_X)_{X\in\mathrm{Reg}\text{-}\Sigma/S}$ and $(\theta'_{X,C_-,C_+})_{(X,C_-,C_+)\in\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S}$ of $\mathcal{H}^{\otimes}_{\text{ict-tm}}$ and $\mathcal{H}^{\Psi,\otimes}$, we need to show that they preserve the sub-functors $\mathcal{H}^{\otimes}_{\text{ict-log}}$ and $\mathcal{H}^{\Upsilon,\otimes}$.

We first consider the case of $(\theta_X)_{X\in\mathrm{Reg}\text{-}\Sigma/S}$. It is enough to show, that every stratum $C \subset X$, the autoequivalence θ_C preserves $\mathcal{H}_{\log}(C/X)$. Note that θ_C preserves $\mathcal{H}_{\mathrm{tame}}(C/X)$ which is the essential image of $\mathcal{H}_{\mathrm{tame}}(X)$ by the inverse image functor; in the next few lines, we will denote by θ_C the induced equivalence on $\mathcal{H}_{\mathrm{tame}}(C/X)$. Let $j : C \hookrightarrow \overline{C}$ be the obvious inclusion. Since $\theta_C \circ j^* \simeq j^* \circ \theta_{\overline{C}}$, we deduce that $\theta_{\overline{C}} \circ j_* \simeq j_* \circ \theta_C$. Now, for $F \in \mathcal{H}_{\log}(C/X)$, the $j_*\mathbf{1}$-module j_*F is dualizable. Since $\theta_{\overline{C}}$ is symmetric monoidal, we deduce that $\theta_{\overline{C}} j_* F \simeq j_* \theta_C F$ is dualizable over $\theta_{\overline{C}} j_* \mathbf{1} \simeq j_*\mathbf{1}$. This show that $\theta_C F$ belongs to $\mathcal{H}_{\log}(C/X)$ as needed.

We now consider the case of $(\theta'_{X,C_-,C_+})_{(X,C_-,C_+)\in\text{Reg-}\Sigma^{\text{dm}}/S}$. Fix $M \in \mathcal{H}_{\log}(\mathrm{N}^{\circ}_{\overline{C}_-}(C_+))$. We need to show that $\theta'_{X,C_-,C_+}(M)$ is also logarithmic at the boundary. We first note that M is unipotent with respect to the projection $p : \mathrm{N}^{\circ}_{\overline{C}_-}(C_+) \to C_+$. Thus, we may assume that $M = p^*N$ with $N \in \mathcal{H}_{\log}(C_+/\overline{C}_+)$. Since $\theta'_{X,C_-,C_+} \circ p^* \simeq p^* \circ \theta'_{X,C_+,C_+}$, we may replace M with N and assume that $C_- = C_+$. We may even replace X with $\overline{C}_+$, and assume that X is connected and $C_+ = X^{\circ}$. Said differently, we are reduced to showing that $\theta'_{X,X^{\circ},X^{\circ}}$ preserves the sub-∞-category $\mathcal{H}_{\log}(X^{\circ}/X)$ of $\mathcal{H}_{\text{tame}}(X^{\circ}/X)$. By Proposition 3.47, this sub-∞-category can be characterised as the largest one satisfying the following properties (see Notations 3.40 and 3.43):

- this sub-∞-category is stable by the tensor product;
- the restriction of the lax monoidal functor $\widetilde{\chi}_C : \mathcal{H}_{\text{tame}}(X^{\circ}/X) \to \mathrm{Mod}_{\chi_C\mathbf{1}}(\mathcal{H}_{\text{tame}}(C/\overline{C}))$ to this sub-∞-category is monoidal for every stratum $C \subset X$.

Thus, to conclude, it is enough to show that there is an equivalence $\chi_C \circ \theta'_{X,X^{\circ},X^{\circ}} \simeq \theta'_{\overline{C},C,C} \circ \chi_C$. This follows from the fact that $\chi_C \simeq q_* \circ \widetilde{\Psi}_C$ where $q : \mathrm{N}^{\circ}_X(C) \to C$ is the obvious projection. □

We can now state the main result of this subsection.

Theorem 4.15. *Assume that $\mathcal{H}^{\otimes}$ satisfies Kummer étale hyperdescent. Then, there is a commutative square of equivalences of group objects in $\mathcal{S}$:*

$$\begin{tikzcd}
\mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}}) \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes}) \arrow[d, "\sim"] \\
\mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-tm}}) \arrow[r, "\sim"] & \mathrm{Auteq}(\mathcal{H}^{\Psi,\otimes}).
\end{tikzcd} \tag{4.15}$$

The vertical equivalences of the square in (4.15) are the ones provided by Corollary 4.14. We now construct the horizontal maps of this square.

Construction 4.16. We call a morphism $f : (Y, D_-, D_+) \to (X, C_-, C_+)$ in $\text{Reg-}\Sigma^{\text{dm}}/S$ inert when $f : Y \to X$ is an isomorphism of stratified S-schemes and $f_*(D_+) = C_+$. We denote by

$$\mathrm{CAT}^{\text{st-inr-cart}}_{\infty/(\text{Reg-}\Sigma^{\text{dm}}/S)} \subset \mathrm{CAT}^{\text{cart}}_{\infty/(\text{Reg-}\Sigma^{\text{dm}}/S)} \tag{4.16}$$

the wide sub-∞-category whose morphisms are those functors respecting Cartesian edges over inert morphisms. Let $p : \text{Reg-}\Sigma^{\text{dm}}/S \to \text{Reg-}\Sigma/S$ be the forgetful functor as in Construction 3.117. The base change functor $p^* : \mathrm{CAT}^{\text{cart}}_{\infty/(\text{Reg-}\Sigma/S)} \to \mathrm{CAT}^{\text{st-inr-cart}}_{\infty/(\text{Reg-}\Sigma^{\text{dm}}/S)}$ admits a right adjoint

$$p^{\text{inr}}_* : \mathrm{CAT}^{\text{st-inr-cart}}_{\infty/(\text{Reg-}\Sigma^{\text{dm}}/S)} \to \mathrm{CAT}^{\text{cart}}_{\infty/(\text{Reg-}\Sigma/S)} \tag{4.17}$$

sending a Cartesian fibration $\mathcal{M} \to \text{Reg-}\Sigma^{\text{dm}}/S$ to the full sub-∞-category of $p_*\mathcal{M}$ spanned by those sections respecting Cartesian edges over inert morphisms. By Corollary 3.119, the functor p^{inr}_* takes the Cartesian fibration classified by $\mathcal{H}^{\Psi,\otimes}$ to the Cartesian fibration classified by $\mathcal{H}^{\otimes}_{\text{ict-tm}}$. The same applies with "Υ" and "ict-log" instead of "Ψ" and "ict-tm". Thus, the functor p^{inr}_* induces a map of group objects in $\mathcal{S}$:

$$\mathrm{Auteq}(\mathcal{H}^{\Psi,\otimes}) \to \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-tm}}) \qquad \text{and} \qquad \mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes}) \to \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}}). \tag{4.18}$$

In fact, the group object $\mathrm{Auteq}(\mathcal{H}^{\Psi,\otimes})$ acts, not only on $\mathcal{H}^{\otimes}_{\text{ict-tm}}$, but also on the counit morphism $p^*\mathcal{H}^{\otimes}_{\text{ict-tm}} \to \mathcal{H}^{\Psi,\otimes}$, which gives an action on the lax 2-functor $\mathcal{H}^{\Psi,\otimes}_+$. The same applies with "Υ"

and “ict-log” instead of “Ψ” and “ict-tm”. Thus, we obtain commutative triangles of maps of group objects in $\mathcal{S}$:

$$
\begin{tikzcd}
& \mathrm{Auteq}(\mathcal{H}^{\Psi,\otimes}) \\
\mathrm{Auteq}(\mathcal{H}^{\Psi,\otimes}) \arrow[r] \arrow[ur, equal] \arrow[dr] & \mathrm{Auteq}(\mathcal{H}_{+}^{\Psi,\otimes}) \arrow[u] \arrow[d] \\
& \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-tm}})
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& \mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes}) \\
\mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes}) \arrow[r] \arrow[ur, equal] \arrow[dr] & \mathrm{Auteq}(\mathcal{H}_{+}^{\Upsilon,\otimes}) \arrow[u] \arrow[d] \\
& \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}})
\end{tikzcd}
\tag{4.19}
$$

where the vertical arrows are given by restriction along the subcategories $\mathrm{Reg}\text{-}\Sigma/S$ and $\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S$ of $\mathrm{Reg}\text{-}\Sigma_{+}^{\mathrm{dm}}/S$.

By Corollary 4.14, Theorem 4.15 follows if we can prove that the map of group objects

$$\mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes}) \to \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}})$$

is an equivalence. This will be obtained as the conjunction of Propositions 4.17 and 4.18 below.

Proposition 4.17. *There is an equivalence of group objects in $\mathcal{S}$:*

$$\mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes}) \xrightarrow{\sim} \mathrm{Auteq}(\mathcal{H}_{+}^{\Upsilon,\otimes}). \tag{4.20}$$

Proof. As explained in Construction 4.16, the obvious map $\mathrm{Auteq}(\mathcal{H}_{+}^{\Upsilon,\otimes}) \to \mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes})$ admits a section, and hence is an epimorphism. Thus, it is enough to show that the kernel

$$\ker(\mathrm{Auteq}(\mathcal{H}_{+}^{\Upsilon,\otimes}) \to \mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes})) \tag{4.21}$$

is contractible. Let $\mathcal{C} = \mathrm{CAT}^{\mathrm{cart}}_{\infty/(\mathrm{Reg}\text{-}\Sigma/S)}$ and $\mathcal{D} = \mathrm{CAT}^{\text{st-inr-cart}}_{\infty/(\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S)}$. The kernel in (4.21) can be identified with the group of autoequivalences of the object

$$\left(\oint_{\mathrm{Reg}\text{-}\Sigma/S} \mathcal{H}^{\otimes}_{\text{ict-log}},\, p^{*}\left(\oint_{\mathrm{Reg}\text{-}\Sigma/S} \mathcal{H}^{\otimes}_{\text{ict-log}}\right) \xrightarrow{\theta} \oint_{\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S} \mathcal{H}^{\Upsilon,\otimes}\right) \tag{4.22}$$

in the ∞-category $\mathcal{C} \times_{p^{*},\mathcal{D}} \mathcal{D}_{/\mathcal{H}^{\Upsilon,\otimes}}$; see Construction 3.117. Since the functor θ induces an equivalence

$$\theta' : \oint_{\mathrm{Reg}\text{-}\Sigma/S} \mathcal{H}^{\otimes}_{\text{ict-log}} \xrightarrow{\sim} p^{\mathrm{inr}}_{*}\left(\oint_{\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S} \mathcal{H}^{\Upsilon,\otimes}\right),$$

the object in (4.22) is a final object of $\mathcal{C} \times_{p^{*},\mathcal{D}} \mathcal{D}_{/\mathcal{H}^{\Upsilon,\otimes}}$. In particular, it has a contractible space of autoequivalences as needed. □

Proposition 4.18. *There is an equivalence of group objects in $\mathcal{S}$:*

$$\mathrm{Auteq}(\mathcal{H}_{+}^{\Upsilon,\otimes}) \xrightarrow{\sim} \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}}). \tag{4.23}$$

Proof. We split the proof in several steps.

Step 1. We will construct a section

$$\mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}}) \to \mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes}_{+}) \tag{4.24}$$

to the obvious map. For this, we need to go through the construction of $\mathcal{H}^{\Upsilon,\otimes}_{+}$, checking that an autoequivalence of $\mathcal{H}^{\otimes}_{\text{ict-log}}$ gives rise in a coherent way to an autoequivalence of $\mathcal{H}^{\Upsilon,\otimes}_{+}$. It will be convenient to denote by $\mathcal{K}$ the classifying space of the group object $\mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}})$. We view $\mathcal{K}$ as an ∞-groupoid, so that we may extend $\mathcal{H}^{\otimes}_{\text{ict-log}}$ into a functor

$$\mathcal{H}^{\otimes}_{\mathcal{K}} : \mathcal{K} \times (\text{Reg-}\Sigma/S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}). \tag{4.25}$$

We apply Construction 3.97 with $\mathcal{H}^{\otimes}_{\mathcal{K}}$ instead of $\mathcal{H}^{\otimes}$. This gives rise to a lax 2-functor

$$\mathcal{H}_{\mathcal{K}}(-;-)^{\otimes} : \mathbb{D}^{1\text{-op},\,2\text{-op}}_{\mathcal{H}_{\mathcal{K}}} \rightsquigarrow \underline{\mathrm{SMCAT}}_{\infty}. \tag{4.26}$$

This can be informally described as in Remark 3.98. We only mention that the objects of $\mathbb{D}_{\mathcal{H}_{\mathcal{K}}}$ are the objects of the ∞-category

$$\mathcal{C}_{\mathcal{H}_{\mathcal{K}}} = \int_{\mathcal{K}\times(\text{Reg-}\Sigma/S)^{\mathrm{op}}} \mathrm{CAlg}(\mathcal{H}_{\mathcal{K}}). \tag{4.27}$$

Given a split torus-embedding T, the logarithmic sheaf $\mathcal{L}_T \in \mathcal{H}(T)$ belongs to $\mathcal{H}_{\text{ict-log}}(T)$ and is naturally fixed by the action of $\mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}})$. This can be easily seen by inspecting Construction 3.24. More is true: the section $\mathcal{L}$ obtained in Construction 3.106 can be extended to a section

$$\mathcal{L} : \mathcal{K} \times \mathrm{TEmb}^{\mathrm{op}} \to \int_{\mathcal{K}\times\mathrm{TEmb}^{\mathrm{op}}} \mathrm{CAlg}(\mathcal{H}_{\mathcal{K}}). \tag{4.28}$$

Adapting Construction 3.108, we thus obtain an oplax 2-functor

$$(\underline{\mathrm{Df}}, \mathcal{L}) : \mathcal{K}^{\mathrm{op}} \times \text{Reg-}\Sigma^{\mathrm{dm}}_{+}/S \to \mathbb{D}_{\mathcal{H}_{\mathcal{K}}}. \tag{4.29}$$

Composing with the lax 2-functor in (4.26), we obtain a lax 2-functor

$$\mathcal{H}_{\mathcal{K}}(\underline{\mathrm{Df}}(-);\mathcal{L})^{\otimes} : \mathcal{K} \times (\text{Reg-}\Sigma^{\mathrm{dm}}_{+}/S)^{\mathrm{op}} \rightsquigarrow \underline{\mathrm{SMCAT}}_{\infty}. \tag{4.30}$$

Finally, passing to a sub-2-functor as in Definition 3.111, yields a lax 2-functor

$$\mathcal{H}^{\Upsilon,\otimes}_{\mathcal{K},+} : \mathcal{K} \times (\text{Reg-}\Sigma^{\mathrm{dm}}_{+}/S)^{\mathrm{op}} \rightsquigarrow \underline{\mathrm{SMCAT}}_{\infty} \tag{4.31}$$

whose restriction to $\text{Reg-}\Sigma^{\mathrm{dm}}_{+}/S$ is $\mathcal{H}^{\Upsilon,\otimes}_{+}$. This gives an action of $\mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}})$ on $\mathcal{H}^{\Upsilon,\otimes}_{+}$ extending the tautological action on $\mathcal{H}^{\otimes}_{\text{ict-log}}$ as needed.

Step 2. Now that we know that the obvious map $\rho : \mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes}_{+}) \to \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}})$ has a section, it is enough to show that its kernel $\ker(\rho)$ is contractible. With the notations as in Construction 3.117, we have an equivalence

$$\mathrm{Auteq}(\mathcal{H}^{\Upsilon,\otimes}_{+}) \simeq \mathrm{Auteq}(\theta) \times_{\mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}}\circ p)} \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}}). \tag{4.32}$$

We claim that the map $\mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}}) \to \mathrm{Auteq}(\mathcal{H}^{\otimes}_{\text{ict-log}} \circ p)$ is an equivalence. Indeed, writing $t_{\amalg}$ for the topology generated by the jointly surjective families of clopen immersions, the functor

$$p^* : \mathrm{Shv}_{t_{\amalg}}(\text{Reg-}\Sigma/S;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})) \to \mathrm{Psh}(\text{Reg-}\Sigma^{\mathrm{dm}}/S;\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})) \tag{4.33}$$

is fully faithful. To prove this, we may show that the unit map $\mathrm{id} \to p_*p^*$ is an equivalence. Using [Lur09a, Proposition 4.3.3.10], this follows from the fact that p is a coCartesian fibration whose fiber at a connected $X \in \text{Reg-}\Sigma/S$ has contractible geometric realisation. This said, we deduce from the formula in (4.32) an equivalence $\mathrm{Auteq}(\mathcal{H}_{+}^{\Upsilon,\otimes}) \simeq \mathrm{Auteq}(\theta)$. Therefore, to show that $\ker(\rho)$ is contractible, we can as well show that $\ker(\rho')$ is contractible, where $\rho' : \mathrm{Auteq}(\theta) \to \mathrm{Auteq}(\mathcal{H}_{\text{ict-log}}^{\otimes} \circ p)$ is the obvious map.

Step 3. Consider the natural transformation θ in (3.96) as a functor from Δ^1 to $\mathrm{CAT}_{\infty/(\text{Reg-}\Sigma^{\text{dm}}/S)}$. Taking the associated coCartesian fibration, we get a diagram

$$\begin{tikzcd} \int_{\Delta^1} \theta \arrow[r, "q"] \arrow[d, "r"] & \text{Reg-}\Sigma^{\text{dm}}/S \\ \Delta^1 & \end{tikzcd}$$

where q is a Cartesian fibration and r is a coCartesian fibration. The Cartesian fibration q is classified by a functor admitting the following informal description.

- It takes an object (X, C_-, C_+) to the ∞-category
$$\int_{\Delta^1} \theta_{X,C_-,C_+}$$
whose objects are pairs (ϵ, M) where $\epsilon \in \{0, 1\}$, and M is an object of $\mathcal{H}_{\text{ict-log}}(X)$, if $\epsilon = 0$, and an object of $\mathcal{H}_{\text{ilog}}(C_+)$, if $\epsilon = 1$.
- It takes a morphism $f : (Y, D_-, D_+) \to (X, C_-, C_+)$ to the functor
$$f^* : \int_{\Delta^1} \theta_{X,C_-,C_+} \to \int_{\Delta^1} \theta_{Y,D_-,D_+}.$$
Over $0 \in \Delta^1$, this is just the pullback functor $f^* : \mathcal{H}_{\text{ict-log}}(X) \to \mathcal{H}_{\text{ict-log}}(Y)$. Over $1 \in \Delta^1$, this the composite functor
$$\mathcal{H}_{\text{ilog}}(C_+) \xrightarrow{\chi_{C'_+}} \mathcal{H}_{\text{ilog}}(C'_+) \to \mathcal{H}_{\text{ilog}}(D_+)$$
where $C'_+ = f_*(D_+)$. Finally, this functor takes a coCartesian edge $(M, \mathrm{id}_{M|_{C_+}})$, at $M \in \mathcal{H}_{\text{ict-log}}(X)$, to the edge $(f^*M, (f^*M)|_{D_+} \to \chi_{C'_+}(M|_{C_+})|_{D_+})$.

Taking the dual coCartesian fibration to q, we obtain a diagram

$$\begin{tikzcd} \mathcal{W} \arrow[r, "q'"] \arrow[d, "r'"] & (\text{Reg-}\Sigma^{\text{dm}}/S)^{\text{op}} \\ \Delta^1 & \end{tikzcd}$$

where q' is a coCartesian fibration and r' is a Cartesian fibration. (The last property follows from the fact that, for every (X, C_-, C_+), the projection $r_{X,C_-,C_+} : \int_{\Delta^1} \theta_{X,C_-,C_+} \to \Delta^1$ is a biCartesian fibration.) Taking the functor classifying the Cartesian fibration r', we obtain a commutative triangle

$$\begin{tikzcd} \mathcal{W}_1 = \int_{(\text{Reg-}\Sigma^{\text{dm}}/S)^{\text{op}}} \mathcal{H}^{\Upsilon,\otimes} \arrow[rr, "\xi"] \arrow[dr] & & \mathcal{W}_0 = \int_{(\text{Reg-}\Sigma^{\text{dm}}/S)^{\text{op}}} \mathcal{H}_{\text{ict-log}}^{\otimes} \circ p \arrow[dl] \\ & (\text{Reg-}\Sigma^{\text{dm}}/S)^{\text{op}} & \end{tikzcd} \tag{4.34}$$

where the slanted arrows are coCartesian fibrations. The fiber of ξ at (X, C_-, C_+) is the functor

$$q_* : \mathcal{H}^{\otimes}_{\mathrm{ilog}}(\mathrm{N}^{\circ}_{\overline{C}_-}(C_+)) \to \mathcal{H}^{\otimes}_{\text{ict-log}}(X)$$

induced by the obvious morphism $q : \mathrm{N}^{\circ}_{\overline{C}_-}(C_+) \to X$. Clearly, we have $\mathrm{Auteq}(\theta) \simeq \mathrm{Auteq}(\xi)$. Therefore, to show that $\ker(\rho')$ is contractible, we may as well show that $\ker(\rho'')$ is contractible, where $\rho'' : \mathrm{Auteq}(\xi) \to \mathrm{Auteq}(\mathcal{W}_0)$ is the obvious map.

Step 4. Let $\mathcal{W}'_0 \subset \mathcal{W}_0$ be the full sub-∞-category spanned by the objects $((X, C_-, C_+), M)$ with M in the essential image of the fully faithful embedding $(\iota_{C_+})_* : \mathcal{H}^{\otimes}_{\mathrm{ilog}}(C_+) \to \mathcal{H}^{\otimes}_{\text{ict-log}}(X)$. (Here, we denote by $\iota_{C_+} : C_+ \hookrightarrow X$ the obvious inclusion.) The following properties are easily checked.

- The functor $\mathcal{W}'_0 \to (\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}}$ is a coCartesian fibration classified by the functor

  $$\mathcal{H}^{\chi,\otimes} : (\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}} \to \mathrm{SMCAT}_{\infty}$$

  sending (X, C_-, C_+) to $\mathcal{H}^{\otimes}_{\mathrm{ilog}}(C_+)$ and $f : (Y, D_-, D_+) \to (X, C_-, C_+)$ to the composition of

  $$\mathcal{H}^{\otimes}_{\mathrm{ilog}}(C_+) \xrightarrow{\chi_{C'_+}} \mathcal{H}^{\otimes}_{\mathrm{ilog}}(C'_+) \to \mathcal{H}^{\otimes}_{\mathrm{ilog}}(D_+).$$

  (As usual, we set $C'_+ = f_*(D_+)$.)
- The functor ξ factors through $\mathcal{W}'_0$ inducing a commutative triangle

$$\begin{tikzcd} \mathcal{W}_1 = \displaystyle\int_{(\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}}} \mathcal{H}^{\Upsilon,\otimes} \arrow[rr, "\xi'"] \arrow[dr] & & \mathcal{W}'_0 = \displaystyle\int_{(\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}}} \mathcal{H}^{\chi,\otimes} \arrow[dl] \\ & (\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}}. & \end{tikzcd} \tag{4.35}$$

  Moreover, this triangle is a morphism of coCartesian fibrations, i.e., ξ' preserves coCartesian edges.

Let $\rho''' : \mathrm{Auteq}(\xi') \to \mathrm{Auteq}(\mathcal{W}'_0)$ be the obvious morphism. Since $\mathcal{W}'_0 \to \mathcal{W}_0$ is fully faithful, the induced functor $\mathrm{CAT}_{\infty/\mathcal{W}'_0} \to \mathrm{CAT}_{\infty/\mathcal{W}_0}$ is also fully faithful. It follows that we have an equivalence $\ker(\rho''') \simeq \ker(\rho'')$. Thus, to prove that $\ker(\rho'')$ is contractible, we may as well show that $\ker(\rho''')$ is contractible.

Step 5. We also denote by $\xi' : \mathcal{H}^{\Upsilon,\otimes} \to \mathcal{H}^{\chi,\otimes}$ the natural transformation deduced from the morphism of coCartesian fibrations in (4.35). It factors as follows

$$\mathcal{H}^{\Upsilon,\otimes} \xrightarrow{\widetilde{\xi}} \mathrm{Mod}_{\xi(\mathbf{1})}(\mathcal{H}^{\chi})^{\otimes} \xrightarrow{\phi} \mathcal{H}^{\chi,\otimes}$$

with ϕ given by the forgetful functors. The natural transformation $\widetilde{\xi}$ is an equivalence. Thus, to show that $\ker(\rho''')$ is contractible, we may as well show that $\ker(\rho'''')$ is contractible, where $\rho'''' : \mathrm{Auteq}(\phi) \to \mathrm{Auteq}(\mathcal{H}^{\chi,\otimes})$ is the obvious map. It follows from the commutative version of [Lur17, Theorem 4.8.5.11] (see also [Lur17, Corollary 4.8.5.21]) that $\ker(\rho'''')$ is equivalent to the group $\mathrm{Auteq}(\xi(\mathbf{1}))$ of autoequivalences of the section

$$\xi(\mathbf{1}) : (\text{Reg-}\Sigma/S)^{\mathrm{op}} \to \int_{(\text{Reg-}\Sigma/S)^{\mathrm{op}}} \mathrm{CAlg}(\mathcal{H}^{\chi}).$$

Thus, we may as well prove that $\mathrm{Auteq}(\xi(\mathbf{1}))$ is contractible. For this, we remark that $\xi(\mathbf{1})$ is the initial object of the full sub-∞-category of

$$\mathrm{Sect}\left(\int_{(\mathrm{Reg}\text{-}\Sigma/S)^{\mathrm{op}}} \mathrm{CAlg}(\mathcal{H}^X)\Big/ (\mathrm{Reg}\text{-}\Sigma/S)^{\mathrm{op}}\right)$$

spanned by those sections taking a morphism of the form $(X, C_-, C'_+) \to (X, C_-, C_+)$, for $C_- \geq C_+ \geq C'_+$, to a coCartesian edge. □

4.3. **Motivic exit-path spaces.**

In this subsection, we introduce some universal motivic sheaves that will play a key role in the proof of our second main theorem, i.e., Theorem 4.37. These are naturally commutative algebras, and could be considered as algebras of functions on motivic exit-path spaces. For later use, we present the construction in a general Voevodsky pullback formalism $\mathcal{H}^{\otimes}$ over an excellent scheme S of characteristic zero. We start by developing a "coCartesian" version of Corollary 3.119 which will be more convenient in this subsection.

Construction 4.19. The equivalence of Cartesian fibrations provided by Corollary 3.119, gives rise to an equivalence of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves on $\mathrm{Reg}\text{-}\Sigma/S$:

$$\mathcal{H}^{\otimes}_{\text{ict-tm}} \xrightarrow{\sim} \overleftarrow{\mho}^{\otimes}_{\mathcal{H}}, \tag{4.36}$$

where $\overleftarrow{\mho}_{\mathcal{H}}$ is the Pr^{L}-valued presheaf given, for $X \in \mathrm{Reg}\text{-}\Sigma/S$, by the full sub-$\infty$-category

$$\overleftarrow{\mho}_{\mathcal{H}}(X) \subset \mathrm{Sect}\left(\overleftarrow{\int}_{\mathcal{P}'_X} \mathcal{H}^{\Psi}\Big/ \mathcal{P}'_X\right) \tag{4.37}$$

spanned by those sections sending an arrow of the form $(C'_-, C_+) \to (C_-, C_+)$, for a sequence of strata $C'_- \geq C_- \geq C_+$ in X, to a Cartesian edge. It follows from [GHN17, Theorem 4.5] that the Cartesian fibration freely generated by $\mathcal{P}'_X$, with marked edges given by $(C'_-, C_+) \to (C_-, C_+)$ as above, is given by the obvious functor $\mathcal{P}''_X \to \mathcal{P}'_X$ introduced in Notation 3.21. Recall that $\mathcal{P}''_X$ is the sub-poset of $(\mathcal{P}_X, \geq) \times (\mathcal{P}_X, \leq) \times (\mathcal{P}_X, \leq)$ whose elements are the triples (C_-, C_0, C_+) of strata in X such that $C_- \geq C_0 \geq C_+$. Thus, we have an equivalence of ∞-categories

$$\overleftarrow{\mho}_{\mathcal{H}}(X) \simeq \mathrm{Fun}^{\mathrm{cart}}_{\mathcal{P}'_X}\left(\mathcal{P}''_X, \overleftarrow{\int}_{\mathcal{P}'_X} \mathcal{H}^{\Psi}\right). \tag{4.38}$$

We may informally describe the image of an object $M \in \mathcal{H}_{\text{ict-tm}}(X)$ in the codomain of the equivalence in (4.38) as follows: it is the morphism of Cartesian fibrations sending a triple (C_-, C_0, C_+) to the object $\widetilde{\Psi}_{C_0, C_+}(M|_{C_0})|_{\mathrm{N}^\circ_{\overline{C}_-}(C_+)}$. (Here and below, we write $\widetilde{\Psi}_{C_0, C_+}$, instead of $\widetilde{\Psi}_{C_+}$, to denote the specialisation functor from $\mathcal{H}_{\mathrm{itame}}(C_0)$ to $\mathcal{H}_{\mathrm{itame}}(\mathrm{N}^\circ_{\overline{C}_0}(C_+))$ associated to the stratum $C_+ \subset \overline{C}_0$.) The aforementioned morphism sends an arrow $(C_-, C'_0, C_+) \to (C_-, C_0, C_+)$, with $C'_0 \leq C_0$, to the morphism induced by

$$\widetilde{\Psi}_{C'_0, C_+}(M|_{C'_0})|_{\mathrm{N}^\circ_{\overline{C}_0}(C_+)} \to \widetilde{\Psi}_{C_0, C_+}(M|_{C_0}).$$

Passing to the dual coCartesian fibrations, we may write the codomain of the equivalence in (4.38) as an ∞-category of morphisms of coCartesian fibrations over $\mathcal{P}'^{\mathrm{op}}_X$. We then repeat the above argument reversely using the free coCartesian fibration generated by $\mathcal{P}'_X$, with marked edges of the form $(C_-, C'_+) \to (C_-, C_+)$. This gives an equivalence of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves

$$\overleftarrow{\mho}^{\otimes}_{\mathcal{H}} \xrightarrow{\sim} \overrightarrow{\mho}^{\otimes}_{\mathcal{H}}, \tag{4.39}$$

where $\overrightarrow{\mho}_{\mathcal{H}}$ is the Pr^{L}-valued presheaf given, for $X \in \text{Reg-}\Sigma/S$, by the full sub-$\infty$-category

$$\overrightarrow{\mho}_{\mathcal{H}}(X) \subset \mathrm{Sect}\left(\int_{\mathscr{P}_X'^{\mathrm{op}}} \mathcal{H}^{\Psi} \middle/ \mathscr{P}_X'^{\mathrm{op}}\right) \tag{4.40}$$

spanned by those sections sending an arrow of the form $(C_-, C_+') \to (C_-, C_+)$, for a sequence of strata $C_- \geq C_+ \geq C_+'$ in X, to a coCartesian edge. Composing the equivalences in (4.36) and (4.39), we obtain an equivalence of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves on $\text{Reg-}\Sigma/S$:

$$\mathcal{H}^{\otimes}_{\text{ict-tm}} \xrightarrow{\sim} \overrightarrow{\mho}^{\otimes}_{\mathcal{H}}. \tag{4.41}$$

Thus, we have proven the following "coCartesian" version of Corollary 3.119.

Corollary 4.20 (Exit-path Theorem)**.** *Denote by $p : \text{Reg-}\Sigma^{\mathrm{dm}}/S \to \text{Reg-}\Sigma/S$ the functor forgetting the demarcation. Then, there is a fully faithful functor*

$$\theta'' : \int_{(\text{Reg-}\Sigma/S)^{\mathrm{op}}} \mathcal{H}^{\otimes}_{\text{ict-tm}} \to p_*\left(\int_{(\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}}} \mathcal{H}^{\Psi,\otimes}\right) \tag{4.42}$$

whose essential image consists of those pairs (X, s), with X a regularly stratified S-scheme and

$$s : \mathscr{P}_X'^{\mathrm{op}} \to \int_{\mathscr{P}_X'^{\mathrm{op}}} \mathcal{H}^{\Psi}$$

a section taking the edges $(C_-, C_+') \to (C_-, C_+)$, with $C_- \geq C_+ \geq C_+'$, to coCartesian edges.

Construction 4.21. We continue denoting by $p : \text{Reg-}\Sigma^{\mathrm{dm}}/S \to \text{Reg-}\Sigma/S$ the forgetful functor. For (X, C_-, C_+) a demarcated regularly stratified S-scheme, we denote by $\mathcal{Q}_{X,C_-,C_+} \subset \mathscr{P}_X'$ the sub-poset whose elements are the pairs (C_-', C_+') such that $C_-' \geq C_- \geq C_+ \geq C_+'$. Clearly, the assignment $(X, C_-, C_+) \mapsto \mathcal{Q}_{X,C_-,C_+}$ defined a sub-functor $\mathcal{Q}$ of $\mathscr{P}' \circ p$. Consider the coCartesian fibrations

$$q : \int_{\text{Reg-}\Sigma^{\mathrm{dm}}/S} \mathcal{Q} \to \text{Reg-}\Sigma^{\mathrm{dm}}/S \qquad \text{and} \qquad p' : \int_{\text{Reg-}\Sigma^{\mathrm{dm}}/S} \mathscr{P}' \circ p \to \text{Reg-}\Sigma^{\mathrm{dm}}/S.$$

We have an obvious functor

$$p^* p_*\left(\int_{(\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}}} \mathcal{H}^{\Psi,\otimes}\right) \simeq p'_*\left(\int_{\left(\int_{\text{Reg-}\Sigma^{\mathrm{dm}}/S} \mathscr{P}' \circ p\right)^{\mathrm{op}}} \mathcal{H}^{\Psi,\otimes}\right) \to q_*\left(\int_{\left(\int_{\text{Reg-}\Sigma^{\mathrm{dm}}/S} \mathcal{Q}\right)^{\mathrm{op}}} \mathcal{H}^{\Psi,\otimes}\right).$$

By straightening and restricting to $\overrightarrow{\mho}$, we obtain a morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves

$$\overrightarrow{\mho}^{\otimes} \circ p \to \widetilde{\mho}^{\otimes} \tag{4.43}$$

where $\widetilde{\mho}$ is the Pr^{L}-valued presheaf given, for $(X, C_-, C_+) \in \text{Reg-}\Sigma^{\mathrm{dm}}/S$, by the full sub-$\infty$-category

$$\widetilde{\mho}(X, C_-, C_+) \subset \mathrm{Sect}\left(\int_{\mathcal{Q}^{\mathrm{op}}_{X,C_-,C_+}} \mathcal{H}^{\Psi} \middle/ \mathcal{Q}^{\mathrm{op}}_{X,C_-,C_+}\right) \tag{4.44}$$

spanned by those sections sending an arrow $(C_-', C_+'') \to (C_-', C_+')$, with $C_-' \geq C_- \geq C_+ \geq C_+' \geq C_+''$, to a coCartesian edge. The category $\mathcal{Q}_{X,C_-,C_+}$ admits a reflexive subcategory $\mathcal{R}_{X,C_-,C_+}$ whose elements are the pairs (C_-', C_+). It follows that we have an equivalence of ∞-categories

$$\widetilde{\mho}(X, C_-, C_+) \simeq \mathrm{Sect}\left(\int_{\mathcal{R}^{\mathrm{op}}_{X,C_-,C_+}} \mathcal{H}^{\Psi} \middle/ \mathcal{R}^{\mathrm{op}}_{X,C_-,C_+}\right). \tag{4.45}$$

Using the equivalence in (4.41), we obtain a morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves

$$\varphi'^{*} : \mathcal{H}^{\otimes}_{\text{ict-tm}} \circ p \to \widetilde{\mho}^{\otimes}_{\mathcal{H}}. \tag{4.46}$$

Informally and modulo the equivalence in (4.45), for a demarcated regularly stratified S-scheme (X, C_-, C_+) the functor φ'^{*}_{X,C_-,C_+} takes an object $M \in \mathcal{H}_{\text{ict-tm}}(X)$ to the section

$$s_M : \mathcal{R}^{\mathrm{op}}_{X,C_-,C_+} \to \int_{\mathcal{R}^{\mathrm{op}}_{X,C_-,C_+}} \mathcal{H}^{\Psi}$$

given by $s_M(C'_-, C_+) = \widetilde{\Psi}_{C'_-,C_+}(M|_{C'_-})$. In particular, letting X_{C_-} be the smallest constructible open neighbourhood of C_- in X, we see that φ'^{*}_{X,C_-,C_+} takes an object supported in $X \smallsetminus X_{C_-}$ to 0. Said differently, φ'^{*}_{X,C_-,C_+} factors through the localisation functor $\mathcal{H}_{\text{ict-tm}}(X) \to \mathcal{H}_{\text{ict-tm}}(X_{C_-})$ yielding a functor φ^{*}_{X,C_-,C_+}. These functors assemble into a morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves. More precisely, let $g : \text{Reg-}\Sigma^{\mathrm{dm}}/S \to \text{Reg-}\Sigma/S$ be the functor given by $g(X, C_-, C_+) = X_{C_-}$. Then the morphism in (4.46) factors through a morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves

$$\varphi^{*} : \mathcal{H}^{\otimes}_{\text{ict-tm}} \circ g \to \widetilde{\mho}^{\otimes}_{\mathcal{H}}. \tag{4.47}$$

The functors φ^{*}_{X,C_-,C_+} admit right adjoints $\varphi_{X,C_-,C_+,*}$. By a standard argument, see for example [AGV20, §3.4], we obtain a section

$$\mathfrak{P}^{\mathcal{H}} = \varphi_* \mathbf{1} : (\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}} \to \int_{(\text{Reg-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}}} \mathrm{CAlg}(\mathcal{H}_{\text{ict-tm}}) \circ g, \tag{4.48}$$

sending $(X, C_-, C_+) \in \text{Reg-}\Sigma^{\mathrm{dm}}/S$ to the commutative algebra $\mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+} = \varphi_{X,C_-,C_+,*}\mathbf{1}$ in $\mathcal{H}_{\text{ict-tm}}(X_{C_-})^{\otimes}$. The reminder of this subsection is devoted to studying these algebras. We may think of $\mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+}$ as an algebra of functions on the exit-path space from C_- to C_+.

To study the commutative algebras $\mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+}$ introduced in Construction 4.21, we need some facts concerning the functors φ^{*}_{X,C_-,C_+} and their right adjoints.

Lemma 4.22. *Let $u : Y \hookrightarrow X$ be a locally closed immersion of regularly stratified S-schemes, such that the stratification of Y is induced from the stratification of X. Let $C'_- \geq C_- \geq C_+$ be strata contained in Y. Consider the commutative square of ∞-categories*

$$\begin{tikzcd} \mathcal{H}_{\text{ict-tm}}(X_{C_-}) \arrow[r, "u'^{*}"] \arrow[d, "\varphi^{*}_{X,C_-,C_+}"] & \mathcal{H}_{\text{ict-tm}}(Y_{C'_-}) \arrow[d, "\varphi^{*}_{Y,C'_-,C_+}"] \\ \widetilde{\mho}_{\mathcal{H}}(X, C_-, C_+) \arrow[r, "u^{*}_{\sigma}"] & \widetilde{\mho}_{\mathcal{H}}(Y, C'_-, C_+) \end{tikzcd} \tag{4.49}$$

where $u' : Y_{C'_-} \to X_{C_-}$ is the morphism induced by u.

(i) If u is an open immersion, the square in (4.49) *is left adjointable.*
(ii) The square in (4.49) *is right adjointable.*

Proof. Once we understand the left and right adjoints of the functor u^{*}_{σ}, the result follows by direct inspection. We split the proof in two parts.

Part 1. Here we prove (i). The left adjoint $u_!^\sigma$ to u_σ^* is given by a relative left Kan extension along the inclusion $\mathcal{R}_{Y,C'_-,C_+} = \mathcal{R}_{X,C'_-,C_+} \subset \mathcal{R}_{X,C_-,C_+}$. Thus, for a section

$$t : (\mathcal{R}_{X,C'_-,C_+})^{\mathrm{op}} \to \int_{(\mathcal{R}_{X,C'_-,C_+})^{\mathrm{op}}} \mathcal{H}^\Psi,$$

the section $u_!^\sigma(t)$ is given informally by

$$u_!^\sigma(t)(C''_-, C_+) = \begin{cases} t(C''_-, C_+) & \text{if } C''_- \geq C'_-, \\ 0 & \text{else.} \end{cases}$$

Said differently, $u_!^\sigma(t)$ coincides with t on $\mathcal{R}_{X,C'_-,C_+}$ and is zero on its complement. Similarly, given $M \in \mathcal{H}_{\text{ict-tm}}(Y_{C'_-})$, the section $\varphi^*_{X,C_-,C_+}(u'_!M)$ coincides with $\varphi^*_{Y,C'_-,C_+}(M)$ on $\mathcal{R}_{X,C'_-,C_+}$ and is zero on its complement. This finishes the proof of (i).

Part 2. Here we prove (ii). The right adjoint u_*^σ of u_σ^* is given by a relative right Kan extension along the inclusion $\mathcal{R}_{Y,C'_-,C_+} \subset \mathcal{R}_{X,C_-,C_+}$. Thus, for a section

$$t : (\mathcal{R}_{Y,C'_-,C_+})^{\mathrm{op}} \to \int_{(\mathcal{R}_{X,C'_-,C_+})^{\mathrm{op}}} \mathcal{H}^\Psi,$$

the section $u_*^\sigma(t)$ is given at $(C''_-, C_+) \in \mathcal{R}_{X,C_-,C_+}$ by

$$u_*^\sigma(t)(C''_-, C_+) = \lim_{E\subset Y,\, E\geq C'_-,\, E\geq C''_-} (p_{E,C''_-})_* t(E, C_+)$$

where $p_{E,C''_-} : \mathrm{N}^\circ_{\overline{E}}(C_+) \to \mathrm{N}^\circ_{\overline{C''_-}}(C_+)$ is the obvious morphism. Thus, for $M \in \mathcal{H}_{\text{ict-tm}}(Y_{C'_-})$, we need to show that the obvious map

$$\widetilde{\Psi}_{C''_-,C_+}((u_*M)|_{C''_-}) \to \lim_{E\subset Y,\, E\geq C'_-,\, E\geq C''_-} (p_{E,C''_-})_* \widetilde{\Psi}_{E,C_+}(M|_E)$$

is an equivalence. Using the natural equivalences $\widetilde{\Psi}_{C''_-,C_+} \circ \chi_{E,C''_-} \simeq (p_{E,C''_-})_* \circ \widetilde{\Psi}_{E,C_+}$, we are reduced to showing that the obvious map

$$(u_*M)|_{C''_-} \to \lim_{E\subset Y,\, E\geq C'_-,\, E\geq C''_-} \chi_{E,C''_-}(M|_E)$$

is an equivalence, which is clear. □

Corollary 4.23. *Let (X, C_-, C_+) be a demarcated regularly stratified S-scheme and $C'_- \geq C_-$ a stratum of X. Then, we have an equivalence of commutative algebras*

$$u'^* \mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+} \simeq \mathfrak{P}^{\mathcal{H}}_{X,C'_-,C_+}. \tag{4.50}$$

Proof. Indeed, by Lemma 4.22(i) and adjunction, we have a natural equivalence

$$u'^* \circ \varphi_{X,C_-,C_+,*} \simeq \varphi_{X,C'_-,C_+,*} \circ u_\sigma^*.$$

The result follows by applying this to the unit object of $\widetilde{\mathfrak{S}}_{\mathcal{H}}(X, C_-, C_+)$. □

Lemma 4.24. *Let $u : Y \hookrightarrow X$ be a closed immersion of regularly stratified S-schemes, such that the stratification of Y is induced from the stratification of X. Let $C_- \geq C_+$ be strata contained in Y.*

The commutative square of ∞-categories

$$\begin{tikzcd} \mathcal{H}_{\text{ict-tm}}(Y_{C_-}) \arrow[r, "u'_*"] \arrow[d, "\varphi^*_{Y,C_-,C_+}"] & \mathcal{H}_{\text{ict-tm}}(X_{C_-}) \arrow[d, "\varphi^*_{X,C_-,C_+}"] \\ \widetilde{\mho}_{\mathcal{H}}(Y, C_-, C_+) \arrow[r, "u^\sigma_*"] & \widetilde{\mho}_{\mathcal{H}}(X, C_-, C_+), \end{tikzcd} \tag{4.51}$$

provided by Lemma 4.22(ii), is right adjointable.

Proof. For a section

$$s : (\mathcal{R}_{Y,C_-,C_+})^{\text{op}} \to \int_{(\mathcal{R}_{Y,C_-,C_+})^{\text{op}}} \mathcal{H}^\Psi,$$

the section $u^\sigma_*(s)$ is given by

$$u^\sigma_*(s)(C'_-, C_+) = \begin{cases} s(C'_-, C_+) & \text{if } C_- \geq C'_-, \\ 0 & \text{else.} \end{cases}$$

Said differently, $u^\sigma_*(s)$ coincides with t on $\mathcal{R}_{Y,C_-,C_+}$ and is zero on its complement. It follows that, for a section

$$t : (\mathcal{R}_{X,C_-,C_+})^{\text{op}} \to \int_{(\mathcal{R}_{X,C_-,C_+})^{\text{op}}} \mathcal{H}^\Psi,$$

we have a cofiber sequence

$$\operatorname*{colim}_{D \in \mathcal{P}_{X \smallsetminus Y}} (j_D)^\sigma_!(j_D)^*_\sigma(t) \to t \to u^\sigma_* u^*_\sigma(t),$$

where $j_D : (X, D, C_+) \to (X, C_-, C_+)$ is the morphism given by the identity of X. It follows by adjuction that there is a fiber sequence

$$u^\sigma_* u^!_\sigma(t) \to t \to \lim_{D \in \mathcal{P}_{X \smallsetminus Y}} (j_D)^\sigma_*(j_D)^*_\sigma(t).$$

Hence, by applying u^*_σ, we obtain the fiber sequence

$$u^!_\sigma(t) \to u^*_\sigma(t) \to \lim_{D \in \mathcal{P}_{X \smallsetminus Y}} u^*_\sigma(t)(j_D)^\sigma_*(j_D)^*_\sigma(t).$$

Similarly, writing $j_D : X_D \to X_{C_-}$ for the obvious inclusions, we have a fiber sequence

$$u'^!(M) \to u'^*(M) \to \lim_{D \in \mathcal{P}_{X \smallsetminus Y}} u'^*(j_D)_*(j_D)^*(M)$$

for every $M \in \mathcal{H}_{\text{ict-tm}}(X_{C_-})$. The result now follows from Lemma 4.22(ii). □

In order to state the key property of the algebras $\mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+}$, we introduce the following notation.

Notation 4.25. Let X be a regularly stratified S-scheme and $C \subset X$ a stratum. Recall that we have a symmetric monoidal functor $\widetilde{\Psi}_C : \mathcal{H}^\otimes_{\text{itame}}(X^\circ) \to \mathcal{H}^\otimes_{\text{itame}}(\mathrm{N}^\circ_X(C))$. We will denote by

$$\pitchfork^{\mathcal{H}}_C : \mathcal{H}^\otimes_{\text{itame}}(\mathrm{N}^\circ_X(C)) \to \mathcal{H}^\otimes_{\text{itame}}(X^\circ) \tag{4.52}$$

the right adjoint to $\widetilde{\Psi}_C$, which is a right-lax monidal functor. Similarly, if (X, C_-, C_+) is a demarcated regularly stratified S-scheme, we denote by $\pitchfork^{\mathcal{H}}_{C_-,C_+}$ the right adjoint to $\widetilde{\Psi}_{C_-,C_+}$.

Theorem 4.26. *Let (X, C_-, C_+) be a demarcated regularly stratified S-scheme. For every stratum $C'_- \geq C_-$, there is a natural equivalence*

$$\iota^!_{C'_-} \mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+} \simeq (\pitchfork^{\mathcal{H}}_{C'_-,C_+} \mathbf{1}) \otimes \iota^!_{C'_-} \mathbf{1}. \tag{4.53}$$

(As usual, we denote by $\iota_{C'_-} : C'_- \hookrightarrow X$ the obvious inclusion.)

Proof. Using Corollary 4.23, we may replace (X, C_-, C_+) with (X, C'_-, C_+) and assume that $C'_- = C_-$. Said differently, it is enough to show that there is a natural equivalence

$$\iota^!_{C_-} \mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+} \simeq (\pitchfork^{\mathcal{H}}_{C_-,C_+} \mathbf{1}) \otimes i^!_{C_-} \mathbf{1}. \tag{4.54}$$

Let Y be the closure of C_- in X and $u : Y \hookrightarrow X$ its inclusion. Combining Lemmas 4.22(ii) and 4.24, we obtain natural equivalences

$$u'^! \circ \varphi_{X,C_-,C_+,*} \circ \varphi^*_{X,C_-,C_+} \simeq \varphi_{Y,C_-,C_+,*} \circ u^!_\sigma \circ \varphi^*_{X,C_-,C_+} \simeq \varphi_{Y,C_-,C_+,*} \circ \varphi^*_{Y,C_-,C_+} \circ u'^!.$$

Applying this to the unit object, we get an equivalence

$$u'^! \mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+} \simeq \mathfrak{P}^{\mathcal{H}}_{Y,C_-,C_+} \otimes u'^! \mathbf{1}.$$

It remains to see that $\mathfrak{P}^{\mathcal{H}}_{Y,C_+,C_-}$ is equivalent to $\pitchfork^{\mathcal{H}}_{C_-,C_+} \mathbf{1}$. This follows immediately from the fact C_- is the open stratum of Y. Indeed, this implies that $\mathcal{R}_{Y,C_-,C_+} = \{(C_-, C_+)\}$ is a singelton so that the functor φ^*_{Y,C_-,C_+} can be identified with $\widetilde{\Psi}_{C_-,C_+}$. □

Definition 4.27. Given a presentable symmetric monoidal ∞-category $\mathcal{C}^\otimes$, we denote by $\mathcal{C}^{\text{liss}} \subset \mathcal{C}$ the full sub-∞-category generated under colimits by the dualizable objects. This is a coreflexive sub-∞-category; the associated coreflexion functor will be denoted by $(-)^{\text{liss}}$ and will be called the lissification functor. This is clearly a right-lax monoidal functor.

Lemma 4.28. *Assume that the unit object of $\mathcal{C}^\otimes$ is compact. Let $\mathcal{A} \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra and let $\mathcal{A}^{\text{liss}}$ be its lissification. Then the base change functor*

$$\mathrm{Mod}_{\mathcal{A}^{\text{liss}}}(\mathcal{C}^{\text{liss}}) \to \mathrm{Mod}_{\mathcal{A}}(\mathcal{C})$$

is fully faithful with essential image the sub-∞-category of $\mathrm{Mod}_{\mathcal{A}}(\mathcal{C})$ generated under colimits by the $\mathcal{A}$-modules freely generated on dualizable objects of $\mathcal{C}$.

Proof. By standard arguments, we reduce to showing that the map

$$\mathrm{Map}_{\mathcal{C}}(M, \mathcal{A}^{\text{liss}} \otimes N) \to \mathrm{Map}_{\mathcal{C}}(M, \mathcal{A} \otimes N)$$

is an equivalence for all dualizable $M, N \in \mathcal{C}$. Replacing M with $M \otimes N^\vee$, we may assume that N is the unit object of $\mathcal{C}$. The result is then clear. □

Notation 4.29. Let (X, C_-, C_+) be a demarcated regularly stratified S-scheme. We denote by

$$\mathcal{H}^{\mathfrak{P}}(X, C_-, C_+) \subset \mathcal{H}_{\text{ict-tm}}(X_{C_-}; \mathfrak{P}_{X,C_-,C_+}) \tag{4.55}$$

the full sub-∞-category generated under colimits by the $\mathfrak{P}_{X,C_-,C_+}$-modules freely generated on dualizable objets of $\mathcal{H}_{\text{ict-tm}}(X_{C_-})$. By Lemma 4.28, writing $\mathfrak{P}^{\mathcal{H},\text{liss}}_{X,C_-,C_+}$ and $\mathcal{H}^{\text{liss}}_{\text{ict-tm}}(X_{C_-})$ instead of $(\mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+})^{\text{liss}}$ and $\mathcal{H}_{\text{ict-tm}}(X_{C_-})^{\text{liss}}$, we have an equivalence of ∞-categories

$$\mathcal{H}^{\mathfrak{P}}(X, C_-, C_+) \simeq \mathrm{Mod}_{\mathfrak{P}^{\mathcal{H},\text{itame}}_{X,C_-,C_+}}(\mathcal{H}^{\text{liss}}_{\text{ict-tm}}(X_{C_-})). \tag{4.56}$$

Construction 4.30. Arguing as in [AGV20, §3.4], we obtain a morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves on $\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S$:

$$\widetilde{\varphi}^* : \mathrm{Mod}_{\mathfrak{P}^{\mathcal{H}}}(\mathcal{H}_{\text{ict-tm}} \circ g)^{\otimes} \to \widetilde{\mho}^{\otimes}_{\mathcal{H}}. \tag{4.57}$$

For a demarcated regularly stratified S-scheme (X, C_-, C_+), the functor $\widetilde{\varphi}^*_{X,C_-,C_+}$ takes a $\mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+}$-module M to the section $\widetilde{s}_M$ given by

$$\widetilde{s}_M(C'_-, C_+) = \widetilde{\Psi}_{C'_-,C_+}(M|_{C'_-}) \otimes_{\widetilde{\Psi}_{C'_-,C_+}(\mathfrak{P}_{X,C_-,C_+}|_{C'_-})} \mathbf{1}.$$

On the other hand, we have a morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves on $\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S$:

$$\widetilde{\mho}^{\otimes}_{\mathcal{H}} \to \mathcal{H}^{\Psi,\otimes} \tag{4.58}$$

constructed as follows. For (X, C_-, C_+) as above, let $\mathcal{R}'_{X,C_-,C_+}$ be the subposet of $\mathcal{Q}_{X,C_-,C_+}$ whose elements are the pairs (C_-, C'_+) with $C'_+ \leq C_+$. There is a morphism of of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves on $\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S$ sending (X, C_-, C_+) to the restriction functor

$$\mathrm{Sect}\left(\int_{\mathcal{Q}^{\mathrm{op}}_{X,C_-,C_+}} \mathcal{H}^{\Psi} \Big/ \mathcal{Q}^{\mathrm{op}}_{X,C_-,C_+}\right) \to \mathrm{Sect}\left(\int_{\mathcal{R}'^{\mathrm{op}}_{X,C_-,C_+}} \mathcal{H}^{\Psi} \Big/ \mathcal{R}'^{\mathrm{op}}_{X,C_-,C_+}\right).$$

The restriction of this functor to $\widetilde{\mho}(X, C_-, C_+)$ lands in the full sub-∞-category spanned by coCartesian sections, which is equivalent to $\mathcal{H}^{\Psi}(X, C_-, C_+)$. This gives rise to the morphism in (4.58). This said, we may compose with the morphism in (4.57) to get the morphism:

$$\mathrm{Mod}_{\mathfrak{P}^{\mathcal{H}}}(\mathcal{H}_{\text{ict-tm}} \circ g)^{\otimes} \to \mathcal{H}^{\Psi,\otimes}, \tag{4.59}$$

which for (X, C_-, C_+) as above, sends a $\mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+}$-module M to $\widetilde{\Psi}_{C_-,C_+}(M|_{C_-}) \otimes_{\widetilde{\Psi}_{C_-,C_+}(\mathfrak{P}_{X,C_-,C_+}|_{C_-})} \mathbf{1}$. We will be mainly interested in the morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves on $\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S$:

$$\phi : \mathcal{H}^{\mathfrak{P},\otimes} \to \mathcal{H}^{\Psi,\otimes} \tag{4.60}$$

obtained by restricting the morphism in (4.59) to the sub-∞-categories introduced in Notation 4.55. We will see below that ϕ is very close to being an equivalence.

Theorem 4.31. *Let (X, C_-, C_+) be a demarcated regularly stratified S-scheme. The functor*

$$\phi_{X,C_-,C_+} : \mathcal{H}^{\mathfrak{P}}(X, C_-, C_+) \to \mathcal{H}^{\Psi}(X, C_-, C_+) \tag{4.61}$$

is fully faithful. It is an equivalence if the inclusion $\overline{C}_+ \hookrightarrow X$ admits a retraction.

Proof. We need to show that, for $M, N \in \mathcal{H}_{\text{ict-tm}}(X_{C_-})$ dualizable, the map

$$\mathrm{Map}(M, \mathfrak{P}_{X,C_-,C_+} \otimes N) \to \mathrm{Map}(\widetilde{\Psi}_{C_-,C_+}(M|_{C_-}), \widetilde{\Psi}_{C_-,C_+}(N|_{C_-})) \tag{4.62}$$

is an equivalence. Since N is dualizable, we may replace M with $M \otimes N^{\vee}$ and assume that $N = \mathbf{1}$. Therefore, using adjunction, we are reduced to showing that

$$\mathrm{Map}(\varphi^*_{X,C_-,C_+}(M), \mathbf{1}) \to \mathrm{Map}(\widetilde{\Psi}_{C_-,C_+}(M|_{C_-}), \mathbf{1}) \tag{4.63}$$

is an equivalence. Denote by

$$t : (\mathcal{R}_{X,C_-,C_+})^{\mathrm{op}} \to \int_{(\mathcal{R}_{X,C_-,C_+})^{\mathrm{op}}} \mathcal{H}^{\Psi}$$

the section $\varphi^*_{X,C_-,C_+}(M)$. Since M is dualizable, we see that for every strata $C''_- \geq C'_- \geq C_-$, the obvious morphism

$$\widetilde{\Psi}_{C'_-,C_+}(M|_{C'_-})|_{\mathrm{N}^{\circ}_{C''_-}(C_+)} \to \widetilde{\Psi}_{C''_-,C_+}(M|_{C''_-})$$

is an equivalence. Thus, t is a coCartesian section. Since the same is true for the unit section, we deduce that the first mapping space in (4.63) is computed in the sub-∞-category of

$$\mathrm{Sect}\left(\int_{\mathcal{R}^{\mathrm{op}}_{X,C_-,C_+}} \mathcal{H}^{\Psi}\middle/ \mathcal{R}^{\mathrm{op}}_{X,C_-,C_+}\right)$$

spanned by the coCartesian sections. Since $\mathcal{R}_{X,C_-,C_+}$ admits a final object (C_-,C_+), the aforementioned sub-∞-category is equivalent to $\mathcal{H}^{\Psi}(X,C_-,C_+)$. This finishes the proof. □

Remark 4.32. The condition that $\overline{C}_+ \hookrightarrow X$ admits a retraction is satisfied locally for the Nisnevich topology when S is the spectrum of a field. Thus, in this case, we see that the morphism in (4.61) induces an equivalence after sheafification for the Nisnevich topology.

For later use, we explicitly state Theorem 4.31 in the special case of the Voevodsky pullback formalism $\mathbf{Sh}_{\mathrm{geo}}$.

Corollary 4.33. *Consider the section*

$$(\mathfrak{P}^{\mathbf{Sh}_{\mathrm{geo}}})^{\mathrm{liss}} : (\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}} \to \int_{(\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}}} \mathrm{CAlg}(\widehat{\mathbf{LS}}_{\mathrm{geo}} \circ g).$$

We have a morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$*-valued presheaves on* $\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k$*:*

$$\phi : \mathrm{Mod}_{(\mathfrak{P}^{\mathbf{Sh}_{\mathrm{geo}}})^{\mathrm{liss}}}(\widehat{\mathbf{LS}}_{\mathrm{geo}} \circ g)^{\otimes} \to \widehat{\mathbf{LS}}^{\Psi,\otimes}_{\mathrm{geo}}$$

which is given by fully faithful functors and which becomes an equivalence after Nisnevich sheafification.

We end this subsection with a comparison result concerning the commutative algebras $\mathfrak{P}^{\mathcal{H}}_{X,C_-,C_+}$ for the two Voevodsky pullback formalisms $\mathbf{MSh}$ and $\mathbf{Sh}_{\mathrm{geo}}$. This comparison result is a key ingredient in the proof of our second main theorem, i.e., Theorem 4.37.

Theorem 4.34. *The Betti realisation* $\mathrm{B}^* : \mathbf{MSh} \to \mathbf{Sh}_{\mathrm{geo}}$ *induces a morphism*

$$\mathrm{B}^*\mathfrak{P}^{\mathbf{MSh}} \to \mathfrak{P}^{\mathbf{Sh}_{\mathrm{geo}}} \tag{4.64}$$

between sections of the coCartesian fibration

$$\int_{(\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}}} \mathrm{CAlg}(\mathbf{Sh}_{\mathrm{geo,\,ict}}) \circ g \to (\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S)^{\mathrm{op}}.$$

Moreover, the morphism in (4.64) *induces equivalences on the* cdh *stalks in the following sense. Let V be the spectrum of a finite rank valuation ring over $\mathcal{O}_S$, and let $\xi_-, \xi_+ \in V$ be two points of V, considered as strata, and such that $\xi_- \geq \xi_+$. The triple (V,ξ_-,ξ_+) can be considered in an essentially unique way as a pro-object in* $\mathrm{Reg}\text{-}\Sigma^{\mathrm{dm}}/S$*. This said, the morphism*

$$\mathrm{B}^*_{V_{\xi_-1}}(\mathfrak{P}^{\mathbf{MSh}}_{V,\xi_-,\xi_+}) \to \mathfrak{P}^{\mathbf{Sh}_{\mathrm{geo}}}_{V,\xi_-,\xi_+}, \tag{4.65}$$

induced by the morphism in (4.64)*, is an equivalence.*

Proof. The construction of the morphism in (4.64) is easy and left to the reader. We only discuss the second part of the statement. For a point $\xi'_- \geq \xi_-$ in V, Theorem 4.26 and passage to the limit yield equivalences

$$\iota^!_{\xi'_-}\,\mathrm{B}^*\,\mathfrak{P}^{\mathbf{MSh}}_{V,\xi_-,\xi_+} \simeq \mathrm{B}^*\,\iota^!_{\xi'_-}\,\mathfrak{P}^{\mathbf{MSh}}_{V,\xi_-,\xi_+} \simeq \mathrm{B}^*\,\pitchfork^{\mathbf{MSh}}_{\xi'_-,\xi_+}\mathbf{1} \otimes \iota^!_{\xi'_-}\mathbf{1}$$
$$\text{and}\quad \iota^!_{\xi'_-}\,\mathfrak{P}^{\mathbf{Sh}_{\mathrm{geo}}}_{V,\xi_-,\xi_+} \simeq \pitchfork^{\mathbf{Sh}_{\mathrm{geo}}}_{\xi'_-,\xi_+}\mathbf{1} \otimes \iota^!_{\xi'_-}\mathbf{1}.$$

Thus, we are left to show that we have an equivalence $\mathrm{B}^*\pitchfork^{\mathbf{MSh}}_{\xi'_-,\xi_+}\mathbf{1} \simeq \pitchfork^{\mathbf{Sh}_{\mathrm{geo}}}_{\xi'_-,\xi_+}\mathbf{1}$. For that, we may assume that $\xi_- = \xi'_-$ is the generic point of V, which we shall denote by η. We may also assume that ξ_+ is the closed point of V, which we shall denote by σ. Remark that we have equivalences of ∞-categories

$$\mathbf{MSh}_{\text{ict-tm}}(\eta/V) \simeq \mathbf{MSh}(\eta) \qquad \text{and} \qquad \mathbf{MSh}_{\text{ict-tm}}(\mathrm{N}^\circ_V(\sigma)) \simeq \mathbf{MSh}(\mathrm{N}^\circ_V(\sigma))_{\mathrm{qun}/\sigma},$$

and similarly for $\mathbf{Sh}_{\mathrm{geo}}$. Therefore, to conclude, it is enough to show that the commutative square

$$\begin{tikzcd} \mathbf{MSh}(\eta) \arrow[r, "\widetilde{\Psi}_\sigma"] \arrow[d, "\mathrm{B}^*"] & \mathbf{MSh}(\mathrm{N}^\circ_V(\sigma))_{\mathrm{qun}/\sigma} \arrow[d, "\mathrm{B}^*"] \\ \mathbf{Sh}_{\mathrm{geo}}(\eta) \arrow[r, "\widetilde{\Psi}_\sigma"] & \mathbf{Sh}_{\mathrm{geo}}(\mathrm{N}^\circ_V(\sigma))_{\mathrm{qun}/\sigma} \end{tikzcd} \tag{4.66}$$

is right adjointable. Using Theorem 1.93, this square can be identified with

$$\begin{tikzcd} \mathbf{MSh}(\eta) \arrow[r, "\widetilde{\Psi}_\sigma"] \arrow[d, "\mathcal{B}\otimes-"] & \mathbf{MSh}(\mathrm{N}^\circ_V(\sigma))_{\mathrm{qun}/\sigma} \arrow[d, "\mathcal{B}\otimes-"] \\ \mathbf{MSh}(\eta;\mathcal{B}) \arrow[r, "\widetilde{\Psi}_\sigma"] & \mathbf{MSh}(\mathrm{N}^\circ_V(\sigma);\mathcal{B})_{\mathrm{qun}/\sigma}. \end{tikzcd} \tag{4.67}$$

The functor $\widetilde{\Psi}_\sigma$ on $\mathbf{MSh}(\eta)$ is symmetric monoidal and takes the commutative algebra $\mathcal{B}|_\eta$ to the commutative algebra $\mathcal{B}|_{\mathrm{N}^\circ_V(\sigma)}$. It follows that the right adjoint of the bottom horizontal functor takes a $\mathcal{B}|_{\mathrm{N}^\circ_V(\sigma)}$-module M to $\pitchfork^{\mathbf{MSh}}_\sigma(M)$ considered as a $\mathcal{B}|_\eta$-module by restriction along $\mathcal{B}|_\eta \to \pitchfork^{\mathbf{MSh}}_\sigma(\mathcal{B}|_{\mathrm{N}^\circ_V(\sigma)})$. This said, the right adjointability of the square in (4.67) would follow if we can show that the obvious morphism

$$A \otimes \pitchfork^{\mathbf{MSh}}_\sigma(B) \to \pitchfork^{\mathbf{MSh}}_\sigma(\widetilde{\Psi}_\sigma(A) \otimes B)$$

is an equivalence for $A = \mathcal{B}|_\eta$. In fact, this is true for any $A \in \mathbf{MSh}(\eta)$. To prove it, we reduce to the case where A is dualizable, and use [Ayo14b, Lemme 2.8]. □

Corollary 4.35. *Consider the sections*

$$(\mathrm{B}^*\mathfrak{P}^{\mathbf{MSh}})^{\mathrm{liss}},\ (\mathfrak{P}^{\mathbf{Sh}_{\mathrm{geo}}})^{\mathrm{liss}} : (\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}} \to \int_{(\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}}} \mathrm{CAlg}(\widehat{\mathbf{LS}}_{\mathrm{geo}} \circ g)$$

and the morphism

$$(\mathrm{B}^*\mathfrak{P}^{\mathbf{MSh}})^{\mathrm{liss}} \to (\mathfrak{P}^{\mathbf{Sh}_{\mathrm{geo}}})^{\mathrm{liss}} \tag{4.68}$$

obtained by lissification from the morphism in (4.64). *The induced morphism of* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_\omega)$*-valued presheaves on* $\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k$*:*

$$\mathrm{Mod}_{(\mathrm{B}^*\mathfrak{P}^{\mathbf{MSh}})^{\mathrm{liss}}}(\widehat{\mathbf{LS}}_{\mathrm{geo}} \circ g)^\otimes \to \mathrm{Mod}_{(\mathfrak{P}^{\mathbf{Sh}_{\mathrm{geo}}})^{\mathrm{liss}}}(\widehat{\mathbf{LS}}_{\mathrm{geo}} \circ g)^\otimes \tag{4.69}$$

becomes an equivalence after cdh*-sheafification. (Here, sheafification is computed in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_\omega)$*.)*

Proof. Using [AGV20, Propositions 2.8.1 & 2.8.2], it is enough to show that the morphism in (4.69) induces equivalences on cdh stalks. Equivalently, we need to show that the morphism of sections in (4.68) induces equivalences on cdh stalks. This follows immediately from Theorem 4.34 since lissification commutes with colimits. □

4.4. **The second main theorem.**

The goal of this subsection is to prove our second main theorem which is Theorem 4.37 below. We start by introducing the spectral $\mathbb{S}$-prestack $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})$.

Definition 4.36. Let k be a field and $\sigma : k \hookrightarrow \mathbb{C}$ a complex embedding. We define the spectral group $\mathbb{S}$-prestack $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})$ as in Definition 2.9. Namely, we apply Construction 1.51 to

- the functor $\mathcal{C} : (\mathrm{SpAFF}^{\mathrm{nc}})^{\mathrm{op}} \to \mathrm{CAT}_{\infty}$ sending $\mathrm{Spec}(\Lambda)$ to the ∞-category
$$\mathrm{Psh}(\mathrm{Sm}/k; \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/})$$
of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/}$-valued presheaves on Sm/k, and
- the natural transformation $\mathrm{pt} \to \mathcal{C}$ sending $\mathrm{Spec}(\Lambda)$ to the functor
$$\widehat{\mathbf{LS}}_{\mathrm{geo}}(-;\Lambda)^{\otimes} : (\mathrm{Sm}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Mod}^{\otimes}_{\Lambda}/}.$$

If we want to stress that $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})$ depends on σ, we will write $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\sigma\text{-geo}})$.

Theorem 4.37 (Main theorem for local systems)**.** *Let k be a field and $\sigma : k \hookrightarrow \mathbb{C}$ a complex embedding. There is an equivalence of spectral group $\mathbb{S}$-prestacks*
$$\mathcal{G}_{\mathrm{mot}}(k,\sigma) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\sigma\text{-geo}}). \tag{4.70}$$
In particular, the right hand side is a spectral affine group scheme.

Remark 4.38. The sought-after equivalence in (4.70) is given by the composition of
$$\mathcal{G}_{\mathrm{mot}}(k,\sigma) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathbf{Sh}^{\otimes}_{\mathrm{geo}}) \to \underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\mathrm{geo}}) \tag{4.71}$$
where the second map exists by the following principle: an autoequivalence of $\mathbf{Sh}^{\otimes}_{\mathrm{geo}}(-;\Lambda)$ preserves dualizable objects and thus restricts to the full sub-functor $\widehat{\mathbf{LS}}^{\otimes}_{\mathrm{geo}}(-;\Lambda)$ spanned by the ind-dualizable objects. In fact, the sought-after equivalence in (4.70) is also the composition of
$$\mathcal{G}_{\mathrm{mot}}(k,\sigma) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathbf{LS}^{\Psi,\otimes}_{\mathrm{geo}}) \to \underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\mathrm{geo}}). \tag{4.72}$$
(See Theorem 4.10.) The proof of Theorem 4.37 can be divided into two independent parts.

- The first part consists in showing that the composition of (4.71) admits a retraction.
- The second part consists in showing that the obvious morphism
$$\underline{\mathrm{Auteq}}(\mathbf{LS}^{\Psi,\otimes}_{\mathrm{geo}}) \to \underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\mathrm{geo}}) \tag{4.73}$$
admits a section.

The first part is relatively easy, and is the subject of Lemma 4.39 below.

Lemma 4.39. *The natural morphism of spectral group $\mathbb{S}$-prestacks $\mathcal{G}_{\mathrm{mot}}(k,\sigma) \to \underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})$ admits a retraction.*

Proof. We will show that $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})$ acts on the Betti spectrum $\mathcal{B}$ (see Notation 1.57), in a way extending the natural action of $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ (see Corollary 1.64). We split the proof in two parts. In the first part, we recall a few facts on the Betti spectrum. In the second part, we give the actual proof. For simplicity, we only treat the case of $\mathbb{S}$-points; the case of Λ-points, for a general $\Lambda \in \mathrm{CAlg}(\mathcal{S}p_{\geq 0})$, is only notationally more complicated.

Part 1. We denote by Γ_{B} the $\mathrm{CAlg}(\mathcal{S}p)$-valued presheaf on Sm/k sending a smooth k-variety X to $\Gamma(X^{\mathrm{an}};\mathbb{S})$, i.e., the cohomology of X^{an} with coefficients in the sphere spectrum. Note that we may define Γ_{B} as $\Omega_T^\infty(\mathcal{B})$, where Ω_T^∞ is the motivic infinite loop space functor. Conversely, $\mathcal{B}$ can be obtained form Γ_{B} as follows. The presheaf Γ_{B} defines a symmetric monoidal functor

$$\Gamma_{\mathrm{B}} : (\mathrm{Sm}/k)^\times \to (\mathcal{S}p^{\mathrm{op}})^\otimes$$

which, by [Rob15, Corollary 2.39], extends uniquely to a colimit-preserving symmetric monoidal functor

$$\mathbf{MSh}(k)^\otimes \to (\mathrm{Pro}(\mathcal{S}p)^{\mathrm{op}})^\otimes.$$

Composing with the righ-lax monoidal functor $\lim : \mathrm{Pro}(\mathcal{S}p)^\otimes \to \mathcal{S}p^\otimes$, we obtain a limit-preserving righ-lax symmetric monoidal functor

$$\mathbf{\Gamma}_{\mathrm{B}} : (\mathbf{MSh}(k)^{\mathrm{op}})^\otimes \to \mathcal{S}p^\otimes.$$

By [Lur09a, Proposition 5.5.2.2], $\mathbf{\Gamma}_{\mathrm{B}}$ is representable by a commutative algebra which is precisely $\mathcal{B}$. This shows that we have an equivalence of spectral group $\mathbb{S}$-prestacks $\underline{\mathrm{Auteq}}(\Gamma_{\mathrm{B}}) \simeq \underline{\mathrm{Auteq}}(\mathcal{B})$. In particular, we see that $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ is also equivalent to $\underline{\mathrm{Auteq}}(\Gamma_{\mathrm{B}})$.

Part 2. The $\mathrm{CAlg}(\mathcal{S}p)$-valued presheaf Γ_{B} on Sm/k can be obtained from the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaf $\widehat{\mathbf{LS}}_{\mathrm{geo}}$ as follows. Consider the morphism of coCartesian fibrations

$$\begin{tikzcd} (\mathrm{Sm}/k)^{\mathrm{op}} \times \mathcal{S}p^\otimes \arrow[rr, "\varsigma^*"] \arrow[dr] & & \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \widehat{\mathbf{LS}}^\otimes_{\mathrm{geo}} \arrow[dl] \\ & (\mathrm{Sm}/k)^{\mathrm{op}} & \end{tikzcd}$$

given, over $X \in \mathrm{Sm}/k$, by the functor $\varsigma_X^* : \mathcal{S}p^\otimes \to \widehat{\mathbf{LS}}_{\mathrm{geo}}(X)^\otimes$ taking a spectrum to the associated constant sheaf on X^{an}. By [Lur17, Proposition 7.3.2.6], the functor ς^* admits a relative right adjoint ς_*. Applying the latter on the unit section, we get a $\mathrm{CAlg}(\mathcal{S}p)$-valued presheaf on Sm/k, which is precisely Γ_{B}. This gives a map of group objects in $\mathcal{S}$:

$$\mathrm{Auteq}(\widehat{\mathbf{LS}}^\otimes_{\mathrm{geo}}) \to \mathrm{Auteq}(\Gamma_B).$$

Clearly, the above construction can be done with $\mathbf{Sh}^\otimes_{\mathrm{geo}}$ instead of $\widehat{\mathbf{LS}}^\otimes_{\mathrm{geo}}$. Thus, we have a commutative diagram of group objects

$$\begin{tikzcd} \mathrm{Auteq}(\mathcal{B}) \arrow[r, "(1)"] \arrow[dr] & \mathrm{Auteq}(\mathbf{Sh}^\otimes_{\mathrm{geo}}) \arrow[d] \arrow[dr, "(2)"] & \\ & \mathrm{Auteq}(\widehat{\mathbf{LS}}^\otimes_{\mathrm{geo}}) \arrow[r] & \mathrm{Auteq}(\Gamma_{\mathrm{B}}), \end{tikzcd}$$

and it is easy to see that the composition of the maps (1) and (2) is the equivalence described in the first part. This finishes the proof of the lemma. □

Thus, to prove Theorem 4.37, it remains to show that the map in (4.73) admits a section. We need to provide a recipe for extending autoequivalences of $\widehat{\mathbf{LS}}^\otimes_{\mathrm{geo}}$ to autoequivalences of $\widehat{\mathbf{LS}}^{\Psi,\otimes}_{\mathrm{geo}}$. To do so, we need the following constructions and the accompanying Proposition 4.43.

Construction 4.40. Let $\mathcal{C}$ be an ∞-category and $\mathcal{M}^{\otimes}$ a $\mathrm{CAlg}(\mathrm{CAT}_\infty)$-valued presheaf on $\mathcal{C}$. Given a morphism $f : Y \to X$ in $\mathcal{C}$, we denote by $f^* : \mathcal{M}(X)^{\otimes} \to \mathcal{M}(Y)^{\otimes}$ the induced functor. We assume that the $\mathcal{M}(X)^{\otimes}$'s are closed, for all $X \in \mathcal{C}$. We want to construct a commutative triangle

$$\begin{tikzcd} \int_{\mathcal{C}^{\mathrm{op}}} (\mathcal{M}^{\mathrm{op}} \times \mathcal{M})^{\otimes} \arrow[rr, "\underline{\mathrm{Hom}}"] \arrow[dr] & & \int_{\mathcal{C}^{\mathrm{op}}} \mathcal{M}^{\otimes} \arrow[dl] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd} \tag{4.74}$$

whose fiber at $X \in \mathcal{C}^{\mathrm{op}}$ is given by the internal Hom bifunctor of $\mathcal{M}(X)^{\otimes}$. We argue as follows. The tensor product bifunctor can be considered as a morphism of $\mathrm{CAlg}(\mathrm{CAT}_\infty)$-valued presheaves

$$\mu : (\mathcal{M} \times \mathcal{M})^{\otimes} \to \mathcal{M}^{\otimes}.$$

Applying the endofunctor $\mathcal{P}$ of $\mathrm{CAlg}(\mathrm{CAT}_\infty)$ we deduce a commutative square of morphisms of $\mathrm{CAlg}(\mathrm{CAT}_\infty)$-valued presheaves

$$\begin{tikzcd} (\mathcal{M} \times \mathcal{M})^{\otimes} \arrow[r, "\mu"] \arrow[d, "\mathrm{y}"] & \mathcal{M}^{\otimes} \arrow[d, "\mathrm{y}"] \\ \mathcal{P}(\mathcal{M} \times \mathcal{M})^{\otimes} \arrow[r, "\mu^*"] & \mathcal{P}(\mathcal{M})^{\otimes}. \end{tikzcd}$$

By [Lur09a, Proposition 7.3.2.6], we have a relative right adjoint functor

$$\begin{tikzcd} \int_{\mathcal{C}^{\mathrm{op}}} \mathcal{P}(\mathcal{M})^{\otimes} \arrow[rr, "\mu_*"] \arrow[dr] & & \int_{\mathcal{C}^{\mathrm{op}}} \mathcal{P}(\mathcal{M} \times \mathcal{M})^{\otimes} \arrow[dl] \\ & \mathcal{C}^{\mathrm{op}}. & \end{tikzcd} \tag{4.75}$$

The natural equivalence of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves $\mathcal{P}(\mathcal{M}\times\mathcal{M})^{\otimes} \simeq \mathrm{Fun}(\mathcal{M}^{\mathrm{op}}, \mathcal{P}(\mathcal{M}))^{\otimes}$ induces an equivalence in $\mathrm{CAlg}(\mathrm{CAT}_{\infty/\mathcal{C}^{\mathrm{op}}})$:

$$\int_{\mathcal{C}^{\mathrm{op}}} \mathcal{P}(\mathcal{M} \times \mathcal{M})^{\otimes} \simeq \mathrm{Fun}_{\mathcal{C}^{\mathrm{op}}}\left(\int_{\mathcal{C}^{\mathrm{op}}} \mathcal{M}^{\mathrm{op}}, \int_{\mathcal{C}^{\mathrm{op}}} \mathcal{P}(\mathcal{M})\right)^{\otimes}.$$

By adjunction, the functor μ_* in (4.75) induces a commutative triangle

$$\begin{tikzcd} \int_{\mathcal{C}^{\mathrm{op}}} (\mathcal{M}^{\mathrm{op}} \times \mathcal{P}(\mathcal{M}))^{\otimes} \arrow[rr] \arrow[dr] & & \int_{\mathcal{C}^{\mathrm{op}}} \mathcal{P}(\mathcal{M})^{\otimes} \arrow[dl] \\ & \mathcal{C}^{\mathrm{op}}. & \end{tikzcd} \tag{4.76}$$

Since the $\mathcal{M}(X)^{\otimes}$'s are closed, the restriction of the above functor to $\int_{\mathcal{C}^{\mathrm{op}}} \mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ factors through $\int_{\mathcal{C}^{\mathrm{op}}} \mathcal{M}$ yielding the triangle in (4.74).

Construction 4.41. For $X \in \mathrm{Sm}/k$, recall that there is an equivalence of ∞-categories

$$\mathrm{D}_X : \mathbf{MSh}_{\mathrm{nis}}(X)^{\mathrm{op}}_{\omega} \xrightarrow{\sim} \mathbf{MSh}_{\mathrm{nis}}(X)_{\omega}$$

given by $\mathrm{D}_X(-) = \underline{\mathrm{Hom}}(-, \mathbf{1})$. Using Construction 4.40, we see that D_X underlies a right-lax monoidal functor which is moreover the fiber of X of a relative functor

$$\begin{tikzcd} \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathbf{MSh}^{\mathrm{op},\otimes}_{\mathrm{nis},\omega} \arrow[rr, "\mathrm{D}"] \arrow[dr, "q'"'] & & \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathbf{MSh}^{\otimes}_{\mathrm{nis},\omega} \arrow[dl, "q"] \\ & (\mathrm{Sm}/k)^{\mathrm{op}} & \end{tikzcd} \tag{4.77}$$

We stress that q' is classified by the functor

$$\mathbf{MSh}^{\mathrm{op},\otimes}_{\mathrm{nis},\omega} : (\mathrm{Sm}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{CAT}_\infty)$$

sending a morphism $f : Y \to X$ to the functor

$$f^* : (\mathbf{MSh}_{\mathrm{nis},\omega}(X))^{\mathrm{op},\otimes} \to (\mathbf{MSh}_{\mathrm{nis},\omega}(Y))^{\mathrm{op},\otimes}$$

deduced from the usual inverse image functor by applying the involution $(-)^{\mathrm{op}}$ of $\mathrm{CAlg}(\mathrm{CAT}_\infty)$. In particular, even when forgetting the monoidal structures, D is not a morphism of coCartesian fibrations and thus not an equivalence although the D_X's are. By indization, we deduce from (4.77) the following commutative triangle of ∞-categories:

$$\begin{tikzcd} \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathrm{Pro}(\mathbf{MSh}_{\mathrm{nis},\omega})^{\mathrm{op},\otimes} \arrow[rr, "\mathrm{D}"] \arrow[dr, "q'"'] & & \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathbf{MSh}^{\otimes}_{\mathrm{nis}} \arrow[dl, "q"] \\ & (\mathrm{Sm}/k)^{\mathrm{op}} & \end{tikzcd} \tag{4.78}$$

where the slanted arrows are coCartesian fibrations.

Remark 4.42. We give here another construction of the functor D in (4.78) which is more convenient for the proof of Proposition 4.43 below. This construction is based on the following observation : for a smooth morphism $f : Y \to X$, the functor D_X sends the motive $\mathrm{M}(Y)$ to $f_*\mathbf{1}$. Since it is a bit long, we split this construction in three steps.

Step 1. As in the proof of Proposition 2.2, we consider the ordinary category D whose objects are pairs $(\langle n\rangle, (f_i : Y_i \to X_i)_{1\le i\le n})$, where $n \ge 0$ is an integer and the f_i's are smooth morphisms between smooth k-varieties. (In the proof of Proposition 2.2 we allowed smooth morphisms between more general S-schemes, but otherwise the description given there applies.) We have obvious functors

$$s : D \to (\mathrm{Sm}/k)^{\mathrm{op},\amalg} \qquad \text{and} \qquad D \to (\mathrm{Sm}/k)^{\mathrm{op},\amalg}$$

sending the object $(\langle n\rangle, (f_i : Y_i \to X_i)_{1\le i\le n})$ to $(\langle n\rangle, (Y_i)_{1\le i\le n})$ and $(\langle n\rangle, (X_i)_{1\le i\le n})$ respectively. We also have a natural transformation $\phi : t \to s$ given at the previously considered object by $\mathrm{id}_{\langle n\rangle}$ and the f_i's. Next, we consider the coCartesian fibration $p : \Xi^\otimes \to (\mathrm{Sm}/S)^{\mathrm{op},\amalg}$ whose fiber at $(\langle n\rangle, (X_i)_{1\le i\le n})$ is the Cartesian product of the ∞-categories $\mathbf{MSh}_{\mathrm{nis}}(X_i)$'s. (See [DG20, Corollary A.12 & Remark A.13].) Pulling back along s and t, we obtain a commutative triangle

$$\begin{tikzcd} \Xi^\otimes_t \arrow[rr, "\phi^*"] \arrow[dr, "p_t"'] & & \Xi^\otimes_s \arrow[dl, "p_s"] \\ & D, & \end{tikzcd}$$

where p_s and p_t are coCartesian fibrations, and ϕ^* preserves coCartesian edges. Informally, over the previously considered object $(\langle n \rangle, (f_i : Y_i \to X_i)_{1 \le i \le n})$, ϕ^* is given by the Cartesian product of the inverse image functors $f_i^* : \mathbf{MSh}_{\mathrm{nis}}(X_i) \to \mathbf{MSh}_{\mathrm{nis}}(Y_i)$ and thus admits a right adjoint given by the product of the functors $f_{i,*}$. By [Lur17, Proposition 7.3.2.6], the functor ϕ^* admits a relative right adjoint ϕ_*. Writing $\mathbf{1}$ for the coCartesian section of p_s given by the monoidal units, we obtain a section $\phi_*\mathbf{1} : D \to \Xi_t^{\otimes}$ of p_t. Equivalently, we have constructed a commutative triangle

$$\begin{tikzcd} D \arrow[rr, "h"] \arrow[dr, "t"'] & & \Xi^{\otimes} \arrow[dl, "p"] \\ & (\mathrm{Sm}/k)^{\mathrm{op},\amalg}. & \end{tikzcd}$$

Taking the base change by the diagonal functor $d : \mathrm{Fin}_* \times (\mathrm{Sm}/k)^{\mathrm{op}} \to (\mathrm{Sch}/S)^{\mathrm{op},\amalg}$, we obtain a commutative triangle

$$\begin{tikzcd} \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} (\mathrm{Sm}/(-))^{\mathrm{op},\amalg} \arrow[rr, "\gamma"] \arrow[dr] & & \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathbf{MSh}_{\mathrm{nis}}^{\otimes} \arrow[dl] \\ & (\mathrm{Sm}/k)^{\mathrm{op}}. & \end{tikzcd} \tag{4.79}$$

The fiber of γ at $X \in \mathrm{Sm}/k$ is the right-lax monoidal functor $\gamma_X : (\mathrm{Sm}/X)^{\mathrm{op},\amalg} \to \mathbf{MSh}_{\mathrm{nis}}(X)^{\otimes}$, sending a smooth morphism $f : Y \to X$ to $f_*\mathbf{1}$.

Step 2. The functor γ in (4.79) does not preserve coCartesian edges. However, we may use γ to construct a morphism of coCartesian fibrations as follows. For $X \in \mathrm{Sm}/k$, we set

$$\Phi(X) = \mathrm{Sect}\left(\int_{((\mathrm{Sm}/k)_{/X})^{\mathrm{op}}} \mathbf{MSh}_{\mathrm{nis}}^{\otimes} \Big/ ((\mathrm{Sm}/k)_{/X})^{\mathrm{op}} \right). \tag{4.80}$$

We can turn the assignment $X \mapsto \Phi(X)$ into a $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaf $\Phi^{\otimes}$ using, for example, Corollary 3.115. There is symmetric monoidal fully faithful embedding $\delta_X : \mathbf{MSh}_{\mathrm{nis}}(X)^{\otimes} \hookrightarrow \Phi(X)^{\otimes}$ whose essential image is spanned by the coCartesian sections. This functor admits a right adjoint ϵ_X given by evaluation on X. It is easy to see that the δ_X's assemble into a morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves $\delta : \mathbf{MSh}_{\mathrm{nis}}^{\otimes} \to \Phi^{\otimes}$. Using [Lur17, Proposition 7.3.2.6], we have a relative right adjoint functor

$$\begin{tikzcd} \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \Phi^{\otimes} \arrow[rr, "\epsilon"] \arrow[dr] & & \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathbf{MSh}_{\mathrm{nis}}^{\otimes} \arrow[dl] \\ & (\mathrm{Sm}/k)^{\mathrm{op}}. & \end{tikzcd} \tag{4.81}$$

Moreover, for $X \in \mathrm{Sm}/k$, γ gives rise to a functor

$$\widetilde{\gamma}_X : (\mathrm{Sm}/X)^{\mathrm{op}} \to \Phi(X)$$

such that, for $Y \in \mathrm{Sm}/X$, $\widetilde{\gamma}_X(Y)$ is the section given by $\widetilde{\gamma}_X(Y)(X') = \gamma_{X'}(Y \times_X X')$. This functor is right-lax monoidal and defines a morphism of SMCAT_∞-valued presheaves

$$\widetilde{\gamma} : (\mathrm{Sm}/(-))^{\mathrm{op},\amalg} \to \Phi^{\otimes}. \tag{4.82}$$

One gets back γ by compositing $\widetilde{\gamma}$ with ϵ.

Step 3. (Here, we implicitly think of monoidal structures as Cartesian fibrations over Fin_* so that we can speak of left-lax monoidal functors.) Although the functors $\widetilde{\gamma}_X : (\mathrm{Sm}/X)^\times \to \Phi(X)^{\mathrm{op},\otimes}$ are only left-lax monoidal, they are strictly compatible with the module structure over $(\mathrm{Sm}/k)^\times$. Moreover, $\widetilde{\gamma}_X$ takes a Nisnevich square to a Cartesian square and the projections $\mathbb{A}^1_Y \to Y$ to equivalences. Using left Kan extension and the aforementioned properties, $\widetilde{\gamma}$ induces a morphism of Pr^{L}-valued presheaves

$$\widetilde{\gamma}' : \mathscr{P}_{\mathbb{A}^1,\mathrm{nis}}(\mathrm{Sm}/(-))^\times \to \mathrm{Pro}(\Phi)^{\mathrm{op},\otimes}.$$

The morphism $\widetilde{\gamma}'$ is naturally a morphism of modules over $\mathscr{P}_{\mathbb{A}^1,\mathrm{nis}}(\mathrm{Sm}/k)^{\mathrm{op},\amalg}$ which acts on the codomain via the composite functor

$$\mathscr{P}_{\mathbb{A}^1,\mathrm{nis}}(\mathrm{Sm}/k)^\times \to \mathbf{MSh}_{\mathrm{nis}}(k)^\otimes \xrightarrow{\mathrm{D}} \mathrm{Pro}(\mathbf{MSh}_{\mathrm{nis},\omega}(k))^{\mathrm{op},\otimes}.$$

In particular, the morphism $\widetilde{\gamma}'$ gives rise to a morphism of Pr^{L}-valued presheaves:

$$\widetilde{\gamma}'' : \mathbf{MSh}_{\mathrm{nis}}(-)^\otimes \to \mathrm{Pro}(\Phi)^{\mathrm{op},\otimes}.$$

It is easy to see that the $\widetilde{\gamma}''_X$, for $X \in \mathrm{Sm}/k$, takes compact motivic sheaves to essentially constant pro-objects in $\Phi(X)$. Thus, passing to compact objects, applying the involution $(-)^{\mathrm{op}}$ and then indization, we obtain a morphism of SMCAT_∞-valued presheaves on Sm/k:

$$\widetilde{\gamma}''' : \mathrm{Pro}(\mathbf{MSh}_{\mathrm{nis},\omega}(-))^\otimes \to \Phi^\otimes.$$

Passing to the associated coCartesian fibrations and compositing with the functor ϵ in (4.81), yields the sought-after commutative triangle in (4.79).

Proposition 4.43. *Consider the sequence of functors over* $(\mathrm{Sm}/k)^{\mathrm{op}}$*:*

$$\int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathrm{Pro}(\mathbf{MSh}_\omega)^{\mathrm{op},\otimes} \xrightarrow{\mathrm{D}} \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathbf{MSh}^\otimes \xrightarrow{\mathrm{B}^*} \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathbf{Sh}^\otimes_{\mathrm{geo}} \xrightarrow{(-)^{\mathrm{liss}}} \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \widehat{\mathbf{LS}}^\otimes_{\mathrm{geo}} \tag{4.83}$$

where D *is the functor obtained in Construction 4.41. Then, the composite functor*

$$\beta : \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathrm{Pro}(\mathbf{MSh}_\omega)^{\mathrm{op},\otimes} \to \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \widehat{\mathbf{LS}}^\otimes_{\mathrm{geo}} \tag{4.84}$$

admits an action of the spectral group $\mathbb{S}$*-stack* $\underline{\mathrm{Auteq}}(\mathbf{LS}^\otimes_{\mathrm{geo}})$*, which is trivial on the domain and which extends the tautological action on the codomain.*

Proof. We will give a direct construction of the functor β which makes it clear that it admits the required action of $\underline{\mathrm{Auteq}}(\mathbf{LS}^\otimes_{\mathrm{geo}})$. This construction is totally parallel to the one described in Remark 4.42; therefore, it will be also clear that the constructed β is equivalent to the composition of the horizontal arrows in (4.83).

Step 1. We use the notations introduced in the first step of the construction given in Remark 4.42. Consider the coCartesian fibration $p : \Xi'^\otimes \to (\mathrm{Sm}/S)^{\mathrm{op},\amalg}$ whose fiber at $(\langle n\rangle, (X_i)_{1\le i\le n})$ is the Cartesian product of the ∞-categories $\widehat{\mathbf{LS}}_{\mathrm{geo}}(X_i)$'s. Pulling back along s and t, we obtain a commutative triangle

$$\begin{tikzcd} \Xi'^\otimes_t \arrow[rr, "\phi^*"] \arrow[dr, "p_t"'] & & \Xi'^\otimes_s \arrow[dl, "p_s"] \\ & D, & \end{tikzcd}$$

where p_s and p_t are coCartesian fibrations, and ϕ^* preserves coCartesian edges. Informally, over the previously considered object $(\langle n\rangle, (f_i : Y_i \to X_i)_{1\leq i\leq n})$, ϕ^* is given by the Cartesian product of the inverse image functors $f_i^* : \widehat{\mathbf{LS}}_{\mathrm{geo}}(X_i) \to \widehat{\mathbf{LS}}_{\mathrm{geo}}(Y_i)$ and thus admits a right adjoint given by the product of the functors $(f_{i,*}(-))^{\mathrm{liss}}$. By [Lur17, Proposition 7.3.2.6], the functor ϕ^* admits a relative right adjoint ϕ_*. Writing $\mathbf{1}$ for the coCartesian section of p_s given by the monoidal units, we obtain a section $\phi_*\mathbf{1} : D \to \Xi_t'^{\otimes}$ of p_t. Equivalently, we have constructed a commutative triangle

$$\begin{tikzcd} D \arrow[rr, "h"] \arrow[dr, "t"'] & & \Xi'^{\otimes} \arrow[dl, "p"] \\ & (\mathrm{Sch}/S)^{\mathrm{op},\amalg}. & \end{tikzcd}$$

Taking the base change by the diagonal functor $d : \mathrm{Fin}_* \times (\mathrm{Sm}/k)^{\mathrm{op}} \to (\mathrm{Sch}/S)^{\mathrm{op},\amalg}$, we obtain a commutative triangle

$$\begin{tikzcd} \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} (\mathrm{Sm}/(-))^{\mathrm{op},\amalg} \arrow[rr, "\gamma"] \arrow[dr] & & \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \widehat{\mathbf{LS}}^{\otimes}_{\mathrm{geo}} \arrow[dl] \\ & (\mathrm{Sm}/k)^{\mathrm{op}}. & \end{tikzcd} \tag{4.85}$$

The fiber of γ at $X \in \mathrm{Sm}/k$ is the right-lax monoidal functor $\gamma_X : (\mathrm{Sm}/X)^{\mathrm{op},\amalg} \to \widehat{\mathbf{LS}}^{\otimes}_{\mathrm{geo}}(X)$, sending a smooth morphism $f : Y \to X$ to $(f_*^{\mathrm{an}}\mathbf{1})^{\mathrm{liss}}$. It is clear that $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})$ acts on γ, and this action is trivial on the domain and extends the tautological action on the codomain. It is also clear that the functor γ in (4.85) can be obtained from the one in (4.79) by composing with the last two functors in (4.83).

Step 2. The functor γ in (4.85) does not preserve coCartesian edges. However, we may use γ to construct a morphism of coCartesian fibrations as follows. For $X \in \mathrm{Sm}/k$, we set

$$\Phi'(X) = \mathrm{Sect}\left(\int_{((\mathrm{Sm}/k)_{/X})^{\mathrm{op}}} \widehat{\mathbf{LS}}^{\otimes}_{\mathrm{geo}} \Big/ ((\mathrm{Sm}/k)_{/X})^{\mathrm{op}}\right). \tag{4.86}$$

As in the second step of the construction described in Remark 4.42, we have a relative right adjoint

$$\begin{tikzcd} \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \Phi'^{\otimes} \arrow[rr, "\epsilon"] \arrow[dr] & & \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \widehat{\mathbf{LS}}^{\otimes}_{\mathrm{geo}} \arrow[dl] \\ & (\mathrm{Sm}/k)^{\mathrm{op}} & \end{tikzcd} \tag{4.87}$$

and a morphism of SMCAT_∞-valued presheaves

$$\widetilde{\gamma} : (\mathrm{Sm}/(-))^{\mathrm{op},\amalg} \to \Phi'^{\otimes}. \tag{4.88}$$

One gets back γ by compositing $\widetilde{\gamma}$ with ϵ. Also, we note that $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})$ acts on $\widetilde{\gamma}$, and this action is trivial on the domain and extends the tautological action on the codomain. Moreover, we have a right-lax monoidal functor $\Phi^{\otimes} \to \Phi'^{\otimes}$ given by $(-)^{\mathrm{liss}} \circ \mathrm{B}^*$ relating the diagram in (4.87) with the one in (4.81) and the morphism in (4.88) with the one in (4.82).

Step 3. As in the third step of the construction described in Remark 4.42, we may use the morphism $\widetilde{\gamma}$ in (4.88) to produce a morphism of SMCAT_∞-valued presheaves

$$\widetilde{\gamma}' : \mathscr{P}_{\mathbb{A}^1,\mathrm{nis}}(\mathrm{Sm}/(-))^{\mathrm{op},\amalg} \to \mathrm{Pro}(\Phi')^{\otimes}.$$

The morphism $\widetilde{\gamma}'$ is naturally a morphism of modules over $\mathscr{P}_{\mathbb{A}^1,\mathrm{nis}}(\mathrm{Sm}/k)^{\mathrm{op},\amalg}$ which acts on the codomain via the composition of

$$\mathscr{P}_{\mathbb{A}^1,\mathrm{nis}}(\mathrm{Sm}/k)^{\mathrm{op},\amalg} \to \mathbf{MSh}_{\mathrm{nis}}(k)^{\mathrm{op},\otimes} \to \mathrm{Pro}(\mathcal{S}p^{\mathrm{op}})^{\otimes}.$$

In particular, the morphism $\widetilde{\gamma}'$ gives rise to a morphism SMCAT_∞-valued presheaves:

$$\widetilde{\gamma}'' : \mathbf{MSh}_{\mathrm{nis}}(-)^{\mathrm{op},\otimes} \to \mathrm{Pro}(\Phi')^{\otimes}.$$

Passing to compact objects, applying the involution $(-)^{\mathrm{op}}$ and then indization, we obtain a morphism of SMCAT_∞-valued presheaves on Sm/k:

$$\widetilde{\gamma}''' : \mathrm{Pro}(\mathbf{MSh}_{\mathrm{nis},\omega}(-))^{\otimes} \to \Phi'^{\otimes}.$$

Passing to the associated coCartesian fibrations and compositing with the functor ϵ in (4.87), yields a commutative triangle

$$\begin{tikzcd} \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathrm{Pro}(\mathbf{MSh}_{\mathrm{nis},\omega})^{\mathrm{op},\otimes} \arrow[rr, "\beta"] \arrow[dr, "q'"'] & & \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \widehat{\mathbf{LS}}^{\otimes}_{\mathrm{geo}} \arrow[dl] \\ & (\mathrm{Sm}/k)^{\mathrm{op}} & \end{tikzcd} \tag{4.89}$$

Clearly, $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})$ acts on β as required, and can be recovered from the functor D in (4.78) by composition with the last two functors in (4.83). ◻

Next, we need a nontrivial property of the algebras $\mathfrak{P}^{\mathbf{MSh}}_{X,C_-,C_+}$ introduced and studied in Subsection 4.3; see Construction 4.21. Roughly speaking, we need to lift the section $\mathfrak{P}^{\mathbf{MSh}}$ along the functor D of Construction 4.41. For simplicity, we write $\mathfrak{P}$ instead of $\mathfrak{P}^{\mathbf{MSh}}$.

Proposition 4.44. *There is a section*

$$\mathfrak{Q} : (\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}} \to \int_{(\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}}} \mathrm{CAlg}(\mathrm{Pro}(\mathbf{MSh}_{\mathrm{nis},\omega})^{\mathrm{op}}) \circ g \tag{4.90}$$

and an equivalence $\mathrm{D}(\mathfrak{Q}) \simeq \mathfrak{P}$.

Proof. To simplify notations, we set $\mathcal{H} = \mathbf{MSh}_{\mathrm{nis}}$. We denote by $\mathcal{K}^{\otimes} : (\mathrm{Sm}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ the functor given by

$$\mathcal{K}^{\otimes}(X) = \mathrm{Pro}(\mathbf{MSh}_{\mathrm{nis}}(X)_{\omega})^{\mathrm{op},\otimes}.$$

For $X \in \mathrm{Sm}\text{-}\Sigma/k$, we define a full sub-$\infty$-category $\mathcal{K}^{\otimes}_{\mathrm{ict\text{-}tm}}(X) \subset \mathcal{K}^{\otimes}(X)$ by

$$\mathcal{K}_{\mathrm{ict\text{-}tm}}(X) = \mathrm{Pro}(\mathbf{MSh}_{\mathrm{nis},\,\mathrm{ct\text{-}tm}}(X))^{\mathrm{op},\otimes}.$$

This defines a sub-functor $\mathcal{K}^{\otimes}_{\mathrm{ict\text{-}tm}} \subset \mathcal{K}^{\otimes}|_{\mathrm{Sm}\text{-}\Sigma/k}$. Since duality preserves constructible tame motivic sheaves, we deduce that the functor

$$\mathrm{D} : \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathcal{K}^{\otimes} \to \int_{(\mathrm{Sm}/k)^{\mathrm{op}}} \mathcal{H}^{\otimes}$$

induces a functor

$$\mathrm{D} : \int_{(\mathrm{Sm}\text{-}\Sigma/k)^{\mathrm{op}}} \mathcal{K}^{\otimes}_{\text{ict-tm}} \to \int_{(\mathrm{Sm}\text{-}\Sigma/k)^{\mathrm{op}}} \mathcal{H}^{\otimes}_{\text{ict-tm}}.$$

We claim that there is a commutative square

$$\begin{tikzcd} \int_{(\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}}} \mathcal{K}^{\otimes}_{\text{ict-tm}} \circ g \arrow[r, "\varphi^*"] \arrow[d, "\mathrm{D}"] & \int_{(\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}}} \widetilde{\mho}^{\otimes}_{\mathcal{K}} \arrow[d, "\mathrm{D}"] \\ \int_{(\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}}} \mathcal{H}^{\otimes}_{\text{ict-tm}} \circ g \arrow[r, "\varphi^*"] & \int_{(\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}}} \widetilde{\mho}^{\otimes}_{\mathcal{H}} \end{tikzcd} \tag{4.91}$$

which is moreover right adjointable. Obviously, this is enough to conclude.

To construct the square in (4.91), we argue as in Construction 4.41. For $(X, C_-, C_+) \in \mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k$, let $\widetilde{\mho}_{\mathcal{H},\omega}(X, C_-, C_+)$ be the full sub-∞-category of $\widetilde{\mho}_{\mathcal{H}}(X, C_-, C_+)$ spanned by those sections taking values in dualizable motivic sheaves (instead of ind-dualizable ones). The functors φ^*_{X,C_-,C_+} : $\mathcal{H}_{\text{ct-tm}}(X_{C_-}) \to \widetilde{\mho}_{\mathcal{H},\omega}(X, C_-, C_+)$ induce a natural transformation $\mathcal{H}^{\otimes}_{\text{ct-tm}} \to \widetilde{\mho}^{\otimes}_{\mathcal{H},\omega}$ which we may consider as a $\mathrm{CAlg}(\mathrm{CAT}_\infty)$-valued presheaf on $\Delta^1 \times \mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k$. We apply Construction 4.40 to this presheaf, and then take indization. In particular, we have

$$\widetilde{\mho}^{\otimes}_{\mathcal{K}} = \mathrm{Pro}(\widetilde{\mho}_{\mathcal{H},\omega})^{\mathrm{op},\otimes}.$$

The right vertical arrow of the square in (4.91) is induced, over (X, C_-, C_+), by the functor

$$\mathrm{D}_X = \underline{\mathrm{Hom}}(-, \mathbf{1}) : (\widetilde{\mho}_{\mathcal{H},\omega})^{\mathrm{op}} \to \widetilde{\mho}_{\mathcal{H},\omega}$$

which can be shown to be an equivalence. In fact, for a fixed (X, C_-, C_+), we have a commutative square

$$\begin{tikzcd} \mathcal{K}_{\text{ict-tm}}(X_{C_-}) \arrow[r, "\varphi^*_{X,C_-,C_+}"] \arrow[d, "\mathrm{D}"] & \widetilde{\mho}_{\mathcal{K}}(X, C_-, C_+) \arrow[d, "\mathrm{D}"] \\ \mathcal{H}_{\text{ict-tm}}(X_{C_-}) \arrow[r, "\varphi^*_{X,C_-,C_+}"] & \widetilde{\mho}_{\mathcal{H}}(X, C_-, C_+) \end{tikzcd}$$

where the vertical functors are equivalences. This shows the right adjointability of the big square in (4.91) and finishes the proof. □

We now have all the ingredients to finish the proof of Theorem 4.37.

Proof of Theorem 4.37. Recall that it remains to construct a section of the morphism in (4.73). Combining Propositions 4.43 and 4.44, we deduce that the section $(\mathrm{B}^*\mathfrak{P})^{\mathrm{liss}}$ is naturally fixed by the action of $\mathrm{Auteq}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})$, i.e., factors into a section

$$\mathfrak{P}' : (\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}} \to \int_{(\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}}} \mathrm{CAlg}(\widehat{\mathbf{LS}}^{\otimes}_{\mathrm{geo}})^{\mathrm{Auteq}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})} \circ g.$$

This shows that there is an action of $\mathrm{Auteq}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})$ on the functor

$$\widehat{\mathbf{LS}}_{\mathrm{geo}}(g(-); \mathfrak{P}')^{\otimes} : (\mathrm{Sm}\text{-}\Sigma^{\mathrm{dm}}/k)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

extending the action on $\widehat{\mathbf{LS}}^{\otimes}_{\mathrm{geo}}$. (Indeed, note that the latter is equivalent to the restriction of the former along the functor sending a connected smooth k-variety X to the triple (X, X, X).) By Corollaries 4.33 and 4.35, the cdh-sheafification of the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$-valued presheaf $\mathrm{Mod}_{\mathfrak{P}'}(\widehat{\mathbf{LS}}_{\mathrm{geo}} \circ g)^{\otimes}$ is equivalent to $\widehat{\mathbf{LS}}^{\Psi,\otimes}_{\mathrm{geo}}$. This shows that $\mathrm{Auteq}(\mathbf{LS}^{\otimes}_{\mathrm{geo}})$ acts also on $\widehat{\mathbf{LS}}^{\Psi,\otimes}_{\mathrm{geo}}$, and that this action extends to tautological one as needed. $\square$

We end this section with a complement concerning the classical motivic Galois group.

Notation 4.45. We denote by $\mathrm{Sm}^{\mathrm{art}}/k \subset \mathrm{Sm}/k$ the full subcategory consisting of those k-varieties whose base change to an algebraic closure of k is a disjoint union of Artin neighbourhoods (see Definition 1.78).

Definition 4.46. Let k be a field and $\sigma : k \hookrightarrow \mathbb{C}$ a complex embedding. We define the (noncommutative) Picard $\mathbb{Z}$-prestack $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\heartsuit,\otimes}_{\mathrm{geo}})$ to be the presheaf of Picard groupoids on AFF sending $\mathrm{Spec}(\Lambda)$, with Λ an ordinary commutative ring, to the Picard groupoid of autoequivalences of the functor $\mathbf{LS}_{\mathrm{geo}}(-;\Lambda)^{\heartsuit,\otimes}$ from $(\mathrm{Sm}^{\mathrm{art}}/k)^{\mathrm{op}}$ to the 2-category of ordinary Λ-linear symmetric monoidal categories. If we want to stress that this depends on the complex embedding σ, we will write $\underline{\mathrm{Auteq}}(\mathbf{LS}^{\heartsuit,\otimes}_{\sigma\text{-geo}})$ instead.

Corollary 4.47. *Let k be a field and $\sigma : k \hookrightarrow \mathbb{C}$ a complex embedding. There is an equivalence of classical Picard $\mathbb{Z}$-prestacks*

$$\mathcal{G}^{\mathrm{cl}}_{\mathrm{mot}}(k,\sigma) \xrightarrow{\sim} \underline{\mathrm{Auteq}}(\mathbf{LS}^{\heartsuit,\otimes}_{\sigma\text{-geo}}).$$

In particular, the right hand side is an affine group scheme.

Proof. We deduce this from Theorem 4.37 as we deduced Corollary 2.13 from Theorem 2.10; instead of using Nori's theorem (i.e., Theorem 1.107) we use Beĭlinson's theorem (i.e., Theorem 1.80). Indeed, note that in Theorem 4.37, one may replace the Zariski hypersheaf $\widehat{\mathbf{LS}}^{\otimes}_{\mathrm{geo}}$ by its restriction to $\mathrm{Sm}^{\mathrm{art}}/k$ since every smooth k-variety admits a hypercover by objects in $\mathrm{Sm}^{\mathrm{art}}/k$. $\square$

By base change to positive characteristic rings, one obtains the following particular case of Corollary 4.47.

Corollary 4.48. *Let k be a field of charactersitic zero, $\overline{k}/k$ an algebraic closure of k and Λ a torsion connected ring. Consider the functor $\overline{\mathcal{E}}(-;\Lambda) : (\mathrm{Sm}^{\mathrm{art}}/k)^{\mathrm{op}} \to \mathrm{CAT}_{\mathrm{ord}}$ sending a k-variety X in $\mathrm{Sm}^{\mathrm{art}}/k$ to the ordinary category of étale locally constant sheaves on $X \otimes_k \overline{k}$ with coefficients in Λ. Then, there is an equivalence of Picard groupoids*

$$\mathcal{G}(\overline{k}/k) \simeq \mathrm{Auteq}_{\mathrm{Psh}(\mathrm{Sm}^{\mathrm{art}}/k;\mathrm{CAlg}(\mathrm{CAT}_{\mathrm{ord}}))}(\overline{\mathcal{E}}(-;\Lambda)^{\otimes}).$$

In particular, the right hand side is discrete.

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