

# A REMARKABLE $\sigma$ -FINITE MEASURE ASSOCIATED WITH LAST PASSAGE TIMES AND PENALISATION PROBLEMS

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*Dedicated to Eckhard Platen for his 60th birthday*

ABSTRACT. In this paper, we give a global view of the results we have obtained in relation with a remarkable class of submartingales, called  $(\Sigma)$ , and which are stated in [12], [9], [11] and [10]. More precisely, we associate to a given submartingale in this class  $(\Sigma)$ , defined on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ , satisfying some technical conditions, a  $\sigma$ -finite measure  $\mathcal{Q}$  on  $(\Omega, \mathcal{F})$ , such that for all  $t \geq 0$ , and for all events  $\Lambda_t \in \mathcal{F}_t$ :

$$\mathcal{Q}[\Lambda_t, g \leq t] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\Lambda_t} X_t]$$

where  $g$  is the last hitting time of zero of the process  $X$ . This measure  $\mathcal{Q}$  has already been defined in several particular cases, some of them are involved in the study of Brownian penalisation, and others are related with problems in mathematical finance. More precisely, the existence of  $\mathcal{Q}$  in the general case solves a problem stated by D. Madan, B. Roynette and M. Yor, in a paper studying the link between Black-Scholes formula and last passage times of certain submartingales. Once the measure  $\mathcal{Q}$  is constructed, we define a family of nonnegative martingales, corresponding to the local densities (with respect to  $\mathbb{P}$ ) of the finite measures which are absolutely continuous with respect to  $\mathcal{Q}$ . We study in detail the relation between  $\mathcal{Q}$  and this class of martingales, and we deduce a decomposition of any nonnegative martingale into three parts, corresponding to the decomposition of finite measures on  $(\Omega, \mathcal{F})$  as the sum of three measures, such that the first one is absolutely continuous with respect to  $\mathbb{P}$ , the second one is absolutely continuous with respect to  $\mathcal{Q}$  and the third one is singular with respect to  $\mathbb{P}$  and  $\mathcal{Q}$ . This decomposition can be generalized to supermartingales. Moreover, if under  $\mathbb{P}$ , the process  $(X_t)_{t \geq 0}$  is a diffusion satisfying some technical conditions, one can state a penalisation result involving the measure  $\mathcal{Q}$ , and generalizing a theorem given in [13]. Now, in the construction of the measure  $\mathcal{Q}$ , we encounter the following problem: if  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  is a filtered probability space satisfying the usual assumptions, then it is usually not possible to extend to  $\mathcal{F}_{\infty}$  (the  $\sigma$ -algebra generated by  $(\mathcal{F}_t)_{t \geq 0}$ ) a coherent family of probability measures  $(\mathbb{Q}_t)$  indexed by  $t \geq 0$ , each of them being defined on  $\mathcal{F}_t$ . That is why we must not assume the usual assumptions in our case. On the other hand, the usual assumptions are crucial in order to obtain the existence of regular versions of paths (typically adapted and continuous or adapted and càdlàg versions) for most stochastic processes of interest, such as the local time of the standard Brownian motion, stochastic integrals, etc. In order to fix this problem, we introduce another augmentation of filtrations, intermediate between the right continuity and the usual conditions, and call it N-augmentation in this paper. This augmentation has also been considered by Bichteler [3]. Most of the important results of the theory of stochastic processes which are generally proved under the usual augmentation still hold under the N-augmentation; moreover this new augmentation allows the extension of a coherent family of probability measures whenever this is possible with the original filtration.

## NOTATION

In this paper,  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  will denote a filtered probability space.  $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$  is the space of continuous functions from  $\mathbb{R}_+$  to  $\mathbb{R}$ .  $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$  is the space of càdlàg functions from  $\mathbb{R}_+$  to  $\mathbb{R}$ . If  $Y$  is a random variable, we denote indifferently by  $\mathbb{P}[Y]$  or by  $\mathbb{E}_{\mathbb{P}}[Y]$  the expectation of  $Y$  with respect to  $\mathbb{P}$ .

### 1. INTRODUCTION

This paper reviews some recent results obtained by Cheridito, Nikeghbali and Platen [4] and by the authors of this paper [12, 9, 11, 10] on the last zero of some remarkable stochastic processes and its relation with a universal  $\sigma$ -finite measure which has many remarkable properties and which seems to be in many places the key object to interpret in a unified way results which do not seem to be related at first sight. The last zero of càdlàg and adapted processes will play an essential role in our discussions: this fact is quite surprising since such a random time is not a stopping time and hence falls outside the domain of applications of the classical theorems in stochastic analysis.

The problem originally came from a paper of Madan, Roynette and Yor [8] on the pricing of European put options where they are able to represent the price of a European put option in terms of the probability distribution of some last passage time. More precisely, they prove that if  $(M_t)_{t \geq 0}$  is a continuous nonnegative local martingale defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfying the usual assumptions, and such that  $\lim_{t \rightarrow \infty} M_t = 0$ , then

$$(K - M_t)^+ = K \mathbb{P}(g_K \leq t | \mathcal{F}_t) \tag{1.1}$$

where  $K \geq 0$  is a constant and  $g_K = \sup\{t \geq 0 : M_t = K\}$ . Formula (1.1) tells that it is enough to know the terminal value of the submartingale  $(K - M_t)^+$  and its last zero  $g_K$  to reconstruct it. Yet a nicer interpretation of (1.1) is suggested in [8]: there exists a measure  $\mathcal{Q}$ , a random time  $g$ , such that the submartingale  $X_t = (K - M_t)^+$  satisfies

$$\mathcal{Q}[F_t \mathbf{1}_{g \leq t}] = \mathbb{E}[F_t X_t], \tag{1.2}$$

for any  $t \geq 0$  and for any bounded  $\mathcal{F}_t$ -measurable random variable  $F_t$ . Indeed, it easily follows from (1.1) that, in this case,  $\mathcal{Q} = K \cdot \mathbb{P}$  and  $g = g_K$ . It is also clear that if a stochastic process  $X$  satisfies (1.2), then it is a submartingale. The problem of finding the class of submartingales which satisfy (1.2) is posed in [8]:

**Problem 1** ([8]): for which nonnegative submartingales  $X$  can we find a  $\sigma$ -finite measure  $\mathcal{Q}$  and the end of an optional set  $g$  such that

$$\mathcal{Q}[F_t \mathbf{1}_{g \leq t}] = \mathbb{E}[F_t X_t]? \tag{1.3}$$

It is also noticed in [8] that other instances of formula (1.2) have already been discovered: for example, in [2], Azéma and Yor proved that for any continuous and uniformly integrable martingale  $M$ , (1.3) holds for  $X_t = |M_t|$ ,  $\mathcal{Q} = |M_\infty| \cdot \mathbb{P}$  and  $g = \sup\{t \geq 0 : M_t = 0\}$ , or equivalently

$$|M_t| = \mathbb{E}[|M_\infty| \mathbf{1}_{g \leq t} | \mathcal{F}_t].$$

Here again the measure  $\mathcal{Q}$  is finite. Recently, another particular case where the measure  $\mathcal{Q}$  is not finite was obtained by Najnudel, Roynette and Yor in their study of Brownian penalisation (see [13]). They prove the existence of the measure  $\mathcal{Q}$  when  $X_t = |Y_t|$  is the

absolute value of the canonical process  $(Y_t)_{t \geq 0}$  under the Wiener measure  $\mathbb{W}$ , on the space  $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$  equipped with the filtration generated by the canonical process. In this case, the measure  $\mathcal{Q}$ , which we shall denote hereafter  $\mathcal{W}$  to remind we are working on the Wiener space, is not finite but  $\sigma$ -finite and is singular with respect to the Wiener measure: it satisfies  $\mathcal{W}(g = \infty) = 0$ , where  $g = \sup\{t \geq 0 : X_t = 0\}$ . However, a closer look at this last example reveals that Problem 1 may lead to some paradox. Indeed, the existence of the measure  $\mathcal{Q}$  implies that the filtration  $(\mathcal{F}_t)_{t \geq 0}$  should not satisfy the usual assumptions. Indeed, if this were the case, then for all  $t \geq 0$ , the event  $\{g > t\}$  would have probability one (under  $\mathbb{W}$ ) and then, would be in  $\mathcal{F}_0$  and, a fortiori, in  $\mathcal{F}_t$ . If one assumes that  $\mathcal{W}$  exists, this implies:

$$\mathcal{W}[g > t, g \leq t] = \mathbb{E}_{\mathbb{W}}[\mathbf{1}_{g > t} X_t],$$

and then

$$\mathbb{E}_{\mathbb{W}}[X_t] = 0,$$

which is absurd. But now, if one does not complete the original probability space, then it is possible to show (see Section 2) that there does not exist a continuous and adapted version of the local time for the Wiener process  $Y$ , which is one of the key processes in the study of Brownian penalisations! More generally, one cannot apply most of the useful results from the general theory of stochastic processes such as the existence of càdlàg versions for martingales, the Doob-Meyer decomposition, the début theorem, etc. Consequently, one has to provide conditions on the underlying filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  under which not only equation (1.3) in Problem 1 can hold, but also under which the existence of regular versions (e.g. continuous and adapted version for the Brownian local time) of stochastic processes of interest do exist:

**Problem 2** What are the natural conditions to impose to  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  in order to have (1.3) and at the same time the existence of regular versions for stochastic processes of interest?

In fact we shall see that the existence of the measure  $\mathcal{Q}$  is related to the problem of extension of a coherent family of probability measures: given a coherent family of probability measures  $(\mathbb{Q}_t)_{t \geq 0}$  with  $\mathbb{Q}_t$  defined on  $\mathcal{F}_t$  (i.e.: the restriction of  $\mathbb{Q}_t$  to  $\mathcal{F}_s$  is  $\mathbb{Q}_s$  for  $t \geq s$ ), does there exist a probability measure  $\mathbb{Q}_\infty$  defined on  $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$  such that the restriction of  $\mathbb{Q}_\infty$  to  $\mathcal{F}_t$  is  $\mathbb{Q}_t$ ? The solution to such extension problems is well-known and very well detailed in the book by Parthasarathy [15], however, it is surprising that such a fundamental problem (if one thinks about the widely used changes of probability measures which are only locally absolutely continuous with respect to a reference probability measure) has rarely been considered together with the existence of càdlàg versions for martingales, the Doob-Meyer decomposition, the début theorem, etc.

In Section 2, we shall propose an alternative augmentation of filtrations, intermediate between the right-continuous version and the usual augmentation, under which most of the properties generally proved under usual conditions (existence of càdlàg versions, existence of Doob-Meyer decomposition, etc.) are preserved, but for which it is still possible to extend compatible families of probability measures. We proposed this augmentation in [9] and discovered later than Bichteler has also proposed it in his book [3]. By using this augmentation, we are able to prove the existence of the measure  $\mathcal{Q}$  under very general assumptions. The

relevant class of submartingales is called  $(\Sigma)$ , and the precise conditions under which we can show that  $\mathcal{Q}$  exists are stated in Section 3. Just before this statement, we give a detailed solution of Problem 1 in the particular case where  $\mathcal{Q}$  is absolutely continuous with respect to  $\mathbb{P}$ , with some applications to financial modeling.

The measure  $\mathcal{Q}$  has some remarkable properties which we shall detail in Section 4. One of its most striking properties is that it allows a unified treatment of many problems of penalisation on the Wiener space which do not seem to be related at first sight. The framework which is generally used is the following (see the book [13] for more details and references): we consider  $\mathbb{W}$ , the Wiener measure on  $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ , endowed with its natural filtration  $(\mathcal{F}_s)_{s \geq 0}$ ,  $(\Gamma_t)_{t \geq 0}$  a family of nonnegative random variables on the same space, such that

$$0 < \mathbb{W}[\Gamma_t] < \infty,$$

and for  $t \geq 0$ , the probability measure

$$\mathbb{Q}_t := \frac{\Gamma_t}{\mathbb{W}[\Gamma_t]} \cdot \mathbb{W}.$$

Under these assumptions, Roynette, Vallois and Yor have proven (see [16]) that for many examples of family of functionals  $(\Gamma_t)_{t \geq 0}$ , there exists a probability measure  $\mathbb{Q}_\infty$  which can be considered as the weak limit of  $(\mathbb{Q}_t)_{t \geq 0}$  when  $t$  goes to infinity, in the following sense: for all  $s \geq 0$  and for all bounded,  $\mathcal{F}_s$ -measurable random variables  $F_s$ , one has

$$\mathbb{Q}_t[F_s] \xrightarrow[t \rightarrow \infty]{} \mathbb{Q}_\infty[F_s].$$

For example, the measure  $\mathbb{Q}_\infty$  exists for the following families of functionals<sup>1</sup>  $(\Gamma_t)_{t \geq 0}$ :

- $\Gamma_t = \phi(L_t)$ , where  $(L_t)_{t \geq 0}$  is the local time at zero of the canonical process  $X$ , and  $\phi$  is a nonnegative, integrable function from  $\mathbb{R}_+$  to  $\mathbb{R}_+$ .
- $\Gamma_t = \phi(S_t)$ , where  $S_t$  is the supremum of  $X$  on the interval  $[0, t]$ , and  $\phi$  is, again, a nonnegative, integrable function from  $\mathbb{R}_+$  to  $\mathbb{R}_+$ .
- $\Gamma_t = e^{\lambda L_t + \mu |X_t|}$ , where  $(L_t)_{t \geq 0}$  is, again, the local time at zero of  $X$ .

In [13], Najnudel, Roynette and Yor obtain a result which gives the existence of  $\mathbb{Q}_\infty$  for a large class of families of functionals  $(\Gamma_t)_{t \geq 0}$ . The proof of this penalisation result involves, in an essential way, the  $\sigma$ -finite measure  $\mathcal{W}$  on the space  $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$  described above. More precisely, they prove that for a relatively large class of functionals  $(\Gamma_t)_{t \geq 0}$ ,

$$\mathbb{Q}_\infty = \frac{\Gamma_\infty}{\mathcal{W}[\Gamma_\infty]} \cdot \mathcal{W}, \tag{1.4}$$

where  $\Gamma_\infty$  is the limit of  $\Gamma_t$  when  $t$  goes to infinity, which is supposed to exist everywhere.

**Problem 3** Can we use our general existence theorem on the measure  $\mathcal{Q}$  in Section 3 to extend the general result on the Brownian penalisation problem and  $\mathcal{W}$  to a larger class of stochastic processes?

At end of Section 4, we shall see that the answer to Problem 3 is positive (which extends (1.4)), if under  $\mathbb{P}$ , the submartingale  $(X_t)_{t \geq 0}$  is a diffusion satisfying some technical

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<sup>1</sup>The discussion around Problem 2 shows that the results described below are not correct because a continuous and adapted version of the Brownian local time does not exist with the natural filtration. The conditions given in Section 2 will remedy this gap.

conditions. In particular, our result can be applied to suitable powers of Bessel processes if the dimension is in the interval  $(0, 2)$  (which includes the case of the reflected Brownian motion). Unlike all our other results, for which we are able to get rid of the Markov and scaling properties that have been used so far in the Brownian studies, the Markov property plays here a crucial role.

## 2. A NEW KIND OF AUGMENTATION OF FILTRATIONS CONSISTENT WITH THE PROBLEM OF EXTENSION OF MEASURES

The discussion in this section follows closely our paper [9]; in particular the proofs which are not provided can be found there.

**2.1. Understanding the problem.** In stochastic analysis, most of the interesting properties of continuous time random processes cannot be established if one does not assume that their trajectories satisfy some regularity conditions. For example, a nonnegative càdlàg martingale converges almost surely, but if the càdlàg assumption is removed, the result becomes false in general. Recall a very simple counter-example: on the filtered probability space  $((\mathcal{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{W})$ , where  $\mathcal{F}_t = \sigma\{X_s, 0 \leq s \leq t\}$ ,  $\mathcal{F} = \sigma\{X_s, s \geq 0\}$ ,  $(X_s)_{s \geq 0}$  is the canonical process and  $\mathbb{W}$  the Wiener measure, the martingale

$$(M_t := \mathbf{1}_{X_t=1})_{t \geq 0},$$

which is a.s. equal to zero for each fixed  $t \geq 0$ , does not converge at infinity. That is the reason why one generally considers a càdlàg version of a martingale. However there are fundamental examples of stochastic processes for which such a version does not exist. Indeed let us define on the filtered probability space  $((\mathcal{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{W})$  described above, the stochastic process  $(\mathcal{L}_t)_{t \geq 0}$  as follows:

$$\mathcal{L}_t = \Phi \left( \liminf_{m \rightarrow \infty} \int_0^t f_m(X_s) ds \right),$$

where  $f_m$  denotes the density of a centered Gaussian variable with variance  $1/m$  and  $\Phi$  is the function from  $\mathbb{R}_+ \cup \{\infty\}$  to  $\mathbb{R}_+$  such that  $\Phi(x) = x$  for  $x < \infty$  and  $\Phi(\infty) = 0$ . The process  $(\mathcal{L}_t)_{t \geq 0}$  is a version of the local time of the canonical process at level zero, which is defined everywhere and  $(\mathcal{F}_t)_{t \geq 0}$ -adapted. It is known that the process:

$$(M_t := |X_t| - \mathcal{L}_t)_{t \geq 0}$$

is an  $(\mathcal{F}_t)_{t \geq 0}$ -martingale. However,  $(M_t)_{t \geq 0}$  does not admit a càdlàg version which is adapted. In other words, there exists no càdlàg, adapted version  $(L_t)_{t \geq 0}$  for the local time at level zero of the canonical process! This property can be proved in the following way: let us consider an Ornstein-Uhlenbeck process  $(U_t)_{t \geq 0}$ , starting from zero, and let us define the process  $(V_t)_{t \geq 0}$  by:

$$V_t = (1 - t)U_{t/(1-t)},$$

for  $t < 1$ , and

$$V_t = 0$$

for  $t \geq 1$ . This process is a.s. continuous: we denote by  $\mathbb{Q}$  its distribution. One can check the following properties:

- For all  $t \in [0, 1)$ , the restriction of  $\mathbb{Q}$  to  $\mathcal{F}_t$  is absolutely continuous with respect to the corresponding restriction of  $\mathbb{W}$ .

- Under  $\mathbb{Q}$ ,  $\mathcal{L}_t \rightarrow \infty$  a.s. when  $t \rightarrow 1$ ,  $t < 1$ .

By the second property, the set  $\{\mathcal{L}_t \xrightarrow[t \rightarrow 1, t < 1]{} \infty\}$  has probability one under  $\mathbb{Q}$ . Since it is negligible under  $\mathbb{P}$ , it is essential to suppose that it is not contained in  $\mathcal{F}_0$ , if we need to have the first property: the filtration must not be completed. The two properties above imply

$$\begin{aligned} \mathbb{Q} \left[ L_{1-2^{-n}} \xrightarrow[n \rightarrow \infty]{} \infty \right] &\geq \mathbb{Q} \left[ \mathcal{L}_{1-2^{-n}} \xrightarrow[n \rightarrow \infty]{} \infty, \forall n \in \mathbb{N}, L_{1-2^{-n}} = \mathcal{L}_{1-2^{-n}} \right] \\ &\geq 1 - \sum_{n \in \mathbb{N}} \mathbb{Q} [L_{1-2^{-n}} \neq \mathcal{L}_{1-2^{-n}}] = 1. \end{aligned}$$

The last equality is due to the fact that for all  $n \in \mathbb{N}$ ,

$$\mathbb{W} [L_{1-2^{-n}} \neq \mathcal{L}_{1-2^{-n}}] = 0,$$

and then

$$\mathbb{Q} [L_{1-2^{-n}} \neq \mathcal{L}_{1-2^{-n}}] = 0,$$

since  $L_{1-2^{-n}}$  and  $\mathcal{L}_{1-2^{-n}}$  are  $\mathcal{F}_{1-2^{-n}}$ -measurable and since the restriction of  $\mathbb{Q}$  to this  $\sigma$ -algebra is absolutely continuous with respect to  $\mathbb{W}$ . We have thus proved that there exist some paths such that  $L_{1-2^{-n}}$  tends to infinity with  $n$ , which contradicts the fact that  $(L_t)_{t \geq 0}$  is càdlàg. From this we also deduce that in general there do not exist càdlàg versions for martingales. Similarly many other important results from stochastic analysis cannot be proved on the most general filtered probability space, e.g. the existence of the Doob-Meyer decomposition for submartingales and the début theorem (see for instance [5] and [6]). In order to avoid this technical problem, it is generally assumed that the filtered probability space on which the processes are constructed satisfies the usual conditions, i.e. the filtration is complete and right-continuous.

But now, if we wish to perform a change of probability measure (for example, by using the Girsanov theorem), this assumption reveals to be too restrictive. Let us illustrate this fact by a simple example. Let us consider the filtered probability space  $(\mathcal{C}(\mathbb{R}_+, \mathbb{R}), \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{W}})$  obtained, from the Wiener space  $(\mathcal{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{W})$  described above, by taking its usual augmentation, i.e.:

- $\tilde{\mathcal{F}}$  is the  $\sigma$ -algebra generated by  $\mathcal{F}$  and its negligible sets.
- For all  $t \geq 0$ ,  $\tilde{\mathcal{F}}_t$  is  $\sigma$ -algebra generated by  $\mathcal{F}_t$  and the negligible sets of  $\mathcal{F}$ .
- $\tilde{\mathbb{W}}$  is the unique possible extension of  $\mathbb{W}$  to the completed  $\sigma$ -algebra  $\tilde{\mathcal{F}}$ .

Let us also consider the family of probability measures  $(\mathbb{Q}_t)_{t \geq 0}$ , such that  $\mathbb{Q}_t$  is defined on  $\tilde{\mathcal{F}}_t$  by

$$\mathbb{Q}_t = e^{X_t - \frac{t}{2}} \cdot \tilde{\mathbb{W}}|_{\tilde{\mathcal{F}}_t}.$$

This family of probability measures is coherent, i.e. for  $0 \leq s \leq t$ , the restriction of  $\mathbb{Q}_t$  to  $\tilde{\mathcal{F}}_s$  is equal to  $\mathbb{Q}_s$ . However, unlike what one would expect, there does not exist a probability measure  $\mathbb{Q}$  on  $\tilde{\mathcal{F}}$  such that its restriction to  $\tilde{\mathcal{F}}_s$  is equal to  $\mathbb{Q}_s$  for all  $s \geq 0$ . Indeed, let us assume that  $\mathbb{Q}$  exists. The event

$$A := \{\forall t \geq 0, X_t \geq -1\}$$

satisfies  $\widetilde{\mathbb{W}}[A] = 0$ , and then  $A \in \widetilde{\mathcal{F}}_0$  by completeness, which implies that  $\mathbb{Q}[A] = 0$ . On the other hand, under  $\mathbb{Q}_t$ , for all  $t \geq 0$ , the process  $(X_s)_{0 \leq s \leq t}$  is a Brownian motion with drift 1, and hence under  $\mathbb{Q}$ . One deduces that:

$$\mathbb{Q}[\forall s \in [0, t], X_s \geq -1] = \widetilde{\mathbb{W}}[\forall s \in [0, t], X_s \geq -s - 1] \geq \widetilde{\mathbb{W}}[\forall s \geq 0, X_s \geq -s - 1].$$

Consequently, by letting  $t$  go to infinity, one obtains:

$$\mathbb{Q}[A] \geq \widetilde{\mathbb{W}}[\forall s \geq 0, X_s \geq -s - 1] > 0,$$

which is a contradiction. Therefore, the usual conditions are not suitable for the problem of extension of coherent probability measures. In fact one can observe that the argument above does not depend on the completeness of  $\widetilde{\mathcal{F}}$ , but only on the fact that  $\widetilde{\mathcal{F}}_0$  contains all the sets in  $\widetilde{\mathcal{F}}$  of probability zero. That is why it still remains available if we consider, with the notation above, the space  $(\mathcal{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{F}, (\mathcal{F}'_t)_{t \geq 0}, \mathbb{W})$ , where for all  $t \geq 0$ ,  $\mathcal{F}'_t$  is the  $\sigma$ -algebra generated by  $\mathcal{F}_t$  and the sets in  $\mathcal{F}$  of probability zero.

**2.2. The N-usual augmentation.** In order to make compatible the general results on the extension of probability measures problem and the existence of regular versions for stochastic processes, we propose an augmentation which is intermediate between right continuity and the usual augmentation, and we call it the *N-augmentation*. As already mentioned, Bichteler has introduced this augmentation before us in his book [3], and has called it the *natural augmentation*. But when we discovered this augmentation, we were not aware of Bichteler's work and this is reflected in the difference in our approaches. Hence the interested reader would benefit by looking at both [3] and [9].

**Definition 2.1** ([9]). Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space. A subset  $A$  of  $\Omega$  is N-negligible with respect to the space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , iff there exists a sequence  $(B_n)_{n \geq 0}$  of subsets of  $\Omega$ , such that for all  $n \geq 0$ ,  $B_n \in \mathcal{F}_n$ ,  $\mathbb{P}[B_n] = 0$ , and

$$A \subset \bigcup_{n \geq 0} B_n.$$

*Remark 2.2.* The integers do not play a crucial rôle in Definition 2.1. If  $(t_n)_{n \geq 0}$  is an unbounded sequence in  $\mathbb{R}_+$ , one can replace the condition  $B_n \in \mathcal{F}_n$  by the condition  $B_n \in \mathcal{F}_{t_n}$ .

Let us now define a notion which is the analog of completeness for N-negligible sets. It is the main ingredient in the definition of what we shall call the N-usual conditions:

**Definition 2.3** ([9]). A filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , is N-complete iff all the N-negligible sets of this space are contained in  $\mathcal{F}_0$ . It satisfies the N-usual conditions iff it is N-complete and the filtration  $(\mathcal{F}_t)_{t \geq 0}$  is right-continuous.

It is natural to ask if from a given filtered probability space, one can define in a canonical way a space which satisfies the N-usual conditions and which is as "close" as possible to the initial space. The answer to this question is positive in the following sense:

**Proposition 2.4** ([9]). Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space, and  $\mathcal{N}$  the family of its N-negligible sets. Let  $\widetilde{\mathcal{F}}$  be the  $\sigma$ -algebra generated by  $\mathcal{N}$  and  $\mathcal{F}$ , and for all  $t \geq 0$ ,  $\widetilde{\mathcal{F}}_t$  the  $\sigma$ -algebra generated by  $\mathcal{N}$  and  $\mathcal{F}_{t+}$ , where

$$\mathcal{F}_{t+} := \bigcap_{u > t} \mathcal{F}_u.$$

Then there exists a unique probability measure  $\tilde{\mathbb{P}}$  on  $(\Omega, \tilde{\mathcal{F}})$  which coincides with  $\mathbb{P}$  on  $\mathcal{F}$ , and the space  $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$  satisfies the N-usual conditions. Moreover, if  $(\Omega, \mathcal{F}', (\mathcal{F}'_t)_{t \geq 0}, \mathbb{P}')$  is a filtered probability space satisfying the N-usual conditions, such that  $\mathcal{F}'$  contains  $\mathcal{F}$ ,  $\mathcal{F}'_t$  contains  $\mathcal{F}_t$  for all  $t \geq 0$ , and if  $\mathbb{P}'$  is an extension of  $\mathbb{P}$ , then  $\mathcal{F}'$  contains  $\tilde{\mathcal{F}}$ ,  $\mathcal{F}'_t$  contains  $\tilde{\mathcal{F}}_t$ , for all  $t \geq 0$  and  $\mathbb{P}'$  is an extension of  $\tilde{\mathbb{P}}$ . In other words,  $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$  is the smallest extension of  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  which satisfies the N-usual conditions: we call it the N-augmentation of  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ .

Once the N-usual conditions are defined, it is natural to compare them with the usual conditions. One has the following result:

**Proposition 2.5.** *Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space which satisfies the N-usual conditions. Then for all  $t \geq 0$ , the space  $(\Omega, \mathcal{F}_t, (\mathcal{F}_s)_{0 \leq s \leq t}, \mathbb{P})$  satisfies the usual conditions.*

*Proof.* The right-continuity of  $(\mathcal{F}_s)_{0 \leq s \leq t}$  is obvious, let us prove the completeness. If  $A$  is a negligible set of  $(\Omega, \mathcal{F}_t, \mathbb{P})$ , there exists  $B \in \mathcal{F}_t$ , such that  $A \subset B$  and  $\mathbb{P}[B] = 0$ . One deduces immediately that  $A$  is N-negligible with respect to  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , and by N-completeness of this filtered probability space,  $A \in \mathcal{F}_0$ .  $\square$

This relation between the usual conditions and the N-usual conditions is the main ingredient to prove that one can replace the usual conditions by the N-usual conditions in most of the classical results in stochastic calculus. For example, the following can be proved:

- If  $(X_t)$  is a martingale, then it admits a càdlàg modification, which is unique up to indistinguishability.
- If  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  is a filtered probability space satisfying the N-usual conditions, and if  $(X_t)_{t \geq 0}$  is an adapted process defined on this space such that there exists a càdlàg version (resp. continuous version)  $(Y_t)_{t \geq 0}$  of  $(X_t)_{t \geq 0}$ , then there exists a càdlàg and adapted version (resp. continuous and adapted version) of  $(X_t)_{t \geq 0}$ , which is necessarily indistinguishable from  $(Y_t)_{t \geq 0}$ .
- Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space satisfying the N-usual conditions, and let  $A$  be a progressive subset of  $\mathbb{R}_+ \times \Omega$ . Then the début of  $A$ , i.e. the random time  $D(A)$  such that for all  $\omega \in \Omega$ :

$$D(A)(\omega) := \inf\{t \geq 0, (t, \omega) \in A\}$$

is an  $(\mathcal{F}_t)_{t \geq 0}$ -stopping time.

- Let  $(X_t)_{t \geq 0}$  be a right-continuous submartingale defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , satisfying the N-usual conditions. We suppose that  $(X_t)_{t \geq 0}$  is of class  $(DL)$ , i.e. for all  $a \geq 0$ ,  $(X_T)_{T \in \mathcal{T}_a}$  is uniformly integrable, where  $\mathcal{T}_a$  is the family of the  $(\mathcal{F}_t)_{t \geq 0}$ -stopping times which are bounded by  $a$  (for example, every nonnegative submartingale is of class  $(DL)$ ). Then, there exist a right-continuous  $(\mathcal{F}_t)_{t \geq 0}$ -martingale  $(M_t)_{t \geq 0}$  and an increasing process  $(A_t)_{t \geq 0}$  starting at zero, such that:

$$X_t = M_t + A_t$$

for all  $t \geq 0$ , and for every bounded, right-continuous martingale  $(\xi_s)_{s \geq 0}$ ,

$$\mathbb{E}[\xi_t A_t] = \mathbb{E} \left[ \int_{(0, t]} \xi_{s-} dA_s \right],$$



where  $\xi_{s-}$  is the left-limit of  $\xi$  at  $s$ , almost surely well-defined for all  $s > 0$ . The processes  $(M_t)_{t \geq 0}$  and  $(A_t)_{t \geq 0}$  are uniquely determined, up to indistinguishability. Moreover, they can be chosen to be continuous if  $(X_t)_{t \geq 0}$  is a continuous process.

- The section theorem holds in a filtered probability space satisfying the N-usual assumptions and hence optional projections are well-defined.

**2.3. Extension of measures and the N-usual augmentation.** We first state a well-known Parthasarathy type condition for the probability measures extension problem.

**Definition 2.6.** Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$  be a filtered measurable space, such that  $\mathcal{F}$  is the  $\sigma$ -algebra generated by  $\mathcal{F}_t$ ,  $t \geq 0$ :  $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$ . We shall say that the property (P) holds if and only if  $(\mathcal{F}_t)_{t \geq 0}$  enjoys the following conditions:

- For all  $t \geq 0$ ,  $\mathcal{F}_t$  is generated by a countable number of sets;
- For all  $t \geq 0$ , there exist a Polish space  $\Omega_t$ , and a surjective map  $\pi_t$  from  $\Omega$  to  $\Omega_t$ , such that  $\mathcal{F}_t$  is the  $\sigma$ -algebra of the inverse images, by  $\pi_t$ , of Borel sets in  $\Omega_t$ , and such that for all  $B \in \mathcal{F}_t$ ,  $\omega \in \Omega$ ,  $\pi_t(\omega) \in \pi_t(B)$  implies  $\omega \in B$ ;
- If  $(\omega_n)_{n \geq 0}$  is a sequence of elements of  $\Omega$ , such that for all  $N \geq 0$ ,

$$\bigcap_{n=0}^N A_n(\omega_n) \neq \emptyset,$$

where  $A_n(\omega_n)$  is the intersection of the sets in  $\mathcal{F}_n$  containing  $\omega_n$ , then:

$$\bigcap_{n=0}^{\infty} A_n(\omega_n) \neq \emptyset.$$

Given this technical definition, one can state the following result:

**Proposition 2.7** ([15], [9]). *Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$  be a filtered measurable space satisfying the property (P), and let, for  $t \geq 0$ ,  $\mathbb{Q}_t$  be a probability measure on  $(\Omega, \mathcal{F}_t)$ , such that for all  $t \geq s \geq 0$ ,  $\mathbb{Q}_s$  is the restriction of  $\mathbb{Q}_t$  to  $\mathcal{F}_s$ . Then, there exists a unique measure  $\mathbb{Q}$  on  $(\Omega, \mathcal{F})$  such that for all  $t \geq 0$ , its restriction to  $\mathcal{F}_t$  is equal to  $\mathbb{Q}_t$ .*

One can easily deduce the following corollary which is often used in practice:

**Corollary 2.8.** *Let  $\Omega$  be  $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ , the space of continuous functions from  $\mathbb{R}_+$  to  $\mathbb{R}^d$ , or  $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$ , the space of càdlàg functions from  $\mathbb{R}_+$  to  $\mathbb{R}^d$  (for some  $d \geq 1$ ). For  $t \geq 0$ , define  $(\mathcal{F}_t)_{t \geq 0}$  as the natural filtration of the canonical process  $Y$ , and  $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$ . Then  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$  satisfies property (P).*

*Proof.* Let us prove this result for càdlàg functions (for continuous functions, the result is similar and proved in [17]). For all  $t \geq 0$ ,  $\mathcal{F}_t$  is generated by the variables  $Y_{rt}$ , for  $r$ , rational, in  $[0, 1]$ , hence, it is countably generated. For the second property, one can take for  $\Omega_t$ , the set of càdlàg functions from  $[0, t]$  to  $\mathbb{R}^d$ , and for  $\pi_t$ , the restriction to the interval  $[0, t]$ . The space  $\Omega_t$  is Polish if one endows it with the Skorokhod metric. Moreover, its Borel  $\sigma$ -algebra is equal to the  $\sigma$ -algebra generated by the coordinates, a result from which one easily deduces the properties of  $\pi_t$  which need to be satisfied. The third property is easy to check: let us suppose that  $(\omega_n)_{n \geq 0}$  is a sequence of elements of  $\Omega$ , such that for all  $N \geq 0$ ,

$$\bigcap_{n=0}^N A_n(\omega_n) \neq \emptyset,$$

where  $A_n(\omega_n)$  is the intersection of the sets in  $\mathcal{F}_n$  containing  $\omega_n$ . Here,  $A_n(\omega_n)$  is the set of functions  $\omega'$  which coincide with  $\omega_n$  on  $[0, n]$ . Moreover, for  $n \leq n'$ , integers, the intersection of  $A_n(\omega_n)$  and  $A_{n'}(\omega_{n'})$  is not empty, and then  $\omega_n$  and  $\omega_{n'}$  coincide on  $[0, n]$ . Therefore, there exists a càdlàg function  $\omega$  which coincides with  $\omega_n$  on  $[0, n]$ , for all  $n$ , which implies:

$$\bigcap_{n=0}^{\infty} A_n(\omega_n) \neq \emptyset.$$

□

*Remark 2.9.* It is easily seen that the conditions of Proposition 2.7 are not satisfied by the space  $\mathcal{C}([0, 1], \mathbb{R})$  endowed with the filtration  $(\mathcal{F}_t)_{t \geq 0}$ , where  $\mathcal{F}_t$  is the  $\sigma$ -algebra generated by the canonical process up to time  $t/(1+t)$ . An explicit counter example is provided in [7].

The next proposition shows how condition (P) combines with the N-usual augmentation:

**Proposition 2.10** ([9]). *Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be the N-augmentation of a filtered probability space satisfying the property (P). Then if  $(\mathbb{Q}_t)_{t \geq 0}$  is a coherent family of probability measures,  $\mathbb{Q}_t$  defined on  $\mathcal{F}_t$ , and absolutely continuous with respect to the restriction of  $\mathbb{P}$  to  $\mathcal{F}_t$ , there exists a unique probability measure  $\mathbb{Q}$  on  $\mathcal{F}$  which coincides with  $\mathbb{Q}_t$  on  $\mathcal{F}_t$ , for all  $t \geq 0$ .*

### 3. A UNIVERSAL $\sigma$ -FINITE MEASURE $\mathbb{Q}$

**3.1. The class  $(\Sigma)$ .** We now have all the ingredients to rigorously answer Problem 1 raised in the Introduction. For this we will first need to introduce a special class of local submartingales which was first introduced by Yor [18] and further studied by Nikeghbali [14] and Cheridito, Nikeghbali and Platen [4].

**Definition 3.1.** Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space. A nonnegative (local) submartingale  $(X_t)_{t \geq 0}$  is of class  $(\Sigma)$ , if it can be decomposed as  $X_t = N_t + A_t$  where  $(N_t)_{t \geq 0}$  and  $(A_t)_{t \geq 0}$  are  $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes satisfying the following assumptions:

- $(N_t)_{t \geq 0}$  is a càdlàg (local) martingale.
- $(A_t)_{t \geq 0}$  is a continuous increasing process, with  $A_0 = 0$ .
- The measure  $(dA_t)$  is carried by the set  $\{t \geq 0, X_t = 0\}$ .

We shall say that  $(X_t)_{t \geq 0}$  is of class  $(\Sigma D)$  if  $X$  is of class  $(\Sigma)$  and of class  $(D)$ .

The class  $(\Sigma)$  contains many well-known examples of stochastic processes (see e.g. [14]) such as nonnegative local martingales,  $|M_t|$ ,  $M_t^+$ ,  $M_t^-$  if  $M$  is a continuous local martingale, the drawdown process  $S_t - M_t$  where  $M$  is a local martingale with only negative jumps and  $S_t = \sup_{u \leq t} M_u$ , the relative drawdown process  $1 - \frac{M_t}{S_t}$  if  $M_0 \neq 0$ , the age process of the standard Brownian motion  $W_t$  in the filtration of the zeros of the Brownian motion, namely  $\sqrt{t - g_t}$ , where  $g_t = \sup\{u \leq t : W_u = 0\}$ , etc. Moreover one notes that if  $X$  is of  $(\Sigma)$ , then  $X_t + M_t$  is also of class  $(\Sigma)$  for any strictly positive càdlàg local martingale  $M$ . Another key property of the class  $(\Sigma)$  is the following stability result which follows from an application of Ito's formula and a monotone class argument.

**Proposition 3.2** ([14]). *Let  $(X_t)_{t \geq 0}$  be of class  $(\Sigma)$  and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a locally bounded Borel function. Let us further assume that  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfies the usual assumptions*

or is  $N$ -complete. Denote  $F(x) = \int_0^x f(y)dy$ . Then the process  $(f(A_t)X_t)_{t \geq 0}$  is again of class  $(\Sigma)$  with decomposition:

$$f(A_t)X_t = f(0)X_0 + \int_0^t f(A_u)dN_u + F(A_t). \quad (3.1)$$

**3.2. A special case related to financial modeling.** Now we state a theorem which gives sufficient conditions under which a process of the class  $(\Sigma)$  which converges to  $X_\infty$  a.s. satisfies (1.3). This result is an extension of a result by Azéma-Yor [2] and Azéma-Meyer-Yor [1]. Indeed, in the case when  $X$  is of class  $(\Sigma D)$ , one could deduce it from part 1 of Theorem 8.1 in [1]. The proof is very simple in this case and we give it.

**Theorem 3.3** ([4]). *Assume that  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfies the usual assumptions or is  $N$ -complete. Let  $(X_t)_{t \geq 0}$  be a process of class  $(\Sigma)$  such that  $\lim_{t \rightarrow \infty} X_t = X_\infty$  exists a.s. and is finite (in particular  $N_\infty$  and  $A_\infty$  exist and are a.s. finite). Let*

$$g := \sup\{t : X_t = 0\} \quad \text{with the convention } \sup \emptyset = 0.$$

(1) *If  $(X_t)_{t \geq 0}$  is of class  $(D)$ , then*

$$X_T = \mathbb{E}[X_\infty 1_{\{g \leq T\}} \mid \mathcal{F}_T] \quad \text{for every stopping time } T. \quad (3.2)$$

(2) *More generally, if there exists a strictly positive Borel function  $f$  such that  $(f(A_t)X_t)_{t \geq 0}$  is of class  $(D)$ , then (3.2) holds.*

(3) *If  $(N_t^+)_{t \geq 0}$  is of class  $(D)$ , then (3.2) holds.*

*Proof.* (1) For a given stopping time  $T$ , denote

$$d_T = \inf\{t > T : X_t = 0\} \quad \text{with the convention } \inf \emptyset = \infty.$$

One checks that  $d_T$  is a stopping time. Since  $X_\infty 1_{\{g \leq T\}} = X_{d_T}$  and  $A_T = A_{d_T}$ , it follows from Doob's optional stopping theorem that

$$\mathbb{E}[X_\infty 1_{\{g \leq T\}} \mid \mathcal{F}_T] = \mathbb{E}[N_{d_T} + A_{d_T} \mid \mathcal{F}_T] = \mathbb{E}[N_T + A_T \mid \mathcal{F}_T] = N_T + A_T = X_T.$$

(2) Assume that there exists a strictly positive Borel function such that  $(f(A_t)X_t)_{t \geq 0}$  is of class  $(D)$ . This property is preserved if one replaces  $f$  by a smaller strictly positive Borel function, hence, one can suppose that  $f$  is locally bounded. Then  $(f(A_t)X_t)_{t \geq 0}$  is of class  $(\Sigma D)$ , and from part (1) of the theorem, we have:

$$f(A_T)X_T = \mathbb{E}[f(A_\infty)X_\infty 1_{\{g \leq T\}} \mid \mathcal{F}_T].$$

But on the set  $\{g \leq T\}$ , we have  $A_\infty = A_T$ , and consequently

$$f(A_T)X_T = f(A_T)\mathbb{E}[X_\infty 1_{\{g \leq T\}} \mid \mathcal{F}_T].$$

The result follows by dividing both sides by  $f(A_T)$  which is strictly positive.

(3) Since  $X \geq 0$  and since  $(N_t^+)_{t \geq 0}$  is of class  $(D)$ , we note that  $(\exp(-A_t)X_t)_{t \geq 0}$  is of class  $(D)$  and the result follows from (2).  $\square$

The above theorem is used in [4] for financial applications. For example, if  $(M_t)_{t \geq 0}$  is a nonnegative local martingale, converging to 0 (which is a reasonable model for stock prices or for portfolios under the benchmark approach), then the drawdown process  $DD_t = \max_{u \leq t} M_u - M_t$  is of class  $(\Sigma)$  as well as the relative drawdown process  $rDD_t = 1 - \frac{M_t}{\max_{u \leq t} M_u}$ .

For example, if  $(M_t)_{t \geq 0}$  is a strict nonnegative local martingale such as the inverse of the 3 dimensional Bessel process, then  $(DD_t)_{t \geq 0}$  is of class  $(\Sigma)$ , satisfies condition (3) of the above

theorem but is not uniformly integrable. Note that Theorem 3.3 applies in this case, even if the local martingale part of the process  $(DD_t)_{t \geq 0}$ , which is equal to  $(-M_t)_{t \geq 0}$ , is not a true martingale.

**3.3. The general case.** A general solution for Problem 1 is provided in [12]; the measure  $\mathcal{W}$  is a special case of this theorem. For the general case, we need to be very careful: the theorem below would be wrong under the usual assumptions and it would also be wrong if the filtration  $(\mathcal{F}_t)_{t \geq 0}$  does not allow the extension of coherent probability measures. The N-usual assumptions are here to ensure in particular that there exists a continuous and adapted version of the process  $A$  which is defined everywhere (think of  $A_t$  being for example the local time at the level 0 of the Wiener process). According to Section 2, typical probability spaces where the following theorem holds are  $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$  and  $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ .

**Theorem 3.4.** *Let  $(X_t)_{t \geq 0}$  be a true submartingale of the class  $(\Sigma)$ : its local martingale part  $(N_t)_{t \geq 0}$  is a true martingale, and  $X_t$  is integrable for all  $t \geq 0$ . We suppose that  $(X_t)_{t \geq 0}$  is defined on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$  satisfying the property (NP), in particular, this space satisfies the N-usual conditions and  $\mathcal{F}$  is the  $\sigma$ -algebra generated by  $\mathcal{F}_t$  for  $t \geq 0$ . Then, there exists a unique  $\sigma$ -finite measure  $\mathcal{Q}$ , defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ , such that for  $g := \sup\{t \geq 0, X_t = 0\}$ :*

- $\mathcal{Q}[g = \infty] = 0$ ;
- For all  $t \geq 0$ , and for all  $\mathcal{F}_t$ -measurable, bounded random variables  $\Gamma_t$ ,

$$\mathcal{Q}[\Gamma_t \mathbf{1}_{g \leq t}] = \mathbb{P}[\Gamma_t X_t].$$

In [12], the measure  $\mathcal{Q}$  is explicitly constructed, in the following way (with a slightly different notation). Let  $f$  be a Borel, integrable, strictly positive and bounded function from  $\mathbb{R}$  to  $\mathbb{R}$ , and let us define the function  $G$  by the formula:

$$G(x) = \int_x^\infty f(y) dy.$$

One can prove that the process

$$\left( M_t^f := G(A_t) - \mathbb{E}_{\mathbb{P}}[G(A_\infty) | \mathcal{F}_t] + f(A_t)X_t \right)_{t \geq 0}, \quad (3.3)$$

is a martingale with respect to  $\mathbb{P}$  and the filtration  $(\mathcal{F}_t)_{t \geq 0}$ . Since  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$  satisfies the N-usual conditions and since  $G(A_t) \geq G(A_\infty)$ , one can suppose that this martingale is nonnegative and càdlàg, by choosing carefully the version of  $\mathbb{E}_{\mathbb{P}}[G(A_\infty) | \mathcal{F}_t]$ . In this case, since  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$  satisfies the property (NP), there exists a unique finite measure  $\mathcal{M}^f$  such that for all  $t \geq 0$ , and for all bounded,  $\mathcal{F}_t$ -measurable functionals  $\Gamma_t$ :

$$\mathcal{M}^f[\Gamma_t] = \mathbb{E}_{\mathbb{P}}[\Gamma_t M_t^f].$$

Now, since  $f$  is strictly positive, one can define a  $\sigma$ -finite measure  $\mathcal{Q}^f$  by:

$$\mathcal{Q}^f := \frac{1}{f(A_\infty)} \cdot \mathcal{M}^f.$$

It is proved in [12] that if the function  $G/f$  is uniformly bounded (this condition is, for example, satisfied for  $f(x) = e^{-x}$ ), then  $\mathcal{Q}^f$  satisfies the conditions defining  $\mathcal{Q}$  in Theorem 3.4, which implies the existence part of this result. The uniqueness part is proved just after in a very easy way: one remarkable consequence of it is the fact that  $\mathcal{Q}^f$  does not depend

on the choice of  $f$ . The measure  $\mathcal{Q}$  has many other interesting properties, which will be detailed in Section 4.

#### 4. FURTHER PROPERTIES OF $\mathcal{Q}$ AND SOME REMARKABLE ASSOCIATED MARTINGALES, AND PENALISATION RESULTS

The properties of  $\mathcal{Q}$  given in this section are stated and proved in [11], except the results concerning penalisation, which are shown in [10]. By the construction of  $\mathcal{Q}$  described above, it is clear that if  $f$  is a Borel, integrable, strictly positive and bounded function from  $\mathbb{R}$  to  $\mathbb{R}$ , and if, with the notation above,  $G/f$  is uniformly bounded, then

$$\mathcal{M}^f = f(A_\infty) \cdot \mathcal{Q}. \quad (4.1)$$

Now, by using (3.3), it is possible to construct  $(M_t^f)_{t \geq 0}$ , and then  $\mathcal{M}^f$ , for all Borel, integrable and nonnegative functions  $f$ . Moreover, it is proved in [11] that (4.1) remains true for any function  $f$  satisfying these weaker assumptions. The relation between the functional  $f(A_\infty)$  and the martingale  $(M_t^f)_{t \geq 0}$  can be generalized as follows:

**Proposition 4.1.** *We suppose that the assumptions of Theorem 3.4 hold, and we take the same notation. Let  $F$  be a  $\mathcal{Q}$ -integrable, nonnegative functional defined on  $(\Omega, \mathcal{F})$ . Then, there exists a càdlàg  $\mathbb{P}$ -martingale  $(M_t(F))_{t \geq 0}$  such that the measure  $\mathcal{M}^F := F \cdot \mathcal{Q}$  is the unique finite measure satisfying, for all  $t \geq 0$ , and for all bounded,  $\mathcal{F}_t$ -measurable functionals  $\Gamma_t$ :*

$$\mathcal{M}^F[\Gamma_t] = \mathbb{P}[\Gamma_t M_t(F)].$$

*The martingale  $(M_t(F))_{t \geq 0}$  is unique up to indistinguishability.*

If  $f$  is Borel, integrable and nonnegative, then  $f(A_\infty)$  is integrable with respect to  $\mathcal{Q}$  and one has

$$M_t(f(A_\infty)) = M_t^f,$$

which gives an explicit expression for  $M_t(f(A_\infty))$ . This explicit form can be generalized to the martingale  $(M_t(F))_{t \geq 0}$  for all nonnegative and  $\mathcal{Q}$ -integrable functionals  $F$ , if the submartingale  $(X_t)_{t \geq 0}$  is uniformly integrable. More precisely, one has the following result:

**Proposition 4.2.** *Let us suppose that the assumptions of Theorem 3.4 are satisfied, and that the process  $(X_t)_{t \geq 0}$  is uniformly integrable. Then,  $X_t$  tends a.s. to a limit  $X_\infty$  when  $t$  goes to infinity, and the measure  $\mathcal{Q}$  is absolutely continuous with respect to  $\mathbb{P}$ , with density  $X_\infty$ . Moreover, a nonnegative functional  $F$  is integrable with respect to  $\mathcal{Q}$  iff  $F X_\infty$  is integrable with respect to  $\mathbb{P}$ , in this case,  $(M_t(F))_{t \geq 0}$  is the càdlàg version of the conditional expectation  $(\mathbb{P}[F X_\infty | \mathcal{F}_t])_{t \geq 0}$ . In particular, it is uniformly integrable, and it converges a.s. and in  $L^1$  to  $F X_\infty$  when  $t$  goes to infinity.*

A more interesting case is when we suppose that  $A_\infty$  is infinite,  $\mathbb{P}$ -almost surely. From now, until the end of this section, we always implicitly make this assumption (which is satisfied, in particular, if  $(X_t)_{t \geq 0}$  is a reflected Brownian motion). The following result gives the asymptotic behaviour of  $(X_t)_{t \geq 0}$  under  $\mathcal{Q}$ , when  $t$  goes to infinity, and the behaviour of  $M_t(F)$ , which is not the same under  $\mathbb{P}$  and under  $\mathcal{Q}$ :

**Proposition 4.3.** *Let us suppose that the assumptions of Theorem 3.4 are satisfied, and that  $A_\infty = \infty$ ,  $\mathbb{P}$ -almost surely. Then,  $\mathcal{Q}$ -almost everywhere,  $X_t$  tends to infinity when  $t$  goes to*

infinity, and for all nonnegative,  $\mathcal{Q}$ -integrable functionals  $F$ , the martingale  $(M_t(F))_{t \geq 0}$  tends  $\mathbb{P}$ -almost surely to zero and  $\mathcal{Q}$ -almost everywhere to infinity. Moreover, one has:

$$\frac{M_t(F)}{X_t} \xrightarrow[t \rightarrow \infty]{} F,$$

$\mathcal{Q}$ -almost everywhere.

By definition, the martingales of the form  $(M_t(F))_{t \geq 0}$  are exactly the local densities of the finite measures which are absolutely continuous with respect to  $\mathcal{Q}$ . This situation is similar to the case of uniformly integrable, nonnegative martingales, which are local densities of finite measures, absolutely continuous with respect to  $\mathbb{P}$ . The following decomposition of nonnegative supermartingales, already proved in [13] in the case of the reflected Brownian motion, involves simultaneously these two kind of martingales:

**Proposition 4.4.** *Let us suppose that the assumptions of Theorem 3.4 are satisfied, and that  $A_\infty = \infty$ ,  $\mathbb{P}$ -almost surely. Let  $Z$  be a nonnegative, càdlàg  $\mathbb{P}$ -supermartingale. We denote by  $Z_\infty$  the  $\mathbb{P}$ -almost sure limit of  $Z_t$  when  $t$  goes to infinity. Then,  $\mathcal{Q}$ -almost everywhere, the quotient  $Z_t/X_t$  is well-defined for  $t$  large enough and converges, when  $t$  goes to infinity, to a limit  $z_\infty$ , integrable with respect to  $\mathcal{Q}$ , and  $(Z_t)_{t \geq 0}$  decomposes as*

$$(Z_t = M_t(z_\infty) + \mathbb{P}[Z_\infty | \mathcal{F}_t] + \xi_t)_{t \geq 0},$$

where  $(\mathbb{P}[Z_\infty | \mathcal{F}_t])_{t \geq 0}$  denotes a càdlàg version of the conditional expectation of  $Z_\infty$  with respect to  $\mathcal{F}_t$ , and  $(\xi_t)_{t \geq 0}$  is a nonnegative, càdlàg  $\mathbb{P}$ -supermartingale, such that:

- $Z_\infty \in L^1_+(\mathcal{F}, \mathbb{P})$ , hence  $\mathbb{P}[Z_\infty | \mathcal{F}_t]$  converges  $\mathbb{P}$ -almost surely and in  $L^1(\mathcal{F}, \mathbb{P})$  towards  $Z_\infty$ .
- $\frac{\mathbb{P}[Z_\infty | \mathcal{F}_t] + \xi_t}{X_t} \xrightarrow[t \rightarrow \infty]{} 0$ ,  $\mathcal{Q}$ -almost everywhere.
- $M_t(z_\infty) + \xi_t \xrightarrow[t \rightarrow \infty]{} 0$ ,  $\mathbb{P}$ -almost surely.

Moreover, the decomposition is unique in the following sense: let  $z'_\infty$  be a  $\mathcal{Q}$ -integrable, nonnegative functional,  $Z'_\infty$  a  $\mathbb{P}$ -integrable, nonnegative random variable,  $(\xi'_t)_{t \geq 0}$  a càdlàg, nonnegative  $\mathbb{P}$ -supermartingale, and let us suppose that for all  $t \geq 0$ ,

$$Z_t = M_t(z'_\infty) + \mathbb{P}[Z'_\infty | \mathcal{F}_t] + \xi'_t.$$

Under these assumptions, if for  $t$  going to infinity,  $\xi'_t$  tends  $\mathbb{P}$ -almost surely to zero and  $\xi'_t/X_t$  tends  $\mathcal{Q}$ -almost everywhere to zero, then  $z'_\infty = z_\infty$ ,  $\mathcal{Q}$ -almost everywhere,  $Z'_\infty = Z_\infty$ ,  $\mathbb{P}$ -almost surely, and  $\xi'$  is  $\mathbb{P}$ -indistinguishable with  $\xi$ .

This result implies the following characterisation of martingales of the form  $(M_t(F))_{t \geq 0}$ :

**Corollary 4.5.** *Let us suppose that the assumptions of Theorem 3.4 are satisfied, and that  $A_\infty = \infty$ ,  $\mathbb{P}$ -almost surely. Then, a càdlàg, nonnegative  $\mathbb{P}$ -martingale  $(Z_t)_{t \geq 0}$  is of the form  $(M_t(F))_{t \geq 0}$  for a nonnegative,  $\mathcal{Q}$ -integrable functional  $F$ , if and only if:*

$$\mathbb{P}[Z_0] = \mathcal{Q} \left( \lim_{t \rightarrow \infty} \frac{Z_t}{X_t} \right). \quad (4.2)$$

Note that, by Proposition 4.4, the limit above necessarily exists  $\mathcal{Q}$ -almost everywhere.

If in Proposition 4.4,  $(Z_t)_{t \geq 0}$  is a nonnegative martingale, the corresponding decomposition can be interpreted as a Radon-Nykodym decomposition of finite measures. Indeed, let us

observe that since the space satisfies the property (NP), there exists a unique finite measure  $\mathbb{Q}_Z$  on  $(\Omega, \mathcal{F})$ , such that for all  $t \geq 0$ , its restriction to  $\mathcal{F}_t$  has density  $Z_t$  with respect to  $\mathbb{P}$ . If one writes the decomposition

$$Z_t = M_t(z_\infty) + \mathbb{P}[Z_\infty | \mathcal{F}_t] + \xi_t,$$

one deduces:

$$\mathbb{Q}_Z = z_\infty \cdot \mathcal{Q} + Z_\infty \cdot \mathbb{P} + \mathbb{Q}_\xi,$$

where the restriction of  $\mathbb{Q}_\xi$  to  $\mathcal{F}_t$  has density  $\xi_t$  with respect to  $\mathbb{P}$ . In [11], it is proved that  $\mathbb{Q}_\xi$  is singular with respect to  $\mathbb{P}$  and  $\mathcal{Q}$ , hence one has a decomposition of  $\mathbb{Q}_Z$  into three parts:

- A part which is absolutely continuous with respect to  $\mathbb{P}$ .
- A part which is absolutely continuous with respect to  $\mathcal{Q}$ .
- A part which is singular with respect to  $\mathbb{P}$  and  $\mathcal{Q}$ .

This decomposition is unique, as a consequence of the uniqueness of the Radon-Nykodym decomposition.

Another interesting problem is to find a relation between the measure  $\mathcal{Q}$  and the last passage times of  $(X_t)_{t \geq 0}$  at a given level, which can be different from zero. The following result proves this relation when  $(X_t)_{t \geq 0}$  is continuous:

**Proposition 4.6.** *Let us suppose that the assumptions of Theorem 3.4 are satisfied, the submartingale  $(X_t)_{t \geq 0}$  is continuous and  $A_\infty = \infty$  almost surely under  $\mathbb{P}$ . For  $a \geq 0$ , let  $g^{[a]}$  be the last hitting time of the interval  $[0, a]$ :*

$$g^{[a]} = \sup\{t \geq 0, X_t \leq a\}.$$

*Then, the measure  $\mathcal{Q}$  satisfies the following formula, available for any  $t \geq 0$ , and for all  $\mathcal{F}_t$ -measurable, bounded variables  $\Gamma_t$ :*

$$\mathcal{Q} [\Gamma_t \mathbf{1}_{g^{[a]} \leq t}] = \mathbb{P} [\Gamma_t (X_t - a)_+]. \quad (4.3)$$

*Moreover,  $((X_t - a)_+)_{t \geq 0}$  is a submartingale of class  $(\Sigma)$  and the  $\sigma$ -finite measure obtained by applying Theorem 3.4 to it is equal to  $\mathcal{Q}$ .*

The relation between the measure  $\mathcal{Q}$  and the last hitting time of a given level  $a$  is one of the main ingredients of the proof of our penalisation result stated below. Since we are able to show this statement only when  $(X_t)_{t \geq 0}$  is a diffusion satisfying some technical conditions, let us give another result, strongly related to penalisations, but available under more general assumptions:

**Proposition 4.7.** *Let us suppose that the assumptions of Theorem 3.4 are satisfied, and  $A_\infty = \infty$ ,  $\mathbb{P}$ -almost surely. Let  $(F_t)_{t \geq 0}$  be a càdlàg, adapted, nonnegative, nonincreasing and uniformly bounded process, such that for some  $a > 0$ , one has for all  $t \geq 0$ ,  $X_t = X_{g^{[a]}}$  on the set  $\{t \geq g^{[a]}\}$ . Then, if  $F_g$  is  $\mathcal{Q}$ -integrable, and if one defines  $F_\infty$  as the limit of  $F_t$  for  $t$  going to infinity (in particular,  $F_\infty = F_{g^{[a]}}$  for  $g^{[a]} < \infty$ ), one has:*

$$\mathbb{P}[F_t X_t] \xrightarrow[t \rightarrow \infty]{} \mathcal{Q}[F_\infty].$$

In [11], there are another version of this result, with slightly different assumptions. In order to obtain a penalisation result, we need to estimate the expectation of  $F_t$  under  $\mathbb{P}$ , instead of the expectation of  $F_t X_t$ . This gives an extra factor, depending on  $t$ , in the asymptotics, which

justifies some restriction on the process  $(X_t)_{t \geq 0}$ . Let us now describe the precise framework which is considered. We define  $\Omega$  as the space of continuous functions from  $\mathbb{R}_+$  to  $\mathbb{R}_+$ ,  $(\mathcal{F}_t^0)_{t \geq 0}$  as the natural filtration of  $\Omega$ , and  $\mathcal{F}^0$ , as the  $\sigma$ -algebra generated by  $(\mathcal{F}_t^0)_{t \geq 0}$ . The probability  $\mathbb{P}^0$ , defined on  $(\Omega, \mathcal{F}^0)$ , satisfies the following condition: under  $\mathbb{P}^0$ , the canonical process is a recurrent diffusion in natural scale, starting from a fixed point  $x_0 \geq 0$ , with zero as an instantaneously reflecting barrier, and such that its speed measure is absolutely continuous with respect to Lebesgue measure on  $\mathbb{R}_+$ , with a density  $m : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ , continuous, and such that  $m(x)$  is equivalent to  $cx^\beta$  when  $x$  goes to infinity, for some  $c > 0$  and  $\beta > -1$ . Moreover, we suppose that there exists  $C > 0$  such that for all  $x > 0$ ,  $m(x) \leq Cx^\beta$  if  $\beta \leq 0$ , and  $m(x) \leq C(1 + x^\beta)$  if  $\beta > 0$ . Let us now define the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  as the N-augmentation of  $(\Omega, \mathcal{F}^0, (\mathcal{F}_t^0)_{t \geq 0}, \mathbb{P}^0)$ :  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfies property (NP) and under  $\mathbb{P}$ , the law of the canonical process is a diffusion with the same parameters as under  $\mathbb{P}^0$ . This diffusion is in natural scale, and one can deduce from this fact and the assumptions above that the canonical process  $(X_t)_{t \geq 0}$  is a submartingale of class  $(\Sigma)$ . In particular,  $X_t$  is integrable for all  $t \geq 0$ . Moreover, the local time  $(L_t)_{t \geq 0}$  of  $(X_t)_{t \geq 0}$  at level zero, is its increasing process. One deduces that Theorem 3.4 applies, which defines the corresponding measure  $\mathcal{Q}$ , and we can check that we are in the situation where  $L_\infty = \infty$ ,  $\mathbb{P}$ -almost surely. Under these assumptions, we can prove the following result:

**Proposition 4.8.** *Let  $(F_t)_{t \geq 0}$  be a càdlàg, adapted, nonnegative, uniformly bounded and nonincreasing process. We assume that there exists  $a > 0$  such that for all  $t \geq 0$ ,  $X_t = X_{g^{[a]}}$  on the set  $\{t \geq g^{[a]}\}$ , we define  $F_\infty$  as the limit of  $F_t$  for  $t$  going to infinity, and we suppose that  $F_\infty$  is integrable with respect to  $\mathcal{Q}$ . Then, there exists  $D > 0$  such that for all  $s \geq 0$  and for all events  $\Lambda_s \in \mathcal{F}_s$ :*

$$t^{1/(\beta+2)} \mathbb{P}[F_t \mathbb{1}_{\Lambda_s}] \xrightarrow[t \rightarrow \infty]{} D \mathcal{Q}[F_\infty \mathbb{1}_{\Lambda_s}].$$

The following penalisation result is a straightforward consequence of Proposition 4.8:

**Proposition 4.9.** *Let  $(F_t)_{t \geq 0}$  be a càdlàg, adapted, nonnegative, uniformly bounded and nonincreasing process. We assume that there exists  $a > 0$  such that for all  $t \geq 0$ ,  $X_t = X_{g^{[a]}}$  on the set  $\{t \geq g^{[a]}\}$ , we define  $F_\infty$  as the limit of  $F_t$  for  $t$  going to infinity, and we suppose that  $0 < \mathcal{Q}[F_\infty] < \infty$ . Then, for all  $t \geq 0$ :*

$$0 < \mathbb{P}[F_t] < \infty,$$

and one can then define a probability measure  $\mathbb{Q}_t$  on  $(\Omega, \mathcal{F})$  by

$$\mathbb{Q}_t := \frac{F_t}{\mathbb{P}[F_t]} \cdot \mathbb{P}.$$

Moreover, the probability measure:

$$\mathbb{Q}_\infty := \frac{F_\infty}{\mathcal{Q}[F_\infty]} \cdot \mathcal{Q}$$

is the weak limit of  $\mathbb{Q}_t$  in the sense of penalisations, i.e. for all  $s \geq 0$ , and for all events  $\Lambda_s \in \mathcal{F}_s$ ,

$$\mathbb{Q}_t[\Lambda_s] \xrightarrow[t \rightarrow \infty]{} \mathbb{Q}_\infty[\Lambda_s].$$



This penalisation result applies, in particular, to the power  $2r$  of a Bessel process of dimension  $2(1-r)$ , for any  $r \in (0, 1)$  (in this case, the parameter  $\beta$  is equal to  $(1/r) - 2$ ). For  $r = 1$ ,  $(X_t)_{t \geq 0}$  is a reflected Brownian motion, and our result is very similar to the penalisation theorem stated in [13] for the Brownian motion.

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