AN ARITHMETIC MODEL FOR THE TOTAL DISORDER PROCESS

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Abstract. We prove a multidimensional extension of Selberg's central limit theorem for the logarithm of the Riemann zeta function on the critical line. The limit is a totally disordered process, whose coordinates are all independent and Gaussian.

1. Introduction

A classical result of Selberg [10] (see also Laurinčikas, [6]) states that the classical continuous determination of the logarithm of the Riemann zeta function is asymptotically normally distributed, in the sense that if Γ is a regular Borel measurable subset of $\mathbb{C}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} \mathbb{1}\left\{ \frac{\log\zeta(\frac{1}{2}+\mathrm{i}t)}{\sqrt{\frac{1}{2}\log\log T}} \in \Gamma \right\} \mathrm{d}t = \frac{1}{2\pi}\int_\Gamma e^{-(x^2+y^2)/2}\,\mathrm{d}x\,\mathrm{d}y$$

where $\mathbb{1}$ is the indicator function, and regular means that the boundary of Γ has zero Lebesgue measure.

If we let

$$L_\lambda(N,u) := \frac{\log\zeta(\frac{1}{2}+\mathrm{i}ue^{N^\lambda})}{\sqrt{\log N}}$$

then Selberg's result implies that

$$\lim_{N\to\infty}\int_1^2 \mathbb{1}\left\{L_\lambda(N,u)\in\Gamma\right\}\,\mathrm{d}u = \mathbb{P}\{G_\lambda\in\Gamma\}$$

where $G_\lambda = G_\lambda^{(1)} + \mathrm{i}G_\lambda^{(2)}$ is a complex-valued Gaussian random variable with mean zero and variance $\lambda/2$, i.e.: $G_\lambda^{(1)}$ and $G_\lambda^{(2)}$ are independent, centered, and $\mathbb{E}[(G_\lambda^{(1)})^2] = \mathbb{E}[(G_\lambda^{(2)})^2] = \lambda/2$.

It is now a natural question, at least from a probabilistic standpoint, to look for an asymptotic distribution for $(L_{\lambda_1}(N,\cdot),\dots,L_{\lambda_k}(N,\cdot))$, for different λ_i's.

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Theorem 1. *For $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$, and for every $(\Gamma_i, i \le k)$ regular,*

$$\lim_{N\to\infty} \int_1^2 \mathbb{1}\left\{L_{\lambda_1}(N,u) \in \Gamma_1, \ldots, L_{\lambda_k}(N,u) \in \Gamma_k\right\} \,\mathrm{d}u = \prod_{j=1}^{k} \mathbb{P}\left\{G_{\lambda_j} \in \Gamma_j\right\}. \tag{1}$$

We now note that if $\left(D_\lambda = D_\lambda^{(1)} + \mathrm{i}D_\lambda^{(2)}, \lambda > 0\right)$ is a totally disordered complex-valued Gaussian process, meaning that $\left(D_\lambda^{(1)}, \lambda > 0\right)$ and $\left(D_\lambda^{(2)}, \lambda > 0\right)$ are two independent Gaussian processes all of whose coordinates are independent with $\mathbb{E}[(D_\lambda^{(1)})^2] = \mathbb{E}[(D_\lambda^{(2)})^2] = \lambda/2$, then the quantity on the right hand side of (1) is

$$\mathbb{P}\left\{D_{\lambda_1} \in \Gamma_1, \ldots, D_{\lambda_k} \in \Gamma_k\right\}.$$

Theorem 1 is an attempt to move from the deterministic set up of the Riemann zeta function, and the "static" central limit theorem of Selberg into a more "dynamic" probabilistic world, where a process appears in the limit. However, this process is quite wild. In the next section, we comment about it, and some of its occurrences in random matrix theory. Finally, in the third section we prove Theorem 1 using the method of moments.

Remark. Our methods apply equally well to any L-function from the Selberg class, but for concreteness and for the sake of simplicity we only state here the result for the Riemann zeta function.

2. Some remarks on total disorder process

2.1. Non-measurability of the total disorder process. The total disorder process is a "wild" process; indeed there is no measurable process $(\lambda, \omega) \mapsto \widetilde{D}_\lambda(\omega)$ which would be a modification of $(D_\lambda, \lambda \ge 0)$, i.e. $\mathbb{P}\{\widetilde{D}_\lambda = D_\lambda) = 1$ for all λ. Indeed, if so, we would get (use Fubini)

$$\int_a^b \widetilde{D}_\lambda \mathrm{d}\lambda = 0 \quad \text{a.s.},$$

hence

$$\widetilde{D}_\lambda = 0 \quad \mathrm{d}\lambda \mathrm{d}\mathbb{P},$$

which is absurd (for some further discussion on the total disorder process, see page 37 of [7]).

2.2. The total disorder process in random matrix theory. The total disorder process has already been observed asymptotically in random matrix theory, although in a different guise. Let $Z_U(\theta) = \det(I - Ue^{-\mathrm{i}\theta})$ be the characteristic polynomial of an $N \times N$ unitary matrix U chosen with Haar measure, then Hughes, Keating and O'Connell [4] prove that

$$\frac{\log Z_U(\theta)}{\sqrt{\frac{1}{2}\log N}}$$

weakly converges to $X(\theta)+\mathrm{i}Y(\theta)$, where $X(\theta), Y(\theta)$ are independent Gaussian processes with covariance structure

$$\mathbb{E}\left[X(\theta_1)X(\theta_2)\right] = \mathbb{E}\left[Y(\theta_1)Y(\theta_2)\right] = \begin{cases} 1 & \text{if } \theta_1 = \theta_2 \\ 0 & \text{otherwise} \end{cases}$$

This was used to provide an explanation for the covariance structure of $C_U(s,t)$, the number of eigenangles of U that lie in the interval (s,t), found earlier by Wieand [12, 13]. A separate explanation was given by Diaconis and Evans [2]. Let

$$\widetilde{C}_U(s,t) := \frac{C_U(s,t) - (t-s)N/2\pi}{\frac{1}{\pi}\sqrt{\log N}}.$$

Wieand proves that for fixed s,t, if the matrices U are chosen with Haar measure from the unitary group, then $\widetilde{C}_U(s,t)$ converges in distribution, as $N \to \infty$, to a standard normal random variable. In fact she goes much further by proving weak convergence of $\widetilde{C}_U(s,t)$ to a certain Gaussian process $C(s,t)$.

Theorem 2 (Wieand). *For $-\pi < s < t \leq \pi$, the finite dimensional distributions of the process $\widetilde{C}_U(s,t)$ converge as $N \to \infty$ to those of a centered Gaussian process $C(s,t)$ with covariance structure*

$$\mathbb{E}\left\{C(s,t)C(s',t')\right\} = \begin{cases} 1 & \textit{if } s = s', t = t' \\ -1 & \textit{if } s = t', t = s' \\ \frac{1}{2} & \textit{if } s = s' \textit{ or if } t = t' \textit{ but not both} \\ -\frac{1}{2} & \textit{if } s = t' \textit{ or if } t = s' \textit{ but not both} \\ 0 & \textit{otherwise} \end{cases}$$

A similar process result had previously been found by Costin and Lebowitz [1] for GUE matrices, and Soshnikov [11] considers a process result for counting the number of eigenangles in an interval with a given minimum displacement. The surprising thing about these correlations is that they imply that even if an interval I contains more than the average number of eigenangles, then any subset of I not sharing a common endpoint with I will usually still contain *its* average number. Also, no matter how close two intervals I and J are, unless they share an endpoint, then C_I and C_J (with obvious notations) are independent.

Of course, the results of Wieand and of Hughes, Keating and O'Connell are strongly related, because

$$\widetilde{C}_U(s,t) = \frac{\mathfrak{Im} \log Z_U(t)}{\sqrt{\log N}} - \frac{\mathfrak{Im} \log Z_U(s)}{\sqrt{\log N}}.$$

3. Proof of theorem 1

We will prove Theorem 1 via the method of moments. The following lemma will play an essential role in our argument.

Lemma 3. *A complex random variable $Z = X + \mathrm{i}Y$ has moments*

$$\mathbb{E}\left[Z^m \overline{Z}^n\right] = \begin{cases} n! 2^n \sigma^{2n} & \text{if } m = n \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

if and only if X and Y are independent and distributed according to the normal law with mean zero and variance σ^2.

Proof. Let $Z = X + \mathrm{i}Y$ where X and Y are independent, centered normal random variables with variance σ^2. Consider the joint moment generating function of Z and $\overline{Z}$: for $(\alpha, \beta) \in \mathbb{C}^2$,

$$\begin{aligned} \mathbb{E}\left[e^{\alpha Z} e^{\beta \overline{Z}}\right] &= \mathbb{E}\left[e^{(\alpha+\beta)X + \mathrm{i}(\alpha-\beta)Y}\right] \\ &= e^{(\alpha+\beta)^2\sigma^2/2 - (\alpha-\beta)^2\sigma^2/2} \\ &= e^{2\alpha\beta\sigma^2} \\ &= \sum_{n=0}^{\infty} n! 2^n \sigma^{2n} \frac{\alpha^n}{n!} \frac{\beta^n}{n!} \end{aligned}$$

which is the two-variable moment generating function of (2). Conversely, assume that (2) holds for the joint moments of Z and $\overline{Z}$. Then working up the above chain of equalities proves that $Z = X + \mathrm{i}Y$ where X and Y are independent, centered gaussians with variance σ^2. ☐

From Lemma 3, if one can show that for any positive integers k and any integers $m_1, \ldots, m_k; n_1, \ldots, n_k$, and if for any $\lambda_1 > \cdots > \lambda_k$

$$\mathbb{E}\left[\prod_{\ell=1}^{k} D_{\lambda_\ell}^{m_\ell} \overline{D_{\lambda_\ell}}^{n_\ell}\right] = \prod_{\ell=1}^{k} n_\ell! \lambda_\ell^{n_\ell} \delta_{m_\ell, n_\ell}$$

then one may conclude that $(D_\lambda, \lambda > 0)$ is a centered complex-valued Gaussian totally disordered process with covariance structure

$$\mathbb{E}\left[D_{\lambda_i} \overline{D_{\lambda_j}}\right] = \begin{cases} \lambda_i & \text{if } \lambda_i = \lambda_j \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbb{E}\left[D_{\lambda_i} D_{\lambda_j}\right] = 0$ for all λ_i, λ_j.

Therefore, Theorem 1 is a consequence of the following

Theorem 4. *Let*

$$L_\lambda(N,u) = \frac{\log\zeta(\frac{1}{2}+\mathrm{i}ue^{N^\lambda})}{\sqrt{\log N}}.$$

If $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ *are fixed, then*

$$\lim_{N\to\infty}\int_1^2 \prod_{j=1}^k L_{\lambda_j}(N,u)^{m_j}\overline{L_{\lambda_j}(N,u)}^{n_j}\,\mathrm{d}u = \prod_{j=1}^k n_j!\lambda_j^{n_j}\delta(m_j,n_j)$$

where $\delta(m_j,n_j)=1$ *if* $m_j=n_j$ *and zero otherwise.*

We need the following theorem of Selberg, [10].

Theorem 5 (Selberg)**.** *If* n *is a positive integer,* $0<a<1$*, and* $T^{a/n} \leq x \leq T^{1/n}$*, then*

$$\frac{1}{T}\int_T^{2T}\left|\log\zeta(\tfrac{1}{2}+\mathrm{i}t) - \sum_{p\leq x}\frac{p^{-\mathrm{i}t}}{\sqrt{p}}\right|^{2n}\,\mathrm{d}t = O\left(n^{4n}e^{An}\right)$$

for some constant A *which depends upon* a*.*

We also need to calculate the moments of certain prime sums.

Lemma 6. *Given* k *a positive integer, let* $\lambda_1 > \cdots > \lambda_k > 0$*. Let*

$$P(\lambda,n) = P(\lambda,n;k,N,u) = \frac{1}{\sqrt{\log N}}\sum_{p\leq\exp\left(\frac{N^\lambda}{40kn}\right)}\frac{p^{-\mathrm{i}ue^{N^\lambda}}}{\sqrt{p}}$$

For any non-negative integers $m_1,\ldots,m_k$ *and* $n_1,\ldots,n_k$*,*

$$\lim_{N\to\infty}\int_1^2\prod_{j=1}^k P(\lambda_j,m_j)^{m_j}\overline{P(\lambda_j,n_j)}^{n_j}\,\mathrm{d}u = \prod_{j=1}^k (n_j)!\,(\lambda_j)^{n_j}\,\delta(m_j,n_j) \quad (3)$$

(Note that for the sake of simplicity the variable u does not appear explicitly in (3) and in some expressions below)

Proof. We wish to expand out

$$\prod_{j=1}^k P(\lambda_j,m_j)^{m_j}$$

as a multiple sum over primes. It is exceedingly complicated. We will introduce the following notation: For $j=1,\ldots,k$, let

$$\mathbf{p}_j = \left(p_{j,1},\ldots,p_{j,m_j}\right)$$

and let

$$\mathcal{P}_j = \mathcal{P}\left(\lambda_j, m_j\right) = \left\{ \mathbf{p}_j \ : \ p_{j,\ell} \text{ is prime } \ , \ p_{j,\ell} \leq \exp\left(\frac{N^{\lambda_j}}{40 k m_j}\right) \right\}$$

Hence

$$\prod_{j=1}^{k} P(\lambda_j, m_j)^{m_j} = (\log N)^{-(m_1+\cdots+m_k)/2} \times \sum_{\mathbf{p}_1 \in \mathcal{P}_1, \ldots, \mathbf{p}_k \in \mathcal{P}_k} \frac{\exp\left(-\mathrm{i}u \sum_{j=1}^{k} e^{N^{\lambda_j}} \log(p_{j,1} \ldots p_{j,m_j})\right)}{\sqrt{\prod_{j=1}^{k} \prod_{\ell_j=1}^{m_j} p_{j,\ell_j}}}$$

Similarly, we let

$$\mathbf{q}_j = \left(q_{j,1}, \ldots, q_{j,n_j}\right)$$

and let

$$\mathcal{Q}_j \equiv \mathcal{P}\left(\lambda_j, n_j\right) = \left\{ \mathbf{q}_j \ : \ q_{j,\ell} \text{ is prime } \ , \ q_{j,\ell} \leq \exp\left(\frac{N^{\lambda_j}}{40 k n_j}\right) \right\}$$

and so

$$\prod_{j=1}^{k} \overline{P(\lambda_j, n_j)}^{n_j} = (\log N)^{-(n_1+\cdots+n_k)/2} \times \sum_{\mathbf{q}_1 \in \mathcal{Q}_1, \ldots, \mathbf{q}_k \in \mathcal{Q}_k} \frac{\exp\left(\mathrm{i}u \sum_{j=1}^{k} e^{N^{\lambda_j}} \log(q_{j,1} \ldots q_{j,n_j})\right)}{\sqrt{\prod_{j=1}^{k} \prod_{\ell_j=1}^{n_j} q_{j,\ell_j}}}$$

Finally, let $\mathbf{p} = \bigcup_{j=1}^{k} \mathbf{p}_j$ and $\mathbf{q} = \bigcup_{j=1}^{k} \mathbf{q}_j$, and let

$$F(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^{k} \exp\left(N^{\lambda_j}\right) \log\left(\frac{q_{j,1} \cdots q_{j,n_j}}{p_{j,1} \ldots p_{j,m_j}}\right)$$

Therefore,

$$\prod_{j=1}^{k} P(\lambda_j, m_j)^{m_j} \overline{P(\lambda_j, n_j)}^{n_j} = (\log N)^{-(m_1+\cdots+m_k+n_1+\cdots+n_k)/2} \times \sum_{\substack{\mathbf{p}_1 \in \mathcal{P}_1, \ldots, \mathbf{p}_k \in \mathcal{P}_k \\ \mathbf{q}_1 \in \mathcal{Q}_1, \ldots, \mathbf{q}_k \in \mathcal{Q}_k}} \frac{\exp\left(\mathrm{i}u F(\mathbf{p}, \mathbf{q})\right)}{\sqrt{\prod_{j=1}^{k} \left(\prod_{\ell_j=1}^{m_j} p_{j,\ell_j}\right) \left(\prod_{\ell_j=1}^{n_j} q_{j,\ell_j}\right)}} \tag{4}$$

We divide the sum up into two parts, depending on whether $F(\mathbf{p}, \mathbf{q})$ equals zero or not. The terms where the sum vanishes we call *diagonal terms*; the other terms are *off-diagonal*. The proof of the lemma will follow from showing that the off-diagonal terms do not contribute in the large-N limit, and using a simple combinatorial enumeration of the diagonal terms, along with the prime number theorem, to estimate the diagonal terms.

3.1. **The diagonal terms.** We will see below in §3.2 that since $\lambda_1 > \cdots > \lambda_k$, then for sufficiently large N, the only way for $F(\mathbf{p},\mathbf{q}) = 0$ is if

$$\exp\left(N^{\lambda_j}\right) \log\left(\frac{q_{j,1}\cdots q_{j,n_j}}{p_{j,1}\cdots p_{j,m_j}}\right) = 0$$

for each $j = 1, \ldots, k$ separately. Thus the diagonal terms are those contained in the sets

$$\mathcal{D}_j := \left\{ (\mathbf{p}_j, \mathbf{q}_j) \ : \ \mathbf{p}_j \in \mathcal{P}_j, \mathbf{q}_j \in \mathcal{Q}_j, \prod_{\ell=1}^{m_j} p_{j,\ell} = \prod_{\ell=1}^{n_j} q_{j,\ell} \right\} \tag{5}$$

Since $p_{j,\ell}$ and $q_{j,\ell}$ are both prime, the set $\mathcal{D}_j$ is empty unless $m_j = n_j$. Under such an assumption, the diagonal terms in (4) are

$$\sum_{(\mathbf{p}_1,\mathbf{q}_1)\in\mathcal{D}_1,\ldots,(\mathbf{p}_k,\mathbf{q}_k)\in\mathcal{D}_k} \frac{1}{\prod_{j=1}^{k}\prod_{\ell_j=1}^{n_j} q_{j,\ell_j}} = \prod_{j=1}^{k}\left(\sum_{(\mathbf{p}_j,\mathbf{q}_j)\in\mathcal{D}_j} \frac{1}{q_{j,1}\cdots q_{j,n_j}} \right)$$

If $q_{j,1}, \ldots, q_{j,n_j}$ are distinct primes, then there are $(n_j)!$ ways of choosing $p_{j,1}, \ldots, p_{j,n_j}$ such that the products are equal. (If the $q_{j,\ell}$ are not distinct, then the result is similar, but with a different combinatorial factor, and the result is at least a couple of logarithms smaller). Hence

$$\begin{aligned}
\sum_{(\mathbf{p}_j,\mathbf{q}_j)\in\mathcal{D}_j} \frac{1}{q_{j,1}\cdots q_{j,n_j}} &= (n_j)! \sum_{\mathbf{q}_j\in\mathcal{Q}_j} \frac{1}{q_{j,1}\cdots q_{j,n_j}} \left(1 + O(\frac{1}{\log^2 N})\right) \\
&= (n_j)! \left(\sum_{q\le\exp\left(\frac{N^{\lambda_j}}{40kn_j}\right)} \frac{1}{q} \right)^{n_j} \left(1 + O(\frac{1}{\log^2 N})\right) \\
&= (n_j)! \left(\log\left(\frac{N^{\lambda_j}}{40kn_j}\right) + O(1) \right)^{n_j} \left(1 + O(\frac{1}{\log^2 N})\right) \\
&= (n_j)!(\lambda_j \log N)^{n_j} \left(1 + O(\frac{1}{\log N})\right) \qquad (6)
\end{aligned}$$

Hence the diagonal contribution to (4) is

$$\left(\prod_{j=1}^{k} (n_j)!(\lambda_j)^{n_j}\delta(m_j, n_j) \right) \left(1 + O(\frac{1}{\log N})\right)$$

which is the right-hand side of (3) in the large-N limit. (The constant implicit in the O-term depends on m_j, n_j, λ_j and k, but these are all constants). Hence the proof of the lemma will be complete if we can show there is no contribution to (3) from the off-diagonal terms, the terms where $F(\mathbf{p},\mathbf{q}) \neq 0$.

3.2. **The off-diagonal terms.** Now we show that the non-diagonal terms of (4) do not contribute to (3) in the limit. Upon integrating (4) for u between 1 and 2, we obtain

$$\sideset{}{'}\sum_{\substack{\mathbf{p}_1\in\mathcal{P}_1,\dots,\mathbf{p}_k\in\mathcal{P}_k\\ \mathbf{q}_1\in\mathcal{Q}_1,\dots,\mathbf{q}_k\in\mathcal{Q}_k}} \frac{\exp\left(\mathrm{i}2F(\mathbf{p},\mathbf{q})\right)-\exp\left(\mathrm{i}F(\mathbf{p},\mathbf{q})\right)}{\mathrm{i}F(\mathbf{p},\mathbf{q})\sqrt{\prod_{j=1}^k\left(\prod_{\ell_j=1}^{m_j}p_{j,\ell_j}\right)\left(\prod_{\ell_j=1}^{n_j}q_{j,\ell_j}\right)}} \tag{7}$$

where $\sum'$ denotes that we are summing only over the non-diagonal terms, those terms where $F(\mathbf{p},\mathbf{q})\neq 0$.

Recall that, without loss of generality, we assumed $\lambda_1>\cdots>\lambda_k$. Assume

$$\log\left(\frac{q_{1,1}\cdots q_{1,n_1}}{p_{1,1}\cdots p_{1,m_1}}\right)\neq 0 \tag{8}$$

Since $p_{1,\ell}\leq\exp\left(\frac{N^{\lambda_1}}{40km_1}\right)$ and $q_{1,\ell}\leq\exp\left(\frac{N^{\lambda_1}}{40kn_1}\right)$, we have

$$\exp\left(N^{\lambda_1}\right)\left|\log\left(\frac{q_{1,1}\cdots q_{1,n_1}}{p_{1,1}\cdots p_{1,m_1}}\right)\right| > \frac{1}{2}\exp\left((1-\frac{1}{40k})N^{\lambda_1}\right)$$

which follows from the fact that if m,n are positive integers, and $m\neq n$, then $|\log(m/n)|>1/(2\min(m,n))$. Furthermore, for $j>1$, for $\mathbf{p}_j\in\mathcal{P}_j$ and $\mathbf{q}_j\in\mathcal{Q}_j$, then for sufficiently large N,

$$\exp\left(N^{\lambda_j}\right)\left|\log\left(\frac{q_{j,1}\cdots q_{j,n_j}}{p_{j,1}\cdots p_{j,m_j}}\right)\right|\leq\frac{1}{40k}\exp\left(N^{\lambda_j}\right)N^{\lambda_j}<\exp\left(2N^{\lambda_j}\right)$$

and so we can conclude that if (8) holds,

$$\begin{aligned}|F(\mathbf{p},\mathbf{q})| &> \exp\left(\left(1-\frac{1}{40k}\right)N^{\lambda_1}\right)-\sum_{j=2}^k\exp\left(2N^{\lambda_j}\right)\\ &> \exp\left(\left(1-\frac{1}{20k}\right)N^{\lambda_1}\right)\end{aligned}$$

for sufficiently large N, since $\lambda_j<\lambda_1$ for all $j>1$.

The contribution of such terms to (7) is clearly bounded by

$$\frac{1}{\exp\left((1-\frac{1}{20k})N^{\lambda_1}\right)}\sum_{\substack{\mathbf{p}_1\in\mathcal{P}_1,\dots,\mathbf{p}_k\in\mathcal{P}_k\\ \mathbf{q}_1\in\mathcal{Q}_1,\dots,\mathbf{q}_k\in\mathcal{Q}_k}}\frac{1}{\sqrt{\prod_{j=1}^k\left(\prod_{\ell_j=1}^{m_j}p_{j,\ell_j}\right)\left(\prod_{\ell_j=1}^{n_j}q_{j,\ell_j}\right)}}$$

$$\leq\exp\left(-(1-\frac{1}{20k})N^{\lambda_1}\right)\exp\left(\sum_{j=1}^k\frac{1}{40k}N^{\lambda_j}\right)\leq\exp\left(-(1-\frac{1}{10k})N^{\lambda_1}\right)$$

once more using the fact that

$$\sum_{j=2}^k\frac{1}{40k}N^{\lambda_j}\leq\frac{1}{20k}N^{\lambda_1}$$

for sufficiently large N.

Therefore, as N tends to infinity, we see that terms which satisfy (8) vanish. Thus, for a non-zero result in the limit, we must have

$$\log\left(\frac{q_{1,1}\cdots q_{1,n_1}}{p_{1,1}\cdots p_{1,m_1}}\right)=0$$

That is, we must have $(\mathbf{p}_1,\mathbf{q}_1)\in\mathcal{D}_1$, where $\mathcal{D}_1$ is defined in (5).

The terms which might possibly contribute to (7) are

$$\sideset{}{'}\sum_{\substack{(\mathbf{p_1},\mathbf{q}_2)\in\mathcal{D}_1\\ \mathbf{p}_2\in\mathcal{P}_2,\dots,\mathbf{p}_k\in\mathcal{P}_k\\ \mathbf{q}_2\in\mathcal{Q}_2,\dots,\mathbf{q}_k\in\mathcal{Q}_k}}\frac{\exp\left(\mathrm{i}2F(\mathbf{p},\mathbf{q})\right)-\exp\left(\mathrm{i}F(\mathbf{p},\mathbf{q})\right)}{\mathrm{i}F(\mathbf{p},\mathbf{q})\sqrt{\prod_{j=1}^{k}\left(\prod_{\ell_j=1}^{m_j}p_{j,\ell_j}\right)\left(\prod_{\ell_j=1}^{n_j}q_{j,\ell_j}\right)}}$$

The same argument as above, shows that the terms with

$$\log\left(\frac{q_{2,1}\cdots q_{2,n_2}}{p_{2,1}\cdots p_{2,m_2}}\right)\neq 0$$

contribute

$$\frac{1}{\exp\left((1-\frac{1}{20k})N^{\lambda_2}\right)}\sum_{\substack{(\mathbf{p_1},\mathbf{q}_2)\in\mathcal{D}_1\\ \mathbf{p}_2\in\mathcal{P}_2,\dots,\mathbf{p}_k\in\mathcal{P}_k\\ \mathbf{q}_2\in\mathcal{Q}_2,\dots,\mathbf{q}_k\in\mathcal{Q}_k}}\frac{1}{\sqrt{\prod_{j=1}^{k}\left(\prod_{\ell_j=1}^{m_j}p_{j,\ell_j}\right)\left(\prod_{\ell_j=1}^{n_j}q_{j,\ell_j}\right)}}$$

$$\leq\exp\left(-(1-\frac{1}{10k})N^{\lambda_2}\right)\sum_{(\mathbf{p_1},\mathbf{q}_2)\in\mathcal{D}_1}\frac{1}{\sqrt{\left(\prod_{\ell=1}^{m_1}p_{1,\ell}\right)\left(\prod_{\ell=1}^{n_1}q_{1,\ell}\right)}}$$

If $(\mathbf{p_1},\mathbf{q}_2)\in\mathcal{D}_1$, then (6) shows that

$$\sum_{(\mathbf{p_1},\mathbf{q}_2)\in\mathcal{D}_1}\frac{1}{\sqrt{\left(\prod_{\ell=1}^{m_1}p_{1,\ell}\right)\left(\prod_{\ell=1}^{n_1}q_{1,\ell}\right)}}\ll(\log N)^{n_j}$$

which is negligible compared to $\exp\left(-(1-\frac{1}{10k})N^{\lambda_2}\right)$. Hence, finally, these terms do not contribute.

Repeating the argument for λ_j, $j=3,4,\dots,k$ we see that any term with

$$\log\left(\frac{q_{j,1}\cdots q_{j,n_j}}{p_{j,1}\cdots p_{j,m_j}}\right)\neq 0$$

has a vanishing contribution to the large-N limit. Therefore, the main term must come from those terms for which

$$\log\left(\frac{q_{j,1}\cdots q_{j,n_j}}{p_{j,1}\cdots p_{j,m_j}}\right)=0$$

for all j. Such terms are the diagonal terms, and their contribution has been calculated above. This completes the proof of Lemma 6. □

Proof of Theorem 4. Recall that

$$L_\lambda(N,u)=\frac{\log\zeta(\frac{1}{2}+\mathrm{i}ue^{N^\lambda})}{\sqrt{\log N}}.$$

and

$$P(\lambda,n;k,N,u)=\frac{1}{\sqrt{\log N}}\sum_{p\le\exp\left(\frac{N^{\lambda_\ell}}{40km_\ell}\right)}\frac{p^{-\mathrm{i}ue^{N^\lambda}}}{\sqrt{p}}$$

Let

$$\begin{aligned}\epsilon(\lambda,n)&=\epsilon(\lambda,n;k,N,u)\\&=L_\lambda(N,u)-P(\lambda,n;k,N,u)\end{aligned}$$

so that, if we write $T=\exp(N^{\lambda_j})$, then changing variables to $t=Tu$,

$$\begin{aligned}&\int_1^2|\epsilon(\lambda_j,m_j)|^{2km_j}\ \mathrm{d}u\\&\qquad=\frac{1}{(\log N)^{km_j}}\frac{1}{T}\int_T^{2T}\left|\log\zeta(\tfrac12+\mathrm{i}t)-\sum_{p\le T^{1/40km_\ell}}\frac{p^{-\mathrm{i}t}}{\sqrt{p}}\right|^{2km_j}\mathrm{d}t\\&\qquad=O\left(\frac{(km_j)^{4km_j}e^{Akm_j}}{(\log N)^{km_j}}\right)\qquad(9)\end{aligned}$$

by Theorem 5. Since the m_j are fixed, this tends to zero as $N\to\infty$.

Consider

$$\int_1^2\left|\prod_{j=1}^k L_{\lambda_j}(N,u)^{m_j}\overline{L_{\lambda_j}(N,u)}^{n_j}-\prod_{j=1}^k P(\lambda_j,m_j)^{m_j}\overline{P(\lambda_j,n_j)}^{n_j}\right|\mathrm{d}u\qquad(10)$$

Writing $L_{\lambda_j}(N,u)$ in terms of $P(\lambda_j,m_j)$ and $\epsilon(\lambda_j,m_j)$, we see that the term inside the modulus signs equals

$$\sum_{\substack{0\le\alpha_1\le m_1,\dots,0\le\alpha_k\le m_k\\0\le\beta_1\le n_1,\dots,0\le\beta_k\le n_k\\\sum\alpha_j+\beta_j\ge1}}\prod_{j=1}^k\binom{m_j}{\alpha_j}\binom{n_j}{\beta_j}P(\lambda_j,m_j)^{m_j-\alpha_j}\epsilon(\lambda_j,m_j)^{\alpha_j}\overline{P(\lambda_j,n_j)}^{n_j-\beta_j}\overline{\epsilon(\lambda_j,n_j)}^{\beta_j}$$

The integral of this in (10) is clearly bounded by

$$\begin{aligned}&\sum_{\substack{0\le\alpha_1\le m_1,\dots,0\le\alpha_k\le m_k\\0\le\beta_1\le n_1,\dots,0\le\beta_k\le n_k\\\sum\alpha_j+\beta_j\ge1}}\left\{\prod_{j=1}^k\binom{m_j}{\alpha_j}\binom{n_j}{\beta_j}\right\}\\&\times\int_1^2\prod_{j=1}^k|P(\lambda_j,m_j)|^{m_j-\alpha_j}|\epsilon(\lambda_j,m_j)|^{\alpha_j}\left|\overline{P(\lambda_j,n_j)}\right|^{n_j-\beta_j}\left|\overline{\epsilon(\lambda_j,n_j)}\right|^{\beta_j}\ \mathrm{d}u\end{aligned}\qquad(11)$$

A version of the generalized Hölder inequality states that

$$\int_1^2 \prod_{j=1}^{k} |A_j||B_j||C_j||D_j| \,\mathrm{d}u$$
$$\leq \prod_{j=1}^{k} \left(\int_1^2 |A_j|^{2kr_j} \,\mathrm{d}u \right)^{1/(2kr_j)} \left(\int_1^2 |B_j|^{2ks_j} \,\mathrm{d}u \right)^{1/(2ks_j)}$$
$$\times \left(\int_1^2 |C_j|^{2kt_j} \,\mathrm{d}u \right)^{1/(2kt_j)} \left(\int_1^2 |D_j|^{2ku_j} \,\mathrm{d}u \right)^{1/(2ku_j)}$$

so long as $\frac{1}{r_j} + \frac{1}{s_j} = 1$ and $\frac{1}{t_j} + \frac{1}{u_j} = 1$ for all $j = 1, \ldots, k$.

Choosing $r_j = m_j/(m_j - \alpha_j)$ and $s_j = m_j/\alpha_j$, and $t_j = n_j/(n_j - \beta_j)$ and $u_j = n_j/\beta_j$, we see that we may bound the above integral by

$$\prod_{j=1}^{k} \left(\int_1^2 |P(\lambda_j, m_j)|^{2km_j} \,\mathrm{d}u \right)^{\frac{m_j - \alpha_j}{2km_j}} \left(\int_1^2 |\epsilon(\lambda_j, m_j)|^{2km_j} \,\mathrm{d}u \right)^{\frac{\alpha_j}{2km_j}}$$
$$\times \left(\int_1^2 \left|\overline{P(\lambda_j, n_j)}\right|^{2kn_j} \,\mathrm{d}u \right)^{\frac{n_j - \beta_j}{2kn_j}} \left(\int_1^2 \left|\overline{\epsilon(\lambda_j, n_j)}\right|^{2kn_j} \,\mathrm{d}u \right)^{\frac{\beta_j}{2kn_j}}$$

From (9), if $\alpha_j \neq 0$,

$$\lim_{N\to\infty} \left(\int_1^2 |\epsilon(\lambda_j, m_j)|^{2km_j} \,\mathrm{d}u \right)^{\frac{\alpha_j}{2km_j}} = 0$$

and from Lemma 6 we have

$$\left(\int_1^2 |P(\lambda_j, m_j)|^{2km_j} \,\mathrm{d}u \right)^{\frac{m_j - \alpha_j}{2km_j}} \ll 1$$

Since the sum in (11) is over those α_j and β_j such that $\sum \alpha_j + \beta_j \geq 1$, there must be at least one j with a non-zero α_j or β_j. Hence, as $N \to \infty$, all the terms in (11) tend to zero. The sum is over a finite number of terms, so we may conclude that

$$\lim_{N\to\infty} \int_1^2 \left| \prod_{j=1}^{k} L_{\lambda_j}(N,u)^{m_j} \overline{L_{\lambda_j}(N,u)}^{n_j} - \prod_{j=1}^{k} P(\lambda_j, m_j)^{m_j} \overline{P(\lambda_j, n_j)}^{n_j} \right| \mathrm{d}u = 0$$

which implies

$$\lim_{N\to\infty} \int_1^2 \prod_{j=1}^{k} L_{\lambda_j}(N,u)^{m_j} \overline{L_{\lambda_j}(N,u)}^{n_j} \,\mathrm{d}u = \lim_{N\to\infty} \prod_{j=1}^{k} P(\lambda_j, m_j)^{m_j} \overline{P(\lambda_j, n_j)}^{n_j}$$

assuming the limits make sense. Therefore Theorem 4 follows from Lemma 6. □

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