NON-SEMISIMPLE INVARIANTS AND HABIRO'S SERIES

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Abstract. In this paper we establish an explicit relationship between Habiro's cyclotomic expansion of the colored Jones polynomial (evaluated at a pth root of unity) and the Akutsu-Deguchi-Ohtsuki (ADO) invariants of the double twist knots. This allows us to compare the Witten-Reshetikhin-Turaev (WRT) and Costantino-Geer-Patureau (CGP) invariants of 3-manifolds obtained by 0-surgery on these knots. The difference between them is determined by the $p-1$ coefficient of the Habiro series. We expect these to hold for all Seifert genus 1 knots.

1. Introduction

In [H] Habiro stated the following result. Given a 0-framed knot K and an N-dimensional irreducible representation of the quantum $\mathfrak{sl}_2$, there exist polynomials $C_n(K;q) \in \mathbb{Z}[q^{\pm 1}]$, $n \in \mathbb{N}$, such that

$$J_K(q^N, q) = \sum_{n=0}^{\infty} C_n(K;q)\,(q^{1+N};q)_n (q^{1-N};q)_n \tag{1.1}$$

is the N-colored Jones polynomial of K. Here $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$ and q is a generic parameter. Replacing q^N by a formal variable x we get

$$J_K(x,q) = \sum_{n=0}^{\infty} C_n(K;q)\,(xq;q)_n (x^{-1}q;q)_n = \sum_{m\geq 0} a_m(K;q)\,\sigma_m(x,q) \tag{1.2}$$

where

$$\sigma_m(x,q) = \prod_{i=1}^{m} \left(x + x^{-1} - q^i - q^{-i}\right) \quad \text{and} \quad C_m(K;q) = (-1)^m q^{-\frac{1}{2}m(m+1)} a_m(K;q)$$

known as *cyclotomic expansion* of the colored Jones polynomial or simply *Habiro's series*. This expression dominates all colored Jones polynomials and converges in the cyclotomic completion

$$\varprojlim_n \mathbb{Z}[q^{\pm 1}][x + x^{-1}]/(\sigma_n(x,q))$$

of the center of $U_q(\mathfrak{sl}_2)$. In details, (1.2) belongs to the cyclotomic completion of the even part of the center after the identification $C^2 - 2 = x + x^{-1}$, where C is the Casimir.

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Habiro's series played a central role in the construction of the unified invariants of homology 3-spheres [H, HaL, BBL] dominating all WRT invariants. In [BCL] it was used to prove integrality of the WRT invariants for all 3-manifolds at all roots of unity.

Given the power of Habiro's series, it is not surprising that they are notoriously difficult to compute. So far, Habiro's cyclotomic expansions were computed explicitly for the following infinite families: the twist knots [Ma], the $(2, 2t+1)$ torus knots [HL], and the double twist knots [LO].

Recently the non-semisimple quantum invariants of links and 3-manifolds attracted a lot of attention. Physicists expect them to play a crucial role in categorification of quantum 3-manifold invariants [GPV]. Mathematicians resolved the problem of nullity of these invariants by introducing the *modified traces* [GPV, BBG].

The aim of this paper is to connect the Habiro cyclotomic expansion with the non-semisimple world. The non-semisimple invariants arise in specializations of the quantum $\mathfrak{sl}_2$ at $q = e_p$, the primitive p^{th} root of unity. The ADO link invariant is obtained in the setting of the unrolled quantum $\mathfrak{sl}_2$ [CGP1]. This group admits p-dimensional irreducible projective modules V_λ whose highest weights λ are given by any complex number. Even through the definition of the unrolled quantum group requires a choice of the square root of e_p (that we denote by e_{2p}), the ADO invariant of a 0-framed knot does not depend on this choice. The representation category of the unrolled $\mathfrak{sl}_2$ is ribbon, and hence, the ADO invariant can be defined by applying the usual Reshetikhin-Turaev construction, i.e. we color the $(1,1)$-tangle T whose closure is K with V_λ, the Reshetikhin-Turaev functor sends then T to an endomorphism of V_λ that is $\mathrm{ADO}_K(e_p^{\lambda+1}, e_p)\mathrm{id}_{V_\lambda}$.

The corresponding non-semisimple 3-manifold invariant for a pair (M, λ), where M is a closed oriented 3-manifold and $\lambda \in H^1(M, \mathbb{C}/2\mathbb{Z})$ is a cohomology class, was defined in [CGP]. If $M = S^3(K)$ is obtained by surgery on a knot K in S^3 with non-zero framing, then M is a rational homology 3-sphere and λ is rational. In this case, the CGP invariant of M was shown to be determined by the Witten-Reshetikhin-Turaev invariant (WRT) in [CGP2]. It remains to analyze the case of 0-framed surgeries with $\lambda \neq 0, 1$. For a 0-framed knot K,

$$\mathrm{CGP}(S^3(K), \lambda) = \sum_{n=0}^{p-1} d^2(\lambda + 2n)\mathrm{ADO}_K(e_p^{\lambda+2n+1}, e_p)$$

where $d(\lambda + 2n)$ is the modified dimension of $V_{\lambda+2n}$.

2. Our results

Recently, in [W] Willetts constructed the knot invariant

$$F_\infty(q, x; K) \in \hat{R} := \varprojlim_n \frac{\mathbb{Z}[q^{\pm 1/2}, x^{\pm 1/2}]}{I_n}$$

where I_n is the ideal generated by

$$\left\{ \prod_{i=0}^{n} (x^{\frac{1}{2}} q^{\frac{l+i}{2}} - x^{-\frac{1}{2}} q^{-\frac{l+i}{2}}) \mid l \in \mathbb{Z} \right\}$$

that dominates the colored Jones polynomials and the ADO invariants of K. In details,

$$F_\infty(q, q^N; K) = J_K(q^N, q) \quad \text{and} \quad F_\infty(e_p, x; K) = \frac{\mathrm{ADO}_K(x, e_p)}{\Delta_K(x^p)} \tag{2.1}$$

where $\Delta_K(x)$ is the Alexander polynomial of K. The famous Melvin-Morton-Rozansky theorem [Ro] follows from the above result at $p = 1$.

We claim that $F_\infty(q, x; K)$ coincide with the Habiro expansion [H, HaL] $J_K(x, q)$ as elements of $\hat{R}$. Hence, the result of Willetts in [W] can be reformulated as follows.

Theorem 2.1. *The universal Habiro series determine the ADO invariants, i.e.*

$$J_K(x, e_p) = \frac{\mathrm{ADO}_K(x, e_p)}{\Delta_K(x^p)}.$$

Example. For the first two knots at the first two roots of unity this looks as follows:

$$J_{3_1}(t, q) = \sum_{m \geq 0} (-1)^m q^{m(m+3)/2} \, \sigma_m(t, q), \quad J_{3_1}(t, 1) = \sum_{m \geq 0} (-1)^m (t + t^{-1} - 2)^m = \frac{1}{t + t^{-1} - 1},$$

$$J_{3_1}(t, -1) = (1 - (t + t^{-1} + 2)) \sum_{m \geq 0} (-1)^m (t^2 + t^{-2} - 2)^m = \frac{\Delta_{3_1}(-t)}{\Delta_{3_1}(t^2)},$$

$$J_{4_1}(t, q) = \sum_{m \geq 0} \sigma_m(t, q), \quad J_{4_1}(t, 1) = \sum_{m \geq 0} (t + t^{-1} - 2)^m = \frac{1}{1 - (t + t^{-1} - 2)} = \frac{1}{\Delta_{4_1}(t)},$$

$$J_{4_1}(t, -1) = (1 + t + t^{-1} + 2) \sum_{m \geq 0} (t^2 + t^{-2} - 2)^m = \frac{\Delta_{4_1}(-t)}{\Delta_{4_1}(t^2)}.$$

A challenging open problem is to find an explicit formula for ADO using the coefficients $a_n(K; e_p)$ of the Habiro series. In the examples above this is possible due to periodicity of these coefficients. Our next theorem generalizes these examples to all double twist knots.

Let $\{K_{(l,m)} \mid l, m \in \mathbb{Z}\}$ be the 2-parameter family of double twist knots such that $K_{(l,m)} = K_{(m,l)}$, $K_{(1,1)} = 3_1$ and $K_{(-1,1)} = 4_1$ depicted in Figure 1.

Theorem 2.2. *Let $K = K_{(l,m)}$ be a double twist knot. Then we have a new expression for its ADO invariant*

$$\mathrm{ADO}_K(x, e_p) = \sum_{n=0}^{p-1} a_n(K; e_p) \, \sigma_n(x, e_p) \ .$$

In addition, for all natural numbers n, k we have

$$a_{n+kp}(K; e_p) = a_n(K; e_p) \, a_k(K; 1) \quad \textit{and} \quad \Delta_K^{-1}(t) = \sum_k a_{kp}(K; e_p)(t + t^{-1} - 2)^k.$$

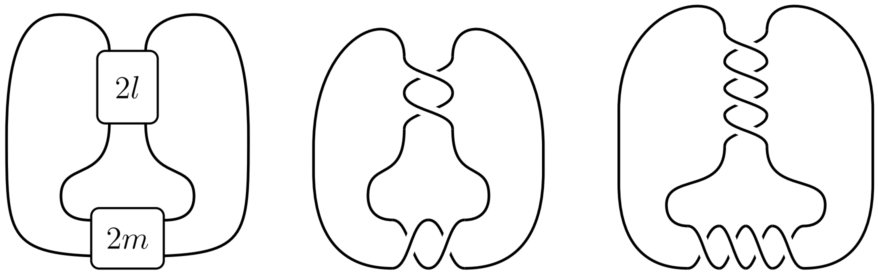

FIGURE 1. The double twist knots: $K_{(l,m)}$, the trefoil and $K_{(2,-2)}$. The integers in the boxes denote the number of half twists.

Our next result establishes a relationship between CGP and WRT invariants of 3-manifolds obtained by 0-surgery on double twist knots. Note that for rational surgery on knots CGP is known to be determined by WRT and the order of the first homology group [CGP2].

Theorem 2.3. *Let $M = S^3(K_{(l,m)})$ where $K_{(l,m)}$ is 0-framed double twist knot and $\lambda \in \mathbb{C}/2\mathbb{Z}$ with $\lambda \neq 0, 1$. For odd $p > 1$ we have*

$$\mathrm{CGP}(M,\lambda) = \frac{1}{\{p\lambda\}^2}\mathrm{WRT}(M) + p\, a_{p-1}(K_{(l,m)}; e_p) \quad \textit{where} \quad \{y\} = e_{2p}^{y} - e_{2p}^{-y}.$$

Corollary 2.4. *The coefficient $a_{p-1}(K_{(l,m)}; e_p)$ of the Habiro series is a topological invariant of the 3-manifold obtained by 0-surgery on the double twist knot $K_{(l,m)}$.*

For $p = 2$, $\mathrm{WRT}(M) = 1$ and $\mathrm{CGP}(M)$ is the Reidemeister torsion of M according to [BCGP, Thm. 6.23]. Hence, one can think about the coefficient $a_{p-1}(K; e_p)$ as a generalization of the Reidemeister torsion. It is an interesting open problem to find a topological interpretation of this invariant. In general we would expect following to hold.

Conjecture 2.5. *Theorems 2.2, 2.3 hold for any Seifert genus 1 knot.*

Examples of knots with higher Seifert genus provide $(2, 2t+1)$ torus knots with $t \geq 2$. Our next result is a computation of the ADO invariants for this family of knots.

Theorem 2.6. *Let $K = T_{(2,2t+1)}$ be a torus knot. Then the ADO invariant is*

$$\mathrm{ADO}_K(x, e_p) = {e_p}^t x^{t(1-p)} \sum_{k_t \geq \cdots \geq k_1 \geq 0}^{p-1} (x\, e_p; e_p)_{k_t}\, x^{k_t} \prod_{i=1}^{t-1} {e_p}^{k_i(k_i+1)} x^{2k_i} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}_{e_p}$$

An interesting open problem is to determine periodicity of $\{a_n(T_{(2,2t+1)}, e_p)\}_{n\geq 0}$ analogous to those for double twist knots in Theorem 2.2. In Appendix we show that $\{a_n(T_{(2,5)}, e_p)\}_{n\geq 0}$ do not satisfy properties listed in Theorem 2.2.

In the last section we compute CGP and WRT invariants for 0-surgeries on $(2, 2t + 1)$ torus knots and compare them. We observe that (up to normalization) CGP can be viewed as a Laurent polynomial in $T^{\pm 1} := (e_p^{\pm\lambda})^p$, such that its evaluation at $T = 1$ reproduces the WRT invariant. In contrast to Theorem 2.3 however, for torus knots the WRT does not form anymore the degree zero part of this Laurent polynomial.

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3. Proofs

In this section we prove our four Theorems.

3.1. **Preliminaries.** We set $[n]_q = \frac{1-q^n}{1-q}$, $[n]_q! = [n]_q \dots [2]_q[1]_q$. The q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

Observe that evaluating at $q = e_p$, the primitive p^{th} root of unity, we have

$$\begin{bmatrix} n+ap \\ k+bp \end{bmatrix}_{e_p} = \begin{bmatrix} n \\ k \end{bmatrix}_{e_p} \binom{a}{b} \tag{3.1}$$

where the last factor is the usual binomial coefficient. Moreover, setting $y = 1$ in the identity

$$x^p + x^{-p} - y^p - y^{-p} = \prod_{i=1}^{p} \left(x + x^{-1} - y e_p^i - y^{-1} e_p^{-i} \right),$$

we get the following relation

$$\sigma_p(x, e_p) = x^p + x^{-p} - 2 = (1 - x^p)(x^{-p} - 1) \tag{3.2}$$

3.2. **Proof of Theorem 2.1.** For $J_K(x,q)$ in (1.2), it is easy to check that its evaluations at $x = 1, q, q^2, \dots$ determine the coefficients $C_n(K;q)$ recursively. Explicitly,

$$C_n(K;q) = -q^{n+1} \sum_{l=1}^{n+1} \frac{(1-q^l)(1-q^{2l})}{(q)_{n+1-l}(q)_{n+1+l}} (-1)^l q^{\frac{1}{2}l(l-3)} J_K(q^l, q) .$$

Moreover, $J_K(x,q) \in \hat{R}$ by (1.2). In [W, Prop. 57] Willetts proved that the evaluations at $x = q^N$ for $N \in \mathbb{N}$ determine $F_\infty(q,x;K)$. Moreover, both invariants coincide at all evaluations, since $F_\infty(q, q^N; K) = J_K(q^N, q)$. Hence,

$$F_\infty(q,x;K) = J_K(x,q) \in \hat{R}$$

and the result follows from the Willetts theorem. $\square$

3.3. **Proof of Theorem 2.2.** Let $K = K_{(l,m)}$ be a double twist knot. For $p = 1$, Theorem 2.1 reads $J_K(t,1) = \frac{1}{\Delta_K(t)}$ and hence using (3.2)

$$\frac{1}{\Delta_K(x^p)} = \sum_{k=0}^{\infty} a_k(1)\sigma_p^k$$

where we set $\sigma_k := \sigma_k(x, e_p)$ and $a_k(e_p) := a_k(K; e_p)$ for brevity. From Theorem 2.1 assuming (3.4) we get

$$J_K(x,e_p) = \sum_{k=0}^{\infty}\left(\sum_{n=0}^{p-1} a_{n+pk}(e_p)\,\sigma_n\right)\sigma_p^k = \sum_{n=0}^{p-1} a_n(e_p)\sigma_n \sum_{k=0}^{\infty} a_k(1)\sigma_p^k. \tag{3.3}$$

Here we used $\sigma_{n+kp} = \sigma_n\sigma_p^k$. The desired symmetries for the Habiro coefficients follow. It remains to prove

$$\mathrm{ADO}_K(e_p^\lambda, e_p) = \sum_{n=0}^{p-1} a_n(e_p)\,\sigma_n(e_p^\lambda, e_p)\ . \tag{3.4}$$

For this purpose, let us first observe that the Alexander polynomial of the double twist knot K is (see, *e.g.*, [Hil])

$$\begin{aligned}\Delta_K(x) &= 1 + lm(x + x^{-1} - 2)\\ &= f_{lm}(-\tfrac{(1-x)^2}{x})\end{aligned} \tag{3.5}$$

where $f_n(z) = 1 - nz$. Hence,

$$\frac{1}{f_n(z)} = 1 + nz + n^2z^2 + n^3z^3 + \dots$$

and

$$\frac{1}{\Delta_K(x)} = \sum_{n=0}^{\infty} (-lm)^n\ (x + x^{-1} - 2)^n = \sum_{n=0}^{\infty} l^n m^n\ (1-x)^n(1-x^{-1})^n \tag{3.6}$$

The Habiro expansion of the colored Jones polynomial for double twist knots is given by Lovejoy and Osburn [LO]. For $l, m > 0$ we have

$$J_{K_{(l,m)}}(x,q) = \sum_{\substack{n=t_m\geq\cdots\geq t_1\geq 0\\ n=s_l\geq\cdots\geq s_1\geq 0}} (xq;q)_n(x^{-1}q;q)_n\, q^n \prod_{i=1}^{l-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1}\\ s_i\end{bmatrix}_q \prod_{j=1}^{m-1} q^{t_j(t_j+1)} \begin{bmatrix} t_{j+1}\\ t_j\end{bmatrix}_q \tag{3.7}$$

(3.8)

$$J_{K_{(l,-m)}}(x,q) = \sum_{\substack{n=t_m\geq\cdots\geq t_1\geq 0\\ n=s_l\geq\cdots\geq s_1\geq 0}} (xq;q)_n(x^{-1}q;q)_n\,(-1)^n q^{-\frac{1}{2}n(n+1)} \prod_{i=1}^{l-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1}\\ s_i\end{bmatrix}_q \prod_{j=1}^{m-1} q^{-t_j(t_{j+1}+1)} \begin{bmatrix} t_{j+1}\\ t_j\end{bmatrix}_q$$

Note that due to the following symmetries: $K_{(l,m)} = K_{(m,l)}$ and $K_{(-l,-m)}$ is the mirror image of $K_{(l,m)}$, the expressions above cover all double twist knots (up to substitution of q by q^{-1}).

Evaluating (3.7) at $q = e_p$ we get

$$
\begin{aligned}
&J_{K_{(l,m)}}(x, e_p)\\
=&\sum_{k=0}^{\infty}\sum_{n=0}^{p-1}(xe_p;e_p)_{kp+n}(x^{-1}e_p;e_p)_{kp+n}e_p^n\\
&\qquad\times\sum_{\substack{kp+n=t_m\geq t_{m-1}\geq\cdots\geq t_1\geq 0\\ kp+n=s_l\geq s_{l-1}\geq\cdots\geq s_1\geq 0}}\prod_{i=1}^{l-1}{e_p}^{s_i(s_i+1)}\begin{bmatrix}s_{i+1}\\ s_i\end{bmatrix}_{e_p}\prod_{j=1}^{m-1}{e_p}^{t_j(t_j+1)}\begin{bmatrix}t_{j+1}\\ t_j\end{bmatrix}_{e_p}\\
=&\sum_{k=0}^{\infty}(1-x^p)^k(1-x^{-p})^k\,(lm)^k\sum_{n=0}^{p-1}(xe_p;e_p)_n(x^{-1}e_p;e_p)_n\,e_p^n\\
&\qquad\times\sum_{\substack{n=t_m\geq t_{m-1}\geq\cdots\geq t_1\geq 0\\ n=s_l\geq s_{l-1}\geq\cdots\geq s_1\geq 0}}\prod_{i=1}^{l-1}{e_p}^{s_i(s_i+1)}\begin{bmatrix}s_{i+1}\\ s_i\end{bmatrix}_{e_p}\prod_{j=1}^{m-1}{e_p}^{t_j(t_j+1)}\begin{bmatrix}t_{j+1}\\ t_j\end{bmatrix}_{e_p}\\
=&\frac{1}{\Delta_{K_m}(x^p)}\sum_{n=0}^{p-1}(xe_p;e_p)_n(x^{-1}e_p;e_p)_n\,e_p^n\sum_{\substack{n=t_m\geq t_{m-1}\geq\cdots\geq t_1\geq 0\\ n=s_l\geq s_{l-1}\geq\cdots\geq s_1\geq 0}}\prod_{i=1}^{l-1}\begin{bmatrix}s_{i+1}\\ s_i\end{bmatrix}_{e_p}\prod_{j=1}^{m-1}{e_p}^{t_j(t_j+1)}\begin{bmatrix}t_{j+1}\\ t_j\end{bmatrix}_{e_p}
\end{aligned}
$$

Here we have used (3.1) and (3.2) (see also (3.15)). Thus applying Theorem 2.1 we get (3.4). The case of $K_{(l,-m)}$ is similar. As a result, for any double twist knot K we have

$$\mathrm{ADO}_K(x, e_p) = \sum_{n=0}^{p-1} a_n(K; e_p)\sigma_n(x, e_p).$$

□

3.4. **Proof of Theorem 2.3.** The non-semisimple invariant is defined as follows

$$\mathrm{CGP}(S^3(K),\lambda) = \sum_{n=0}^{p-1} d^2(\lambda+2n)\mathrm{ADO}_K(e_p^{\lambda+2n+1}, e_p) \quad \text{where} \quad d(y) = \frac{\{y+1\}}{\{py\}} \tag{3.9}$$

is the modified dimension and $\{y\} = e_{2p}^{y} - e_{2p}^{-y}$. Inserting the new expression for the ADO invariant and exchanging the sums we get

$$\mathrm{CGP}(S^3(K),\lambda) = \frac{1}{\{p\lambda\}^2}\sum_{m=0}^{p-1} a_m(e_p)\sum_{n=0}^{p-1}\{\lambda+2n+1\}^2\sigma_m(e_p^{\lambda+2n+1}, e_p) \tag{3.10}$$

On the other hand, for odd p the WRT invariant can be written as follows (see [Le])

$$
\begin{aligned}
\mathrm{WRT}(S^3(K)) &= \sum_{\substack{0<n<2p\\ n:\ \mathrm{odd}}} \left(e_{2p}^{n} - e_{2p}^{-n}\right)^2 J_K(x = e_p^{-n}, e_p) \qquad (3.11)\\
&= \sum_{m=0}^{(p-3)/2} a_m(e_p)\sum_{n=0}^{p-1}\{2n+1\}^2\sigma_m(e_p^{2n+1}, e_p)
\end{aligned}
$$

(see,*e.g.* [Le]). The usual normalization of the WRT can be obtained by multiplying with $\{1\}^{-2}$. Both invariants can be computed explicitly using so-called Laplace transform method. Observe that up to normalization both invariants can be written as

$$\sum_m a_m(e_p)\left(\sum_{n=0}^{p-1}\{z+m\}\{z+m-1\}\dots\{z+1\}\{z\}^2\{z-1\}\dots\{z-m+1\}\{z-m\}\right)$$

where $z=\lambda+2n+1$ for the CGP and $z=2n+1$ for the WRT invariants. The expression in the brackets is a monic polynomial of degree $m+1$ in $e_p^z+e_p^{-z}$. Moreover, for an odd p and any $a\in\mathbb{Z}$ s.t. $0\le|a|\le p$ we have

$$\sum_{n=0}^{p-1} e_p^{(\lambda+2n+1)a}=\begin{cases}0 & \text{if } a\neq 0,\pm p\\ p & \text{if } a=0\\ pe_p^{\pm\lambda p} & \text{if } a=\pm p\end{cases} \tag{3.12}$$

The contribution from terms with $m<(p-1)/2$ and $a=0$ in CGP is exactly the WRT invariant of $M=S^3(K)$, i.e.

$$\mathrm{WRT}(M)=-2p\sum_{m=0}^{(p-3)/2}(-1)^m a_m(e_p)\begin{bmatrix}2m+1\\ m\end{bmatrix}_{e_p} e_p^{-\frac{1}{2}m(m+1)}$$

(compare [Le, Prop.3.1]). The next coefficients for $a=0$ and $(p-1)/2\le m<p-1$ are zero, since the q-binomial in this case contains $\{p\}=0$. The contribution for $a=\pm p$ and $m=p-1$ gives $pa_{p-1}(e_p^{\lambda p}+e_p^{-\lambda p})$ and for $a=0$ and $m=p-1$

$$(-2)p\,a_{p-1}(e_p)\,(-1)^{p-1}\begin{bmatrix}2p-1\\ p\end{bmatrix}_{e_{2p}}=-2pa_{p-1}(e_p).$$

Using that $\{p\lambda\}^2=e_p^{\lambda p}+e_p^{-\lambda p}-2$ we get the result. □

3.5. **Proof of Theorem 2.6.** It is known that the N-colored Jones polynomial for knot K satisfies the recurrence relation. For instance, for torus knot $K=T_{(s,t)}$, it is [Hi2]

$$J_{T_{(s,t)}}(q^N,q)=\frac{q^{\frac{1}{2}(s-1)(t-1)(1-N)}}{1-q^{-N}}\left(1-q^{s(1-N)-1}-q^{t(1-N)-1}+q^{(s+t)(1-N)}\right)+\frac{1-q^{2-N}}{1-q^{-N}}q^{st(1-N)-1}J_{T_{(s,t)}}(q^{N-2},q) \tag{3.13}$$

For a case of $K=T_{(2,2t+1)}$, we have a q-hypergeometric series [Hi]

$$J_K(x,q)=(qx)^t\sum_{k_t\ge\cdots\ge k_2\ge k_1\ge 0}(qx;q)_{k_t}x^{k_t}\prod_{i=1}^{t-1}q^{k_i(k_i+1)}x^{2k_i}\begin{bmatrix}k_{i+1}\\ k_i\end{bmatrix}_q\in\hat{R}, \tag{3.14}$$

where the N-colored Jones polynomial is given by setting $x=q^{-N}$. Following [W], we get the ADO invariant by putting $q=e_p$ in this expression and then by multiplying the result with the Alexander polynomial. Note that

$$\Delta_K(x) = x^{-t}\frac{1+x^{2t+1}}{1+x}.$$

As an application of (3.1), we have

$$\sum_{p+m=k_t\geq\cdots\geq k_1\geq 0}\prod_{i=1}^{t-1}{e_p}^{k_i(k_i+1)}x^{2k_i}\begin{bmatrix}k_{i+1}\\ k_i\end{bmatrix}_{e_p} = \frac{1-x^{2tp}}{1-x^{2p}}\sum_{m=k_t\geq\cdots\geq k_1\geq 0}\prod_{i=1}^{t-1}{e_p}^{k_i(k_i+1)}x^{2k_i}\begin{bmatrix}k_{i+1}\\ k_i\end{bmatrix}_{e_p}. \tag{3.15}$$

Then we see that

$$J_K(x,e_p) = \frac{1}{\Delta_K(x^p)}{e_p}^t x^{t(1-p)}\sum_{k_t\geq\cdots\geq k_1\geq 0}^{p-1}(xe_p;e_p)_{k_t}x^{k_t}\prod_{i=1}^{t-1}{e_p}^{k_i(k_i+1)}x^{2k_i}\begin{bmatrix}k_{i+1}\\ k_i\end{bmatrix}_{e_p}$$

This implies the statement of the theorem. □

Example. In the case of $p=2$, *i.e.* $e_2=-1$, we have $\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right]_{-1}=1$ when $0\leq b\leq a\leq 1$. Then we get the known result that the ADO polynomial coincides with the Alexander polynomial as follows.

$$\begin{aligned}\mathrm{ADO}_K(x,-1) &= (-1)^t x^{-t}\sum_{k_t\geq\cdots\geq k_1\geq 0}^{1}(-x;-1)_{k_t}x^{k_t}x^{2(k_1+\cdots+k_{t-1})}\\ &= (-x)^{-t}\left(1+(1+x)x\frac{1-x^{2t}}{1-x^2}\right)\\ &= \Delta_K(-x)\end{aligned}$$

Remark on Theorem 2.6. Setting $x=q^{-N}$ in the recurrence relation (3.13), we get

$$J_K(x,q) = \frac{(xq)^{\frac{1}{2}(s-1)(t-1)}}{1-x}\left(1-q^{s-1}x^s-q^{t-1}x^t+q^{s+t}x^{s+t}\right)+\frac{1-xq^2}{1-x}q^{st-1}x^{st}J_K(xq^2,q)$$

whose solution is a q-series as

$$J_{T_{(s,t)}}(x,q) = \frac{(qx)^{\frac{1}{2}(s-1)(t-1)}}{1-x}\sum_{n\geq 0}\chi_{s,t}(n)\,q^{\frac{n^2-(st-s-t)^2}{4st}}x^{\frac{1}{2}(n-(st-s-t))} \tag{3.16}$$

where

$$\chi_{s,t}(n) = \begin{cases}1, & \text{for } n = st\pm(s+t) \mod 2st,\\ -1, & \text{for } n = st\pm(s-t) \mod 2st,\\ 0, & \text{otherwise.}\end{cases}$$

It is conjectured [GHN+] that this gives the ADO invariant

$$\begin{aligned}\mathrm{ADO}_{T_{(s,t)}}(x,e_p) &= \Delta_{T_{(s,t)}}(x^p)\,J_{T_{(s,t)}}(x,e_p)\\ &= \frac{x^{\frac{1}{2}-\frac{1}{2}(s-1)(t-1)p}(1-x^p)}{(1-x)(1-x^{sp})(1-x^{tp})}e_p^{\frac{1}{4}(st-\frac{s}{t}-\frac{t}{s})}\sum_{l=0}^{2stp}\chi_{s,t}(l)e_p^{\frac{l^2}{4st}}x^{\frac{l}{2}}\end{aligned} \tag{3.17}$$

Numerical computations for some p's in the case of $T_{(2,2t+1)}$ support the equality, Theorem 2.6 and (3.17).

For a mirror image of $T_{(2,2t+1)}$, the coefficients of the Habiro expansion (1.2) were determined in [HL] as

(3.18)
$$a_n(\overline{T_{(2,2t+1)}};q) = (-1)^n q^{\frac{1}{2}n(n+1)+n+1-t} \sum_{n+1=k_t\geq k_{t-1}\geq\cdots\geq k_1\geq 1} \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1}+k_i-i+2\sum_{j=1}^{i-1}k_j \\ k_{i+1}-k_i \end{bmatrix}_q .$$

This is written in q-binomial, and can be evaluated at $q=e_p$. Though, contrary to the double twist knots, $a_n(K;e_p)$ do not have a simple periodicity. See Appendix.

4. CGP versus WRT for 0-surgeries on torus knots

Throughout this section $K = T_{(2,2t+1)}$ is a torus knot and M is a 3-manifold obtained by 0-surgery on K. We can compute the WRT invariant of M by inserting (3.14) into the definition

$$\text{WRT}(M) = \sum_{\substack{0<n<2p \\ n:\text{odd}}} \left(e_{2p}^n - e_{2p}^{-n}\right)^2 J_K(q^{-n};e_p)\Big|_{q=e_p} \tag{4.1}$$

which gives

(4.2)
$$\begin{aligned} \text{WRT}(M) &= \frac{e_p^t}{2}\sum_{k=0}^{p-1}(-1)^k e_p^{\frac{2t+1}{2}k^2+\frac{2t-1}{2}k} \sum_{\substack{0<n<2p \\ n:\text{odd}}} e_p^{(1-t-(2t+1)k)n}\left(1-e_p^{-n}\right)\left(1-e_p^{2k+1}e_p^{-2n}\right) \\ &= \frac{1}{2}\sum_{k=0}^{p-1}(-1)^k e_p^{\frac{2t+1}{2}k^2+\frac{2t-1}{2}k-(2t+1)k}\sum_{n=0}^{p-1} e_p^{-2n(t+(2t+1)k)}\left(e_p^{2n+1}-1\right)\left(1-e_p^{2k-4n-1}\right) \end{aligned}$$

The CGP invariant of M can be computed using the ADO given in Theorem 2.6. A simpler expression of the CGP invariant can be obtained from (3.17). In our case, (3.17) can be rewritten as

$$\text{ADO}_K(x,e_p) = \frac{e_p^t x^{(1-p)t}}{(1-x)(1+x^p)}\sum_{k=0}^{p-1}(-1)^k e_p^{\frac{2t+1}{2}k^2+\frac{2t-1}{2}k}x^{(2t+1)k}(1-e_p^{2k+1}x^2)$$

Inserting this into the definition of the CGP, we have

$$\begin{aligned}\text{(4.3)}\qquad \mathrm{CGP}(S^3(K),\lambda) &= \sum_{n=0}^{p-1} d^2(\lambda+2n)\,\mathrm{ADO}_K(e_n^{-(\lambda+2n+1)}, e_p)\\ &= \frac{e_p^{(p-1)t\lambda}}{(e_{2p}^{p\lambda}-e_{2p}^{-p\lambda})^2(1+e_p^{-p\lambda})}\sum_{k=0}^{p-1}(-1)^k e_p^{\frac{2t+1}{2}k^2+\frac{2t-1}{2}k-(\lambda+1)(2t+1)k}\\ &\quad\times\sum_{n=0}^{p-1} e_p^{-2n(k(2t+1)+t)}(e_p^{\lambda+2n+1}-1)(1-e_p^{2k-1-2\lambda-4n})\end{aligned}$$

Observe that for $\lambda = 0$ (4.3) coincides with (4.2) if we forget about the normalizing factors in front of the sum. Furthermore, in both expressions the sum over n can be computed using (3.12). The ath power of e_p^{2n} has a nonzero contribution only if $p \mid a$. Now for each $(e_p^n)^{kp}$ with $k \in \mathbb{Z}$ in (4.2), there will be a corresponding term $(e_p^{\lambda+2n+1})^{kp}$ contributing to (4.3) with (up to normalization) the same coefficient. Setting $(e_p^{k\lambda})^p := T^k$, we observe that (up to normalization) CGP is a Laurent polynomial in $T^{\pm 1}$, such that evaluated at $T = 1$ it coincides with WRT.

Appendix A.

Let $K = \overline{T_{(2,5)}}$, a mirror image of $T_{(2,5)}$. The coefficient of the Habiro expansion

$$\text{(A.1)}\qquad a_n(q) = (-1)^n q^{\frac{1}{2}n^2+\frac{3}{2}n-1}\sum_{k=1}^{n+1} q^{k^2}\begin{bmatrix} n+k \\ 2k-1\end{bmatrix}_q$$

is obtained by setting $t = 2$ in (3.18). Using $k = \ell p + j$, we get by (3.1)

$$\begin{aligned}a_{mp}(e_p) &= (-1)^m e_p^{-1}\left(e_p + \sum_{\ell=0}^{m-1}\sum_{j=1}^{p} e_p^{j^2}\begin{bmatrix}(m+\ell)p+j \\ 2(\ell p+j)-1\end{bmatrix}_{e_p}\right)\\ &= (-1)^m\left(1+\sum_{\ell=0}^{m-1}\binom{m+\ell}{2\ell} + e_p^{-1}\sum_{\ell=0}^{m-1}\binom{m+\ell}{2\ell+1}\sum_{j=\lfloor\frac{p}{2}\rfloor+1}^{p-1} e_p^{j^2}\begin{bmatrix} j \\ 2j-1-p\end{bmatrix}_{e_p}\right)\end{aligned}$$

Especially we have

$$\text{(A.2)}\qquad a_p(e_p) = -2 - \sum_{j=\lfloor\frac{p}{2}\rfloor+1}^{p-1} e_p^{j^2-1}\begin{bmatrix} j \\ 2j-1-p\end{bmatrix}_{e_p}.$$

By definition (A.1), we have

$$\begin{aligned}a_{n-1}(1) &= (-1)^{n-1}\sum_{\ell=0}^{n}\binom{n+\ell}{2\ell+1}\\ a_{2m}(-1) &= (-1)^m\left(1+\sum_{\ell=0}^{m-1}\binom{m+\ell}{2\ell}\right)\end{aligned}$$

Combining these identities, we obtain

$$a_{mp}(e_p) = a_{2m}(-1) + a_{m-1}(1)\,(2 + a_p(e_p)) \tag{A.3}$$

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