arXiv:q-alg/9708029v1 26 Aug 1997

The Casson-Walker-Lescop Invariant as a Quantum 3-manifold Invariant

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(July 24, 1997)

Abstract

Let $Z(M)$ be the 3-manifold invariant of Le, Murakami and Ohtsuki. We give a direct computational proof that the degree 1 part of $Z(M)$ satisfies $Z_1(M) = \frac{(-1)^{b_1(M)}}{2}\lambda_M$, where $b_1(M)$ denotes the first Betti number of M and where λ_M denotes the Lescop generalization of the Casson-Walker invariant of M. Moreover, if $b_1(M) = 2$, we show that $Z(M)$ is determined by λ_M.

*Research supported in part by the CNRS.
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1 Introduction and Statement of Results

A *universal* invariant of 3-manifolds, denoted by $Z(M)$, taking values in the graded (completed) algebra of Feynman diagrams, $A(\emptyset)$, was introduced by T. Le, J. Murakami and T. Ohtsuki, [LMO]. The map $M \mapsto Z(M)$, when restricted to integral homology spheres, was shown by Le in [L1] to be the universal invariant of finite type (in the sense of T. Ohtsuki, [O]). In particular, this map induces an isomorphism from the vector space, which has as basis the set of diffeomorphism classes of oriented homology spheres (completed with respect to the Ohtsuki filtration), to $A(\emptyset)$. Thus the invariant $Z(M)$ is a rich source of information, for homology spheres.

On the other hand, in [H], $Z(M)$ was computed for 3-manifolds M whose first betti number, $b_1(M)$, is greater than or equal to 3. More precisely, it was shown that $Z(M) = 1$, if $b_1(M) > 3$, and that $Z(M) = \Sigma_n \lambda_M^n \gamma_n$, if $b_1(M) = 3$. Here γ_n denotes a certain nontrivial element of $A_n(\emptyset)$, and λ_M denotes the Lescop invariant of M, [Ls]. Lescop's invariant is a generalization to all oriented 3-manifolds of the Casson-Walker invariant of (rational) homology 3-spheres.

More recently, S. Garoufalidis and the first author, [GH], computed the universal invariant of a 3-manifold M, satisfying $H_1(M, \mathbf{Z}) = \mathbf{Z}$, in terms of the Alexander polynomial of M.

In this paper, we fill in the missing gap for $b_1(M) = 2$. Namely, we have the following:

Theorem 1 *There exist non-zero $\mathcal{H}_n \in A_n(\emptyset)$, such that for all closed oriented M satisfying $b_1(M) = 2$, one has $Z(M) = \Sigma_n \lambda_M^n \mathcal{H}_n$, where λ_M denotes the Lescop invariant of M.*

In order to describe the element $\mathcal{H}_n$ of $A_n(\emptyset)$, we first recall that in [LMO], maps $\iota_n \colon A_{n\ell+i}(\coprod_{i=1}^{\ell} S^1) \to A_i(\emptyset)$ were defined for all $i \geq 0$. (We set $\iota_n = 0$ otherwise.) We denote by $p_\ell \colon A(\coprod_{i=1}^{\ell} I) \to A(\coprod_{i=1}^{\ell} S^1)$ the quotient mapping. We set $\mathcal{H}_0 = 1$, and $\mathcal{H}_n = \iota_n(p_2(\chi(\frac{H_{12}^n}{2^n\, n!}))) \in A_n(\emptyset)$, where H_{12} is the chinese character of degree 3 which is given by the 'H' diagram below.

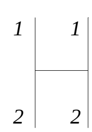


Figure 1 *The chinese character H_{12}.*

Note that $A_1(\emptyset)$ is 1-dimensional and that $A_1(\emptyset)^{\otimes n}$ is a direct summand of $A_n(\emptyset)$. Moreover, it is easily seen from the definition of ι_n that the image of $\mathcal{H}_n$ in $A_1(\emptyset)^{\otimes n}$ is nonzero. Hence $\mathcal{H}_n$ is nonzero.

In [HM], G. Masbaum and the first author investigated the relation between finite type invariants and Milnor's $\overline{\mu}$-invariants of string-links. It was shown, for example, that the coefficient of H_{12} in (the tree-like part of) the Kontsevich expansion (the universal finite type invariant of tangles) of a (string) link with vanishing linking numbers, is *half* the Milnor invariant μ_{1122} in the classical index notation. (In fact, [HM] was motivated by early attempts by the first author to establish Theorem 1.)

This recent advance makes it possible to prove theorem 1 and to give a direct proof of the fact, first proven in [LMO], that the linear term in the LMO expansion is, up to a factor, the Casson-Walker-Lescop invariant. Namely we prove

Theorem 2 $Z_1(M) = \frac{(-1)^{b_1(M)}}{2}\lambda_M$.

Acknowledgements: The authors wish to thank Christine Lescop, for her help in establishing Theorem 2, Dylan Thurston, for suggesting an approach which led to our proof of Theorem 1, and Gregor Masbaum for his encouragement to complete this project. This work began during a visit of the first author to the University of Berne during June of 1997. He thanks Christine Riedtman and the Berne Mathematics Department for their hospitality. He also wishes to thank the CNRS for its support, and the University of Nantes for delegating his time towards research during the academic year 96-97.

2 The universal finite type invariant

This section provides a brief review of some known facts about the universal finite type 3-manifold invariant.

A *tangle* in a manifold M is a smooth compact 1-manifold X, and a smooth embedding $T\colon (X, \partial X) \to (M^3, \partial M^3)$, transverse to the boundary. If $X = \coprod_{i=1}^{\ell} I$, and $M = D^2 \times I$ and the smooth embedding is a fixed standard embedding on the boundary, the tangle is called a *string link*. A *framed tangle* is a tangle with a framing which restricts to a predetermined, fixed framing on the boundary. (The use of the term framing is slightly abusive here, as we actually only require a single non-vanishing section of the normal bundle.)

Note that tangles may have empty boundary, so that links are special cases of tangles. By gluing a string link σ, along the boundary, with the trivial string link, we have the notion of the *closure* $\hat{\sigma}$ of σ.

We refer the reader to [B1] [B2] for the definition of the (graded, completed) $\mathbf{Q}$-coalgebra $\mathcal{A}(X)$ of Feynman diagrams on the 1-manifold X. Elements of $\mathcal{A}(X)$ are represented by linear combinations of vertex-oriented diagrams $X \cup \Gamma$, subject to the AS and IHX relations. Here Γ is a uni-trivalent graph, whose univalent vertices lie in the interior of X. It is customary to refer to the trivalent vertices of Γ as *internal vertices* and to the univalent vertices of Γ as *external vertices*. It is also customary to refer to the components of X as *solid*, and to the components of Γ as *dashed*. The space $\mathcal{A}(X)$ is graded by the degree, where the degree of a diagram is half the number of vertices of Γ. The degree n part of $\mathcal{A}(X)$ will be denoted by $\mathcal{A}_n(X)$.

We will denote $\mathcal{A}(\coprod_{i=1}^{\ell} I)$ by $\mathcal{A}(\ell)$. Note that $\mathcal{A}(\ell)$ has a juxtaposition product (and is thus a Hopf algebra) given by stacking diagrams. An inclusion of I in X defines an injection of $\mathcal{A}(1)$ into $\mathcal{A}(X)$, and an action of $\mathcal{A}(1)$ on $\mathcal{A}(X)$. For $X = S^1$, such an inclusion induces an isomorphism of $\mathcal{A}(1)$ with $\mathcal{A}(S^1)$, which thus inherits a product (and Hopf algebra) structure. $\mathcal{A}(\emptyset)$ has a product given by disjoint union.

In [LM2], (see also [LM1]), T. Le and J. Murakami constructed an invariant $Z(T) \in \mathcal{A}(X)$, a version of the Kontsevich integral, for any framed q-tangle $T\colon X \to \mathbf{R}^2 \times I$. (This was denoted by $\hat{Z}_f(T)$ in [LM2] and should not be confused with its precursor $Z_f(T)$.)

In [LMO], Le, Murakami and Ohtsuki give a 3-manifold invariant, called the *LMO invariant*, defined as follows:

$$Z_n(M) = \left[\frac{\iota_n(\check{Z}(L))}{(\iota_n(\check{Z}(U_+)))^{\sigma_+}(\iota_n(\check{Z}(U_-)))^{\sigma_-}}\right]^{(n)} \in \mathcal{A}_n(\emptyset). \tag{1}$$

Here $\xi^{(n)}$ denotes the degree n part of $\xi \in \mathcal{A}(\emptyset)$. $L \subset S^3$ is a framed link such that surgery on L gives the 3-manifold M. $\check{Z}(L) = \nu^{\otimes \ell} Z(L)$ is obtained by successively taking the connected sum of $Z(L)$ with ν along each component of L, where ν is the value of Z on the unknot with zero framing. $U_\pm$ is the trivial knot with ± 1-framing. $\sigma_\pm$ is the dimension of the positive and negative eigenspaces of the linking matrix for the framed link $L\colon \coprod_{i=1}^{\ell} S^1 \to \mathbf{R}^3$.

$$\iota_n\colon \mathcal{A}(\textstyle\coprod_{i=1}^{\ell} S^1) \to \mathcal{A}(\emptyset)$$

is a map defined in [LMO]. The map ι_n, although rather complicated, is more transparent when evaluated on Chinese characters, (see below).

Recall that there is a natural isomorphism χ of $\mathcal{A}(\ell)$ with $\mathcal{B}(\ell)$, the $\mathbf{Q}$ (Hopf) algebra of so-called *Chinese characters*, i.e., uni-trivalent (dashed) graphs (modulo AS and IHX relations), whose trivalent vertices are oriented, and whose univalent vertices are labelled by elements of the set $\{1, \ldots, \ell\}$. χ is given by mapping a chinese character to the average of all of the ways of attaching its labelled edges to the ℓ strands. χ is comultiplicative, but *not* multiplicative.

One has the following formula, (see [L2]): For $i \geq 0$, the composite

$$\mathcal{B}_{n\ell+i}(\ell) \xrightarrow{\chi} \mathcal{A}_{n\ell+i}(\ell) \xrightarrow{p_\ell} \mathcal{A}_{n\ell+i}(\coprod_{i=1}^{\ell} S^1) \xrightarrow{\iota_n} \mathcal{A}_i(\emptyset)$$

is zero on those characters which do not have exactly $2n$ univalent vertices of each label, and on characters which do have exactly $2n$ univalent vertices of each label is given by the formula

$$\xi \mapsto O_{-2n}(<\xi>),$$

where $<\xi>$ denotes the sum of all ways of joining the univalent vertices of ξ, having the same label, in pairs, and O_{-2n} is the map which sets circle components equal to $-2n$.

3 Proof of Theorem 1

Let M be a 3-manifold obtained by surgery on an algebraically split link L in S^3 (i.e., L has vanishing pairwise linking numbers). We begin by expressing the Lescop invariant of M in terms of the Milnor invariants of L, assuming that $b_1(M) = 2$. See [C] for geometric interpretations of the Milnor invariants μ_{ijk} and μ_{iijj} of a link.

For an abelian group H, let $|H|$ denote its cardinality if this is finite and zero otherwise.

Lemma 3 *Let L be an ℓ-component algebraically split link with framing μ_{ii} on the i-th component, such that $\mu_{11} = \mu_{22} = 0$ and $\mu_{ii} \neq 0, i > 2$. Let $M = S^3(L)$ be the 3-manifold obtained by surgery on L.*

Then

$$\lambda_M = |\prod_{i=3}^{\ell} \mu_{ii}| \left(\sum_{i=3}^{\ell} \frac{\mu_{12i}^2}{\mu_{ii}} + \mu_{1122} \right).$$

Proof: Recall that in case $b_1(M) = 2$ (see [Ls], T5.2), one has that

$$\lambda_M = -|\mathrm{Torsion}(H_1(M))| \; \mathrm{lk}_M(\gamma, \gamma_+).$$

Here, γ denotes the framed curve obtained by intersecting two surfaces whose homology classes generate $H_2(M)$. γ_+ denotes the parallel (using the framing) of γ. Note that $|\mathrm{Torsion}(H_1(M))| = |\prod_{i=3}^{\ell} \mu_{ii}|$. Thus it remains to show that $\mathrm{lk}_M(\gamma, \gamma_+) = \sum_{i=3}^{\ell} \frac{\mu_{12i}^2}{\mu_{ii}} + \mu_{1122}$.

Since L is algebraically split, we may find Seifert surfaces for each component avoiding the other components. The surfaces bounding the first two components may be capped off in M with disks, to yield closed surfaces representing the generators of $H_2(M, \mathbf{Z})$. Thus, we must compute $\mathrm{lk}_M(\gamma, \gamma_+)$, where γ denotes the intersection of these two Seifert surfaces. It follows from Cochran, [C], that $\mathrm{lk}_{S^3}(\gamma, \gamma_+) = -\mu_{1122}$.

From the geometric interpretation of triple Milnor invariants in terms of the intersection of three Seifert surfaces, one has that $\mathrm{lk}_{S^3}(\gamma, L_i) = \mathrm{lk}_{S^3}(\gamma_+, L_i) = -\mu_{12i}$. Let m_i, resp. l_i, denote the meridian, resp. longitude, of the i-th component. Note that since we are doing surgery on the i-th component with framing μ_{ii}, the cycle

$\mu_{ii}m_i + l_i$ bounds in the complement of γ_+ and all other components of L. Therefore one obtains the formula

$$\begin{aligned}
\mathrm{lk}_M(\gamma,\gamma_+) &= -\sum_{i=3}^{\ell} \mu_{12i}\, \mathrm{lk}_M(m_i,\gamma_+) + \mathrm{lk}_{S^3}(\gamma,\gamma_+) \\
&= -\sum_{i=3}^{\ell} \frac{\mu_{12i}}{-\mu_{ii}}\, \mathrm{lk}_{S^3}(l_i,\gamma_+) - \mu_{1122} \\
&= -\left(\sum_{i=3}^{\ell} \frac{\mu_{12i}^2}{\mu_{ii}} + \mu_{1122}\right).
\end{aligned}$$

□

Proof of Theorem 1: It is sufficient to compute $Z_n(M)$ for the case when the linking matrix of L is diagonal. (Indeed, given any symmetric bilinear form over the integers, one can find a diagonal matrix D having non-zero determinant, such that the given form becomes diagonalizable after taking the direct sum with D (see [M], lemma 2.2). Let L' denote a link whose linking matrix is D. Then $L \sqcup L'$ is equivalent to an algebraically split link L'', via handle sliding ([M], corollary 2.3). Let $M'' = S^3(L'')$ and $M' = S^3(L')$. The formula for connected sum (see [LMO]) gives

$$Z_n(M'') = Z_n(M)|H_1(M')|^n.$$

On the other hand, one has that

$$\lambda_{M''} = \lambda_M |H_1(M')|.$$

Since $|H_1(M')| = |\det D| \neq 0$, it follows that the theorem holds for M, provided it holds for M''.)

From now on suppose that L is an ℓ-component algebraically split link with framing μ_{ii} on the i-th component, such that $\mu_{11} = \mu_{22} = 0$ and $\mu_{ii} \neq 0, i > 2$.

It follows from [LMO] (which can also be seen from what follows) that the numerator in the definition of $Z_n(M)$, $\iota_n(\nu^{\otimes \ell} Z(L))$, vanishes in degrees $< n$. Thus we obtain that

$$Z_n(M) = (-1)^{n\sigma_+}\ \iota_n(\nu^{\otimes \ell} Z(L))^{(n)}.$$

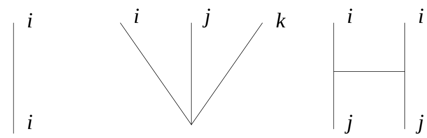
One has (see [LM2]) that $Z(L) = p_\ell(Z(\sigma)\nu_\ell)$, where σ is any string link whose closure is L. Here $\nu_\ell \in \mathcal{A}(\ell)$ is obtained from $\nu = \nu_1$ by the operator which takes a diagram on the interval to the sum of all lifts of vertices to each of the ℓ intervals.

We claim that

$$\iota_n(p_\ell(\nu^{\otimes \ell} Z(L)))^{(n)} = \iota_n(p_\ell(Z(\sigma)\nu_\ell \nu^{\otimes \ell}))^{(n)}$$

$$= \sum_{\substack{m_i \\ m_3+\ldots+m_\ell+m_H=n}} \left(\prod_{k=3}^{\ell} \frac{\mu_{12k}^{2m_k}}{(2m_k)!} \frac{(\mu_{kk}/2)^{n-m_k}}{(n-m_k)!} \right) \frac{(\mu_{1122}/2)^{m_H}}{m_H!} \iota_n(p_\ell(\chi(\xi_{m_3,\ldots,m_\ell,m_H}))),$$

where $\xi_{m_3,\ldots,m_\ell,m_H} = W_{123}^{2m_3} \cdots W_{12\ell}^{2m_\ell} H_{12}^{m_H} I_3^{n-m_3} \cdots I_\ell^{n-m_\ell}$, and where the chinese caracters I_i, W_{ijk} and H_{ij} are drawn below.



To see this, note that the coefficients of the chinese caracters I_i, W_{ijk} and H_{ij} in the Kontsevich integral are $\mu_{ii}/2$, $-\mu_{ijk}$ and $\mu_{iijj}/2$, respectively, (see [HM]). (N.b., since $\mu_{ij} = 0$, $i \neq j$, the Milnor invariants μ_{ijk} and μ_{iijj} are well defined integers.) It follows that one has

$$\chi^{-1}(Z(\sigma)) = \exp\left(\sum_{k=3}^{\ell} (\mu_{ii}/2) I_i - \sum_{k=3}^{\ell} \mu_{12k} W_{12k} + (\mu_{1122}/2) H_{12} + \epsilon \right).$$

(The chinese character I_{ij}, i.e. the interval whose vertices are labelled with i and j, does not appear in the above expansion, since $\mu_{ij} = 0$, $i \neq j$.) Here the exponential is taken with respect to the disjoint union multiplication in $B(\ell)$. ϵ consists of terms which are either not trees, are trees of degree > 3, or are trees of degree 3 involving some index other than 1 and 2. These terms will not be involved in the computation, as they contribute too many vertices (see below). A similar formula holds for $\chi^{-1}(Z(\sigma)\nu_\ell \nu^{\otimes \ell})$.

The result claimed above now follows easily, upon remarking that the chinese characters $\xi_{m_3,\ldots,m_\ell,m_H}$, with $m_3 + \ldots + m_\ell + m_H = n$, are the only terms from the exponential which have $2n$ vertices of each label and have at most $2n$ internal vertices. (Indeed, since the characters I_{ij}, I_1 and I_2 do not appear, the only way to obtain $2n$ vertices of label 1 and 2, and at most $2n$ internal vertices is to use the characters W_{12k} and H_{12}. Moreover, using these will yield exactly $2n$ internal vertices. Since no other internal vertices are then allowed, the remaining vertices labelled k must be obtained using the character I_k, and since the vertices of I_k come in pairs, it follows that W_{12k} must occur an even number of times.)

Denote by $j_n^k : \mathcal{B}(k) \to \mathcal{B}(k-1)$ the map which is zero, if there are not exactly $2n$ univalent vertices labelled k, and otherwise is given by the sum over all $(2n-1)!!$ ways of joining the $2n$ univalent vertices labelled k in pairs, and then replacing every circle component by $-2n$. Then $\iota_n(p_\ell(\chi(x))) = j_n^1 \circ \cdots \circ j_n^\ell\,(x)$, for $x \in \mathcal{B}(\ell)$.

The theorem is an immediate consequence of the following formula

$$\frac{j_n^3 \circ j_n^4 \circ \cdots \circ j_n^\ell\,(\xi_{m_3,\ldots,m_\ell,m_H})}{\prod_{k=3}^{\ell} 2^{n-m_k}\,(n-m_k)!\,(2m_k)!} = \frac{(-1)^{n\ell}}{\prod_{k=3}^{\ell} 2^{m_k}\, m_k!} H_{12}^n. \tag{2}$$

Indeed, using (2), which follows from repeated use of Lemma 4 below, we obtain that

$$Z_n(M) = (-1)^{n\sigma_+}\ \iota_n(\nu^{\otimes \ell} Z(L))^{(n)}$$

$$= (-1)^{n\sigma_+ + n\ell} (\prod_{i=3}^{\ell} \mu_{ii})^n \sum_{\substack{m_i \\ m_3+\ldots+m_\ell+m_H=n}} \left(\prod_{k=3}^{\ell} \frac{(\mu_{12k}^2/2\mu_{kk})^{m_k}}{(m_k)!} \right) \frac{(\mu_{1122}/2)^{m_H}}{m_H!} j_n^1 \circ j_n^2(H_{12}^n)$$

$$= \frac{|\prod_{i=3}^{\ell} \mu_{ii}|^n}{2^n\, n!} \left(\sum_{k=3}^{\ell} \frac{\mu_{12k}^2}{\mu_{kk}} + \mu_{1122} \right)^n j_n^1 \circ j_n^2(H_{12}^n)$$

$$= \lambda_M^n \iota_n(p_2(\chi(\frac{H_{12}^n}{2^n\ n!}))).$$

□

Lemma 4 *For $i \geq 3$, set $W_{12i} = W$, $H_{12} = H$ and $I_i = I$. Then*

$$\frac{j_n^i\ (W^{2m} I^{n-m})}{(2m)!\ 2^{n-m}\ (n-m)!} = \frac{(-1)^n}{2^m\ m!} H^m. \tag{3}$$

Proof: Let J_n^i be as in the definition of j_n^i, but without the requirement that we must have exactly $2n$ vertices to be joined. To compute $J_n^i(W^{2m} I^k)$, let us fix an i-labeled univalent vertex of I^k. This vertex can either be paired with one of the $2m$ i-labeled vertices of W^{2m}, or with one of the remaining $2k-1$ vertices of I^k (one of which results in a circle component). Replacing the circle component by $-2n$, we have that

$$J_n^i(W^{2m} I^k) = (2m + 2k - 2 - 2n)\ J_n^i(W^{2m} I^{k-1}),$$

and hence inductively we have

$$j_n^i(W^{2m} I^{n-m}) = (-2)^{n-m}(n-m)!\ J_n^i(W^{2m}). \tag{4}$$

Similarly, we have that

$$J_n^i(W^{2m}) = -(2m-1)\ J_n^i(W^{2m-2})H,$$

and hence inductively that

$$J_n^i(W^{2m}) = (-1)^m (2m-1)!!\ H^m. \tag{5}$$

Hence, combining equations (4) and (5), we obtain

$$j_n^i(W^{2m} I^{n-m}) = (-1)^n 2^{n-m}(n-m)!(2m-1)!!\ H^m,$$

which is equivalent to equation (3), since $(2m-1)!!\ 2^m m! = (2m)!$. □

4 Proof of Theorem 2

We will denote by $o(n)$ terms of degree greater than or equal to n.

In low degrees, the Kontsevich expansion of the trivial knot can be computed by applying the Alexander-Conway weight system. One obtains the formula

$$\chi^{-1}(\nu) = 1 + \frac{\phi_1}{48} + o(4).$$

From this it follows (see below) that

$$\iota_1(\check{Z}(U_\pm)) = \mp 1 + \frac{\Theta}{16} + o(2).$$

The graphs ϕ_i and Θ are depicted in Figure 2.

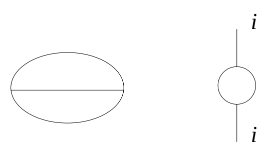


Figure 2 *The graph* Θ *and the chinese character* ϕ_i.

Proof of Theorem 2: Let $M = S^3(L)$, where L is an ℓ-component algebraically split link which is the closure of a string link σ. (The general case reduces to this one by the same argument as in the proof of Theorem 1, using the connected sum formulas for Z_1 and λ, see [LMO], [Ls].)

We wish to calculate

$$Z_1(M) = \left[\frac{\iota_1(p_\ell(Z(\sigma)\nu_\ell \nu^{\otimes \ell}))}{(-1+\frac{\Theta}{16})^{\sigma_+}(1+\frac{\Theta}{16})^{\sigma_-}} \right]^{(1)}.$$

Denote by ϵ terms in $\chi^{-1}(\log(Z(\sigma))$ which either have more than 2 internal vertices, or have 2 internal vertices but an odd number of external vertices with a given label. (Such terms do not contribute to the end expression.) Then we have

$$Z(\sigma) = \exp\left(\chi\left(\sum_{i=1}^{\ell} \frac{\mu_{ii}}{2} I_i - \sum_{i<j<k} \mu_{ijk} W_{ijk} + \sum_{i<j} \frac{\mu_{iijj}}{2} H_{ij} - \sum_i \frac{a_1^{(i)}}{2}\phi_i + \epsilon\right)\right).$$

The coefficient $a_1^{(i)}$ can be calculated by applying the Alexander-Conway weight system (see e.g., [GH]). One has that $a_1^{(i)} = \frac{1}{2}\Delta''(L_i)(1)$, where $\Delta(K)(t)$ denotes the Alexander polynomial of K (normalized to be symmetric in t and t^{-1} and taking the value 1 at $t = 1$). In the above expansion of the exponential, one uses the juxtaposition product of $\mathcal{A}(\ell)$. Note that if we use this product to induce a second product structure on chinese characters via χ, denoted by $\cdot_\times$, then, for example, one has that

$$I_k \cdot_\times I_k = I_k^2 + \frac{1}{6}\phi_k.$$

One deduces that

$$\chi^{-1}(Z(\sigma)\nu_\ell \nu^{\otimes \ell}) = \exp\left(\sum_{i=1}^{\ell} \frac{\mu_{ii}}{2} I_i - \sum_{i<j<k} \mu_{ijk} W_{ijk} + \sum_{i<j} \frac{\mu_{iijj}}{2} H_{ij} + \sum_i b^{(i)}\phi_i + \epsilon\right),$$

where

$$b^{(i)} = 2\frac{1}{48} + \frac{1}{6}\frac{1}{2!}\left(\frac{\mu_{ii}}{2}\right)^2 - \frac{a_1^{(i)}}{2} = \frac{2+\mu_{ii}^2 - 24a_1^{(i)}}{48}.$$

Here the exponential is expanded using the disjoint union product in $\mathcal{B}(\ell)$. (Here again, ϵ denotes terms which either have more than 2 internal vertices, or have 2 internal vertices, but an odd number of external vertices with a given label.)

It follows that one has

$$\iota_1(\check{Z}(L)) = (-1)^\ell \left(\prod_{i=1}^{\ell} \mu_{ii} + c(L)\Theta\right) + o(2), \qquad (6)$$

where the coefficient of Θ, $c(L)$, is given by

$$c(L) = -\sum_i b^{(i)} \prod_{j\neq i} \mu_{jj} + \sum_{i<j} \frac{\mu_{iijj}}{2} \prod_{k\neq i,j} \mu_{kk} + \sum_{i<j<k} \frac{\mu_{ijk}^2}{2} \prod_{t\neq i,j,k} \mu_{tt}.$$

Hence

$$\begin{aligned} Z_1(M) &= \left[(-1)^{\sigma_+}\left(1 + \frac{(\sigma_+ - \sigma_-)}{16}\Theta\right)(-1)^\ell\left(\prod_{i=1}^{\ell}\mu_{ii} + c(L)\Theta\right)\right]^{(1)} \\ &= (-1)^{b_1(M)+\sigma_-}\left(\frac{(\sigma_+ - \sigma_-)}{16}\prod_{i=1}^{\ell}\mu_{ii} + c(L)\right)\Theta \\ &= \frac{(-1)^{b_1(M)}}{2}\lambda_M\Theta, \end{aligned}$$

where

$$\lambda_M = \frac{|H_1(M)|\,(\sigma_+ - \sigma_-)}{8} + (-1)^{\sigma_-}2c(L).$$

Note that λ_M is indeed Lescop's generalization of the Casson-Walker invariant. In the special case of an algebraically split link in the 3-sphere, Lescop's surgery formula ([Ls] 1.4.8) is given as follows:

Let I be a subset of $\{1,\ldots,\ell\}$. Denote by L_I the link obtained from L by forgetting the components whose subscripts do not belong to I. Then

$$\lambda_M = \frac{|H_1(M)|\,(\sigma_+ - \sigma_-)}{8} +$$
$$(-1)^{\sigma_-}\left[\sum_i \left(\zeta(L_{\{i\}}) - \frac{1}{24}(\mu_{ii}^2+1)\right)\prod_{j\neq i}\mu_{jj} + \sum_{I,\,|I|\neq 0,1}\zeta(L_I)\prod_{j\neq I}\mu_{jj}\right].$$

Note that in the algebraically split case, the values of the ζ function are given as follows.

$$\zeta(L_I) = \begin{cases} a_1^{(i)} - \frac{1}{24} & : \quad I=\{i\} \\ \mu_{iijj} & : \quad I=\{i,j\} \\ \mu_{ijk}^2 & : \quad I=\{i,j,k\} \\ 0 & : \quad |I|>4 \end{cases}$$

This can easily be established using the results in ([Ls], chapter 5). □

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