Spin Topological Quantum Field Theories

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Abstract: Starting from the quantum group $U_q(\mathfrak{sl}(2,\mathbb{C}))$, we construct operator invariants of 3-cobordisms with spin structure, satisfying the requirements of a topological quantum field theory and refining the Reshetikhin–Turaev and Turaev–Viro models. We establish the relationship between these two refined models.

1 Introduction

This paper is devoted to the refinement of the quantum invariants of 3-manifolds taking into account spin structures. The invariants of Reshetikhin–Turaev type, corresponding to the quantum group $U_q(\mathfrak{sl}(2,\mathbb{C}))$ and determined by a spin structure on a closed 3-manifold, were first constructed by Blanchet [Bl], Kirby–Melvin [KM] and Turaev [Tu]. The idea of the construction was the following: Using a presentation of a closed 3-manifold M by surgery along a link L, one can identify a spin structure s on M with a characteristic sublink K of L (see section 3.2 for the definition). The Reshetikhin-Turaev invariant $\tau(M)$ is defined as a sum over all colourings (with some coefficients) of the coloured link invariant of L. The refined Reshetikhin-Turaev invariant $\tau(M,s)$ is defined analogously, where the sum is taken over odd colourings of K and even colourings of $L-K$ only. It turns out that

$$\tau(M) = \sum_s \tau(M,s)\,.$$

A refinement of the Turaev–Viro invariant $Z(M)$ of a closed 3-manifold M was done in two steps. First, a state sum $Z(M,h)$ for $h \in H^1(M,\mathbb{Z}_2)$ was defined in [TV], such that

$$Z(M) = \sum_h Z(M,h).$$

Then Roberts [R] constructed an invariant $Z(M,s,h)$ of a closed oriented 3-manifold M equipped with a spin structure s and $h \in H^1(M,\mathbb{Z}_2)$, such that

$$Z(M,h) = \sum_s Z(M,h,s).$$

As is well-known (see [Wi], [At]), a theory of quantum invariants of closed 3-manifolds is a part of topological quantum field theory (TQFT), which associates vector spaces to closed surfaces and linear operators to 3-cobordisms. In this article, topological quantum field theories extending the quantum invariants of closed 3-manifolds with spin structure will be referred to as 'spin' TQFT's.

The first spin TQFT was constructed by Blanchet and Masbaum in [BM]. They use an algebraic technique of [BHMV] in order to extend the invariants of [Bl], [KM] and [Tu]. Among the results of [BM] are the dimension formula for modules associated to closed connected surfaces with spin structure and the transfer map from the Reshetikhin–Turaev theory to the spin TQFT.

In this paper we give a different, geometric construction of a spin TQFT extending the refined Reshetikhin–Turaev invariants. Our construction is parallel to the one given in [T, Chapter 4]. Whence we briefly recall the construction of Turaev in section 3.1. We represent the vector space $V_{(\Sigma_g,\sigma)}$ associated to a closed oriented surface Σ_g of genus g with spin structure σ as a (subspace of a) vector space generated by 'special' colourings of some ribbon graph G^g (see Fig.1). The graph G^g is chosen in such a way that its regular neighborhood is a handlebody of genus g. 'Special' colourings is a subset of admissible colourings of G^g, depending on σ. We show that

$$V_\Sigma = \oplus_\sigma V_{(\Sigma,\sigma)},$$

where V_Σ is a vector space associated to Σ in the standard Reshetikhin–Turaev TQFT.

We define the operator invariant $\tau(M,s)$ of the spin 3-cobordism (M,s) as follows: First, to each connected component Σ_j of genus g_j of the boundary of M we glue a regular neighborhood of the graph G^{g_j}, containing this graph. This results in a closed 3-manifold $\tilde{M}$ with some ribbon graph, say G, sitting inside. The graph G is a disjoint union of the graphs inside the handlebodies. Using a surgery presentation of $\tilde{M}$ along a link L, we show that there is a one-to-one correspondence between spin structures on M and characteristic sublinks of $L \cup G$ (see section 3.2 for the definition). Finally, we define $\tau(M,s)$ as a refined Reshetikhin-Turaev invariant of the pair $(\tilde{M},G)$, where one sums over odd colourings of the characteristic sublink (determined by s) and over even colourings of the other components of L. Note that $\tau(M,s)$ is an element of the vector space generated by the 'special' colourings of G. We study gluing properties of $\tau(M,s)$ and give an explicit formula for the projector

$$\tau^\sigma : V_\Sigma \to V_{(\Sigma,\sigma)} \,.$$

In addition, we show, that for connected Σ, the dimension of $V_{(\Sigma,\sigma)}$ coincides with the dimension calculated in [BM]. The Reshetikhin-Turaev invariant of a 3-cobordism M splits as a sum of the refined invariants, i.e.

$$\tau(M) = \oplus_\sigma \sum_s \tau(M,s)\,,$$

where the sum is taken over s such that $s|_{\partial M} = \sigma$.

In section 4 we construct a spin TQFT extending Roberts' invariants. In order to do this, we use a modified state sum operator $Z(M,G)$ of a 3-cobordism M together with a 3-valent graph G, which is a subcomplex of a triangulation of ∂M (see [KS], [BD1] and [BD2]). This operator is equal to the Turaev–Viro state sum of M with a triangulation of the boundary ∂M given by the graph dual to G. The advantage is that $Z(M,G)$ is a homotopy invariant of the graph G, which can be effectively calculated.

In [BD2] an isomorphism was constructed between the vector space V_{Σ_g} of Turaev–Viro TQFT and the vector space associated to the two copies of the graph G^g. Refining this construction, we define the vector space $V_{\Sigma_g}(\sigma,\mathrm{h})$ associated to a closed oriented surface Σ_g with spin structure σ and first cohomology class h, such that

$$V(\Sigma_g) = \oplus_{\sigma,\,\mathrm{h}}\ V_{\Sigma_g}(\sigma, \mathrm{h}).$$

Then we construct the state sum operator $Z(M, s, h)$ of a spin 3-cobordism (M, s) with $h \in H^1(M, \mathbb{Z}_2)$.

Finally, we show that

$$V_\Sigma(\sigma, \mathrm{h}) = V_{(\Sigma,\,\sigma)} \otimes V_{(-\Sigma,\,\sigma+\mathrm{h})}$$

and

$$Z(M, s, h) = \tau(M, s) \otimes \tau(-M, s + h),$$

where a negative sign means the orientation reversal. This proves that the operator $Z(M, s, h)$ gives rise to an (anomaly free non-degenerate) TQFT on compact oriented 3-cobordisms equipped with a spin structure and a first $\mathbb{Z}_2$-cohomology class.

2 Initial data and notation

In this section we define basic algebraic data, which will be used in the construction of invariants.

Let A be a primitive root of unity of order $4r$, where $r \in \mathbb{N}$ and $r = 0 \pmod 4$. Consider the set $I = \{0, 1, 2, ..., r - 2\}$. For each $i \in I$, we fix complex numbers ω_i and q_i, such that

$$\omega_i^2 = (-1)^i [i + 1] \quad \text{and} \quad q_i^2 = (-1)^i A^{i^2+2i}, \tag{2.1}$$

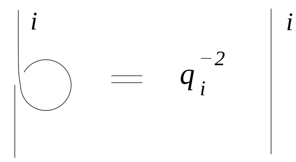
where

$$[n] = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}, \quad \text{for } n \in \mathbb{N}.$$

Furthermore, we choose a complex number ω, such that

$$\omega^2 = \sum_{i \in I} \omega_i^4 = \frac{-2r}{(A^2 - A^{-2})^2}\,. \tag{2.2}$$

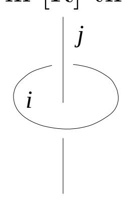
These data come from the modular category provided by 'good' representations of the quantum group $U_q(\mathfrak{sl}(2, \mathbb{C}))$ (see [RT]), where $A^4 = q$. In this article we enumerate irreducible representations of $U_q(\mathfrak{s}l(2,\mathbb{C}))$ by doubled spins $i \in I$. We recall that ω_i^2 is equal to the quantum dimension of the i^{th} representation and the ribbon graph invariant, defined in [RT], is multiplied by q_i^{-2} under one twist on an i-coloured ribbon:



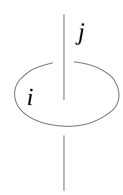
A triple $(i,j,k) \in I^3$ is called admissible if $i+j+k$ is even and

$$i \le j+k,\ \ j \le i+k,\ \ k \le i+j,\ \ i+j+k \le 2(r-2)\,. \tag{2.3}$$

We finish this section by collecting relations which will be of importance in the sequel. It was shown in [R] that

$$\sum_{i=0,\, i\,\mathrm{even}}^{r-2} \omega_i^2$$


$$= \frac{\omega^2}{2}(\delta_{j,0} + \delta_{j,r-2})\,, \tag{2.4}$$

$$\sum_{i=1,\, i\,\mathrm{odd}}^{r-2} \omega_i^2$$


$$= \frac{\omega^2}{2}(\delta_{j,0} - \delta_{j,r-2})\,. \tag{2.5}$$

Moreover,

$$\omega^2 = 2 \sum_{i=0,\, i\ \mathrm{even}}^{r-2} \omega_i^4 = 2 \sum_{i=1,\, i\,\mathrm{odd}}^{r-2} \omega_i^4 .$$

In addition, we have

$$q_{r-2-i}^2 = (-1)^{i+1} q_i^2\,, \qquad \omega_{r-2-i}^2 = \omega_i^2\,. \tag{2.6}$$

It follows that

$$\Delta = \sum_{i\in I} q_i^2 \omega_i^4 = \sum_{i\,\mathrm{odd}} q_i^2 \omega_i^4\ .$$

Finally,

$$\Delta\,\bar{\Delta} = \omega^2\,, \tag{2.7}$$

where $\bar{\Delta} = \sum_i q_i^{-2}\omega_i^4$.

3 Spin Reshetikhin–Turaev TQFT

We begin this section by recalling the standard construction of a TQFT given by Reshetikhin and Turaev ([RT] and [T, Chapter 4]). After a brief review on spin structures, we discuss a refinement of this construction determined by a spin structure on a 3-cobordism.

3.1 Standard model

Consider a compact oriented 3-cobordism M with boundary $\partial M = (-\partial_- M) \cup \partial_+ M$, where $\partial_- M$ and $\partial_+ M$ are the bottom and top bases of M, respectively, and minus means the orientation reversal. Assume that the boundary of M is parametrized, i.e., each connected component $\Sigma \subset \partial M$ is supplied with an orientation preserving homeomorphism $\phi : \Sigma_g \to \Sigma \subset \partial_+ M$ or $-\phi : -\Sigma_g \to -\Sigma \subset \partial_- M$, where Σ_g and $-\Sigma_g$ are the boundaries of a standard oriented handlebody (H_g^+, G^g) and an oppositely oriented handlebody $(H_g^-, \bar{G}^g)$, respectively.

The handlebody (H_g^+, G^g) is defined as a regular neighborhood in $\mathbb{R}^3$ of the graph G^g, depicted in Fig.1, together with the graph itself sitting inside.

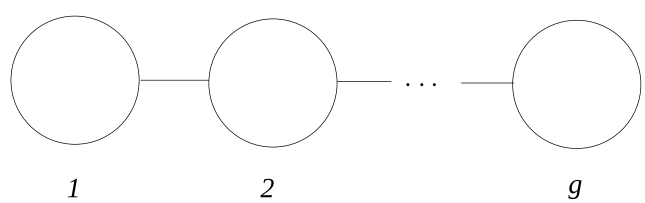


Fig.1 *The 3-valent graph G^g*

The mirror image of (H_g^+, G^g) with respect to a horizontal plane in $\mathbb{R}^3$ defines the oppositely oriented handlebody $(H_g^-, \bar{G}^g)$.

By an admissible colouring $e = \{e_1, e_2, ..., e_{3g-3}\}$ of G^g, we mean an assignment of a colour (from I) to each line of G^g, so that the three colours of lines, meeting in a 3-vertex, form an admissible triple in the sense of Section 2. We will denote the e-coloured 3-valent graph by G^g_e.

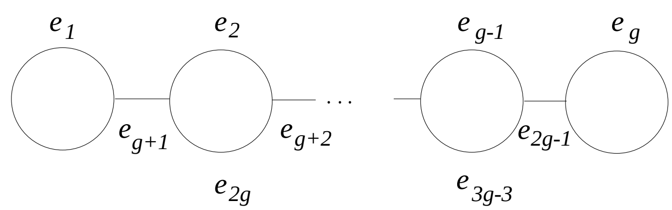


Fig.2 *The coloured 3-valent graph* G^g_e

We note that the admissible colourings of G^g provide a basis of the vector space V_Σ associated by the Reshetikhin–Turaev TQFT to a closed parametrized surface Σ of genus g. Their number is equal to the dimension of V_Σ given by the Verlinde formula. To a non-connected surface one associates the tensor product of the vector spaces belonging to connected components.

The construction of a 3-cobordism invariant is as follows: To each connected component of $\partial_- M$ of genus g one glues a copy of (H^+_g, G^g) along the given parametrization and analogously one glues the oppositely oriented handlebody to each connected component of $\partial_+ M$. The result is a closed 3-manifold $\tilde{M}$ with a ribbon graph, say $G^+ \cup G^-$, sitting inside. The graph $G^+ \cup G^-$ is the disjoint union of graphs $\bar{G}^g$ and G^g inside the standard handlebodies. Now the invariant of the 3-cobordism M is defined as an invariant of the pair $(\tilde{M}, G^+ \cup G^-)$. More precisely, this invariant in the basis, given by the admissible colourings of $G^+ \cup G^-$, can be written as follows:

$$\tau(M)_{ee'} = (\Delta\omega^{-1})^{\sigma(L)}\omega^{-m-1+\frac{\chi(\partial_+ M)}{2}}\omega_e\omega_{e'}\sum_c \omega_c^2 \, Z(L_c \cup G^+_e \cup G^-_{e'}) \,, \quad (3.1)$$

where

$$\omega_e = \prod_i \omega_{e_i},$$

e (resp. e') is a colouring of G^+ (resp. G^-), $L \subset S^3$ is an m-component surgery link for $\tilde{M}$, $c = \{c_1, c_2, ..., c_m\} \in I^m$ is a colouring of L, $\sigma(L)$ is the signature of the linking matrix, χ is the Euler characteristic and $Z(G_e)$ denotes the invariant of a coloured ribbon graph G_e in S^3 as defined in [RT].

We set

$$\tau(M) = \oplus_{ee'}\, \tau(M)_{ee'} : V_{\partial_- M} \to V_{\partial_+ M}\,. \tag{3.2}$$

It was shown in [T] that the linear operator $\tau(M)$ determines a TQFT. In particular, this means that gluing of cobordisms is described by composing operators and that

$$\tau(\Sigma \times [0,1]) = id_{V_\Sigma}\,.$$

This construction can be naturally generalized to 3-cobordisms between punctured surfaces. The only significant modification requires the notion of a standard handlebody.

Consider the handlebody $H_g^+(p)$, whose boundary is an oriented surface Σ_g with a set $p = \{p_1, p_2, ..., p_n\}$ of distinguished points (punctures). Attach to each puncture a colour from the set $a = \{a_1, a_2, ..., a_n\}$ and embed the graph $G^g(a)$ depicted below

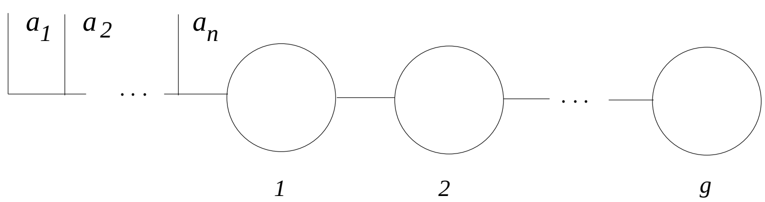


Fig.3 *The graph* $G^g(a)$

in $H_g^+(p)$, so that its 1-vertices lie on Σ_g and coincide with the punctures p and the remainder of the graph forms a deformation retract of H_g^+. The resulting pair $(H_g^+(p), G^g(a))$ is a punctured standard handlebody. A construction of a TQFT is quite analogous to the one described above and will not be repeated here. We mention only that the vector space associated by this TQFT to the punctured surface $\Sigma_g(p)$ is generated by colourings of the graph $G^g(a)$.

3.2 Spin structures on manifolds

A spin structure on an n-dimensional manifold N is a homotopy class of a trivialization of the tangent bundle of N over the 1-skeleton which extends over the 2-skeleton (see [Ki]). The number of different spin structures on N (if it is not zero) is equal to the number of elements in $H_1(N)$[1]. Moreover, the whole set of spin structures on N (if it is not empty) is obtained by adding elements of $H^1(N)$ to any fixed spin structure.

There exist two spin structures on a circle: the bounding spin structure (which extends over a disc) and the non-bounding or Lie spin structure. A spin structure σ on a connected surface Σ defines a quadratic form $q_\sigma : H_1(\Sigma) \to \mathbb{Z}_2$, such that for any embedded closed curve γ, $q_\sigma(\gamma) = 0$, if $\sigma|_\gamma$ is bounding, and $q_\sigma(\gamma) = 1$ otherwise (see [Jo]). To determine a spin structure on a surface, it is sufficient to say which simple closed curves in a canonical homology basis (as in Fig.4) are spin bounding and which are not.

One can also think of a spin structure on a manifold M as being a first cohomology class of an oriented frame bundle $F(M)$, whose restriction to each fibre is non-trivial. If M is 3-dimensional, this class can be evaluated on a framed knot in M, representing a 1-cycle in $F(M)$ (the rest of a true frame can be reconstructed using the orientation of M). This cohomology class is equal to 1 on a trivial knot in M with zero framing.

Let us denote by $\mathrm{Spin}(M)$ a set of spin structures on a 3-manifold M. Suppose that M is obtained by surgery on a framed m-component link L. Denote by $S^3\backslash L$ the 3-sphere S^3 with a regular neighborhood of L removed. Then one can identify $\mathrm{Spin}(M)$ with a subset of $\mathrm{Spin}(S^3\backslash L)$, consisting of all spin structures which are equal to 1 on each component L_i of L.

Taking into account that

$$\mathrm{Spin}(S^3\backslash L) = s_0 + H^1(S^3\backslash L),$$

where s_0 is a spin structure on $S^3\backslash L$, induced by the unique spin structure on S^3, we observe that any spin structure on $S^3\backslash L$ is completely determined by its values on the meridians $\{m_i\}_{i=1}^m$ of the regular neighborhood of L. One can evaluate a cohomology class $s \in \mathrm{Spin}(S^3\backslash L)$ on a framed knot γ in $S^3\backslash L$ as follows:

$$s(\gamma) = 1 + \gamma\cdot\gamma + \sum_{j=1}^{m}(\gamma\cdot L_j)(1 + s(m_j)),$$

[1]Throughout this paper all (co)homology groups will have $\mathbb{Z}_2$-coefficients.

where $\gamma \cdot L_j = \mathrm{lk}(\gamma, L_j)$ is the linking number and $\gamma \cdot \gamma$ is the framing on γ. Imposing the condition

$$s(L_i) = 1 \qquad \text{for} \qquad i = 1, 2, ..., m,$$

we obtain that any spin structure $s \in \mathrm{Spin}(M)$ defines a sublink $K \subset L$, such that for any component L_i of L

$$L_i \cdot K = L_i \cdot L_i \,. \tag{3.3}$$

The sublink K satisfying (3.3) is called a characteristic sublink of L. It consists of all the components L_i of L, such that s is non-bounding on the meridian m_i of L_i or, in other words, $s(m_i) = 0$. We define a characteristic coefficient $c_i \in \mathbb{Z}_2$ of the component L_i of L equal to one if $L_i \in K$ and zero otherwise.

For a 3-cobordism M with parametrized boundary, one can identify $\mathrm{Spin}(M)$ with a subset of

$$\mathrm{Spin}(S^3 \backslash (L \cup G^+ \cup G^-)) = s_0 + H^1(S^3 \backslash (L \cup G^+ \cup G^-)),$$

consisting of all spin structures which are equal to 1 on L (see Section 3.1 for the definition of $G^+ \cup G^-$). A basis in $H_1(S^3 \backslash (L \cup G^+ \cup G^-))$ is given by meridians $\{m_i\}$ of L together with meridians $\{b_i\}$ of (a regular neighborhood of) $G^+ \cup G^-$. Denoting by $\{a_i\}$ the longitudes of $G^+ \cup G^-$, we have that

$$s(L_i) = 1 + L_i \cdot L_i + \sum_j (L_i \cdot L_j)(1 + s(m_j)) + \sum_j (L_i \cdot a_j)(1 + s(b_j)) \,,$$

where $s \in \mathrm{Spin}(S^3 \backslash (L \cup G^+ \cup G^-))$. It follows that there exists a one-to-one correspondence between the solutions of the following equations

$$L_i \cdot (K + A) = L_i \cdot L_i, \quad 1 \leq i \leq m,$$

where $K \subset L$ and $A \subset \cup_i a_i$, and spin structures on a 3-cobordism M, which do not extend over the meridians of K and A. We will call K a characteristic sublink of $L \cup G^+ \cup G^-$.

3.3 Spin Reshetikhin–Turaev model

In this section we construct a spin TQFT by refining the model described in section 3.1.

Definition of invariants

We start by modifying the notion of a standard handlebody.

Consider the handlebody H_g^+ with the boundary $\partial H_g^+ = \Sigma_g$ as depicted below.

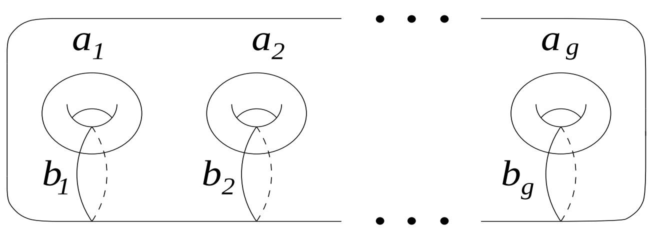


Fig.4 *The canonical homology basis on* Σ_g

Associate to each meridian b_i of Σ_g a number $\mathfrak{s}_i \in \mathbb{Z}_2$ and denote by $\mathfrak{s}$ the sequence of these numbers, i.e.

$$\mathfrak{s} = \{\mathfrak{s}_1, \mathfrak{s}_2, ..., \mathfrak{s}_g\} \in \mathbb{Z}_2^g .$$

Then we embed the graph G^g (see Fig.1) in H_g^+ as its deformation retract. The resulting triple $(H_g^+, G^g, \mathfrak{s})$ will be called a standard handlebody. The oppositely oriented handlebody $(H_g^-, \bar{G}^g, \mathfrak{s})$ is defined by a mirror image of $(H_g^+, G^g, \mathfrak{s})$.

Let $E_\mathfrak{s}$ be a subset of admissible colourings of the graph G^g subject to the following relation:

- a colour $e_i \in I$, $1 \le i \le g$, is even, if $\mathfrak{s}_i = 0$, and odd otherwise.

In the sequel we will call the elements of $E_\mathfrak{s}$ *special colourings* of the graph G^g.

By a parametrized surface $(\Sigma, \mathfrak{s})$ of genus g we understand an oriented closed connected surface of genus g supplied with an orientation preserving homeomorphism

$$\phi : \Sigma_g \to \Sigma$$

and a sequence $\mathfrak{s}$ of $\mathbb{Z}_2$-numbers associated to $\phi(b_i)$, $1 \le i \le g$. We denote by $V_{(\Sigma,\mathfrak{s})}$ the vector space associated to the parametrized surface $(\Sigma, \mathfrak{s})$, which is generated by the special colourings $E_{\mathfrak{s}}$ of the graph G^g. Clearly,

$$V_\Sigma = \oplus_{\mathfrak{s}} V_{(\Sigma,\mathfrak{s})},$$

where V_Σ denotes as before the vector space associated to Σ in the standard Reshetikhin-Turaev model and the direct sum is taken over 2^g possible choices of $\mathfrak{s}$. To disjoint unions of surfaces we associate the tensor product of vector spaces.

Consider a spin 3-cobordism (M, s) with parametrized boundary $\partial M = (-\partial_- M) \cup \partial_+ M$, where s is a spin structure on M. Let us enumerate the connected components of ∂M by an index j, $1 \le j \le n$. Suppose that the first l of them belong to $\partial_- M$ and the remaining to $\partial_+ M$. Choose a sequence $\mathfrak{s}_j$ of $\mathbb{Z}_2$-numbers associated to the j^{th} connected component Σ_j of ∂M in such a way, that

$$(\mathfrak{s}_j)_i = q_{s|_{\Sigma_j}}(\phi_j(b_i)), \quad 1 \le i \le g_j,$$

where $\phi_j : \Sigma_{g_j} \to \Sigma_j$ is the parametrization homeomorphism.

After gluing (along the parametrizations) of $(H^+_{g_j}, G^{g_j}, \mathfrak{s}_j)$, $1 \le j \le l$, and $(H^-_{g_j}, \bar{G}^{g_j}, \mathfrak{s}_j)$, $l < j \le n$, to connected components of $\partial_- M$ and $\partial_+ M$, respectively, we obtain a closed manifold $\tilde{M}$ with the graph, say $G^+ \cup G^-$, sitting inside. Denote by L an m-component surgery link for $\tilde{M}$. In general, the spin structure s does not extend over $\tilde{M}$, but it determines a spin structure on $S^3 \backslash (L \cup G^+ \cup G^-)$. Now we choose a characteristic sublink K of $L \cup G^+ \cup G^-$, consisting of all the components L_i of L, such that s is non-bounding on the corresponding meridians. Set

$$\tau(M, s)_{ee'} = (\Delta\omega^{-1})^{\sigma(L)} \omega^{-m-1+\frac{\chi(\partial_+ M)}{2}} \omega_e\, \omega_{e'}$$

$$\sum_{c \text{ odd}} \omega_c^2 \sum_{b \text{ even}} \omega_b^2\, Z(K_c \cup (L-K)_b \cup G^+_e \cup G^-_{e'}) \;, \tag{3.4}$$

where $e \in E_{\mathfrak{s}_+}$ and $e' \in E_{\mathfrak{s}_-}$ are special colourings of G^+ and G^-, respectively. Here

$$\mathfrak{s}_+ = \cup_{j=l+1}^n \mathfrak{s}_j, \qquad \mathfrak{s}_- = \cup_{j=1}^l \mathfrak{s}_j$$

and we denote by c and b the colourings of K and $L-K$, respectively. A colouring is called even (resp. odd), if all its values are even (resp. odd).

We define the linear operator

$$\tau(M,s) : V_{(\partial_- M,\,\mathfrak{s}_-)} \to V_{(\partial_+ M,\,\mathfrak{s}_+)}$$

corresponding to the spin cobordism (M,s) by taking a direct sum over all special colourings of G^+ and G^-, i.e.,

$$\tau(M,s) = \oplus_{ee'} \tau(M,s)_{ee'}\,, \qquad e \in E_{\mathfrak{s}_+},\;\; e' \in E_{\mathfrak{s}_-}. \tag{3.5}$$

Theorem 1 *$\tau(M,s)$ is a topological invariant of a compact spin 3-cobordism (M,s) with parametrized boundary.*

We say that two spin cobordisms (M,s) and (M',s') with parametrized boundary are spin homeomorphic if there exists a spin homeomorphism $f : (M,s) \to (M',s')$ which preserves the parametrized bases (or, in other words, whose restriction to the boundary commutes with the parametrizations).

Lemma 2 *Two spin cobordisms (M,s) and (M',s') with parametrized boundary are spin homeomorphic if and only if $(L,K,G^+ \cup G^-)$ and $(L',K',G'^+ \cup G'^-)$ can be related by (a sequence of) the following refined Kirby move(s).*

Add to L an unknotted component L_i with framing $\varepsilon = \pm 1$ and characteristic coefficient

$$c_i = 1 + \mathrm{lk}(L_i, K) + \mathrm{lk}_{\mathrm{odd}}(L_i, G^+ \cup G^-) \tag{3.6}$$

and change simultaneously the part of $L \cup G^+ \cup G^-$, lying in a regular neighborhood of a disc bounded by L_i, by giving a twist (right or left handed, depending on the sign of ε). The last term in (3.6) denotes the linking number of L_i with the odd coloured lines of the graph $G^+ \cup G^-$.

Proof of Theorem 1: One have to show that (3.4) is invariant under the refined Kirby move. It is not difficult to verify by direct calculation (see also [KM] or [Bl]) that adding of an odd (resp. even) coloured unknotted ε-framed component to L, linked with even (resp. odd) number of odd coloured strings [2], and twisting of these strings, will multiply the second line in (3.4) by Δ (if $\varepsilon = -1$) or by $\bar{\Delta}$ (if $\varepsilon = 1$) and the first line by Δ^{-1} (if $\varepsilon = -1$) or by $\omega^{-2}\Delta$ (if $\varepsilon = 1$). The claim follows now from (2.7). $\square$

The construction described above can be straightforwardly generalized to the case, when the surfaces $\partial_\pm M$ are provided with punctures coloured by $a_\pm$. The corresponding operator invariant is denoted by $\tau(a_+, M, s, a_-)$.

Presentation of spin cobordisms by special ribbon graphs

In (3.4) we represented a spin 3-cobordism (M, s) by some special ribbon graph $K \cup (L - K) \cup G^+ \cup G^-$ in S^3. We recall that K is the odd coloured, characteristic sublink of $L \cup G^+ \cup G^-$ and the colourings of G^+ and G^- are determined by $\mathfrak{s}_+$ and $\mathfrak{s}_-$, respectively. It turns out that this construction is invertible. This means that each such special ribbon graph gives rise to a 3-cobordism M with certain spin structure s. Starting from the special ribbon graph, one can construct (M, s) as follows:

One removes tubular neighborhoods of G^+ and G^- from S^3. This results in a 3-cobordism E with bottom base Σ^- and top base Σ^+. We provide E with orientation induced by right-handed orientation in S^3 and bases with orientations, such that $\partial E = (-\Sigma^-) \cup \Sigma^+$. We choose the parametrizations, which send the a-cycles of Σ_g to the loops on $\Sigma^\pm$ homotopic to the circles of the graphs $G^\pm$. Now remove from E a regular neighborhood of L. Choose a spin structure s on $E \backslash L$, which is non-bounding on the meridians of K and on the meridians of the odd coloured lines of $G^+ \cup G^-$. Glue solid tori back to $E \backslash L$ along the homeomorphisms determined by framing. This results in an oriented 3-cobordism, say M, with spin structure s and parametrized boundary.

Gluing properties

We will show that the operator $\tau(M, s)$ defines a non-degenerate spin TQFT.

Theorem 3 *If the spin 3-cobordism (M, s) is obtained from (M_1, s_1) and (M_2, s_2) by gluing along a homeomorphism $f : \Sigma \to \Sigma'$ which preserves spin*

[2] Fusion preserves the parity of colours.

structures and commutes with parametrizations, then

$$\tau(M,s)_{ee'} = k \sum_{e''\in E_{\mathfrak{s}}} \tau(M_2,s_2)_{ee''}\, \tau(M_1,s_1)_{e''e'}\,, \tag{3.7}$$

where $\Sigma = \partial_+ M_1$, $\Sigma' = \partial_- M_2$ *are parametrized connected surfaces and* $k = (\Delta\omega^{-1})^{\sigma(L)-\sigma(L_1)-\sigma(L_2)}$ *is an anomaly factor.*

Proof: We can represent M_1 and M_2 by special ribbon graphs $K_1 \cup (L_1 - K_1) \cup \bar{G}^g \cup G_1^-$ and $K_2 \cup (L_2 - K_2) \cup G_2^+ \cup G^g$, respectively, where g is the genus of Σ. Putting the special ribbon graph representing M_2 on the top of the graph for M_1 and summing over e_i'' ($e'' \in E_{\mathfrak{s}}$) with $i > g$, we obtain a special ribbon graph

$$K_2 \cup K_1 \cup (L_2 - K_2) \cup (L_1 - K_1) \cup \Omega \cup G_2^+ \cup G_1^-\,, \tag{3.8}$$

where by Ω we denote the g annuli, which remain of G^g and $\bar{G}^g$ after the summation. The graph (3.8) is, in fact, a special ribbon graph representing M (see [T, p.175] for more details). Its characteristic sublink consists of $K_1 \cup K_2$ together with the odd coloured annuli of Ω. □

Remark: Theorem 3 can be straightforwardly generalized to the case, when $\Sigma \subset \partial_+ M_1$, $\Sigma' \subset \partial_- M_2$.

If we glue 3-cobordisms along non-connected surfaces, the situation becomes more complicated, because a spin structure on the resulting manifold is not uniquely determined by the spin structure on 3-cobordisms glued together. In this case we have the following theorem:

Theorem 4 *If the spin 3-cobordism* (M,s) *is obtained from* (M_1,s_1) *and* (M_2,s_2) *by gluing along a homeomorphism* $f : \partial_+ M_1 \to \partial_- M_2$ *which preserves spin structures and commutes with parametrizations, then*

$$\sum_s \tau(M,s)_{ee'} = k \sum_{e''} \tau(M_2,s_2)_{ee''} \tau(M_1,s_1)_{e''e'}\,, \tag{3.9}$$

where the sum on the left hand side is taken over spin structures such that $s|_{M_1} = s_1$ *and* $s|_{M_2} = s_2$.

Proof: Assume that $\partial_+ M_1$ consists of n connected components of genera g_1, g_2, ... and g_n. Now the special ribbon graph representing M can be obtained from (3.8) by replacing Ω with a family of Ω_i, $1 \leq i \leq n$, where by Ω_i we denote g_i annuli, and then by encircling Ω_i, $1 \leq i \leq n-1$, by an unknotted annulus (see [T, p.177] for more details). Using fusion rules, (2.4) and (2.5) one can split this graph for M into two parts. The first one consists of a disjoint union of the special ribbon graphs representing M_1 and M_2. The second part contains terms where the special ribbon graph for M_1 and M_2 are connected by $(r-2)$-coloured lines. The sign of these terms depends on the choice of a spin structure s on M, whose restrictions to M_1 and M_2 are equal to s_1 and s_2, respectively. Taking the sum over all 2^{n-1} such s, we obtain (3.9). $\square$

In the next lemma we calculate the invariant of a spin 3-manifold obtained from two other spin manifolds by gluing along a non-connected surface.

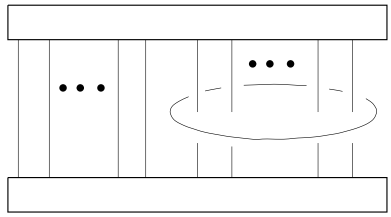
Lemma 5 *Let (M, s_i), $i \in \mathbb{Z}_2$, be spin 3-cobordisms obtained from (M_1, s_1) and (M_2, s_2) by gluing along a homeomorphism $f : \partial_+ M_1 \to \partial_- M_2$ which preserves spin structures and commutes with parametrizations. Here $\partial_+ M = \Sigma_1 \cup \Sigma_2$, $s_i|_{M_1} = s_1$, $s_i|_{M_2} = s_2$, s_0 is bounding and s_1 is not bounding on the additional cycle, which appears after gluing along a non-connected surface. Then*

$$\tau(M, s_i)_{ee'} = k/2 \left[\sum_{e''} \tau(M_2, s_2)_{ee''} \tau(M_1, s_1)_{e''e'} + \right.$$

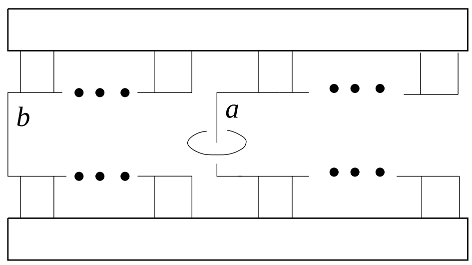
$$\left. + \ (-1)^i \sum_{e''} \tau(M_2, s_2, r-2, r-2)_{ee''} \ \tau(r-2, r-2, M_1, s_1)_{e''e'} \right], \quad (3.10)$$

where, in the second term, Σ_1 and Σ_2 are supposed to have an (r-2)-coloured puncture.

Proof: As explained in the proof of theorem 4, the special ribbon graph representing M looks as follows:



where the rectangles designate the remainder of the ribbon graph. The circle is odd coloured for s_1 and even for s_0. Using fusion rules, one can change this graph in the following way:



where one takes a sum over colourings of the new lines with quantum dimensions as coefficients. It follows from (2.4) or (2.5) that the colour a could be either 0 or $r-2$. If $a=0$ (resp. $a=r-2$), b should be equal to 0 (resp. $r-2$) too, and we get the first (resp. second) term in (3.10). □

Vector spaces associated to surfaces with spin structure

Due to Theorem 3, for a spin 3-cobordism (M,s) whose boundary $\partial M = \Sigma$ is a parametrized surface of genus g with spin structure $\sigma = s|_\Sigma$,

$$\tau(M,s)_e = \sum_{e' \in E_\mathfrak{s}} \tau(\Sigma \times [0,1], \sigma \cup \sigma)_{ee'}\, \tau(M,s)_{e'}\,. \tag{3.11}$$

One can now define the vector space $V_{(\Sigma,\sigma)}$, associated by the spin TQFT to the surface Σ with spin structure σ, as the support of the projector

$$\tau(\Sigma \times [0,1], \sigma \cup \sigma) : V_{(\Sigma,\mathfrak{s})} \to V_{(\Sigma,\mathfrak{s})}.$$

Assume (without loss of generality) that the parametrization of Σ in the cylinder is given by the identity homeomorphism. Then the special ribbon graph in Fig.5

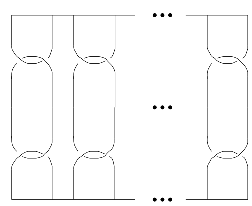


Fig.5 *The special ribbon graph corresponding to a cylinder*

represents the 3-cobordism $\Sigma \times [0,1]$ (see [T, p.173] for more details). It consists of two copies of the graph G^g linked with g annuli $\Gamma_1\,,\ \Gamma_2\,,\ \ldots,\ \Gamma_g$. A special colouring of G^g is determined by $q_\sigma(b_i)$, $1 \le i \le g$, and the parity of colours on Γ_i by $q_\sigma(a_i)$. Using (2.4), (2.5) and fusion rules one can calculate that

$$\tau(\Sigma \times [0,1], \sigma \cup \sigma)_{ee'} = \frac{1}{2^g}(\delta_{e_1\, e'_1} + (-1)^{c_1}\delta_{e_1\, \hat{e}'_1})\ldots(\delta_{e_g\, e'_g} + (-1)^{c_g}\delta_{e_g\, \hat{e}'_g}) \prod_{i>g} \delta_{e_i\, e'_i}\,, \tag{3.12}$$

where $c_i = q_\sigma(a_i)$ and $\hat{e}_i = r - 2 - e_i$. For simplicity we will write τ^σ for $\tau(\Sigma \times [0,1], \sigma \cup \sigma)$ in what follows.

One can easily establish that the operators τ^σ form a family of 4^g orthogonal projectors on the vector spaces $V_{(\Sigma,\,\sigma)}$, i.e.

$$\sum_{e'} \tau^{\sigma_1}_{ee'}\, \tau^{\sigma_2}_{e'e''} = \begin{cases} 0, & \text{if } \sigma_1 \neq \sigma_2 \\ \tau^{\sigma_1}_{ee''}, & \text{if } \sigma_1 = \sigma_2 \end{cases}$$

and

$$V_\Sigma = \oplus_{i=1}^{4^g} V_{(\Sigma,\,\sigma_i)}. \tag{3.13}$$

As usual, we associate the tensor product of vector spaces to the disjoint union of surfaces.

Clearly,

$$\tau(M, s) : V_{(\partial_- M,\, s_-)} \to V_{(\partial_+ M,\, s_+)},$$

where $s_\pm = s|_{\partial_\pm M}$.

Lemma 6 *The Reshetikhin–Turaev invariant of a 3-cobordism M with parametrized boundary splits as a sum of the refined invariants corresponding to different spin structures, i.e.*

$$\tau(M) = \oplus_{s_\pm} \sum_s \tau(M, s)\,, \tag{3.14}$$

where the sum is over all spin structures s on M such that $s|_{\partial_\pm M} = s_\pm$.

Proof: The claim follows from the fact that the contribution to $\tau(M)$ from odd coloured, non-characteristic sublinks of $L \cup G^+ \cup G^-$ is equal to zero. The explicit computations are quite analogous to the one given in [Bl] or [KM], and they will not be repeated here. □

Dimension of vector spaces

The Reshetikhin-Turaev TQFT yields a representation of the mapping class group (MCG). The matrix elements for generators of MCG are listed, e.g., in [KSV]. In the spin TQFT, the MCG generates transformations between vector spaces associated to different spin structures with the same Arf-invariant. We recall that the Arf-invariant of a quadratic form q_σ (corresponding to spin structure σ on Σ_g) is defined as follows:

$$\mathrm{Arf}(\sigma) = \sum_{i=1}^{g} q_\sigma(a_i) q_\sigma(b_i)\,,$$

where a_i, b_i is the symplectic homology basis depicted in Fig.4.

As a result, the dimension of $V_{(\Sigma,\sigma)}$ depends only on the Arf-invariant of σ, but not on σ itself. On Σ there exist $2^{g-1}(2^g+1)$ spin structures with Arf-invariant equal to zero and $2^{g-1}(2^g-1)$ with Arf-invariant equal to one.

Theorem 7 *For a closed surface Σ of genus g with spin structure σ,*

$$\mathrm{dim} V_{(\Sigma,\,\sigma)} = \frac{1}{4^g}\,[\,\mathrm{dim} V_\Sigma + (r/2)^{g-1}(2^g-1)\,],\ \text{ if } \ \mathrm{Arf}(\sigma) = 0\,, \tag{3.15}$$

$$\mathrm{dim} V_{(\Sigma,\,\sigma)} = \frac{1}{4^g}[\,\mathrm{dim} V_\Sigma - (r/2)^{g-1}(2^g+1)\,],\ \text{ if } \ \mathrm{Arf}(\sigma) = 1\,, \tag{3.16}$$

where $\mathrm{dim} V_\Sigma$ *is given by the Verlinde formula.*

The dimensions of spin modules were first calculated in [BHMV] using a rather developed algebraic technique. Here we will use simple geometric arguments, which refine Lickorish's calculations in [Li].

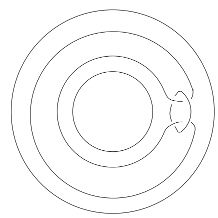
Proof: The dimension of the vector space $V_{(\Sigma,\,\sigma)}$ can be calculated as follows

$$\dim V_{(\Sigma,\,\sigma)} = \mathrm{tr}\ \tau(\Sigma \times [0,1], \sigma \cup \sigma). \tag{3.17}$$

Theorem 4 implies that

$$\operatorname{tr} \tau(\Sigma \times [0,1], \sigma \cup \sigma) = \tau(S^1 \times \Sigma, s_0) + \tau(S^1 \times \Sigma, s_1)\,, \tag{3.18}$$

where $s_i|_\Sigma = \sigma$, s_0 is bounding and s_1 is not bounding on S^1. A surgery diagram for $S^1 \times \Sigma$ can be obtained by taking g copies of the annulus containing a link, which is depicted below,

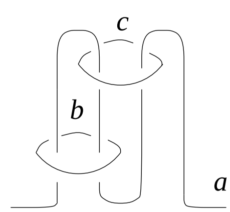


(3.19)

threading un unknotted closed curve l though the annuli and finally taking the resultant link of $2g+1$ components.

Denote a colour of l by a. Then the invariant $\tau(S^1 \times \Sigma, s_i)$ can be calculated in the following way: One takes g times expression (3.20),

$$\sum_{b\,c} \omega_b^2 \omega_c^2$$



(3.20)

closes an a-coloured line, sums over a with ω_a^2 as coefficients, (note that a is even for s_0 and odd for s_1) and multiplies by ω^{-2g-2}.

Consider at first the case when $\mathrm{Arf}(\sigma) = 0$. Then one can suppose that all colours (except of a) are even. Applying fusion rules, (2.4) and the following formula

$$\begin{vmatrix} r/2-1 & r-2 & r/2-1 \\ r/2-1 & r-2 & r/2-1 \end{vmatrix} = -\,\omega_{r/2-1}^{-2}$$

(see (4.5) in [BD1] for the graphic and [TV] for the analytic definition of 6j-symbols), one can reduce (3.20) to the a-coloured line multiplied by

$$\frac{\omega^4}{4\,\omega_a^4}(1 + \delta_{a,\,r/2-1})\,,$$

where the last term contributes to $\tau(S^1\times\Sigma, s_1)$ only, because $r/2-1$ is odd. Taking into account all coefficients, we obtain that

$$\tau(S^1\times\Sigma, s_0) = \frac{\omega^{2g-2}}{4^g}\sum_{a\,\mathrm{even}}\omega_a^{4-4g}\,,$$

$$\tau(S^1\times\Sigma, s_1) = \frac{\omega^{2g-2}}{4^g}\Big(\sum_{a\,\mathrm{odd}}\omega_a^{4-4g} + \frac{2^g-1}{\omega_{r/2-1}^{4g-4}}\Big),$$

which after substituting in (3.18) and using (2.1) and (2.2) implies (3.15).

The dimension of $V_{(\Sigma,\sigma)}$ with $\mathrm{Arf}(\sigma)=1$ can be calculated analogously or determined from the formula:

$$\dim V_\Sigma = 2^{g-1}(2^g+1)\dim V_{(\Sigma,\,\sigma_0)} + 2^{g-1}(2^g-1)\dim V_{(\Sigma,\,\sigma_1)}\,,$$

where $\mathrm{Arf}(\sigma_0)=0$ and $\mathrm{Arf}(\sigma_1)=1$. □

4 Refined Turaev–Viro TQFT

The aim of this section is to refine the construction of Turaev–Viro 3-cobordism invariants as given in [BD1], [BD2] and define the state sum operator $Z(M,s,h)$, satisfying the requirements of a TQFT, where s is a spin structure on M and $h\in H^1(M)$. We start by recalling the construction of [BD1,2].

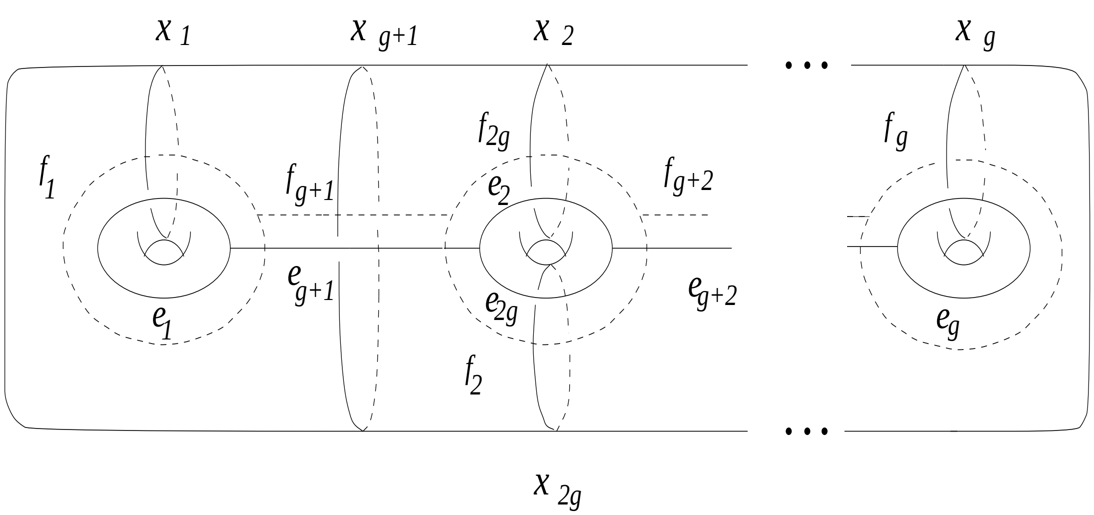
4.1 Standard model

The Turaev-Viro state sum is defined for any compact triangulated 3-manifold M as follows: One puts colours on 1-simplexes of M and associates 6j-symbols to coloured tetrahedra. Then the Turaev-Viro invariant is given by a sum over all colourings of 1-simplexes in the interior of M of the product of 6j-symbols (with some coefficients). The vector space $V(\Sigma)$ associated to a triangulated surface Σ is defined as a direct sum over all colourings of the tensor product of vector spaces belonging to coloured triangles of Σ modulo some equivalence relation.

As was already mentioned in the introduction, we will use a modified state sum operator $Z(M,G)$, where G is a 3-valent ribbon graph on ∂M.

The operator $Z(M,G)$ was defined in [BD1] (see also [KS]) in such a way, that it is equal to the Turaev-Viro state sum for M, where the triangulation of ∂M is given by the graph dual to G [3]. Moreover, $Z(M,G)$ is a homotopy invariant of the graph G. In [BD2] an isomorphism was constructed between $V(\Sigma)$ and the vector space generated by colourings of two copies of the graph, depicted in Fig.1.

The cobordism M_g^+ providing this isomorphism we will call a standard handlebody. M_g^+ is a cylinder $\Sigma_g \times [0,1]$, where Σ_g is a closed oriented surface of genus g standardly embedded in $\mathbb{R}^3$. Furthermore, M_g^+ contains an arbitrary 3-valent graph $\mathcal{G}^g$, sitting on $\Sigma_g = \Sigma_g \times \{1\}$, and the coloured graph $G_e^g \cup \bar{G}_f^g \cup m_x$, depicted below, on $-\Sigma_g = \Sigma_g \times \{0\}$,



(4.1)

where $m = \{m_1, ..., m_{3g-3}\}$ is the ordered set of meridians coloured by $x = \{x_1, ..., x_{3g-3}\}$ and $e = \{e_1, ..., e_{3g-3}\}$, $f = \{f_1, ..., f_{3g-3}\}$ are admissible colourings of G^g and $\bar{G}^g$, respectively. We note that the f-coloured graph is drawn on the backward side of Σ_g.

[3] In this article we suppose that the graph G is large enough in order that its dual defines a triangulation of ∂M.

The state sum K_{ef} of the standard handlebody is given by the formula:

$$K_{ef} = \omega^{g-1}\omega_e\omega_f \sum_x \prod_{i=1}^{3g-3} \frac{\omega_{x_i}^2}{\omega^2}\, Z(M_g^+, G_e^g \cup \bar{G}_f^g \cup m_x \cup \mathcal{G}^g), \qquad (4.2)$$

where the sum is over colourings of the meridians. This state sum defines a linear operator

$$K_{ef} : V_g^L(e) \otimes V_g^R(f) \to V(\Sigma_g),$$

where $V_g^L(e) \otimes V_g^R(f)$ is the vector space associated to the graph $G_e^g \cup \bar{G}_f^g$. It turns out, that the mirror image M_g^- of M_g^+ yields an inverse operator

$$L_{ef} : V(\Sigma_g) \to V_g^L(e) \otimes V_g^R(f),$$

which satisfies the following equation (see [BD2] for more details):

$$L_{e'f'}K_{ef} = \delta_{e,e'}\delta_{f,f'}\ \mathbb{1}_{\,V_g^L(e)\otimes V_g^R(f)}.$$

Taking into account that the dimensions of $\oplus_{ef}\{V_g^L(e) \otimes V_g^R(f)\}$ and $V(\Sigma_g)$ coincide, we obtain that

$$K = \oplus_{ef} K_{ef}$$

is an isomorphism and admissible colourings of $G_e^g \cup \bar{G}_f^g$ provide a basis of $V(\Sigma_g)$.

From now on we fix the standard handlebodies M_g^+ and M_g^- together with the graphs on their boundaries. We say that an oriented triangulated surface Σ is parametrized, if it is supplied with a simplicial map $\phi : (\mathcal{G}^g)^* \to X$, where by $(\mathcal{G}^g)^*$ we denote the triangulation of Σ_g, given by the graph dual to $\mathcal{G}^g$, and X is a triangulation of Σ. The parametrization of $-\Sigma$ is given by the map $-\phi : (\bar{\mathcal{G}}^g)^* \to -X$.

Consider a 3-cobordism M with parametrized boundary $\partial M = (-\partial_- M) \cup \partial_+ M$. Let us glue the standard handlebodies to the connected components of $\partial_\pm M$ along the parametrizations. The state sum of the resulting manifold with a 3-valent graph on the boundary defines an invariant of the 3-cobordism M in the basis mentioned above. More precisely,

$$Z(M)_{ef,e'f'} = \omega^{\frac{-\chi(\partial M)}{2}} \omega_e\omega_f\omega_{e'}\omega_{f'} \sum_{xy} \prod_{i\,j} \frac{\omega_{x_i}^2\,\omega_{y_j}^2}{\omega^2\,\omega^2}\, Z(M, G_{ef}^+ \cup G_{e'f'}^- \cup m_x \cup m_y)\,, \qquad (4.3)$$

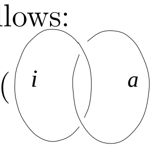
where $G^+_{ef} = G^+_e \cup \bar{G}^+_f$ and $G^-_{ef} = G^-_e \cup \bar{G}^-_f$ are the disjoint unions of the graphs $\bar{G}^g_e \cup G^g_f$ and $G^g_e \cup \bar{G}^g_f$, sitting on the boundaries of the standard handlebodies M^-_g and M^+_g, respectively. Representing M by surgery on an m-component link L and using the technique developed in [BD1] and [BD2], one can rewrite (4.3) in terms of the link invariants:

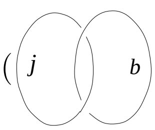
$$Z(M)_{ef,e'f'} = \frac{\omega_e \omega_{e'}}{\omega^{m+1-\chi(\partial_+ M)/2}} \sum_c \omega_c^2 \; Z(L_c \cup G^+_e \cup G^-_{e'}) \times$$
$$\times \frac{\omega_f \omega_{f'}}{\omega^{m+1-\chi(\partial_- M)/2}} \sum_b \omega_b^2 \; Z(\bar{L}_b \cup \bar{G}^+_f \cup \bar{G}^-_{f'}) \tag{4.4}$$

or

$$Z(M)_{ef,\,e'f'} = \tau(M)_{ee'} \; \tau(-M)_{f'f} \, . \tag{4.5}$$

Example: Consider a solid torus $D^2 \times S^1$. Due to (4.4) the corresponding state sum can be written as follows:

$$Z_{ij}(D^2 \times S^1) = \frac{\omega_i}{\omega^2} \sum_a \omega_a^2 Z\Big($$


$$\Big) \; \frac{\omega_j}{\omega^2} \sum_b \omega_b^2 Z\Big($$


$$\Big). \tag{4.6}$$

Recall that Euler characteristic of an empty set is equal to zero.

We split the sums in (4.6) into the sums over even and odd colours, i.e.

$$Z_{ij}(D^2 \times S^1) = Z_{ij}(D^2 \times S^1, s_0, 0) + Z_{ij}(D^2 \times S^1, s_1, 0) +$$
$$+ Z_{ij}(D^2 \times S^1, s_0, h) + Z_{ij}(D^2 \times S^1, s_1, h), \tag{4.7}$$

where the first (resp. second) term corresponds in (4.6) to the case, when both a and b are even (resp. odd), in the third term a is even and b odd, and inversely in the forth term. Using (2.4) and (2.5) one can calculate

$$Z_{ij}(D^2 \times S^1, s_0, 0) = \frac{1}{4}(\delta_{i,0} + \delta_{i,\,r-2})(\delta_{j,0} + \delta_{j,\,r-2}),$$
$$Z_{ij}(D^2 \times S^1, s_1, 0) = \frac{1}{4}(\delta_{i,0} - \delta_{i,\,r-2})(\delta_{j,0} - \delta_{j,\,r-2}),$$
$$Z_{ij}(D^2 \times S^1, s_0, h) = \frac{1}{4}(\delta_{i,0} + \delta_{i,\,r-2})(\delta_{j,0} - \delta_{j,\,r-2}),$$
$$Z_{ij}(D^2 \times S^1, s_1, h) = \frac{1}{4}(\delta_{i,0} - \delta_{i,\,r-2})(\delta_{j,0} + \delta_{j,\,r-2}).$$

Finally, we have

$$Z_{ij}(D^2 \times S^1) = \delta_{i,0} \; \delta_{j,0} \, .$$

4.2 Refined Turaev–Viro model

In this section we refine the construction of [BD2].

Definition of invariants

We start by modifying the notion of a standard handlebody. As before, consider the cylinder $\Sigma_g \times [0,1]$ with the graph $\mathcal{G}^g \in \Sigma_g$ and the graph (4.1) on $-\Sigma_g$. Denote by b_i a closed 1-dimensional subcomplex of the graph $\mathcal{G}^g$, representing the i^{th} meridian of Σ_g, $1 \leq i \leq g$. We recall that the graph dual to $\mathcal{G}^g$ provides a triangulation of Σ_g. Associate a $\mathbb{Z}_2$-number to the meridian b_i of Σ_g, $1 \leq i \leq g$, and denote by $\mathfrak{s}$ a sequence of these numbers. Let $\mathfrak{h}$ be a fixed subset of $\{b_i\}$. These data define a standard handlebody $(M_g^+, \mathfrak{s}, \mathfrak{h})$.

We say that (e, f) is a special colouring of the graph $G^g \cup \bar{G}^g$, if the following conditions are satisfied:

- colours e_i and f_i, $1 \leq i \leq g$, are even, if $b_i \notin \mathfrak{h}$ and $\mathfrak{s}_i = 0$;
- colours e_i and f_i, $1 \leq i \leq g$, are odd, if $b_i \notin \mathfrak{h}$ and $\mathfrak{s}_i = 1$;
- a colour e_i is even and f_i is odd, $1 \leq i \leq g$, if $b_i \in \mathfrak{h}$ and $\mathfrak{s}_i = 0$;
- a colour e_i is odd and f_i is even, $1 \leq i \leq g$, if $b_i \in \mathfrak{h}$ and $\mathfrak{s}_i = 1$.

We denote the set of all special colourings by $E(\mathfrak{s}, \mathfrak{h})$. The state sum for the standard handlebody is given by the formula:

$$K_{ef}(\mathfrak{s}, \mathfrak{h}) = \omega^{g-1} \omega_e \omega_f \sum_x \prod_{i=1}^{3g-3} \frac{\omega_{x_i}^2}{\omega^2} \, Z(M_g^+, G_e^g \cup \bar{G}_f^g \cup m_x \cup \mathcal{G}^g), \tag{4.8}$$

and

$$K(\mathfrak{s}, \mathfrak{h}) = \oplus_{e\,f} \, K_{ef}(\mathfrak{s}, \mathfrak{h}), \quad (e, f) \in E(\mathfrak{s}, \mathfrak{h}). \tag{4.9}$$

This defines an inclusion

$$K(\mathfrak{s}, \mathfrak{h}) : V_{\Sigma_g}(\mathfrak{s}, \mathfrak{h}) \to V(\Sigma_g),$$

where

$$V_{\Sigma_g}(\mathfrak{s}, \mathfrak{h}) = \oplus_{e\,f} \, \{V_g^L(e) \otimes V_g^R(f)\}, \quad (e, f) \in E(\mathfrak{s}, \mathfrak{h}).$$

The oppositely oriented handlebody is given in the usual way as the mirror image of $(M_g^+, \mathfrak{s}, \mathfrak{h})$. The corresponding state sum $L(\mathfrak{s}, \mathfrak{h})$ yields a projector

$$L(\mathfrak{s}, \mathfrak{h}) : V(\Sigma_g) \to V_{\Sigma_g}(\mathfrak{s}, \mathfrak{h}).$$

It is not difficult to verify by direct calculation that

$$L(\mathfrak{s}', \mathfrak{h}')K(\mathfrak{s}, \mathfrak{h}) = \begin{cases} 0, & \text{if } \mathfrak{s}' \neq \mathfrak{s} \text{ and } \mathfrak{h}' \neq \mathfrak{h}; \\ id_{V_{\Sigma_g}(\mathfrak{s},\mathfrak{h})}, & \text{if } \mathfrak{s}' = \mathfrak{s} \text{ and } \mathfrak{h}' = \mathfrak{h}. \end{cases} \tag{4.10}$$

By a parametrized triangulated surface $(\Sigma, \mathfrak{s}, \mathfrak{h})$ of genus g we mean a parametrized oriented surface Σ with triangulation X provided with a sequence $\mathfrak{s}$ of $\mathbb{Z}_2$-numbers associated to the meridians $\phi(b_i)$, $1 \leq i \leq g$, and a fixed subset $\phi(\mathfrak{h})$ of the meridians, where $\phi : (\mathcal{G}^g)^* \to X$ is the parametrization of Σ. To the parametrized surface $(\Sigma, \mathfrak{s}, \mathfrak{h})$ we associate a vector space $V_\Sigma(\mathfrak{s}, \mathfrak{h})$, generated by special colourings $E(\mathfrak{s}, \mathfrak{h})$ of the graph $G^g \cup \bar{G}^g$.

Consider a 3-cobordism (M, s, h) with parametrized boundary $\partial M = (-\partial_- M) \cup \partial_+ M$, where s is a spin structure on M and $h \in H^1(M)$. Let us enumerate the connected components of ∂M by an index j, $1 \leq j \leq n$. Suppose that the first l of them belong to $\partial_- M$ and the remaining to $\partial_+ M$. Choose a sequence $\mathfrak{s}_j$ of $\mathbb{Z}_2$-numbers and a set $\mathfrak{h}_j$ on the j^{th} connected component Σ_j of ∂M, such that

$$(\mathfrak{s}_j)_i = q_{s|_{\Sigma_j}}(\phi_j(b_i)), \;\; 1 \leq i \leq g_j, \tag{4.11}$$

and $\mathfrak{h}_j$ consists of the meridians b_i, such that h is non-trivial on the homology class $[\phi_j(b_i)] \in H_1(M)$. Here ϕ_j is the parametrization of Σ_j.

One glues (along the parametrizations) $(M_{g_j}^+, \mathfrak{s}_j, \mathfrak{h}_j)$, $1 \leq j \leq l$, and $(M_{g_j}^-, \mathfrak{s}_j, \mathfrak{h}_j)$, $l < j \leq n$, to the connected components of $\partial_- M$ and $\partial_+ M$, respectively. The resulting manifold can be represented by surgery on S^3 with n handlebodies removed and with a graph (given by the image of (4.1) under parametrization) sitting on the boundary of each handlebody (see [BD2] for more details). We set

$$Z(M, s, h)_{ef,e'f'} = \omega^{-\chi(\partial M)/2} \omega_e \omega_f \omega_{e'} \omega_{f'} \sum_{xyz} \prod_{i\,j,k} \frac{\omega_{x_i}^2\, \omega_{y_j}^2\, \omega_{z_k}^2}{\omega^2\, \omega^2\, \omega^2}$$

$$\sum_{a\,b\,a'\,b'} Z(\tilde{S}^3, L_{ab} \cup m_z \cup G^+_{ef} \cup G^-_{e'f'} \cup m_x \cup m_y) \prod_{i=1}^{m} S_{a_i\,a'_i}\, S_{b_i\,b'_i}\; Z_{a'_i\,b'_i}(D^2 \times S^1, s_i, h_i)\,, \tag{4.12}$$

where

$$(e,f) \in E(\mathfrak{s},\mathfrak{h}), \quad (e',f') \in E(\mathfrak{s}',\mathfrak{h}'),$$

L is an m-component surgery link; $\tilde{S}^3$ is S^3 with neighborhoods of L, G^+ and G^- removed; $L_{ab} \cup m_z$ is the coloured graph on the boundary of a neighborhood of L [4]; S_{ij} is an invariant of the Hopf link (normalized by ω^{-1}), or equivalently, an element of MCG interchanging cycles in the canonical homology basis of a torus; s_i and h_i are the restrictions of s and h on the neighborhood of L_i. The state sums of a solid torus with additional structures are listed in the example of section 4.1, where s_0 (resp. s_1) denotes the spin structure, which is (not) bounding on S^1.

Taking into account that

$$\sum_{a'} S_{aa'}(\delta_{a',0} + \delta_{a',r-2}) = \begin{cases} \omega^{-1}\omega_a^2, & \text{if } a \text{ is even} \\ 0, & \text{if } a \text{ is odd} \end{cases}$$

$$\sum_{a'} S_{aa'}(\delta_{a',0} - \delta_{a',r-2}) = \begin{cases} 0, & \text{if } a \text{ is even} \\ \omega^{-1}\omega_a^2, & \text{if } a \text{ is odd} \end{cases}$$

and repeating the computation given in the proof of Theorem 2 in [BD2], one obtains that

$$Z(M,s,h)_{ef,\,e'f'} = \tau(M,s)_{ee'}\; \tau(-M, s+h)_{f'f}\,. \tag{4.13}$$

As a result, the operator $Z(M,s,h)$, defined by (4.12), extends the Roberts' invariant to an anomaly free non-degenerate TQFT.

Gluing property

Corollary 8 *If the 3-cobordism (M,s,h) is obtained from (M_1,s_1,h_1) and (M_2,s_2,h_2) by gluing along a homeomorphism $f : \partial_+ M_1 \to \partial_- M_2$ which*

[4] More precisely, $L_{ab} \cup m_z = \cup_{i=1}^{m}(L_{a_i b_i} \cup m_{z_i})$, where $L_{a_i b_i}$ consists of two (a_i- and b_i-coloured) lines homotopic to L_i, where one of them overcrosses and the other one undercrosses meridian m_{z_i}.

preserves structure and commutes with parametrizations, then

$$\sum_{s,h} Z(M,s,h)_{ef,e'f'} = \sum_{e''f''} Z(M_2,s_2,h_2)_{ef,e''f''}\, Z(M_1,s_1,h_1)_{e''f'',e'f'}\,,$$

where the sum on the left hand side is taken over all s and h, such that $s|_{M_1}=s_1$, $s|_{M_2}=s_2$ and $h|_{M_1}=h_1$, $h|_{M_2}=h_2$.

Vector spaces associated to surfaces with structure

Due to (4.13), for a closed connected surface Σ with spin structure σ and $\mathrm{h}\in H^1(\Sigma)$,

$$Z(\Sigma\times[0,1],\sigma\cup\sigma',\mathrm{h}\cup\mathrm{h}')_{ef,e'f'} = \begin{cases} 0, & \text{if } \sigma\neq\sigma' \text{ and } \mathrm{h}\neq\mathrm{h}';\\ \tau^{\sigma}_{ee'}\tau^{\sigma+\mathrm{h}}_{f'f}, & \text{if } \sigma=\sigma' \text{ and } \mathrm{h}=\mathrm{h}'.\end{cases}$$

Taking a direct sum over all special colourings we obtain an operator $Z(\Sigma\times[0,1],\sigma,\mathrm{h})$. We define the vector space $V_\Sigma(\sigma,\mathrm{h})$ to be the support of this operator. This vector space is associated by the spin TQFT of Turaev–Viro type to the closed oriented connected surface Σ provided with spin structure σ and first cohomology class h. Clearly,

$$V(\Sigma)=\oplus_{\sigma,\,\mathrm{h}}V_\Sigma(\sigma,\mathrm{h}),$$

$$\mathrm{dim}V_\Sigma(\sigma,\mathrm{h}) = \mathrm{dim}V_{(\Sigma,\,\sigma)}\,\mathrm{dim}V_{(\Sigma,\,\sigma+\mathrm{h})}$$

and

$$Z(M,s,h): V_{\partial_- M}(s_-,\mathrm{h}_-)\to V_{\partial_+ M}(s_+,\mathrm{h}_+)\,,$$

where $s_\pm = s|_{\partial_\pm M}$ and $\mathrm{h}_\pm = h|_{\partial_\pm M}$.

It follows from the results of Section 3.3, that

$$Z(M)=\oplus_{s_\pm,\,\mathrm{h}_\pm}\sum_{s,h} Z(M,s,h),$$

where the sum is over s and h, such that $s|_{\partial_\pm M}=s_\pm$ and $h|_{\partial_\pm M}=\mathrm{h}_\pm$. Moreover,

$$Z(M,h)=\oplus_{s_\pm}\sum_{s} Z(M,s,h)$$

is an invariant of a 3-cobordism M with first cohomology class h, which can be defined as follows (see [TV]): Let us introduce a function $a : I \to \mathbb{Z}_2$, such that

$$a(i) = i \pmod{2}\,.$$

Then for any admissible triple (i, j, k)

$$a(i) + a(j) + a(k) = 0.$$

Therefore, each colouring of a triangulated 3-manifold M composed with a is a 1-cocycle of M. For any $h \in H^1(M)$, $Z(M, h)$ is equal to the Turaev-Viro invariant, where one sums over all colourings which induce cocycles representing h.

5 Concluding remarks

In this article we restrict our attention to the case $r = 0 \pmod{4}$, because it corresponds to the invariants with the richest topological structure. The case $r = 2 \pmod{4}$ can be treated by quite similar methods, but it leads to invariants of 3-cobordisms with a first $\mathbb{Z}_2$-cohomology class only. For odd r so far no refined invariants are known.

It would be interesting to find out whether refined quantum invariants determined by additional topological structures on 3-manifolds could be defined for higher quantum groups. We leave this question for future investigation.

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