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ABELIAN TQFTS AND SCHRÖDINGER LOCAL SYSTEMS

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Abstract. In this paper we construct an action of 3-cobordisms on the finite dimensional Schrödinger representations of the Heisenberg group by Lagrangian correspondences. In addition, we review the construction of the abelian Topological Quantum Field Theory (TQFT) associated with a q-deformation of $U(1)$ for any root of unity q. We prove that for 3-cobordisms compatible with Lagrangian correspondences, there is a normalization of the associated Schrödinger action that reproduces the abelian TQFT. Restricting to mapping cylinders, our construction yields two projective representations of the mapping class group. We show that their linearizations do not coincide by analysing the corresponding 2-cocycles.

To the memory of Vaughan Jones,
the founder of quantum topology.

1. Introduction

The discovery of the Jones polynomial revolutionized low–dimensional topology. The new link invariants constructed by Jones, Kauffman, HOMFLY-PT, Reshetikhin–Turaev etc. were extended to mapping class group representations, later shown to be asymptotically faithful, and to 3-manifold invariants. These developments have reached their peak in constructions of Topological Quantum Field Theories (TQFTs) [30, 10]. The scope of ideas initiated by Vaughan Jones built the foundations for the new domain of mathematics – the quantum topology. One of the main open problems in quantum topology is to understand the topological nature of quantum invariants.

In the 90s, Lawrence [25] initiated a program aiming at homological interpretation of quantum invariants. In 2001 Bigelow was able to read the Jones polynomial from the intersection pairing on the twisted homology of the configuration space $\mathrm{Conf}_n(\mathbb{D}^2_m)$ of n points in m-punctured disc $\mathbb{D}^2_m$. This construction led to a family of representations (indexed by n) of the braid group B_m, that recovers for $n=1$ the Burau representation. A spectacular achievement was the proof by Bigelow [8] and Krammer [24] that this braid group representation for $n=2$ is faithful, showing the linearity of the braid group. Bigelow's construction was extended later to other quantum link invariants [9, 1, 2].

Recently homological mapping class group representations were constructed by the second author together with Palmer and Shaukat [12]. The idea here was to use a *Heisenberg* cover of the space $\mathrm{Conf}_n(\Sigma)$ of unordered n configurations in a 1-punctured surface Σ, whose group of deck transformations is the *Heisenberg group* $\mathscr{H}(\Sigma)$. Recall that $\mathscr{H}(\Sigma)=\mathbb{Z}\times H_1(\Sigma,\mathbb{Z})$ has the group law

$$(k,x)(l,y)=(k+l+\ x.y, x+y)$$

where $x.y$ is the intersection pairing. Since the surface braid group $B_n(\Sigma):=\pi_1(\mathrm{Conf}_n(\Sigma))$ surjects onto $\mathscr{H}(\Sigma)$, the Heisenberg cover $\widetilde{\mathrm{Conf}}_n(\Sigma)$ is determined by the kernel of this map.

Since the group of deck transformations $\mathscr{H}(\Sigma)$ acts on the chain groups of the Heisenberg cover, any module M over $\mathbb{C}[\mathscr{H}(\Sigma)]$ can be used as a local system to define a twisted homology $H_\bullet(\mathrm{Conf}_n(\Sigma), M)$ with coefficients in M as homology of the complex

$$C_\bullet(\widetilde{\mathrm{Conf}}_n(\Sigma)) \otimes_{\mathbb{C}[B_n(\Sigma)]} M$$

where $\mathbb{C}[B_n(\Sigma)]$ is the group algebra of the surface braid group. An interesting choice of M provides a finite dimensional Schrödinger representation $W_q(L)$ of a finite quotient of $\mathscr{H}(\Sigma)$, which depends on a choice of a Lagrangian $L \subset H_1(\Sigma, \mathbb{Z})$ and a root of unity q. If the order of q is odd, the resulting mapping class group representations were recently shown to contain the quantum representations arising from the non-semisimple TQFT for the small quantum $\mathfrak{sl}_2$ by De Renzi and Martel [15]. In particular, they defined the action of the quantum $\mathfrak{sl}_2$ on the Schrödinger homology explicitly and showed that it commutes with the action of the mapping class group.

To complete Lawrence–Bigelow program we are lacking homological interpretation of quantum 3-manifold invariants and of the action of 3-cobordism on the Schrödinger homologies. This paper is a first step in this direction. Here we construct the symplectic action of 3-cobordisms on the Schrödinger local systems by Lagrangian correspondences. In addition, we show that on a certain subcategory of extended 3-cobordisms and after a suitable normalization this action recovers the abelian TQFT.

Abelian TQFTs are functorial extensions of 3-manifold invariants constructed by Murakami–Ohtsuki–Okada from linking matrices [29]. Their connections with theta functions and Schrödinger representations, in the case when the quantum parameter (called t in these papers) is a root of unity of order divisible by 4, were extensively studied by Gelca and collaborators [20, 19, 18, 17]. Here we work with an arbitrary root of unity. We show that interesting cases are if the order is either odd or divisible by 4. In the latter case we complete the work of Gelca and al. by constructing TQFTs via *modularization functor*. In addition, we discuss refined TQFTs corresponding to the choice of the spin structure or a first cohomology class on 3-manifolds.

Our preferred cobordism category is the Crane–Yetter category 3Cob of connected oriented 3-cobordism between connected 1-punctured surfaces with the boundary connected sum as a monoidal structure. This category has a beautiful algebraic presentation: it is monoidally generated by the Habiro Hopf algebra object — the punctured torus [7, 13]. By the result of [6], for any finite unimodular ribbon category $\mathscr{C}$, there exits a monoidal TQFT functor $F : 3\mathrm{Cob}^\sigma \to \mathscr{C}$ defined by sending the punctured torus to the end of $\mathscr{C}$. Here $3\mathrm{Cob}^\sigma$ is the category of *extended* 3-cobordisms, whose objects are connected 1-punctured surfaces equipped with a choice of Lagrangian, and morphisms are 3-cobordisms equipped with natural numbers called weights. The composition includes a correction term given by a Maslov index. Note that if $\mathscr{C}$ is the category of modules over a unimodular ribbon Hopf algebra H, then $\mathrm{end}(\mathscr{C}) = (H, \triangleright)$ where $\triangleright$ denotes the adjoint action.

Let us define a subcategory $3\mathrm{Cob}^{\mathrm{LC}}$ of $3\mathrm{Cob}^\sigma$ having the same objects, but a smaller set of morphisms. A cobordism $C = (C, 0)$ belongs to $3\mathrm{Cob}^{\mathrm{LC}}((\Sigma_-, L_-), (\Sigma_+, L_+))$ if and only if

$$L_C.L_- = L_+ \quad \text{where} \quad L_C = \mathrm{Ker}\,(i_* : H_1(\partial C, \mathbb{Z}) = H_1(-\Sigma_-, \mathbb{Z}) \oplus H_1(\Sigma_+, \mathbb{Z}) \to H_1(C, \mathbb{Z}))$$

is the *Lagrangian correspondence* determined by C which acts on $L_- \subset H_1(\Sigma_-, \mathbb{Z})$ by

$$L_C.L_- = \{y \in H_1(\Sigma_+) \,|\, \exists\, x \in L_-,\ (x, y) \in L_C\}.$$

In this subcategory all anomalies vanish. We get a linear representation of the subgroup of the mapping class group fixing a Lagrangian. The full mapping class group is replaced by a

groupoid whose objects are Lagrangian and morphisms are compatible mapping classes. This *action groupoid* is a subcategory in $3\mathrm{Cob}^{\mathrm{LC}}$.

Assume $q \in \mathbb{C}$ is a primitive p-th root of unity of order $p \geq 3$. Let $p' = p$ if p is odd, and $p' = p/2$ otherwise. If $p \not\equiv 2 \pmod 4$, we can define a finite quotient of the Heisenberg group $\mathscr{H}(\Sigma)$ as

$$\mathscr{H}_p(\Sigma) = \mathbb{Z}_p \times H_1(\Sigma, \mathbb{Z}_{p'})$$

where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ through this paper. Given a Lagrangian submodule $L \subset H_1(\Sigma, \mathbb{Z})$, let $L_p = L \otimes \mathbb{Z}_{p'} \subset H_1(\Sigma, \mathbb{Z}_{p'})$, and $\widetilde{L}_p = \mathbb{Z}_p \times L_p \subset \mathscr{H}_p(\Sigma)$ be a maximal abelian subgroup. Denote by $\mathbb{C}_q$ a 1-dimensional representation of $\widetilde{L}_p$, where (k, x) acts by q^k. Then inducing from $\mathbb{C}_q$ we obtain

$$W_q(L) = \mathbb{C}[\mathscr{H}_p(\Sigma)] \otimes_{\mathbb{C}[\widetilde{L}_p]} \mathbb{C}_q$$

a p'^g-dimensional *Schrödinger* representation of $\mathscr{H}_p(\Sigma)$. Note that as $\mathbb{C}[\mathscr{H}_p(\Sigma)]$-module $W_q(L)$ is generated by $\mathbf{1} \in \mathbb{C}_q$. Given a cobordism in the category $3\mathrm{Cob}^{\mathrm{LC}}$, $C : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$, $L_+ = L_C.L_-$, we have a Schrödinger representation $W(L_C)$ of the Heisenberg group $\mathscr{H}(\partial C)$ which can be considered as a $(\mathbb{C}[\mathscr{H}(\Sigma_+)], \mathbb{C}[\mathscr{H}(\Sigma_-)])$-bimodule, after identifying the subgroup $\mathscr{H}(-\Sigma_-) \subset \mathscr{H}(\partial C)$ with $\mathscr{H}(\Sigma_-)^{op}$, and defining a right action of $\mathscr{H}(\Sigma_-)$ on $W_q(L_C)$ as the left action of the same element of $\mathscr{H}(-\Sigma_-)$.

The main results of this paper can be formulated as follows.

Theorem 1. *Assume $p \not\equiv 2 \pmod 4$. For any cobordism C from (Σ_-, L_-) to (Σ_+, L_+) in $3\mathrm{Cob}^{\mathrm{LC}}$ there exists an isomorphism of $\mathbb{Z}[\mathscr{H}(\Sigma_+)]$-modules*

$$\psi_C : W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}_p(\Sigma_-)]} W_q(L_-) \xrightarrow{\sim} W_q(L_+)$$

sending $\mathbf{1} \otimes \mathbf{1}$ to $\mathbf{1}$.

By composing the map $W_q(L_-) \to W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}_p(\Sigma_-)]} W_q(L_-)$ with the isomorphism ψ_C we associate to a cobordism C a map between the Shrödinger representations of the input and output surfaces. We are able to normalise this map so that it is functorial, producing a functor isomorphic to the abelian TQFT. The normalising coefficient, denoted by $Z(\check{C})$, is actually the Murakami–Ohtsuki–Okada invariant of a closed 3-manifold $\check{C}$ obtained from C by gluing of two standard handlebodies $(H_\pm, L_\pm)$ with $\partial H_\pm = \Sigma_\pm$ and $L_\pm$ generated by meridians, along diffeomorphisms identifying the Lagrangians.

Theorem 2. *The map between Schrödinger local systems induced by $C : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$*

$$\begin{aligned} F_C : W_q(L_-) &\to W_q(L_+) \\ w &\mapsto Z(\check{C})\, \psi_C(\mathbf{1} \otimes w) \in W_q(L_+) \end{aligned}$$

extends to a functor $F : 3\mathrm{Cob}^{\mathrm{LC}} \to \mathrm{Vect}_{\mathbb{C}}$ which is equivalent to the abelian TQFT at q on $3\mathrm{Cob}^{\mathrm{LC}}$.

Observe that the normalization coefficient $Z(\check{C}) = 0$ if and only if there exists $\alpha \in H^1(\check{C}, \mathbb{Z}_{p'})$ with non vanishing triple product $\alpha \cup \alpha \cup \alpha$ [29, Thm. 3.2], however the Schrödinger action is always non trivial.

We plan to use these results to construct an action of cobordisms on Schrödinger homology and provide a homological interpretation of the Kerler–Lyubashenko TQFTs. Our long term goal will be to use infinite dimensional Schrödinger representations to construct TQFTs with generic quantum parameter q, rather than at a root of unity. An existence of such TQFTs was predicted by physicists. They are expected to play a crucial role in the categorification of

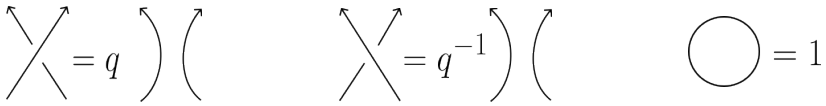

FIGURE 1. $U(1)$ skein relations.

quantum 3-manifold invariants [22]. Lagrangian Floer homology may serve as an inspiration for this purpose.

The paper is organized as follows. In Section 2 we review representation theoretical and skein constructions of abelian TQFTs, we discuss modularization functors, refinements as well as the action of the mapping class group and its extensions. In Section 3 we define Schrödinger local systems and compare two different mapping class group actions on them. In Section 4 we prove the two main theorems.

2. Abelian TQFTs

2.1. **Algebraic approach.** Let $q \in \mathbb{S}^1 \subset \mathbb{C}$ be a primitive p-th root of 1 and $p \geq 3$ is an integer. Let $p' = p$ if p is odd and $p' = p/2$ if p even. Consider the group algebra $H = \mathbb{C}[K]/(K^p - 1)$ of the cyclic group. This algebra can be identified with the Cartan part of the quantum $\mathfrak{sl}_2$ at q by extending the group monomorphism

$$U(1) \to SL(2, \mathbb{C})$$
$$z \to \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$$

For this reason, abelian TQFTs are also called $U(1)$ TQFTs.

The algebra H has a natural Hopf algebra structure with a grouplike generator, i.e. $\Delta(K) = K \otimes K$, $S(K) = K^{-1}$. Moreover, H is a ribbon Hopf algebra with R-matrix and its inverse given by

$$R = \frac{1}{p} \sum_{0 \leq i,j \leq p-1} q^{-ij} K^i \otimes K^j, \qquad R^{-1} = \frac{1}{p} \sum_{0 \leq i,j \leq p-1} q^{ij} K^{-i} \otimes K^{-j},$$

the ribbon elements

$$v = \frac{1}{p} \sum_{0 \leq i,j \leq p-1} q^{i(j-i)} K^j, \qquad v^{-1} = \frac{1}{p} \sum_{0 \leq i,j \leq p-1} q^{i(i-j)} K^{-j}$$

and the trivial pivotal structure.

Similarly to the $U_q(\mathfrak{sl}_2)$ case, the representation category $H-$mod has p simple modules V_k for $0 \leq k \leq p-1$. However, here V_k is the 1-dimensional representation determined by its character $K \mapsto q^k$. Also in our case, the fusion rules are very simple: $V_i \otimes V_j = V_{i+j}$ where the index $i+j$ is taken module p. Hence, all objects V_j are *invertible*, meaning that for each j there exists $k = p - j$ such that $V_j \otimes V_k = V_0$, where V_0 is the tensor unit of $H-$mod. The R-matrix is acting by q^{kl} on $V_k \otimes V_l$.

2.2. **Skein approach.** For explicit computations, it is more convenient to work with a skein theoretic construction.

Consider the skein relations depicted in Figure 1. Given a 3-manifold M, a skein module $S(M)$ is a $\mathbb{C}$-vector space generated by links in M modulo the skein relations. For a surface F it is custom to denote by $S(F)$ the skein $S(F \times [0,1])$. We will usually identify a coloring

of a component K of a framed link with an element of the skein $S(A)$, where the annulus A is embedded along K by using the framing. For example, V_j-coloring is represented by an element y^j where y is the core of A. Here we use the usual algebra structure on $S(A)$ to identify y^j with j parallel copies of y. The Kirby color is

$$\Omega = \sum_{j=0}^{p-1} y^j \in S(A).$$

The Ω-colored $(+1)$-framed unknot gets value

$$G = \sum_{k=0}^{p-1} q^{k^2} = \begin{cases} \varepsilon\sqrt{p}\,(1+\sqrt{-1}) & \text{if } p \equiv 0 \pmod 4 \\ \pm\sqrt{p} & \text{if } p \equiv 1 \pmod 4, \\ 0 & \text{if } p \equiv 2 \pmod 4 \\ \pm\sqrt{-p} & \text{if } p \equiv 3 \pmod 4 \end{cases} \tag{1}$$

by the well-known Gauss formula, where ε is a 4-th root of 1. For p odd, we write $|G| = \eta^{-1}$, $\kappa = \eta G$. For $p = 0 \pmod 4$, we explain in the next section, why the sums for G and Ω should be taken till $p' - 1$, and we denote them by

$$g = \sum_{k=0}^{p'-1} q^{k^2} \quad \text{and} \quad \omega = \sum_{j=0}^{p'-1} y^j. \tag{2}$$

Using that in our case $q^{k^2} = q^{(k+p')^2}$, we obtain $|g| = \eta^{-1} = \frac{|G|}{2} = \sqrt{p'}$. Hence, for all p except $p' = 2 \pmod 4$, we can define the invariant of a closed 3-manifold M obtained by surgery on S^3 along a framed n component link L as follows

$$Z(M) = \kappa^{-\mathrm{sign}(L)} \eta \langle \eta\omega, \dots, \eta\omega \rangle_L$$

where $\mathrm{sign}(L)$ is the signature of the linking matrix and $\langle x_1, \dots, x_n \rangle_L$ denotes the evaluation of of L whose ith component is colored by x_i in the skein $S(\mathbb{R}^3)$.

The normalization is chosen in such a way that

$$Z(S^2 \times S^1) = \eta^2 \sum_{i=0}^{p'-1} \langle y^i \rangle = 1 \quad \text{and} \quad Z(S^3) = \eta = 1/\sqrt{p'}.$$

The right Dehn twist along a curve γ is represented by coloring the curve γ with

$$\eta\omega_- = \eta \sum_{j=0}^{p'-1} q^{-j^2} y^j.$$

If $p \equiv 0 \pmod 4$, then we split $\omega = \omega_0 + \omega_1$ into odd and even colors. Then we define an additional topological structure on M that determines a $\mathbb{Z}/2\mathbb{Z}$-grading on the components of L, and thus a $\mathbb{Z}/2\mathbb{Z}$-grading on their colorings. In particular, for $p' \equiv 4 \pmod 8$, we construct an invariant of the pair (M, s)

$$Z(M, s) = \kappa^{-\mathrm{sign}(L)} \eta \langle \eta\omega_{s_1}, \dots, \eta\omega_{s_n} \rangle_L$$

where $(s_1, \dots, s_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ satisfying

$$\sum_{j=1}^{n} L_{ij} s_j = L_{ii} \pmod 2$$

determines a characteristic sublink of L corresponding to the spin structure s on M. Analogously, for $p' \equiv 0 \pmod 8$ we construct invariants of a pair (M, h)

$$Z(M,h) = \kappa^{-\text{sign}(L)}\eta\langle \eta\omega_{h_1}, \dots, \eta\omega_{h_n}\rangle_L$$

where $(h_1, \dots, h_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ satisfying

$$\sum_{j=1}^{n} L_{ij}h_j = 0 \pmod 2$$

determines the first cohomology class $h \in H^1(M, \mathbb{Z}/2\mathbb{Z})$. In both cases, $Z(M)$ is the sum of the refined invariants over all choices of the additional structure.

2.3. **Modularization and refinements.** A $\mathbb{C}$-linear ribbon category with a finite number of dominating simple objects is called *premodular*. If in addition, the monodromy S-matrix is invertible, then the category is modular. In our case the S-matrix, whose (i, j) component is the invariant q^{2ij} of the (i, j)-colored Hopf link, is invertible only for odd p, and in this case $H-$mod is *modular*, providing an abelian TQFT by standard constructions [30] or [10].

We call a premodular category $\mathscr{C}$ *modularizable*, if there exists a braided monoidal essentially surjective functor from $\mathscr{C}$ to a modular category, sending the subcategory of transparent objects to the tensor unit. In [14, Prop. 4.2] Bruguières gave a simple criterion for a premodular category to be modularizable, see also [28]. In particular, such category cannot contain *transparent* objects with twist coefficient -1. Recall that an object is called transparent, if it has trivial braiding with any other object. Observe that the row in the S-matrix corresponding to the transparent object is colinear with the one for the tensor unit.

If p is even, $H-$mod is a premodular category. The object $V_{p'}$ is transparent and has twist coefficient $q^{p'^2}$, which is 1 if p' is even and -1 is p' is odd. Using results of [14], we deduce that in the case when $p \equiv 0 \pmod 4$, $H-$mod is modularizable. The resulting modular category has p' simple objects, that are all invertible. The new Kirby color is given in (2). Hence, we have $\eta = |g|^{-1} = (\sqrt{p'})^{-1}$ in all cases when invariant is defined.

Furthermore, if $p' \equiv 4 \pmod 8$, the object $V_{p'/2}$ has twist coefficient -1. From [4] we deduce that our category in this case is actually *spin modular*, hence providing an abelian spin TQFT for 3-cobordisms equipped with a spin structure. Analogously, if $p' \equiv 0 \pmod 8$, we can construct a refined TQFT that gives rise to invariants of 3-cobordisms equipped with first cohomology classes over $\mathbb{Z}/2\mathbb{Z}$. We refer to [4] for details about the construction of the refined invariants and their properties.

In the case $p \equiv 2 \pmod 4$, $H-$mod is not modularizable. The best we can do in this case to obtain 3-manifold invariants is to consider the degree 0 subcategory with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading given by the action of $K^{p'}$. The corresponding invariants will coincide with those obtained with the quantum parameter of odd order equal to p'.

To construct a map associated by an abelian TQFT with a 3-cobordism $C : \Sigma_- \to \Sigma_+$, we first need to choose parametrizations of surfaces $\Sigma_\pm$, i.e. diffeomorphisms $\phi_\pm : \Sigma_{g_\pm} \to \Sigma_\pm$ where Σ_g is the standard genus g surface. If $p \not\equiv 2 \pmod 4$, the TQFT vector space associated with Σ_g has dimension p'^g. A basis $\{y^{\mathbf{i}}, \mathbf{i} = (i_1, \dots, i_g), 0 \le i_j \le p' - 1\}$ is given by p' colorings of g cores of the 1-handles of a bounding handlebody H_g. The $(\mathbf{i}, \mathbf{j})$-matrix element of the TQFT map is constructed as follows: We glue the standard handlebodies $H_{g_\pm}$ to C along the parametrizations. Inside H_- we put the link $y^{\mathbf{j}}$ and inside H_+ the link $y^{\mathbf{i}}$. The result is a closed 3-manifold

$\check{M} = S^3(L)$ with a collection of cirles $c^+ \cup c^-$ inside, then

$$Z(C)^{\mathbf{i}}_{\mathbf{j}} := \kappa^{-\mathrm{sign}(L)}\eta^{g_+}\langle \eta\omega, \dots, \eta\omega, y^{\mathbf{i}}, y^{\mathbf{j}}\rangle_{L\cup c_+\cup c_-}.$$

By using the universal construction [10], this map can also be computed by gluing just one handlebody $(H_{g_-}, y^{\mathbf{j}})$ to C and by evaluating the result in the skein of $C \cup H_{g_-}$. The parametrization reduces in this approach to the choice of Lagrangian $L \in H_1(\Sigma, \mathbb{Z})$, which is equal to $\ker : H_1(\Sigma, \mathbb{Z}) \to H_1(H_g, \mathbb{Z})$, and its complement $L^\vee$. Since all curves representing elements of L are trivial in the skein of H_g, the basis curves $y^{\mathbf{j}}$ of the TQFT vector space are parametrized by a basis of $L^\vee$.

In this paper we will be particularly interested in the Crane–Yetter category 3Cob of connected 3-cobordisms between connected 1-punctured surfaces. In this category the monoidal product is given by the boundary connected sum rather than by the disjoint union, thus leading to a rich algebraic structure [13]. By the result of [6], for any finite unimodular ribbon category $\mathscr{C}$, there exits a TQFT functor $F : 3\mathrm{Cob}^\sigma \to \mathscr{C}$ defined by sending the 1-punctured torus to the end of $\mathscr{C}$. In our case, for odd p

$$\mathrm{end}(H\text{−mod}) = \oplus_{j=0}^{p-1} V_j.$$

Modularization creates an isomorphism $V_k \cong V_k \otimes V_{p'}$, hence for $p \equiv 0 \pmod 4$ we have

$$\mathrm{end}(H\text{−mod}) = \oplus_{j=0}^{p'-1} V_j.$$

In both cases, the vector space associated by F to a genus g surface with one boundary component has dimension p'^g.

For even p', refined TQFTs on $3\mathrm{Cob}^\sigma$ can be constructed along the lines of [5]. On the standard cobordism category this was done in [3, 11].

2.4. **Extended cobordisms and Lagrangian correspondence.** Let us recall that the skein or Reshetikhin–Turaev TQFT constructions give rise to projective representations of the mapping class group and the gluing formula has a so-called *framing anomaly* which can be resolved by using *extended cobordisms*. The later are given by a pair: a 3-cobordism between surfaces equipped with Lagrangian subspaces in the first homology group and a natural number. This approach leads to a representation of a certain central extension of the mapping class group.

If p is odd, then the framing anomaly κ, defined as the argument of the Gauss sum g in (1), is a 4-th root of 1. From [21, Remark 6.9] we can deduce that the corresponding TQFT contains a native representation of the mapping class group. This is because, the central generator of the extension acts by $\kappa^4 = 1$, hence the index 4 subgroup described in [21] is the trivial extension. Recall that the metaplectic group Mp_{2g} is the non trivial double cover of the symplectic group Sp_{2g}. The metaplectic mapping class group is the pull back of this double cover using the symplectic action. In the case $p \equiv 0 \pmod 4$ the framing anomaly κ is a primitive 8-th root of unity and the above argument shows that the TQFT contains a native representation of the metaplectic mapping class group.

To avoid anomaly issues in general, we will work with a subcategory $3\mathrm{Cob}^{\mathrm{LC}}$ of the category of connected *extended* 3-cobordisms between connected 1-punctured surfaces with Lagrangians. Objects of $3\mathrm{Cob}^{\mathrm{LC}}$ are pairs: a connected 1-punctured surface Σ and a Lagrangian subspace $L \subset H_1(\Sigma, \mathbb{Z})$. Recall that a Lagrangian is a maximal submodule with vanishing intersection pairing. A 3-cobordism $C : \Sigma_- \to \Sigma_+$ defines a *Lagrangian correspondence*

$$L_C = \mathrm{Ker}\,(i_* : H_1(\partial C, \mathbb{Z}) = H_1(-\Sigma_-, \mathbb{Z}) \oplus H_1(\Sigma_+, \mathbb{Z}) \to H_1(C, \mathbb{Z}))\,.$$

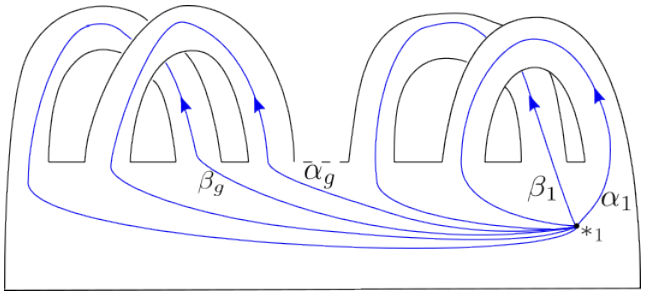

FIGURE 2. Model for Σ.

This Lagrangian correspondence gives an action of C on the Lagrangian subspace $L_- \subset H_1(\Sigma_-, \mathbb{Z})$ by

$$C.L_- = \{y \in H_1(\Sigma_+) \,|\, \exists\, x \in L_-,\ (x, y) \in L_C\}.$$

The cobordism C belongs to $3\mathrm{Cob}^{\mathrm{LC}}((\Sigma_-, L_-), (\Sigma_+, L_+))$ if and only if $L_C.L_- = L_+$. If we restrict to mapping cylinders we obtain the so called *action groupoid* of the mapping class group action on Lagrangian subspaces.

Restriction of the TQFT functor to $3\mathrm{Cob}^{\mathrm{LC}}$ kills all Maslov indices needed to compute framing anomalies in gluing formulas (compare [21, Sec.2].

3. Schrödinger local systems on surface configurations

3.1. **Heisenberg group as a quotient of the surface braid group.** Let Σ be an oriented surface of genus g with one boundary component. For $n \geq 2$, the unordered configuration space of n points in Σ is

$$\mathrm{Conf}_n(\Sigma) = \{\{c_1, \ldots, c_n\} \subset \Sigma \mid c_i \neq c_j \text{ for } i \neq j\}.$$

The surface braid group is then defined as $B_n(\Sigma) = \pi_1(\mathrm{Conf}_n(\Sigma), *)$. To construct a presentation, we fix based loops, $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ on Σ, as depicted in Figure 2. The base point $*_1$ on Σ belongs to the base configuration $*$ in $\mathrm{Conf}_n(\Sigma)$. By abuse of notation, we use α_r, β_s also for the loops in $\mathrm{Conf}_n(\Sigma)$ where only the first point is moving along the corresponding curve. We write composition of loops from right to left. The braid group $B_n(\Sigma)$ has generators $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ together with the classical braid generators $\sigma_1, \ldots, \sigma_{n-1}$, and relations:

$$\begin{cases} [\sigma_i, \sigma_j] = 1 & \text{for } |i-j| \geq 2, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1, \\ [\zeta, \sigma_i] = 1 & \text{for } i > 1 \text{ and all } \zeta \text{ among the } \alpha_r, \beta_s, \\ [\zeta, \sigma_1\zeta\sigma_1] = 1 & \text{for all } \zeta \text{ among the } \alpha_r, \beta_s, \\ [\zeta, \sigma_1^{-1}\eta\sigma_1] = 1 & \text{for all } \zeta \neq \eta \text{ among the } \alpha_r, \beta_s, \text{ with} \\ & \{\zeta, \eta\} \neq \{\alpha_r, \beta_r\}, \\ \sigma_1\beta_r\sigma_1\alpha_r\sigma_1 = \alpha_r\sigma_1\beta_r & \text{for all } r. \end{cases} \tag{3}$$

We denote by $x.y$ the standard intersection form on $H_1(\Sigma, \mathbb{Z})$. The Heisenberg group $\mathscr{H}(\Sigma)$ is the central extension of the homology group $H_1(\Sigma, \mathbb{Z})$ induced by the 2-cocycle $(x, y) \mapsto x.y$. As a set $\mathscr{H}(\Sigma)$ is equal to $\mathbb{Z} \times H_1(\Sigma, \mathbb{Z})$, with the group structure

$$(k, x)(l, y) = (k + l + x.y, x + y). \tag{4}$$

We use the notation a_r, b_s for the homology classes of α_r, β_s, respectively. Let us denote by $[\sigma_1, B_n(\Sigma)]$ the normal subgroup of the surface braid group $B_n(\Sigma)$ generated by the commutators $\{[\sigma_1, x],\ x \in B_n(\Sigma)\}$. From the presentation above we obtain the following (see [12] for more details).

Proposition 3. *For each $g \geq 0$ and $n \geq 2$, the quotient*

$$B_n(\Sigma)/[\sigma_1, B_n(\Sigma)] \xrightarrow{\sim} \mathscr{H}(\Sigma)$$

is isomorphic to the Heisenberg group. An isomorphism is induced by the surjective homomorphism

$$\phi\colon B_n(\Sigma) \longrightarrow \mathscr{H}(\Sigma)$$

sending each σ_i to $u = (1, 0)$, α_r to $\tilde{a}_r = (0, a_r)$, β_s to $\tilde{b}_s = (0, b_s)$.

It follows that any representation of the Heisenberg group $\mathscr{H}(\Sigma)$ is also a representation of the surface braid group $B_n(\Sigma) = \pi_1(\mathrm{Conf}_n(\Sigma), *)$ and hence provides a local system on the configuration space $\mathrm{Conf}_n(\Sigma)$.

Let us denote by $\mathrm{Aut}^+(\mathscr{H}(\Sigma))$ the group of automorphisms of $\mathscr{H}(\Sigma))$ acting by identity on the center. By [12, Lemma 15] we have the following split short exact sequence

$$1 \to H^1(\Sigma, \mathbb{Z}) \xrightarrow{j} \mathrm{Aut}^+(\mathscr{H}(\Sigma)) \xrightarrow{l} \mathrm{Sp}(H_1(\Sigma)) \to 1$$

where $j(c) = [(k, x) \to (k + c(x), x)]$ and $\mathrm{Sp}(H_1(\Sigma))$ is the symplectic group preserving the intersection pairing. The homomorphism l has a section

$$s : g \mapsto [(k, x) \mapsto (k, g(x))] \tag{5}$$

providing a semi-direct decomposition $\mathrm{Aut}^+(\mathscr{H}(\Sigma)) \cong \mathrm{Sp}(H_1(\Sigma)) \ltimes H^1(\Sigma; \mathbb{Z})$.

Let us denote by $\mathrm{Mod}(\Sigma)$ the mapping class group. Its action on $H_1(\Sigma, \mathbb{Z})$ preserves the symplectic form, and hence using the section s from (5) we get a *symplectic action* of the mapping class group on the Heisenberg group, where $f \in \mathrm{Mod}(\Sigma)$ acts by

$$(k, x) \mapsto (k, f_*(x)). \tag{6}$$

On the other hand, the quotient map $\phi : B_n(\Sigma) \to \mathscr{H}(\Sigma)$ induces a different action of $\mathrm{Mod}(\Sigma)$ on $\mathscr{H}(\Sigma)$. The following proposition is proved in [12, Section 3].

Proposition 4. *For $f \in \mathrm{Mod}(\Sigma)$, there exists a unique homomorphism $f_{\mathscr{H}}\colon \mathscr{H}(\Sigma) \to \mathscr{H}(\Sigma)$ such that the following square commutes:*

$$\begin{array}{ccc} B_n(\Sigma) & \xrightarrow{f_{B_n(\Sigma)}} & B_n(\Sigma) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \mathscr{H}(\Sigma) & \xrightarrow{f_{\mathscr{H}}} & \mathscr{H}(\Sigma) \end{array} \tag{7}$$

We obtain an action of $\mathrm{Mod}(\Sigma)$ on the Heisenberg group $\mathscr{H}(\Sigma)$ given by

$$\begin{gathered} \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Aut}^+(\mathscr{H}(\Sigma)) \\ f \mapsto f_{\mathscr{H}} : (k, x) \mapsto (k + \theta_f(x), f_*(x)) \end{gathered} \tag{8}$$

where the map $\theta : \mathrm{Mod}(\Sigma) \to H^1(\Sigma, \mathbb{Z})$ sending f to $\theta_f \in \mathrm{Hom}(H_1(\Sigma), \mathbb{Z})$ is called *crossed homomorphism*, satisfying $\theta(fg) = \theta(f) + f_*(\theta(g))$. Clearly, both actions coincide on $\mathrm{Sp}(H_1(\Sigma))$, i.e. $l(f_{\mathscr{H}}) = f_*$.

3.2. **Finite dimensional Schrödinger representations.** Let us fix an integer $p \geq 3$, $p \not\equiv 2 \bmod 4$, with $p' = p$ if p is odd and $p' = p/2$ if p even. Then we can define a finite quotient of the Heisenberg group $\mathscr{H}(\Sigma)$ as follows

$$\mathscr{H}_p(\Sigma) = \mathbb{Z}_p \times H_1(\Sigma, \mathbb{Z}_{p'}).$$

Given a Lagrangian submodule $L \subset H_1(\Sigma, \mathbb{Z})$, let $L_p = L \otimes \mathbb{Z}_{p'}$ and $\widetilde{L}_p = \mathbb{Z}_p \times L_p \subset \mathscr{H}_p(\Sigma)$ is a maximal abelian subgroup. Let q be a primitive p-th root of unity. Denote by $\mathbb{C}_q$ a 1-dimensional representation of $\mathscr{H}_p(\Sigma)$, where (k, x) acts by q^k. Then inducing from $\mathbb{C}_q$ we obtain

$$W_q(L) = \mathbb{C}[\mathscr{H}_p(\Sigma)] \otimes_{\mathbb{C}[\widetilde{L}_p]} \mathbb{C}_q$$

a p'^g-dimensional *Schrödinger representation* of the finite Heisenberg group $\mathscr{H}_p(\Sigma)$.

The following finite dimensional version of the famous Stone–von Neumann theorem holds.

Theorem 5 (Stone–von Neumann)**.** *For q a root of unity of order p, $p \geq 3$, $p \not\equiv 2$ mod 4, $W_q(L)$ is the unique irreducible unitary representation of $\mathscr{H}_p(\Sigma)$, up to unitary isomorphism, where the central generator $u = (1, 0)$ acts by q.*

A proof for even p can be found in [20, Theorem 2.4]. The odd case works similarly. The Schrödinger representation $W_q(L)$ can be twisted by an automorphim $\tau \in \mathrm{Aut}^+(\mathscr{H}(\Sigma))$. We denote by ${}_\tau W_q(L)$ this twisted representation where $h \in \mathscr{H}(\Sigma)$ acts by $\tau(h)$. The above Stone–von Neumann theorem provides an isomorphism $W_q(L) \cong {}_\tau W_q(L)$ defined up to a complex number of absolute value 1.

Using the Stone–von Neumann theorem, for a mapping class f we obtain a unitary isomorphism $\mathcal{S}_{\mathscr{H}}(f) : W_q(L) \xrightarrow{\sim} {}_{f_{\mathscr{H}}} W_q(L)$ defined, up to a scalar in $\mathbb{S}^1 \subset \mathbb{C}$, by the following commutative diagram

$$\begin{array}{ccc} W_q(L) & \xrightarrow{\mathcal{S}_{\mathscr{H}}(f)} & {}_{f_{\mathscr{H}}} W_q(L) \\ {\scriptstyle \rho_W(k,x)}\downarrow & & \downarrow{\scriptstyle \rho_W(f_{\mathscr{H}}(k,x))} \\ W_q(L) & \xrightarrow{\mathcal{S}_{\mathscr{H}}(f)} & {}_{f_{\mathscr{H}}} W_q(L) \end{array}$$

where $\rho_W : \mathscr{H}(\Sigma) \to U(W_q(L)$ is the Shrödinger representation. This provides an homomorphism

$$\mathcal{S}_{\mathscr{H}} : \mathrm{Mod}(\Sigma) \to \mathrm{PU}(W_q(L)), \quad \text{where} \quad \mathrm{PU}(W_q(L)) = \mathrm{U}(W_q(L))/\mathbb{S}^1$$

is the projective unitary group.

Denote by ${}_{f_*} W_q(L)$ the Shrödinger representation twisted with the symplectic action, we also have an isomorphism $\mathcal{S}(f) : W_q(L) \xrightarrow{\sim} {}_{f_*} W_q(L)$ defined, up to a scalar in $\mathbb{S}^1 \subset \mathbb{C}$, by the condition

$$\rho_W(k, f_*(x)) \circ \mathcal{S}(f) = \mathcal{S}(f) \circ \rho_W(k, x), \text{ for any } (k, x) \in \mathscr{H}_p(\Sigma). \tag{9}$$

This provides another homomorphism

$$\mathcal{S} : \mathrm{Mod}(\Sigma) \to \mathrm{PU}(W_q(L))$$

The following theorem was essentially proven by Gelca with collaborators [20, Theorem 8.1], [19], [17, Chapter 7].

Theorem 6. *The homomorphism $\mathcal{S} : \mathrm{Mod}(\Sigma) \to \mathrm{PU}(W_q(L))$ given by the symplectic action is isomorphic to the one resulting from the abelian TQFT described in Section 2.*

In general, any projective representation of a group G can be linearised on an appropriate central extension. Given an homomorphism $R : G \to PGL(V)$, where V is a complex vector space, a choice of lift (as a set map) $\tilde{R} : G \to GL(V)$ defines a defect map $c : G \times G \to \mathbb{C}^*$, by $\tilde{R}(gg') = c(g, g')\tilde{R}(g)\tilde{R}(g')$. In the case of a projective unitary representation the map c takes values in $\mathbb{S}^1$. It is well known from basic group cohomology theory that c is a 2-cocycle defining a central extension of G on which R can be linearised. This central extension is classified by the class $[c] \in H^2(G, \mathbb{C}^*)$. If this class can be reduced to a subgroup, then the linearisation already arises on a smaller extension. If $[c] = 0$, the minimal extension is G itself.

Projective actions of $\mathrm{Mod}(\Sigma)$ on Schrödinger representations are naturally equipped with such cohomology classes, determined by the Stone–von Neumann isomorphisms. We will show that the extension which linearises the projective representation $\mathcal{S}_{\mathscr{H}}$ is non trivial by computing its classifying class in odd case.

From now on in this section we suppose that p is odd. As explained in Section 2.4, the homomorphism $\mathcal{S}$ can be linearised and we use the same notation for a linearisation $\mathcal{S} : \mathrm{Mod}(\Sigma) \to U(W_q(L))$. A key observation is that, for a mapping class f, the automorphism $f_{\mathscr{H}} : \mathscr{H}(\Sigma) \to \mathscr{H}(\Sigma)$ is equal to the symplectic one composed with an inner automorphism

$$f_{\mathscr{H}}(k, x) = (k + \theta_f(x), f_*(x)) = (0, f_*(t_f))(k, f_*(x)(0, -f_*(t_f)),$$

where $2t_f \in H_1(\Sigma, \mathbb{Z}_p)$ is the Poincaré dual of θ_f. Here we use that 2 is invertible modulo p and that $f_*(t_f).f_*(x) = t_f.x$. Acting on $W_q(L)$ we get the following commutative diagram

$$\begin{array}{ccccc} W_q(L) & \xrightarrow{\mathcal{S}(f)} & {}_{f_*}W_q(L) & \xrightarrow{\rho_W(0,f_*(t_f))} & {}_{f_{\mathscr{H}}}W_q(L) \\ \downarrow{\scriptstyle \rho_W(k,x)} & & \downarrow{\scriptstyle \rho_W(k,f_*(x))} & & \downarrow{\scriptstyle \rho_W(f_{\mathscr{H}}(k,x))} \\ W_q(L) & \xrightarrow{\mathcal{S}(f)} & {}_{f_*}W_q(L) & \xrightarrow{\rho_W(0,f_*(t_f))} & {}_{f_{\mathscr{H}}}W_q(L) \end{array}$$

Hence the two projective actions are related as follows

$$\mathcal{S}_{\mathscr{H}}(f) = \rho_W(0, f_*(t_f)) \circ \mathcal{S}(f) = \mathcal{S}(f) \circ \rho_W(0, t_f) \ .$$

We can now compute the cocycle from the intertwinning isomorphism

$$W_q(L) \cong {}_{(fg)_{\mathscr{H}}}W_q(L) = {}_{g_{\mathscr{H}}}({}_{f_{\mathscr{H}}}W_q(L)) \ .$$

$$\begin{aligned} \mathcal{S}_{\mathscr{H}}(f) \circ \mathcal{S}_{\mathscr{H}}(g) &= \mathcal{S}(f) \circ \rho_W(0, t_f) \circ \mathcal{S}(g) \circ \rho_W(0, t_g) \\ &= \mathcal{S}(f) \circ \mathcal{S}(g) \circ \rho_W(0, g_*^{-1}(t_f)) \circ \rho_W(0, t_g) \\ &= \mathcal{S}(f) \circ \mathcal{S}(g) \circ \rho_W(g_*^{-1}(t_f).t_g, g_*^{-1}(t_f) + t_g) \\ &= q^{g_*^{-1}(t_f).t_g} \mathcal{S}_{\mathscr{H}}(fg) \end{aligned}$$

Here we used that the crossed homomorphism property $\theta_{fg} = \theta_g + g^*(\theta_f)$ implies for the Poincaré dual $t_{fg} = t_g + g_*^{-1}(t_f)$. Using $t_{gg^{-1}} = t_{g^{-1}} + g_*(t_g) = 0$, we get that the cocycle is equal to $q^{c(f,g)}$ where $c(f, g) = g_*^{-1}(t_f).t_g = t_f.g_*(t_g) = t_f.t_{g^{-1}}$.

Morita studied in [27] the intersection cocycle $(f, g) \mapsto c_{\mathrm{Mor}}(f, g) = t_{f^{-1}}.t_g = c(g, f)$ which represents $12c_1$ where c_1 is the Chern class generating $H^2(\mathrm{Mod}(\Sigma), \mathbb{Z}) = \mathbb{Z}$ for surfaces of genus at least 3. The Meyer cocycle $\tau(f, g)$ is the signature of the oriented 4-dimensional manifold defined as the surface bundle over the pair of pants with monodromy f and g on 2 boundary components. This definition is symmetric in f and g so that we have $\tau(f, g) = \tau(f, g)$. From Morita work we have that $[c_{\mathrm{Mor}}] = 3[\tau] = 12c_1$. By switching the variable we get $[c] = 3[\tau] = 12c_1$. It follows that

for odd p the projective action $\mathcal{S}_{\mathscr{H}} : \mathrm{Mod}(\Sigma) \to PU(W_q(L)$ cannot be linearised on the mapping class group while the symplectic action does.

4. Proofs

In the previous section we have shown that finite dimensional Schrödinger representations provide local systems on surface configuration spaces. Here we will show that morphisms of $3\mathrm{Cob}^{\mathrm{LC}}$ act naturally on these local systems by extending the symplectic action of mapping cylinders.

Let $C \in 3\mathrm{Cob}^{\mathrm{LC}}((\Sigma_-, L_-), (\Sigma_+, L_+))$ be a cobordism compatible with Lagrangian correspondence. This means that the kernel of the inclusion map $H_1(\partial C, \mathbb{Z}) \to H_1(C, \mathbb{Z})$ is a Lagrangian submodule $L_C \subset H_1(\partial C, \mathbb{Z}) \cong H_1(-\Sigma_-, \mathbb{Z}) \oplus H_1(\Sigma_+, \mathbb{Z})$ and $L_C.L_- = L_+$. Then we have three Heisenberg groups $\mathscr{H}(\Sigma_-)$, $\mathscr{H}(\Sigma_+)$ and $\mathscr{H}(\partial C)$, and respective Shrödinger representations $W_q(L_-)$, $W_q(L_+)$ and $W_q(L_C)$ for a p-th root of unity q, with $p \geq 3$, $p \not\equiv 2 \pmod 4$.

Using that $\partial C = -\Sigma_- \cup_{S^1} \Sigma_+$ and the inclusions $H_1(-\Sigma_-, \mathbb{Z}) \to H_1(\partial C, \mathbb{Z})$, $H_1(\Sigma_+, \mathbb{Z}) \to H_1(\partial C, \mathbb{Z})$ we have commuting actions of $\mathscr{H}(-\Sigma_-)$ and $\mathscr{H}(\Sigma_+)$ on $W_q(L_C)$. Actually, $W_q(L_C)$ can be viewed as a $(\mathbb{C}[\mathscr{H}(\Sigma_+)], \mathbb{C}[\mathscr{H}(\Sigma_-)])$-bimodule, after identifying the group $\mathscr{H}(-\Sigma_-)$ with $\mathscr{H}(\Sigma_-)^{op}$, and defining a right action of $\mathscr{H}(\Sigma_-)$ on $W_q(L_C)$ as the left action of the same element of $\mathscr{H}(-\Sigma_-)$. Then we can form the tensor product $W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}(\Sigma_-)]} W_q(L_-)$ and compare it with $W_q(L_+)$.

Recall that the Shrödinger representation $W_q(L)$ is induced from $\mathbb{C} = \mathbb{C}_q$ considered as a 1-dimensional representation of $\widetilde{L}_p = \mathbb{Z}_p \oplus (L \otimes \mathbb{Z}_{p'}) \subset \mathscr{H}_p(\Sigma)$, and we denote by $\mathbf{1}$ the canonical generator of $W_q(L)$ as $\mathbb{C}[\mathscr{H}(\Sigma)]$-module. Moreover, throughout this section to *simplify notation* we denote $L \otimes \mathbb{Z}_{p'}$ by L.

Any Lagrangian $L^\vee \subset H_1(\Sigma, \mathbb{Z})$ complementary to L provides a basis for $W_q(L)$ indexed by $\mathsf{L}^\vee$. Given $b \in \mathsf{L}^\vee$ we denote by v_b the corresponding basis vector. In this basis the left action of the finite Heisenberg group is as follows.

- The central generator $u = (1, 0)$ acts by $v_b \mapsto q v_b$.
- For $y \in \mathsf{L}^\vee$, $(0, y)$ acts by translation: $v_b \mapsto v_{b+y}$.
- For $x \in \mathsf{L}$, $(0, x)$ acts by $v_b \mapsto q^{2x.b}\, v_b$.

In the last step we used the rule $(0, x)(0, b) = (x.b, x + b) = (0, b)(2x.b, x)$.

Our main results provide a new model for the abelian TQFT based on Schrödinger local systems. Let us prove our main theorems.

Proof of Theorem 1. Any morphism in $3\mathrm{Cob}^{\mathrm{LC}}$ can be decomposed into mapping cylinders and elementary index 1 or 2 surgeries. This is a consequence of the existence of a Morse datum [16, 23]. We will first prove the isomorphism $W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}_p(\Sigma_-)]} W_q(L_-) \cong W_q(L_+)$ for the elementary cobordisms and then argue that the bimodule associated with a composition of cobordisms is the expected tensor product of bimodules. In addition, for every elementary cobordism we compute the induced map $\check{F}_C : W_q(L_-) \to W_q(L_+)$, that will be used to prove Theorem 2. The reader interested in Theorem 1 only may skip this part.

For a mapping cylinder $C_f : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$, where the diffeomorphism $f : \Sigma_- \to \Sigma_+$ sends L_- to $f_*(L_-) = L_+$, we have

$$L_{C_f} = \{(-x, f_*(x)),\ x \in H_1(\Sigma_-, \mathbb{Z})\}$$

We choose a Lagrangian $L_-^\vee$ complementary to L_-. Then $L_+^\vee = f_*(L_-^\vee)$ is complementary to L_+. The submodule

$$L_{C_f}^\vee = L_- \oplus L_+^\vee \subset H_1(\partial C_f, \mathbb{Z})$$

is Lagrangian and complementary to L_{C_f}. Indeed, if $(-x, f_*(x))$ belongs to $L_{C_f}^\vee$, then $x \in L_-$ and $f_*(x) \in L_+^\vee \cap L_+ = \{0\}$, showing that $L_{C_f}^\vee \cap L_{C_f} = \{0\}$. Recall that for all kinds of Lagrangians L, the notation L means $L \otimes \mathbb{Z}_{p'}$. A $\mathbb{C}$-basis b_y^+ for $W_q(L_+)$ is labelled by elements $y \in \mathsf{L}_+^\vee$.

We have bases $\{B_z, z \in \mathsf{L}_{C_f}^\vee\}$ for $W_q(L_{C_f})$, and $\{b_x^-, x \in \mathsf{L}_-^\vee\}$ for $W_q(L_-)$. As a vector space the tensor product is generated by

$$\{B_z \otimes b_x, z \in \mathsf{L}_{C_f}^\vee, x \in \mathsf{L}_-^\vee\}$$

with relations coming from the action by elements in $\mathscr{H}_p(\Sigma_-)$. We write $z \in \mathsf{L}_{C_f}^\vee$ as $z = (z_-, z_+)$, $z_- \in \mathsf{L}_-$, $z^+ \in \mathsf{L}_+^\vee = f_*(\mathsf{L}_-^\vee)$.

For an element $y \in \mathsf{L}_-$ we get the relation

$$q^{2y.x} B_{(z_-,z_+)} \otimes b_x^- = B_{(z_-,z_+)}(0,y) \otimes b_x^- = (0,(y,0)) B_{(z_-,z_+)} \otimes b_x^- = B_{(z_-+y,z_+)} \otimes b_x^-$$

This reduces the set of generators to $\{B_{(0,z_+)} \otimes b_x^-, x \in \mathsf{L}_-^\vee, z_+ \in \mathsf{L}_+^\vee\}$.

For an element $x \in \mathsf{L}_-^\vee$ we get another relation

$$\begin{aligned} B_{(0,z_+)} \otimes b_x^- =& B_{(0,z_+)}(0,x) \otimes \mathbf{1} = (0,(x,0)) B_{(0,z_+)} \otimes \mathbf{1} \\ =& (0,(0,f_*(x))(0,(x,-f_*(x)) B_{(0,z_+)} \otimes \mathbf{1} = q^{-2f_*(x).z_+} B_{(0,z_+ + f_*(x))} \otimes \mathbf{1} \end{aligned} \tag{10}$$

where the intersection is written on the positively oriented Σ_+. This further reduces the generators to $\{B_{(0,z_+)} \otimes \mathbf{1}, z_+ \in \mathsf{L}_+^\vee\}$. Since any relation coming from any element in $\mathscr{H}(\Sigma_-)$ can be deduced from the previously written ones, we get that $\{B_{(0,y)} \otimes \mathbf{1}, y \in \mathsf{L}_+^\vee\}$ represents a $\mathbb{C}$-basis for the tensor product $W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}_p(\Sigma_-)]} W_q(L_-)$. It follows that the $\mathbb{C}[\mathscr{H}(\Sigma_+)]$-module map

$$\psi_C : W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}_p(\Sigma_-)]} W_q(L_-) \to W_q(L_+)$$

which sends $\mathbf{1} \otimes \mathbf{1}$ to $\mathbf{1}$ is an isomorphism. Moreover, the map

$$\begin{aligned} \check{F}_C : W_q(L_-) &\to W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}_p(\Sigma_-)]} W_q(L_-) \cong W_q(L_+) \\ b_x^- &\mapsto \psi_C(\mathbf{1} \otimes b_x^-) \end{aligned}$$

sends a basis vector b_x^- to $b_{f_*(x)}^+$, for any $x \in \mathsf{L}_-^\vee$, by using (10) with $z_+ = 0$.

In the case of an elementary cobordism $C : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$ corresponding to an index 1 surgery, the genus increases by 1. The Lagrangian correspondence is

$$L_C = \{(-x_-, x_+),\ x_- \in H_1(\Sigma_-, \mathbb{Z})\} \oplus \mathbb{Z}(0, \mu)$$

where μ is a meridian of the new handle and x_+ is the class x_- pushed in Σ_+. Let λ be a longitude for the new handle. We choose a Lagrangian $L_-^\vee$ complementary to L_-. By pushing through the cobordism, we may also consider $L_-^\vee$ as a subspace in $H_1(\Sigma_+, \mathbb{Z})$. The span of $L_-^\vee$ and λ gives a Lagrangian $L_+^\vee$ complementary to L_+. Then $L_C^\vee = L_-^\vee \oplus L_+^\vee$ is complementary to L_C and the previous argument constructs the isomorphism. Here the map

$$\check{F}_C : W_q(L_-) \to W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}_p(\Sigma_-)]} W_q(L_-) \cong W_q(L_+)$$

sends a basis vector b_x^- to b_x^+, where $x \in \mathsf{L}_-^\vee = L_-^\vee \otimes \mathbb{Z}_{p'}$.

Let us consider an elementary cobordism $C : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$ corresponding to an index 2 surgery on a curve γ. Let δ be a curve in Σ_- such that $\gamma.\delta = 1$. The curves γ and δ determine a genus one subsurface Σ_1. Outside Σ_1 the cobordism is trivial. Denote by $\Sigma \subset \Sigma_-$ the complement of Σ_1 which we consider also as a subsurface of Σ_+. We arrange the splitting so that $\Sigma_- = \Sigma \natural \Sigma_1$

is a boundary connected sum. Then all Lagrangian subspaces and Schrödinger modules split. Over Σ the cobordism is trivial and the expected result is clear, so that it is enough to compute in the genus 1 case, $\Sigma_- = \Sigma_1$ and $\Sigma_+ = D^2$. The Lagrangian L_- is generated by a simple curve m. A complementary Lagrangian $L_-^\vee$ is generated by l with $m.l = 1$. We have $\gamma = \alpha m + \beta l$, $\gcd(\alpha, \beta) = 1$. The Lagrangian correspondence is

$$L_C = \mathbb{Z}(\gamma, 0) \quad \text{with complement} \quad L_C^\vee = \mathbb{Z}(\delta, 0)$$

where $\delta = um + vl$, $\alpha v - \beta u = 1$. Then $m = v\gamma - \beta\delta$, $l = -u\gamma + \alpha\delta$. We have bases $B_{k\delta}$ and $b_{\nu l}$, $0 \le k, \nu < p'$ for $W(L_C)$ and $W(L_-)$, respectively. Using l we get the relation

$$B_{k\delta} \otimes b_{(\nu+1)l} = B_{k\delta}(0, -u\gamma + \alpha\delta) \otimes b_{\nu l} = (0, \alpha\delta)(u\alpha, -u\gamma) B_{k\delta} \otimes b_{\nu l} = q^{u\alpha + 2ku} B_{(\alpha+k)\delta} \otimes b_{\nu l}$$

where we used intersection on $-\Sigma_-$. This reduces the set of generators to $B_{k\delta} \otimes \mathbf{1}$, $0 \le k < p'$. The relation coming from m then gives

$$B_{k\delta} \otimes \mathbf{1} = B_{k\delta}(0, v\gamma - \beta\delta) \otimes \mathbf{1} = (0, -\beta\delta)(\beta v, v\gamma) B_{k\delta} \otimes \mathbf{1} = q^{\beta v - 2kv} B_{(k-\beta)\delta} \otimes \mathbf{1}. \tag{11}$$

If the surgery curve γ is in L_-, we can choose $m = \gamma$, $l = \delta$. The last relation gives $B_{k\delta} \otimes \mathbf{1} = q^{-2k} B_{k\delta} \otimes \mathbf{1}$. Hence we have $B_{k\delta} \otimes \mathbf{1} = 0$ for $0 < k < p'$ and the tensor product is $\mathbb{C}$-generated by $\mathbf{1} \otimes \mathbf{1}$. The equalities

$$B_{k\delta} \otimes \mathbf{1} = \begin{cases} 1 & \text{if} \quad k \equiv 0 \pmod{p'} \\ 0 & \text{else} \end{cases}$$

define an isomorphism $W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}_p(\Sigma_-)]} W_q(L_-) \cong \mathbb{C}_q$. In particular,

$$\check{F}_C(b_{kl}) = \begin{cases} 1 & \text{if} \quad k \equiv 0 \pmod{p'} \\ 0 & \text{else} \end{cases}$$

If the surgery curve γ is not in L_- then $\beta \neq 0$. Let $d = \gcd(\beta, p')$, then the order of β modulo p' is $a = \frac{p'}{d}$. Hence, relation (11) reduces the generators to $\{B_{k\delta} \otimes \mathbf{1}, 0 \le k < d\}$. Finally, the action of $(0, am)_{-\Sigma_-}$ gives the following relation

$$B_{k\delta} \otimes \mathbf{1} = (0, av\gamma - a\beta\delta) B_{k\delta} \otimes \mathbf{1} = (0, -a\beta\delta)(a^2\beta v, av\gamma) B_{k\delta} \otimes \mathbf{1} = q^{-2kav} B_{k\delta} \otimes \mathbf{1}$$

since $q^{a^2\beta} = 1$ and the intersection pairing is taken on $-\Sigma_-$. It follows $B_{k\delta} \otimes \mathbf{1} = 0$ unless k is divisible by d, hence $\mathbf{1} \otimes \mathbf{1}$ generates $W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}_p(\Sigma_-)]} W_q(L_-) \cong \mathbb{C}_q$, as expected.

We are left with computing $\check{F}_C$. The action of $(0, a\gamma) = (0, a\alpha m + a\beta l)$ gives

$$\mathbf{1} \otimes b_{kl} = \mathbf{1} \otimes (0, a\beta l)(a^2\alpha\beta, a\alpha m) b_{kl} = q^{-2ka\alpha} \mathbf{1} \otimes b_{kl}$$

implying that $\mathbf{1} \otimes b_{kl} = 0$ if $d \nmid k$. If $d \mid k$ we set $k\alpha = k'\beta$ and compute

$$\mathbf{1} \otimes b_{kl} = \mathbf{1}(0, -ku\gamma + k\alpha\delta) \otimes \mathbf{1} = (0, k\alpha\delta)(k^2 u\alpha, -ku\gamma)\mathbf{1} \otimes \mathbf{1} = q^{k^2 u\alpha} B_{k\alpha\delta} \otimes \mathbf{1} = q^{kk'\beta u} B_{k'\beta\delta} \otimes \mathbf{1}.$$

From the action of $(0, k'm)$ we get

$$\begin{aligned} \mathbf{1} \otimes b_{kl} =& q^{kk'\beta u}(0, k'v\gamma - k'\beta\delta) B_{k'\beta\delta} \otimes \mathbf{1} = q^{kk'\beta u}(-(k')^2 v\beta, k'v\gamma)(0, -k'\beta\delta) B_{k'\beta\delta} \otimes \mathbf{1} \\ =& q^{kk'\beta u - kk'\alpha v} \mathbf{1} \otimes \mathbf{1} = q^{-kk'} \mathbf{1} \otimes \mathbf{1} = q^{-\alpha k^2/\beta} \mathbf{1} \otimes \mathbf{1}. \end{aligned}$$

We deduce that

$$\check{F}_C(b_{kl}) = \begin{cases} 0 & \text{if } d \nmid k \\ q^{-\alpha k^2/\beta} & \text{else.} \end{cases}$$

In order to complete the proof, we need to consider a composition of elementary cobordisms. A modification of the previous proof shows that for a cobordism $C' : (\Sigma_-, L_-) \to (\Sigma, L)$ and

an elementary cobordism $C : (\Sigma, L) \to (\Sigma_+, L_+)$, both in $3\mathrm{Cob}^{\mathrm{LC}}$, we have an isomorphism of $(\mathbb{Z}[\mathscr{H}(\Sigma_+), \mathbb{Z}[\mathscr{H}(\Sigma_-])$-bimodules

$$\psi_C^{C'} : W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}_p(\Sigma)]} W_q(L_{C'}) \xrightarrow{\sim} W_q(L_{C' \cup_\Sigma C})$$

sending $\mathbf{1} \otimes \mathbf{1}$ to $\mathbf{1}$. The key point is that $W_q(L_{C'})$ can be written as a direct sum of copies of the Schrödinger representation of $\mathscr{H}(\Sigma)$. The general case of the theorem is then obtained by induction on the length of the decomposition, where the inductive step follows from the three cases we computed above. □

It remains to show that the above construction recovers the abelian TQFT.

Proof of Theorem 2. We will use the skein model from Section 2. Following [20, Theorem 4.5] the Heisenberg group algebra $\mathbb{C}[\mathscr{H}(\Sigma)]$ can be identified with the $U(1)$-skein algebra $S(\Sigma)$. This makes the TQFT vector space $V(\Sigma, L)$ to a module over $\mathbb{C}[\mathscr{H}(\Sigma)]$. Actually, it is isomorphic to the Schrödinger representation, see [20, Theorem 4.7] for even p.

A priori, the Stone-von Neumann theorem provides the isomorphism up to a complex number in $\mathbb{S}^1 \subset \mathbb{C}$. Here we prefer to construct the isomorphism explicitly. Let us denote by $S_p(\Sigma)$ the reduced $U(1)$ skein module, where q is specified to the pth root of unity. Then $S_p(\Sigma)$ is identified with $\mathbb{C}[\mathscr{H}_p(\Sigma)]$ by sending a simple closed curve γ with blackboard framing to

$$(0, [\gamma]) \in \mathbb{Z}_p \times H_1(\Sigma, \mathbb{Z}_{p'}) = \mathscr{H}_p(\Sigma)$$

Let H be a handlebody with boundary Σ such that L is the kernel of the inclusion $H_1(\Sigma, \mathbb{Z}) \hookrightarrow H_1(H, \mathbb{Z})$. Then the TQFT vector space $V(\Sigma, L)$ is the quotient of $S_p(\Sigma)$ by the subspace generated by $\gamma - 1$ where γ is a simple curve that bounds in H or equivalently such that $[\gamma] \in \mathsf{L}$. Using the isomorphism $S(\Sigma) \cong \mathbb{C}[\mathscr{H}(\Sigma)]$, we deduce that the quotients $V(\Sigma, L)$ and $W_q(L)$ are isomorphic.

A basis $\{b_x, x \in \mathsf{L}^\vee\}$ for $W_q(L)$ can be represented by skein elements $\{y_x, x \in \mathsf{L}^\vee\}$ in H providing a basis for $V(\Sigma, L)$. Here for an embedded curve x in Σ, the element y_x is obtained by pushing x in H with blackboard framing and then by taking its skein class. For example, the element y_{3x} correspond to the three parallel copies of y_x obtained by using the blackboard framing. We are now able to compare $F_C = Z(\check{C})\check{F}_C$ with the TQFT map on elementary cobordisms.

Let us consider a mapping cylinder $C_f : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$ with $g_- = g_+ = g$. A basis for the TQFT vector space identified with the Shrödinger representation $W_q(L_-)$ is represented by a handlebody H_-, with $\partial H_- = \Sigma_-$ and with the cores l_i, $1 \leq i \leq g$, of its handles colored by y^k, $0 \leq k \leq p'$. The TQFT map is represented by gluing the mapping cylinder C_f to the handlebody H_-. This results in a handlebody H_+ with boundary Σ_+. Moreover when pushing the colored curve l across the cylinder we get a curve parallel to $f(l)$. Hence, the TQFT map sends y^k in H_- to $f(y^k)$ in H_+, matching F_{C_f}. Note that $\check{C}_f$ is a connected sum of g copies of $S^2 \times S^1$, since f preserves the Lagrangians. Hence, in our normalization $Z(\check{C}_f) = 1$.

In the case of an index 1 surgery $C : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$, the TQFT map is represented by the inclusion of a handlebody $H_- \hookrightarrow H_+ = H_- \cup_{\Sigma_-} C$, where $\partial H_- = \Sigma_-$ and

$$\ker\left(H_1(\Sigma, \mathbb{Z}) \hookrightarrow H_1(H_-, \mathbb{Z})\right) = L_-.$$

This inclusion map matches again F_C with $Z(\check{C}) = 1$.

In the case of an index 2 surgery on a curve γ, we only need to consider the case where Σ_- is a genus 1 surface. Then the TQFT map $Z(C) : V(\Sigma_-, L_-) \to \mathbb{C}_q$ is given by the evaluation of the skein element (H_-, x) inside $M_\gamma = (H_- \cup_{\Sigma_-} C) \cup_{S^2} D^3$. If $\gamma = m$, $M_\gamma = S^1 \times S^2$ and the

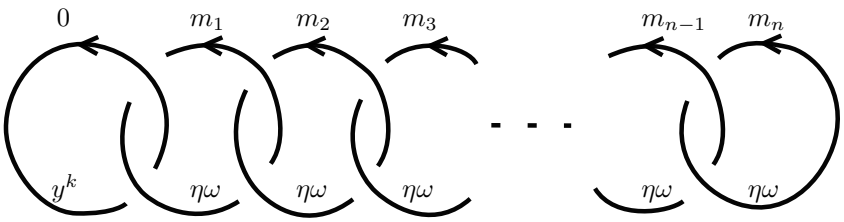

FIGURE 3. Surgery link for the lens space where the upper indices correspond to the framings and the lower ones to the colors.

evaluation reduces to a Hopf link with one Kirby-colored component, which is zero unless $x = 0$, when it is 1.

If $\gamma = l$, $M_\gamma = S^3$ and the evaluation is 1 for all $x = y^k$. Hence, in both cases we recover F_C.

More generally, for $\gamma = \alpha m + \beta l$ with $\beta \neq 0$, the manifold M_γ is the lens space $L(\beta, \alpha)$. Let us choose a continued fraction decomposition $\beta/\alpha = [m_1, \dots, m_n]$ as in [26]. Then a surgery link L for M_γ is the length n Hopf chain with framings m_i. Hence, the TQFT map sends y^k to the following number

$$Z(C)_k = \kappa^{-\mathrm{sign}(L)}\, \eta^n \sum_{j_1,\dots,j_n=1}^{p'} q^{\sum_{i=1}^n m_i j_i^2}\, q^{2kj_1}\, q^{2\sum_{i=1}^{n-1} j_i j_{i+1}}$$

Since a recursive computation of this sum was done in [26], we present here just the result.

$$Z(C)_k = \begin{cases} 0 & d \nmid k \\ q^{-\frac{\alpha k^2}{\beta}} Z(L(\beta,\alpha)) & \text{else} \end{cases}$$

where $d = \gcd(\beta, p')$. This coincides with F_C on this cobordism with $Z(\check{C}) = Z(L(\beta,\alpha))$. Since the TQFT map for any cobordism is a composition of maps for elementary ones and the same works for F_C, for any cobordism C we have that the TQFT map

$$Z(C) : V(\Sigma_-) \cong W_q(L_-) \to V(\Sigma_+) \cong W_q(L_+)$$

is equal, up to a coefficient, to the inclusion map $W_q(L_-) \to W_q(L_C) \otimes_{\mathbb{C}[\mathscr{H}_p(\Sigma)]} W_q(L_-)$ composed with the isomorphism from Theorem 1. Closing with handlebodies compatible with the Lagrangians we get that the coefficient is $Z(\check{C})$ which completes the proof. □

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