CYCLOTOMIC EXPANSIONS FOR $\mathfrak{gl}_N$ LINK INVARIANTS VIA INTERPOLATION MACDONALD POLYNOMIALS

ANNA BELIAKOVA AND EUGENE GORSKY

Abstract. In this paper we construct a new basis for the cyclotomic completion of the center of the quantum $\mathfrak{gl}_N$ in terms of the interpolation Macdonald polynomials. Then we use a result of Okounkov to provide a dual basis with respect to the quantum Killing form (or Hopf pairing). The main applications are: 1) cyclotomic expansions for the $\mathfrak{gl}_N$ Reshetikhin–Turaev link invariants and the universal $\mathfrak{gl}_N$ knot invariant; 2) an explicit construction of the unified $\mathfrak{gl}_N$ invariants for integral homology 3-spheres using universal Kirby colors. These results generalize those of Habiro for $\mathfrak{sl}_2$. In addition, we give a simple proof of the fact that the universal $\mathfrak{gl}_N$ invariant of any evenly framed link and the universal $\mathfrak{sl}_N$ invariant of any 0-framed algebraically split link are Γ-invariant, where $\Gamma = Y/2Y$ with the root lattice Y.

1. Introduction

In a series of papers [1, 2, 15] Habiro, the first author et al. defined *unified* invariants of homology 3-spheres that belong to the Habiro ring and dominate Witten–Reshetikhin–Turaev (WRT) invariants. Unified invariants provide an important tool to study structural properties of the WRT invariants. In [3, 5] they were used to prove integrality of the $\mathfrak{sl}_2$ WRT invariants for all 3-manifolds at all roots of unity.

The theory of unified invariants for $\mathfrak{sl}_2$ is based on *cyclotomic expansions* for the colored Jones polynomial and for the universal knot invariant constructed as follows. Given a framed oriented link L in the 3-sphere, we open its components to obtain a bottom tangle T, presented by a diagram D (see Figure 1). For a ribbon Hopf algebra $U_q\mathfrak{g}$, the *universal* link invariant $J_L(\mathfrak{g}; q)$ is obtained by spliting D intro elementary pieces: crossings, caps and cups and then by associating to them $R^{\pm 1}$-matrices, and pivotal elements, respectively.

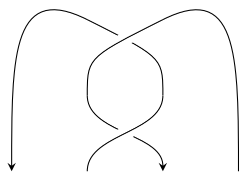

Figure 1. An example of the clasp bottom tangle

For a knot K, $J_K(\mathfrak{g}; q)$ belongs to (some completion of) the center $\mathcal{Z}(U_q\mathfrak{g})$. In the easiest case $\mathfrak{g} = \mathfrak{sl}_2$, the center is generated by the Casimir C. For a 0-framed knot K, Habiro showed that there are coefficients $a_m(K) \in \mathbb{Z}[q^{\pm 1}]$ such that

$$J_K(\mathfrak{sl}_2; q) = \sum_{m=0}^{\infty} a_m(K)\, \sigma_m \quad \text{with} \quad \sigma_m = \prod_{i=1}^{m} \left(C^2 - (q^i + q^{-i} + 2)\right). \tag{1}$$

Replacing C^2 in (1) by its value $q^n+q^{-n}+2$ on the n-dimensional irreducible representation V_{n-1}, we get the n-colored Jones polynomial of K (normalized to 1 for the unknot)

$$J_K(V_{n-1},q)=\sum_{m=0}^{\infty}(-1)^m q^{-\frac{m(m+1)}{2}}a_m(K)\,(q^{1+n};q)_m(q^{1-n};q)_m \tag{2}$$

where $(a;q)_m=(1-a)(1-aq)\dots(1-aq^{m-1})$. Equation (2) is known as a cyclotomic expansion of the colored Jones polynomial. Thus, Habiro's series (1) dominates all colored Jones polynomials of K. To prove the fact that $J_K(\mathfrak{sl}_2;q)$ belongs to the even part of $\mathcal{Z}(U_q\mathfrak{sl}_2)$, generated by C^2, Habiro used the whole power of the theory of bottom tangles developed in [16].

In this paper we give a simple proof for the "evenness" of the universal invariant of algebraically split links for all quantum groups of type A. Recall that $U_q\mathfrak{g}$ has a natural action of a finite group $\Gamma=Y/2Y$ where Y is the root lattice of $\mathfrak{g}$. For $\mathfrak{g}=\mathfrak{gl}_N$, $\Gamma=\mathbb{Z}_2^N$ and for $\mathfrak{g}=\mathfrak{sl}_N$, $\Gamma=\mathbb{Z}_2^{N-1}$.

Theorem 1.1. *The universal $\mathfrak{gl}_N$ invariant of any evenly framed link is Γ-invariant. The universal $\mathfrak{sl}_N$ invariant of any 0-framed algebraically split link is Γ-invariant.*

The quantum group $U_q\mathfrak{gl}_N$ admits a finite dimensional irreducible representation $V(\lambda)$ with highest weight v^λ for any partition $\lambda=(\lambda_1\geq\dots\geq\lambda_N)$ with N parts and $v^2=q$. To prove Theorem 1.1 we extend the Reshetikhin-Turaev invariants to tangles colored with representations $L(\zeta)\otimes V(\lambda)$ where $L(\zeta)$ is a one-dimensional representation of $U_q\mathfrak{gl}_N$ for $\zeta\in\Gamma$. Then the claim follows from the comparison of the $\mathfrak{gl}_N$ Reshetikhin-Turaev link invariants colored with $L(\zeta)\otimes V(\lambda)$ and $V(\lambda)$.

The next main result of the paper establishes an explicit basis in the Γ-invariant part of the center $\mathcal{Z}$ of $U_q\mathfrak{gl}_N$. It generalizes Habiro's basis $\{\sigma_m\,|\,m\in\mathbb{N}\}$ for the even part of $\mathcal{Z}(U_q\mathfrak{sl}_2)$.

Theorem 1.2. *There exists a family of central elements $\sigma_\lambda\in\mathcal{Z}$ labeled by partitions λ with at most N parts with the following properties:*

(a) *σ_λ is Γ-invariant and annihilates $L(\zeta)\otimes V(\mu)$ for all $\zeta\in\Gamma$ and partitions μ with at most N parts not containing λ;*
(b) *σ_λ does not annihilate $V(\lambda)$ and acts on it by an explicit scalar (see Theorem 8.2).*

The proof uses the theory of interpolation Macdonald polynomials developed in [23, 24, 29, 30, 31, 32, 36]. This theory allows one to reconstruct a symmetric function $f(x_1,\dots,x_N)$ from its values at special points $x_i=q^{-\mu_i-N+i}$ where μ is an arbitrary partition with at most N parts. The connection between the center of $U_q\mathfrak{gl}_N$ and symmetric functions goes through the quantum Harish-Chandra isomorphism, and we interpret $f(q^{-\mu_1-N+1},\dots,q^{-\mu_N})$ as the scalar by which the element of the center f acts on the irreducible representation $V(\mu)$. Interpolation Macdonald polynomials then correspond to a natural basis in the center of $U_q\mathfrak{gl}_N$.

The polynomials σ_λ yield a basis in the Γ-invariant parts of both the center $\mathcal{Z}$ and its completion (a function in the completion is determined by its values on all finite-dimensional representations). We use a formula of Okounkov [29] to give explicit expansion of a given central element z in the basis σ_λ in terms of the scalars by which z acts on all finite-dimensional representations $V(\lambda)$. This leads to an expansion of the universal knot invariant in the basis σ_λ, where the coefficients are related to Reshetikhin-Turaev invariants of the same knot colored by $V(\mu)$ via an explicit triangular matrix $(d_{\lambda,\mu})$ which does not depend on the knot.

Theorem 1.3. *For any evenly framed knot K, there exist Laurent polynomials $a_\lambda(K) \in \mathbb{Z}[q, q^{-1}]$ such that the universal invariant of K has the following expansion:*

$$J_K(\mathfrak{gl}_N; q) = \sum_\lambda a_\lambda(K)\, \sigma_\lambda \ . \tag{3}$$

Moreover, the coefficients $a_\lambda(K)$ can be computed in terms of the Reshetikhin-Turaev invariants as follows:

$$a_\lambda(K) = \sum_{\mu \subset \lambda} d_{\lambda,\mu}(q^{-1})\ J_K(V(\mu), q)$$

where the coefficients $d_{\lambda,\mu}(q)$ are defined in Theorem 10.17.

We prove Theorem 1.3 as Proposition 8.7. We would like to emphasize that the fact that $a_\lambda(K)$ are Laurent polynomials in q is highly nontrivial. Indeed, we have computed the tables of coefficients $d_{\lambda,\mu}(q)$ for $\mathfrak{gl}_2$ in Section 11.4 and these are complicated rational functions, so a priori $a_\lambda(K)$ are rational functions as well. Theorem 1.3 thus encodes certain divisibility properties for the linear combinations of colored invariants of K. We refer to Section 11.5 for the explicit computation of the coefficients $a_\lambda(K)$ for the figure eight knot.

We call (3) a *cyclotomic expansion* of the universal $\mathfrak{gl}_N$ knot invariant. The name *cyclotomic* is justified by the fact that (3) has well-defined evaluations at any root of unity by Lemma 10.29 below. Note that for $N = 2$ and a 0-framed knot, our expansion does not coincide with that of Habiro, simply because if an element $z \in U_q\mathfrak{gl}_2$ is central and Γ-invariant, it does not imply z has a decomposition in even powers of the Casimir. Therefore, our cyclotomic expansion is rather a generalization of F_∞ in [37] or [4, eq.(3.14)], both having interesting application in the theory of non semisimple invariants of links and 3-manifolds.

For our next application, assume M is an integral homology 3-sphere obtained by ε-surgery on an ℓ-component algebraically split 0-framed link L with $\varepsilon \in \{\pm 1\}^\ell$. Following Habiro–Le, we define an $\mathfrak{gl}_N$ unified invariant $I(M)$ as

$$I(M) = \langle\, r^{\otimes \varepsilon}, J_L(\mathfrak{gl}_N; q)\,\rangle$$

where r is the $\mathfrak{gl}_N$ ribbon element and $\langle \cdot, \cdot \rangle$ is the Hopf pairing. In the case of $\mathfrak{sl}_N$ Habiro–Le proved [19] that the unified invariant belongs to a cyclotomic completition of the polynomial ring

$$\widehat{\mathbb{Z}[q]} := \varprojlim_n \frac{\mathbb{Z}[q]}{((q;q)_n)}$$

known as *Habiro ring*. Using interpolation, we are able to express $I(M)$ in terms of special linear combinations of Reshetikhin–Turaev invariants of L, called *Kirby colors*. For this we diagonalize the Hopf pairing, i.e. find a basis P_μ that is orthonormal to σ_λ and orthogonal to $V(\lambda)$ with respect to the Hopf pairing. This allows us to give explicit formulas for the universal Kirby colors $\omega_\pm$ (see (25)) in the basis P_μ and to prove the following result.

Theorem 1.4. *The unified invariant*

$$I(M) = J_L(\omega_{\epsilon_1}, \dots, \omega_{\epsilon_\ell}) \ \in\ \widehat{\mathbb{Z}[q]}$$

belongs to the Habiro ring and dominates $\mathfrak{gl}_N$ WRT invariants of $M_\pm$ at all roots of unity. Moreover, $I(M)$ is equal to the $\mathfrak{sl}_N$ Habiro–Le invariant of $M_\pm$.

To prove that $I(M)$ is equal to the $\mathfrak{sl}_N$ Habiro–Le invariant we show the equality of the universal $\mathfrak{gl}_N$ and $\mathfrak{sl}_N$ invariants for 0-framed algebraically split links, and the fact that the $\mathfrak{gl}_N$ and $\mathfrak{sl}_N$ twist forms $x \mapsto \langle r^{\pm 1}, x\rangle$ on them coincide. It follows that $I(M)$ belongs to the Habiro ring. Then we establish invariance of Kirby colors $\omega_\pm$ under Hoste moves (a version of Fenn–Rourke moves between algebraically split links) in Lemma 9.1, and finally deduce the equality $I(M) = J_L(\omega_{\epsilon_1}, \dots, \omega_{\epsilon_\ell})$.

The main advantage of Theorem 1.4 compared to Habiro–Le approach is the interpretation of $I(M)$ as the Reshetikhin–Turaev invariant of L colored by ω_ε. This leads to various striking divisibility results and allows us to extend our cyclotomic expansion to links.

Corollary 1.5. *Given an ℓ component algebraically split 0-framed link L, then for all but finitely many partitions λ_i with $1 \le i \le \ell$, there exist positive integers $n = n(\lambda_i, N)$, such that*

$$J_L(P'_{\lambda_1}, \dots, P'_{\lambda_\ell}) \in (q;q)_n\, \mathbb{Z}[q, q^{-1}]$$

where $P'_\lambda = v^{|\lambda|} \dim_q V(\lambda)\, P_\lambda$ is a scalar multiple of P_λ.

This is a generalization of the famous integrability theorem in [15, Thm. 8.2]. The authors do not know any direct proof of Corollary 1.5 without using the theory of unified invariants. Based on Corollary 9.4 we obtain a cyclotomic expansion for the Reshetikhin-Turaev invariants of L:

$$J_L(\lambda_1, \dots, \lambda_\ell) = v^{\sum_i |\lambda_i|} \sum_{\mu_i \subset \lambda_i} \prod_{j=1}^{l} c_{\lambda_j, \mu_j}(q^{-1})\, J_L(P'_{\mu_1}, \dots, P'_{\mu_\ell}) \tag{4}$$

where the matrix $[c_{\lambda,\mu}(q)]_{\lambda,\mu} := \left[F_\lambda(q^{-\mu_i - N + i})\right]_{\lambda,\mu}$ is the inverse of $[d_{\lambda,\mu}(q)]_{\lambda,\mu}$. This generalizes equation (8.2) in [15].

In addition, in the case of knot surgeries we give a direct proof of the fact that

$$I(M_\pm) = J_L(\omega_\pm) \ \in \ \widehat{\mathbb{Z}[v]}$$

by using our cyclotomic expansion and the interpolation theory.

Finally, we would like to comment on potential ideas for categorification of these results. The ring of symmetric polynomials in N variables is naturally categorified by the category of annular $\mathfrak{gl}_N$-webs, with morphisms given by annular foams [6, 33, 34, 13, 11]. By the work of the second author and Wedrich [13], one can interpret it as a symmetric monoidal Karoubian category generated by one object E corresponding to a single essential circle. The symmetric polynomials are then categorified by the Schur functors of E.

We expect the categorified interpolation polynomials to correspond to interpolation Macdonald polynomials where q plays the role of quantum grading and t of the homological grading (after some change of variables). We recall the general definitions and properties of these polynomials from [29] in Appendix. The key obstacle for categorification of interpolation polynomials is that they are not homogeneous. Therefore one needs to enrich the category and allow additional morphisms between E and identity.

On the other hand, the conjectures of the second author, Neguț and Rasmussen ([12], see [10, 11] for further discussions) relate a version of the annular category to the derived category of the Hilbert scheme of points on the plane. The interpolation Macdonald polynomials appear in that context as well [7].

The paper is organized as follows. After recalling the definitions, we compare the Reshetikhin–Turaev invariants of tangles colored by $V(\lambda)$ and $L(\zeta) \otimes V(\lambda)$ in Section 4. In the next two sections we summarize known results about the center of $U_q\mathfrak{gl}_N$, define its completion and prove Theorem 1.1 in Section 6.2. The remaining results are proven in Sections 8, 9 assuming some facts about interpolation.

In the last sections we develop the theory of the interpolation Macdonald polynomials, starting from the one variable case. We define multi-variable interpolation polynomials, state and prove their properties in Section 10.2. Next, we solve the interpolation problem in two ways, one using the approach of Okounkov (Theorem 10.17), and another using Hopf pairing (see (38)). We study divisibility of $F_\lambda(q^{a_1},\dots,q^{a_n})$ by quantum factorials in Section 10.5 (see Lemma 10.29). Section 11 is focused on various stability properties of the interpolation polynomials such as adding a column to a partition λ (Proposition 11.8) and changing N for a fixed Young diagram λ. In particular, in Proposition 11.5 we describe a HOMFLY-PT analogue of the interpolation polynomials depending on an additional parameter $A = q^N$. We provide lots of examples and tables of interpolation polynomials, especially for $\mathfrak{gl}_2$. In Appendix A we collect some additional known facts about the interpolation Macdonald polynomials and the Habiro ring.

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2. Notations and conventions

2.1. **q-binomial formulas.** Throughout the paper we will use the following notations for the q-series. The q-Pochhammer symbols are defined as

$$(a;q)_m = \prod_{i=0}^{m-1}(1-aq^i),\ (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),\ m \geq 0.$$

It is easy to see that

$$(a;q)_{m+k} = (a;q)_m(aq^m;q)_k,\ (a;q)_m = \frac{(a;q)_\infty}{(aq^m;q)_\infty}.$$

We will use two normalizations for q-binomial coefficients defined as follows:

$$\{a\}_q = 1-q^a,\ [a]_q = \frac{\{a\}_q}{\{1\}_q},\ [a]_q! = [1]_q\cdots[a]_q,\ \binom{a}{b}_q = \frac{[a]_q!}{[b]_q![a-b]_q!}\ .$$

Note that

$$[a]_q = \frac{(q;q)_a}{(1-q)^a},\ \binom{a}{b}_q = \frac{(q;q)_a}{(q;q)_b(q;q)_{a-b}}.$$

Finally, the q-binomial formula gives

$$(a;q)_m = \sum_{j=0}^{m}(-1)^j q^{\frac{j(j-1)}{2}}\binom{m}{j}_q a^j.$$

Let us also define symmetric q-numbers. For this we chose v such that $v^2=q$ and set

$$\{a\} = v^a - v^{-a},\quad [a] := \frac{\{a\}}{\{1\}},\quad \begin{bmatrix} a \\ b \end{bmatrix} = \frac{\{a\}!}{\{b\}!\{a-b\}!}\ .$$

We will use all these formulas throughout the paper without a reference.

2.2. **Partitions.** We will work with partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N)$ which we will identify with the corresponding Young diagrams in French notation, where the rows have length λ_i. Transpose diagram to λ is denoted by λ', and $|\lambda| = \sum \lambda_i$. Given a box in a Young diagram, we define its arm, co-arm, leg and co-leg as in Figure 2.

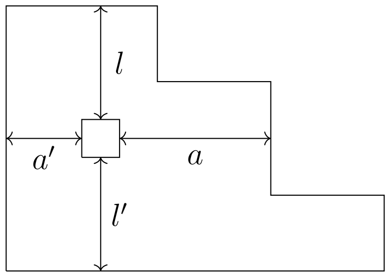


FIGURE 2. Arm, co-arm, leg and co-leg

We define the hook length as $h(\square) = a(\square) + l(\square) + 1$, and the content $c(\square) = a' - l'$. Let

$$n(\lambda) = \sum (i-1)\lambda_i = \sum_{\square} l'(\square) = \sum_{\square} l(\square),$$

then

$$n(\lambda') = \sum \frac{\lambda_i(\lambda_i - 1)}{2} = \sum_{\square} a'(\square) = \sum_{\square} a(\square).$$

The content of λ is defined as

$$c(\lambda) = \sum_{\square} c(\square) = n(\lambda') - n(\lambda).$$

Let $\bar{\lambda}_i = \lambda_i + N - i$ for $1 \leq i \leq N$, then we have the following identity

$$\prod_{\square \in \lambda} (1 - t^{h(\square)}) = \frac{\prod_{i \geq 1} \prod_{j=1}^{\bar{\lambda}_i} (1 - t^j)}{\prod_{i<j} \left(1 - t^{\bar{\lambda}_i - \bar{\lambda}_j}\right)} \tag{5}$$

and we define

$$D_N(\lambda) = \sum_{i=1}^{N} \frac{(\bar{\lambda}_i)(\bar{\lambda}_i - 1)}{2} = \sum_i \frac{\lambda_i(\lambda_i - 1)}{2} + \sum_i (N-i)\lambda_i + \sum_{i=1}^{N} \binom{N-i}{2} \tag{6}$$

$$= n(\lambda') + (N-1)|\lambda| - n(\lambda) + \binom{N}{3} = c(\lambda) + (N-1)|\lambda| + \binom{N}{3}. \tag{7}$$

3. Quantum groups

3.1. **Quantum $\mathfrak{gl}_N$.** The quantum group $\mathcal{U} = U_q\mathfrak{gl}_N$ is a $\mathbb{C}(v)$-algebra generated by $E_1, \ldots, E_{N-1}$, $F_1, \ldots, F_{N-1}$, $K_1^{\pm 1}, \ldots, K_N^{\pm 1}$ satisfying the following relations:

$$K_i E_i = v E_i K_i,\ K_i F_i = v^{-1} F_i K_i,\ K_{i+1} E_i = v^{-1} E_i K_{i+1},\ K_{i+1} F_i = v F_i K_{i+1} \tag{8}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_{i+1} K_i^{-1}}{v - v^{-1}},\ [K_i, K_j] = 0, \tag{9}$$

$$E_i^2 E_j - [2] E_i E_j E_i + E_j E_i^2 = 0 \text{ if } |i-j| = 1 \text{ and } [E_i, E_j] = 0 \text{ otherwise} \tag{10}$$

and analogously for F_i, where $v^2 = q$. To simplify the notation we set $\mathcal{K}_i := K_i K_{i+1}^{-1}$. Then the Hopf algebra structure on $\mathcal{U}$ (i.e. coproduct, antipode and counit) can be defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + \mathcal{K}_i \otimes E_i,\ \Delta(F_i) = 1 \otimes F_i + F_i \otimes \mathcal{K}_i^{-1},\ \Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1},$$

$$S(K_i^{\pm 1}) = K_i^{\mp 1},\ S(E_i) = -E_i \mathcal{K}_i^{-1},\ S(F_i) = -\mathcal{K}_i F_i$$

$$\varepsilon(K_i^{\pm 1}) = 1, \varepsilon(E_i) = \varepsilon(F_i) = 0.$$

Usually $\mathcal{U}$ is considered as a subalgebra of $\mathcal{U}_h$ that is an h-adically complete $\mathbb{C}[[h]]$-algebra topologically generated by E_i, F_i and H_j for $1 \leq i \leq N-1$ and $1 \leq j \leq N$ with

$$v = \exp h/2, \quad K_i = v^{H_i} = \exp hH_i/2$$

satisfying (9), (10) and

$$H_i E_i - E_i H_i = E_i,\ \ H_i F_i - F_i H_i = -F_i,\ \ H_{i+1} E_i - E_i H_{i+1} = -E_i,\ \ H_{i+1} F_i - F_i H_{i+1} = F_i$$

replacing (8). Rewriting the defining relations in terms of the generators

$$e_i = E_i(v - v^{-1}), \quad F_i^{(n)} = \frac{F_i^n}{[n]!} \quad \text{and} \quad K_j \quad \text{for} \quad 1 \leq i \leq N-1, \quad 1 \leq j \leq N$$

we obtain an integral version $\mathcal{U}_{\mathbb{Z}}$ as a Hopf algebra over $\mathbb{Z}[v, v^{-1}] \subset \mathbb{C}(v) \subset \mathbb{C}[[h]]$.

The quantum group $\mathfrak{gl}_N$ has a fundamental representation $\mathbb{C}^N$ with basis $v_1, \ldots, v_N$ such that

$$K_i v_j = v^{\delta_{ij}} v_j,\ E_i v_j = \begin{cases} v_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}, F_i v_j = \begin{cases} v_{i+1} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

It generates a braided monoidal category with simple objects $V(\lambda)$, where λ is a partition with at most N parts. These are highest weight modules where K_i act on the highest weight vector by v^{λ_i}. The fundamental representation corresponds to $\lambda = (1)$. The representations $V(\lambda)$ have integral basis where $\mathcal{U}_{\mathbb{Z}}$ acts by $\mathbb{Z}[v, v^{-1}]$-valued matrices.

3.2. **Ribbon structure.** The Hopf algebra $\mathcal{U}_h$ admits a ribbon Hopf algebra structure (see e.g. [8, Cor. 8.3.16]). The universal R-matrix has the form $\mathcal{R} = D\Theta$ where the diagonal part D and the quasi-R-matrix are defined as follows

$$D = v^{\sum_{i=1}^N H_i \otimes H_i} \quad \text{and} \quad \Theta = \sum_{\mathbf{n} \in \mathbb{N}^{N-1}} F_{\mathbf{n}} \otimes e_{\mathbf{n}}$$

where for any sequence of non-negative integers $\mathbf{n} = (n_1, \ldots, n_{N-1})$, the elements $e_{\mathbf{n}}$ and $F_{\mathbf{n}}$ are defined by equations (66) and (67) in [19] and form topological bases of the positive and negative parts in the triangular decomposition of $\mathcal{U}_{\mathbb{Z}}$. The inverse matrix $\mathcal{R}^{-1} = \iota(\Theta) D^{-1}$ is obtained by applying the involution $\iota : v \to v^{-1}$.

The ribbon element and its inverse have the form

$$r = \sum_{\mathbf{n}} F_{\mathbf{n}}\, \mathcal{K}_{\mathbf{n}}\, r_0\, e_{\mathbf{n}} \quad \text{and} \quad r^{-1} = \sum_{\mathbf{n}} \iota(F_{\mathbf{n}})\, \mathcal{K}_{-\mathbf{n}}\, r_0^{-1}\, \iota(e_{\mathbf{n}}) \tag{11}$$

where $r_0 = K_{-2\rho} v^{-\sum_i H_i^2}$ and $K_{-2\rho} = \prod_{i=1}^N K_i^{2i-N-1}$ is the pivotal element. Here for any sequence of integers $\mathbf{n} \in \mathbb{Z}^{N-1}$ we set $\mathcal{K}_{\mathbf{n}} = \prod_i \mathcal{K}_i^{n_i}$, and denote by

$$\rho = \left(\frac{N-1}{2}, \frac{N-3}{2}, \ldots, \frac{1-N}{2} \right) = \frac{1-N}{2}(1, \ldots, 1) + (N-1, N-2, \ldots, 0)$$

the half sum of all positive roots. Using the central element $K = \prod_{i=1}^N K_i$, we can write the previous definitions as follows:

$$r_0^{-1} = K^N \prod_{i=1}^N K_i^{-2i}\, v^{\sum_i H_i(H_i+1)}, \quad K_{-2\rho} = K^{-N-1} \prod_{i=1}^N K_i^{2i}\ .$$

The central element r^{-1} acts on $V(\lambda)$ by the multiplication with

$$\theta_{V(\lambda)} = v^{(\lambda,\lambda+2\rho)} = v^{N|\lambda|} q^{c(\lambda)}$$

where $(\lambda,\mu) = \sum_{i=1}^N \lambda_i\mu_i$, $c(\lambda)$ is the content of λ and $v^2 = q$.

3.3. **Even part of $\mathcal{U}$.** The algebra $\mathcal{U}$ has a natural grading by $\Gamma = \mathbb{Z}_2^N = \{\pm 1\}^N$ where $\zeta = (\zeta_1, \dots, \zeta_N) \in \Gamma$ acts on K_i by ζ_i, on E_i by 1 and on F_i by $\zeta_i\zeta_{i+1}$. It is easy to see that the defining relations are preserved under this action. Following [19], we call an element of $\mathcal{U}_N$ *even* or Γ-invariant if it is preserved under the action of Γ.

Let us denote by $\mathcal{U}_{\mathbb{Z}}^{\mathrm{ev}}$ a $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\mathcal{U}_{\mathbb{Z}}$ generated by e_i, $F_i^{(n)}\mathcal{K}_i$ and K_j^2 for $1 \le i \le N-1$ and $1 \le j \le N$. It is easy to check that $\mathcal{U}_{\mathbb{Z}}^{\mathrm{ev}}$ is Γ-invariant.

The action of Γ descends on the category $Rep(\mathcal{U})$ of all finite-dimensional representations. Given $\zeta = (\zeta_1, \dots, \zeta_N) \in \Gamma$, we can define a one-dimensional representation $L(\zeta)$ where E_i and F_i act by zero, and K_i act by ζ_i. We can also define representation $V(\lambda) \otimes L(\zeta)$ where K_i act on the highest weight vector by $\zeta_i v^{\lambda_i}$.

Lemma 3.1. *The action of $\mathcal{U}$ on $V(\lambda) \otimes L(\zeta)$ agrees with the Γ-twisted action of $\mathcal{U}$ on $V(\lambda)$.*

Proof. Indeed, $\Delta(F_i) = 1 \otimes F_i + F_i \otimes \mathcal{K}_i^{-1}$, so F_i acts on $V(\lambda) \otimes L(\zeta)$ via $F_i \otimes \mathcal{K}_i^{-1} = F_i\zeta_i\zeta_{i+1}$. Similarly, E_i acts on $V(\lambda) \otimes L(\zeta)$ via $E_i \otimes 1 = E_i$ and K_i acts via $K_i \otimes K_i = K_i\zeta_i$. □

3.4. **The subalgebra $U_q\mathfrak{sl}_N$.** We define $U_q\mathfrak{sl}_N$ as a subalgebra of $\mathcal{U}$ generated by E_i, F_i and $\mathcal{K}_i^{\pm 1} := K_i^{\pm 1} K_{i+1}^{\mp 1}$ for $1 \le i \le N-1$. The Hopf algebra $U_q\mathfrak{sl}_N$ also admits an integral version $\mathcal{U}_{\mathbb{Z}}\mathfrak{sl}_N$ generated by

$$e_i, \quad F_i^{(n)} \quad \text{and} \quad \mathcal{K}_i^{\pm 1}$$

over $\mathbb{Z}[q,q^{-1}]$. The braiding $\mathcal{R} = D'\Theta$ with Θ as for $\mathfrak{gl}_N$, but different diagonal part

$$D' = v^{\sum_{i=1}^{N-1} \frac{\mathcal{H}_i \otimes \mathcal{H}_i}{2}} \quad \text{where} \quad \mathcal{H}_i = H_i - H_{i+1}\ .$$

The ribbon element is defined by (11) with $r_0 = K_{-2\rho} \prod_{i=1}^{N-1} v^{-\mathcal{H}_i^2/2}$. The pivotal element $K_{-2\rho}$ does not change. Note that the Γ-invariant part of $U_q\mathfrak{sl}_N$ generated by e_i, $F_i^{(n)}\mathcal{K}_i$ and $\mathcal{K}_j^2$ for $1 \le i, j \le N-1$ has a smaller Cartan part than its $\mathfrak{gl}_N$ analogue.

Example 3.2. For $N = 2$ the product K_1K_2 is central. By denoting $\mathcal{K} = K_1K_2^{-1}, E = E_1, F = F_1$ we get the standard presentation for $U_q(\mathfrak{sl}_2)$:

$$\mathcal{K}E = v^2 E\mathcal{K},\ \mathcal{K}F = v^{-2}F\mathcal{K},\ [E,F] = \frac{\mathcal{K} - \mathcal{K}^{-1}}{v - v^{-1}}.$$

3.5. **Universal invariant.** Lawrence, Reshetikhin, Ohtsuki and Kauffman constructed quantum group valued *universal* link invariants. As it was already mentioned in the introduction, the universal invariant of a link is defined by splitting a diagram of its bottom tangle into elementary pieces and by associating R-matrices and pivotal elements to them. For more details and references we recommend to consult [16, Sec. 7.3]. However, we admit here the convention from [19, Sec. 2.7] and write the contributions from left to right along the orientation of each component.

4. Ribbon structure on $Rep(\mathcal{U})$

The aim of this section is to compare the Reshetikhin-Turaev invariants of a bottom tangle whose components are colored with $V(\lambda)$ and $V(\lambda)\otimes L(\zeta)$. This will be later used to prove Theorem 1.1.

Let us denote by $\mathcal{R}_{\mathbb{Q}}$ the representation ring of $Rep(\mathcal{U})$ over $\mathbb{Q}(v)$. Given an l component link L, Reshetikhin–Turaev functor associated with Lie algebra $\mathfrak{g}$ provides a $\mathbb{Q}(v)$-multilinear map

$$\begin{aligned} J_L : \mathcal{R}_{\mathbb{Q}} \times \cdots \times \mathcal{R}_{\mathbb{Q}} &\to \mathbb{Q}(v) \\ (\mu_1, \dots, \mu_l) &\mapsto \bigotimes_i \mathrm{Tr}_q^{V(\mu_i)} \left(J_L(\mathfrak{g}; q)\right) =: J_L(\mathfrak{g}; \mu_1, ..., \mu_l) \end{aligned}$$

normalized to $\prod_i \dim_q(V(\mu_i))$ for the 0-framed $(\mu_1, \dots, \mu_l)$-colored unlink. In cases when $\mathfrak{g}$ is fixed in the context, we will remove it from the notation for simplicity.

Note that in the case of a knot, we have $J_K(\lambda) = \dim_q(V(\lambda))J_K(V(\lambda), q)$ where the last invariant is the colored Jones polynomial used in Introduction and normalized to be 1 for the unknot.

The universal R-matrix defines a braiding between the representations $V(\lambda)$. We can extend this braiding to $Rep(\mathcal{U})$ as follows. Clearly, $L(\zeta)\otimes L(\zeta') \simeq L(\zeta\zeta')$ and we define the braiding between $L(\zeta)$ and $L(\zeta')$ to be trivial. Let V be a finite-dimensional representation of $\mathcal{U}$ where the eigenvalues of K_i are integral powers of v. Given $\zeta \in \Gamma$ we consider a $\mathbb{C}$-linear map $T_V(\zeta) : V \to V$ which acts by $\prod \zeta_i^{a_i}$ on the weight subspace of V where K_i acts as v^{a_i}.

Lemma 4.1. *The maps*

$$c_{\zeta,V} := \mathrm{swap} \circ (\mathrm{Id} \otimes T_V(\zeta)) : L(\zeta) \otimes V \to V \otimes L(\zeta)$$

with inverses

$$c_{V,\zeta} := \mathrm{swap} \circ (T_V(\zeta) \otimes \mathrm{Id}) : V \otimes L(\zeta) \to L(\zeta) \otimes V$$

define a braiding on $Rep(\mathcal{U})$.

Proof. First, let us check that $\mathrm{swap} \circ (\mathrm{Id} \otimes T_V(\zeta))$ intertwines the actions of $\mathcal{U}$ on both sides. Indeed, let $v \in V$ be a vector with weight $(v^{a_1}, \dots, v^{a_N})$, then $E_i v$ has weight $(v^{a_1}, \dots, v^{a_i+1}, v^{a_{i+1}-1}, \dots, v^{a_N})$ while $F_i v$ has weight $(v^{a_1}, \dots, v^{a_i-1}, v^{a_{i+1}+1}, \dots, v^{a_N})$.

Let $\bullet$ denote the basis vector in $L(\zeta)$, then

$$\begin{aligned} c_{\zeta,V} E_i(\bullet \otimes v) &= c_{\zeta,V}\left(\zeta_i\zeta_{i+1} \bullet \otimes E_i(v)\right) = \zeta_1^{a_1} \cdots \zeta_i^{a_i} \zeta_{i+1}^{a_{i+1}} \cdots \zeta_N^{a_N} E_i(v) \otimes \bullet, \\ c_{\zeta,V} F_i(\bullet \otimes v) &= c_{\zeta,V}\left(\bullet \otimes F_i(v)\right) = \zeta_1^{a_1} \cdots \zeta_i^{a_i-1} \zeta_{i+1}^{a_{i+1}+1} \cdots \zeta_N^{a_N} F_i(v) \otimes \bullet, \\ c_{\zeta,V} K_i(\bullet \otimes v) &= c_{\zeta,V}\left(\zeta_i \bullet \otimes K_i(v)\right) = \zeta_1^{a_1} \cdots \zeta_i^{a_i+1} \cdots \zeta_N^{a_N} K_i(v) \otimes \bullet, \end{aligned}$$

while

$$\begin{aligned} E_i c_{\zeta,V}(\bullet \otimes v) &= E_i(\zeta_1^{a_1} \cdots \zeta_N^{a_N} v \otimes \bullet) = \zeta_1^{a_1} \cdots \zeta_N^{a_N} E_i(v) \otimes \bullet, \\ F_i c_{\zeta,V}(\bullet \otimes v) &= F_i(\zeta_1^{a_1} \cdots \zeta_N^{a_N} v \otimes \bullet) = \zeta_1^{a_1} \cdots \zeta_i^{a_i-1} \zeta_{i+1}^{a_{i+1}+1} \cdots \zeta_N^{a_N} F_i(v) \otimes \bullet, \\ K_i c_{\zeta,V}(\bullet \otimes v) &= K_i(\zeta_1^{a_1} \cdots \zeta_N^{a_N} v \otimes \bullet) = \zeta_1^{a_1} \cdots \zeta_i^{a_i+1} \cdots \zeta_N^{a_N} K_i(v) \otimes \bullet. \end{aligned}$$

Next, we observe that $T_V(\zeta)T_V(\zeta') = T_V(\zeta\zeta')$ and $T_{U\otimes V}(\zeta) = T_U(\zeta)\otimes T_V(\zeta)$, so $c_{\zeta,V}$ indeed defines a braiding. Even more concretely, we get the braiding as the composition

(12)
$$c_{L(\zeta)\otimes V, L(\zeta')\otimes U} : L(\zeta)\otimes V\otimes L(\zeta')\otimes U \xrightarrow{c_{V,\zeta'}} L(\zeta)\otimes L(\zeta')\otimes V\otimes U = L(\zeta')\otimes L(\zeta)\otimes V\otimes U \xrightarrow{c_{V,U}}$$
$$L(\zeta')\otimes L(\zeta)\otimes U\otimes V \xrightarrow{c_{\zeta,U}} L(\zeta')\otimes U\otimes L(\zeta)\otimes V.$$

□

The representations $L(\zeta)$ are self-dual, and it is easy to see that the braiding $c_{\zeta,V}$ is compatible with changing V to V^*. Therefore, $Rep(\mathcal{U})$ with objects $L(\zeta)\otimes V$ form a pivotal braided monoidal category.

The quantum dimension of $L(\zeta)$ equals to the trace of the action of the pivotal element, which is $(\prod_i \zeta_i)^{N+1}$. The twist coefficient $\theta_{L(\zeta)}$ is defined as the action of the ribbon element on $L(\zeta)$, and is given by $(\prod_i \zeta_i)^N$.

Lemma 4.2. *$Rep(\mathcal{U})$ is a ribbon category with twist $\theta_{L(\zeta)\otimes V} = \theta_{L(\zeta)}\theta_V$.*

Proof. By definition $\theta_{L(\zeta)\otimes V} = c_{\zeta,V}\theta_{L(\zeta)}\theta_V c_{V,\zeta} = \theta_{L(\zeta)}\theta_V$. □

4.1. **Braiding in** $Rep(U_q\mathfrak{sl}_N)$**.** In this section, we study the action of Γ and the corresponding braiding for $U_q\mathfrak{sl}_N$, starting from $N=2$. Similarly to the previous section, $U_q\mathfrak{sl}_2$ has a one dimensional representation $L(-1)$ where E and F act by 0 and $\mathcal{K}$ acts by -1. The action of $U_q\mathfrak{sl}_2$ on $L(-1)\otimes V$ is equivalent to $\mathbb{Z}_2$-twisted action on V where $\mathbb{Z}_2$ scales E by 1 and $F,\mathcal{K}$ by -1.

One can attempt to define a braiding for $U_q\mathfrak{sl}_2$. Since E and F shift the weights by 2, it is easy to see that the analogue of T_V should act by $(\sqrt{-1})^a$ on a subspace with weight v^a, and it does not square to identity. Nevertheless, it squares to $\pm\mathrm{id}$ on each irreducible representation. This means that braiding relations on $Rep(U_q\mathfrak{sl}_2)$ hold up to sign.

To pin down this sign, we define the *sign automorphism* Σ_V which acts by $(-1)^a$ on a subspace with weight v^a. Since E,F shift the weight by $\pm v^2$, Σ_V commutes with the action of $U_q\mathfrak{sl}_2$ on V. The operator Σ_V acts on the irreducible representation $V(n)$ by a scalar $(-1)^n$. Also, it is easy to see that $\Sigma_{V\oplus W} = \Sigma_V\oplus\Sigma_W$ and $\Sigma_{V\otimes W} = \Sigma_V\otimes\Sigma_W$.

Lemma 4.3. *The operators T_V and Σ_V satisfy the following properties:*

(a) *We have*
$$T_V^2 = \Sigma_V,\ c_{L(-1),V} = c^{-1}_{L(-1),V}(1\otimes\Sigma_V) = (\Sigma_V\otimes 1)c^{-1}_{L(-1),V}$$

(b) *Let $c_{V,W}: V\otimes W\to W\otimes V$ be the braiding, then*
$$c_{V,W}(\Sigma_V\otimes 1) = (1\otimes\Sigma_V)c_{V,W},\ c_{V,W}(1\otimes\Sigma_W) = (\Sigma_W\otimes 1)c_{V,W}$$

(c) *We have $c_{L(-1),V\otimes W} = c_{L(-1),V}\circ c_{L(-1),W}$.*

(d) *The braiding with $L(-1)$ satisfies Yang-Baxter equation, that is, the following diagram commutes:*
$$\begin{tikzcd}
L(-1)\otimes V\otimes W \arrow[r, "c_{L(-1),V}"] \arrow[d, "c_{V,W}"] & V\otimes L(-1)\otimes W \arrow[r, "c_{L(-1),W}"] & V\otimes W\otimes L(-1) \arrow[d, "c_{V,W}"] \\
L(-1)\otimes W\otimes V \arrow[r, "c_{L(-1),W}"] & W\otimes L(-1)\otimes V \arrow[r, "c_{L(-1),V}"] & W\otimes V\otimes L(-1)
\end{tikzcd}$$

Proof. Part (a) is clear. To prove (b), observe that the action of $U_q\mathfrak{sl}_2 \otimes U_q\mathfrak{sl}_2$ on $V \otimes W$ commutes with both $\Sigma_V \otimes 1$ and $1 \otimes \Sigma_V$, and the R-matrix is an element of the completion of $U_q\mathfrak{sl}_2 \otimes U_q\mathfrak{sl}_2$.

Given a pair of vectors $u \in V, w \in W$ such that $Ku = v^i u$ and $Kw = v^j w$, we get $K(u \otimes w) = v^{i+j} u \otimes w$, so $T_{V \otimes W} = T_V \otimes T_W$. Since $c_{L(-1),V} = \mathrm{swap} \circ (\mathrm{Id} \otimes T_V)$, we get the desired relation. Finally, (d) follows from (c). $\square$

We can generalize the above results to representations of $U_q\mathfrak{sl}_N$ as follows. For $\zeta \in \mathbb{Z}_2^{N-1}$ there is a one-dimensional representation $L(\zeta)$ of $U_q(\mathfrak{sl}_N)$ where E_i, F_i act by 0 and $\mathcal{K}_i = K_i K_{i+1}^{-1}$ act by ζ_i ($1 \le i \le N-1$). Given a representation V where all weights of $\mathcal{K}_i$ are integral powers of v, we can define an operator $T_{\zeta,V} : V \to V$ which acts by $\zeta^{A^{-1}\mathbf{a}}$ on a subspace where $\mathcal{K}_i$ acts by v^{a_i}. Here A is the Cartan matrix for $\mathfrak{sl}_N$ given by

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix} \tag{13}$$

and $\mathbf{a} = (a_1, \dots, a_{N-1})$. Note that $\det(A) = N$, so A^{-1} has rational entries with denominator N and one needs to choose an N-th root of (-1) to define $\zeta^{A^{-1}\mathbf{a}}$. Define $\Sigma_{\zeta,V} = T_{\zeta,V}^2$.

Lemma 4.4. *The operators $T_{\zeta,V}$ and $\Sigma_{\zeta,V}$ satisfy the following properties:*

(a) $T_{\zeta,V} E_i = \zeta_i E_i T_{\zeta,V}, T_{\zeta,V} F_i = \zeta_i F_i T_{\zeta,V}$
(b) $\Sigma_{\zeta,V}$ *commutes with the action of $U_q\mathfrak{sl}_N$ on V*
(c) *The map $c_{L(\zeta),V} = \mathrm{swap} \circ (\mathrm{Id} \otimes T_{\zeta,V}) : L(\zeta) \otimes V \to V \otimes L(\zeta)$ is a morphism of $U_q\mathfrak{sl}_N$-representations*
(d) *The maps $T_{\zeta,V}$ and $\Sigma_{\zeta,V}$ satisfy all equations in Lemma 4.3 with $L(-1)$ changed to $L(\zeta)$.*

Proof. (a) The operator F_i changes the weight $\mathbf{a} = (a_1, \dots, a_{N-1})$ by Ae_i, so if $\mathcal{K}_i v = v^{a_i} v$ then

$$T_{\zeta,V} F_i(v) = \zeta^{A^{-1}(\mathbf{a}+Ae_i)} F_i v = \zeta^{A^{-1}\mathbf{a}+e_i} F_i v = \zeta_i F_i T_{\zeta,V}(v).$$

The proof for E_i is similar. Part (b) immediately follows from (a).

For (c), we observe that the action of E_i on $L(\zeta) \otimes V$ is the same as the action on V, while the actions of $F_i, \mathcal{K}_i$ are twisted by ζ_i. On the other hand, the action of F_i on $V \otimes L(\zeta)$ is the same as the action on V, while the actions of $E_i, \mathcal{K}_i$ are twisted by ζ_i. Therefore by (a) the operator $c_{L(\zeta),V}$ intertwines the actions of $U_q\mathfrak{sl}_N$ on $L(\zeta) \otimes V$ and $V \otimes L(\zeta)$.

Finally, the proof of the rest of Lemma 4.3 extends to $U_q\mathfrak{sl}_N$ verbatim. $\square$

Remark 4.5. The above construction of $T_{\zeta,V}$ and $\Sigma_{\zeta,V}$ can be extended to an arbitrary semisimple Lie algebra with Cartan matrix A. The action of $\Sigma_{\zeta,V}$ can be interpreted in terms of projection of the weight lattice to its quotient by the root lattice.

We draw a tangle colored by a representation $V = V(\lambda)$ using solid lines, and a tangle colored by $L(\zeta)$ by dotted lines. If a component is colored by $L(\zeta) \otimes V$, we draw a dotted line on the left of a solid line and parallel to it. The crossings between solid and dotted lines correspond to $c^{\pm}_{L(\zeta),V}$ depicted in Figure 3. Note that unlike $\mathfrak{gl}_N$ case, $c_{L(\zeta),V}$ does not square to identity and we have to distinguish under- and over-crossings between solid and dotted lines. This allows us to define Reshetikhin-Turaev invariants for framed tangles colored by representations of $U_q\mathfrak{sl}_N$ of the form $L(\zeta) \otimes V(\lambda)$.

Using the notations as in Figure 3, we can visualize the statement of Lemma 4.4 in Figure 4.

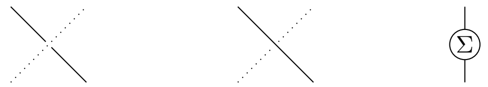


FIGURE 3. The operators $c_{L(\zeta),V}$, $c^{-1}_{L(\zeta),V}$ and Σ_ζ.

Theorem 4.6. *(a) Let L be an algebraically split 0-framed link with ℓ components. Then for arbitrary partitions $\lambda_1, \ldots, \lambda_\ell$ and $\zeta_1, \ldots, \zeta_\ell \in \Gamma$ the following identity of Reshetikhin–Turaev invariants holds:*

$$J_L\left(\mathfrak{sl}_N; V(\lambda_1) \otimes L(\zeta_1), \ldots, V(\lambda_\ell) \otimes L(\zeta_\ell)\right) = J_L\left(\mathfrak{sl}_N; V(\lambda_1), \ldots, V(\lambda_\ell)\right) \cdot \dim_q L(\zeta_1) \cdots \dim_q L(\zeta_\ell), \tag{14}$$

where $\dim_q L(\zeta_i) = \mathrm{Tr}_q^{L(\zeta_i)}(1) = \pm 1$.

(b) Let L be an arbitrary link with evenly framed components, if N is odd. Then (14) *holds for $\mathfrak{gl}_N$ Reshetikhin–Turaev invariants.*

Proof. (a) We use the results of Lemmas 4.3 and 4.4 and the above diagrammatic notation. By Lemma 4.3(a), we can change crossings between dotted and solid lines at a cost of placing Σ_ζ on solid lines. By doing this iteratively, we can make all dotted lines to be above solid lines. At this stage, each solid component of L acquires several copies of Σ_ζ and Σ_ζ^{-1} at various places of the link diagram. The number of these copies (with signs) equals the linking number between this component and the dotted part which is even by our assumption. By Lemma 4.3(b) we can combine all these copies of Σ_ζ together and cancel out. Finally, using Lemma 4.3(d), we can separate the dotted and solid links. By changing the crossings in the dotted link, we transform it to the 0-framed unlink. Therefore the invariant of the solid link equals $J_L\left(\mathfrak{sl}_N; V(\lambda_1), \ldots, V(\lambda_\ell)\right)$ while the invariant of the dotted link equals $\dim_q L(\zeta_1) \cdots \dim_q L(\zeta_\ell)$.

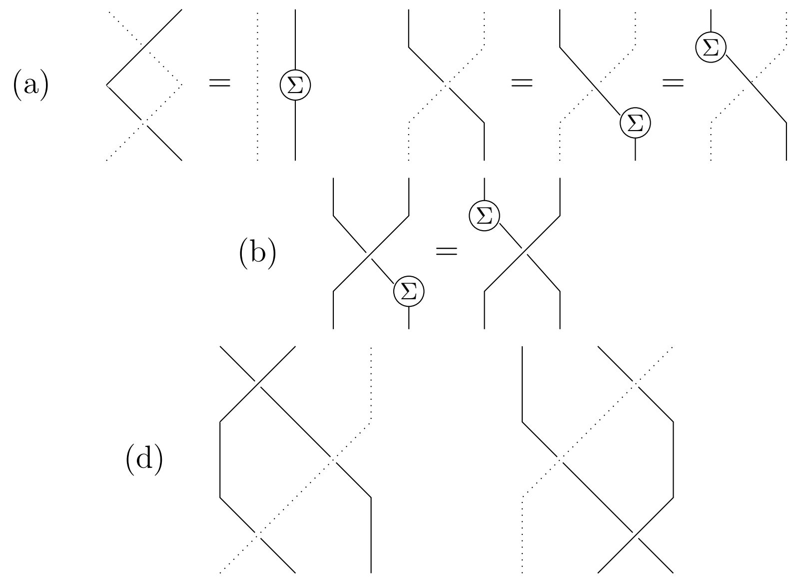


FIGURE 4. Diagrammatics for Lemma 4.4

The proof of (b) is similar, except that Σ_V is trivial for all V. As before we can unknot dotted components. Now the ribbon element acts on $L(\zeta)$ by $\theta_{L(\zeta)} = (\prod_i \zeta_i)^N$, and hence, any (even if N is odd) number of them acts by 1. The result follows.

□

5. Center of $\mathcal{U}$

Let $\mathcal{Z}$ be the center of $\mathcal{U}_{\mathbb{Z}}$. In this section we recall the main facts known about $\mathcal{Z}$.

5.1. **Harish-Chandra isomorphism.** Let $(\mathcal{U}^0_{\mathbb{Z}})^{S_N} := \mathbb{Z}[v, v^{-1}][K_1^{\pm 1}, \dots, K_N^{\pm 1}]^{S_N}$ be the Cartan part of $\mathcal{U}_{\mathbb{Z}}$ invariant under the Weyl group action. After a multiplication by an appropriate power of the central element $K := \prod_{i=1}^N K_i$, each element of $(\mathcal{U}^0)^{S_N}$ can be viewed as a symmetric function in N variables. This allows to identify $(\mathcal{U}^0_{\mathbb{Z}})^{S_N}$ with the ring of symmetric functions divided by powers of the elementary symmetric polynomial $e_N = K$. In the classical case, this ring can be identified with the center using the Harish-Chandra isomorphism. After quantization, the image of the Harish-Chandra homomorphism belongs to

$$Sym = \mathbb{Z}[v^{\pm 1}, e_N^{-1}][x_1, \dots, x_N]^{S_N}$$

where $x_i = K_i^2$ (compare e.g. [20, Ch. 6]). In this section we will furthermore identify Sym with the Grothendieck ring $\mathcal{R}$ of $Rep(\mathcal{U})$ with coefficients $\mathbb{Z}[v^{\pm 1}]$.

First, the character map

$$ch : \mathcal{R} \to Sym$$

sends a representation U to its character $ch(U)$. Clearly, $ch(U \oplus V) = ch(U) + ch(V)$ and $ch(U \otimes V) = ch(U)ch(V)$, so ch is a ring homomorphism. The character of $V(\lambda)$ equals the Schur function $s_\lambda(x_1, \dots, x_N)$, while the character of $L(\zeta)$ equals $\zeta_1 \cdots \zeta_N$.

The Harish-Chandra map

$$hc : \mathcal{Z} \to Sym$$

is defined as follows. Let ϕ be a central element in $\mathcal{U}$, it acts in the Verma module $\Delta(\lambda)$ by some scalar $\phi|_{\Delta(\lambda)}$. We define $hc(\phi)$ to be the polynomial in Sym defined by the condition

$$hc(\phi)(q^{\rho+\lambda}) = \phi|_{\Delta(\lambda)} \quad \text{for all} \quad \lambda$$

where $\rho = \left(\frac{N-1}{2}, \frac{N-3}{2}, \dots, \frac{1-N}{2}\right)$. Note that the product $\phi\phi'$ acts on $\Delta(\lambda)$ by the product of the corresponding scalars, so hc is also a ring homomorphism. It is known to be an isomorphism (see e.g. [20, Ch. 6]).

Finally, the map $\xi : \mathcal{R} \to \mathcal{Z}$ is defined by $\xi = hc^{-1} \circ ch$. It is a composition of two ring homomorphisms and hence a ring homomorphism too. Hence, we get the commutative diagram:

$$\begin{tikzcd}
\mathcal{R} \arrow[rr, "\xi"] \arrow[dr, "ch"'] & & \mathcal{Z} \arrow[dl, "hc"] \\
& Sym &
\end{tikzcd}$$

In Lemma 5.3 we will show that ξ actually coincides with the Drinfeld map.

Example 5.1. The central element $K = K_1 \cdots K_N$ acts on $V(\lambda)$ by a scalar $v^{\sum \lambda_i}$. Since $\sum \rho_i = 0$, we get $hc(K_1 \cdots K_N) = y_1 \cdots y_N$.

Example 5.2. The center of $U_q\mathfrak{sl}_2$ is generated by the Casimir element:

$$C = (v - v^{-1})^2 FE + v\mathcal{K} + v^{-1}\mathcal{K}^{-1}$$

It acts on a representation V_m by $v^{m+1}+v^{-m-1}$, so $hc(C)=y+y^{-1}$ (note that $v^\rho=v$ in this case). On the other hand, $ch(V_1)=y+y^{-1}$, so $\xi(V_1)=C$, where V_1 the 2-dimensional representation.

Similarly, we can consider the corresponding central element in $U_q\mathfrak{gl}_2$ defined by

$$C_{\mathfrak{gl}_2}=(v-v^{-1})^2FE+vK_1K_2^{-1}+v^{-1}K_1^{-1}K_2.$$

It acts on a representation $V(\lambda)$ by a scalar

$$v^{1+\lambda_1-\lambda_2}+v^{-1-\lambda_1+\lambda_2}=\frac{y_1}{y_2}+\frac{y_2}{y_1},\qquad y_1=v^{1/2+\lambda_1},y_2=v^{-1/2+\lambda_2},$$

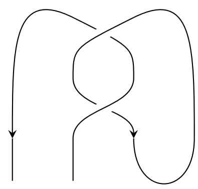
so $hc(C_{\mathfrak{gl}_2})=\frac{y_1}{y_2}+\frac{y_2}{y_1}=\frac{y_1^2+y_2^2}{y_1y_2}=e_2^{-1}(y_1,y_2)(x_1+x_2)$.

5.2. **Hopf pairing.** The Hopf pairing $\langle U,V\rangle$ of two representations $U,V\in\mathcal{R}$ is defined as the Reshetikhin–Turaev invariant of the Hopf link with components labeled by U and V. This is a symmetric bilinear pairing on $\mathcal{R}$. The map ξ is related to the Hopf pairing as follows:

Lemma 5.3. *The Hopf pairing on $\mathcal{R}$ can be computed as*

$$\langle U,V\rangle=\mathrm{Tr}_q^U(\xi(V)).$$

Proof. Consider the Drinfeld map D [9] which sends a representation V to a central element corresponding to the universal invariant of the following tangle:

$D(V):=$  V

By e.g. [14, eq. (20)] (see also [19, Proposition 8.19] and references therein) the eigenvalue of $D(V)$ on the irreducible representation $V(\lambda)$ equals $ch(q^{\lambda+\rho})$ where ch is the character of V. By the definition of the Harish-Chandra map, this means that $hc(D(V))=ch(V)$, and

$$D(V)=hc^{-1}(ch(V))=\xi(V),$$

so ξ agrees with the Drinfeld map. Now $\langle U,V\rangle=\mathrm{Tr}_q^U(D(V))=\mathrm{Tr}_q^U(\xi(V))$ or more precisely,

$$\langle V(\lambda),V(\mu)\rangle=s_\lambda(q^{\mu+\rho})s_\mu(q^\rho)\quad\text{where}\quad\dim_q V(\mu)=s_\mu(q^\rho).$$

□

Using the Drinfeld isomorphism ξ we can extend the Hopf pairing to the center by setting

$$\langle z_1,z_2\rangle:=\langle\xi^{-1}(z_1),\xi^{-1}(z_2)\rangle\quad\text{for any}\quad z_1,z_2\in\mathcal{Z}\ .$$

6. Cyclotomic completion and the universal invariant

The universal invariant of a link belongs a priori to a (completed) tensor product of copies of $\mathcal{U}_h$, rather than $\mathcal{U}$ or $\mathcal{U}_\mathbb{Z}$, due to the diagonal part of the R-matrix. The aim of this section is to define a certain completion of $\mathcal{U}_\mathbb{Z}$ and its tensor powers, such that the universal $\mathfrak{gl}_N$ invariant of evenly framed links belongs to it. Since the action of Γ extends to the completion, this will allow us to speak about Γ-invariance of $J_L(\mathfrak{gl}_N;q)$.

6.1. **Cyclotomic completion of $\mathcal{U}_{\mathbb{Z}}$.** Given $n \in \mathbb{N}$, we define a family of two-sided ideals $\mathcal{U}_{\mathbb{Z}}^{(n)}$ as the minimal filtration such that $\mathcal{U}_{\mathbb{Z}}^{(n)}\mathcal{U}_{\mathbb{Z}}^{(m)} \subset \mathcal{U}_{\mathbb{Z}}^{(m+n)}$ and

$$(q;q)_n, \quad e_i^n, \quad f_n(K_j^2) \in \mathcal{U}_{\mathbb{Z}}^{(n)}$$

for any $1 \le i \le N-1$ and $1 \le j \le N$ where $f_n(x) = (x;q)_n$. In other words, $\mathcal{U}_{\mathbb{Z}}^{(n)}$ is the two-sided ideal generated by the products

$$(q;q)_a \, e_{\mathbf{m}} \, f_{c_1}(K_1^2) \cdots f_{c_N}(K_N^2), \quad \text{with} \quad a + \sum_i m_i + \sum_i c_i = n \, . \tag{15}$$

Lemma 6.1. *We have*

$$\Delta\left(f_n\left(K_i^2\right)\right) = \sum_{a=0}^{n} \binom{n}{a}_q f_a(K_i^2) \otimes K_i^{2(n-a)} f_{n-a}(K_i^2).$$

Proof. We prove Lemma by induction in n. For $n=0$ it is clear. The induction step follows from the identities

$$f_{n+1}(K_i^2) = f_n(K_i)(1 - q^n K_i^2)$$

and

$$\Delta(1 - q^n K_i^2) = 1 \otimes 1 - q^n K_i^2 \otimes K_i^2 = (1 - q^a K_i^2) \otimes q^{n-a} K_i^2 + 1 \otimes (1 - q^{n-a} K_i^2).$$

□

Proposition 6.2. *a) $\mathcal{U}_{\mathbb{Z}}^{(n)}$ is the left ideal generated by* (15).

b) $\mathcal{U}_{\mathbb{Z}}^{(n)}$ form a Hopf algebra filtration, that is $\Delta \mathcal{U}_{\mathbb{Z}}^{(n)} \subset \sum_{i+j=n} \mathcal{U}_{\mathbb{Z}}^{(i)} \otimes \mathcal{U}_{\mathbb{Z}}^{(j)}$.

c) Assume that $\lambda_i \le k$ for all i. Given arbitrary m, there exists $n = n(k,m)$ such that the elements of $\mathcal{U}_{\mathbb{Z}}^{(n)}$ act on the integral basis of $V(\lambda)$ by matrices divisible by $(q;q)_m$.

Proof. a) Observe that by Lemma 10.5 we get $f_n(q^s K_i^2) \in \mathcal{U}_{\mathbb{Z}}^{(n)}$ for all integer s. Now the statement follows from the identities

$$f_n(K_i^2) F_i^{(s)} = F_i^{(s)} f_n(q^{-s} K_i^2), \ f_n(K_{i+1}^2) F_i^{(s)} = F_i^{(s)} f_n(q^s K_{i+1}^2)$$

and

$$f_n(K_i^2) e_i^s = e_i^s f_n(q^s K_i^2), \ f_n(K_{i+1}^2) e_i^s = e_i^s f_n(q^{-s} K_{i+1}^2).$$

b) Follows from the identity

$$\Delta(e_j^m) = \sum_{i=0}^{m} \binom{m}{i}_q e_j^{m-i} \mathcal{K}^i \otimes e_j^i$$

and Lemma 6.1.

c) By (a), it is sufficient to check the statement for e_i^n and $f_n(K_i^2)$. If $\lambda_i \le k$ then for $n > k$ e_i^n annihilates $V(\lambda)$, while $f_n(K_i^2)$ acts on a vector with weight $(v^{\lambda_1}, \dots, v^{\lambda_N})$ by $f_n(q^{\lambda_i}) = (q^{\lambda_i};q)_n$ which is divisible by $(q;q)_n$. □

By Proposition 6.2(b), the filtration

$$\mathcal{U}_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}}^{(0)} \supset \mathcal{U}_{\mathbb{Z}}^{(1)} \supset \dots \mathcal{U}_{\mathbb{Z}}^{(n)} \supset \dots$$

is a Hopf algebra filtration of $\mathcal{U}_{\mathbb{Z}}$ with respect to a descending filtration of ideals $I_n = ((q;q)_n)$ in $\mathbb{Z}[v, v^{-1}]$ in the sense of [18, Sec. 4]. Hence, the completion

$$\widehat{\mathcal{U}} := \varprojlim_n \frac{\mathcal{U}_{\mathbb{Z}}}{\mathcal{U}_{\mathbb{Z}}^{(n)}}$$

is a complete Hopf algebra over the ring

$$\widehat{\mathbb{Z}[v]} := \varprojlim_n \frac{\mathbb{Z}[v]}{((q;q)_n)}.$$

We refer to [18, Section 4] for details. Analogously, we define the Γ-invariant subalgebra

$$\widehat{\mathcal{U}^{\mathrm{ev}}} := \varprojlim_n \frac{\mathcal{U}_{\mathbb{Z}}^{\mathrm{ev}}}{\mathcal{U}_{\mathbb{Z}}^{(n)}}$$

as a complete Hopf algebra over the Habiro ring $\widehat{\mathbb{Z}[q]}$. Let us now extend the completion to the tensor powers of $\mathcal{U}_{\mathbb{Z}}$. For this we define the filtration for $\mathcal{U}_{\mathbb{Z}}^{\otimes l}$ for $l \geq 1$ as follows

$$\mathcal{F}_n(\mathcal{U}_{\mathbb{Z}}^{\otimes l}) = \sum_{i=1}^{l} \mathcal{U}_{\mathbb{Z}}^{\otimes i-1} \otimes \mathcal{U}_{\mathbb{Z}}^{(n)} \otimes \mathcal{U}_{\mathbb{Z}}^{\otimes l-i}$$

and the completed tensor product $\mathcal{U}_{\mathbb{Z}}^{\hat{\otimes} l}$ with respect to this filtration will be the image of the homomorphism

$$\varprojlim_n \frac{\mathcal{U}_{\mathbb{Z}}^{\otimes l}}{\mathcal{F}_n(\mathcal{U}_{\mathbb{Z}}^{\otimes l})} \to \mathcal{U}_h^{\otimes k}$$

where on the right hand side we use the h-adically completed tensor product.

6.2. **Hopf pairing and universal invariants.** Let us denote by $c \in \mathcal{U}_h \otimes \mathcal{U}_h$ the double braiding or the universal invariant of the clasp tangle in Figure 1, given by

$$c = (S \otimes \mathrm{id})\, \mathcal{R}_{21}\mathcal{R}\ .$$

The main point about this element is that it is dual to the Hopf pairing or the quantum Killing form (compare [19, Sec. 4]). Hence, after writing $c = \sum_i c(i) \otimes c'(i)$ the Hopf pairing is defined by setting

$$\langle c(i), c'(j) \rangle := \delta_{ij} \tag{16}$$

Restricting to the Cartan part this gives us (compare [19, Lemma 3.12])

$$D^{-2} = \prod_{i=1}^{N} q^{-H_i \otimes H_i} = \prod_{i=1}^{N} \sum_{n_i} (-1)^{n_i} \frac{h^{n_i}}{n_i!} H_i^{n_i} \otimes H_i^{n_i} \tag{17}$$

and hence, $\langle H_i^n, H_j^m \rangle = \delta_{ij}\delta_{nm}(-1)^n \frac{n!}{h^n}$. We deduce that $\langle K_i^2, K_j^2 \rangle = q^{-1}$ or, more generally,

$$\langle K_i^{2a}, K_j^{2b} \rangle = \delta_{ij} q^{-ab}$$

defines the Hopf pairing on the Γ-invariant part of the Cartan. In Section 10 we construct another basis for the Cartan given by $\prod_{i=1}^{N} f_{n_i}(K_i^2)$ such that $\langle f_n, f_m \rangle = \delta_{nm}(-1)^n q^{-n}(q;q)_n$. In this new basis, we can rewrite the Cartan part of the clasp element as follows:

$$D^{-2} = \sum_{\mathbf{n} \in \mathbb{N}^N} \prod_{i=1}^{N} \frac{(-1)^{n_i} q^{n_i}}{(q;q)_{n_i}} f_{n_i}(K_i^2) \otimes f_{n_i}(K_i^2) \tag{18}$$

For $\mathfrak{sl}_N$ similar computations will give

$$(D')^{-2} = \sum_{\mathbf{n} \in \mathbb{N}^{N-1}} \prod_{i=1}^{N-1} \frac{(-1)^{n_i} q^{n_i}}{(q;q)_{n_i}} f_{n_i}(\mathcal{K}_i) \otimes f_{n_i}(\mathcal{K}_i^2)$$

(compare Section B.1 in [19]).

Let us denote by

$$\mathrm{Inv}\,(\mathcal{U}) = \{u \in \mathcal{U} \,|\, x \triangleright u = \epsilon(x) u \quad \forall x \in \mathcal{U}\}$$

the invariant part of $\mathcal{U}$ under the adjoint action $x \triangleright u := x_{(1)} u S(x_{(2)})$ in Sweedler notation. The main advantage of the usage of bottom tangles in the definition of $J_L(\mathfrak{gl}_N;q)$ is that in this case $J_L(\mathfrak{gl}_N;q) \in \mathrm{Inv}\,(\mathcal{U})$ (compare [15, Sec.4.3]). As a corollary, we get the following:

Proposition 6.3. *Given an l-component evenly framed link L, the universal invariant $J_L(\mathfrak{gl}_N;q)$ is a well defined element of* $\mathrm{Inv}\left(\widehat{\mathcal{U}}^{\hat{\otimes} l}\right)$.

Proof. By definition, J_K is obtained by multiplying together elementary pieces, such as $F_{\mathbf{n}}$, $e_{\mathbf{n}}$, $K_{2\rho}^{\pm 1}$, $D^{\pm 1}$, and by then taking a sum over all indices. The linking between different components and framing will make appear powers of $D^{\pm 2}$ that we can decompose using the basis elements $f_n(K_i^2)$ of the completion by (18). Note that we can collect all diagonal contributions of each component by using formulas like

$$D(E_i \otimes 1)D^{-1} = E_i \otimes \mathcal{K}_i \quad \text{and} \quad D(1 \otimes F_j)D^{-1} = \mathcal{K}_j^{-1} \otimes F_j \ .$$

Since framing is assumed to be even, we will have an even number of D-parts. Hence using (18) and the explicit form of the quasi R-matrix Θ, we get the claim. $\square$

Remark 6.4. For $\mathfrak{sl}_N$ we can build the same completion after replacing K_i with $\mathcal{K}_i$. Then the arguments in the proof of Proposition 6.3 will show us that for any algebraically split link the universal invariants belongs to this completion.

Proof of Theorem 1.1.

Using Proposition 6.3 and remark above, we can define the action of Γ on each component of $J_L(\mathfrak{g};q)$ separately. We will denote by $J_L^{\zeta_1,\dots,\zeta_\ell}(\mathfrak{g};q)$ the result of this action. Then we have

$$J_L\left(V(\lambda_1)\otimes L(\zeta_1),\dots,V(\lambda_\ell)\otimes L(\zeta_\ell)\right) =$$

$$\bigotimes_{i=1}^{l} \mathrm{Tr}_q^{V(\lambda_i)\otimes L(\zeta_i)}\left(J_L(\mathfrak{g};q)\right) = \bigotimes_{i=1}^{l} \mathrm{Tr}_q^{V(\lambda_i)}\left(J_T^{\zeta_1,\dots,\zeta_\ell}(\mathfrak{g};q)\right)\cdot \dim_q L(\zeta_1)\cdots\dim_q L(\zeta_\ell).$$

The second equation follows from Lemma 3.1. By Theorem 4.6 we conclude that

$$J_L^{\zeta_1,\dots,\zeta_\ell}\left(\lambda_1,\dots,\lambda_\ell\right) = J_L\left(\lambda_1,\dots,\lambda_\ell\right)$$

for all $\lambda_1,\dots,\lambda_\ell$ under the assumptions of Theorem 1.1, therefore $J_L(\mathfrak{g};q) = J_L^{\zeta_1,\dots,\zeta_\ell}(\mathfrak{g};q)$ and hence, $J_L(\mathfrak{g};q)$ is Γ-invariant under the same assumptions.

$\square$

Corollary 6.5. *For any ℓ-component evenly framed link L, $J_L(\mathfrak{gl}_N;q)$ belongs to the Γ-invariant part of* $\mathrm{Inv}\left(\widehat{\mathcal{U}}^{\hat{\otimes}\ell}\right)$. *Moreover, for every 0-framed algebraically split link L,*

$$J_L(\mathfrak{gl}_N;q) = J_L(\mathfrak{sl}_N;q)\ .$$

Proof. The first statement is the direct consequence of Theorem 1.1. The second one follows from the fact that the only difference in the definitions of both invariants is in the diagonal part of the R-matrix, that does not contribute since the linking matrix vanishes and the rules for moving of D and D' along a component of the link coincide. $\square$

6.3. **Twist forms.** Let us denote by $\widehat{\mathcal{Z}}$ the center of $\widehat{\mathcal{U}}$. In what follows, we will be particularly interested in the following twist forms

$$\mathcal{T}_{\pm} : \widehat{\mathcal{Z}} \to \widehat{\mathcal{Z}} \quad \text{given by} \quad \mathcal{T}_{\pm}(z) := \langle r^{\pm 1}, z\rangle$$

the Hopf pairing with the ribbon element. On the Γ-invariant Cartan part they are easy to compute, given the Hopf pairing between the generators H_i in Section 6.2 . We have

$$\mathcal{T}_{\pm}(K_{2\mathbf{a}}) = \langle r_0^{\pm 1}, K_{2\mathbf{a}}\rangle = v^{\pm(\mathbf{a},2\rho-\mathbf{a})} \in \mathbb{Z}[v,v^{-1}] \tag{19}$$

for any $\mathbf{a} \in \mathbb{Z}^N$. Now equation (16) allows to extend the twists form to $\widehat{\mathcal{U}}^{\mathrm{ev}}$ as follows:

$$\mathcal{T}_\pm(F_{\mathbf{m}}\mathcal{K}_{\mathbf{m}}K_{2\mathbf{a}}e_{\mathbf{n}}) = \delta_{\mathbf{m},\mathbf{n}}q^{(\rho,\sum_i n_i\alpha_i)}v^{\pm(\mathbf{a},2\rho-\mathbf{a})} \in \mathbb{Z}[v,v^{-1}]$$

where $\alpha_i = e_i - e_{i+1}$ are the simple roots. Observe that after restriction to $\mathcal{U}_q^{\mathrm{ev}}\mathfrak{sl}_N$, i.e. replacing $K_{2\mathbf{a}}$ with $\mathcal{K}_{2\mathbf{b}}$ in the above formula, the result *belong to* $\mathbb{Z}[q,q^{-1}]$ and coincide with [19, eq. (102)] for any $\mathbf{b} \in \mathbb{Z}^{N-1}$.

7. Habiro's basis for $\mathcal{Z}(U_q\mathfrak{sl}_2)$

In this section we summarize Habiro's results for $\mathfrak{sl}_2$ in the way suitable for our generalization.

Habiro [15] defined a remarkable family of central elements in $\mathcal{Z}(U_q\mathfrak{sl}_2)$:

$$\sigma_m := \prod_{i=1}^m \left(C^2 - (v^i + v^{-i})^2\right) = \prod_{i=1}^m (C - v^i - v^{-i})(C + v^i + v^{-i}). \tag{20}$$

Since C acts on the $(j+1)$-dimensional representation V_j by a scalar $v^{j+1} + v^{-j-1}$, the polynomial σ_m is completely characterized by the following properties:

(a) (Parity) σ_m is $\Gamma = \mathbb{Z}_2$-invariant.
(b) (Vanishing) σ_m annihilates the representations V_j for $j < m$.
(c) (Normalization) σ_m acts on the representation V_m by a scalar

$$\prod_{i=1}^m \left((v^{m+1} + v^{-m-1})^2 - (v^i + v^{-i})^2\right) . \tag{21}$$

Note that parity implies that σ_m also annihilates the representations $L(-1)\otimes V_j$ for $j < m$. By using the Harish-Chandra isomorphism, we can alternatively consider the polynomials

$$T_m(y) := hc(\sigma_m) := \prod_{i=1}^m (yv^i - y^{-1}v^{-i})(yv^{-i} - y^{-1}v^i) = (-1)^m \prod_{i=1}^m q^{-i}(1 - y^2q^i)(1 - y^{-2}q^i)$$

which are characterized by the following properties:

(a) (Parity) T_m is $\mathbb{Z}_2$-invariant, that is, $T_m(-y) = T_m(y)$
(b) (Vanishing) $T_m(\pm v^{j+1}) = 0$ for $j < m$
(c) (Normalization) $T_m(v^{m+1})$ is given in (21).

Habiro proved that $\{\sigma_m\}_{m\geq 0}$ form a basis in (a certain completion of) the Γ-invariant part of the center. Hence, the elements $S_m = \xi^{-1}(\sigma_m)$, given by

$$S_m := \prod_{i=1}^m (V_1 - v^i - v^{-i})(V_1 + v^i + v^{-i})$$

form a basis of $\mathcal{R}$. We will show that

$$P_n = \prod_{i=0}^{n-1} (V_1 - v^{2i+1} - v^{-2i-1}) \in \mathcal{R}$$

is a dual basis to $\{S_m\}_{m\geq 0}$ with respect to the Hopf pairing. The following is a slight reformulation of [15, Prop. 6.3].

Lemma 7.1. *We have*

$$\langle P_n, S_m\rangle = \frac{\{2n+1\}!}{\{1\}}\delta_{n,m} .$$

Proof. Clearly, one has

$$\xi(P_n) = \prod_{i=0}^{n-1}(C - v^{2i+1} - v^{-2i-1})$$

which annihilates V_{2i} for $i < n$. We have the following cases:

1) For $n < m$ we have $\langle P_n, S_m\rangle = \mathrm{Tr}_q^{P_n}(\sigma_m)$. Since P_n is in span of V_i for $i \le n$ and σ_m annihilates all these, we get $\langle P_n, S_m\rangle = 0$.

2) For $m < n$ we have $\langle P_n, S_m\rangle = \mathrm{Tr}_q^{S_m}(\xi(P_n))$. Since S_m is in span of V_{2i} for $i \le n$ and

$$\langle P_n, V_{2i}\rangle = \{i+n\}\dots\{i-n+1\}[2i+1]\ .$$

Hence P_n annihilates all these, we get $\langle P_n, S_m\rangle = 0$.

3) Finally, for $n = m$ we observe that P_n has a unique copy of V_n and

$$\langle P_n, S_n\rangle = \langle V_n, S_n\rangle = \mathrm{Tr}_q^{V_n}(\sigma_n)$$

which is easy to compute. □

We can use the above results to compute the coefficients in the decomposition of any central element into $\{\sigma_m\}_{m\ge 0}$.

Lemma 7.2. *Let ϕ be a $\mathbb{Z}_2$-invariant element in $\mathcal{Z}(U_q\mathfrak{sl}_2)$ which acts on V_j by a scalar ϕ_j. Then*

$$\phi = \sum a_n\sigma_n, \ \textit{where}\ \ a_n = \sum_{i=0}^{n}(-1)^{n-i}\frac{\{2i+2\}\{i+1\}}{\{n+i+2\}!\{n-i\}!}\,\phi_i\ .$$

Proof. We have ([15, Lemma 6.1])

$$P_n = \sum_{i=0}^{n}(-1)^{n-i}\frac{[2i+2]}{[n+i+2]}\begin{bmatrix} 2n+1 \\ n+1+i\end{bmatrix} V_i.$$

If $\phi = \sum a_m\sigma_m$ then

$$a_n = \frac{\{1\}}{\{2n+1\}!}\mathrm{Tr}_q^{P_n}(\phi) = \frac{\{1\}}{\{2n+1\}!}\sum_{i=0}^{n}(-1)^{n-i}\frac{[2i+2]}{[n+i+2]}\begin{bmatrix} 2n+1 \\ n+1+i\end{bmatrix}\mathrm{Tr}_q^{V_i}(\phi) =$$

$$\sum_{i=0}^{n}(-1)^{n-i}\frac{\{2i+2\}\{1\}}{\{n+i+2\}!\{n-i\}!}\dim_q(V_i)\phi_i.$$

Using $\dim_q(V_i) = [i+1]$ we obtain the result. □

Habiro proved that for any 0-framed knot K, there exist $a_n(K) \in \mathbb{Z}[q, q^{-1}]$ such that

$$J_K(\mathfrak{sl}_2; q) = \sum_{n\ge 0} a_n(K)\,\sigma_n$$

known as a *cyclotomic* expansion of the colored Jones polynomial of the knot K.

8. New basis for the center of $\widehat{\mathcal{U}}$

Recall that $\widehat{\mathcal{Z}}$ is the center of the completion $\widehat{\mathcal{U}}$. In this section we construct the basis $\{\sigma_\lambda\}_\lambda$ of the Γ-invariant part of $\widehat{\mathcal{Z}}$. Furthermore, we explicitly define its dual $\{P_\lambda\}_\lambda$ with respect to the Hopf pairing. This allows us to construct the cyclotomic expansion of $J_K(\mathfrak{gl}_N; q)$ for any 0-framed knot K.

The proof uses the existence and properties of interpolation Macdonald polynomials [29] which are summarized in the following theorem.

Theorem 8.1. *There is a family of symmetric polynomials $F_\lambda(x_1,\dots,x_N;q)$ such that:*

(a) *F_λ is in the span of Schur functions s_μ for $\mu \le \lambda$ with the leading term*

$$F_\lambda = (-1)^{|\lambda|+\binom{N}{2}} q^{D_N(\lambda)} s_\lambda + \dots \ .$$

(b) $F_\lambda(q^{-\mu_1-N+1}, \dots, q^{-\mu_N}) = 0$ *unless μ contains λ.*

(c) $F_\lambda(q^{-\lambda_1-N+1}, \dots, q^{-\lambda_N}) = (-1)^{\binom{N}{2}} q^{n(\lambda)+\binom{N}{3}} \prod_{\square\in\lambda}(1-q^{-h(\square)})$.

(d) *Any function F in the completion can be written as*

$$F(x_1, \dots, x_N) = \sum_{\lambda,\ \mu\subset\lambda} d_{\mu,\lambda}(q) F(q^{-\mu_1-N+1}, \dots, q^{-\mu_N}) F_\lambda(x_1, \dots, x_N; q) \tag{22}$$

where $d_{\lambda,\mu}$ are explicit coefficients prescribed by Theorem 10.17.

We discuss the definition and give more details on interpolation Macdonald polynomials in Section 10.

Theorem 8.2. *There exists a family of central elements $\sigma_\lambda \in \mathcal{Z}$ with the following properties:*

(a) *σ_λ is Γ-invariant and annihilates $L(\zeta)\otimes V(\mu)$ for all μ not containing λ and $\zeta \in \Gamma$.*

(b) *$hc(\sigma_\lambda)$ is in the span of $s_\mu(x_1, \dots, x_N)$ for $\mu \le \lambda$, with the leading term*

$$hc(\sigma_\lambda) = (-1)^{|\lambda|+\binom{N}{2}} v^{(N-1)|\lambda|} q^{D_N(\lambda)} s_\lambda + \dots \ .$$

(c) *σ_λ acts on $V(\lambda)$ by a scalar*

$$\sigma_\lambda|_{V(\lambda)} = (-1)^{\binom{N}{2}} q^{-n(\lambda)-\binom{N}{3}} \prod_{\square\in\lambda}(1-q^{h(\square)}) \ .$$

Proof. Define $\sigma_\lambda = hc^{-1}(g_\lambda)$, where $g_\lambda(x_1, \dots, x_N) = F_\lambda(v^{N-1}x_1, \dots, v^{N-1}x_N; q^{-1})$. Then σ_λ is clearly Γ-invariant and

$$\sigma_\lambda|_{L(\zeta)\otimes V(\mu)} = g_\lambda(\zeta_i \cdot v^{\mu_i+\rho_i}) = F_\lambda(q^{(\mu_1+N-1)}, \dots, q^{\mu_N}; q^{-1}).$$

Indeed, if $y_i = \zeta_i \cdot v^{\mu_i+\rho_i} = \zeta_i v^{(\mu_i - \frac{N-1}{2} + N - i)}$ then $v^{N-1}y_i^2 = q^{(\mu_i+N-i)}$.

Now $F_\lambda(q^{(\mu_1+N-1)}, \dots, q^{\mu_N}; q^{-1})$ vanishes unless μ contains λ, and has the nonzero value prescribed by the previous theorem for $\mu = \lambda$. $\square$

Let us define $\mathcal{R}_\mathbb{Q} := \mathcal{R} \otimes \mathbb{Q}(v)$ by extending the coefficient ring $\mathbb{Z}[v^{\pm1}]$ of $\mathcal{R}$ to the rational functions in v.

Theorem 8.3. *Define the following formal elements of $\mathcal{R}_\mathbb{Q}$*

$$P_\lambda = \sum_{\mu\subset\lambda} \frac{d_{\lambda,\mu}(q^{-1})}{\dim_q V(\mu)}\, V(\mu) \in \mathcal{R}_\mathbb{Q},$$

then one has

$$\langle P_\lambda, \sigma_\nu\rangle := \mathrm{Tr}_q^{P_\lambda}(\sigma_\nu) = \delta_{\lambda,\nu} \ . \tag{23}$$

Proof. First, let us write the interpolation formula (22) for $F = F_\nu$:

$$F_\nu(x_1, \dots, x_N; q) = \sum_{\mu\subset\lambda} d_{\lambda,\mu}(q) F_\nu(q^{-\mu_1-N+1}, \dots, q^{-\mu_N}; q) F_\lambda(x_1, \dots, x_N; q),$$

so

$$\sum_{\mu\subset\lambda} d_{\lambda,\mu}(q) F_\nu(q^{-\mu_1-N+1}, \dots, q^{-\mu_N}; q) = \delta_{\lambda,\nu}.$$

By changing q to q^{-1} we get

$$\sum_{\mu\subset\lambda} d_{\lambda,\mu}(q^{-1})F_\nu(q^{\mu_1+N-1},\dots,q^{\mu_N};q^{-1})=\delta_{\lambda,\nu}. \tag{24}$$

Now $\mathrm{Tr}_q^{V(\mu)}(\sigma_\nu)=\dim_q(V(\mu))\, g_\nu(q^{\mu_1+N-1},\dots,q^{\mu_N})$, hence

$$\mathrm{Tr}_q^{P_\lambda}(\sigma_\nu)=\sum_{\mu\subset\lambda}\frac{d_{\lambda,\mu}(q^{-1})}{\dim_q V(\mu)}\mathrm{Tr}_q^{V(\mu)}(\sigma_\nu)=\ \delta_{\lambda,\nu}\ .$$

□

Next, we would like to study the integrality properties of the universal knot invariant.

Lemma 8.4. *(a) Let $\sigma\in\mathcal{U}_{\mathbb{Z}}^{ev}$. Then $\sigma=(K_1\cdots K_N)^{-2s}\sum a_\lambda\sigma_\lambda$ with $a_\lambda\in\mathbb{Z}[q,q^{-1}]$.*

(b) Given k and m, there exists $n=n(k,m)$ such that for all Γ-invariant central elements σ in the ideal $\mathcal{U}_{\mathbb{Z}}^{(n)}$ the coefficients a_λ are divisible by $(q;q)_m$ for $|\lambda|\le k$.

Proof. (a) Recall that Harish-Chandra transform hc identifies the Γ-invariant part of the center of $\mathcal{U}_{\mathbb{Z}}$ with the space of symmetric functions in $x_1,\dots,x_N$ with coefficients in $\mathbb{Z}[q,q^{-1}]$. Since F_λ is a polynomial with top degree part equal to the Schur polynomial (up to a monomial in q), we can write $(x_1\cdots x_N)^s f(x_1,\dots,x_N)=\sum_\lambda a_\lambda F_\lambda(x_1,\dots,x_N;q^{-1})$ and the result follows.

(b) If σ is in the ideal $\mathcal{U}_{\mathbb{Z}}^{(n)}$ for sufficiently large n, then by Proposition 6.2 its matrix elements in the integral basis of $V(\lambda)$ are divisible by $(q;q)_m$. By definition of Harish-Chandra transform, this implies that the values $f(q^{-\lambda_1-N+1},\dots,q^{-\lambda_N})$ are divisible by $(q;q)_m$ and hence by the interpolation formula (22) the coefficients a_λ are divisible by $(q;q)_m$ as well. □

Corollary 8.5. *The center of the completion $\widehat{\mathcal{U}}$ is isomorphic to the completion of the space of symmetric polynomials with coefficients in $\widehat{\mathbb{Z}[v]}$ with respect to the basis F_λ.*

Proof. By Lemma 8.4 any element of the center of $\widehat{\mathcal{U}}$ can be written as an infinite series $\sum a_\lambda F_\lambda$ with coefficients in $\widehat{\mathbb{Z}[v]}$, up to a factor $(x_1\cdots x_N)^{-s}$. By Corollary 11.10 the multiplication by $(x_1\cdots x_N)^{-s}$ preserves the space of such series. □

Corollary 8.6. *Any $\sigma\in\widehat{\mathcal{U}^{ev}}$ can be written as an infinite sum $\sigma=\sum a_\lambda\sigma_\lambda$ with coefficients $a_\lambda=\mathrm{Tr}_q^{P_\mu}(\sigma)\in\widehat{\mathbb{Z}[q]}$.*

Proposition 8.7. *The universal knot invariant admits an expansion*

$$J_K(\mathfrak{gl}_N;q)=\sum_\lambda a_\lambda(K)\sigma_\lambda \quad \textit{with} \quad a_\lambda(K)=\sum_{\mu\subset\lambda} d_{\lambda,\mu}(q^{-1})\,J_K(V(\mu),q)\in\mathbb{Z}[q,q^{-1}]$$

called a cyclotomic expansion of the universal $\mathfrak{gl}_N$ knot invariant.

Proposition 8.7 implies Theorem 1.3 in Introduction. Note that the knot invariant $J_K(V(\mu),q)$ is normalized to be 1 for the unknot.

Proof. By Corollary 6.5, $J_K(\mathfrak{gl}_N;q)$ is a central element in $\widehat{\mathcal{U}^{\mathrm{ev}}}$, so it can be written as $\sigma=\sum a_\lambda\sigma_\lambda$ with coefficients $a_\lambda\in\widehat{\mathbb{Z}[q]}$. On the other hand, the value of J_K on any representation V_λ is in $\mathbb{Z}[q,q^{-1}]$, so by the interpolation formula (22) the coefficients a_λ can be written as rational functions with numerators in $\mathbb{Z}[q,q^{-1}]$ and cyclotomic denominators. By Proposition 12.1 this implies that $a_\lambda\in\mathbb{Z}[q,q^{-1}]$. The explicit formula for a_λ is obtained by taking Hopf pairing with P_μ and observing that $\mathrm{Tr}_q^{V(\mu)}(J_K(\mathfrak{gl}_N;q))=\dim_q(V(\mu))J_K(V(\mu),q)$ according to our normalization. □

The last result shows that $a_\lambda(K)=\mathrm{Tr}_q^{P_\lambda}(J_K(\mathfrak{gl}_N;q))\in\mathbb{Z}[q^{\pm1}]$, even through the coefficients $d_{\lambda,\mu}(q)$ are rational functions in q (compare Example 10.23).

9. Unified invariants of integral homology 3-spheres

This section is devoted to our main application of the previous results — a construction of the unified invariants for integral homology 3-spheres. We start with few auxiliary results.

Let us denote by

$$P'_\lambda = v^{-|\lambda|}\dim_q V(\lambda)\sum_{\mu\subset\lambda}\frac{d_{\lambda,\mu}(q^{-1})}{\dim_q V(\mu)}\,V(\mu)\in\mathcal{R}_{\mathbb{Q}}$$

and define

$$\omega_\pm=\sum_\lambda(-1)^{|\lambda|+\binom{N}{2}}q^{\mp c(\lambda)}q^{w_\pm(\lambda)}P'_\lambda\in\widehat{\mathcal{R}}_{\mathbb{Q}}\quad\text{with}\quad\begin{array}{ll}w_+(\lambda)&=D_N(\lambda)\\ w_-(\lambda)&=D_N(\lambda)+N|\lambda|\end{array}\tag{25}$$

where $c(\lambda)$ is the content of λ. The next Lemma implies that $\omega_\pm$ is the universal Kirby color for (±1)-surgery.

Lemma 9.1. *For any $x\in\widehat{\mathcal{R}}_{\mathbb{Q}}$, we have*

$$\langle\omega_\pm,x\rangle=J_{U_\mp}(x)=\langle r^{\pm1},\xi(x)\rangle\tag{26}$$

where $J_{U_\pm}(x)$ is the Reshetikhin–Turaev invariant of the (±1)-framed unknot colored by x.

Proof. It is enough to check (26) for the basis elements $x=V(\nu)$. We compute

$$\langle P'_\lambda,V(\nu)\rangle=v^{-|\lambda|}\dim_q V(\lambda)\sum_{\mu\subset\lambda}\frac{d_{\lambda,\mu}(q^{-1})}{\dim_q V(\mu)}\,\langle V(\mu),V(\nu)\rangle=v^{-|\lambda|}\dim_q V(\lambda)\sum_{\mu\subset\lambda}d_{\lambda,\mu}(q^{-1})s_\nu(q^{\mu_i+N-i})$$

$$=\dim_q V(\lambda)\sum_{\mu\subset\lambda}d_{\lambda,\mu}(q^{-1})C_\nu F_\nu(q^{\mu_i+N-i})=C_\lambda\delta_{\lambda,\nu}\dim_q V(\lambda)$$

where we used Lemma 5.3, equation (24) and the expansion $s_\nu=(-1)^{|\lambda|+\binom{N}{2}}q^{-D_N(\lambda)}v^{(1-N)|\lambda|}F_\nu+$ lower terms and hence,

$$C_\lambda=(-1)^{|\lambda|+\binom{N}{2}}q^{-D_N(\lambda)}v^{-N|\lambda|}\ .$$

Using this computation it is easy to check that

$$\langle\omega_\pm,V(\nu)\rangle=v^{\mp N|\nu|}q^{\mp c(\nu)}\dim_q V(\nu)=v^{\mp(\nu,\nu+2\rho)}\dim_q V(\nu)=\mathrm{Tr}_q^{V(\nu)}(r^{\pm1})\tag{27}$$

is equal to $J_{U_\mp}(V(\nu))$. □

From the following computation for $V'(\nu)=\frac{V(\nu)}{\dim_q(V(\nu))}$

$$\langle\omega_+\omega_-,V'(\nu)\rangle=\langle\omega_+,V'(\nu)\rangle\langle\omega_-,V'(\nu)\rangle=v^{(\nu,\nu+2\rho)}v^{-(\nu,\nu+2\rho)}=\langle 1,V'(\nu)\rangle$$

we see that ω_+ and ω_- are inverse to each other in the algebra $\widehat{\mathcal{R}}_{\mathbb{Q}}$ isomorphic to $\widehat{\mathcal{Z}}_{\mathbb{Q}}$.

A direct consequence of the above Lemma and the fusion rules is the following result.

Theorem 9.2. *Let $L\cup K=L_1\cup L_2\cdots\cup L_l\cup K$ be an $(l+1)$ component algebraically split 0-framed link such that K is the unknot. We denote by $L_{(K,\pm1)}$ the framed link in S^3 obtained from L by ±1-surgery along K, then for any $p_1,\dots,p_l\in\mathcal{R}$*

$$J_{L\cup K}(p_1,\dots,p_l,\omega^{\pm1})=J_{L_{(K,\pm1)}}(p_1,\dots,p_l).$$

Proof. The proof is given in [15, Thm. 9.4]. □

9.1. **Construction of the unified invariants.** Without loss of generality, we can assume that an integral homology 3-sphere M is obtained by ε-surgery on an ℓ component algebraically split link L, where $\varepsilon \in \{\pm 1\}^\ell$. For $\mathfrak{sl}_N$ Habiro and Le defined a unified invariant of M as follows

$$I^{\mathrm{HL}}(M) := \mathcal{T}'_\varepsilon(J_L(\mathfrak{sl}_N; q)) \in \widehat{\mathbb{Z}[q]} \quad \text{where} \quad \mathcal{T}'_\varepsilon = \bigotimes_{i=1}^{\ell} \mathcal{T}'_{\varepsilon_i}$$

is the $\mathfrak{sl}_N$ twist form. They also proved that $I^{\mathrm{HL}}(M)$ belongs to the Habiro ring [19].

For $\mathfrak{gl}_N$ we define the unified invariant of M similarly

$$I(M) := \mathcal{T}_\varepsilon(J_L(\mathfrak{gl}_N; q)) \in \widehat{\mathbb{Z}[q]} \quad \text{where} \quad \mathcal{T}_\varepsilon = \bigotimes_{i=1}^{\ell} \mathcal{T}_{\varepsilon_i}$$

by using the $\mathfrak{gl}_N$ twist forms.

Theorem 9.3. *For any integral homology* 3*-sphere* M*,*

$$I(M) = J_L(\omega_{\varepsilon_1}, \dots, \omega_{\varepsilon_l}) \in \widehat{\mathbb{Z}[q]}$$

Moreover, its evaluation at any root of unity coincides with the $\mathfrak{sl}_N$ *Witten–Reshetikhin–Turaev invariant of* M*.*

This implies Theorem 1.4 from Introduction.

Proof. By Corollary 6.5, for any algebraically split 0-framed link L we have

$$J_L(\mathfrak{gl}_N; q) = J_L(\mathfrak{sl}_N; q).$$

Hence, as explained in Section 6.3, the $\mathfrak{gl}_N$ and $\mathfrak{sl}_N$ twist forms on J_L do coincide. This implies $I(M) = I^{\mathrm{HL}}(M)$. Since the Habiro–Le invariant is known to belong to the Habiro ring and to evaluate at a root of unity to the Witten-Reshetikhin-Turaev (WRT) one, it remain to show $I(M) = J_L(\omega_{\epsilon_1}, \dots, \omega_{\epsilon_\ell})$. We prove this claim in two steps.

Step 1: Assume $\ell = 1$, then $J_L(\omega_\pm) = I(M)$ by Lemma 9.1.

Step 2: For any $x = x_1 \otimes \cdots \otimes x_\ell \in \mathrm{Inv}(\widehat{\mathcal{U}}_{\mathbb{Z}}^{\hat{\otimes}\ell})$ we define a_k for $k = 0, 1, \dots, \ell$ and $b_k = 1, \dots, \ell$ as follows:

$$a_k = \prod_{i=1}^{k} \langle r^{\epsilon_i}, x_i \rangle \prod_{j=k+1}^{\ell} \langle \omega_{\epsilon_j}, x_j \rangle, \quad b_k = x_k \prod_{i=1}^{k-1} \langle r^{\epsilon_i}, x_i \rangle \prod_{j=k+1}^{\ell} \langle \omega_{\epsilon_j}, x_j \rangle.$$

Then

$$a_{k-1} = \langle \omega_{\epsilon_k}, b_k \rangle \quad \text{and} \quad a_k = \langle r^{\epsilon_k}, b_k \rangle.$$

where we identify $\omega_\pm$ with their image under ξ for simplicity. Since $b_k \in \mathcal{Z}_{\mathbb{Q}}$, we have $a_k = a_{k-1}$ by Step 1 for $k = 1, 2, \dots, \ell$. Hence, we have $a_0 = a_\ell$ which is our claim. □

Theorem 9.3 has striking consequences. Indeed, for any $\lambda \in \mathcal{R}$ let us denote

$$\mathcal{P}_\lambda = \mathrm{Span}_{\mathbb{Z}[q^{\pm 1}]}\{P'_\mu \,|\, \lambda \subset \mu\}, \quad \mathcal{P} = \mathcal{P}_\emptyset \quad \text{and} \quad \widehat{\mathcal{P}} := \varprojlim_{\lambda} \frac{\mathcal{P}}{\mathcal{P}_\lambda}.$$

For any framed link L, the Reshetikhin–Turaev functor provides a $\mathbb{Q}(v)$-multilinear map $J_L : \mathcal{R}_{\mathbb{Q}} \times \cdots \times \mathcal{R}_{\mathbb{Q}} \to \mathbb{Q}(v)$. For any algebraically split 0-framed link L, Theorem 9.3 implies that its restriction to $\mathcal{P}$ provides a $\mathbb{Z}[q^{\pm 1}]$-multilinear map

$$J_L : \mathcal{P} \times \cdots \times \mathcal{P} \to \mathbb{Z}[q, q^{-1}]$$

inducing

$$J_L : \widehat{\mathcal{P}} \times \cdots \times \widehat{\mathcal{P}} \to \widehat{\mathbb{Z}[q]}\ .$$

This leads to a generalization of the famous integrability theorem in [15, Thm. 8.2].

Corollary 9.4. *Given an ℓ component algebraically split* 0*-framed link L, then for all but finitely many partitions λ_i with $1 \leq i \leq \ell$, there exist positive integers $n = n(\lambda_i, N)$, such that*

$$J_L(P'_{\lambda_1}, \ldots, P'_{\lambda_\ell}) \in (q;q)_n \mathbb{Z}[q, q^{-1}] \ .$$

It would be interesting to have a direct proof of Corollary 9.4 without using the theory of unified invariants.

Based on Corollary 9.4 we can give a cyclotomic expansion of the Reshetikhin–Turaev invariant of L as follows:

$$J_L(\lambda_1, \ldots, \lambda_\ell) = v^{\sum_i |\lambda_i|} \sum_{\mu_i \subset \lambda_i} \prod_{j=1}^{\ell} c_{\lambda_j, \mu_j}(q^{-1})\, J_L(P'_{\mu_1}, \ldots, P'_{\mu_\ell}) \tag{28}$$

where the matrix $[c_{\lambda,\mu}(q)]_{\lambda,\mu} := \left[F_\lambda(q^{-\mu_i - N + i})\right]_{\lambda,\mu}$ is the inverse of $[d_{\lambda,\mu}(q)]_{\lambda,\mu}$ by Theorem 10.17 below. This generalizes equation (8.2) in [15].

9.2. **Few direct arguments.** Our proof of the fact that $I(M)$ belongs to the Habiro ring is based on the result that $I^{\mathrm{HL}}(M) \in \widehat{\mathbb{Z}[q]}$ proven in [19] on more than 100 pages. Given the complexity of their argument, we decided to collect here different facts that can be shown without reference to [19].

Theorem 9.5. *Assume $M_\pm$ is obtained by* (± 1)*-surgery on the knot K, then*

$$I(M_\pm) = J_K(\omega_\pm) \ \in \ \widehat{\mathbb{Z}[v]}$$

belongs to the Habiro ring.

Proof. By Theorem 1.3 we know

$$J_K(\mathfrak{gl}_N; q) = \sum_\mu a_\mu(K) \sigma_\mu \quad \text{ with } \quad a_\mu(K) \in \mathbb{Z}[q, q^{-1}]$$

The fact that $I(M_\pm)$ belongs to the Habiro ring easily follows from the claim that $\mathcal{T}_\pm(\sigma_\mu)$ is divisible by $(v;v)_m$ for some m depending on μ and N. Let us prove this claim. By (19) the Hopf pairing with $r^{\pm 1}$ replaces an element $x_i^{k_i}$ with $v^{Q_\pm(k_i)}$ where $Q_\pm$ is a quadratic form. By Lemma 10.27 we can rewrite σ_μ as a linear combination of $\prod_{i=1}^d f_{n_i}(q^{s_i} x_i)$ such that $\sum_i n_i = |\mu|$, $d = N(N+1)/2$ and $s_i \in \mathbb{Z}$. Moreover, each $f_n(q^a x_i)$ is divisible by $f_n(v^a y_i)$ where $y_i^2 = x_i$ and hence belongs to the ideal I_n of $\mathbb{Z}[v^{\pm 1}, y_i^{\pm 1}]$ characterized in Proposition 2.1 of [3]. The result follows now from [3, Theorem 2.2]. The number m we are looking for is $\left\lfloor \frac{|\mu|}{N(N+1)} \right\rfloor$.

□

Combining previous results we obtain an explicit expression for the unified invariant for knot surgeries:

$$I_{M_\pm} = J_K(\omega_\pm) = \mathrm{Tr}_q^{\omega_\pm} J_K(\mathfrak{gl}_N; q) = \sum_\lambda (-1)^{|\lambda| + \binom{N}{2}} q^{\mp c(\lambda)} q^{w_\pm(\lambda)}\, J_K(P'_\lambda) \tag{29}$$

Assuming that $I(M) = J_L(\omega_{\varepsilon_1}, \ldots, \omega_{\varepsilon_l})$ is well defined, its topological invariance can be shown directly as follows. Since $I(M)$ depends only on the isotopy class of L, it remains to check its invariance under Hoste moves (a version of Fenn-Rourke moves between algebraically split links). Without loss of generality, we can assume that the last component is an unknot, then the statement follows from Theorem 9.2.

Assuming $I(M)$ belongs to the Habiro ring, and as such has well defined evaluations at roots of unity [17, Thm. 6.3] we can use the same trick as above to show that for any root of unity ζ

$$\mathrm{ev}_\zeta I(M) = \mathrm{WRT}(M, \zeta).$$

Let us recall that the WRT invariant is obtained from $J_L(\mathfrak{sl}_N; q)$ by taking trace along each component with the Kirby color

$$\Omega_\pm = \frac{\sum_\lambda v^{\pm(\lambda,\lambda+2\rho)} \dim_q V(\lambda)}{\sum_\mu v^{\pm(\mu,2\rho+\mu)} \dim_q^2 V(\mu)} V(\lambda)$$

where the sums are taken over all $\lambda, \mu \in \mathcal{R}^{\mathrm{fin}} = \{\lambda | \dim_\zeta V(\lambda) \neq 0\}$ and v^2 is evaluated to ζ. Hence, we need to show that for any x in the ad-invariant part of the completed ℓth tensor power of $\widehat{\mathcal{U}}_\mathbb{Z}$, we have

$$\mathrm{Tr}_q^{\Omega_\varepsilon}(x) \overset{\zeta}{=} \mathrm{Tr}_q^{\omega_\varepsilon}(x) \qquad \forall x \in \mathrm{Inv}(\widehat{\mathcal{U}}_\mathbb{Z}^{\hat{\otimes}\ell})$$

where $\overset{\zeta}{=}$ means the equality after evaluation $v^2 = \zeta$. We will prove this fact in two steps.

Step 1: Assume $\ell = 1$, in this case $\mathrm{Inv}\,\widehat{\mathcal{U}}_\mathbb{Z} = \widehat{\mathcal{Z}}$ with basis given by $z_\lambda = \xi(V(\lambda))$. Since $\Omega_\pm$ is invariant under Hoste moves, we have

$$\mathrm{Tr}_q^{\Omega_\pm}(z_\nu) = \langle \Omega_\pm, V(\nu) \rangle = \mathrm{ev}_\zeta \left(v^{\mp(\nu,\nu+2\rho)} \dim_q V(\nu) \right)$$

where we interpret the left hand side as a Hopf link with components colored by $\Omega_\pm$ and $V(\nu)$, and the right hand side is the result of the sliding. Comparing this computation with (27), we deduce that at roots of unity the actions of $\Omega_\pm$ and $\omega_\pm$ do coincide on $\xi(\mathcal{R}^{\mathrm{fin}})$, and they vanish on $x \in \widehat{\mathcal{Z}} \setminus \xi(\mathcal{R}^{\mathrm{fin}})$ after evaluation.

Step 2: Define a_k for $k = 0, 1, \ldots, \ell$ and $b_k = 1, \ldots, \ell$ as follows:

$$a_k = \bigotimes_{j=1}^{k} \mathrm{Tr}_q^{\Omega_{\varepsilon_j}} \otimes \bigotimes_{j=k+1}^{\ell} \mathrm{Tr}_q^{\omega_{\varepsilon_j}}(x), \quad b_k = \bigotimes_{j=1}^{k-1} \mathrm{Tr}_q^{\Omega_{\varepsilon_j}} \otimes 1 \otimes \bigotimes_{j=k+1}^{\ell} \mathrm{Tr}_q^{\omega_{\varepsilon_j}}(x).$$

Then

$$a_{k-1} = \mathrm{Tr}_q^{\omega_{\varepsilon_k}}(b_k) \quad \text{and} \quad a_k = \mathrm{Tr}_q^{\Omega_{\varepsilon_k}}(b_k).$$

Since $b_k \in \mathcal{Z}_\mathbb{Q}$, we have $a_k \overset{\zeta}{=} a_{k-1}$ by Step 1 and Lemma 9.1 for $k = 1, 2, \ldots, \ell$. Hence, we have $a_0 \overset{\zeta}{=} a_\ell$ which is our claim.

10. Interpolation polynomials

In this section we summarize the theory of interpolation Macdonald polynomials.

10.1. **One variable case.** Consider the space of polynomials in one variable x over $\mathbb{C}(q)$ with the following bilinear form

$$(x^k, x^m) = q^{-km}.$$

Let us define polynomials $f_m(x), m = 0, 1, \ldots$ by the equation $f_0(x) = 1$ and

$$f_m(x) = (x; q)_m = (1 - x) \cdots (1 - xq^{m-1}) \quad \text{for} \quad m \geq 1. \tag{30}$$

Clearly, $f_m(x)$ is a degree m polynomial with leading term $(-1)^m q^{\frac{m(m-1)}{2}} x^m$, so $\{f_m\}_{m\geq 0}$ form a basis in $\mathbb{Z}_{q,q^{-1}}[x]$. Our next aim is to show that this basis is orthogonal. Observe that $f_m(q^{-k}) = 0$ for $k < m$.

Lemma 10.1. *We have* $(f_m(x), f_k(x)) = \delta_{km} q^{-m} (q; q)_m$

Proof. First, observe that $(g(x), x^k) = g(q^{-k})$ for any polynomial $g(x)$. Therefore for $m > k$ we have $(f_m(x), x^k) = f_m(q^{-k}) = 0$, so $(f_m(x), g(x)) = 0$ for any polynomial $g(x)$ of degree strictly less than m. In particular, $(f_m(x), f_k(x)) = 0$ for $m > k$ and

$$(f_m(x), f_m(x)) = (-1)^m q^{\frac{m(m-1)}{2}}(f_m(x), x^m) = (-1)^m q^{\frac{m(m-1)}{2}} f_m(q^{-m}) =$$
$$(-1)^m q^{\frac{m(m-1)}{2}}(1-q^{-m})\cdots(1-q^{-1}) = q^{-m}(1-q^m)\cdots(1-q).$$

□

Lemma 10.2. *The transition matrix between the monomial basis x^a and the basis $f_b(x)$ has the following form:*

$$x^a = \sum_{b\le a} k_{a,b} f_b(x), \quad k_{a,b} = (-1)^b q^{-ab+\frac{b(b+1)}{2}} \binom{a}{b}_q. \tag{31}$$

Proof. To find the coefficients we compute the pairing $(f_b(x), x^a)$, then using orthogonality we obtain

$$k_{a,b} = \frac{(f_b(x), x^a)}{(f_b(x), f_b(x))} = \frac{f_b(q^{-a})}{(f_b(x), f_b(x))}.$$

For $a \ge b$ from Lemma 10.1 we get

$$(f_b(x), f_b(x)) = q^{-b}(q;q)_b,$$

while

$$f_b(q^{-a}) = (1-q^{-a})\cdots(1-q^{-a+b-1}) = (-1)^b q^{-ab+\frac{b(b-1)}{2}}(1-q^a)\cdots(1-q^{a-b+1})$$
$$= (-1)^b q^{-ab+\frac{b(b-1)}{2}} \frac{(q;q)_a}{(q;q)_{a-b}}.$$

and the equation follows. □

Our next goal is to expand arbitrary polynomial $f(x)$ in the basis $f_m(x)$. This can be done in two different ways. First, we can expand $f(x)$ in the monomial basis and apply (31). Alternatively, we can apply *Newton interpolation method*: if $f(x) = \sum a_m f_m(x)$ then

$$f(q^{-j}) = \sum_{m\ge j} a_m f_m(q^{-j}),$$

which is a triangular system of equations for the unknown coefficients a_m. Thus knowing $f(q^{-j})$ one can at least theoretically reconstruct the coefficients a_m. This can be made explicit by the following:

Lemma 10.3. *We have*

$$f(x) = \sum_{m=0}^{\infty} a_m f_m(x),\ a_m = \frac{1}{(f_m, f_m)} \sum_{j=0}^{m} (-1)^j q^{\frac{j(j-1)}{2}} \binom{m}{j}_q f(q^{-j}). \tag{32}$$

Proof. By q-binomial theorem we have

$$f_m(x) = \sum_{j=0}^{m} (-1)^j q^{\frac{j(j-1)}{2}} \binom{m}{j}_q x^j. \tag{33}$$

Now

$$a_m = \frac{(f, f_m)}{(f_m, f_m)} = \frac{1}{(f_m, f_m)} \sum_{j=0}^{m} (-1)^j q^{\frac{j(j-1)}{2}} \binom{m}{j}_q (f, x^j).$$

Finally, $(f, x^j) = f(q^{-j})$. □

Remark 10.4. Equation (33) can be interpreted as an explicit inverse of the matrix in (31).

One can consider completion $\widehat{\mathbb{Z}_q[x]}$ of the space of polynomials with respect to the basis $f_m(x)$. In this completion, infinite sums $\sum_{m=0}^{\infty} a_m f_m(x)$ are allowed. Newton interpolation method and (32) identify this completion with the space of distributions on the interpolation nodes $1, q^{-1}, \dots$.

We will need the following lemma.

Lemma 10.5. *We have*

$$(x-q^s)(x-q^{s+1})\cdots(x-q^{s+m-1}) = \\ \sum_{j=0}^{m} (-1)^j q^{-jm+\binom{j+1}{2}} \binom{m}{j}_q (1-q^{s+j})\cdots(1-q^{s+m-1}) f_j(x).$$

Proof. We prove it by induction in m. For $m=1$ we get

$$x-q^s = -(1-x)+(1-q^s) = -f_1 + (1-q^s) f_0.$$

For the step of induction we observe

$$(x-q^{s+m})f_j(x) = -q^{-j}(1-q^j x)f_j(x) + (q^{-j}-q^{s+m})f_j(x) \\ = -q^{-j} f_{j+1}(x) + q^{-j}(1-q^{s+m+j}) f_j(x). \tag{34}$$

Using (34), it is easy to identify the coefficient at $f_j(x)$ in

$$(x-q^{s+m}) \sum (-1)^j q^{-jm+\binom{j+1}{2}} \binom{m}{j}_q (1-q^{s+j})\cdots(1-q^{s+m-1}) f_j(x)$$

as

$$-q^{-j+1}(-1)^{j-1} q^{-(j-1)m+\binom{j}{2}} \binom{m}{j-1}_q (1-q^{s+j-1})\cdots(1-q^{s+m-1}) \\ + q^{-j} q^{-jm+\binom{j+1}{2}} \binom{m}{j}_q (1-q^{s+j})\cdots(1-q^{s+m-1})(1-q^{s+m+j}) \\ = -q^{-j(m+1)+\binom{j+1}{2}} (1-q^{s+j})\cdots(1-q^{s+m-1}) \\ \times \left[q^{m-j+1} \binom{m}{j-1}_q (1-q^{s+j-1}) + \binom{m}{j}_q (1-q^{s+m+j}) \right].$$

It remains to notice that

$$q^{m-j+1} \binom{m}{j-1}_q (1-q^{s+j-1}) + \binom{m}{j}_q (1-q^{s+m+j})$$

$$= \left[q^{m-j+1} \binom{m}{j-1}_q + \binom{m}{j}_q \right] - q^{s+m} \left[\binom{m}{j-1}_q + q^j \binom{m}{j}_q \right]$$

$$= \binom{m+1}{j}_q - q^{s+m} \binom{m+1}{j}_q = (1-q^{s+m}) \binom{m+1}{j}_q.$$

$\square$

Remark 10.6. If we set a formal variable $y = q^s$ in Lemma 10.5, then we get the identity

$$(x-y)(x-qy)\cdots(x-yq^{m-1}) = \sum_{j=0}^{m}(-1)^j q^{-jm+\binom{j+1}{2}}\binom{m}{j}_q f_{m-j}(yq^j)f_j(x).$$

This is a q-analogue of the binomial identity

$$(x-y)^m = \sum_{j=0}^{m}(-1)^j\binom{m}{j}(1-y)^{m-j}(1-x)^j.$$

10.2. **Multi-variable case: polynomials.** Let us generalize the above results to the case of N variables. The pairing has the form

$$(x_1^{a_1}\cdots x_N^{a_N}, x_1^{b_1}\cdots x_N^{b_N}) = q^{-\sum a_i b_i} = (x_1^{a_1}, x_1^{b_1})\cdots(x_N^{a_N}, x_N^{b_N}).$$

Note that for $\mathbf{x} = (x_1, \ldots, x_N)$

$$(g(\mathbf{x}), x_1^{b_1}\cdots x_N^{b_N}) = g(q^{-b_1}, \ldots, q^{-b_N}).$$

Consider the products

$$f_{k_1,\ldots,k_N}(\mathbf{x}) = f_{k_1}(x_1)\cdots f_{k_N}(x_N).$$

Since $f_k(x)$ give a basis in $\mathbb{C}(q)[x]$, the polynomials $f_{k_1,\ldots,k_N}$ give a basis in $\mathbb{C}(q)[x_1, \ldots, x_N]$. Clearly,

$$(f_{k_1,\ldots,k_N}, x_1^{b_1}\cdots x_N^{b_N}) = 0 \text{ unless } b_i \geq k_i \text{ for all } i.$$

Lemma 10.7. *We have $(f_{k_1,\ldots,k_N}, f_{m_1,\ldots,m_N}) = 0$ unless $k_i = m_i$ for all i.*

Proof. Suppose that $k_i > m_i$ for some i. Since $f_{m_1,\ldots,m_N}$ contains only monomials of the form $x_1^{b_1}\cdots x_N^{b_N}$ with $b_i \leq m_i$, we have $(f_{k_1,\ldots,k_N}, x_1^{b_1}\cdots x_N^{b_N}) = 0$ for all such monomials and hence $(f_{k_1,\ldots,k_N}, f_{m_1,\ldots,m_N}) = 0$. □

Next, we would like to describe the basis in symmetric polynomials. It will be labeled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N)$ with at most N parts. We define

$$F_\lambda(\mathbf{x}) = \frac{\det(f_{\lambda_i+N-i}(x_j))}{\prod_{i<j}(x_i - x_j)}. \tag{35}$$

Clearly, the numerator in (35) is antisymmetric in x_i, so it is divisible by $\prod_{i<j}(x_i - x_j)$ and the ratio is a symmetric function. It is easy to see that $F_\lambda(\mathbf{x})$ is a non-homogeneous polynomial of degree $|\lambda|$, and the top degree component equals $(-1)^{|\lambda|+\binom{N}{2}}q^{D_N(\lambda)}s_\lambda$ where s_λ is the Schur function and $D_N(\lambda)$ is defined by (6). The function $F_\lambda(\mathbf{x})$ is known as a special case of a factorial Schur function [26, 27, 28], it is also a specialization of nonsymmetric Macdonald polynomials described below.

Lemma 10.8. *Suppose that $b_1 > \ldots > b_N$. Then $F_\lambda(q^{-b_1}, \ldots, q^{-b_N}) = 0$ unless $b_i \geq \lambda_i + N - i$ for all i.*

Proof. Suppose that $b_j < \lambda_j + N - j$ for some j, then for all $i \leq j$ and $\ell > j$ one has $\lambda_i + N - i \geq \lambda_j + N - j > b_j \geq b_\ell$, so $f_{\lambda_i+N-i}(q^{-b_\ell}) = 0$. This implies $\det[f_{\lambda_i+N-i}(q^{-b_\ell})]_{i,\ell=1}^N = 0$. On the other hand, since $b_i \neq b_j$ the denominator $\prod_{i<j}(q^{-b_i} - q^{-b_j})$ does not vanish. □

Corollary 10.9. *If μ is another partition then we can define $b_i = \mu_i + N - i$, and conclude that $F_\lambda(q^{-\mu_i-N+i}) = 0$ unless $\mu_i \geq \lambda_i$ for all i, that is, partition μ contains λ.*

Example 10.10. Suppose that $\lambda = (1)$, then $F_{(1)}$ is a symmetric function of degree 1 with leading term $(-1)^{1+\binom{N}{2}} q^{D_N(1)} s_{(1)} = q^{D_N(1)} \sum x_i$. We have $D_N(1) = N - 1 + \binom{N}{3}$, so $F_{(1)}(x_1, \ldots, x_N) = (-1)^{1+\binom{N}{2}} q^{N-1+\binom{N}{3}} \sum x_i + c$. To find the constant c, we observe that by Corollary 10.9 we get $F_{(1)}(q^{-N+1}, q^{-N+2}, \ldots, 1) = 0$, so

$$c = (-1)^{\binom{N}{2}} q^{N-1+\binom{N}{3}} (q^{-N+1} + q^{-N+2} + \ldots + 1) = (-1)^{\binom{N}{2}} q^{\binom{N}{3}} [N]_q.$$

Lemma 10.11. *We have*

$$F_\lambda(q^{-\lambda_i - N + i}) = (-1)^{\binom{N}{2}} q^{n(\lambda)+\binom{N}{3}} \prod_{\square \in \lambda} (1 - q^{-h(\square)}),$$

where $h(\square)$ is the hook length of a box $\square$ in the Young diagram corresponding to λ.

Proof. Since the sequence $\lambda_i + N - i$ is strictly decreasing, we have $f_{\lambda_j+N-j}(q^{-\lambda_i-N+i}) = 0$ for $j > i$ and

$$f_{\lambda_i+N-i}(q^{-\lambda_i-N+i}) = \{\lambda_i + N - i\}_{q^{-1}}!$$

and

$$\det(f_{\lambda_j+N-j}(q^{-\lambda_i-N+i})) = \prod_i \{\lambda_i + N - i\}_{q^{-1}}!\ .$$

On the other hand,

$$\prod_{i<j} (q^{-\lambda_i-N+i} - q^{-\lambda_j-N+j}) = (-1)^{\binom{N}{2}} q^{-\sum(\lambda_j+N-j)(j-1)} \prod_{i<j} (1 - q^{-\lambda_i+i+\lambda_j-j}).$$

and the statement follows now from formula (5) and the identity

$$\sum (\lambda_j + N - j)(j - 1) = n(\lambda) + \binom{N}{3}.$$

$\square$

Example 10.12. For arbitrary N and $\lambda = (1)$ we computed in Example 10.10 that

$$F_{(1)} = (-1)^{1+\binom{N}{2}} q^{\binom{N}{3}} (q^{N-1}(x_1 + \ldots + x_N) - [N]_q).$$

Hence,

$$F_{(1)}(q^{-N}, q^{-N+2}, \ldots, 1) = q^{\binom{N}{3}} (q^{N-1}(q^{-N} + q^{-N+2} + \ldots + 1) - [N]_q) =$$

$$(-1)^{1+\binom{N}{2}} q^{\binom{N}{3}} (q^{-1} - 1) = (-1)^{\binom{N}{2}} q^{\binom{N}{3}} (1 - q^{-1}).$$

We summarize the above results in the following proposition:

Proposition 10.13. [29] *There exists a unique collection of nonhomogeneous symmetric polynomials $F_\lambda(x_1, \ldots, x_N)$ with the following properties:*

- *$F_\lambda(x_1, \ldots, x_N)$ has degree $|\lambda|$.*
- *$F_\lambda(q^{-\mu_i-N+i}) = 0$ for all partitions μ not containing λ.*
- *$F_\lambda(q^{-\lambda_i-N+i}) = (-1)^{\binom{N}{2}} q^{n(\lambda)+\binom{N}{3}} \prod_{\square\in\lambda}(1 - q^{-h(\square)})$.*

We will denote the value $F_\lambda(q^{-\lambda_i-N+i}) = (-1)^{\binom{N}{2}} q^{n(\lambda)+\binom{N}{3}} \prod_{\square\in\lambda}(1 - q^{-h(\square)})$ by $c_{\lambda,\lambda}$.

Lemma 10.14. *Suppose that q is a root of unity. Then $c_{\lambda,\lambda}$ vanishes for all but finitely many partitions λ.*

Proof. Observe that $\prod_{\square\in\lambda}(1-q^{-h(\square)})$ is divisible by $\prod_i[\lambda_i-\lambda_{i+1}]_q!$ and

$$\sum_{i=1}^{N} i(\lambda_i-\lambda_{i+1})=|\lambda|.$$

This means that for some i we must have

$$i(\lambda_i-\lambda_{i+1})\ge\frac{|\lambda|}{N},\ \lambda_i-\lambda_{i+1}\ge\frac{|\lambda|}{iN}\ge\frac{|\lambda|}{N^2},$$

and $c_{\lambda,\lambda}$ is divisible by $(1-q)\cdots(1-q^{\lfloor\frac{|\lambda|}{N^2}\rfloor})$. If $q^s=1$ then it vanishes for $|\lambda|\ge sN^2$. $\square$

Remark 10.15. A partition is called an *s-core* if none of its hook lengths is divisible by s. The s-core partitions play an important role in representation theory of symmetric groups in finite characteristic, and of Hecke algebras at roots of unity [21]. If $q^s=1$ then clearly $c_{\lambda,\lambda}(q)\neq 0$ if and only if λ is an s-core. Although there are infinitely many s-cores, Lemma 10.14 shows that there are finitely many s-cores with at most N rows.

For example, for $s=2$ the 2-cores are "staircase partitions" $\lambda=(k,k-1,\ldots,1)$, and the maximal 2-core with at most N rows has size $N+(N-1)+\ldots+1=\binom{N+1}{2}$.

10.3. **Multi-variable case: interpolation.** One can use the polynomials F_λ to solve the following interpolation problem.

Problem 10.16. *Find a symmetric function $f=\sum a_\lambda F_\lambda$ given its values $f(q^{-\mu_i-N+i})$ for all μ.*

We have

$$f(q^{-\mu_i-N+i})=\sum a_\lambda F_\lambda(q^{-\mu_i-N+i})$$

This is a linear system on a_λ with the triangular matrix

$$\mathsf{C}=\left[c_{\lambda,\mu}\right]_{\lambda,\mu},\ c_{\lambda,\mu}(q):=F_\lambda(q^{-\mu_i-N+i}) \tag{36}$$

It is clear from Proposition 10.13 that to find a_λ for a given λ it is sufficient to know all coefficients $c_{\mu,\nu}$ for $\mu\subset\nu\subset\lambda$.

In [29] Okounkov computed the inverse matrix $D=\mathsf{C}^{-1}$ which allows one to explicitly compute the coefficients a_λ.

Theorem 10.17. [29] *Define $c^*_{\lambda,\mu}(q)=c_{\lambda,\mu}(q^{-1})$ and $\mathrm{cont}(\lambda)=n(\lambda)-n(\lambda')$. Then*

$$D=\left[d_{\lambda,\mu}\right]_{\lambda,\mu},\ d_{\lambda,\mu}=(-1)^{|\mu|-|\lambda|}q^{\mathrm{cont}(\lambda)-\mathrm{cont}(\mu)}\frac{c^*_{\lambda,\mu}}{c_{\mu,\mu}c^*_{\lambda,\lambda}}$$

and

$$a_\mu=\sum_{\lambda\subset\mu}d_{\lambda,\mu}f(q^{-\lambda_i-N+i})=\frac{1}{c_{\mu,\mu}}\sum_{\lambda\subset\mu}(-1)^{|\mu|-|\lambda|}q^{\mathrm{cont}(\lambda)-\mathrm{cont}(\mu)}\frac{c^*_{\lambda,\mu}}{c^*_{\lambda,\lambda}}f(q^{-\lambda_i-N+i}).$$

Example 10.18. If $\lambda=\mu$ then clearly $d_{\lambda,\mu}=\frac{1}{c_{\lambda,\lambda}}$.

Example 10.19. We have $F_{(\emptyset)}=(-1)^{\binom{N}{2}}q^{\binom{N}{3}}$, so

$$c_{(\emptyset),(\emptyset)}=c_{(\emptyset),(1)}=(-1)^{\binom{N}{2}}q^{\binom{N}{3}},\quad c^*_{(\emptyset),(\emptyset)}=c^*_{(\emptyset),(1)}=(-1)^{\binom{N}{2}}q^{-\binom{N}{3}}.$$

Since

$$c_{(1),(1)}=(-1)^{\binom{N}{2}}q^{\binom{N}{3}}(1-q^{-1})=(-1)^{\binom{N}{2}+1}q^{\binom{N}{3}-1}(1-q),$$

we get

$$d_{(\emptyset),(1)}=\frac{(-1)^{\binom{N}{2}}q^{-\binom{N}{3}+1}}{(1-q)}.$$

So the first two terms of interpolation series have the following form:

$$f(x_1,\ldots,x_N)=(-1)^{\binom{N}{2}}q^{-\binom{N}{3}}f(q^{1-N},q^{2-N},\ldots,1)F_{(\emptyset)}(\mathbf{x})+$$
$$\frac{(-1)^{\binom{N}{2}+1}q^{-\binom{N}{3}+1}}{1-q}\left[-f(q^{1-N},q^{2-N},\ldots,1)+f(q^{-N},q^{2-N},\ldots,1)\right]F_{(1)}(\mathbf{x})+\ldots$$

Example 10.20. For $N=1$ and $a\geq b$ we have

$$c_{(b),(a)}=f_b(q^{-a})=(1-q^{-a})\cdots(1-q^{-a+b-1})$$

hence

$$c^*_{(b),(a)}=(1-q^{a})\cdots(1-q^{a-b+1})$$

Now

$$\frac{c^*_{(b),(a)}}{c^*_{(b),(b)}}=\frac{(1-q^{a})\cdots(1-q^{a-b+1})}{(1-q^b)\cdots(1-q)}=\binom{a}{b}_q,$$

and

$$d_{(b),(a)}=(-1)^{a-b}q^{\frac{b(b-1)}{2}-\frac{a(a-1)}{2}}\frac{c^*_{(b),(a)}}{c_{(a),(a)}c^*_{(b),(b)}}=\frac{(-1)^{a-b}}{c_{(a),(a)}}q^{\frac{b(b-1)}{2}-\frac{a(a-1)}{2}}\binom{a}{b}_q,$$

which matches (32).

Example 10.21. *Let $N=2$, $\lambda=(1)$ and $\mu=(3,2)$. We have $F_\lambda=q(x_1+x_2)-(1+q)$, so*

$$c_{\lambda,\mu}=F_\lambda(q^{-4},q^{-2})=(-q-1+q^{-1}+q^{-3}),\ c^*_{\lambda,\mu}=q^3+q-1-q^{-1},$$

and using Lemma 10.11

$$c_{\lambda,\lambda}=-(1-q^{-1}),\ c^*_{\lambda,\lambda}=-(1-q),$$

$$c_{\mu,\mu}=-q^2(1-q^{-1})^2(1-q^{-2})(1-q^{-3})(1-q^{-4})=q^{-9}(1-q)^2(1-q^2)(1-q^3)(1-q^4).$$

Now

$$d_{\lambda,\mu}=q^{-2}\frac{c^*_{\lambda,\mu}}{c_{\mu,\mu}c^*_{\lambda,\lambda}}=-q^6\frac{q^4+q^2-q-1}{(1-q)^3(1-q^2)(1-q^3)(1-q^4)}.$$

10.4. **Hopf pairing.** We have a symmetric bilinear form $(\cdot,\cdot)$ on $\mathbb{Z}[x_1,\ldots,x_N]^{S_N}$ defined by its values on Schur polynomials

$$(s_\lambda,s_\mu)=s_\lambda(q^{-\mu_1-N+1},\ldots,q^{-\mu_N})s_\mu(q^{-N+1},\ldots,1).$$

It is closely related to the Hopf pairing $\langle\cdot,\cdot\rangle$ for $\mathcal{R}=\mathit{Rep}(\mathcal{U})$ defined in Section 5.2. Note that

$$(f,s_\mu)=f(q^{-\mu_1-N+1},\ldots,q^{-\mu_N})s_\mu(q^{-N+1},\ldots,1).$$

for any symmetric function f.

Proposition 10.22. *We have*

$$(F_\lambda,F_\nu)=\delta_{\lambda,\nu}q^{-|\lambda|+2\binom{N}{3}}\prod_{\square\in\lambda}(1-q^{N+c(\square)}),\tag{37}$$

so the Hopf pairing is diagonal in the basis $\{F_\lambda\}_\lambda$.

Proof. We have

$$(F_\lambda,s_\mu)=F_\lambda(q^{-\mu_1-N+1},\ldots,q^{-\mu_N})s_\mu(q^{-N+1},\ldots,1)=0$$

unless $\lambda\subset\mu$. On the other hand, F_ν can be expanded in s_μ for $\mu\preceq\nu$, so (F_λ,F_ν) vanishes unless there exists $\mu\preceq\nu$ such that $\lambda\subset\mu$, in particular, $\lambda\preceq\nu$.

Since the Hopf pairing is symmetric, (F_λ, F_ν) vanishes unless $\lambda \preceq \nu$ and $\nu \preceq \lambda$, so $\lambda = \nu$. Finally,

$$(F_\lambda, F_\lambda) = (-1)^{|\lambda|+\binom{N}{2}} q^{D_N(\lambda)} (F_\lambda, s_\lambda) = (-1)^{|\lambda|+\binom{N}{2}} q^{D_N(\lambda)} F_\lambda(q^{-\lambda_1-N+1},\ldots,q^{-\lambda_N}) s_\lambda(q^{-N+1},\ldots,1).$$

Now

$$F_\lambda(q^{-\lambda_1-N+1},\ldots,q^{-\lambda_N}) = (-1)^{\binom{N}{2}} q^{n(\lambda)+\binom{N}{3}} \prod_{\square\in\lambda}(1-q^{-h(\square)})$$

while

$$s_\lambda(q^{-N+1},\ldots,1) = q^{-n(\lambda)} \prod_{\square\in\lambda} \frac{(1-q^{-N-c(\square)})}{(1-q^{-h(\square)})}.$$

hence

$$F_\lambda(q^{-\lambda_1-N+1},\ldots,q^{-\lambda_N}) s_\lambda(q^{-N+1},\ldots,1) = (-1)^{\binom{N}{2}} q^{\binom{N}{3}} \prod_{\square\in\lambda}(1-q^{-N-c(\square)}) =$$

$$(-1)^{|\lambda|+\binom{N}{2}} q^{-N|\lambda|-c(\lambda)+\binom{N}{3}} (1-q^{N+c(\square)}).$$

On the other hand, $D_N(\lambda) = c(\lambda) + (N-1)|\lambda| + \binom{N}{3}$. □

This provides us with a different perspective for the interpolation problem. Suppose that we have a Schur expansion for F_λ:

$$F_\lambda = \sum_{\mu\preceq\lambda} b_{\lambda,\mu} s_\mu.$$

Then for an arbitrary symmetric function $f(x_1,\ldots,x_N)$ we can write

$$f = \sum_\lambda \frac{(f, F_\lambda)}{(F_\lambda, F_\lambda)} F_\lambda = \sum_\lambda \sum_{\mu\preceq\lambda} b_{\lambda,\mu} \frac{(f, s_\mu)}{(F_\lambda, F_\lambda)} F_\lambda = \sum_\lambda \sum_{\mu\preceq\lambda} \frac{b_{\lambda,\mu} s_\mu(q^{-N-i})}{(F_\lambda, F_\lambda)} f(q^{-\mu_i-N+i}) F_\lambda,$$

and the interpolation coefficient is equal to

$$d_{\lambda,\mu} = \frac{b_{\lambda,\mu} s_\mu(q^{-N+i})}{(F_\lambda, F_\lambda)}. \tag{38}$$

Example 10.23. For $N = 2$ and $\lambda = (3,2)$ we have

$$F_{(3,2)} = q^2(1-x_1)(1-qx_1)(1-x_2)(1-qx_2)(q^3(x_1+x_2)-(1+q)) =$$
$$q^7 s_{3,2} - q^6(1+q)s_{3,1} - q^4(1+q+q^2+q^3)s_{2,2} + q^6 s_{3,0} + q^3(1+q+q^2+q^3)(1+q)s_{2,1} -$$
$$q^3(1+q+q^2+q^3)s_{2,0} - q^2(1+q+q^2+q^3)(1+q)s_{1,1} + (q^5+q^4+2q^3+q^2)s_{1,0} - (q^3+q^2).$$

Also

$$(F_{3,2}, F_{3,2}) = -q^{-5}(1-q^4)(1-q^3)(1-q^2)^2(1-q)$$

Therefore the interpolation coefficient for $\lambda = (3,2)$ and $\mu = (1,0)$ equals

$$d_{(3,2),(1,0)} = (q^5+q^4+2q^3+q^2)\frac{s_{1,0}(q^{-1},1)}{(F_{3,2},F_{3,2})} =$$

$$-\frac{(q^5+q^4+2q^3+q^2)(1+q^{-1})}{q^{-5}(1-q^4)(1-q^3)(1-q^2)^2(1-q)} = -\frac{q^6(q^4+q^2-q-1)}{(1-q^4)(1-q^3)(1-q^2)(1-q)^3}.$$

This agrees with Example 10.21.

10.5. **Divisibility.** Given a polynomial $f(x)$, define

$$\partial_{xy}(f) := \frac{f(x)-f(y)}{x-y}.$$

Observe that

$$\partial_{xy}(fg) = \frac{f(x)-f(y)}{x-y}g(x) + f(y)\frac{g(x)-g(y)}{x-y} = \partial_{xy}f \cdot g(x) + f(y)\cdot \partial_{xy}(g).$$

More generally, we have

$$\begin{aligned}\partial_{xy}(f_1\cdots f_k) = \partial_{xy}(f_1)f_2(x)\cdots f_k(x) + f_1(y)\partial_{xy}(f_2)f_3(x)\cdots f_k(x) + \dots \\ + f_1(y)f_2(y)\cdots \partial_{xy}(f_k).\end{aligned} \tag{39}$$

Example 10.24. For $f_n(x) = (1-x)\cdots(1-q^{n-1}x)$, note that $\partial_{xy}(1-q^ix) = -q^i$, so we get

$$\begin{aligned}F_{n,0}(x,y) = \partial_{x,y}f_{n+1}(x) &= \sum_{i=0}^{n}(1-y)\cdots(1-q^{i-1}y)[\partial_{x,y}(1-q^ix)](1-q^{i+1}x)\cdots(1-q^nx) \\ &= \sum_{i=0}^{n} f_i(y)\cdot(-q^i)f_{n-i}(q^{i+1}x).\end{aligned}$$

Example 10.25. For example,

$$F_{1,0}(x,y) = q(x+y)-(1+q) = q(y-1)+(qx-1) = -[qf_1(y)+f_1(qx)].$$

Similarly,

$$\begin{aligned}F_{2,0}(x,y) &= -q^3(x^2+xy+y^2)+(q+q^2+q^3)(x+y)-(1+q+q^2) \\ &= -[(1-qx)(1-q^2x)+q(1-q^2x)(1-y)+q^2(1-y)(1-qy)] \\ &= -[f_2(qx)+qf_1(x)f_2(y)+q^2f_2(y)].\end{aligned}$$

Corollary 10.26. *For all integers a and b the value $F_{n,0}(q^a,q^b)$ is divisible by $\left(\lfloor\frac{n}{2}\rfloor\right)_q!$*

Proof. Let $k = \lfloor\frac{n}{2}\rfloor$. In the above equation either $i \geq k$ or $n-i\geq k$, so each term in the sum is either divisible by $f_k(q^{i+1+a})$ or by $f_k(q^b)$, so by q-binomial theorem it is divisible by $(k)_q!$ □

More generally, let $\partial_i = \partial_{x_i,x_{i+1}}$ then it is well known that ∂_i satisfy braid relations, so one can define ∂_w for any permutation w. Furthermore,

$$F_\lambda(x_1,\dots,x_N) = \partial_{w_0}[f_{\lambda_1+N-1}(x_1)\cdots f_{\lambda_N}(x_N)],$$

where $w_0 = (N\ N-1\ \dots 1)$ is the longest element in S_N.

Lemma 10.27. *For all λ one can write $F_\lambda(x_1,\dots,x_N)$ as the sum where each term has the form*

$$f_{j_1}(q^{s_1}x_{m_1})\cdots f_{j_d}(q^{s_d}x_{m_d}),\ \text{where } j_1+\dots+j_d = |\lambda| \text{ and } d = \binom{N+1}{2}. \tag{40}$$

Here the indices m_i might repeat arbitrarily.

Proof. From (39) and Example 10.24 it is clear that ∂_i applied to a product (40) with ℓ factors produces a sum of similar products with $\ell+1$ factors. We start from a product of

N factors, and ∂_w is a composition of $\binom{N}{2}$ operators ∂_i, so the terms in the resulting sum have $N+\binom{N}{2}=\binom{N+1}{2}$ factors. Also, each ∂_i decreases the degree by 1, so

$$j_1+\ldots+j_d=\sum(\lambda_i+N-i)-\binom{N}{2}=|\lambda|.$$

□

Remark 10.28. A more careful analysis of this proof leads to a combinatorial formula for F_λ where the terms are labeled by semistandard tableaux, but we do not need it here. This is a q-analogue of the expansion of a Schur function in the monomial basis.

Lemma 10.29. *For any sequence of integers $a_1,\ldots,a_N$ the value $F_\lambda(q^{a_1},\ldots,q^{a_N})$ is divisible by $(k)_q!$ where $k=\left\lfloor\frac{|\lambda|}{\binom{N+1}{2}}\right\rfloor$.*

Proof. In each term (40) there are $d=\binom{N+1}{2}$ indices $j_1,\cdots,j_d$ which add up to $|\lambda|$, so at least one of these indices is greater than $|\lambda|/d$. It remains to notice that $f_j(q^a)$ is divisible by $(q)_j!$ for all integers a. □

The following lemma gives a rough description of the expansion

$$F_\lambda(x_1,\ldots,x_N)=\sum_{m_1,\ldots,m_N} b_{m_1,\ldots,m_k} f_{m_1}(x_1)\cdots f_{m_N}(x_N). \tag{41}$$

of the symmetric interpolation polynomial F_λ in terms of nonsymmetric ones.

Lemma 10.30. *Given k, for sufficiently large $|\lambda|$ for all terms of the expansion* (41) *either the coefficient $b_{m_1,\ldots,m_k}$ is divisible by $(k)_q!$ or there exists $m_i\ge k$ for some $1\le i\le N$.*

Proof. We follow the same logic as in Lemma 10.29. For $|\lambda|>2k\binom{N+1}{2}$ every term (40) is divisible by $f_{2k}(q^s x_i)$ for some s and i. By Lemma 10.5 this can be further decomposed into terms which are divisible by $(j)_q!f_{2k-j}(x_i)$, and either j or $2k-j$ is greater than or equal to k. Overall, we presented

$$F_\lambda(x_1,\ldots,x_N)=A(k)_q!+\sum B_i f_k(x_i)$$

for some polynomials A and B_i. It remains to notice that the polynomial $B_i f_k(x_i)$ can be presented as the sum of $f_{m_1}(x_1)\cdots f_{m_N}(x_N)$ where $m_i\ge k$. □

11. Stability of interpolation and the case $N=2$

11.1. **Stability of interpolation matrices.** In this section study the dependence of the interpolation polynomials on N.

As above, if partition λ has less than N parts we can complete it with zeroes. We denote by $F_{\lambda;N}(x_1,\ldots,x_N)$ the corresponding polynomial in N variables.

Lemma 11.1. *Let λ be a partition with at most N parts. Then*

$$F_{\lambda;N}(x_1,\ldots,x_{N-1},1)=\begin{cases}(-1)^{N-1}q^{\binom{N-1}{2}}F_{\lambda;N-1}(qx_1,\ldots,qx_{N-1}) & \text{if } \lambda_N=0\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let μ be a partition with at most $N-1$ parts. Then by Proposition 10.13

$$F_{\lambda;N}(q^{-\mu_1-N+1},\ldots,q^{-\mu_{N-1}-1},1)=0$$

unless μ contains λ. If $\lambda_N>0$ then this never happens and $F_{\lambda;N}(x_1,\ldots,x_{N-1},1)=0$. If $\lambda_N=0$ we write $L(x_1,\ldots,x_{N-1})=F_{\lambda;N-1}(qx_1,\ldots,qx_{N-1})$. We have

$$L(q^{-\mu_1-N+1},\ldots,q^{-\mu_{N-1}-1})=F_{\lambda;N-1}(q^{-\mu_1-(N-1)+1},\ldots,q^{-\mu_{N-1}})$$

which vanishes unless μ contains λ, so by Proposition 10.13 $F_{\lambda;N}(x_1,\ldots,x_{N-1},1)$ is proportional to $L(x_1,\ldots,x_{N-1})$. Finally, at $\mu=\lambda$ we can use Lemma 10.11 to determine the coefficient. $\square$

Remark 11.2. We can also prove the lemma using the explicit determinantal formula. Indeed, $f_{\lambda_i+N-i}(1)=0$ unless $f_{\lambda_i+N-i}=0$ which is equivalent to $i=N$ and $\lambda_N=0$. Therefore for $\lambda_N\neq 0$ the last row in the matrix $f_{\lambda_i+N-i}(x_j)$ vanishes (where $x_N=1$), and $F_{\lambda;N}(x_1,\ldots,x_{N-1},1)=0$. For $\lambda_N=0$ we have

$$F_{\lambda;N}(x_1,\ldots,x_{N-1},1)=\frac{\det\left[f_{\lambda_i+N-i}(x_j)\right]_{i,j=1}^{N-1}}{\prod_{i<j\le N-1}(x_i-x_j)\prod_{i\le N-1}(x_i-1)}.$$

Note that $f_{k+1}(x)=(1-x)f_k(qx)$, so

$$f_{\lambda_i+N-i}(x_j)=(1-x_j)f_{\lambda_i+(N-1)-i}(qx_j)$$

Therefore

$$F_{\lambda;N}(x_1,\ldots,x_{N-1},1)=\frac{\prod_{i=1}^{n}(1-x_i)\det\left[f_{\lambda_i+(N-1)-i}(qx_j)\right]_{i,j=1}^{N-1}}{\prod_{i<j\le N-1}(x_i-x_j)\prod_{i\le N-1}(x_i-1)}=$$

$$(-1)^{N-1}q^{\binom{N-1}{2}}F_{\lambda;N-1}(qx_1,\ldots,qx_{N-1}).$$

Corollary 11.3. Let $c^{(N)}_{\lambda,\mu}$ be the coefficient defined in previous section for symmetric functions in N variables. Then the expressions

$$(-1)^{\binom{N}{2}}q^{-\binom{N}{3}}c^{(N)}_{\lambda,\mu},\ (-1)^{\binom{N}{2}}q^{\binom{N}{3}}c^{(N)*}_{\lambda,\mu},\ (-1)^{\binom{N}{2}}q^{\binom{N}{3}}d^{(N)}_{\lambda,\mu}$$

are independent of N (provided that λ and μ have at most N parts).

Example 11.4. For one-row partitions $\lambda=(b)$ and $\mu=(a)$ the interpolation coefficients are given by the formulas in Example 10.20 up to a monomial factor.

The above results allow us to describe Schur expansion of interpolation polynomials:

Proposition 11.5. We have

$$F^{(N)}_{\lambda}=(-1)^{\binom{N}{2}}q^{\binom{N}{3}}\sum_{\mu\subset\lambda}\overline{b_{\lambda,\mu}}A^{|\mu|}\prod_{\square\in\lambda\setminus\mu}(1-Aq^{c(\square)})s^{(N)}_{\lambda} \tag{42}$$

where $A=q^N$ and the coefficients

$$\overline{b_{\lambda,\mu}}=(-1)^{\binom{N}{2}}q^{\binom{N}{3}}d^{(N)}_{\lambda,\mu}q^{-|\lambda|-|\mu|-n(\mu)}\prod_{\square\in\mu}(1-q^{h(\square)})$$

do not depend on N.

Proof. It follows from (38) that

$$F_{\lambda}=\sum b_{\lambda,\mu}s_{\mu},\ b_{\lambda,\mu}=\frac{d_{\lambda,\mu}(F_{\lambda},F_{\lambda})}{s_{\mu}(q^{-N+i})}.$$

Since $d_{\lambda,\mu}$ vanishes unless $\mu\subset\lambda$, the same is true for $b_{\lambda,\mu}$. By Corollary 11.3 the product $\overline{d_{\lambda,\mu}}=(-1)^{\binom{N}{2}}q^{\binom{N}{3}}d_{\lambda,\mu}$ does not depend on N, and we can use the formulas

$$(F_{\lambda},F_{\lambda})=q^{-|\lambda|+2\binom{N}{3}}\prod_{\square\in\lambda}(1-Aq^{c(\square)}),$$

$$s_{\mu}(q^{-N+i})=q^{n(\mu)-(N-1)|\mu|}\prod_{\square\in\mu}\frac{(1-Aq^{c(\square)})}{(1-q^{h(\square)})}.$$

to write

$$b_{\lambda,\mu}=(-1)^{\binom{N}{2}}q^{-\binom{N}{3}}\overline{d_{\lambda,\mu}}\cdot q^{-|\lambda|+2\binom{N}{3}-n(\mu)+N|\mu|-|\mu|}\prod_{\square\in\lambda}(1-Aq^{c(\square)})\prod_{\square\in\mu}\frac{(1-q^{h(\square)})}{(1-Aq^{c(\square)})}.$$

The result follows. $\square$

Corollary 11.6. The one-row interpolation polynomials have the following Schur expansion:

$$F^{(N)}_{(m)}=(-1)^{\binom{N}{2}}q^{\binom{N}{3}}\sum_{\mu\subset\lambda}(-1)^j q^{\frac{j(j-3)}{2}}A^j\frac{(1-Aq^j)\cdots(1-Aq^{m-1})}{(1-q)\cdots(1-q^{m-j})}h^{(N)}_j=$$

$$(-1)^{\binom{N}{2}}q^{\binom{N}{3}}\sum_{\mu\subset\lambda}(-1)^j q^{\frac{j(j-3+2N)}{2}}\binom{N+m-1}{m-j}_q h^{(N)}_j.$$

Here $h^{(N)}_j=s^{(N)}_{(j)}$ are complete symmetric functions in N variables.

Proof. For $\lambda=(m)$ and $N=1$ we have

$$f_m(x)=\sum_{j=0}^{m}(-1)^j q^{\frac{j(j-1)}{2}}\binom{m}{j}_q x^j.$$

By writing $A=q$ and $\mu=(j)$ we get

$$A^{|\mu|}\prod_{\square\in\lambda\setminus\mu}(1-Aq^{c(\square)})=q^j(1-q^{j+1})\cdots(1-q^m),$$

so

$$\overline{b_{(m),(j)}}=(-1)^j\frac{q^{\frac{j(j-3)}{2}}}{(1-q)\cdots(1-q^{m-j})}.$$

$\square$

Remark 11.7. The HOMFLY-PT limit of interpolation polynomials in Proposition 11.5 appears to be related to the results and conjectures in [22], it would be interesting to find a precise connection.

11.2. **Adding a column.** It is well known that in symmetric functions in N variables one has the identity

$$s_{\lambda+1^N}=x_1\cdots x_N\cdot s_\lambda.$$

Here $\lambda+1^N=(\lambda_1+1,\ldots,\lambda_N+1)$ and the corresponding Young diagram is obtained from the Young diagram for λ by adding a vertical column.

For interpolation polynomials we have two different generalizations of this identity: the first relates $F_{\lambda+1^N}$ to F_λ and the second describe the action of the multiplication by $x_1\cdots x_N$.

Proposition 11.8. *We have* $F_{\lambda+1^N}(x_1,\ldots,x_N)=q^{\binom{N}{2}}\prod_{i=1}^N(1-x_i)F_\lambda(qx_1,\ldots,qx_N)$. *More generally,*

$$F_{\lambda+k^N}(x_1,\ldots,x_N)=q^{k\binom{N}{2}}\prod_{i=1}^{N}f_k(x_i)F_\lambda(q^kx_1,\ldots,q^kx_N). \tag{43}$$

Proof. We have $f_{m+1}(x)=(1-x)f_m(qx)$, therefore

$$\det\left[f_{\lambda_i+1+N-i}(x_j)\right]=\det\left[(1-x_j)f_{\lambda_i+N-i}(qx_j)\right]=\prod_{j=1}^{N}(1-x_j)\det\left[f_{\lambda_i+N-i}(x_j)\right].$$

Since each factor $(x_i - x_j)$ in the denominator gets multiplied by q after changing $x_i \to qx_i$, this implies the first equation. Now (43) can be obtained by applying it k times. $\square$

Let e_i denote the i-th basic vector in $\mathbb{Z}^N$ with 1 at i-th position and 0 at other positions. Given $I \subset \{1, \ldots, n\}$, we define $e_I = \sum_{i \in I} e_i$.

Proposition 11.9. *We have*

$$x_1 \cdots x_N F_\lambda(x_1, \ldots, x_N) = q^{-|\lambda| - \binom{N}{2}} \sum_{I \subset \{1, \ldots, n\}} (-1)^{|I|} F_{\lambda + e_I}(x_1, \ldots, x_N).$$

Here we use the convention that $F_{\lambda + e_I} = 0$ unless the entries of $\lambda + e_I$ are non-increasing (that is, $\lambda + e_I$ is a partition).

Proof. We have $f_{m+1}(x) = f_m(x)(1 - q^m x)$, so

$$x f_m(x) = q^{-m}(f_m(x) - f_{m+1}(x)).$$

Therefore

$$x_1 \cdots x_N \det\left[f_{\lambda_i + N - i}(x_j)\right] = \det\left[x_j f_{\lambda_i + N - i}(x_j)\right] =$$

$$\det\left[q^{-\lambda_i - N + i}(f_{\lambda_i + N - i}(x_j) - f_{\lambda_i + 1 + N - i}(x_j))\right].$$

$\square$

Corollary 11.10. *Consider the completion of the space of symmetric functions with coefficients in $\mathbb{Z}[q, q^{-1}]$ with respect to the basis F_λ. Then the operator of multiplication by $x_1 \cdots x_N$ is invertible in this completion and its inverse is given by the equation*

$$(x_1 \cdots x_N)^{-1} F_\lambda(x_1, \ldots, x_N) = q^{\binom{N}{2}} \sum_{v \in \mathbb{Z}^N_{\geq 0}} q^{|\lambda| + v} F_{\lambda + v}(x_1, \ldots, x_N).$$

Proof. Define the operators A_i by $A_i(F_\lambda) = F_{\lambda + e_i}$, and $p_i(F_\lambda) = q^{\lambda_i} F_\lambda$ for $i = 1, \ldots, N$. Clearly, $[A_i, A_j] = [p_i, p_j] = [A_i, p_j]$ for $i \neq j$ and by Proposition 11.9 we have

$$x_1 \cdots x_N = q^{-\binom{N}{2}} \prod_i (1 - A_i) p_i^{-1},$$

hence

$$(x_1 \cdots x_N)^{-1} = q^{\binom{N}{2}} \prod_i p_i (1 + A_i + A_i^2 + \ldots).$$

$\square$

Example 11.11. For $N = 1$ and $\lambda = (0)$ we get a curious identity

$$x^{-1} = \sum_{m=0}^{\infty} f_m(x) q^m$$

We can check this identity directly, by computing the values of both sides at q^{-j} for all j. Denote

$$u_j = \sum_{m=0}^{\infty} f_m(q^{-j}) q^m = \sum_{m=0}^{j} f_m(q^{-j}) q^m.$$

Then $u_{j+1} = 1 + q(1 - q^{-j-1}) u_j$ and $u_0 = 1$, so it is easy to see that $u_j = q^j$.

11.3. **Interpolation polynomials for $\mathfrak{gl}_2$.** In this subsection we describe the interpolation polynomials for $\mathfrak{gl}_2$ explicitly. By definition, we have polynomials $F_\lambda(x_1, x_2)$ where $\lambda_1 \geq \lambda_2$:

$$F_{\lambda_1,\lambda_2}(x_1, x_2) = \frac{1}{x_1 - x_2} \begin{vmatrix} f_{\lambda_1+1}(x_1) & f_{\lambda_1+1}(x_2) \\ f_{\lambda_2}(x_1) & f_{\lambda_2}(x_2) \end{vmatrix}$$

Let us consider the case $\lambda_2 = 0$ first, and write $\lambda_1 = k$. Then

$$F_{k,0}(x_1, x_2) = \frac{1}{x_1 - x_2} \det \begin{vmatrix} f_{k+1}(x_1) & f_{k+1}(x_2) \\ 1 & 1 \end{vmatrix} = \frac{f_{k+1}(x_1) - f_{k+1}(x_2)}{x_1 - x_2}.$$

Let

$$h_i(x_1, x_2) = \frac{x_1^{i+1} - x_2^{i+1}}{x_1 - x_2}.$$

Recall that $f_{k+1}(x) = \sum_{j=0}^{k+1} (-1)^j q^{\frac{j(j-1)}{2}} \binom{k+1}{j}_q x^j$, so

$$F_{k,0}(x_1, x_2) = \sum_{j=1}^{k+1} (-1)^j q^{\frac{j(j-1)}{2}} \binom{k+1}{j}_q h_{j-1}(x_1, x_2),$$

compare with Corollary 11.6. We just replace each x^j in the expression for $f_{k+1}(x)$ by $h_{j-1}(x_1, x_2)$.

Example 11.12. We have

$$f_1(x) = 1 - x,\ f_2(x) = (1 - x)(1 - qx) = 1 - (1 + q)x + qx^2,$$

$$f_3(x) = (1 - x)(1 - qx)(1 - q^2x) = 1 - (1 + q + q^2)x + (q + q^2 + q^3)x^2 - q^3x^3$$

so

$$F_{0,0}(x_1, x_2) = -1,\ F_{1,0}(x_1, x_2) = q(x_1 + x_2) - (1 + q),$$

$$F_{2,0} = -q^3(x_1^2 + x_1x_2 + x_2^2) + (q + q^2 + q^3)(x_1 + x_2) - (1 + q + q^2).$$

By Proposition 11.8 we have

$$F_{\lambda_1,\lambda_2}(x_1, x_2) = q^{\lambda_2} f_{\lambda_2}(x_1) f_{\lambda_2}(x_2) F_{\lambda_1-\lambda_2,0}(q^{\lambda_2}x_1, q^{\lambda_2}x_2).$$

In particular, for $(\lambda_1, \lambda_2) = (k, k)$ we have

$$F_{k,k}(x_1, x_2) = q^k f_k(x_1) f_k(x_2).$$

Also, by Lemma 11.1 we get

$$F_{\lambda_1,\lambda_2}(x_1, 1) = \begin{cases} -f_{\lambda_1}(qx_1) & \text{if } \lambda_2 = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

11.4. **Interpolation tables for $\mathfrak{gl}_2$.** For the reader's convenience, we have computed the polynomials $F_\lambda(x_1, x_2)$ and the corresponding interpolation matrices using `Sage` [35].

First, we present F_λ in Schur basis:

$$F_0=-1,\quad F_1=qs_1-(q+1),\quad F_2=-q^3s_2+(q^3+q^2+q)s_1-(q^2+q+1),$$
$$F_{1,1}=-qs_{1,1}+qs_1-q=-q(1-x_1)(1-x_2)$$
$$F_3=q^6s_3-(q^6+q^5+q^4+q^3)s_2+(q^5+q^4+2q^3+q^2+q)s_1-(q^3+q^2+q+1)$$
$$F_{2,1}=q^3s_{2,1}-q^3s_2-(q^3+q^2+q)s_{1,1}+(q^3+q^2+q)s_1-(q^2+q)$$
$$F_{3,1}=-q^6s_{3,1}+q^6s_3+(q^6+q^5+q^4+q^3)s_{2,1}-(q^6+q^5+q^4+q^3)s_2-$$
$$(q^5+q^4+2q^3+q^2+q)s_{1,1}+(q^5+q^4+2q^3+q^2+q)s_1-(q^3+q^2+q)$$
$$F_{2,2}=-q^4s_{2,2}+(q^4+q^3)s_{2,1}-q^3s_2-(q^4+q^3+q^2)s_{1,1}+(q^3+q^2)s_1-q^2$$
$$F_{3,2}=q^7s_{3,2}-(q^7+q^6)s_{3,1}-(q^7+q^6+q^5+q^4)s_{2,2}+q^6s_3+(q^7+2q^6+2q^5+2q^4+q^3)s_{2,1}-$$
$$(q^6+q^5+q^4+q^3)s_2-(q^6+2q^5+2q^4+2q^3+q^2)s_{1,1}+(q^5+q^4+2q^3+q^2)s_1-(q^3+q^2)$$
$$F_{3,3}=-q^9s_{3,3}+(q^9+q^8+q^7)s_{3,2}-(q^8+q^7+q^6)s_{3,1}-(q^9+q^8+2q^7+q^6+q^5)s_{2,2}+q^6s_3+$$
$$(q^8+2q^7+2q^6+2q^5+q^4)s_{2,1}-(q^6+q^5+q^4)s_2-(q^7+q^6+2q^5+q^4+q^3)s_{1,1}+(q^5+q^4+q^3)s_1-q^3$$

Next, we list the values of the evaluations $c_{\lambda,\mu}=F_\lambda(q^{-\mu_1-1},q^{-\mu_2})$ for various λ and μ in Tables 1, 2, 3 below. The resulting matrix $\mathsf{C}=(c_{\lambda,\mu})$ is upper-triangular, with diagonal entries prescribed by Lemma 10.11. Zero entries correspond to pairs (λ,μ) where μ does not contain λ. The entry corresponding to $(\lambda,\mu)=((1),(3,2))$ is marked in bold, it is divisible by $1-q$ but does not factor any further.

Using either Theorem 10.17 or equation (38), one can easily reconstruct the inverse matrix $D=\mathsf{C}^{-1}$, and we list part of it in Table 4 (see Examples 10.21 and 10.23 for more computations).

Note that by Corollary 11.3 this determines the coefficients $c_{\lambda,\mu}$ and $d_{\lambda,\mu}$ for $\lambda\subset\mu\subset(3,3)$ and arbitrary N.

11.5. **Link invariants for $\mathfrak{gl}_2$.** We can use the interpolation tables to expand the invariants of simple knots in the basis F_λ. Indeed, the colored $\mathfrak{gl}_2$ invariants are determined by the colored $\mathfrak{sl}_2$ invariants (that is, colored Jones polynomial) by the formula

$$J_K(V(\lambda_1,\lambda_2),q)=J_K(V_{\lambda_1-\lambda_2},q).$$

The coefficients $a_\lambda(K)$ are then determined by Theorem 1.3

$$a_\lambda(K)=\sum_{\mu\subset\lambda}d_{\lambda,\mu}(q^{-1})J_K(V(\mu),q).$$

For example, for the figure eight knot we have the following values of the colored Jones polynomial:

$$J_K(V_0,q)=1=J_K(V(1,1),q),\ J_K(V_1,q)=J_K(V(2,1),q)=1+q^2+q^{-2}-q-q^{-1},$$
$$J_K(V_2,q)=1+q^3+q^{-3}-q-q^{-1}+(q^3+q^{-3}-q-q^{-1})(q^3+q^{-3}-q^2-q^{-2}).$$

Using the values of $d_{\lambda,\mu}$ from Table 4 (and changing q to q^{-1}) we obtain

$$a_0(K)=-J_K(V_0,q)=-1,\ a_1(K)=-\frac{q^{-1}}{1-q^{-1}}J_K(V_0,q)+\frac{q^{-1}}{1-q^{-1}}J_K(V_1,q)=q^{-2}(q^3-1),$$

$$a_2(K)=-\frac{q^{-2}}{(1-q^{-1})(1-q^{-2})}J_K(V_0,q)+\frac{q^{-2}}{(1-q^{-1})^2}J_K(V_1,q)-$$
$$\frac{q^{-3}}{(1-q^{-1})(1-q^{-2})}J_K(V_2,q)=q^{-6}(-q^9+q^5+q^4-q^3-1),$$

$$a_{1,1}(K) = -\frac{q^{-3}}{(1-q^{-1})(1-q^{-2})}J_K(V_0,q) + \frac{q^{-2}}{(1-q^{-1})^2}J_K(V_1,q) - \frac{q^{-2}}{(1-q^{-1})(1-q^{-2})}J_K(V(1,1),q) = q^{-2}(q^2+q+1),$$

$$a_{2,1}(K) = -\frac{q^{-4}}{(1-q^{-1})^2(1-q^{-3})}J_K(V_0,q) + \frac{q^{-3}}{(1-q^{-1})^3}J_K(V_1,q) - \frac{q^{-4}}{(1-q^{-1})^2(1-q^{-2})}J_K(V_2,q) - \frac{q^{-3}}{(1-q^{-1})^2(1-q^{-2})}J_K(V(1,1),q) + \frac{q^{-4}}{(1-q^{-1})^2(1-q^{-3})}J_K(V(2,1),q) = q^{-6}(-q^8-q^7-q^6-q^5+q^4+2q^3+q^2+q+1).$$

Using Tables 1, 2, 3, one can similarly compute the values of $a_\lambda(K)$ for all $\lambda \subset (3,3)$ and verify that these are indeed Laurent polynomials in q.

$\lambda\backslash\mu$	(0)	(1)	(2)	(1,1)
(0)	-1	-1	-1	-1
(1)	0	$q^{-1}(1-q)$	$q^{-2}(1-q^2)$	$q^{-1}(1-q^2)$
(2)	0	0	$-q^{-3}(1-q)(1-q^2)$	0
(1,1)	0	0	0	$-q^{-2}(1-q)(1-q^2)$
(3)	0	0	0	0
(2,1)	0	0	0	0
(3,1)	0	0	0	0
(2,2)	0	0	0	0
(3,2)	0	0	0	0
(3,3)	0	0	0	0

TABLE 1. Evaluations of interpolation polynomials: matrix $C = (c_{\lambda,\mu})$

$\lambda\backslash\mu$	(3)	(2,1)	(3,1)
(0)	-1	-1	-1
(1)	$q^{-3}(1-q^3)$	$q^{-2}(1-q^3)$	$q^{-3}(1-q^4)$
(2)	$-q^{-5}(1-q^2)(1-q^3)$	$-q^{-3}(1-q)(1-q^3)$	$-q^{-5}(1-q^2)(1-q^4)$
(1,1)	0	$-q^{-3}(1-q)(1-q^3)$	$-q^{-4}(1-q)(1-q^4)$
(3)	$q^{-6}(1-q)(1-q^2)(1-q^3)$	0	$q^{-6}(1-q)(1-q^2)(1-q^4)$
(2,1)	0	$q^{-4}(1-q)^2(1-q^3)$	$q^{-6}(1-q)(1-q^2)(1-q^4)$
(3,1)	0	0	$-q^{-7}(1-q)^2(1-q^2)(1-q^4)$
(2,2)	0	0	0
(3,2)	0	0	0
(3,3)	0	0	0

TABLE 2. Matrix $C = (c_{\lambda,\mu})$, continued

12. APPENDIX

Here we collect some useful definitions and facts about Habiro's ring and interpolation Macdonald polynomials.

$\lambda\backslash\mu$	(2,2)	(3,2)	(3,3)
(0)	-1	-1	-1
(1)	$q^{-2}(1+q)(1-q^2)$	$\mathbf{q^{-3}(-q^4-q^3+q^2+1)}$	$q^{-3}(1+q)(1-q^3)$
(2)	$-q^3(1-q^2)(1-q^3)$	$-q^{-5}(1-q^3)(1-q^4)$	$-q^{-5}(1+q)(1-q^3)^2$
(1,1)	$-q^{-4}(1-q^2)(1-q^3)$	$-q^{-5}(1-q^2)(1-q^4)$	$-q^{-6}(1-q^3)(1-q^4)$
(3)	0	$q^{-6}(1-q)(1-q^3)(1-q^4)$	$q^{-6}(1-q^2)(1-q^3)(1-q^4)$
(2,1)	$q^{-5}(1-q^2)^2(1-q^3)$	$q^{-7}(1-q^2)(1-q^3)(1-q^4)$	$q^{-8}(1+q)(1-q^2)(1-q^3)(1-q^4)$
(3,1)	0	$-q^{-8}(1-q)(1-q^2)(1-q^3)(1-q^4)$	$-q^{-9}(1-q^2)(1-q^3)^2(1-q^4)$
(2,2)	$-q^{-6}(1-q)(1-q^2)^2(1-q^3)$	$-q^{-8}(1-q)(1-q^2)(1-q^3)(1-q^4)$	$-q^{-10}(1-q^2)(1-q^3)^2(1-q^4)$
(3,2)	0	$q^{-9}(1-q)^2(1-q^2)(1-q^3)(1-q^4)$	$q^{-11}(1-q^2)^2(1-q^3)^2(1-q^4)$
(3,3)	0	0	$-q^{-12}(1-q)(1-q^2)^2(1-q^3)^2(1-q^4)$

TABLE 3. Matrix $C = (c_{\lambda,\mu})$, continued

$\lambda\backslash\mu$	(0)	(1)	(2)	(1,1)	(2,1)
(0)	-1	$-\frac{q}{1-q}$	$-\frac{q^2}{(1-q)(1-q^2)}$	$-\frac{q^3}{(1-q)(1-q^2)}$	$-\frac{q^4}{(1-q)^2(1-q^3)}$
(1)	0	$\frac{q}{1-q}$	$\frac{q^2}{(1-q)^2}$	$\frac{q^2}{(1-q)^2}$	$\frac{q^3}{(1-q)^3}$
(2)	0	0	$-\frac{q^3}{(1-q)(1-q^2)}$	0	$-\frac{q^4}{(1-q)^2(1-q^2)}$
(1,1)	0	0	0	$-\frac{q^2}{(1-q)(1-q^2)}$	$-\frac{q^3}{(1-q)^2(1-q^2)}$
(2,1)	0	0	0	0	$\frac{q^4}{(1-q)^2(1-q^3)}$

TABLE 4. Interpolation matrix $D = (d_{\lambda,\mu}) = C^{-1}$

12.1. **Habiro's ring.** The Habiro ring [17] is defined as

$$\widehat{\mathbb{Z}[q]} := \varprojlim_n \frac{\mathbb{Z}[q]}{((q;q)_n)}$$

Any element of $\widehat{\mathbb{Z}[q]}$ can be presented (not uniquely) as infinite series

$$f(q) = \sum_{n=0}^{\infty} f_n\,(q;q)_n, \quad f_n \in \mathbb{Z}[q].$$

Evaluations of such $f(q)$ at all roots of unity are well defined, since if $q^s = 1$ one has $f(q) = \sum_{n=0}^{s-1} f_n(q)_n$. It is easy to expand every $f(q) \in \widehat{\mathbb{Z}[q]}$ into formal power series in $(q-1)$, denoted by $T(f)$ and called the Taylor series of $f(q)$ at $q=1$. One important property of the Habiro ring is that any $f \in \widehat{\mathbb{Z}[q]}$ is uniquely determined by its Taylor series. In other words, the map $T : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[[q-1]]$ is injective [17, Thm 5.4]. In particular, $\widehat{\mathbb{Z}[q]}$ is an integral domain. Moreover, every $f \in \widehat{\mathbb{Z}[q]}$ is determined by the values of f at any infinite set of roots of unity of prime power order. Because of these properties, Habiro ring is also known as a ring of analytic functions at roots of unity.

Since $\cap_{n\geq 0} I_n = 0$ with $I_n = (q;q)_n\mathbb{Z}[q]$, the natural map $\mathbb{Z}[q] \to \widehat{\mathbb{Z}[q]}$ is injective. The image of q under this map is invertible, and the inverse is given by

$$q^{-1} = \sum_{n=1}^{\infty} q^n (q;q)_n,$$

compare with Example 11.11. This implies that there is an injective map $\mathbb{Z}[q,q^{-1}] \to \widehat{\mathbb{Z}[q]}$. The following result is proved in [17, Proposition 7.5], but we give a slightly different proof here for the reader's convenience. We will denote by $\Phi_n(q)$ the nth cyclotomic polynomial $\Phi_n(q) = \prod_{(a,n)=1}(q - \zeta_n^a)$ where ζ_n is any primitive nth root of unity.

Proposition 12.1. *Suppose that $f(q) \in \widehat{\mathbb{Z}[q]}$ and $f(q)h(q) \in \mathbb{Z}[q, q^{-1}]$ for some product of cyclotomic polynomials $h(q) = \Phi_{n_1}(q) \cdots \Phi_{n_r}(q)$. Then $f(q) \in \mathbb{Z}[q, q^{-1}]$.*

Proof. Let us denote $g(q) = f(q)h(q) \in \mathbb{Z}[q, q^{-1}]$, we prove the statement by induction in r. For $r = 1$ we get $h(q) = \Phi_n(q)$ and $g(q) = f(q)\Phi_n(q)$, so for any primitive n-th root of unity ζ_n we have $g(\zeta_n) = f(\zeta_n)\Phi_n(\zeta_n) = 0$, so $g(q) = \alpha(q)\Phi_n(q)$ for some $\alpha \in \mathbb{Z}[q, q^{-1}]$. This implies $(f(q) - \alpha(q))\Phi_n(q) = 0$, and since $\widehat{\mathbb{Z}[q]}$ is an integral domain we get $f(q) = \alpha(q)$.

For $r > 1$ we get

$$f(q)\Phi_{n_1}(q) \cdots \Phi_{n_r}(q) \in \mathbb{Z}[q, q^{-1}],$$

so by the above

$$f(q)\Phi_{n_1}(q) \cdots \Phi_{n_{r-1}}(q) \in \mathbb{Z}[q, q^{-1}],$$

and by the assumption of induction $f(q) \in \mathbb{Z}[q, q^{-1}]$. □

12.2. **Interpolation Macdonald polynomials.** We consider partitions with at most N parts.

Theorem 12.2. [23, 24, 29, 30, 31, 32, 36] *There exists unique up to scalar factors family of symmetric polynomials $I_\lambda(x_1, \ldots, x_N; q, t)$ with the following properties:*

(a) $I_\lambda(q^{-\mu_i}t^{N-i}) = 0$ *unless μ contains λ*
(b) $I_\lambda(q^{-\lambda_i}t^{N-i}) \neq 0$
(c) *I_λ is a nonhomogeneous polynomial of degree $|\lambda|$, and its degree $|\lambda|$ part is proportional to the Macdonald polynomial $P_\lambda(x_1, \ldots, x_N; q, t)$.*

The polynomials I_λ are called interpolation Macdonald polynomials. In fact, the properties (a) and (b) already uniquely determine I_λ (up to a scalar), and their existence follows from the fact that $q^{-\lambda_i}t^{N-i}$ for a nondegenerate grid in the sense of [31]. Part (c) is then a deep property of these polynomials.

It is easy to see that at $q = t$ interpolation Macdonald polynomials I_λ specialize to F_λ. Unlike F_λ, there is no determinant formula for I_λ but there is a different combinatorial formula [30].

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Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
Email address: `anna@math.uzh.ch`

Department of Matematics, University of California, Davis, One Shields Avenue, Davis CA 95616, USA
Email address: `egorskiy@math.ucdavis.edu`