Quantum Link Homology via Trace Functor I

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Abstract

Motivated by topology, we develop a general theory of traces and shadows in an endobicategory, which is a bicategory $\mathbf{C}$ with an endobifunctor $\Sigma\colon \mathbf{C} \longrightarrow \mathbf{C}$. Applying this framework to the bicategory of Chen–Khovanov bimodules with identity as Σ we reproduce Asaeda–Przytycki–Sikora (APS) homology for links in a thickened annulus. If Σ is a reflection, we obtain the APS homology for links in a thickened Möbius band. Both constructions can be deformed by replacing Σ with an endofunctor Σ_q such that $\Sigma_q\alpha := q^{-\deg\alpha}\Sigma\alpha$ for any 2-morphism α and identity otherwise, where q is a fixed invertible scalar. We call the resulting invariant the *quantum link homology.* We prove in the annular case that this homology carries an action of $\mathcal{U}_q(\mathfrak{sl}_2)$, which intertwines the action of cobordisms. In particular, the quantum annular homology of an n–cable admits an action of the braid group, which commutes with the quantum group action and factors through the Jones skein relation. This produces a nontrivial invariant for surfaces knotted in four dimensions. Moreover, a direct computation for torus links shows that the rank of quantum annular homology groups does depend on the quantum parameter q. Hence, our quantum link homology has a richer structure.

Contents

1 Introduction

In the diagrammatic calculus assigned to a pivotal category, a trace of an endomorphism f corresponds to closing of the $(1,1)$–tangle representing f. In higher categories there are more possibilities to close a morphism. For a cobordism with corners, apart from the described *vertical* trace, there is also a *horizontal* trace shown in Figure 1.

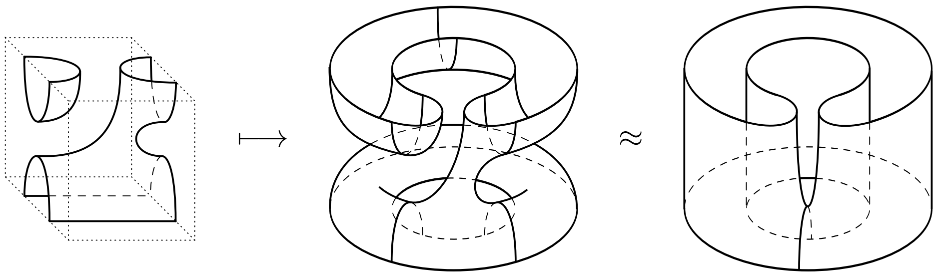

Figure 1: A horizontal trace of a cobordism with corners.

In this paper, inspired by topology, we further generalize the vertical and horizontal traces of a bicategory. Recall that a surface bundle M over a circle with monodromy ϕ is constructed from a thickened surface $F\times I$ by gluing $F\times\{0\}$ to $F\times\{1\}$ along the diffeomorphism ϕ:

$$M := {}^{F\times I}\big/_{(p,1)\sim(\phi(p),0)} \tag{1.1}$$

To mimic this construction on a more abstract level, we consider a pair: a category $\mathscr{C}$ and an endofunctor $\Sigma\colon \mathscr{C} \longrightarrow \mathscr{C}$. Then the *trace* of $(\mathscr{C}, \Sigma)$ is defined as

$$\operatorname{Tr}(\mathscr{C}, \Sigma) := \coprod_{x \in \operatorname{Ob}(\mathscr{C})} \mathscr{C}(x, \Sigma x) \Big/ g \circ f \sim \Sigma f \circ g \tag{1.2}$$

where f and g run through all pairs of composable morphisms $\Sigma x \xleftarrow{g} y \xleftarrow{f} x$ in $\mathscr{C}$. If Σ is the identity functor, we recover the usual trace of $\mathscr{C}$, called also its 0th Hochschild–Mitchell homology. For a graded $\mathscr{C}$ we quantize the above construction by taking $\Sigma_q f = q^{-\deg f}\Sigma f$, which results in the *quantum trace* $\operatorname{Tr}_q(\mathscr{C}, \Sigma)$. In addition, we investigate the quantum analogue of the Hochschild–Mitchell homology.

A geometric example of our trace is the identification of the Kauffman Bracket Skein Module $\mathcal{S}(M)$ of the surface bundle M, defined in (1.1), with $\operatorname{Tr}(\mathcal{S}(F \times I), \phi_*)$. Here, $\mathcal{S}(F \times I)$ is viewed as a category whose objects are points in F and morphisms are tangles in $F \times I$ modulo the Kauffman skein relations, and $\phi_*(T) = (\phi^{-1} \times \mathrm{id})(T)$.

Given a set S, a *trace function* on $(\mathscr{C}, \Sigma)$ valued in S is a collection of functions $\operatorname{tr}_x\colon \mathscr{C}(x, \Sigma x) \longrightarrow S$ indexed by objects $x \in \mathscr{C}$, such that the relation in (1.2) is satisfied. Clearly, $\operatorname{Tr}(\mathscr{C}, \Sigma)$ is the universal trace function, in the sense that any trace function factorizes through it. We observe that any natural transformation of endofunctors induces a new trace function. For example, fixing a framed $(1,1)$–tangle T we can define a natural transformation of the category of tangles in the thickened plane by precomposing any (m,n)–tangle with the n–cabling of T. In particular, one can compose any (n,n)–tangle with a full twist before closing. Hence, the trace of the category of tangles encodes information not only about a simple annular closure $\widehat{T}$ of an (n,n)–tangle T, but also on all satellites of $\widehat{T}$.

This paper is devoted to a categorification of these constructions. We define a notion of a *preshadow* of an endobicategory $(\mathbf{C}, \Sigma)$, generalizing shadows introduced by Ponto and Schulman [PS13]. A preshadow categorifies the notion of a trace function. We prove that any preshadow factors through the *horizontal trace*, introduced in [BHLZ14], under the mild assumption that $\mathbf{C}$ has left duals. Hence, for bicategories with duals the horizontal trace $\operatorname{hTr}(\mathbf{C}, \Sigma)$ is a universal preshadow.

We show that the horizontal trace can be viewed as a categorification of the trace of a category in the following sense. Let $\Pi\colon n\mathbf{Cat} \longrightarrow (n-1)\mathbf{Cat}$ be the decategorification functor, which forgets the n–morphisms and identifies isomorphic $(n-1)$–morphisms. Then there is a natural bijection between the sets $\Pi(\operatorname{hTr}(\mathbf{C}, \Sigma))$ and $\operatorname{Tr}(\Pi\mathbf{C}, \Pi\Sigma)$ for any small endobicategory $(\mathbf{C}, \Sigma)$ with both left and right duals.

Applying the construction (1.2) to morphism categories in $\mathbf{C}$, we get the vertical trace $\operatorname{vTr}(\mathbf{C}, \Sigma)$. There is a functor from the vertical to the horizontal trace of an endobicategory, which is full and faithful, but not necessarily surjective on objects.

Our basic examples of bicategories are $\mathbf{Cob}$ and $\mathbf{Tan}(F)$. The former has objects collections of points on a line, collections of arcs and circles in $\mathbb{R} \times I$, called flat tangles, as 1–morphisms, and cobordisms in $(\mathbb{R} \times I) \times I$ as 2–morphisms. In the latter objects are marked points on a surface F, 1–morphisms are oriented tangles in $F \times I$ and 2–morphisms are tangle cobordisms in $(F \times I) \times I$. Given a 3–manifold M, let $\mathscr{L}inks(M)$ be the category, whose objects are oriented links in M and morphisms are link cobordisms in $M \times I$. Our first result is a computation of the horizontal trace of $\mathbf{Tan}(F)$ with the endofunctor induced by the diffeomorphism ϕ. A similar result was proven in [QR15] for $F = \mathbb{R}^2$ and Σ the identity functor.

Theorem A. *Let M be a surface bundle with fiber F and monodromy $\phi \in \mathit{Diff}(F)$. Then there is an equivalence of categories*

$$\mathrm{hTr}(\mathbf{Tan}(F), \phi_*) \simeq \mathscr{L}\!\mathit{inks}(M) \tag{1.3}$$

where $\phi_*(S) := (\phi^{-1} \times \mathrm{id} \times \mathrm{id})(S)$ for a cobordism $S \subset F \times I \times I$.

An analogous result holds for **Cob**. A real line admits only two diffeomorphisms up to an isotopy, so that the analogue of M is either an annulus $\mathbb{A}$ or a Möbius band $\mathbb{M}$. This way we interpret the Bar-Natan categories $\mathscr{B}\mathscr{N}(\mathbb{A})$ and $\mathscr{B}\mathscr{N}(\mathbb{M})$ as horizontal traces.

Our next goal is to use Theorem A to construct new invariants of links in a solid torus. For this purpose we examine the homology of tangles constructed by Chen and Khovanov [CK14], which is a projective bifunctor $\widetilde{\mathbf{F}}_{CK}$ from **Tan** to the derived bicategory of diagrammatic bimodules **DB**. Here, "projective" means that the bifunctor is defined on 2–morphisms only up to a sign. Chen and Khovanov constructed a family of diagrammatic algebras A^n, commonly called *arc algebras*, and a chain complex $C_{CK}(T)$ of graded (A^n, A^m)–bimodules for each (m, n)–tangle T. A tangle cobordism induces a chain map between these complexes that is defined only up to a sign. The algebra A^n categorifies the n–th tensor power of the fundamental representation V_1 of $\mathscr{U}_q(\mathfrak{sl}_2)$. It was first introduced by Braden [Bra02] using generators and relations to describe the category of perverse sheaves on Grassmannians. The arc algebras and their representations were independently studied by Brundan and Stroppel [BS11, BS10].

To construct an annular link invariant it is enough to compose $\widetilde{\mathbf{F}}_{CK}$ with a preshadow on the bicategory of diagrammatic bimodules **DB**—because the latter has duals, every preshadow factors through the horizontal trace of **DB**. An immediate choice of a preshadow is the Hochschild homology HH. We quantize the above by taking the quantum horizontal trace and *quantum Hochschild homology* qHH, a one parameter deformation of HH that factors through the quantum horizontal trace. Then we construct the right vertical arrow in the following diagram

$$\begin{tikzcd}
\mathbf{Tan} \arrow[r, mapsto, "\mathrm{hTr}"] \arrow[d, "\widetilde{\mathbf{F}}_{CK}"'] & \mathrm{hTr}(\mathbf{Tan}) \arrow[r, "\simeq"] & \mathscr{L}\!\mathit{inks}(\mathbb{A}) \arrow[d, dashed] \\
\mathscr{D}(\mathbf{DB}) \arrow[r, mapsto, "\mathrm{hTr}_q"] & \mathrm{hTr}_q(\mathscr{D}(\mathbf{DB})) \arrow[r, "qHH"] & \mathscr{D}(\mathscr{U}_q(\mathfrak{sl}_2))
\end{tikzcd} \tag{1.4}$$

where we write shortly $\mathscr{L}\!\mathit{inks}(\mathbb{A})$ for $\mathscr{L}\!\mathit{inks}(\mathbb{A}\times\mathbb{R})$. Indeed, we get a well-defined homology for links, but the chain map assigned to a cobordisms is defined only up multiplication by $\pm q^{\pm 1}$: the sign comes from the fact that $\widetilde{\mathbf{F}}_{CK}$ is merely a projective functor, and the overall power of q is not well-defined, because the bottom arrow satisfies the quantum trace relation. Hence, the chain map depends on the presentation of an annular cobordism as a horizontal closure of a surface with corners. We again call this behavior "projective".

Theorem B. *Assume an annular link $\widehat{T}$ is a closure of an (n, n)-tangle T. Then the homotopy type of $qHH(C_{CK}(T); A^n)$ is a triply graded invariant of $\widehat{T}$, which is projectively functorial with respect to annular link cobordisms. Moreover, it admits an action of the quantum group $\mathscr{U}_q(\mathfrak{sl}_2)$ that commutes with the differential and the maps induced by annular link cobordisms intertwine this action.*

There are many papers devoted to fixing the sign issue to get a strictly functorial Khovanov homology [Vo15, Bla10, CMW09, Ca07]. We believe that these approaches can be used to fix the overall sign and make the diagram (1.4) commute up to scaling by $q^{\pm 1}$. Then one has to replace $\mathscr{L}inks(\mathbb{A})$ with an analogue of the quantum trace of **Tan**. The naive way is to linearize both **Tan** and $\mathscr{L}inks(\mathbb{A})$, and then take the quantum horizontal trace as defined for linear categories. However, we believe that a better approach is to decorate annular link cobordisms with an additional topological data that would satisfy an analogue of the quantum trace relation.

Setting $q = 1$ in Theorem B we obtain a relationship between the Hochschild homology of the Chen–Khovanov complex $C_{CK}(T)$ for an (n,n)–tangle T and the Asaeda–Przytycki–Sikora complex $CKh_{\mathbb{A}}(\widehat{T})$ for the annular closure $\widehat{T}$ of T.

Theorem C. *Let $\widehat{T}$ be the annular closure of an (n,n)–tangle T. Then there is an isomorphism of chain complexes*

$$CKh_{\mathbb{A}}(\widehat{T}) \cong HH(C_{CK}(T); A^n) \tag{1.5}$$

natural with respect to the chain maps associated to tangle cobordisms. The annular grading in $CKh_{\mathbb{A}}(\widehat{T})$ corresponds to the weight decomposition of $C_{CK}(T)$.

This proves Conjecture 1.1 formulated by Auroux, Grigsby, and Wehrli in [AGW15]. Note that it was first shown in [GLW15] that the annular link homology carries an action of $\mathfrak{sl}_2$ that commutes with the action of cobordisms.

We prove Theorem B by explicitly constructing the right vertical map in (1.4). We do this following the general recipe for Khovanov homology as described in [BN05]. There, a link L in a thickened surface F is assigned a formal complex $[\![L]\!]$ in the Bar-Natan skein category $\mathscr{BN}(F)$, whose objects are curves in F and morphisms are cobordisms in $F \times I$. To construct actual homology one then applies to $[\![L]\!]$ a certain TQFT functor $\mathscr{F}\colon \mathscr{BN}(F) \longrightarrow \mathscr{M}od_{\Bbbk}$, where $\Bbbk$ is a fixed ring of scalars.

The construction of Chen and Khovanov also fits in this framework, but because flat tangles in a stripe can also be composed, the Bar-Natan category is actually a bicategory, denoted here by **BN**, and the TQFT functor is a bifunctor $\mathbf{F}_{CK}\colon \mathbf{BN} \longrightarrow \mathbf{DB}$ valued in the bicategory of diagrammatic bimodules. Following the idea of the proof of Theorem A we show that the horizontal trace of **BN** coincides with $\mathscr{BN}(\mathbb{A})$. This motivates us to define the *quantum skein category* $\mathscr{BN}_q(\mathbb{A})$ as $\mathrm{hTr}_q(\mathbf{BN})$. The bifunctor $\mathbf{F}_{CK}$ induces then a functor on $\mathscr{BN}_q(\mathbb{A})$ valued in $\mathrm{hTr}_q(\mathbf{DB})$, which after composing with the quantum Hochschild homology produces a desired TQFT functor $\mathscr{F}_{\mathbb{A}_q}$.

Once we know the functor exists, we reconstruct it in another way, more suitable for computations. The starting point is an equivalence between a certain quotient $\mathscr{BBN}(\mathbb{A})$ of the "classical" category $\mathscr{BN}(\mathbb{A})$ and the Temperley–Lieb category at $q = 1$. The main ingredient in proving Theorems B and C is to obtain a similar connection between the Temperley–Lieb category at any q with the quantum skein category $\mathscr{BN}_q(\mathbb{A})$. Indeed, we construct a commuting diagram of functors

$$\begin{tikzcd} \mathscr{TL} \arrow[rr, "\simeq"] \arrow[dr, "\mathscr{F}_{TL}"'] & & \mathscr{BBN}_q(\mathbb{A}) \arrow[dl, "\mathscr{F}_{\mathbb{A}_q}"] \\ & \mathrm{g}\mathscr{R}ep(\mathscr{U}_q(\mathfrak{sl}_2)) & \end{tikzcd} \tag{1.6}$$

where $\mathscr{BBN}_q(\mathbb{A})$ is a quotient of $\mathscr{BN}_q(\mathbb{A})$, through which $\mathscr{F}_{\mathbb{A}_q}$ factors, the horizontal functor is an equivalence of categories, and $\mathscr{F}_{TL}$ is the Reshetikhin–Turaev realization of the Temperley–Lieb diagrams as $\mathscr{U}_q(\mathfrak{sl}_2)$ intertwiners between tensor powers of V_1, the fundamental representation of the quantum group. To show that (1.6) commutes one needs to find an isomorphism

$$V_1^{\otimes n} \cong qHH(A^n).$$

When $q = 1$, the existence of the isomorphism follows from the invariance result for Hochschild homology due to Keller [Ke98]: if A is a finite dimensional algebra, $E \subset A$ is a separable subalgebra such that $A = E \oplus \operatorname{rad}(A)$, each simple A–module is one-dimensional, and A has finite global dimension, then $HH(E) \cong HH(A)$. It was proven by Brundan and Stroppel that arc algebras have finite global dimension [BS11]. We reprove the Keller's result for quantum and—more generally—for twisted Hochschild homology of an algebra A by identifying the latter with quantum Hochschild–Mitchell homology of the endocategory of finite dimensional representations of A with an appropriate endofunctor. The advantage of replacing algebras and bimodules with representation categories and functors is that the invariance of quantum Hochschild homology follows then from a more general result: a map induced on homology by a functor $\mathscr{F}$ depends only on the trace class of the identity transformation on $\mathscr{F}$. To reprove Keller's result it is then enough to show that the (A, A)–bimodules A and $A \otimes_E E$ coincide in the Grothendieck group of the category of (A, A)–bimodules, because every category $\mathscr{C}$ admits a *Chern character* that relates $K_0(\mathscr{C})$ with the Hochschild–Mitchell homology of $\mathscr{C}$. This is proven in Proposition 4.11 for graded algebras of finite global dimension.

Once the relation between our TQFT functor $\mathscr{F}_{\mathbb{A}_q}$ and the Temperley–Lieb functor $\mathscr{F}_{TL}$ is established, we can directly show that setting $q = 1$ recovers the TQFT functor described by Asaeda, Przytycki, and Sikora to define the annular homology. Thence, Theorem C follows. This motivates us to call our new link invariant the *quantum annular link homology*. We compute it explicitly for $(2, n)$ torus links in Section 5, showing that it is richer than the APS annular link homology. Hence, our homology is a non-trivial one parameter deformation of the APS categorification of the Kauffman Bracket Skein Module of the annulus.

Let K be an annular closure of a framed $(1, 1)$–tangle T. There is an action of the category of oriented tangles on the quantum annular homology of cablings of K: cups and caps act via introducing or removing two cables, whereas a crossing switches two of them. The action was first observed in [GLW15] in the non-quantized setting. We compute this action and show that it factors through the Jones skein relation. In the following we use the notation $S_T(T')$ for the chain map associated to a tangle T'.

Theorem D. *The homomorphism $S_T(T')_*$ depends only on the annular closure K of T. It is functorial in T' up to a sign, intertwines the action of $\mathscr{U}_q(\mathfrak{sl}_2)$, and factors through the Jones skein relation:*

$$q^2 S_T(\nwarrow\!\!\!\!\nearrow)_* - q^{-2} S_T(\nwarrow\!\!\!\!\nearrow)_* = (q - q^{-1}) S_T(\curvearrowleft\curvearrowright)_*. \tag{1.7}$$

In particular, the n–strand braid group acts on the quantum annular homology $Kh_{\mathbb{A}_q}(\widehat{T}^n)$ of the n–cabling of a framed $(1, 1)$–tangle T.

In particular, our quantum annular link homology assigns the Jones polynomial $J_L(q)$ to a surface of revolution $\mathbb{S}^1 \times L \subset \mathbb{S}^1 \times R^3$. In the APS setting this invariant is trivial.

Recall that we have identified links in a thickened Möbius band with closures of tangles. Therefore, one may hope to get a parallel quantization of the APS homology for links with diagrams on a Möbius band. Indeed, such a construction is described at the end of Section 5. The basic ingredient is the *twisted* quantum Hochschild homology, defined for a pair of a graded algebra A and its automorphism φ. Here, we consider the automorphism ρ_* of the arc algebras induced by a horizontal flip of the diagrams. Again, setting $q = 1$ we recover the original APS homology.

We end the paper with two generalizations of the annular homology. We have already observed that any preshadow applied to $C_{CK}(T)$ produces an annular link invariant. In particular, we can pre-compose an (n,n)–tangle T with and n–cabling of a framed $(1,1)$–tangle, such as a sequence of full twists. Geometrically, these theories correspond to different embeddings of the annulus into $\mathbb{R}^3$. In the other generalization we again fix a $(1,1)$–tangle T and we assign to an essential circle in an annulus the quantum annular homology of either $\widehat{T}$ or its mirror image. We then use the duality between the homology of a link and its mirror image to define the differential in the chain complex.

This paper is the beginning of our program aimed to quantize the APS homology for any surface. Because every surface can be constructed from a polygon by identifying its edges, in our second paper we will look on tangles, whose endpoints are placed on more than two lines. We will then associate diagrammatic multimodules *á la* Chen–Khovanov to such tangles. With a closed analogy to the annular case, the identification of the edges of the polygon will correspond to a partial shadow on multimodules.

Recently, Queffelec and Rose introduced an annular $\mathfrak{sl}_n$ link homology using the horizontal trace of foams [QR15]. Their bicategory is graded and we believe their homology can be quantized using our framework. An analogue of the Chen–Khovanov bifunctor $\widetilde{\mathbf{F}}_{CK}$ is given by Soergel bimodules, and it is a challenging task to compute their quantum Hochschild homology.

Outline

The paper is organized as follows. In the first section we review constructions of Khovanov homology used in this paper: the general framework of Bar-Natan, the Asaeda–Przytycki–Sikora homology for links in a thickened annulus or in a thickened Möbius band, and the Chen–Khovanov homology for tangles.

Section 3 is devoted to the construction of categorical traces in endobicategories. In Section 3.3 we discuss preshadows with a number of examples, including the quantization and twisting of Hochschild homology of bimodules. The proof of Theorem A is the substantial part of Section 3.5.

The quantum twisted Hochschild homology for algebras is discussed in details in Section 4. Here we introduce the quantum Hochschild–Mitchell homology of linear categories and prove its invariance. Proposition 4.11 is treated in Section 4.4.

We quantize the APS homology in Section 5. Theorems B and C are proven in Section 5.3, whereas Section 5.5 is devoted to prove Theorem D. The two generalizations of the annular homology are described in Section 6.

Finally, we gathered in Appendix basic definitions and results concerning bicategories and homological algebra, including a crash course on derived categories and Grothendieck groups.

Notation

Throughout the paper we fix a commutative unital ring $\Bbbk$, to which we refer as the *ring of scalars*. Main examples are $\mathbb{Z}$, $\mathbb{Z}[q^{\pm 1}]$, but occasionally we require it to be either a field or a ring of finite characteristic, such as $\mathbb{Z}_2[q^{\pm 1}]$. The symbol $\mathscr{M}od_\Bbbk$ is used for the category of all $\Bbbk$–modules.

All algebras are assumed to be $\Bbbk$–algebras modeled on flat $\Bbbk$–modules. We write $\mathscr{M}od(A)$ for the category of finitely generated left modules over an algebra A, and $\mathscr{R}ep(A)$ for its subcategory of projective modules. The symbol $\mathscr{B}imod(A,B)$ stands for the category of (A,B)–bimodules for any algebras A and B, and $\mathscr{R}ep(A,B)$ for the subcategory of those bimodules that are finitely generated and projective as left A–modules. Because both A and B are $\Bbbk$–algebras, there are two actions of $\Bbbk$ on an (A,B)–bimodule M. We require that both actions are equal.

Acknowledgements

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2 Khovanov Homology

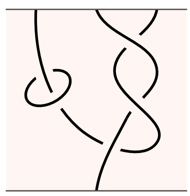
2.1 Knots and tangles

Let M be an oriented smooth 3–manifold. A proper 1–submanifold $T \subset M$ is called a *tangle*. We call it a *link* if it has no boundary, and a *knot* if in addition it has one component. All tangles and links in this paper are assumed to be oriented unless stated otherwise. An *isotopy of tangles* T and T' is a smooth map $\Phi\colon M \times I \longrightarrow M$ such that each $\Phi_t := \Phi(-,t)$ is a diffeomorphism fixed at the boundary, $\Phi_0 = \mathrm{id}$, and $\Phi_1(T) = T'$. If T and T' are oriented, then we require that the orientation is preserved by Φ_1.

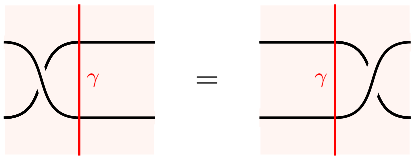
Denote by $-T$ the tangle T with reversed orientation of all its components. A *tangle cobordism* from a tangle T_0 to T_1 is an oriented surface $S \subset M \times I$ with boundary $\partial S = T_0 \times \{0\} \cup -T_1 \times \{1\} \cup (\partial T_0 \times I)$. We shall consider cobordisms only up to an isotopy, in which case they form a category: composition is given by gluing cobordisms, and the identity morphism on a tangle T is represented by the cylinder $T \times I \subset M \times I$.

When $M = F \times I$ is a thickened surface, then isotopy classes of oriented tangles in M form a category, with the product induced by stacking, $M \cup M \cong M$, and tangles with tangle cobordisms form a 2–category.

Notation. We shall write $\mathit{Links}(M)$ for the set of isotopy classes of oriented links in M, and $\mathscr{L}\mathit{inks}(M)$ for the category of oriented links in M and cobordisms between them. Isotopy classes of oriented tangles in a thickened surface $F \times I$ form a category $\mathscr{T}\mathit{an}(F)$ with the composition induced by stacking, and similarly cobordisms between tangles in $F \times I$ form a 2–category $\mathbf{Tan}(F)$. We write simply $\mathscr{T}\mathit{an}$ and $\mathbf{Tan}$ when $F = \mathbb{R} \times I$.



Assume M is a line bundle over a surface F and consider the projection $M \longrightarrow F$ onto the zero section. It maps a generic tangle T to an immersed collection of intervals and circles $\tilde{T} \subset F$ with only finitely many multiple points, each a transverse intersection of two arcs. A *diagram* of T is constructed from $\tilde{T}$ by breaking one of the arcs at each double point as follows. When F is oriented, then fibers of M admit a canonical orientation and we break the lower arc at each double point of $\tilde{T}$ (see an example above for $F = \mathbb{R} \times I$). In case F is nonorientable, choose a minimal collection of curves γ that cuts F into an orientable surface. Then there is a normal field over $F - \gamma$ and we can construct the diagram as before. Then Reidemeister moves and planar isotopies relate diagrams between isotopic tangles if a crossing is switched when moved through γ:



This follows, because the normal field is reversed at points of γ.

2.2 The formal bracket

Let us start with a brief reminder of the formal Khovanov bracket following [BN05]. In this section F stands for a surface, possibly with boundary.

Choose a diagram $D \subset F$ of an oriented tangle $T \subset F \times \mathbb{R}$ and let n stand for the number of crossings in D. To compute the formal Khovanov bracket of T one begins with creating the n-dimensional *cube of resolutions* $\mathcal{I}(D)$ of D defined as follows:

- each vertex ξ of $\mathcal{I}(D)$ is decorated with the resolution D_ξ of D, i.e. the collection of circles and proper intervals in F obtained from D by forgetting the orientation and replacing each i–th crossing $\times$ by its horizontal $\asymp$ or vertical $)($ smoothing for $\xi_i = 0$ and $\xi_i = 1$ respectively, and
- the edges are decorated by cobordisms with a unique saddle point connecting the two resolutions, directed from the resolution with less vertical smoothings to the one with more of them.

We can view $\mathcal{I}(D)$ as a commutative cubical diagram in $\mathcal{C}ob(F)$, the category whose objects are (non-oriented) flat tangles in F and morphisms the isotopy classes of surfaces in $F \times I$. The degree of a surface $S \subset F \times I$ is given by the formula

$$\deg S := \chi(S) - \frac{\#B}{4} \tag{2.1}$$

where B is the set of corners and $\chi(S)$ the Euler characteristic of S. Alternatively, $\deg S$ counts the critical points of the natural height function $h\colon S \longrightarrow I$ with signs: a point of index μ contributes $(-1)^\mu$ towards deg. Thence, to ensure each morphism in $\mathcal{I}(D)$ has degree 0, we introduce formal degree shifts[1] and place at each vertex ξ the shifted resolution $D_\xi\{|\xi|\}$, where $|\xi| := \xi_1 + \dots + \xi_n$.

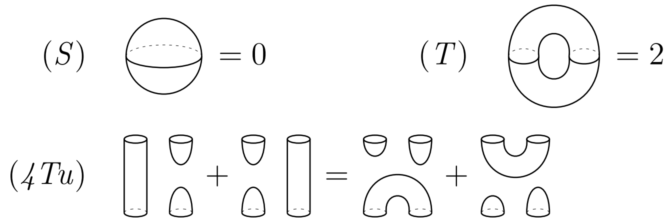
Let $\mathcal{C}ob(F)^\oplus$ be the additive closure of $\mathcal{C}ob(F)$. The *formal Khovanov bracket* $[\![D]\!]$ of the tangle diagram D is the complex in $\mathcal{C}ob(F)^\oplus$ obtained from the cube by distributing signs over some edges to make every square anticommute, then taking direct sums along diagonal sections of the cube, and finally applying suitable degree shifts:

$$[\![D]\!]^i := \bigoplus_{|\xi| = i + n_-} D_\xi\{i + n_+ - n_-\} \tag{2.2}$$

where $n_\pm$ stand for the amount of positive or negative crossings in D.

[1] Formally, objects in $\mathcal{C}ob(F)$ are now symbols $T\{i\}$ formed by a flat tangle $T \subset F$ and $i \in \mathbb{Z}$.

Theorem 2.1 (cf. [BN05]). *The homotopy type of the complex $[\![D]\!]$ is a tangle invariant after imposing on $\mathscr{C}ob(F)^{\oplus}$ the following local relations*[2]



introduced first by Bar-Natan [BN05].

Let $\overcrossing$, $\undercrossing$, $\smoothing$, and $\hsmoothing$ be link diagrams, which are the same except a small disk, in which they look as indicated by the pictures. Because the formal Khovanov bracket categorifies the Jones polynomial [BN05], it is natural to expect the existence of exact sequences connecting the formal brackets of the four diagrams, lifting the Jones relation. The sequences were first observed by Viro [Vi02] for knots in $\mathbb{R}^3$, and his construction immediately translates to the general setting.

Proposition 2.2 (cp. [Vi02]). *Let $\overcrossing$, $\undercrossing$, $\smoothing$, $\hsmoothing$ be four link diagrams as above, where we choose any orientation for $\hsmoothing$. There are short exact sequences of graded complexes*

$$0 \longrightarrow [\![\hsmoothing]\!]\{3e+2\}[e+1] \longrightarrow [\![\overcrossing]\!] \longrightarrow [\![\smoothing]\!]\{1\} \longrightarrow 0,$$
$$0 \longrightarrow [\![\smoothing]\!]\{-1\} \longrightarrow [\![\undercrossing]\!] \longrightarrow [\![\hsmoothing]\!]\{3e-2\}[e-1] \longrightarrow 0,$$

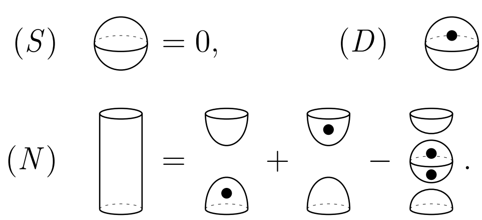
and an exact sequence

$$0 \longrightarrow [\![\smoothing]\!]\{1\}[1] \longrightarrow [\![\undercrossing]\!]\{2\}[1] \longrightarrow [\![\overcrossing]\!]\{-2\}[-1] \longrightarrow [\![\smoothing]\!]\{-1\}[-1] \longrightarrow 0,$$

where $e = n_-(\hsmoothing) - n_-(\overcrossing)$.

Proof. It follows directly from the construction of the formal Khovanov bracket that, up to degree shifts and orientations of links, $[\![\overcrossing]\!]$ is the cone complex of a chain map $[\![\smoothing]\!] \longrightarrow [\![\hsmoothing]\!]$ induced by a saddle move. The first exact sequence follows and likewise the second one. The last sequence is then obtained by gluing the other two along $[\![\hsmoothing]\!]$. $\square$

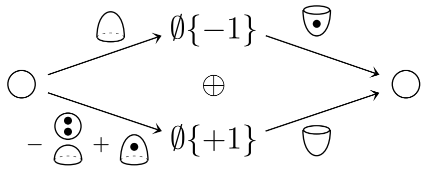
To retrieve homology groups from $[\![D]\!]$ one has to replace $\mathscr{C}ob(F)^{\oplus}$ with some abelian category $\mathscr{A}$. This is done by applying to the above construction a graded TQFT functor $\mathscr{F}\colon \mathscr{C}ob(F) \longrightarrow \mathscr{A}$ which preserves the relations S, T, and $4Tu$. Most known functors of this type factorize through the universal Bar-Natan skein category $\widetilde{\mathscr{BN}}(F)$ [BN05], the additive linear category generated by circles and proper intervals in F as objects, and formal linear combinations of cobordisms in $F \times I$ decorated by dots (each dot decreases the degree of a cobordism by 2) as morphisms, subject to the local relations



[2] *Locality* means that each picture represents a part of a cobordism inside a ball in $F \times I$.

The last relation is commonly referred as the *neck cutting relation*, because it allows to reduce any surface to a linear combination of surfaces of genus zero. Both T and $4Tu$ follows, and a contractible circle can be replaced with a pair of shifted copies of an empty set. In particular, the empty set generates $\widetilde{\mathcal{BN}}(F)$ when F is either $\mathbb{R}^2$ or $\mathbb{S}^2$.

Proposition 2.3 (Delooping, cf. [BN07])**.** *There is a pair of mutually inverse isomorphisms in $\widetilde{\mathcal{BN}}(F)$*

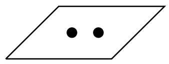


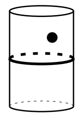
(2.3)

for every circle $\bigcirc$ bounding a disk in F.

If $\mathcal{F}\colon \mathcal{C}ob(F) \longrightarrow \mathcal{A}$ produces an invariant finite dimensional chain complex, $\mathcal{F}(\bigcirc)$ must be a rank two module over $\mathcal{F}(\emptyset)$ [Kh06]. In particular, $\mathcal{F}$ factors through $\widetilde{\mathcal{BN}}(F)$ if $\mathcal{F}(\bigcirc)$ is a free module, so that little information is lost when we decide to work with the universal Bar-Natan category instead of $\mathcal{C}ob(F)$.

The TQFT functors we consider in this paper factor through certain quotients of $\widetilde{\mathcal{BN}}(F)$. Consider the following two relations:

(TD)  $= 0$

(B) γ  $= 0$ if $[\gamma] \neq 0$ in $\pi_1(F \times I)$.

The first one asserts that two dots annihilate any surface—which together with the neck cutting relation implies further that a single dot annihilates any surface of positive genus—and the second one prohibits any surface that contains a nontrivial curve to carry a dot. We write $\mathcal{BN}(F)$ when only TD is imposed and $\mathcal{BBN}(F)$ when both of them. $\mathcal{BN}(F)$ is commonly called the Bar-Natan skein category and it was first defined in [BN05]. For the annulus it was extensively analyzed by Russell [Rus09]. The relation B was introduced by Boerner [Boe08] for any surface F and we call $\mathcal{BBN}(F)$ the *Boerner–Bar-Natan skein category.*

Functoriality

The construction of the formal bracket $[\![T]\!]$ is functorial up to signs: given a cobordism S between tangle diagrams T and T' there is a chain map $[\![S]\!]\colon [\![T]\!] \longrightarrow [\![T']\!]$ defined up to the factor ± 1, such that $[\![SS']\!] = \pm[\![S]\!] \circ [\![S']\!]$. Hence, the same type of functoriality holds for any TQFT functor $\mathcal{F}\colon \mathcal{C}ob(F) \longrightarrow \mathcal{A}$ satisfying the relations S, T, and $4Tu$.

The chain map $[\![S]\!]$ is computed from a *movie presentation* of S, a sequence of generic sections $S_t = S \cap (F \times \{t\})$ called *movie clips*, such that the part $S|_{[t,t']}$ of S between two consecutive clips S_t and $S_{t'}$ is either one of the Reidemeister moves, a saddle cobordism, a cap, or a cup [CS98]. There is a well-defined chain map for each of the parts and $[\![S]\!]$ is defined as the composition of these pieces. A cobordism S admits many movie presentations and it is proven that up to sign $[\![S]\!]$ does not depend on the presentation chosen [BN05]. Unfortunately, direct computation shows that the sign of $[\![S]\!]$ does depend on the presentation [Jac04].

There are a few approaches to attack the sign issue. In case of $F = \mathbb{R}^2$ one can use the deformation of the Khovanov homology due to Lee [Le05] to define *canonical generators*, which are preserved by $[\![S]\!]$ up to sign [Ras05]. We can then redefine $[\![S]\!]$ so that the generators are actually preserved. This approach was used in [GLW15] to fix signs in certain cases.

A different idea is to replace $\mathscr{C}ob(F)$ with another category. This was done successfully when $F = \mathbb{R}^2$ first by Clark, Morrison and Walker using cobordisms with seams and coefficients in the ring of Gaussian integers $\mathbb{Z}[i]$ [CMW09], then by Blanchet using nodal foams [Bla10], and recently by Vogel using *mixed cobordisms*: locally oriented cobordisms with certain disorientation curves [Vo15]. The last approach is closest to the original construction—there is a forgetful functor from mixed cobordisms to $\mathscr{C}ob(\mathbb{R}^2)$—and we believe it can be extended to any surface F. There is also a strictly functorial construction for $(2m, 2n)$–tangles due to Caprau [Ca07], which is defined over Gaussian integers. It assigns to a tangle a direct summand of the corresponding invariant due to Chen and Khovanov, which is used in this paper.

Conjecture 2.4. *For every surface F there is a category of mixed cobordisms $\mathscr{MC}ob(F)$ with a forgetful functor $\mathscr{U}\colon \mathscr{MC}ob(F) \longrightarrow \mathscr{C}ob(F)$, and a functorial Khovanov bracket $[\![-]\!]'$ valued in a certain quotient of the additive closure of $\mathscr{MC}ob(F)$ such that for any tangle $T \subset F \times \mathbb{R}$ the complexes $\mathscr{U}([\![T]\!]')$ and $[\![T]\!]$ are isomorphic.*

The rest of this section is devoted to review a number of TQFT functors that produce invariant chain complexes.

2.3 Khovanov homology for links in $\mathbb{R}^3$

In what follows we fix a commutative unital ring $\Bbbk$. Let R be a commutative $\Bbbk$–algebra and choose a Frobenius algebra A of rank 2. This datum determines a TQFT functor with $\mathscr{F}(\emptyset) := R$ and $\mathscr{F}(\bigcirc) := A$, and it produces an invariant chain complex for link diagrams on a plane [Kh06]. For instance, Khovanov's functor $\mathscr{F}_{Kh}\colon \mathscr{C}ob(\mathbb{R}^2) \longrightarrow \mathscr{M}od_{\Bbbk}$ is defined this way by taking $R := \Bbbk$ and equipping $A := Rw_+ \oplus Rw_-$ with the structure maps

$$m\colon A \otimes A \longrightarrow A \qquad \begin{cases} w_+ \otimes w_+ \longmapsto w_+, \\ w_\pm \otimes w_\mp \longmapsto w_-, \\ w_- \otimes w_- \longmapsto 0, \end{cases} \tag{2.4}$$

$$\Delta\colon A \longrightarrow A \otimes A \qquad \begin{cases} w_+ \longmapsto w_+ \otimes w_- + w_- \otimes w_+, \\ w_- \longmapsto w_- \otimes w_-, \end{cases} \tag{2.5}$$

$$\eta\colon \Bbbk \longrightarrow A \qquad \left\{ 1 \longmapsto w_+, \right. \tag{2.6}$$

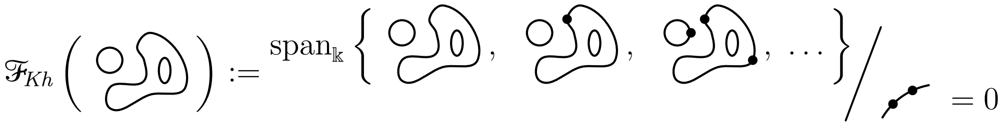
$$\epsilon\colon A \longrightarrow \Bbbk \qquad \begin{cases} w_+ \longmapsto 0, \\ w_- \longmapsto 1. \end{cases} \tag{2.7}$$

The functor is graded if we set $\deg w_\pm := \pm 1$. We shall write $Kh(D)$ for the homology of $CKh(D) := \mathscr{F}_{Kh}[\![D]\!]$, where D is a diagram of a link L; it is called the *Khovanov homology* of the link L.

Khovanov's functor factorizes through $\mathscr{BN}(\mathbb{R}^2)$, where a dot is understood as multiplication by w_-. In particular, both generators are images of $1 \in \Bbbk$ under cup cobordisms

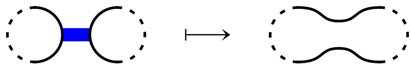
$$\mathscr{F}_{Kh}\left(\quad\right) : 1 \longmapsto w_+ \qquad\qquad \mathscr{F}_{Kh}\left(\quad\right) : 1 \longmapsto w_- \tag{2.8}$$

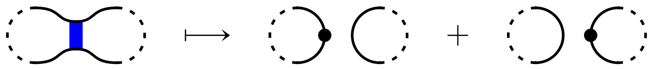
which motivates the following graphical description of $\mathcal{F}_{Kh}$. Given a *collection of curves* $\Gamma \subset \mathbb{R}^2$ we identify $\mathcal{F}_{Kh}(\Gamma)$ with the $\Bbbk$–module freely generated by all diagrams obtained from Γ by decorating some curves with dots, imposing the relation that two dots on a single curve annihilate the diagram.[3] For example,



The generators w_+ and w_- of the algebra A are represented by the circle without and with a dot respectively. To redefine $\mathcal{F}_{Kh}$ on a *cobordism* S we use the following rules:

- if S creates a circle, then $\mathcal{F}_{Kh}(S)$ modifies a diagram by inserting the new circle with no dot on it,
- if S contracts a circle, then $\mathcal{F}_{Kh}(S)$ removes the circle from a diagram if it was decorated by a dot, or takes the diagram to 0 otherwise, and
- we use the following local surgery formulas to define $\mathcal{F}_{Kh}(S)$ if S is a merge or a split


(2.9)


(2.10)

where the blue thick arcs visualize the saddle of S.

Notice that a merge of two curves is zero, when each curve carries a dot, as the surgery (2.9) produces a curve with two dots. Likewise, a split of a curve with a dot results in one diagram, as one of the two terms at the right hand side of (2.10) vanishes.

2.4 Annular link homology

There are two types of closed curves in the annulus: *trivial curves* bounding balls in $\mathbb{A}$, and *essential curves*, parallel to the core of $\mathbb{A}$. The value of an annular TQFT functor $\mathcal{F}\colon \mathcal{BN}(\mathbb{A}) \longrightarrow \mathcal{Mod}_\Bbbk$ on trivial curves is determined by Bar-Natan's relations, but not the value on essential curves.

The first construction of an annular TQFT functor is due to Asaeda, Przytycki, and Sikora [APS04]. The APS functor $\mathcal{F}_\mathbb{A}\colon \mathcal{BN}(\mathbb{A}) \longrightarrow \mathcal{Mod}_\Bbbk$ assigns to a trivial and an essential curve the free $\Bbbk$–modules

$$W := \mathrm{span}_\Bbbk\{w_+, w_-\}, \qquad\qquad V := \mathrm{span}_\Bbbk\{v_+, v_-\}, \tag{2.11}$$

respectively, with the degree defined on generators as

$$\deg w_\pm = \pm 1, \qquad\qquad \deg v_\pm = 0. \tag{2.12}$$

This degree is denoted by j' in [GLW15] and differs from the one used in [Rob13]. In addition, the modules admit the *annular grading*, denoted adeg and defined as

$$\operatorname{adeg} w_\pm = 0, \qquad\qquad \operatorname{adeg} v_\pm = \pm 1. \tag{2.13}$$

[3] Clearly, those diagrams in which each circle carries at most one dot form a free basis for $\mathcal{F}_{Kh}(\Gamma)$.

One can define $\mathcal{F}_{\mathbb{A}}$ by comparing it to $\mathcal{F}_{Kh}$. Indeed, the spaces V and W are isomorphic as $\Bbbk$–modules, but the modified grading (2.12) induces a filtration on $\mathcal{F}_{Kh}$. The functor $\mathcal{F}_{\mathbb{A}}$ can be constructed as the graded associate functor [Rob13], although $\mathcal{F}_{Kh}$ does not behave well with respect to adeg. For completeness we write down the maps corresponding to the elementary saddle moves. A *merge* is assigned one of the maps

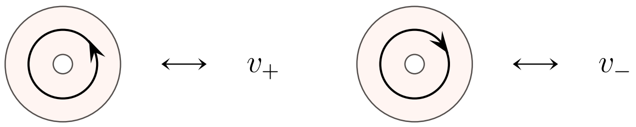
$$\begin{array}{l} \underline{W \otimes W \longrightarrow W} \\ w_+ \otimes w_+ \longmapsto w_+ \\ w_\pm \otimes w_\mp \longmapsto w_- \\ w_- \otimes w_- \longmapsto 0 \end{array} \qquad \begin{array}{l} \underline{V \otimes W \longrightarrow V} \\ v_\pm \otimes w_+ \longmapsto v_\pm \\ v_\pm \otimes w_- \longmapsto 0 \\ \end{array} \qquad \begin{array}{l} \underline{V \otimes V \longrightarrow W} \\ v_\pm \otimes v_\pm \longmapsto 0 \\ v_\pm \otimes v_\mp \longmapsto w_- \\ \end{array}$$

depending on the curves involved, whereas for *splits* we choose one of

$$\begin{array}{l} \underline{W \longrightarrow W \otimes W} \\ w_- \longmapsto w_- \otimes w_- \\ w_+ \longmapsto w_+ \otimes w_- + w_- \otimes w_+ \end{array} \qquad \begin{array}{l} \underline{V \longrightarrow V \otimes W} \\ v_\pm \longmapsto v_\pm \otimes w_- \\ \end{array} \qquad \begin{array}{l} \underline{W \longrightarrow V \otimes V} \\ w_- \longmapsto 0 \\ w_+ \longmapsto v_+ \otimes v_- + v_- \otimes v_+ \end{array}$$

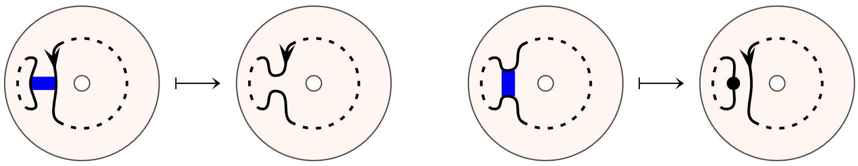
The value of $\mathcal{F}_{\mathbb{A}}$ on caps and cups is unchanged.

The graphical description of $\mathcal{F}_{Kh}$ can be extended to the annular case. Trivial curves can again carry dots, but the essential ones cannot, because the merge cobordism takes $v_\pm \otimes w_-$ to zero. Therefore, we shall visualize the two generators of V by choosing an orientation of the essential curve, anticlockwise for v_+ and clockwise for v_-:

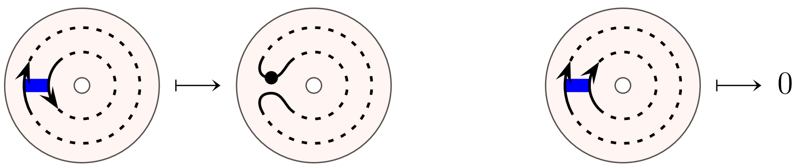


(2.14)

We use the usual surgery formulas for merging a trivial curve to an essential one or splitting it off, keeping in mind that an essential curve cannot carry dots:

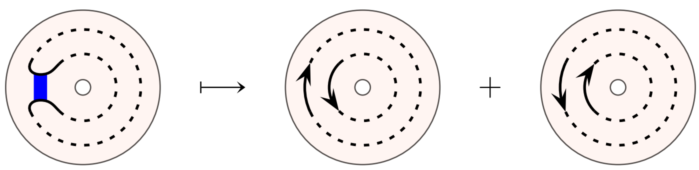

(2.15)

Two essential curves can be merged together only if they have opposite orientations, in which case we decorate the resulting trivial curve with a dot, and otherwise we have zero:



(2.16)

Finally, a surgery from a trivial curve to two essential ones is assigned the map


(2.17)

which can be viewed as performing the surgery on the trivial curve considered with both orientations at the same time.

Action of $\mathfrak{sl}_2$

It has been recently observed in [GLW15] that the annular link homology admits an action of $\mathfrak{sl}_2$ if we consider W as a trivial representation and V is identified with the fundamental one or its dual

$$V_1 = \mathrm{span}_{\Bbbk}\{v_+, v_-\} \qquad\qquad V_1^* = \mathrm{span}_{\Bbbk}\{v_+^*, v_-^*\}$$

depending on the nestedness of the associated essential curve, i.e. V_1 and V_1^* are assigned alternatively. The following tables describe the action of $\mathfrak{sl}_2$.

$$\begin{array}{ll} \multicolumn{2}{c}{\text{The action on } V_1} \\ \hline Ev_+ = 0 & Ev_- = v_+ \\ Fv_+ = v_- & Fv_- = 0 \end{array} \qquad\qquad \begin{array}{ll} \multicolumn{2}{c}{\text{The action on } V_1^*} \\ \hline Ev_-^* = 0 & Ev_+^* = -v_-^* \\ Fv_-^* = -v_+^* & Fv_+^* = 0 \end{array} \tag{2.18}$$

There is an obvious isomorphism of $\mathfrak{sl}_2$–modules $V_1 \cong V_1^*$, which identifies $v_\pm$ with $\pm v_\mp^*$. However, the action on the annular chain complex is defined using instead the $\Bbbk$–linear isomorphism $V \cong V_1^*$ sending $v_\pm$ to $v_\mp^*$, so that the action depends on the position of V in the tensor product. For instance, two essential curves are assigned $V \otimes V \cong V_1^* \otimes V_1$ with $\mathfrak{sl}_2$ acting in the following way:

$$\begin{aligned} E(v_+ \otimes v_+) &= 0 & F(v_+ \otimes v_+) &= v_- \otimes v_+ - v_+ \otimes v_- \\ E(v_+ \otimes v_-) &= -v_+ \otimes v_+ & F(v_+ \otimes v_-) &= v_- \otimes v_- \\ E(v_- \otimes v_+) &= v_+ \otimes v_+ & F(v_- \otimes v_+) &= -v_- \otimes v_- \\ E(v_- \otimes v_-) &= v_+ \otimes v_- - v_- \otimes v_+ & F(v_- \otimes v_-) &= 0 \end{aligned}$$

It follows that the maps $V \otimes V \longrightarrow W$ and $W \longrightarrow V \otimes V$ intertwine the action and the annular TQFT functor is upgraded to $\mathscr{F}_{\mathbb{A}} \colon \mathscr{BN}(\mathbb{A}) \longrightarrow \mathrm{g}\mathscr{Rep}(\mathfrak{sl}_2)$. In particular, $\mathfrak{sl}_2$ acts on the triply graded annular homology.

Remark. The action admits the following graphical description: each clockwise oriented curve in a diagram w contributes to Ew a diagram obtained from w by reversing the curve, and scaling it by (-1) if it is separated from the outer boundary by an odd number of curves. Likewise for F we reverse orientations of anticlockwise oriented curves.

We shall now reconstruct the action in a more conceptual way. Choose an invertible $q \in \Bbbk$ and recall the graded Temperly–Lieb category $\mathrm{g}\mathscr{TL}$:

- objects are symbols $B\{d\}$ with B a finite collection of points on a line and $d \in \mathbb{Z}$,
- there are no non-zero morphisms between $B\{d\}$ and $B'\{d'\}$ unless $d = d'$, in which case the morphisms are formal linear combinations of loopless flat tangles, i.e. collection of disjoint intervals with endpoints $B \sqcup B'$, and
- composition is defined by stacking pictures one on top of the other and replacing each closed loop created this way by $(q + q^{-1})$.

Consider an operation that takes a flat tangle F into the surface $\mathbb{S}^1 \times F \subset \mathbb{A} \times I$. A closed loop in F corresponds to a toroidal component of the surface, which is evaluated to 2 in $\mathscr{BN}(\mathbb{A})$ due to the neck cutting relation. Hence, there is a well-defined functor $\mathbb{S}^1 \times (-) \colon \mathrm{g}\mathscr{TL}|_{q=1} \longrightarrow \mathscr{BN}(\mathbb{A})$. Finally, recall that $\mathscr{BBN}(\mathbb{A})$ is the quotient of $\mathscr{BN}(\mathbb{A})$ by the Boerner's relation, which forces a dot to annihilate every surface with an essential circle in its boundary.

Proposition 2.5. *The functor $\mathbb{S}^1 \times (-) \colon \mathrm{g}\mathscr{TL}|_{q=1} \longrightarrow \mathscr{BBN}(\mathbb{A})$ is an equivalence of categories.*

Proof. Each object in $\mathscr{BBN}(\mathbb{A})$ is isomorphic to a collection of essential curves due to delooping, whereas the neck cutting relation implies that morphisms between such collections are generated by incompressible surfaces, i.e. annuli [AF07]. Hence, the functor $\mathbb{S}^1 \times (-)$ is full and essentially surjective. Faithfulness follows, because the annuli are linearly independent in $\mathscr{BBN}(\mathbb{A})$, see [Rus09]. $\square$

The category $\mathrm{g}\mathscr{TL}|_{q=1}$ is equivalent to the full subcategory of $\mathrm{g}\mathscr{R}ep(\mathfrak{sl}_2)$ generated by tensor powers of V_1. One can check that the triangle of functors

$$\begin{tikzcd} \mathrm{g}\mathscr{TL}|_{q=1} \arrow[r, "\mathbb{S}^1\times(-)"] \arrow[dr] & \mathscr{BBN}(\mathbb{A}) \arrow[d, "\mathscr{F}_{\mathbb{A}}"] \\ & \mathrm{g}\mathscr{R}ep(\mathfrak{sl}_2) \end{tikzcd} \tag{2.19}$$

commutes and that the induced action of $\mathfrak{sl}_2$ on the annular chain complex coincides with the one described above.

2.5 Homology for links in a thickened Möbius band

Beyond links in a thickened annulus, we shall also consider in this paper links in the twisted line bundle over the Möbius band ("twisted" means that the monodromy along the orientation reversing curve is $-id$, so that the bundle is orientable). Let us recall the construction of the APS functor in this case.

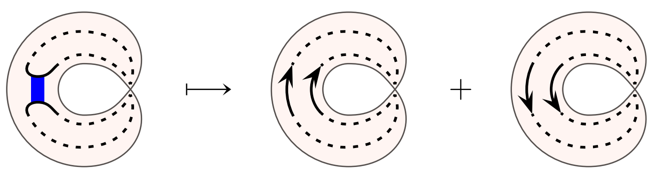
A Möbius band $\mathbb{M}$ admits three types of curves: *trivial curves* bounding disks, *separating curves* cutting $\mathbb{M}$ into a Möbius band and an annulus, and *nonseparating ones.* The APS functor $\mathscr{F}_{\mathbb{M}} \colon \mathscr{BN}(\mathbb{M}) \longrightarrow \mathscr{M}od_{\Bbbk}$ assigns to them the free $\Bbbk$–modules

$$W := \mathrm{span}_{\Bbbk}\{w_+, w_-\}, \quad V := \mathrm{span}_{\Bbbk}\{v_+, v_-\}, \quad \text{and} \quad U := \mathrm{span}_{\Bbbk}\{u_+, u_-\}, \tag{2.20}$$

with the degree function vanishing on both V and U, and $\deg w_\pm = \pm 1$ as usual. There are more types of saddle cobordisms in $\mathbb{M} \times I$ than in $\mathbb{A} \times I$, and $\mathscr{F}_{\mathbb{M}}$ vanishes on those without trivial circles in the boundary. Otherwise, it is defined as in the annular case for merges and splits (where both V and U can play the role of the "annular" V) and one of the maps

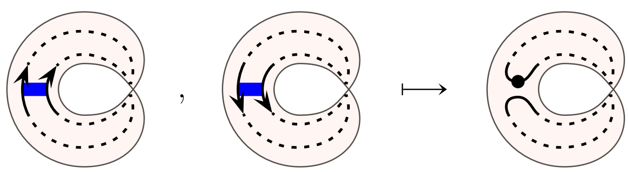
$$\begin{array}{l} W \longrightarrow V \\ \hline w_+ \longmapsto v_+ + v_- \\ w_- \longmapsto 0 \end{array} \qquad\qquad \begin{array}{l} V \longrightarrow W \\ \hline v_\pm \longmapsto w_- \\ \end{array}$$

for a saddle cobordism between a trivial circle and a separating one. We can represent the latter graphically as



(2.21)

and



(2.22)

We shall write $CKh_{\mathbb{M}}(D) := \mathcal{F}_{\mathbb{M}}[\![D]\!]$ for the chain complex for a link diagram D on $\mathbb{M}$, and $Kh_{\mathbb{M}}(D)$ for its homology. A resolution of D can have at most one non-separating curve—such a curve cuts the band into an annulus. In particular, U only appears in $CKh_{\mathbb{M}}(D)$ when D meets any cross section of $\mathbb{M}$ in an odd number points. If so, $CKh_{\mathbb{M}}(D) \cong \overline{CKh}_{\mathbb{M}}(D) \otimes U$, where we write $\overline{CKh}_{\mathbb{M}}(D)$ for the chain complex of D computed with $\Bbbk$ assigned to non-separating curves instead of U.

2.6 Chen–Khovanov homology for tangles

Tangle diagrams in a thickened stripe $\mathbb{R}\times I$ form a category. Thence, the formal Khovanov bracket of a tangle is a chain complex built over the bicategory $\mathbf{Cob} = \mathbf{Cob}(\mathbb{R} \times I)$ of points on a line, flat tangles in a stripe $\mathbb{R} \times I$, and surfaces in $(\mathbb{R} \times I) \times I$. This bicategory is graded with the degree of a surface S defined in (2.1). To preserve this richer structure, the homology for tangles is constructed by Chen and Khovanov [CK14] using a 2–functor $\mathbf{F}_{CK}\colon \mathbf{Cob} \longrightarrow \mathbf{gBimod}$ valued in the bicategory of graded bimodules (see Appendinx A.1). We begin with describing the $\Bbbk$–modules assigned to tangles, then the algebras assigned to points, and finally reconstructing the bimodule structure.

Cup diagrams with platforms

A *crossingless matching* between $2n$ points in a line is a collection of n disjoint arcs attached to the points. We shall draw the arcs in the lower half-plane $\mathbb{R} \times \mathbb{R}_{-}$ and refer to them as a *cup diagram*. Following [CK14] we generalize cup diagrams to allow semi-infinite arcs, each attached to one point only and going left or right towards infinity. This can be visualized by drawing two vertical platforms going out of the horizontal line, one to the left and one to the right of all the points, and attaching semi-infinite arcs to them. In particular, odd number of points are allowed. We shall call the points on the line *termini* to distinguish them from the endpoints on the platforms. Figure 2 presents all cup diagrams with three termini.

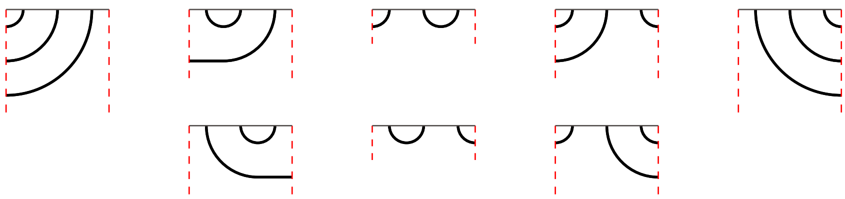


Figure 2: The generalized cup diagrams with three termini.

Let $\mathcal{GM}^n$ be the set of such diagrams with n termini. We define the *weight* of a diagram $a \in \mathcal{GM}^n$ as $\operatorname{wt}(a) := r - \ell$, where r and ℓ count respectively the arcs terminating on the right and on the left platform. In what follows we shall write $\mathcal{GM}^n(\lambda) \subset \mathcal{GM}^n$ for the subset of diagrams of weight λ. Notice that $\mathcal{GM}^n(\lambda)$ is empty unless λ has the same parity as n.

Dually we define the set $\mathcal{GM}_n$ of *cap diagrams with platforms* with arcs drawn in the upper half-plane. The reflection along the horizontal line induces a bijection of sets

$$\mathcal{GM}^n(\lambda) \ni a \longmapsto a^! \in \mathcal{GM}_n(\lambda) \tag{2.23}$$

for every n and λ.

An extension of $\mathscr{F}_{Kh}$

A pair of cup diagrams $a \in \mathscr{GM}^m(\lambda)$ and $b \in \mathscr{GM}^n(\lambda)$ can be used to produce a planar closure $b^! T a$ of any flat tangle $T \in \mathbf{Cob}(m,n)$. The closure is constructed by gluing a to the bottom of T and $b^!$ to the top, then turning the platforms towards themselves and identifying the endpoints of the arcs from inside out. In case $m \neq n$ there will be unmatched endpoints, the same number at each side, because a and b have equal weights. We connect them with half-circles, see Figure 3.

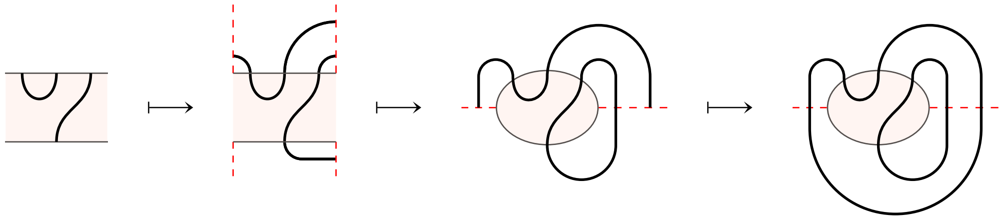


Figure 3: The construction of a planar closure of a $(1,3)$–tangle.

The functor $\mathscr{F}_{Kh}\colon \mathscr{C}ob(\mathbb{R}^2) \longrightarrow \mathscr{M}od_{\Bbbk}$ is extended to collections of curves with platforms by assigning to such a collection a $\Bbbk$–module generated by all possible decorations of the curves with dots as before, but with more restrictions:

1) a diagram vanishes when it contains a curve intersecting any of the platform twice,
2) a dot annihilates a diagram when placed on a curve that intersects a platform, and
3) as before, two dots on one curve annihilate the diagram.

A diagram is *nonadmissible* if one of the above situations happens, see Figure 4. Surgeries (2.9) and (2.10) on nonadmissible diagrams produce nonadmissible ones, so that $\mathscr{F}_{Kh}(S)$ is well-defined for any surface S, see also [CK14]. The Chen–Khovanov functor assigns to a flat tangle $T \in \mathbf{Cob}(m,n)$ the $\Bbbk$–module

$$\mathbf{F}_{CK}(T) := \bigoplus_{\lambda} \mathbf{F}_{CK}(T;\lambda), \quad \text{with} \quad \mathbf{F}_{CK}(T;\lambda) := \bigoplus_{\substack{a\in\mathscr{GM}^m(\lambda)\\ b\in\mathscr{GM}^n(\lambda)}} \mathscr{F}_{Kh}(b^! T a), \tag{2.24}$$

and to a cobordism S between flat tangles T_0 and T_1 the $\Bbbk$–linear map

$$\mathbf{F}_{CK}(S) := \bigoplus_{\lambda} \mathbf{F}_{CK}(S;\lambda), \quad \text{with} \quad \mathbf{F}_{CK}(S;\lambda) := \bigoplus_{\substack{a\in\mathscr{GM}^m(\lambda)\\ b\in\mathscr{GM}^n(\lambda)}} \mathscr{F}_{Kh}(b^! S a), \tag{2.25}$$

where $b^! S a$ stands for the surface $(b^! \times I) \cup S \cup (a \times I)$.

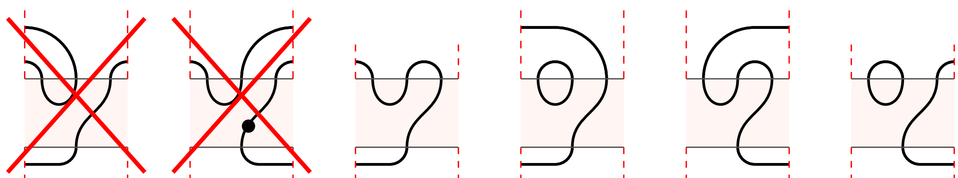
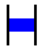
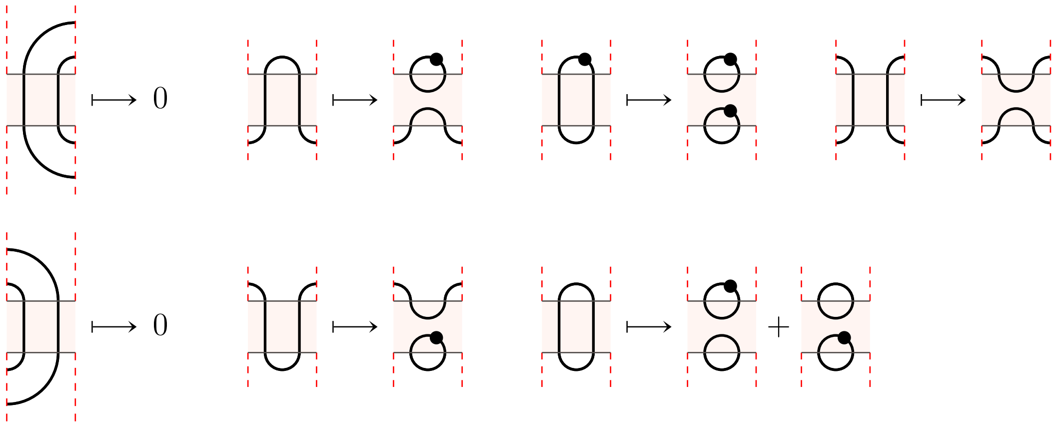


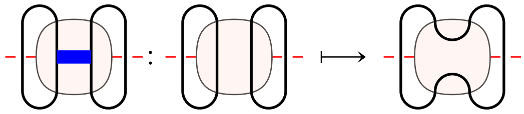
Figure 4: Examples of generators of $\mathbf{F}_{CK}(\,\,)$. The first two diagrams are nonadmissible, because they contain either a turnback or a dot on an open arc.

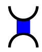
Remark. The closures of in Figure 4 are drawn without identifying the platforms of cup diagrams. This makes the pictures cleaner, but also easier to describe the module structure on $\mathbf{F}_{CK}(T)$ once the Chen–Khovanov algebras are introduced.

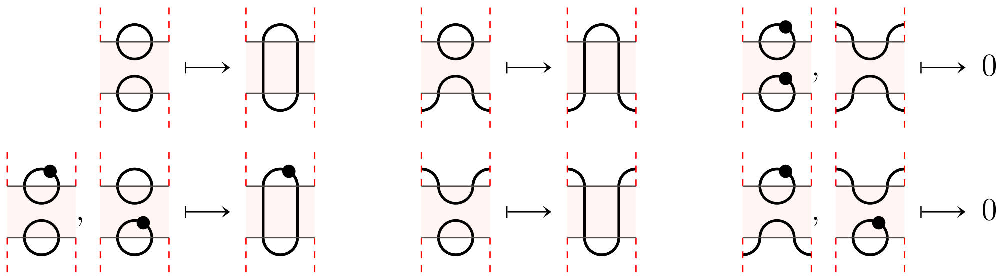
Example 2.6. Consider the saddle cobordism $S :=$  :  $\longrightarrow$  between the identity $(2,2)$–tangle and the tangle consisting of a cap followed by a cup. The module $\mathbf{F}_{CK}($  $)$ has seven generators, on which $\mathbf{F}_{CK}(S)$ takes the following values:



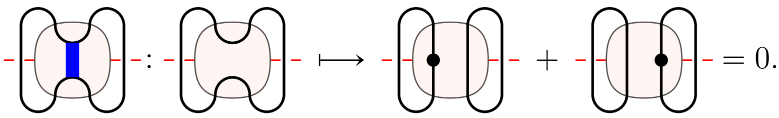
For example, the top right component of $\mathbf{F}_{CK}(S)$ is a merge when the platforms are identified:



Example 2.7. Consider now the cobordism $S :=$  :  $\longrightarrow$  going in the other direction. The module $\mathbf{F}_{CK}($  $)$ has eight generators, on which $\mathbf{F}_{CK}(S)$ is defined as below:



The two arcs in the top right corner are mapped to zero, because the corresponding closure of S in this case is a split with each circle in its output touching a platform:

 (2.26)

Arc algebras and diagrammatic bimodules

Let $c \in \mathscr{GM}^n$ be a generalized cup diagram and write S_c for the cobordism from $c \sqcup c^!$ to $2n$ vertical lines obtained by a sequence of n surgeries, one per arc in c, see Figure 5. The collection of such cobordisms defines linear maps

$$\mu_{T',T} \colon \mathbf{F}_{CK}(T') \otimes \mathbf{F}_{CK}(T) \longrightarrow \mathbf{F}_{CK}(T'T), \tag{2.27}$$

one per a pair of tangles $T \in \mathbf{Cob}(m,n)$ and $T' \in \mathbf{Cob}(n,k)$. Explicitly, $\mu_{T',T}(x \otimes y) = 0$ for $x \in \mathbf{F}_{CK}(d^! T' c)$ and $y \in \mathbf{F}_{CK}(b^! T a)$ unless $b = c$, in which case $\mu = \mathscr{F}_{Kh}(d^! S_c a)$.

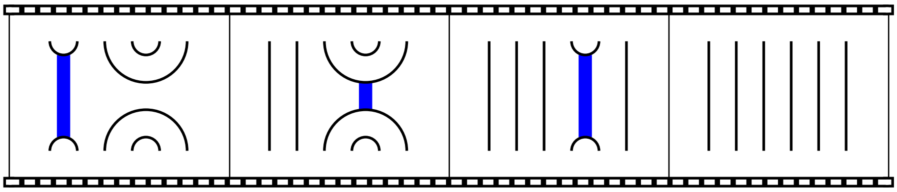


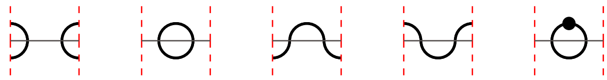
Figure 5: A sequence of surgeries replacing a disjoint union of a cup diagram and its vertical flip with vertical lines.

The *Chen–Khovanov algebra* A^n is the module assigned to the tangle formed by n vertical lines, with $x \cdot y := \mu(x \otimes y)$. It admits a *weight decomposition*

$$A^n = \bigoplus_{\lambda} A^n(\lambda), \tag{2.28}$$

which is related to that from [CK14] by setting $A^{n-k,k} = A^n(n-2k)$. There is a unique primitive idempotent $e_c \in A^n$ for each closure $c \in \mathscr{GM}^n$ given by the diagram $c^! c$ with no dots. The idempotents are mutually orthogonal, and their sum is a unit in A^n.

Example 2.8. The algebra A^2 has generators in weights -2, 0, and 2. Both $A^2(-2)$ and $A^2(2)$ are one dimensional, whereas $A^2(0)$ has five generators:



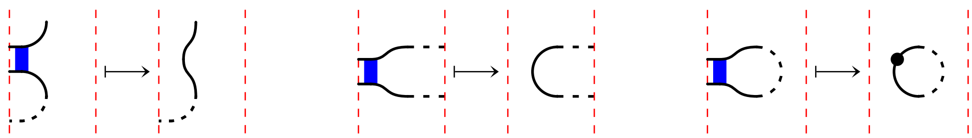
(2.29)

of which the first two are idempotents and the other square to zero. Furthermore,

(2.30)

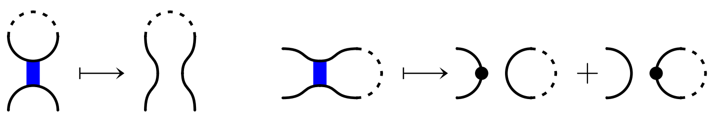
The product in A^n can be described explicitly using generalized surgeries as in [BS11]. Because we do not identify platforms when drawing diagrams, each diagram has four platforms drawn vertically. The product $x \cdot y$, when nonzero, can be then computed graphically by stacking x over y, connecting the platforms in between, and following the two steps below.

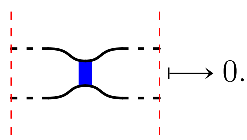
Step I: surgeries at platforms. Replace two opposite arcs touching one of the platforms with a vertical line and decorate with a dot each closed loop created that way:



(2.31)

Step II: surgeries on half-circles. When no arc at inner platforms is left, perform surgeries on the remaining arcs using the usual surgery formulas (2.9) and (2.10), except that a merge of two open arcs is zero and a diagram with a dot on an open arc vanishes (in particular, the first term in the result of the second surgery may vanish):





(2.32)

The merge of two arcs vanishes in the second step, because the two arcs have endpoints on the outer platforms and they belong to the same circle when the platforms are identified (compare with (2.26)).

It follows from the construction that the maps $\mu_{T',T}$ are natural with respect to tangle cobordisms. Thence, the following result holds, see also [CK14].

Proposition 2.9. *$\mathbf{F}_{CK}(T)$ is an (A^n, A^m)–bimodule for any flat tangle $T \in \mathbf{Cob}(m,n)$, where the actions of the algebras are given by μ. Moreover, (2.27) descend to natural isomorphisms of bimodules*

$$\mathbf{F}_{CK}(T') \underset{A^n}{\otimes} \mathbf{F}_{CK}(T) \xrightarrow{\cong} \mathbf{F}_{CK}(T'T), \tag{2.33}$$

so that there is a 2–functor $\mathbf{F}_{CK}\colon \mathbf{Cob} \longrightarrow \mathbf{Bimod}$.

Throughout the paper we call $\mathbf{F}_{CK}(T)$ a *diagrammatic bimodule* and we shall write **DB** for the additive bicategory generated by diagrammatic bimodules. They are called *geometric* in [CK14]. Each $\mathbf{F}_{CK}(T)$ has a two sided dual $\mathbf{F}_{CK}(T^!)$, where $T^!$ is the vertical flip of T. It is also known that each weight component $\mathbf{F}_{CK}(T;\lambda)$ is indecomposable when T contains no loops [BS10, Theorem 4.14], and otherwise $\mathbf{F}_{CK}(T;\lambda) \cong \mathbf{F}_{CK}(\widetilde{T};\lambda) \otimes \Bbbk^{2\ell}$ where $\widetilde{T}$ is the tangle T with ℓ loops removed. Therefore, **DB** has duals and is closed under direct summands.

Grading

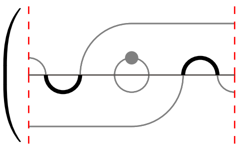
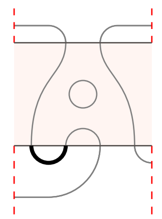
The grading on A^n is defined in [CK14] by shifting by n the grading induced by the functor $\mathscr{F}_{Kh}$. This does not work well for bimodules assigned to tangles, though. Regarding a cup diagram $c \in \mathscr{G}\mathscr{M}^n$ as a flat $(0,n)$–tangle, there is an isomorphism of graded bimodules $\mathbf{F}_{CK}(c^!) \otimes \mathbf{F}_{CK}(c) \cong e_c A^n e_c$, but $e_c A^n e_c$ is graded differently from $\mathbf{F}_{CK}(c^! c)$.

In [BS11, BS10] a grading is computed differently. It agrees with that from [CK14] for arc algebras, and it is coherent with tensor products of tangle bimodules. On the other hand, it depends on a Morse decomposition of a tangle, although it is well understood how the degree changes under planar isotopies, see [BS10, Lemma 2.4]. This motivates the following definition.

Let T be a flat tangle with ℓ loops and c arcs connecting bottom endpoints. Given a diagram $x \in \mathbf{F}_{CK}(T)$ orient all its curves counterclockwise. The platforms and boundary lines of T split some curves of x into vertical lines, caps, and cups. Let a be the number of cups and caps with clockwise orientation. Then the degree of x is given by the formula,

$$\deg x := \ell + c - a - 2d, \tag{2.34}$$

where d is the number of dots. For example,

$\deg \Bigg($  $\Bigg) = -4,$ and $\deg \Bigg($  $\Bigg) = 1,$ (2.35)

where the arcs with clockwise orientation are thick. A quick look on the surgery formulas (2.9) and (2.10) reveals that for any tangle cobordism S the map $\mathbf{F}_{CK}(S)$ is homogeneous of degree $\deg S$. Furthermore, $\mu_{T',T}\colon \mathbf{F}_{CK}(T') \otimes \mathbf{F}_{CK}(T) \longrightarrow \mathbf{F}_{CK}(T'T)$ preserves the degree, see [BS10, Theorem 3.5 (iii)], so that (2.33) is an isomorphism of graded bimodules. However, $\mathbf{F}_{CK}(T^!)$ is dual to $\mathbf{F}_{CK}(T)$ only up to a degree shift.

Chain complex and homology

Assume now that $T \in \mathbf{Tan}(m,n)$ is an oriented tangle in a thickened stripe with m points on the bottom and n on top. The *Chen–Khovanov complex* $C_{CK}(T) := \mathbf{F}_{CK}[\![T]\!]$ is the complex of (A^n, A^m)–bimodules, obtained from the formal bracket by applying the bifunctor $\mathbf{F}_{CK}$. We refer to the homology $Kh(T) := H(C_{CK}(T))$ as the *Chen–Khovanov homology*. It is a triply graded theory: beyond the homological and quantum grading $Kh(T)$ admits also a weight decomposition.

3 Generalized traces

3.1 Trace functions on endocategories

Choose a category $\mathscr{C}$ with an endofunctor $\Sigma\colon \mathscr{C} \longrightarrow \mathscr{C}$; we will refer to the pair $(\mathscr{C}, \Sigma)$ as an *endocategory*. This additional structure allows us to twist the usual definition of a trace on $\mathscr{C}$. For simplicity, we will often write $\mathscr{C}$ for the endocategory $(\mathscr{C}, \mathrm{Id})$.

Definition 3.1. Fix a set S and an endocategory $(\mathscr{C}, \Sigma)$. A *trace function* $\mathrm{tr}\colon (\mathscr{C}, \Sigma) \longrightarrow S$ is a collection of functions $\mathrm{tr}_x\colon \mathscr{C}(x, \Sigma x) \longrightarrow S$, one for each object $x \in \mathscr{C}$, such that $\mathrm{tr}_x(g \circ f) = \mathrm{tr}_y(\Sigma f \circ g)$ for every pair of morphisms $\Sigma x \xleftarrow{g} y \xleftarrow{f} x$.

We recover the usual definition of a symmetric trace when $\Sigma = \mathrm{Id}$. To make a distinction, we shall sometimes refer to generalized trace functions as *twisted traces* or *traces with a monodromy*, see Example 3.7 for an explanation.

Every trace function tr is Σ–invariant: $\mathrm{tr}_x(f) = \mathrm{tr}_x(\Sigma f)$ by taking $g = \mathrm{id}_x$ in the definition. Reversing the order of composition leads to a *dual trace function*, a collection of functions $\mathrm{tr}_x\colon \mathscr{C}(\Sigma x, x) \longrightarrow S$ satisfying $\mathrm{tr}_x(g \circ f) = \mathrm{tr}_y(f \circ \Sigma g)$. If Σ is invertible, then dual trace functions are precisely the trace functions on $(\mathscr{C}, \Sigma^{-1})$.

Fix a commutative unital ring $\Bbbk$. When $\mathscr{C}$ is $\Bbbk$–linear, we usually take a $\Bbbk$–module as S and require that each $\mathrm{tr}_x\colon \mathscr{C}(x, \Sigma x) \longrightarrow S$ is $\Bbbk$–linear. Suppose $\mathscr{C}$ is *pregraded*, i.e. the morphism sets of $\mathscr{C}$ are graded $\Bbbk$–modules

$$\mathscr{C}(x,y) = \bigoplus_{d\in\mathbb{Z}} \mathscr{C}_d(x,y), \tag{3.1}$$

and the degree is additive with respect to the composition and preserved by Σ, i.e. $\deg \Sigma f = \deg f$ if f is homogeneous. We can then deform Σ into Σ_q satisfying

$$\Sigma_q f := q^{-d} \Sigma f \tag{3.2}$$

for any $f \in \mathscr{C}_d(x,y)$ and fixed $q \in \Bbbk$.

Definition 3.2. A *quantum trace function* tr_q on $(\mathscr{C}, \Sigma)$ is a trace function on $(\mathscr{C}, \Sigma_q)$. In other words, we deform the trace condition into $\mathrm{tr}_q(f \circ g) = q^{-|g|}\,\mathrm{tr}_q(\Sigma g \circ f)$ for a homogeneous g.

The following observation is an immediate consequence of the above definition.

Proposition 3.3. *Assume $\Sigma = \mathrm{Id}$. Then $(1 - q^d)\,\mathrm{tr}_q(f) = 0$ for every quantum trace function tr_q and a homogeneous morphism f of degree d.*

A pregraded category $\mathscr{C}$ is *graded* if it admits an autoequivalence $x \longmapsto x\{1\}$, called a *degree shift*, and natural isomorphisms of $\Bbbk$–modules

$$\mathscr{C}_d(x, y) \cong \mathscr{C}_{d+1}(x, y\{1\}) \cong \mathscr{C}_{d-1}(x\{1\}, y). \tag{3.3}$$

A quantum trace function tr_q on a graded category $\mathscr{C}$ is precisely a trace function on the subcategory $\mathscr{C}_0$ of graded morphisms, satisfying $\operatorname{tr}_q(f\{1\}) = q \operatorname{tr}_q(f)$. It is understood that Σ commutes with the degree shift functor, i.e. there is a natural isomorphism $\Sigma(x\{1\}) \cong (\Sigma x)\{1\}$.

Choose an algebra A and an (A, A)–bimodule M. The *module of coinvariants* of M is defined as the quotient $\Bbbk$–module

$$\operatorname{coInv}(M) := M\big/_{[M, A]}, \tag{3.4}$$

where $[M, A] := \operatorname{span}_\Bbbk\{am - ma \mid m \in M, a \in A\}$. The following example extends the usual notion of the trace of an endomorphism of a projective A–module.

Example 3.4. A bimodule $M \in \mathscr{R}ep(A, A)$ induces an endofunctor $\Sigma P := M \otimes_A P$ on the category $\mathscr{R}ep(A)$ and we define a trace function tr_M as follows. For a homomorphism $f\colon A^n \longrightarrow M^n$ write

$$\operatorname{tr}_M(f) := \sum_{i=1}^{n} \bar{f}_{ii} \in \operatorname{coInv}(M), \tag{3.5}$$

where (f_{ij}) is the matrix of f with coefficients in M, and for each $m \in M$ we write $\overline{m}$ for its image in $\operatorname{coInv}(M)$. One checks that the trace relation $\operatorname{tr}_M(g \circ f) = \operatorname{tr}_M(\Sigma f \circ g)$ holds for each $f\colon A^n \longrightarrow A^m$ and $g\colon A^m \longrightarrow M^n$. Because each projective module is a direct summand of a free module, the formula (3.5) extends to all morphisms in $\mathscr{R}ep(A)$.

Example 3.5. Choose an algebra automorphism $\varphi \in \operatorname{Aut}(A)$ and a left A–module M. We construct the twisted module ${}_\varphi M$ by twisting the action of A by φ, i.e. $a \cdot m := \varphi(a)m$. Thence, a homomorphism $f\colon M \longrightarrow {}_\varphi M$ is the same as a $\Bbbk$–linear map $f\colon M \longrightarrow M$ satisfying $f(am) = \varphi(a)f(m)$. Clearly, $\varphi^{-1}\colon A \longrightarrow {}_{\varphi^{-1}}A$ is A–linear and ${}_{\varphi^{-1}}A \otimes_A {}_\varphi M \cong M$. Hence, we can define a twisted trace function

$$\operatorname{tr}_\varphi(f) := \operatorname{tr}(\varphi^{-1} \otimes f) \in \operatorname{coInv}({}_\varphi A) \tag{3.6}$$

where $\operatorname{tr}(\varphi^{-1} \otimes f)$ is the symmetric trace of the A–linear map $\varphi^{-1} \otimes f$ from $A \otimes_A M \cong M$ to ${}_{\varphi^{-1}}A \otimes_A {}_\varphi M \cong M$. It is a special case of Example 3.4.

Example 3.6. Assume A is graded and fix $q \in \Bbbk$. We define the quantum trace of a degree zero endomorphism $f\colon P \longrightarrow P$ of a graded projective A–module P as the sum

$$\operatorname{tr}_q(f) := \sum_{n \in \mathbb{Z}} q^n \operatorname{tr}(f|_{P_n}), \tag{3.7}$$

where $P = \bigoplus_{n \in \mathbb{Z}} P_n$ is the graded decomposition of P, and each P_n is considered as a module over the subalgebra $A_0 \subset A$. Then for homogeneous morphism $P \xleftarrow{g} Q \xleftarrow{f} P$, where $\deg f = k$ and $\deg g = -k$, we compute

$$\operatorname{tr}_q(g \circ f) = \sum_{n \in \mathbb{Z}} q^n \operatorname{tr}(g \circ f|_{P_n}) = \sum_{n \in \mathbb{Z}} q^{-k} q^{n+k} \operatorname{tr}(f \circ g|_{Q_{n+k}}) = q^{-k} \operatorname{tr}_q(g \circ f), \tag{3.8}$$

so that tr_q satisfies the quantum trace relation.

Another example of a twisted trace comes from low dimensional topology. Recall that a 3–manifold M is a *surface bundle over a circle*, when it is a fiber bundle over $\mathbb{S}^1$ with a surface F as its fiber. Each surface bundle can be described as a mapping torus

$$M := {F \times I}\Big/{(p,1) \sim (\phi(p), 0)} \tag{3.9}$$

of some diffeomorphism $\phi \in \mathit{Diff}(F)$. The diffeomorphism is unique up to an isotopy, and it is called the *monodromy* of M.

Example 3.7. Let M be a surface bundle with fiber F and monodromy $\phi \in \mathit{Diff}(F)$. Given a tangle $T \subset F \times I$ with input B and output $\phi^{-1}(B)$ we can construct its closure $\widehat{T} := \pi(T)$ in M, where $\pi\colon F \times I \longrightarrow M$ is the natural quotient map. This operation satisfies the trace relation with respect to the endofunctor ϕ_* on $\mathcal{T}an(F)$ that takes a tangle $T \subset F \times I$ into $(\phi^{-1} \times \mathrm{id})(T)$.

Proposition 3.8. *Let $\eta\colon \Sigma' \longrightarrow \Sigma$ be a natural transformation of endofunctors of a category $\mathscr{C}$ and choose a trace function (resp. quantum trace function) $\mathrm{tr}\colon (\mathscr{C}, \Sigma) \longrightarrow S$. Then $\eta^*\mathrm{tr}_x(f) := \mathrm{tr}_x(\eta_x \circ f)$ is a trace function (resp. quantum trace if each η_x has degree zero) on $(\mathscr{C}, \Sigma')$.*

Proof. Staightforward and left as an exercise. ☐

Example 3.9. Let $\mathrm{tr}\colon \mathcal{T}an(\mathbb{R}^2) \longrightarrow \mathit{Links}(\mathbb{R}^3)$ be the natural closure of T, defined by identifying $\mathbb{R}^2 \times I$ with a solid torus $\mathbb{R}^2 \times \mathbb{S}^1$ and embedding it into $\mathbb{R}^3$ in a standard way. In what follows we shall consider only tangles with endpoints on the horizontal line $\mathbb{R} \times \{0\}$. Choose a framed $(1,1)$–tangle T_0 and let $\eta_n \in \mathcal{T}an(\mathbb{R}^2)$ be the n–cabling of T_0. Then $\{\eta_n\}$ is a natural transformation of the identity functor on $\mathcal{T}an(\mathbb{R}^2)$, and $\eta^*\,\mathrm{tr}$ takes a tangle T to the satellite link with companion $\mathrm{tr}(T_0)$.

3.2 The universal trace of an endocategory

Each small endocategory $(\mathscr{C}, \Sigma)$ admits a universal trace function $\mathrm{tr}^u\colon (\mathscr{C}, \Sigma) \longrightarrow \mathrm{Tr}(\mathscr{C}, \Sigma)$, called *the trace of* $(\mathscr{C}, \Sigma)$, defined as the quotient

$$\mathrm{Tr}(\mathscr{C}, \Sigma) := {\coprod_{x \in \mathrm{Ob}(\mathscr{C})} \mathscr{C}(x, \Sigma x)}\Big/{g \circ f \sim \Sigma f \circ g} \tag{3.10}$$

$$\begin{tikzcd}
\mathscr{C}(x,\Sigma x) \arrow[r, "\mathrm{tr}_x"] \arrow[d, "\mathrm{tr}^u_x"'] & S \\
\mathrm{Tr}(\mathscr{C},\Sigma) \arrow[ur, dashed, "\exists! u"'] &
\end{tikzcd}$$

where f and g run through all pairs of morphisms $\Sigma x \xleftarrow{g} y \xleftarrow{f} x$ in $\mathscr{C}$. Universality means that each trace function $\mathrm{tr}\colon (\mathscr{C}, \Sigma) \longrightarrow S$ admits a unique function $u\colon \mathrm{Tr}(\mathscr{C}, \Sigma) \longrightarrow S$ such that $\mathrm{tr}_x = u \circ \mathrm{tr}^u_x$ for every $x \in \mathscr{C}$, see the diagram to the left.

If $(\mathscr{C}, \Sigma)$ is $\Bbbk$–linear, then we assume as before that S is a $\Bbbk$–module and each function $\mathrm{tr}_x\colon \mathscr{C}(x, \Sigma x) \longrightarrow S$ is $\Bbbk$–linear. In this setting the trace of $(\mathscr{C}, \Sigma)$ is the $\Bbbk$–module

$$\mathrm{Tr}(\mathscr{C}, \Sigma) := {\bigoplus_{x \in \mathrm{Ob}(\mathscr{C})} \mathscr{C}(x, \Sigma x)}\Big/{\mathrm{span}_{\Bbbk}\{g \circ f - \Sigma f \circ g\}}. \tag{3.11}$$

Likewise, for a fixed $q \in \Bbbk$ there is the *universal quantum trace* $\mathrm{Tr}_q(\mathscr{C}, \Sigma) := \mathrm{Tr}(\mathscr{C}, \Sigma_q)$ with the defining relation $g \circ f = q^{-|f|}\Sigma f \circ g$ for a homogeneous f.

We shall simply write $\mathrm{Tr}(\mathscr{C})$ and $\mathrm{Tr}_q(\mathscr{C})$ when $\Sigma = \mathrm{Id}$. The latter is a graded $\Bbbk$–module and its degree d submodule $\mathrm{Tr}_q(\mathscr{C}; d)$ is annihilated by $(1 - q^d)$. In particular, it vanishes when $\Bbbk$ is a field and $q^d \neq 1$.

Example 3.10. Let M be a surface bundle with a fiber F and monodromy ϕ. The closure operation from Example 3.7 is the universal trace of $\mathcal{T}an(F)$. Clearly, each link in M is a closure of a certain tangle, so that $Links(M)$ is a quotient set of $\mathrm{Tr}(\mathcal{T}an(F), \phi_*)$. Suppose that closures $\widehat{T}$ and $\widehat{T}'$ are isotopic in M. Because each isotopy can be expressed as a sequence of isotopies supported in small 3–balls, we may assume that the isotopy fixes some fiber $F' \subset M$. Then the cuts $\widehat{T}$ and $\widehat{T}'$ along F' are isotopic in $F \times I$ and the trace relation implies the images of T and T' in $\mathrm{Tr}(\mathcal{T}an(F), \phi_*)$ coincide.

Example 3.11. The Kauffman Bracket Skein Module $\mathcal{S}(M)$ of an oriented 3–manifold M is a $\mathbb{Z}[A, A^{-1}]$–module generated by isotopy classes of framed tangles in M (we require the isotopies to fix ∂M if nonempty) modulo the local relations

$$\times = A \asymp + A^{-1})(, \text{ and} \tag{3.12}$$
$$\bigcirc = -A^2 - A^{-2}, \tag{3.13}$$

in which all the parts of tangles are picked by some ball. In particular, the circle in the second relation bounds a disk. As in the previous example, $\mathcal{S}(F \times I)$ is a category when F is a surface, and $\mathrm{Tr}(\mathcal{S}(F \times I), \phi_*) = \mathcal{S}(M)$ where M is the surface bundle with fiber F and monodromy ϕ.

The trace is a functor from the category of small (resp. $\Bbbk$–linear) endocategories to the category of sets (resp. $\Bbbk$–modules). It is easy to check that it preserves small products and small coproducts. Write $\mathscr{C}^{\oplus}$ for the additive closure of $\mathscr{C}$ and $\mathrm{Kar}(\mathscr{C})$ for its Karoubi envelope. The properties listed below are immediate generalizations of the corresponding properties of the usual trace of a category, see e.g. [BHLZ14].

Proposition 3.12. *The natural inclusions of endocategories*

$$\mathscr{C} \longrightarrow \mathscr{C}^{\oplus}, \qquad x \longmapsto (x) \tag{3.14}$$
$$\mathscr{C} \longrightarrow \mathrm{Kar}(\mathscr{C}), \qquad x \longmapsto \mathrm{id}_x \tag{3.15}$$

induce natural isomorphisms of the universal traces.

Proof. For the case of the additive closure we compute

$$\begin{aligned} \mathrm{tr}^u(\Sigma\underline{x} \xleftarrow{(f_{ij})} \underline{x}) &= \sum_i \mathrm{tr}^u(\Sigma\underline{x} \xleftarrow{(f_{ij})} \underline{x} \xleftarrow{\pi_i} x_i \xleftarrow{\iota_i} \underline{x}) \\ &= \sum_i \mathrm{tr}^u(\Sigma x_i \xleftarrow{\Sigma\iota_i} \Sigma\underline{x} \xleftarrow{(f_{ij})} \underline{x} \xleftarrow{\pi_i} x_i) = \sum_i \mathrm{tr}^u(\Sigma x_i \xleftarrow{f_{ii}} x_i) \end{aligned}$$

where $\underline{x} \xrightarrow{\pi_i} x_i$ and $x_i \xrightarrow{\iota_i} \underline{x}$ are the canonical projections and inclusions. The case of the Karoubi envelope is proven likewise:

$$\begin{aligned} \mathrm{tr}^u(\Sigma e \xleftarrow{f} e) &= \mathrm{tr}^u(\Sigma e \xleftarrow{f} e \xleftarrow{e} x \xleftarrow{e} e) \\ &= \mathrm{tr}^u(\Sigma x \xleftarrow{\Sigma e} \Sigma e \xleftarrow{f} e \xleftarrow{e} x) = \mathrm{tr}^u(\Sigma x \xleftarrow{f} x) \end{aligned}$$

where $e \in \mathscr{C}(x, x)$ is an idempotent such that $\Sigma e \circ f \circ e = f$ in $\mathscr{C}$. ◻

Corollary 3.13. *Choose a* $\Bbbk$*–algebra* A *with* $\varphi \in \mathrm{Aut}(A)$*. The trace from Example 3.5 is universal, i.e.* $\mathrm{Tr}(\mathscr{R}ep(A), \varphi_*) \cong \mathrm{coInv}({}_{\varphi}A)$ *for the endofunctor* $\varphi_*(M) := {}_{\varphi}M$*.*

Proof. It follows from the equivalence of categories $\mathscr{R}ep(A) \simeq \mathrm{Kar}(\mathscr{A}^{\oplus})$, where $\mathscr{A}$ is the full subcategory of $\mathscr{R}ep(A)$ with A as its unique object. $\square$

Let $\mathscr{C}' \subset \mathscr{C}$ be a Σ–invariant subcategory (i.e. Σ acts as the identity functor on $\mathscr{C}'$). Then $\mathrm{Tr}(\mathscr{C}', \Sigma) = \mathrm{Tr}(\mathscr{C}')$ resulting in a natural map $\mathrm{Tr}(\mathscr{C}') \longrightarrow \mathrm{Tr}(\mathscr{C}, \Sigma)$. In particular, there is a $\Bbbk$–linear map $\mathrm{Tr}(\mathscr{C}_0) \longrightarrow \mathrm{Tr}_q(\mathscr{C})$ when $\mathscr{C}$ is pregraded.

Example 3.14. Assume that Σ *stabilizes*, i.e. for each morphism f there is $N > 0$ such that $\Sigma^{N+1} f = \Sigma^N f$; write $\Sigma^\infty f := \Sigma^N f$. Then $[f]_{\mathrm{Tr}} = [\Sigma^\infty f]_{\mathrm{Tr}}$, so that $\mathrm{Tr}(\mathscr{C}, \Sigma) = \mathrm{Tr}(\Sigma^\infty \mathscr{C}, \Sigma) = \mathrm{Tr}(\Sigma^\infty \mathscr{C})$, where the last equality holds, because $\Sigma^\infty \mathscr{C}$ is Σ–invariant.

Corollary 3.15. *Choose a Σ–invariant subcategory $\mathscr{C}' \subset \mathscr{C}$. There is a homomorphism of abelian groups*

$$h\colon K_0^{sp}(\mathscr{C}') \longrightarrow \mathrm{Tr}(\mathscr{C}, \Sigma), \qquad [x]_{\cong} \longmapsto [\mathrm{id}_x]_{\mathrm{Tr}}, \tag{3.16}$$

called the Chern character*, which is a homomorphism of $\mathbb{Z}[q, q^{-1}]$–modules if $\mathscr{C}$ is graded. In general, it is neither injective nor surjective.*

Remark. There is a *dual universal trace* $\mathrm{Tr}^{op}(\mathscr{C}, \Sigma)$, which is generated by morphisms $\Sigma x \longrightarrow x$ modulo the relation $g \circ f \sim f \circ \Sigma g$. It is naturally isomorphic to $\mathrm{Tr}(\mathscr{C}, \Sigma^{-1})$ when Σ is invertible. All results from this section have their analogues for the dual trace.

3.3 Categorified traces

A categorified notion of a symmetric trace function appeared in [PS13] under the name *shadow*, and later a slightly more general version was introduced in [HPT15]. Here we adapt the ideas to endobicategories, weakening the definition at the same time. Thence the name *preshadow*. A brief review of bicategories is included in Appendix A.1.

An *endobicategory* $(\mathbf{C}, \Sigma)$ consists of a bicategory $\mathbf{C}$ and a bifunctor $\Sigma\colon \mathbf{C} \longrightarrow \mathbf{C}$. Hence, there are natural isomorphisms $\mathfrak{m}\colon \Sigma g \circ \Sigma f \overset{\cong}{\Longrightarrow} \Sigma(g \circ f)$ and $\mathfrak{i}\colon \Sigma(\mathrm{id}_x) \overset{\cong}{\Longrightarrow} \mathrm{id}_{\Sigma x}$, and Σ preserves dual morphisms up to 2-isomorphisms.

Definition 3.16. A *preshadow* on an endobicategory $(\mathbf{C}, \Sigma)$ valued in a category $\mathscr{T}$ is a collection of functors $\langle\!\langle - \rangle\!\rangle_x\colon \mathbf{C}(x, \Sigma x) \longrightarrow \mathscr{T}$, one for each object $x \in \mathbf{C}$, and morphisms $\theta_{f,g}\colon \langle\!\langle g \circ f \rangle\!\rangle_x \longrightarrow \langle\!\langle \Sigma f \circ g \rangle\!\rangle_y$ in $\mathscr{T}$, one for each pair of 1–morphisms $\Sigma x \xleftarrow{g} y \xleftarrow{f} x$, such that for any 2–morphisms $\alpha\colon f \Longrightarrow f'$ and $\beta\colon g \Longrightarrow g'$

$$\theta_{f',g'} \circ \langle\!\langle \beta \circ \alpha \rangle\!\rangle_x = \langle\!\langle \Sigma\alpha \circ \beta \rangle\!\rangle_y \circ \theta_{f,g} \tag{3.17}$$

and the following diagrams commute

$$\begin{tikzcd}
\langle\!\langle (h \circ g) \circ f \rangle\!\rangle_x \arrow[r, "\theta"] \arrow[d, "\langle\!\langle \mathfrak{a}^{-1} \rangle\!\rangle"'] & \langle\!\langle \Sigma f \circ (h \circ g) \rangle\!\rangle_y \arrow[r, "\langle\!\langle \mathfrak{a} \rangle\!\rangle"] & \langle\!\langle (\Sigma f \circ h) \circ g \rangle\!\rangle_y \arrow[d, "\theta"] \\
\langle\!\langle h \circ (g \circ f) \rangle\!\rangle_x \arrow[r, "\theta"] & \langle\!\langle \Sigma(g \circ f) \circ h \rangle\!\rangle_z \arrow[r, "\langle\!\langle (\mathfrak{m} \circ \mathbf{1}) * \mathfrak{a})^{-1} \rangle\!\rangle"] & \langle\!\langle \Sigma g \circ (\Sigma f \circ h) \rangle\!\rangle_z
\end{tikzcd} \tag{3.18}$$

$$\begin{tikzcd}
\langle\!\langle k \circ \mathrm{id}_x \rangle\!\rangle_x \arrow[r, "\theta"] \arrow[dr, "\langle\!\langle \mathfrak{r} \rangle\!\rangle"'] & \langle\!\langle \Sigma(\mathrm{id}_x) \circ k \rangle\!\rangle_x \arrow[r, "\langle\!\langle \mathfrak{i} \circ \mathbf{1} \rangle\!\rangle"] & \langle\!\langle \mathrm{id}_{\Sigma x} \circ k \rangle\!\rangle_x \arrow[dl, "\langle\!\langle \mathfrak{l} \rangle\!\rangle"] \\
& \langle\!\langle k \rangle\!\rangle_x &
\end{tikzcd} \tag{3.19}$$

for all $\Sigma x \xleftarrow{h} z \xleftarrow{g} y \xleftarrow{f} x$ and $\Sigma x \xleftarrow{k} x$. A preshadow is called a *shadow* if each $\theta_{f,g}$ is an isomorphism. We refer to preshadows as *symmetric* when $\Sigma = \mathrm{Id}$. If each morphism category $\mathbf{C}(x,y)$ is $\Bbbk$–linear, we assume that both $\mathcal{T}$ and the functors $\langle\!\langle - \rangle\!\rangle_x$ are $\Bbbk$–linear.

Our definition of a symmetric shadow coincides with the categorified trace as defined in [HPT15] when $\mathbf{C}$ has only one object, i.e. it is a monoidal category. In comparison to [PS13] we do not require $\theta_{f,g} \circ \theta_{g,f} = \mathrm{id}$.

A $\Bbbk$–linear bicategory $\mathbf{C}$ is *pregraded* if each $\mathbf{C}(x,y)$ is pregraded, the degree is additive with respect to the horizontal composition, and the canonical 2–isomorphisms (A.1) are homogeneous of degree zero. In such a case we require that Σ preserves the degree of 2–morphisms and that $\mathfrak{m}$ and $\mathfrak{i}$ are homogeneous of degree 0. As in the case of categories, there is a bifunctor Σ_q that agrees with Σ on objects and 1–morphisms, but

$$\Sigma_q \alpha := q^{-|\alpha|} \Sigma \alpha \tag{3.20}$$

for any homogeneous 2–morphism α.

Definition 3.17. A *quantum preshadow* on a pregraded endobicategory $(\mathbf{C}, \Sigma)$ is a preshadow on $(\mathbf{C}, \Sigma_q)$. In other words, the condition (3.17) is twisted by an appropriate power of q, but diagrams (3.18) and (3.19) are unchanged.

We can construct new preshadows from a given one as we did for traces on categories.

Proposition 3.18. *Let $\eta\colon \Sigma' \longrightarrow \Sigma$ be a natural transformation of endobifunctors on $\mathbf{C}$ and choose a preshadow (resp. quantum preshadow) $(\langle\!\langle - \rangle\!\rangle, \theta)$ on $(\mathbf{C}, \Sigma)$. Then there is a preshadow (resp. quantum preshadow if each η_f has degree zero) $(\eta^*\langle\!\langle - \rangle\!\rangle, \eta^*\theta)$ with*

$$\eta^*\langle\!\langle f \rangle\!\rangle_x := \langle\!\langle \eta_x \circ f \rangle\!\rangle_x, \qquad \eta^*\langle\!\langle \alpha \rangle\!\rangle_x := \langle\!\langle \mathrm{id} \circ \alpha \rangle\!\rangle_x,$$

for a 1–morphism $f\colon x \longrightarrow \Sigma x$ and a 2–morphism $\alpha\colon f \Longrightarrow f'$, and the morphism $\eta^*\theta_{f,g}\colon \eta^*\langle\!\langle g \circ f \rangle\!\rangle_x \longrightarrow \eta^*\langle\!\langle \Sigma' f \circ g \rangle\!\rangle_y$ defined as the composition

$$\langle\!\langle \eta_x \circ g \circ f \rangle\!\rangle_x \xrightarrow{\theta} \langle\!\langle \Sigma f \circ \eta_x \circ g \rangle\!\rangle_y \xrightarrow{\langle\!\langle \eta_f \circ \mathrm{id} \rangle\!\rangle} \langle\!\langle \eta_y \circ \Sigma' f \circ g \rangle\!\rangle_y.$$

Proof. Left as an exercise. ☐

We end this section with several examples of shadows. In what follows A and B are $\Bbbk$–algebras, **Bimod** is the bicategory of finitely generated bimodules, and $\mathbf{Rep} \subset \mathbf{Bimod}$ is the subbicategory of those bimodules that are finitely generated and projective as left modules, see Appendix.

Example 3.19. The module of coinvariants of an (A,A)–bimodule, as defined in (3.4), is a shadow on **Bimod**: given bimodules $M \in \mathcal{B}imod(A,B)$ and $N \in \mathcal{B}imod(B,A)$ there is a natural isomorphism $\theta_{M,N}\colon \mathrm{coInv}(M \otimes_B N) \longrightarrow \mathrm{coInv}(N \otimes_A M)$, which takes $[m \otimes n]$ to $[n \otimes m]$. Notice that $\theta_{M,N} \circ \theta_{N,M} = \mathrm{id}$.

Example 3.20. Let M be a chain complex of (A,A)–bimodules. The *Hochschild homology $HH(M;A)$* of M is defined as the derived of the coinvariant module functor, i.e. it is the homology of the chain complex

$$CH(M;A) := \mathrm{coInv}(M \underset{A}{\otimes} R_\bullet(A)) = M \underset{A^e}{\otimes} R_\bullet(A), \tag{3.21}$$

where $R_\bullet(A)$ is the bar resolution of A (see Example A.9) and $A^e := A \otimes A^{op}$.[4] This is a symmetric shadow on $\mathscr{D}^-(\mathbf{Bimod})$ with an involutive isomorphism

$$HH(M \underset{B}{\hat{\otimes}} N; A) \xrightarrow{\cong} HH(N \underset{A}{\hat{\otimes}} M; B) \tag{3.22}$$

induced by the isomorphism $\mathrm{coInv}(M \otimes_B N) \cong \mathrm{coInv}(N \otimes_A M)$. The passage to the derived category is not necessary when we restrict to bimodules from **Rep**. Indeed, the derived tensor product of such bimodules coincides with the usual one, so that the Hochschild homology is already a shadow on **Rep**.

Consider the bicategory **gBimod** of graded bimodules and fix an invertible $q \in \Bbbk$. Following Example 3.5 we write ${}_qM$ for a given graded (A, B)–bimodule M with a twisted left action: $a \cdot m := q^{-d}am$ where $a \in A_d$. If M is a graded (A, A)–bimodule, we define its *quantum module of coinvariants* as

$$\mathrm{coInv}_q(M) := \mathrm{coInv}({}_qM) = {}^{M}\!\big/_{[M, A]_q}, \tag{3.23}$$

where $[M, A]_q := \mathrm{span}_\Bbbk\{q^{-d}am - ma \mid a \in A_d, m \in M\}$. Likewise, we define the *quantum Hochschild homology* of a chain complex M of graded (A, A)–bimodules as the homology of $qCH(M; A) := CH({}_qM; A)$. Explicitly, the last term in the Hochschild differential is replaced with

$$d_n(m \otimes a_1 \otimes \cdots \otimes a_n) := q^{-|a_n|}a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}. \tag{3.24}$$

Proposition 3.21. *The quantum module of coinvariants is a quantum shadow on the bicategory of graded bimodules* **gBimod***, and the quantum Hochschild homology is a quantum shadow on both* **gRep** *and the derived bicategory* $\mathscr{D}^-(\mathbf{gBimod})$*.*

Proof. Because both coInv and HH are shadows, it is enough to construct an isomorphism of graded bimodules ${}_qM \otimes_B N \cong {}_qN \otimes_A M$ for every $M \in \mathrm{g}\mathscr{B}imod(A, B)$ and $N \in \mathrm{g}\mathscr{B}imod(B, A)$. For that use the isomorphisms

$$\begin{aligned} {}_qA \underset{A}{\otimes} M &\xrightarrow{\cong} {}_qM, \quad & a \otimes m &\longmapsto am, \\ M \underset{B}{\otimes} {}_qB &\xrightarrow{\cong} {}_qM, \quad & m \otimes b &\longmapsto q^{|m|}mb. \end{aligned}$$

In particular, $\theta_{M,N} \colon \mathrm{coInv}_q(M \otimes_B N) \xrightarrow{\cong} \mathrm{coInv}_q(N \otimes_A M)$ takes $[m \otimes n]$ to $q^{-|n|}[n \otimes m]$ for homogeneous $m \in M$ and $n \in N$. ☐

Another examples of shadows are given by twisting. Let $\widetilde{\mathbf{Bimod}}$ be the bicategory obtained from **Bimod** by replacing objects with pairs (A, φ) consisting of an algebra A and its automorphism φ. Regarding a bimodule $M \in \mathscr{B}imod(A, B)$ as a 1–morphism from (B, ψ) to (A, φ) define ΣM by twisting the actions of the algebras by the automorphisms:

$$a \cdot m \cdot b \;:=\; \varphi^{-1}(a)m\psi^{-1}(b) \tag{3.25}$$

for any $m \in M$, $a \in A$, and $b \in B$. Then Σ is an endobifunctor on $\widetilde{\mathbf{Bimod}}$. We construct a graded version $(\mathbf{g}\widetilde{\mathbf{Bimod}}, \Sigma)$ likewise.

[4] The *opposite ring* A^{op} has the same set of elements as A, but the opposite product: $a \cdot b := ba$.

Given a pair $(A, \varphi) \in \widetilde{\mathbf{Bimod}}$ define a *twisted module of coinvariants* of an (A, A)–bimodule M as

$$\operatorname{coInv}_\varphi(M) := \operatorname{coInv}({}_\varphi M) \cong \operatorname{coInv}({}_\varphi A \underset{A}{\otimes} M), \tag{3.26}$$

where the defining relation is replaced by $ma = \varphi(a)m$. Likewise, if M is a chain complex of (A, A)–bimodules, then its *twisted Hochschild homology* is given by the formula

$$HH(M; A, \varphi) := HH({}_\varphi M, A). \tag{3.27}$$

As before, both constructions admit quantized versions when $\widetilde{\mathbf{Bimod}}$ is replaced with analogously defined $\mathbf{g}\widetilde{\mathbf{Bimod}}$.

Proposition 3.22. *The twisted modules of coinvariants and the twisted Hochschild homology are shadows on endobicategories $(\widetilde{\mathbf{Bimod}}, \Sigma)$ and $(\mathcal{D}^-(\widetilde{\mathbf{Bimod}}), \Sigma)$ respectively. The quantized versions are quantum shadows on $(\mathbf{g}\widetilde{\mathbf{Bimod}}, \Sigma)$ and $(\mathcal{D}^-(\mathbf{g}\widetilde{\mathbf{Bimod}}), \Sigma)$.*

Proof. Given bimodules $M \in \mathscr{B}imod(A, B)$ and $N \in \mathscr{B}imod(B, A)$ there is an isomorphism $\operatorname{coInv}_\varphi(M \otimes_B N) \cong \operatorname{coInv}_\varphi(\Sigma N \otimes_A M)$ that takes $[m \otimes n]$ to $[n \otimes m]$. The quantum case follows the same way as in the proof of Proposition 3.21. ☐

3.4 The horizontal trace as a universal preshadow

We shall now generalize the horizontal trace [BHLZ14] to endobicategories. The main result states that the horizontal trace is a universal preshadow if the bicategory has right duals.

Definition 3.23. The *horizontal trace* of an endobicategory $(\mathbf{C}, \Sigma)$ is the category $\operatorname{hTr}(\mathbf{C}, \Sigma)$, whose objects are 1–morphisms $f\colon x \longrightarrow \Sigma x$ in $\mathbf{C}$ and morphisms from $f\colon x \longrightarrow \Sigma x$ to $g\colon y \longrightarrow \Sigma y$ are equivalence classes $[p, \alpha]$ of squares

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & \Sigma x \arrow[d, "\Sigma p"] \\ y \arrow[r, "g"'] & \Sigma y \end{tikzcd} \tag{3.28}$$

where $\alpha\colon \Sigma p \circ f \Longrightarrow g \circ p$ is a 2–morphism in $\mathbf{C}$, modulo the relation

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, bend right=50, "p"'] \arrow[d, bend left=50, "p'"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow\alpha"] & \Sigma x \arrow[d, "\Sigma p"] \\ y \arrow[r, "g"'] & \Sigma y \end{tikzcd} \sim \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & \Sigma x \arrow[d, bend right=50, "\Sigma p"'] \arrow[d, bend left=50, "\Sigma p'"] \arrow[d, phantom, "\overset{\Sigma\tau}{\Leftarrow}"] \\ y \arrow[r, "g"'] & \Sigma y \end{tikzcd} \tag{3.29}$$

for 1–morphisms $p, p'\colon x \longrightarrow y$, and 2–morphisms $\alpha\colon \Sigma p \circ f \Longrightarrow g \circ p'$ and $\tau\colon p' \Longrightarrow p$.

The defining relation (3.29) can be written algebraically as

$$(p, (\mathbf{1}_g \circ \tau) * \alpha) \sim\ (p', \alpha * (\Sigma\tau \circ \mathbf{1}_f)). \tag{3.30}$$

The composition of two morphisms $[p, \alpha]\colon f \longrightarrow g$ and $[q, \beta]\colon g \longrightarrow h$ is defined by stacking the squares (3.28), one on top of the other:

$$[q, \beta] \circ [p, \alpha] := [q \circ p, (\beta \circ \mathbf{1}_p) * (\mathbf{1}_{\Sigma q} \circ \alpha)],$$

and the identity morphism on $f\colon x \longrightarrow \Sigma x$ is given by $\mathrm{id}_f := [\mathrm{id}_x, \mathbf{1}_f]$. Unitarity and associativity of the composition follows from (3.29) with an appropriate composition of associators and unitors as τ. Hence, $\mathrm{hTr}(\mathbf{C},\Sigma)$ is indeed a category.

If $(\mathbf{C},\Sigma)$ is pregraded, then we define the *quantum horizontal trace* $\mathrm{hTr}_q(\mathbf{C},\Sigma) := \mathrm{hTr}(\mathbf{C},\Sigma_q)$, where Σ_q is given as in (3.20). We shall use the simplified notation $\mathrm{hTr}(\mathbf{C})$ and $\mathrm{hTr}_q(\mathbf{C})$ when $\Sigma = \mathrm{Id}$.

Consider the obvious functors $\mathbf{C}(x,\Sigma x) \longrightarrow \mathrm{hTr}(\mathbf{C},\Sigma)$ and morphisms $\theta^{\mathrm{h}}_{f,g} := [f,\mathfrak{a}]$ represented by squares

$$\begin{tikzcd} x \arrow[r, "g\circ f"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow\mathfrak{a}"] & \Sigma x \arrow[d, "\Sigma f"] \\ y \arrow[r, "\Sigma f\circ g"'] & \Sigma y \end{tikzcd} \tag{3.31}$$

We check directly that this datum describes a preshadow $\langle\!\langle - \rangle\!\rangle^{\mathrm{h}}\colon (\mathbf{C},\Sigma) \longrightarrow \mathrm{hTr}(\mathbf{C},\Sigma)$. Our goal is to prove that it is a universal preshadow on $(\mathbf{C},\Sigma)$, but first let us formalize what a factorization of a preshadow is.

Definition 3.24. Let $\langle\!\langle - \rangle\!\rangle\colon (\mathbf{C},\Sigma) \longrightarrow \mathcal{T}$ and $\langle\!\langle - \rangle\!\rangle'\colon (\mathbf{C},\Sigma) \longrightarrow \mathcal{T}'$ be two preshadows. A *morphism of preshadows* $(T,\eta)\colon \langle\!\langle - \rangle\!\rangle \longrightarrow \langle\!\langle - \rangle\!\rangle'$ is a functor $T\colon \mathcal{T} \longrightarrow \mathcal{T}'$ and a collection of natural transformations $\eta_x\colon T\circ\langle\!\langle - \rangle\!\rangle_x \longrightarrow \langle\!\langle - \rangle\!\rangle'_x$ such that $\eta_y \circ T(\theta_{f,g}) = \theta'_{f,g}\circ\eta_x$ for any $\Sigma x \xleftarrow{g} y \xleftarrow{f} x$. We say that $\langle\!\langle - \rangle\!\rangle'$ *factorizes through* $\langle\!\langle - \rangle\!\rangle$ when each η_x is a natural isomorphism.

Definition 3.25. A natural transformation $\epsilon\colon T \longrightarrow T'$ is a *transformation of preshadow morphisms* (T,η) and (T',η') when $\eta_x = \eta'_x * (\epsilon \circ \mathbf{1}_{\langle\!\langle - \rangle\!\rangle})$. We say it is an *equivalence of morphisms* if ϵ is a natural isomorphism.

The following diagrams visualize the condition for ϵ to be a transformation between morphisms of preshadows (T,η) and (T',η'):

$$\begin{tikzcd} \mathbf{C}(x,\Sigma x) \arrow[r, bend left, "\langle\!\langle - \rangle\!\rangle_x"] \arrow[r, bend right, "\langle\!\langle - \rangle\!\rangle'_x"'] \arrow[r, phantom, "\swarrow\eta'"] & \mathcal{T} \arrow[d, bend right, "T'"'] \arrow[d, bend left, "T"] \arrow[d, phantom, "\overset{\epsilon}{\Leftarrow}"] \\ & \mathcal{T}' \end{tikzcd} = \begin{tikzcd} \mathbf{C}(x,\Sigma x) \arrow[r, bend left, "\langle\!\langle - \rangle\!\rangle_x"] \arrow[r, bend right, "\langle\!\langle - \rangle\!\rangle'_x"'] \arrow[r, phantom, "\swarrow\eta"] & \mathcal{T} \arrow[d, bend left, "T"] \\ & \mathcal{T}' \end{tikzcd} \tag{3.32}$$

Theorem 3.26. *If* $\mathbf{C}$ *has left duals, then every preshadow on* $(\mathbf{C},\Sigma)$ *factorizes through* $\mathrm{hTr}(\mathbf{C},\Sigma)$ *uniquely up to an equivalence. Moreover,* $\langle\!\langle - \rangle\!\rangle^{\mathrm{h}}$ *is a shadow when* $\mathbf{C}$ *has right duals.*

Proof. Given a preshadow $\langle\!\langle - \rangle\!\rangle\colon (\mathbf{C},\Sigma) \longrightarrow \mathcal{T}$ we construct a functor $T\colon \mathrm{hTr}(\mathbf{C},\Sigma) \longrightarrow \mathcal{T}$ by taking $x \xrightarrow{f} \Sigma x$ to $\langle\!\langle f \rangle\!\rangle$ and a morphism $[p,\alpha]\colon f \longrightarrow g$ to the composition

$$\begin{aligned} \langle\!\langle f \rangle\!\rangle \xrightarrow{\langle\!\langle \mathbf{1}\circ coev \rangle\!\rangle} \langle\!\langle f\circ {}^*p\circ p \rangle\!\rangle &\xrightarrow{\theta_{p,f\circ {}^*p}} \langle\!\langle \Sigma p\circ f\circ {}^*p \rangle\!\rangle \\ &\xrightarrow{\langle\!\langle \alpha\circ\mathbf{1} \rangle\!\rangle} \langle\!\langle g\circ p\circ {}^*p \rangle\!\rangle \xrightarrow{\langle\!\langle \mathbf{1}\circ ev \rangle\!\rangle} \langle\!\langle g \rangle\!\rangle \end{aligned} \tag{3.33}$$

where for clarity we omitted associators and unitors. Notice that $\langle\!\langle - \rangle\!\rangle = T\circ\langle\!\langle - \rangle\!\rangle^{\mathrm{h}}$ and $\theta_{f,g} = T(\theta^{\mathrm{h}}_{f,g})$.

For uniqueness, choose a morphism of shadows $(T',\eta)\colon \langle\!\langle - \rangle\!\rangle^{\mathrm{h}} \longrightarrow \langle\!\langle - \rangle\!\rangle$ such that each η_x is an isomorphism. It can be regarded as a natural isomorphism $\eta\colon T' \longrightarrow T$, because

$\langle\!\langle f\rangle\!\rangle^{\mathrm{h}} = f$ and $Tf = \langle\!\langle f\rangle\!\rangle$; naturality follows from the definition of T on morphisms (3.33) and an observation that a similar sequence determines $T'([p,\alpha])$.

For the second statement consider a morphism in $\mathrm{hTr}(\mathbf{C},\Sigma)$ represented by

$$\begin{tikzcd} y \arrow[r, "g"] \arrow[d, "f^*"'] \arrow[dr, "\mathrm{id}_y" description] & \Sigma x \arrow[r, "\Sigma f"] \arrow[dr, "\mathrm{id}_{\Sigma x}" description] & \Sigma y \arrow[d, "\Sigma f^*"] \\ x \arrow[r, "f"'] & y \arrow[r, "g"'] & \Sigma x \end{tikzcd} \tag{3.34}$$

where we fill the triangles with the coevaluation and evaluation 2–morphisms, and the middle square with a suitable composition of unitors. It is a two-sided inverse of $\theta^{\mathrm{h}}_{f,g}$ due to the relations (A.5) between ev and $coev$. ☐

Remark. There is a dual construction of the horizontal trace, where we reverse the 2–morphism in (3.28). Such a category provides a universal preshadow when $\mathbf{C}$ has right duals, and it is a shadow when $\mathbf{C}$ has left duals.

When $\Sigma = \mathrm{id}$, then one can think of $\mathrm{hTr}(\mathbf{C})$ as the category of pictures on a vertical cylinder. An object is a circle with a basepoint decorated with some endomorphism, and a morphism is a tube with a vertical string connecting the basepoints, which cuts the tube into a square (3.28). The string can be deformed (compare with (3.29)), but its endpoints must remain fixed. In particular, performing a twist on the tube changes the morphism, contrary to [PS13].

The symmetric horizontal trace is sometimes called the *annularization functor* [MW10]. Its generalization to any surface is known as *factorization homology* [BZBJ15].

3.5 Computation of horizontal traces

It is generally a hard problem to identify the horizontal trace $\mathrm{hTr}(\mathbf{C},\Sigma)$ for a given bicategory $\mathbf{C}$, but the answer is very natural for the bicategory of tangles and alike. We begin with a proof of the first result stated in the introduction

Theorem A. *Let M be a surface bundle with fiber F and monodromy $\phi \in \mathit{Diff}(F)$. Then there is an equivalence of categories*

$$\mathrm{hTr}(\mathbf{Tan}(F),\phi_*) \simeq \mathscr{L}\!\mathit{inks}(M) \tag{3.35}$$

where $\phi_*(S) := (\phi^{-1}\times\mathrm{id}\times\mathrm{id})(S)$ for a cobordism $S \subset F\times I\times I$.

Proof. Let $\pi\colon F\times I \longrightarrow M$ be the natural quotient map and consider the fiber $F_0 := \pi(F\times\{0\})$ along which M can be cut open to $F\times I$. Objects of $\mathrm{hTr}(\mathbf{Tan}(F),\phi_*)$ can be identified with links in M transverse to F_0, while morphisms are represented by link cobordisms in $M\times I$, transverse to the *membrane* $F_0\times I$. The surface can be deformed by an ambient isotopy that fixes the membrane, and the trace relation allows us to isotope the embedding of the membrane (although $F_0\times\{i\}$ is fixed for $i=0,1$).

There is an obvious functor $\mathrm{hTr}(\mathbf{Tan}(F),\phi_*) \longrightarrow \mathscr{L}\!\mathit{inks}(M)$ that forgets the membrane. By the transversality argument it is essentially surjective on objects (each link is isotopic to a link transverse to F_0) and full on morphisms (each surface between links transverse to F_0 can be isotoped to be transverse to the standard membrane $F_0\times I$). It remains to show that if two surfaces $S, S' \subset F\times I\times I$ represent isotopic cobordisms $\widehat{S}$ and $\widehat{S}'$ in $M\times I$, then their images in $\mathrm{hTr}(\mathbf{Tan}(F),\phi_*)$ coincide.

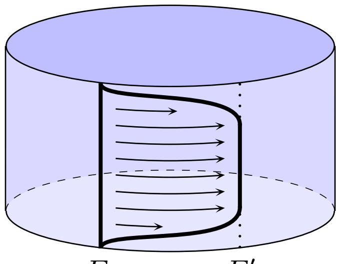


Figure 6: A visualization of the isotopy pushing $F_0 \times I$ onto $F' \times I$. Each point of the cylinder represents a fiber of the F–bundle $M \times I \longrightarrow \mathbb{S}^1 \times I$. The thick straight line is the standard membrane $F_0 \times I$, whereas the curve is its isotopic deformation. The flat part of the curve corresponds to the piece of the deformed membrane contained in $F' \times I$.

Assume there is an isotopy φ_t of $M \times I$ taking $\widehat{S}$ to $\widehat{S}'$ with support[5] $\operatorname{supp}(\varphi)$ disjoint from $M \times \partial I$ and a membrane $F' \times I$ for some fiber $F' \subset M$. It is enough to consider only such isotopies, because every two isotopic surfaces in $M \times I$ are connected by a sequence of them. If $F' = F_0$, then S and S' are already isotopic in $F \times I \times I$ and we are done. Otherwise, let $p \in \mathbb{S}^1$ be the point over which F' lives and consider a bump function $\beta\colon I \longrightarrow \mathbb{S}^1$ with $\beta(0) = \beta(1) = 1$ and $\beta(t) = p$ for $t \in [\epsilon, 1-\epsilon]$ for some $\epsilon > 0$ such that $\operatorname{supp}(\varphi) \subset M \times [\epsilon, 1-\epsilon]$, see Fig. 6. The preimage in M of the graph of β is a membrane isotopic to $F_0 \times I$; it can be visualized as pushing the interior of $F_0 \times I$ onto $F' \times I$. Because the new membrane is disjoint from the support of φ, the cuts of $\widehat{S}$ and $\widehat{S}'$ along it are isotopic. This proves the faithfulness, because the cuts represent in $\mathrm{hTr}(\mathbf{Tan}(F), \phi_*)$ the same morphisms as the surfaces S and S'. □

Each orientation preserving diffeomorphism ϕ of $\mathbb{R}^2$ is isotopic to identity, so that $\mathrm{hTr}(\mathbf{Tan}, \phi_*)$ is equivalent to the category of links in a solid torus. Because tangles in a thickened plane $\mathbb{R}^2 \times I$ can be represented by diagrams on the stripe $\{0\} \times \mathbb{R} \times I$, is it worth to consider those diffeomorphisms that preserve the line $\{0\} \times \mathbb{R}$. There are two of them:

- the identity, in which case the stripe is closed to an annulus $\mathbb{A}$, and
- the rotation by 180 degrees, for which the image of the stripe is a Möbius band $\mathbb{M}$.

The solid torus is a trivial line bundle over $\mathbb{A}$ and a twisted one over $\mathbb{M}$ respectively. Hence, given an invariant of tangles computed from their diagrams, there are two ways to get invariants of links in a solid torus.

Corollary 3.27. *There are equivalences of categories*

$$\mathrm{hTr}(\mathbf{Tan}) \simeq \mathrm{hTr}(\mathbf{Tan}, \rho_*) \simeq \mathscr{L}\!\mathit{inks}(\mathbb{S}^1 \times \mathbb{R}^2),$$

where $\rho \in \mathrm{Aut}(\mathbb{R}^2)$ is the half–rotation. Thence, a bifunctor $\mathbf{I}\colon \mathbf{Tan} \longrightarrow \mathbf{C}$ induces invariants of links in a solid torus

$$\begin{aligned} \mathrm{hTr}(\mathbf{I})\colon \mathscr{L}\!\mathit{inks}(\mathbb{R}^2 \times \mathbb{S}^1) &\longrightarrow \mathrm{hTr}(\mathbf{C}), \ \textit{and} \\ \mathrm{hTr}(\mathbf{I}, \rho_*)\colon \mathscr{L}\!\mathit{inks}(\mathbb{R}^2 \times \mathbb{S}^1) &\longrightarrow \mathrm{hTr}(\mathbf{C}, \Sigma), \end{aligned}$$

where Σ is an endofunctor of $\mathbf{C}$ satisfying $\Sigma \circ \mathbf{I} \cong \mathbf{I} \circ \rho_*$. In particular, $\langle\!\langle \mathrm{hTr}(\mathbf{I}) \rangle\!\rangle$ is a link invariant for any symmetric preshadow $\langle\!\langle - \rangle\!\rangle$ on $\mathbf{C}$.

[5] A *support* of an isotopy is the closure of the set of points that are not stationary under the isotopy.

A similar argument can be used to compute the horizontal trace of $\mathbf{Cob} := \mathbf{Cob}(\mathbb{R}\times I)$ (resp. $\mathbf{BN} := \mathbf{BN}(\mathbb{R}\times I)$), the bicategory of flat tangles in the stripe $\mathbb{R}\times I$ and cobordisms between them (resp. linear combinations of cobordisms with dots modulo Bar-Natan's relations). We only state the result, leaving the details to the reader.

Proposition 3.28. *There are equivalences of categories*

$$\mathrm{hTr}(\mathbf{Cob}) \simeq \mathscr{C}ob(\mathbb{A}) \qquad\qquad \mathrm{hTr}(\mathbf{Cob}, \sigma_*) \simeq \mathscr{C}ob(\mathbb{M}) \tag{3.36}$$

$$\mathrm{hTr}(\mathbf{BN}) \simeq \mathscr{BN}(\mathbb{A}) \qquad\qquad \mathrm{hTr}(\mathbf{BN}, \sigma_*) \simeq \mathscr{BN}(\mathbb{M}). \tag{3.37}$$

where $\mathbb{A}$ is the annulus, $\mathbb{M}$ the Möbius band, and $\sigma(t) = -t$ is the reflection of $\mathbb{R}$.

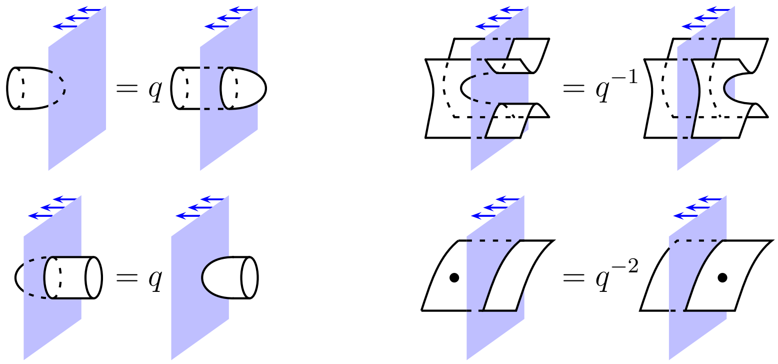
Recall that $\mathbf{BN}$ is a graded bicategory and each cobordism S is a homogeneous morphism of degree

$$\deg S = \chi(S) - \frac{\#B}{4} - 2d, \tag{3.38}$$

where B is the set of corners of S and d the number of dots. Thence, we can deform the categories $\mathscr{BN}(\mathbb{A})$ and $\mathscr{BN}(\mathbb{M})$ by taking quantum horizontal traces of $\mathbf{BN}$, obtaining their *quantizations*

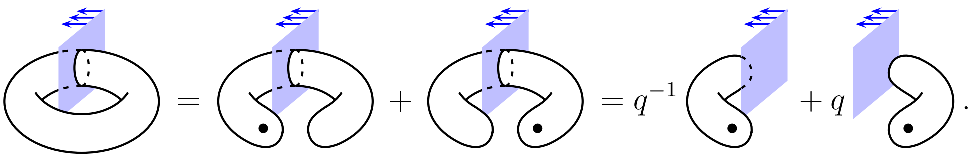
$$\mathscr{BN}_q(\mathbb{A}) := \mathrm{hTr}_q(\mathbf{BN}) \quad \text{and} \quad \mathscr{BN}_q(\mathbb{M}) := \mathrm{hTr}_q(\mathbf{BN}, \sigma_*). \tag{3.39}$$

These categories admit the following description. In each case, the identified boundaries of $(\mathbb{R} \times I) \times I$ distinguish a membrane in the resulting solid torus, and the orientation of the core of $\mathbb{A}$ (resp. $\mathbb{M}$) equips the membrane with a coorientation. Isotopic cobordisms are identified whenever the isotopy between them fixes the membrane. Otherwise, we scale the target cobordism according to the following rules:



(3.40)

In particular, a torus wrapped once around the annulus evaluates to $q + q^{-1}$:



(3.41)

3.6 Decategorification

An n–category $\mathbf{C}$ can be truncated to an $(n-1)$–category by forgetting its highest level morphisms and identifying isomorphic $(n-1)$–morphisms. Thence, we have a functor $\Pi\colon n\mathbf{Cat} \longrightarrow (n{-}1)\mathbf{Cat}$. Clearly, this construction applies to endo–n–categories as well. We are mostly interested in the case $n = 1, 2$.

Applying the functor Π to a shadow $\langle\!\langle - \rangle\!\rangle\colon \mathbf{C} \longrightarrow \mathscr{T}$ results in a trace function $\Pi\langle\!\langle - \rangle\!\rangle\colon \Pi\mathbf{C} \longrightarrow \Pi\mathscr{T}$. We start with an observation that each trace function on $\Pi\mathbf{C}$ can be lifted to a shadow if $\mathbf{C}$ is small.

Lemma 3.29. *Choose a small endobicategory* $(\mathbf{C},\Sigma)$ *and a trace* $\mathrm{tr}\colon (\Pi\mathbf{C},\Pi\Sigma) \longrightarrow S$ *for some set* S*. There exists a shadow* $T\colon (\mathbf{C},\Sigma) \longrightarrow \mathcal{S}$ *such that* $\Pi\mathcal{S} = S$ *and* $\Pi T = \mathrm{tr}$*.*

Proof. In what follows we write $\mathrm{dom}(\alpha) := f$ and $\mathrm{cod}(\alpha) := f'$ for the domain and codomain of a 2–morphism $\alpha\colon f \Longrightarrow f'$. Let $\tilde{\mathcal{S}}$ be the category with $\mathrm{Ob}(\tilde{\mathcal{S}}) = S$ and morphisms $\tilde{\mathcal{S}}(s,t)$ the finite sequences $(\alpha_n,\dots,\alpha_1)$ of 2–morphisms $\alpha_i \in \mathbf{C}(x_i,x_i)$ satisfying $\mathrm{tr}(\mathrm{dom}(\alpha_{i+1})) = \mathrm{tr}(\mathrm{cod}(\alpha_i))$ for $i = 1,\dots,n-1$, such that $s = \mathrm{tr}(\mathrm{dom}(\alpha_1))$ and $t = \mathrm{tr}(\mathrm{cod}(\alpha_n))$. Composition of morphisms is defined as concatenation of sequences. The category $\mathcal{S}$ is a quotient of $\tilde{\mathcal{S}}$ by the relations

$$(\dots,\beta,\alpha,\dots) \sim (\dots,\beta*\alpha,\dots) \quad \text{whenever } \beta*\alpha \text{ exists, and} \tag{3.42}$$

$$(\dots,\mathbf{1},\dots) \sim (\dots,\dots). \tag{3.43}$$

They allow us to reduce a given sequence of 2–morphism to a sequence of noncomposable morphisms, none of which is the identity. An easy application of the Bergman Diamond Lemma [Be78] shows that this reduced sequence is unique. In particular,

$$(\beta_m,\dots,\beta_1)\circ(\alpha_1,\dots,\alpha_n) \sim () \tag{3.44}$$

implies $m = n$ and each $\beta_i \circ \alpha_i = \mathbf{1}$ if both sequences are reduced. Hence, each α_i is a 2–isomorphism in $\mathbf{C}$ if $(\alpha_1,\dots,\alpha_n)$ is an isomorphism in $\mathcal{S}$, which implies that $\mathrm{tr}(\mathrm{dom}(\alpha_i)) = \mathrm{tr}(\mathrm{cod}(\alpha_i))$ and

$$\mathrm{dom}(\alpha_n,\dots,\alpha_1) = \mathrm{tr}(\mathrm{dom}(\alpha_1)) = \mathrm{tr}(\mathrm{cod}(\alpha_n)) = \mathrm{cod}(\alpha_n,\dots,\alpha_1). \tag{3.45}$$

Therefore, $\mathcal{S}$ does not have isomorphisms between different objects, and $\Pi\mathcal{S} = S$. The desired shadow $T\colon (\mathbf{C},\Sigma) \longrightarrow \mathcal{S}$ is defined as $T(f) := \mathrm{tr}(f)$ and $T(\alpha) := (\alpha)$. ☐

The above result allows us to formally prove that the horizontal trace is a categorification of the universal trace of a category. Although it is proven only for small bicategories, we believe that a slight modification of our argument works also for locally small ones.

Theorem 3.30. *There is a natural bijection* $\Pi(\mathrm{hTr}(\mathbf{C},\Sigma)) \approx \mathrm{Tr}(\Pi\mathbf{C},\Pi\Sigma)$ *for each small endobicategory* $(\mathbf{C},\Sigma)$ *with both left and right duals. In other words, there is a diagram of functors*

$$\begin{tikzcd} \mathscr{E}ndo\mathscr{B}icat^* \arrow[r, "\mathrm{hTr}"] \arrow[d, "\Pi"'] & \mathscr{E}ndo\mathscr{C}at \arrow[d, "\Pi"] \\ \mathscr{C}at \arrow[r, "\mathrm{Tr}"] & \mathscr{S}et \end{tikzcd} \tag{3.46}$$

that commutes up to a natural isomorphism, where $\mathscr{E}ndo\mathscr{B}icat^*$ *is the category of endobicategories with left and right duals.*

Proof. Every trace function $\mathrm{tr}\colon (\Pi\mathbf{C},\Pi\Sigma) \longrightarrow S$ comes from a shadow $T\colon (\mathbf{C},\Sigma) \longrightarrow \mathcal{S}$ due to Lemma 3.29, and Theorem 3.26 gives us a functor $\mathrm{hTr}(\mathbf{C},\Sigma) \longrightarrow \mathcal{S}$ that descends to a factorization of traces $\varphi\colon \Pi(\mathrm{hTr}(\mathbf{C},\Sigma)) \longrightarrow S$. Hence, $\Pi(\mathrm{hTr}(\mathbf{C},\Sigma))$ is the universal trace on $(\Pi\mathbf{C},\Pi\Sigma)$ and the thesis follows. ☐

3.7 A connection with the vertical trace

Another way to construct a trace of an endobicategory $(\mathbf{C}, \Sigma)$ is to apply the trace construction to each morphism category. Thence, the *vertical trace* $\mathrm{vTr}(\mathbf{C}, \Sigma)$ has the same objects as $\mathbf{C}$ and morphisms sets are given by $\mathrm{Tr}(\mathbf{C}(x, y), \Sigma)$ for any $x, y \in \mathbf{C}$. Composition in $\mathrm{vTr}(\mathbf{C}, \Sigma)$ is induced by the horizontal composition functors. It is unital and associative, because the trace class of a morphism is invariant under conjugation.

Likewise for a trace of a category, there is a dual version of the vertical trace, $\mathrm{vTr}^{op}(\mathscr{C}, \Sigma)$, which morphisms are equivalence classes of 2–morphisms of type $\Sigma f \Longrightarrow f$. When Σ fixes objects of $\mathbf{C}$, then there exists a functor $\mathfrak{i}\colon \mathrm{vTr}^{op}(\mathbf{C}, \Sigma) \longrightarrow \mathrm{hTr}(\mathbf{C}, \Sigma)$ defined by "expanding" objects to identity morphisms:

$$\mathfrak{i}(x) := (x \xrightarrow{\mathrm{id}_x} x) \qquad \text{and} \qquad \mathfrak{i}\left(\begin{tikzcd} x \arrow[d, bend right, "f"'] \arrow[d, bend left, "\Sigma f"] \arrow[d, phantom, "\overset{\sigma}{\Longleftarrow}"] \\ y \end{tikzcd}\right) := \begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\sigma}{\swarrow}"] & x \arrow[d, "\Sigma f"] \\ y \arrow[r, "\mathrm{id}_y"] & y \end{tikzcd}$$

It is easy to see that $\mathfrak{i}$ is full and faithful, but not necessarily surjective on objects. Thus, $\mathrm{hTr}(\mathbf{C}, \Sigma)$ contains more information about $\mathbf{C}$ than $\mathrm{vTr}^{op}(\mathbf{C}, \Sigma)$.

Finally, notice that both traces preserve higher level equivalences.

Lemma 3.31. *Assume that a bifunctor $F\colon (\mathbf{C}, \Sigma) \longrightarrow (\mathbf{D}, \Sigma')$ of endobicategories restricts to equivalences of categories $\mathbf{C}(x, y) \xrightarrow{\simeq} \mathbf{D}(x, y)$. Then both $\mathrm{vTr}(F)$ and $\mathrm{hTr}(F)$ are full and faithful.*

Proof. The statement is clear for $\mathrm{vTr}(F)$, because Tr takes equivalences of categories to bijections of sets. For $\mathrm{hTr}(F)$ notice first that each $p \in \mathbf{D}(Fx, \Sigma Fy)$ is isomorphic to some Fq for $q \in \mathbf{C}(x, \Sigma y)$, so that $[p, \sigma] = [Fq, F\eta]$ for a certain 2–morphism η in $\mathbf{C}$ (because F is surjective on 2-morphisms). Now it is clear, that $(Fq, F\eta) \sim (Fq', F\eta')$ if and only if the pairs are related by moves (3.29) with all morphisms and 2–morphisms from the image of F. Hence, $(q, \eta) \sim (q', \eta')$ in $\mathbf{C}$, showing that $\mathrm{hTr}(F)$ is faithful. $\square$

4 Quantum Hochschild homology

4.1 Simplicial and cyclic modules

We begin with a brief review of simplicial modules, a convenient framework to work with the Hochschild homology of endocategories. This will allow us to simplify later a few arguments when proving properties of the homology.

Fix a commutative unital ring $\Bbbk$. A *simplicial* $\Bbbk$*–module* M is a sequence of $\Bbbk$–modules $\{M_n\}_{n\geqslant 0}$ together with families of *face maps* $\{d_i\colon M_n \longrightarrow M_{n-1}\}_{0\leqslant i\leqslant n}$ and *degeneracy maps* $\{s_j\colon M_n \longrightarrow M_{n+1}\}_{0\leqslant j\leqslant n}$, one for each n, satisfying

$$d_i d_j = d_{j-1} d_i \quad \text{for } i < j, \tag{4.1}$$
$$s_i s_j = s_j s_{i-1} \quad \text{for } i > j, \tag{4.2}$$
$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j, \\ \mathrm{id} & \text{for } i = j, j+1, \\ s_j d_{i-1} & \text{for } i > j+1. \end{cases} \tag{4.3}$$

A collection of $\Bbbk$–linear maps $f_n\colon M_n \longrightarrow N_n$ that commutes with face and degeneracy maps is called a *simplicial map* and is denoted by $f\colon M \longrightarrow N$.

To every simplicial module M we can associate a chain complex (M, ∂) with $\partial_n := \sum_{i=0}^{n}(-1)^i d_i$. A simplicial map $f\colon M \longrightarrow N$ is then a chain map between the associated chain complexes.

Example 4.1. The quantum Hochschild complex $qCH(M; A, \varphi)$ of an (A, A)–bimodule M twisted by $\varphi \in \mathrm{Aut}(A)$ arises from a simplicial module that we denote with the same symbol. The face and degeneracy maps are given by the formulas

$$d_i(m \otimes a_1 \otimes \ldots \otimes a_n) := \begin{cases} ma_1 \otimes a_2 \otimes \ldots \otimes a_n & \text{if } i = 0, \\ \quad m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & \text{if } 0 < i < n, \\ q^{-|a_n|}\varphi(a_n)m \otimes a_1 \otimes \ldots \otimes a_{n-1} & \text{if } i = n, \end{cases} \tag{4.4}$$

$$s_j(m \otimes a_1 \otimes \ldots \otimes a_n) := m \otimes a_1 \otimes \ldots \otimes a_j \otimes 1 \otimes a_{j+1} \otimes \ldots \otimes a_n. \tag{4.5}$$

Setting $q = 1$ and $\varphi = \mathrm{id}$ recovers the usual simplicial module underlying the Hochschild homology. We shall write $qCH(A; \varphi)$ and $qHH(A; \varphi)$ when $M = A$. As before, φ will be omitted from the notation when it is the identity map.

Choose simplicial $\Bbbk$–modules M and N. A *simplicial homotopy* $h\colon M \longrightarrow N$ is a collection of families $\{h_k\colon M_n \longrightarrow N_{n+1}\}_{0\leqslant k\leqslant n}$ of $\Bbbk$–linear maps, one for each n, satisfying

$$d_i h_k = \begin{cases} h_{k-1} d_i & \text{for } i < k, \\ d_k h_k & \text{for } i = k+1, \\ h_k d_{i-1} & \text{for } i > k+1, \end{cases} \qquad s_j h_k = \begin{cases} h_{k+1} s_j & \text{for } j \leqslant k, \\ h_k s_{j+1} & \text{for } j > k. \end{cases} \tag{4.6}$$

Notice that $f := d_0 h_0$ and $g := d_{n+1} h_n$ are simplicial maps and $h := \sum_k (-1)^k h_k$ is a chain homotopy between them. A simplicial module M is *(simplicially) contractible* if id_M is (simplicially) homotopic to the zero map.

Degeneracy maps of a simplicial module M are not involved in the construction of the chain complex (M, ∂). Therefore, it makes sense to consider sequences $\{M_n\}_{n\geqslant 0}$ admitting only face maps; they are called *presimplicial modules*. In the analogy to simplicial modules, we define *presimplicial maps* and *presimplicial homotopies*, which induce respectively chain maps and chain homotopies on associated chain complexes.

A family $\{t_n\colon M_n \longrightarrow M_n\}$ of linear maps is *semicyclic* if

$$d_i t_n = \begin{cases} d_n & \text{for } i = 0, \\ t_{n-1} d_{i-1} & \text{for } i > 0, \end{cases} \qquad s_j t_n = \begin{cases} t_{n+1}^2 s_n & \text{for } j = 0, \\ t_{n+1} s_{j-1} & \text{for } j > 0. \end{cases} \tag{4.7}$$

Notice that $\{t_n\}$ is neither simplicial nor a chain map. Let $T_n := t_n^{n+1}$. Then $T\colon M \longrightarrow M$ is a simplicial map. We say that $\{t_n\}$ is *cyclic* if $T = \mathrm{id}$.

Lemma 4.2. *The map $T\colon M \longrightarrow M$ is chain homotopic to the identity.*

Proof. Define $\sigma_n := t_{n+1} s_n$, so that

$$d_i \sigma_n = \begin{cases} \mathrm{id} & \text{for } i = 0, \\ \sigma_{n-1} d_{i-1} & \text{for } 0 < i < n, \\ t_n & \text{for } i = n. \end{cases} \tag{4.8}$$

We claim $h_n = \sum_{j=0}^{n}(-1)^{jn}\sigma_n t_n^j$ is the desired chain homotopy. First, write

$$h_{n-1}\partial_n = \sum_{j=0}^{n-1}\sum_{i=0}^{n}(-1)^{i+j(n-1)}\sigma_{n-1}t_{n-1}^j d_i \tag{4.9}$$

$$\partial_{n+1}h_n = \sum_{i=0}^{n+1}\sum_{j=0}^{n}(-1)^{i+jn}d_i\sigma_n t_n^j \tag{4.10}$$

and notice the following cancellation in (4.10):

$$(-1)^{n+1+jn}d_{n+1}\sigma_n t_n^j = -(-1)^{(j+1)n}t_n^{j+1} = -(-1)^{(j+1)n}d_0\sigma_n t_n^{j+1} \tag{4.11}$$

so that

$$\sum_{j=0}^{n}(-1)^{jn}(d_0 - (-1)^n d_{n+1})\sigma_n t_n^j = d_0\sigma_n - d_{n+1}\sigma_n t_n^n = \mathrm{id} - t_n^{n+1}. \tag{4.12}$$

To finish the computation put the remaining terms of ∂h as well as the terms of $h\partial$ in the lexicographic order with respect to i then j, to create $n(n+1)$ pairs:

$$\begin{array}{ccccccccccccc} d_1\sigma_n & < & d_2\sigma_n & < & \cdots & < & d_n\sigma_n & < & d_1\sigma_n t_n & < & d_2\sigma_n t_n & < & \cdots \\ \updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow & & \updownarrow & & \\ \sigma_{n-1}d_0 & < & \sigma_{n-1}d_1 & < & \cdots & < & \sigma_{n-1}d_{n-1} & < & \sigma_{n-1}d_n & < & \sigma_{n-1}t_{n-1}d_0 & < & \cdots \end{array} \tag{4.13}$$

We shall show that each pair contributes nothing to $\partial h + h\partial$.

The term $d_{i+1}\sigma_n t_n^j$ is at position $jn+i+1$ in the upper sequence of (4.13) and it appears in (4.10) with sign $(-1)^{jn+i+1}$. We compute

$$d_{i+1}\sigma_n t_n^j = \sigma_{n-1}d_i t_n^j = \begin{cases} \sigma_{n-1}t_{n-1}^{j-1}d_{i-j+n+1} & \text{if } 0 \leqslant i < j, \\ \sigma_{n-1}t_{n-1}^{j}d_{i-j} & \text{if } j \leqslant i < n, \end{cases} \tag{4.14}$$

obtaining a term at the position $jn+i+1$ in the lower sequence of (4.13), which appears in (4.9) with sign $(-1)^{j(n-1)+i-j} = (-1)^{jn+i}$. Hence, none of the pairs in (4.13) contributes to $\partial h + h\partial$ and the thesis follows. $\square$

Example 4.3. The module $qCH(A;\varphi)$ as defined in Example 4.1 is semicyclic with $t_n(a_0 \otimes \cdots \otimes a_n) := q^{-|a_n|}\varphi(a_n) \otimes a_0 \otimes \cdots \otimes a_{n-1}$. Clearly, $T = q^{-d}\varphi_*$ on the degree d component $qCH(A,\varphi;d)$, where $\varphi_*(a_0 \otimes \cdots \otimes a_n) := \varphi(a_0) \otimes \cdots \otimes \varphi(a_n)$. Due to Lemma 4.2 the map $(\varphi_* - q^d)$ vanishes on $qHH(A,\varphi;d)$. In particular, $qHH(A;d) = 0$ when $\Bbbk$ is a field, $\varphi = \mathrm{id}$, and q is not a d-th root of unity.

4.2 The semicyclic set of an endocategory

We assign to a $\Bbbk$–linear endocategory $(\mathscr{C}, \Sigma)$ a semicyclic module $\mathscr{CH}(\mathscr{C}, \Sigma)$ with

$$\mathscr{CH}_n(\mathscr{C}, \Sigma) := \bigoplus_{x_0,\dots,x_n \in \mathrm{Ob}\mathscr{C}} \mathscr{C}(x_0, \Sigma x_n) \otimes \mathscr{C}(x_1, x_0) \otimes \cdots \otimes \mathscr{C}(x_n, x_{n-1}) \tag{4.15}$$

and structure maps

$$d_i(f_0 \otimes \ldots \otimes f_n) := \begin{cases} f_0 \otimes \ldots \otimes (f_i \circ f_{i+1}) \otimes \ldots \otimes f_n & \text{if } i < n, \\ (\Sigma f_n \circ f_0) \otimes f_1 \otimes \ldots \otimes f_{n-1} & \text{if } i = n, \end{cases} \tag{4.16}$$

$$s_j(f_0 \otimes \ldots \otimes f_n) := f_0 \otimes \ldots \otimes f_j \otimes \mathrm{id}_{x_j} \otimes f_{j+1} \otimes \ldots \otimes f_n, \tag{4.17}$$

$$t_n(f_0 \otimes \cdots \otimes f_n) := \Sigma f_n \otimes f_0 \otimes \cdots \otimes f_{n-1}. \tag{4.18}$$

We call it the *Hochschid–Mitchell module of* $(\mathscr{C}, \Sigma)$. The *Hochschild–Mitchell homology of* $(\mathscr{C}, \Sigma)$, denoted by $\mathscr{HH}(\mathscr{C}, \Sigma)$, is the homology of the associated chain complex.

Assume that $(\mathscr{C}, \Sigma)$ is pregraded. We define the *quantum Hochschild–Mitchell module* $q\mathscr{CH}(\mathscr{C}, \Sigma) := \mathscr{CH}(\mathscr{C}, \Sigma_q)$ and *homology* $q\mathscr{HH}(\mathscr{C}, \Sigma) := \mathscr{HH}(\mathscr{C}, \Sigma_q)$, where as before $\Sigma_q(f) := q^{-|f|}\Sigma f$ for a homogeneous morphism f and a fixed $q \in \Bbbk$. Both are graded modules with degree d components written as $q\mathscr{CH}(\mathscr{C}, \Sigma; d)$ and $q\mathscr{HH}(\mathscr{C}, \Sigma; d)$ respectively.

We shall omit Σ from the notation when it is the identity functor. In such a case we recover the usual Hochschild–Mitchell homology of a category [Mit72].

In what follows we fix endocategories $(\mathscr{C}, \Sigma)$, $(\mathscr{C}', \Sigma')$. Choose a functor $\mathscr{F} \colon \mathscr{C} \longrightarrow \mathscr{C}'$ together with a natural isomorphism $\omega \colon \mathscr{F}\Sigma \longrightarrow \Sigma'\mathscr{F}$. We shall refer to the pair $(\mathscr{F}, \omega)$ as a *functor of endocategories*; ω is usually omitted in the notation. Such a functor induces a simplicial map $\mathscr{F}_* \colon \mathscr{CH}(\mathscr{C}, \Sigma) \longrightarrow \mathscr{CH}(\mathscr{C}', \Sigma')$

$$f_0 \otimes \ldots \otimes f_n \longmapsto (\omega \circ \mathscr{F} f_0) \otimes \mathscr{F} f_1 \otimes \ldots \otimes \mathscr{F} f_n, \tag{4.19}$$

where ω is used to fix the codomain of $\mathscr{F} f_0$.

Proposition 4.4. *The simplicial map Σ_* induces the identity on homology. In particular, $(q^d - 1)$ annihilates $q\mathscr{HH}(\mathscr{C}; d)$ if $\mathscr{C}$ is pregraded and $\Sigma = \mathrm{Id}$.*

Proof. It follows from Lemma 4.2, because $\Sigma_* = T$ on $\mathscr{CH}_n(\mathscr{C}, \Sigma)$. ☐

The following is an immediate generalization of the analogous properties of the usual Hochschild–Mitchell modules.

Proposition 4.5. *The Hochschild–Mitchell complex is homotopy invariant under additive closure and idempotent completion: the canonical inclusions*

$$I_\oplus \colon \mathscr{C} \longrightarrow \mathscr{C}^\oplus \quad \text{and} \quad I_{\mathrm{Kar}} \colon \mathscr{C} \longrightarrow \mathrm{Kar}(\mathscr{C})$$

induce a simplicial and a presimplicial homotopy equivalence respectively.

Proof. Consider the simplicial map $T \colon \mathscr{CH}(\mathscr{C}^\oplus, \Sigma^\oplus) \longrightarrow \mathscr{CH}(\mathscr{C}, \Sigma)$

$$f^0 \otimes \cdots \otimes f^n \longmapsto \sum_{i_0, \ldots, i_n} f^0_{i_0, i_1} \otimes f^1_{i_1, i_2} \otimes \cdots \otimes f^n_{i_n, i_0}.$$

Clearly, $T \circ I_\oplus = \mathrm{Id}_{\mathscr{C}}$. The other composition is homotopic to the identity by the simplicial homotopy

$$h^k(f^0 \otimes \cdots \otimes f^n) := \sum_{i_0, \ldots, i_k} (f^0 \circ in_{i_0}) \otimes f^1_{i_1, i_0} \otimes \cdots \otimes f^k_{i_k, i_{k-1}} \otimes pr_{i_k} \otimes f^{k+1} \otimes \cdots \otimes f^n,$$

where for any sequence $\underline{x} = (x_1, \ldots, x_r)$ and $1 \leqslant k \leqslant r$ we write $in_k \colon x_k \longrightarrow \underline{x}$ and $pr_k \colon \underline{x} \longrightarrow x_k$ for the canonical inclusion and projection morphisms.

To prove the invariance under the idempotent completion notice first, that there is a presimplicial map $\mathscr{U}_*$ associated to the semifunctor[6] $\mathscr{U} \colon \mathrm{Kar}(\mathscr{C}, \Sigma) \longrightarrow (\mathscr{C}, \Sigma)$ that

[6] A semifunctor preserves the composition, but not identity morphisms.

takes an idempotent $e \in \operatorname{End}(x)$ to x, and a morphism between idempotents to the underlying morphisms of $\mathscr{C}$. Again, $\mathscr{U} \circ I_{\mathrm{Kar}} = \mathrm{Id}_{\mathscr{C}}$, and the other composition induces a presimplicial map, which is homotopic to the identity by the presimplicial homotopy

$$h^k(f_0 \otimes \cdots \otimes f_n) := (f_0 \circ e_0) \otimes f_1 \otimes \cdots \otimes f_i \otimes e_i \otimes f_{i+1} \otimes \cdots \otimes f_n.$$

Simpliciality is lost, because $\mathscr{U}$ does not preserve identity morphisms. □

Corollary 4.6. *The modules $qCH(A,\varphi)$ and $q\mathscr{C}\mathscr{H}(\mathrm{g}\mathscr{R}ep(A),\varphi_*)$ are homotopy equivalent for any graded algebra A and its graded automorphism $\varphi \in \operatorname{Aut}(A)$.*

Proof. Let $\mathscr{A}$ be the full subcategory of $\mathrm{g}\mathscr{R}ep(A)$ with a unique object A. Because $\operatorname{End}_A(A) \cong A$, we can regard φ as a functor on $\mathscr{A}$. Notice that $\mathrm{g}\mathscr{R}ep(A) \cong \operatorname{Kar}(\mathscr{A}^{\oplus})$ and the functor $\varphi\colon \mathscr{A} \longrightarrow \mathscr{A}$ extends to $\varphi_*\colon \mathrm{g}\mathscr{R}ep(A) \longrightarrow \mathrm{g}\mathscr{R}ep(A)$. The thesis follows now from Proposition 4.5. □

The proof of Proposition 4.5 shows that the inclusion of categories $\mathscr{C} \longrightarrow \mathscr{C}^{\oplus}$ induces a deformation retraction of the corresponding simplicial modules. We shall now construct a similar deformation for the categories of matrices of a fixed size. For that fix $r \in \mathbb{Z}_+$ and let $Mat_r(\mathscr{C})$ be the category of r-dimensional matrices:

- objects are r-tuples $\underline{x} = (x_1, \ldots, x_r) \in \operatorname{Ob}(\mathscr{C})^r$,
- morphisms between $\underline{x}$ and $\underline{y}$ are $r \times r$ matrices (f_{ij}) with $f_{ij}\colon x_i \longrightarrow y_j$, and
- the composition is given by the product rule of matrices.

An endofunctor Σ on $\mathscr{C}$ extends naturally to $Mat_r(\mathscr{C})$ by setting $\Sigma(f_{ij}) := (\Sigma f_{ij})$. We distinguish two subcategories of $Mat_r(\mathscr{C})$ by restricting the class of morphisms:

1) $UT_r(\mathscr{C}) \subset Mat_r(\mathscr{C})$ has only upper triangular matrices as morphisms, i.e. $f_{ij} = 0$ for $i > j$, and
2) $Diag_r(\mathscr{C}) \subset Mat_r(\mathscr{C})$ has only diagonal matrices as morphisms: $f_{ij} = 0$ if $i \neq j$.

Both subcategories are preserved by Σ. Clearly $Diag_r(\mathscr{C}) \subset UT_r(\mathscr{C})$, and there is a functor $\mathscr{R}\colon UT_r(\mathscr{C}) \longrightarrow Diag_r(\mathscr{C})$ that forgets non-diagonal entries in matrices.

Lemma 4.7. *The functor $\mathscr{R}\colon UT_r(\mathscr{C}) \longrightarrow Diag_r(\mathscr{C})$ induces a deformation retract between the Hochschild–Mitchell modules.*

Proof. For any object $\underline{x} = (x_1, \ldots, x_r)$ denote by e_i the idempotent projecting $\underline{x}$ onto x_i. Then the collection of maps

$$h^k(f_0 \otimes \cdots \otimes f_n) := \sum_{i=1}^{r} (f_0 \circ e_i) \otimes \mathscr{R}f_1 \otimes \cdots \otimes \mathscr{R}f_k \otimes e_i \otimes f_{k+1} \otimes \cdots \otimes f_n \tag{4.20}$$

constitute the desired simplicial homotopy, because $\sum_i (\Sigma e_i \circ f_0 \circ e_i) = \mathscr{R}f_0$. □

4.3 Invariance

One can directly check that equivalent functors induce homotopy equivalent simplicial maps. In this section we prove a stronger statement: the induced maps are homotopy equivalent if the trace classes of identity transformations of the functors coincide.

As before we fix endocategories $(\mathscr{C},\Sigma)$ and $(\mathscr{C}',\Sigma')$. The category $[\mathscr{C},\mathscr{C}']$ of functors $(\mathscr{F},\omega)\colon \mathscr{C} \longrightarrow \mathscr{C}'$ and natural transformations admits an endofunctor $\boldsymbol{\Sigma}$, defined as

$$\boldsymbol{\Sigma}\mathscr{F} := \mathscr{F}\Sigma \quad \text{and} \quad \boldsymbol{\Sigma}\eta := (\omega')^{-1} \circ \Sigma'\eta \circ \omega \tag{4.21}$$

for functors $(\mathscr{F}, \omega)$, $(\mathscr{G}, \omega')$ and a natural transformation $\eta \colon \mathscr{F} \longrightarrow \mathscr{G}$. In other words, $(\boldsymbol{\Sigma}\eta)_x \colon \mathscr{F}\Sigma x \longrightarrow \mathscr{G}\Sigma x$ is determined by the square

$$\begin{tikzcd}
\mathscr{F}\Sigma x \arrow[r, dashed, "(\boldsymbol{\Sigma}\eta)_x"] \arrow[d, "\omega_x"'] & \mathscr{G}\Sigma x \arrow[d, "\omega'_x"] \\
\Sigma'\mathscr{F}x \arrow[r, "\Sigma'\eta_x"] & \Sigma'\mathscr{G}x
\end{tikzcd} \tag{4.22}$$

We say that η is $\boldsymbol{\Sigma}$-*invariant* when $(\boldsymbol{\Sigma}\eta)_x = \eta_{\Sigma x}$.

The construction (4.19) of a simplicial map out of a functor extends to natural transformations. Namely, given a natural transformation $\alpha \colon \mathscr{F} \longrightarrow \mathscr{F}$ of a functor $\mathscr{F} \colon \mathscr{C} \longrightarrow \mathscr{C}'$ we define a simplicial map $\alpha_* \colon \mathscr{CH}(\mathscr{C}, \Sigma) \longrightarrow \mathscr{CH}(\mathscr{C}', \Sigma')$ by the formula

$$f_0 \otimes \cdots \otimes f_n \longmapsto (\omega \circ \alpha \circ \mathscr{F}f_0) \otimes \mathscr{F}f_1 \otimes \cdots \otimes \mathscr{F}f_n. \tag{4.23}$$

The only nontrivial equality $d_n\alpha_* = \alpha_* d_n$ follows from naturality of α.

Proposition 4.8. *The homotopy class of α_* depends only on the trace class of α.*

Proof. Choose functors $(\mathscr{F}, \omega)$ and $(\mathscr{G}, \omega')$. For natural transformations $\eta \colon \mathscr{F} \longrightarrow \mathscr{G}$ and $\nu \colon \mathscr{G} \longrightarrow \mathscr{F}$ we check directly that the family of maps

$$h^k(f_0 \otimes \ldots \otimes f_n) := (\omega \circ \nu \circ \mathscr{G}f_0) \otimes \mathscr{G}f_1 \otimes \ldots \otimes \mathscr{G}f_k \otimes \eta \otimes \mathscr{F}f_{k+1} \otimes \ldots \otimes \mathscr{F}f_n.$$

defines a simplicial homotopy. Clearly, $d^{n+1}h^n = (\boldsymbol{\Sigma}\eta \circ \nu)_*$ and naturality of η implies that $d^0h^0 = (\nu \circ \eta)_*$. ☐

Assume now that $\mathscr{C}'$ is additive. We say that a sequence of functors

$$\mathscr{F}' \xrightarrow{\iota} \mathscr{F} \xrightarrow{\pi} \mathscr{F}'' \tag{4.24}$$

is *semisplit* if the sequence

$$0 \longrightarrow \mathscr{F}'x \xrightarrow{\iota_x} \mathscr{F}x \xrightarrow{\pi_x} \mathscr{F}''x \longrightarrow 0 \tag{4.25}$$

splits for every object x. Every split exact sequence is semisplit, but not vice versa.

Proposition 4.9. *Choose a commutative diagram in $[\mathscr{C}, \mathscr{C}']$*

$$\begin{tikzcd}
\mathscr{F}' \arrow[r] \arrow[d, "\alpha'"] & \mathscr{F} \arrow[r] \arrow[d, "\alpha"] & \mathscr{F}'' \arrow[d, "\alpha''"] \\
\mathscr{F}' \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{F}''
\end{tikzcd} \tag{4.26}$$

with semisplit rows. Then α_* is homotopic to $\alpha'_* + \alpha''_*$. In particular, $\mathscr{F}_* = \mathscr{F}'_* + \mathscr{F}''_*$ for any semisplit exact sequence $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$ of functors.

Proof. For each object $x \in \mathrm{Ob}(\mathscr{C})$ choose a decomposition $\mathscr{F}x \cong \mathscr{F}'x \oplus \mathscr{F}''x$, so that we can represent α and every morphism $f \colon \mathscr{F}x \longrightarrow \mathscr{F}x$ of the form $f = \mathscr{F}g$ respectively by upper-triangular matrices

$$\begin{pmatrix} \alpha' & * \\ 0 & \alpha'' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f' & * \\ 0 & f'' \end{pmatrix}. \tag{4.27}$$

In particular, we can view $\mathscr{F}$ as a functor $\mathscr{C} \longrightarrow UT_2(\mathscr{D})$ and α as its natural transformation. It follows now from Lemma 4.7 that α_* is homotopic to $(\alpha' \oplus \alpha'')_* \simeq \alpha'_* + \alpha''_*$. ☐

4.4 An application to Hochschild homology of algebras

Choose algebras A and B with automorphisms $\varphi \in \operatorname{Aut}(A)$ and $\psi \in \operatorname{Aut}(B)$. Recall, that we have defined ΣM for $M \in \mathcal{B}imod(A,B)$ as the bimodule with twisted actions:

$$a \cdot m \cdot b := \varphi^{-1}(a) m \psi^{-1}(b) \qquad \text{for } a \in A, b \in B, m \in M. \tag{4.28}$$

If there is an isomorphism $\theta\colon M \xrightarrow{\cong} \Sigma M$ of bimodules, then

$$M \underset{B}{\otimes} \psi_*(P) \cong \varphi_*(\Sigma M \underset{B}{\otimes} P) \cong \varphi_*(M \underset{B}{\otimes} P) \tag{4.29}$$

for any $P \in \mathcal{M}od(B)$. Hence, (M,θ) can be viewed as a functor between $(\mathcal{M}od(B), \psi_*)$ and $(\mathcal{M}od(A), \varphi_*)$. In particular, there is a simplicial map $M_*\colon CH(B,\psi) \longrightarrow CH(A,\varphi)$ when $M \in \mathrm{g}\mathcal{R}ep(A,B)$. There is also a graded version with CH replaced by the quantum module qCH.

Let $\mathrm{g}\mathcal{B}imod(A,\varphi;B,\psi)$ be a category with objects pairs (M,θ_M) as above and morphisms between (M,θ_M) and (N,θ_N) all bimodule maps. A bimodule map $f\colon M \longrightarrow N$ is Σ–invariant if and only if it intertwines θ_M with θ_N, and not all maps are Σ–invariant. This category is closed under kernels and cokernels, so that it is abelian. Likewise, let $\mathrm{g}\mathcal{R}ep(A,\varphi;B,\psi)$ be the subcategory generated by pairs (M,θ) with $M \in \mathrm{g}\mathcal{R}ep(A,B)$. Following the usual convention we shall write $G_0(A,\varphi;B,\psi)$ and $K_0(A,\varphi;B,\psi)$ respectively for the Grothendieck groups of these categories.

Theorem 4.10. *The homotopy class of $M_*\colon qCH(B,\psi) \longrightarrow qCH(A,\varphi)$ induced by a bimodule $M \in \mathrm{g}\mathcal{R}ep(A,\varphi;B,\psi)$ depends only on the class $[M] \in K_0(A,\varphi;B,\psi)$.*

Proof. It follows from Proposition 4.9, because each exact sequence in $\mathrm{g}\mathcal{R}ep(A,\varphi;B,\psi)$ semisplits when bimodules are regarded as tensor functors. □

Let now $A = \bigoplus_{d\in\mathbb{N}} A_d$ be positively graded. The degree-zero subalgebra $A_0 \subset A$ is preserved by any graded automorphism $\varphi \in \operatorname{Aut}(A)$. Hence, the inclusion and projection maps induce simplicial maps $qCH(A_0,\varphi) \longrightarrow qCH(A,\varphi)$ and $qCH(A,\varphi) \longrightarrow qCH(A_0,\varphi)$ respectively. One of the compositions is clearly the identity map, but not the other. These maps are not homotopy equivalences in general. For example, it is known that the algebra of dual numbers $\Bbbk[x]/(x^2)$, where $\deg x = 2$, has unbounded Hochschild homology, whereas its degree 0 part does not. The situation changes dramatically when we assume A has finite global dimension: under certain conditions on A the inclusion $A_0 \longrightarrow A$ induces an isomorphism on Hochschild homology [Ke98]. Here we reprove this result for twisted homology and its quantization.

Proposition 4.11. *Choose a positively graded algebra $A = \bigoplus_d A_d$ of finite dimension such that each simple A–module is one dimensional. If A has finite global dimension, then the inclusion $A_0 \longrightarrow A$ induces an isomorphism $qHH(A_0, \varphi|_{A_0}) \longrightarrow qHH(A,\varphi)$ for any graded automorphism $\varphi \in \operatorname{Aut}(A)$.*

Proof. We can consider A and A_0 as an (A_0,A)– and (A,A_0)–bimodules respectively. Both belongs to $\mathrm{g}\mathcal{R}ep(A,\varphi;A,\varphi)$ with the isomorphisms $A \cong \Sigma A$ and $A_0 \cong \Sigma A_0$ given by φ. Clearly, $A \otimes_A A_0 \cong A_0$ as (A_0,A_0)–bimodules and in the view of Theorem 4.10 it is enough to show that the images of A and $A_0 \otimes_{A_0} A$ coincides in $K_0(A,\varphi;A,\varphi)$. It follows from the isomorphisms of (A,A)–bimodules

$$A_{\geqslant d}/A_{>d} \cong (A_0 \underset{A_0}{\otimes} A_{\geqslant d})/(A_0 \underset{A_0}{\otimes} A_{>d}),$$

that $[A] = [A_0 \otimes_{A_0} A]$ in $G_0(A,\varphi;A,\varphi)$. It is now enough to show that the homomorphism $K_0(A,\varphi,A,\varphi) \longrightarrow G_0(A,\varphi;A,\varphi)$ is invertible. For that notice first that $A^e := A \otimes A^{op}$ has finite global dimension: each simple A^e–module is of the form $L' \otimes L^*$ for certain simple A–modules L' and L, so that it has a uniformly bounded projective resolution. Next, the derived category of $\mathrm{g}\mathcal{B}imod(A,\varphi;A,\varphi)$ is modelled on pairs (C,θ), where C is a complex of graded (A,A)–bimodules and $\theta\colon C \longrightarrow \Sigma C$ is a quasi-isomorphism. It follows that each $(M,\theta) \in \mathrm{g}\mathcal{B}imod(A,\varphi;A,\varphi)$ has a bounded projective resolution $(P(M),P(\theta))$ and $[M] = [P(M)]$ in $K_0(A,\varphi;A,\varphi)$. $\square$

5 Quantization of link homology

5.1 Representations of $\mathcal{U}_q(\mathfrak{sl}_2)$

As usual fix a commutative unital ring $\Bbbk$ and an invertible $q \in \Bbbk$. By definition, $\mathcal{U}_q(\mathfrak{sl}_2)$ is the unital associative $\Bbbk$–algebra with generators E, F, K, K^{-1} and relations

$$\begin{aligned} KE &= q^2 EL, & KK^{-1} &= 1 = K^{-1}K, \\ KF &= q^{-2}FK, & K - K^{-1} &= (q-q^{-1})(EF - FE). \end{aligned}$$

It is a Hopf algebra with the comultiplication $\Delta\colon \mathcal{U}_q(\mathfrak{sl}_2) \longrightarrow \mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$, the counit $\epsilon\colon \mathcal{U}_q(\mathfrak{sl}_2) \longrightarrow \Bbbk[q,q^{-1}]$, and the antipode $S\colon \mathcal{U}_q(\mathfrak{sl}_2) \longrightarrow \mathcal{U}_q(\mathfrak{sl}_2)$ defined by

$$\Delta(E) = E \otimes K + 1 \otimes E, \qquad \epsilon(E) = 0, \qquad S(E) = -EK^{-1}, \tag{5.1}$$

$$\Delta(F) = F \otimes 1 + K^{-1} \otimes F, \qquad \epsilon(F) = 0, \qquad S(F) = -KF \tag{5.2}$$

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \qquad \epsilon(K^{\pm 1}) = 1, \qquad S(K^{\pm 1}) = K^{\mp 1}. \tag{5.3}$$

Using this Hopf algebra structure, we can regard the category of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ as a monoidal category with duals. The unit in this monoidal category is given by the trivial representation $V_0 = \Bbbk$, on which $\mathcal{U}_q(\mathfrak{sl}_2)$ acts by multiplication by $\epsilon(X)$ for any $X \in \mathcal{U}_q(\mathfrak{sl}_2)$.

We write $V_1 := \mathrm{span}_\Bbbk\{v_1, v_{-1}\}$ and $V_1^* := \mathrm{span}_\Bbbk\{v_1^*, v_{-1}^*\}$ for the fundamental representation and its dual. We identify both with the rank two module $V := \mathrm{span}_\Bbbk\{v_+, v_-\}$ using the isomorphisms

$$\begin{cases} v_1 \longmapsto v_+ \\ v_{-1} \longmapsto v_- \end{cases} \quad \text{and} \quad \begin{cases} v_1^* \longmapsto v_- \\ v_{-1}^* \longmapsto q^{-1}v_+ \end{cases}. \tag{5.4}$$

This equips V with two actions of $\mathcal{U}_q(\mathfrak{sl}_2)$ that differ by signs:

$$\begin{array}{ll} \multicolumn{2}{c}{\text{The action on } V \cong V_1} \\ \hline Ev_+ = 0 & Ev_- = v_+ \\ Fv_+ = v_- & Fv_- = 0 \\ Kv_+ = qv_+ & Kv_- = q^{-1}v_- \end{array} \qquad \begin{array}{ll} \multicolumn{2}{c}{\text{The action on } V \cong V_1^*} \\ \hline Ev_+ = 0 & Ev_- = -v_+ \\ Fv_+ = -v_- & Fv_- = 0 \\ Kv_+ = qv_+ & Kv_- = q^{-1}v_- \end{array} \tag{5.5}$$

The duality between V_1 and V_1^* comes with the evaluation and coevaluation maps

$$\begin{array}{ll} \multicolumn{2}{c}{ev\colon V \otimes V \longrightarrow \Bbbk} \\ \hline v_+ \otimes v_+ \longmapsto 0 & v_+ \otimes v_- \longmapsto q \\ v_- \otimes v_- \longmapsto 0 & v_- \otimes v_+ \longmapsto 1 \end{array} \qquad \begin{array}{c} coev\colon \Bbbk \longrightarrow V \otimes V \\ \hline 1 \longmapsto v_+ \otimes v_- + q^{-1} v_- \otimes v_+ \end{array} \tag{5.6}$$

that intertwine the action $\mathcal{U}_q(\mathfrak{sl}_2)$ if $V \otimes V$ is identified with either $V_1 \otimes V_1^*$ or $V_1^* \otimes V_1$.

Recall that the Temperly–Lieb category $\mathcal{TL}$ is a $\Bbbk$–linear additive category with objects finite collections of points and morphisms generated by flat tangles modulo the relation $\bigcirc = q + q^{-1}$. There is a functor $\mathcal{F}_{TL}\colon \mathcal{TL} \longrightarrow \mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_2))$, which assigns $V^{\otimes n}$ to a collection of n points, whereas on caps and cups it is defined using the evaluation and evaluation homomorphisms defined above. It is known that $\mathcal{F}_{TL}$ is faithful [Th99].

5.2 Quantum Hochschild homology of A^n

The Chen–Khovanov bifunctor categorifies the Temperly–Lieb functor [CK14]. With our notation, the result can be stated as follows.

Theorem 5.1 (cf. [CK14])**.** *Each indecomposable left projective A^n-module is isomorphic to $P_a\{i\}$ for a unique cup diagram $a \in \mathcal{GM}^n$ and $i \in \mathbb{Z}$. Moreover,*

$$K_0(A^n) \otimes_{\mathbb{Z}} \mathbb{C} \cong V^{\otimes n} \tag{5.7}$$

and for a flat tangle T the tensor functor $\mathbf{F}_{CK}(T) \otimes (-)$ descends to $\mathcal{F}_{TL}(T)$, the action of T on the above representation.

For completeness let us recall the isomorphism. Choose a cup diagram $a \in \mathcal{GM}^n$ and orient its arcs counter-clockwise. Decorate each terminum with $+$ or $-$ depending on whether it is respectively a source or a target of an arc. This determines a sequence $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$. The isomorphism (5.7) takes the projective P_a to the basis vector $v_{\underline{\epsilon}} := v_{\epsilon_1} \heartsuit \ldots \heartsuit v_{\epsilon_n}$, computed according to the rules

$$\begin{aligned} v_- \heartsuit w &:= v_- \otimes w \\ w \heartsuit v_+ &:= w \otimes v_+ \\ w \heartsuit v_+ \heartsuit v_- \heartsuit w' &:= (\mathrm{id} \otimes b \otimes \mathrm{id})(w \heartsuit w'), \end{aligned} \tag{5.8}$$

where $b = v_+ \otimes v_- + q^{-1} v_- \otimes v_+$ is the value of the coevaluation homomorphism on 1. The vectors $v_{\underline{\epsilon}}$ are closely related to the Lusztig canonical basis of $V_1^{\otimes n}$, see [FK97].

Remark. The isomorphism in [CK14] was constructed using $b = v_+ \otimes v_- + q v_- \otimes v_+$ instead. This is because of a different convention for grading: our grading is opposite to that from [CK14], which requires to replace q with q^{-1}.

$K_0(A^n)$ is naturally isomorphic with $E^n \subset A^n$, the degree zero subalgebra generated by idempotents. Because $E^n \cong \Bbbk^{2^n}$ as a ring, it follows that $qHH_0(E^n) \cong E^n$ and higher homology groups vanish.

Proposition 5.2. *The inclusion $E^n \subset A^n$ induces an isomorphism on quantum Hochschild homology. In particular $qHH_0(A^n) \cong E^n$ and $qHH_{>0}(A^n) = 0$.*

Proof. Due to the Universal Coefficient Theorem it is enough to prove the proposition for each finite field $\Bbbk = \mathbb{Z}_p$. This follows from Proposition 4.11, because A^n has finite global dimension over any field [BS11] and simple modules are one-dimensional. □

Corollary 5.3. *The inclusion of algebras $A^n \otimes A^m \subset A^{m+n}$ induces isomorphisms on the quantum Hochschild homology.*

Proof. Precompose the inclusion with $E^{n+m} \cong E^n \otimes E^m \subset A^n \otimes A^m$. □

It is more convenient to think of $qHH(A^n)$ as the quantum Hochschild–Mitchell homology of $\mathrm{g}\mathscr{R}ep(A^n)$. There is a homomorphism $K_0(A^n) \longrightarrow q\mathscr{H}\mathscr{H}(\mathrm{g}\mathscr{R}ep(A^n))$, called the Chern character, that takes a projective P to the trace of the identity id_P.

Corollary 5.4. *The Chern character $K_0(A^n) \longrightarrow q\mathscr{H}\mathscr{H}(\mathrm{g}\mathscr{R}ep(A^n))$ is an isomorphism of $\mathbb{Z}[q, q^{-1}]$–modules. In particular, the map on quantum Hochschild homology induced by $\mathbf{F}_{CK}(T)$ agrees with $\mathscr{F}_{TL}(T)$ for any flat tangle T.*

Proof. Follows from Proposition 5.2 and naturality of the Chern character. □

5.3 Quantum annular homology

Recall that $\mathscr{B}\mathscr{N}_q(\mathbb{A}) := \mathrm{hTr}_q(\mathbf{BN})$ is a quantization of the Bar-Natan skein category of the annulus. Applying hTr_q to the Chen–Khovanov functor induces then an additive functor $\mathrm{hTr}_q(\mathbf{F}_{CK})\colon \mathscr{B}\mathscr{N}_q(\mathbb{A}) \longrightarrow \mathrm{hTr}_q(\mathbf{DB})$, and because diagrammatic bimodules admit duals, we can define $\mathscr{F}_{\mathbb{A}_q}\colon \mathscr{B}\mathscr{N}_q(\mathbb{A}) \longrightarrow \mathscr{M}od_{\Bbbk}$ by composing $\mathrm{hTr}_q(\mathbf{F}_{CK})$ with the quantum Hochschild homology, see Proposition 3.21 and Theorem 3.26. In particular,

$$\mathscr{F}_{\mathbb{A}_q}(\widehat{T}) := qHH(\mathbf{F}_{CK}(T); A^n) \tag{5.9}$$

for an annular closure $\widehat{T}$ of a flat (n, n)–tangle T. If $S \subset \mathbb{A} \times I$ is a cobordism between closures of an (n, n)–tangle $\widehat{T}$ and an (n', n')–tangle $\widehat{T}'$, then the map $\mathscr{F}_{\mathbb{A}_q}(S)$ is given by the following sequence

$$\begin{aligned} qHH(\mathbf{F}_{CK}(T); A^n) &\xrightarrow{coev_*} qHH(\mathbf{F}_{CK}(T) \underset{A^n}{\otimes} {}^*\mathbf{F}_{CK}(T_0) \underset{A^{n'}}{\otimes} \mathbf{F}_{CK}(T_0); A^n) \\ &\xrightarrow{\theta} qHH(\mathbf{F}_{CK}(T_0) \underset{A^n}{\otimes} \mathbf{F}_{CK}(T) \underset{A^n}{\otimes} {}^*\mathbf{F}_{CK}(T_0); A^{n'}) \\ &\xrightarrow{\mathbf{F}_{CK}(\widetilde{S})_*} qHH(\mathbf{F}_{CK}(T') \underset{A^{n'}}{\otimes} \mathbf{F}_{CK}(T_0) \underset{A^n}{\otimes} {}^*\mathbf{F}_{CK}(T_0); A^{n'}) \\ &\xrightarrow{ev_*} qHH(\mathbf{F}_{CK}(T'); A^{n'}) \end{aligned} \tag{5.10}$$

where $T_0 = S \cap (\mu \times I)$ and $\widetilde{S}$ is the surface S cut open along the membrane and regarded as a cobordism from $T_0 T$ to $T' T_0$, see Section 3.4. We shall now give a more conceptual construction of the functor.

Let $\mu = \{1\} \times \mathbb{R} \subset \mathbb{S}^1 \times \mathbb{R}$ be the arc formed by identifying the boundaries of $\mathbb{R} \times I$. We shall call it the *seam* of $\mathbb{A}$. Write $c_m \subset \mathbb{A}$ for a collection of m parallel essential circles, each intersecting μ exactly once. Consider the full additive subcategory $\mathscr{E} \subset \mathscr{B}\mathscr{N}_q(\mathbb{A})$ generated by the objects c_m for $m \geqslant 0$.

Lemma 5.5. *The inclusion $\mathscr{E} \longrightarrow \mathscr{B}\mathscr{N}_q(\mathbb{A})$ is an equivalence of categories: for any object $c \in \mathscr{B}\mathscr{N}_q(\mathbb{A})$ there is an object $c' \in \mathscr{E}$ and a canonical isomorphism $i_c\colon c \longrightarrow c'$.*

Proof. Let Γ be a collection of curves in $\mathbb{A}$. If $\alpha \subset \Gamma$ is an embedded arc such that $\partial\alpha = \mu \cap \Gamma$, then we shall say that Γ is *retractible* if there is an embedded disk $D \subset \mathbb{A}$ whose interior is disjoint from Γ and whose boundary is formed by the arc α and the subarc of μ that lies between the two endpoints of α. We shall call a retractible arc *positive* or *negative*, depending on whether the coorientation of μ points into or out of D, see Figure 7.

If Γ contains a retractible arc α, then we can reduce the geometric intersection number of Γ with μ by 2 by using an isotopy that pushes α across D and takes it off μ. The

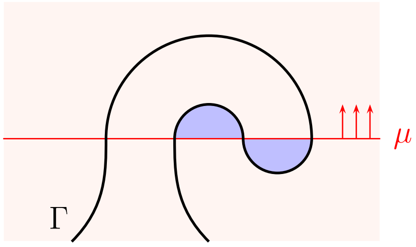


Figure 7: An example of a positive (to the left) and a negative (to the right) retractible arc, each with the retracting disk. The top arc is not retractible, but it will be after the positive arc is retracted.

cobordism $S \subset \mathbb{A} \times I$ traced out by such an isotopy will represent an isomorphism i_α in $\mathscr{BN}_q(\mathbb{A})$, whose inverse is given by S reflected along $\mathbb{A} \times \{\frac{1}{2}\}$ and multiplied by q^{-1} or q depending on whether α is positive of negative.

To prove the lemma, note that Γ can be transformed into some c_m by repeatedly applying delooping isomorphisms and isomorphisms i_α associated to isotopies of negative retractible arcs. It follows that m is the algebraic intersection number of Γ with μ and it does not depend on the whole procedure. $\square$

It follows that $\mathscr{F}_{\mathbb{A}_q}$ is completely determined by its restriction to $\mathscr{E}$. Clearly,

$$\mathscr{F}_{\mathbb{A}_q}(c_n) = qHH(A^n) \cong E^n \cong V^{\otimes n}, \tag{5.11}$$

see Proposition 5.2. Using the Bar-Natan relations and the trace relation we can reduce any cobordism S to a linear combination of surfaces of genus zero, each being a collection of disjoint annuli carrying or not a dot and intersecting the membrane $\mu \times I$ in one arc. In the view of the isomorphism $\mathscr{F}_{\mathbb{A}_q}(c_n) \otimes \mathscr{F}_{\mathbb{A}_q}(c_m) \cong \mathscr{F}_{\mathbb{A}_q}(c_{n+m})$ it is then enough to compute $\mathscr{F}_{\mathbb{A}_q}$ only for connected surfaces: a vertical annulus, and an annulus with both boundaries on the top or on the bottom boundary of $\mathbb{A} \times I$.

Lemma 5.6. *$\mathscr{F}_{\mathbb{A}_q}(\mathbb{S}^1 \times \cup) \colon \Bbbk \longrightarrow V^{\otimes 2}$ and $\mathscr{F}_{\mathbb{A}_q}(\mathbb{S}^1 \times \cap) \colon V^{\otimes 2} \longrightarrow \Bbbk$ are the coevaluation and evaluation maps. Furthermore, $\mathscr{F}_{\mathbb{A}_q}(S) = 0$ for any annulus S marked with a dot.*

Proof. The two maps are dual to each other—in particular, one determines the other—so that it is enough to compute $\mathscr{F}_{\mathbb{A}_q}(\mathbb{S}^1 \times \cup)$. When cut open, the surface $\mathbb{S}^1 \times \cup$ is the identity cobordism on $\cup$ and the sequence (5.10) simplifies to

$$\begin{aligned} \Bbbk &\xrightarrow{coev_*} qHH({}^*\mathbf{F}_{CK}(\cup) \otimes \mathbf{F}_{CK}(\cup); \Bbbk) \\ &\xrightarrow{\theta} qHH(\mathbf{F}_{CK}(\cup) \otimes {}^*\mathbf{F}_{CK}(\cup); A^2) \xrightarrow{ev_*} qHH(A^2) \end{aligned}$$

where ${}^*\mathbf{F}_{CK}(\cup) = \mathrm{Hom}_{A^2}(\mathbf{F}_{CK}(\cup), A^2)$ is the bimodule of left A^2–linear maps. As a left A^2–module, $\mathbf{F}_{CK}(\cup)$ is generated by a single element $x :=$ in degree 0, and the generator ${}^*x \in {}^*\mathbf{F}_{CK}(\cup)$ takes it to the idempotent $e =$ $\in A^2$. Hence,

$$ev_*(\theta(coev_*(1))) = ev_*(\theta([x^* \otimes x])) = ev_*([x \otimes x^*]) = [e],$$

which is identified with $v_+ \otimes v_- + q^{-1} v_- \otimes v_+ = \mathscr{F}_{TL}(\cup)(1)$ under the sequence of isomorphisms $E^2 \cong K_0(A^2) \cong V^{\otimes 2}$. $\square$

We shall now justify the adjective "quantized" for the skein category. Consider an operation that takes a flat tangle T into the surface $\mathbb{S}^1 \times T \subset \mathbb{A} \times I$. We used it in Proposition 2.5 to show that the graded Temperly–Lieb category $\mathcal{TL}$, when specialized at $q = 1$, is equivalent to the Boerner–Bar-Natan category of the annulus $\mathcal{BBN}(\mathbb{A})$, i.e. the quotient of $\mathcal{BN}(\mathbb{A})$ by the Boerner's relation (a single dot annihilates any surface different from a disk and a sphere). The value of q is fixed because every torus in $\mathcal{BN}(\mathbb{A})$ evaluates to 2, and so it does in $\mathcal{BBN}(\mathbb{A})$. However, in $\mathcal{BN}_q(\mathbb{A})$ a torus evaluates to $q + q^{-1}$, so that the Cartesian product with a circle is a functor for any value of q.

Theorem 5.7. *The functor $\mathbb{S}^1 \times (-) \colon \mathcal{TL} \longrightarrow \mathcal{BBN}_q(\mathbb{A})$ is an equivalence of categories, where $\mathcal{BBN}_q(\mathbb{A})$ is the quotient of $\mathcal{BN}_q(\mathbb{A})$ by the Boerner's relation.*

Proof. The image of $\mathbb{S}^1 \times (-)$ in $\mathcal{BBN}_q(\mathbb{A})$ is precisely the image of the subcategory $\mathcal{E} \subset \mathcal{BN}_q(\mathbb{A})$ in $\mathcal{BBN}_q(\mathbb{A})$. Because of Lemma 5.5, it thus follows that the functor $\mathbb{S}^1 \times (-)$ is full and essentially surjective. Finally, consider the following diagram of functors

$$\begin{tikzcd} \mathcal{TL} \arrow[rr, "\mathbb{S}^1\times(-)"] \arrow[dr, "\mathcal{F}_{TL}"'] & & \mathcal{BBN}_q(\mathbb{A}) \arrow[dl, "\mathcal{F}_{\mathbb{A}_q}"] \\ & \mathrm{g}\mathcal{Rep}(\mathcal{U}_q(\mathfrak{sl}_2)) & \end{tikzcd} \tag{5.12}$$

It commutes up to a natural isomorphism due to Lemma 5.6. This ends the proof, because $\mathcal{F}_{TL}$ is faithful. ☐

An immediate consequence of the above theorem is the existence of an invariant homology for annular links with an action of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$, as stated in the introduction.

Theorem B. *Assume an annular link $\widehat{T}$ is a closure of an (n, n)-tangle T. Then the homotopy type of $qHH(C_{CK}(T); A^n)$ is a triply graded invariant of $\widehat{T}$, which is projectively functorial with respect to annular link cobordisms. Moreover, it admits an action of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ that commutes with the differential and the maps induced by annular link cobordisms intertwine this action.*

Proof. By the definition of $\mathcal{F}_{\mathbb{A}_q}$ we have that $qHH(C_{CK}(T); A^n) = \mathcal{F}_{\mathbb{A}_q}[\![\widehat{T}]\!]$. The functor $\mathcal{F}_{\mathbb{A}_q}$ takes values in the category of representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ due to Theorem 5.7. The thesis follows from projective functoriality of the formal Khovanov bracket. ☐

It was conjectured in [AGW15] that Hochschild homology of the Chen–Khovanov complexes recovers the annular chain complex, with a proof for the algebras $A^{n,1}$, the next–to–highest weight subalgebra of A^n. The conjecture follows from Theorem 5.7.

Theorem C. *Let $\widehat{T}$ be the annular closure of an (n, n)–tangle T. Then there is an isomorphism of chain complexes*

$$CKh_{\mathbb{A}}(\widehat{T}) \cong HH(C_{CK}(T); A^n) \tag{5.13}$$

natural with respect to the chain maps associated to tangle cobordisms. The annular grading in $CKh_{\mathbb{A}}(\widehat{T})$ corresponds to the weight decomposition of $C_{CK}(T)$.

Proof. The functor $\mathscr{F}_{\mathbb{A}_q}$ agrees with $\mathscr{F}_{\mathbb{A}}$ when $q = 1$. Thence, the diagram

$$\begin{tikzcd} \mathrm{hTr}(\mathbf{BN}) \arrow[r, "\widehat{(-)}"] \arrow[d, "\mathrm{hTr}(\mathbf{F}_{CK})"'] & \mathscr{BN}(\mathbb{A}) \arrow[d, "\mathscr{F}_{\mathbb{A}}"] \\ \mathrm{hTr}(\mathbf{DB}) \arrow[r, "HH"] & \mathrm{g}\mathscr{R}ep(\mathfrak{sl}_2) \end{tikzcd} \tag{5.14}$$

commutes up to a natural isomorphism. ☐

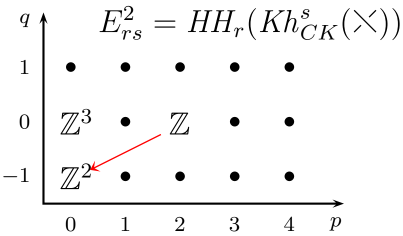
This motivates the following name for the invariant from Theorem B.

Definition 5.8. The *quantum annular homology* $Kh_{\mathbb{A}_q}(L)$ of a link L is the homology of the *quantum annular chain complex* $CKh_{\mathbb{A}_q}(D) := \mathscr{F}_{\mathbb{A}_q}[\![D]\!]$, where D is a diagram of L.

Proposition 5.9. *There is a spectral sequence with* $E^2_{rs} = qHH_r(Kh^s_{CK}(T); A^n)$ *that converges to* $Kh_{\mathbb{A}_q}(\hat{T})$ *for every* (n,n)*–tangle* T.

Proof. Consider the bicomplex $C_{ij} := {}_qC^i_{CK}(T) \otimes R_j(A^n)$, where $R_\bullet(A^n)$ is the bar resolution of A^n and the subscript q at the front twists the left action of A^n on the chain complex. The filtration with respect to j induces a spectral sequence with $E^2_{rs} = qHH_r(Kh^s_{CK}(T))$, whereas the one with respect to i induces a spectral sequence with $E^\infty_{rs} = E^2_{rs} = H_s(qHH_r(C_{CK}(T); A^n))$. Since $E^2_{rs} \neq 0$ only when $r = 0$, $H_s(TotC_{ij}) \cong E^2_{0s} = Kh^s_{\mathbb{A}_q}(\hat{T})$. ☐

The spectral sequence from Proposition 5.9 may not collapse immediately. The second page for $T = \times$ has three non-trivial entries and there is a non-trivial differential that kills two generators, see the diagram to the right for $\Bbbk = \mathbb{Z}$ and $q = 1$. The third page agrees with the annular homology of the link $\hat{T}$, the homology of $(W \longrightarrow V^{\otimes 2})$ with W in homological degree -1.

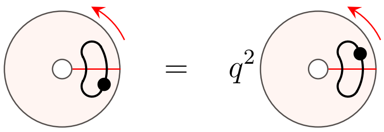


5.4 Homology for $(2, n)$ torus links

Consider the subcategory $\mathscr{E}_2 \subset \mathscr{BN}_q(\mathbb{A})$ generated by objects intersecting the seam μ in exactly two points. Note that every such object is of the form $\Gamma_I \cup \Gamma_N$, where Γ_N is a (possibly empty) union of trivial circles not intersecting μ, and Γ_I is either a trivial circle intersecting μ in two points, or a pair of essential circles, each intersecting μ in a single point.

In what follows we shall write W or W_I for the module assigned to a trivial circle depending on whether it is disjoint from μ or not. W is freely generated by w_+ and w_-, the images of $1 \in \Bbbk$ under the maps induced by a cup cobordism disjoint from $\mu \times I$, without and with a dot respectively. To pick generators for W_I consider a cup cobordism that intersects $\mu \times I$ in a single arc α and define w_+, w^-_- and w^+_- as the images of $1 \in \Bbbk$ under the maps induced by the cobordism respectively without any dot, a dot on the negative side of $\mu \times I$, and a dot on the positive side of $\mu \times I$. All three generate W_I, but they are not linearly independent: $w^-_- = q^2 w^+_-$, because a dot has degree -2.

We can represent elements of the modules graphically as in Section 2.4: the generators of V are visualized by orienting the essential circles, and those of W and W_I are given as the trivial circle without or with a dot. Then the relation in W_I between w_-^- and w_-^+ follows from



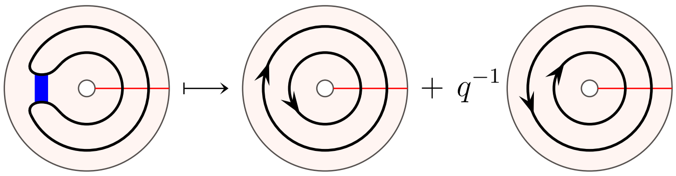
(5.15)

Here we choose the counter-clockwise orientation of the core of the annulus, so that the seam μ is cooriented upwards. Because (5.6) are not symmetric, the essential circles must be ordered. We choose the left–to–right ordering read from the seam.

Capping of the trivial circle touching the seam vanishes on w_+ and takes $w_-^\pm$ to $q^{\mp 1}$, as the result is a sphere without or with a dot respectively, but intersecting the membrane. We then isotope it off the membrane using (3.40), which introduces a power of q. It is now straightforward to compute the saddle cobordisms in $\mathscr{E}_2$ using the comparison with $\mathscr{F}_{TL}$. A merge of two essential circles followed by capping off the trivial circle is the evaluation map, which implies the following surgery rules :

(5.16)

The other saddle cobordism is even easier to find out. For degree reasons it must vanish if the trivial circle carries a dot, and otherwise it is the coevaluation map:



(5.17)

Fix $n > 0$ and let $T_{2,n}$ denote the annular $(2, n)$ torus link: the annular closure of the braid σ^{-n}, where σ is the positive generator of the 2–strand braid group.

Proposition 5.10. *The quantum annular Khovanov homology of the annular* $(2, n)$ *torus link is given by*

$$Kh^{i,j}_{\mathbb{A}_q}(T_{2,n}) = \begin{cases} V_2 & \text{if } i = 0 \text{ and } j = -n, \\ V_0/(q^2 + (-1)^i) & \text{if } -n+1 \leqslant i \leqslant -1 \text{ and } j = 2i - n, \\ K(q^2 - (-1)^i) & \text{if } -n \leqslant i \leqslant -2 \text{ and } j = 2i - n + 2, \\ V_0 & \text{if } i = -n \text{ and } j = -3n, \\ 0 & \text{else,} \end{cases} \tag{5.18}$$

where $V_2 := (V \otimes V)/\mathrm{span}_{\Bbbk}\{v_+ \otimes v_- + q^{-1} v_- \otimes v_+\}$ *and* $K(a) := \{v \in V_0 \mid av = 0\}$ *for any* $a \in \Bbbk$.

Proof. Let $D \subset \mathbb{A}$ be a standard diagram for $T_{2,n}$ such that cutting D along μ results in a diagram for σ^{-n}. Then each resolution of D belong to $\mathscr{E}_2$. We introduce the notations $u_q, l_q \colon W_I \longrightarrow W_I$ for the maps that put a dot on the positive and negative side of the circle respectively, and $w_q \colon W_I \longrightarrow V \otimes V$ for the split map. Explicitly,

$$u_q(w_+) = w_-^+, \qquad\qquad u_q(w_-) = 0, \tag{5.19}$$

$$l_q(w_+) = w_-^- = q^2 w_-^+, \qquad l_q(w_-) = 0, \tag{5.20}$$
$$w_q(w_+) = v_+ \otimes v_- + q^{-1} v_- \otimes v_+, \qquad w_q(w_-^\pm) = 0. \tag{5.21}$$

Let $\{m\}$ denote the grading shift functor which raises the j–degree by m. Arguing as in [Kh99, Proposition 26], one can show that $CKh_{\mathbb{A}_q}(D)$ is quasi-isomorphic to the chain complex

$$\begin{aligned} 0 \longrightarrow W_I\{-3n+1\} \xrightarrow{\partial^{-n}} W_I\{-3n+3\} \xrightarrow{\partial^{-n+1}} \dots \\ \xrightarrow{\partial^{-3}} W_I\{-n-3\} \xrightarrow{\partial^{-2}} W_I\{-n-1\} \xrightarrow{\partial^{-1}} V_1 \otimes V_1^*\{-n\} \longrightarrow 0 \end{aligned}$$

where $\partial^{-1} = w_q$ and $\partial^i = u_q - (-1)^i l_q$ for $-n \le i \le -2$. One can write the above complex more explicitly by writing each W_I as a direct sum

$$W_I = \mathrm{span}_{\Bbbk}\{w_+, w_-\} = V_0\{+1\} \oplus V_0\{-1\},$$

and by noting that the map $u_q - (-1)^i l_q$ is given by

$$u_q - (-1)^i l_q = \begin{pmatrix} 0 & 0 \\ 0 & 1-(-1)^i q^2 \end{pmatrix}$$

with respect to this direct sum decomposition. It then follows that the above complex is isomorphic to a direct sum of complexes

$$\begin{aligned} &\Big(0 \longrightarrow V_0 \xrightarrow{\tilde{w}_q} V_1 \otimes V_1^* \longrightarrow 0\Big)\{-n\} \\ &\Big(0 \longrightarrow V_0 \xrightarrow{1+(-1)^i q^2} V_0 \longrightarrow 0\Big)\{2i-n\} \qquad \text{for } -n+1 \le i \le -1 \\ &\Big(0 \longrightarrow V_{0,q} \longrightarrow 0\Big)\{-3n\} \end{aligned}$$

where, in each of these complexes, the bidegree of the rightmost nonzero term is supported on the diagonal $j = 2i - n$, and where $\tilde{w}_q$ is the coevaluation map $\tilde{w}_q\colon V_0 \longrightarrow V_1 \otimes V_1^*$ given by $\tilde{w}_q(1) = w_q(w_+) = v_+ \otimes v_- + v_- \otimes v_+$. The proposition now follows by passing to homology. ☐

If $\Bbbk$ is a field and $q^2 \neq \pm 1$, then $q^2 + (-1)^i$ is invertible. It follows from the above proof that in such a case

$$Kh^{i,j}_{\mathbb{A}_q}(D) = \begin{cases} V_2 & \text{if } (i,j) = (0,-n), \\ V_0 & \text{if } (i,j) = (-n,-3n), \\ 0 & \text{else.} \end{cases} \tag{5.22}$$

On the other hand, the quantum homology contains additional copies of V_0 if $q^2 = \pm 1$, see Table 1. This illustrates that quantum annular homology is in general richer than the non-quantized theory, which corresponds to the case $q = 1$.

5.5 Action of tangles on cablings of an $(1,1)$–tangle

The action of a braid group on the annular homology of a cabling of a framed annular knot K was studied in [GLW15]. In what follows we compute the action on the quantum annular homology with the help of traces when $K = \widehat{T}$ is an annular closure of a framed $(1,1)$–tangle T.

$j \backslash i$	-5	-4	-3	-2	-1	0
-5						V_2
-7				$V_0^\dagger$	$V_0^\dagger$	
-9			$V_0^\ddagger$	$V_0^\ddagger$		
-11		$V_0^\dagger$	$V_0^\dagger$			
-13	$V_0^\ddagger$	$V_0^\ddagger$				
-15	V_0					

Table 1: The quantum annular homology for the torus knot $T(2,5)$. The representations marked by a dagger ($\dagger$) only occur if $q^2 = 1$, and the ones marked by a double dagger ($\ddagger$) only occur if $q^2 = -1$. The unmarked representations are always there.

The action can be easily extended to all oriented tangles. Given a finite sequence $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_k)$ with $\epsilon_i \in \{-,+\}$ we write $T^{\underline{\epsilon}}$ for the $\underline{\epsilon}$–oriented k–cabling of T: each i-th cable is oriented parallel or opposite to T depending on whether ϵ_i is $+$ or $-$ respectively. In case each $\epsilon_i = +$ we shall write simply T^k for the cabling.

It follows immediately from the definition of the formal Khovanov bracket that

$$[\![T^{\underline{\epsilon}}]\!] \cong [\![T^{\underline{\epsilon}'}]\!][2(k-1)w]\{6(k-1)w\} \tag{5.23}$$

where both $\underline{\epsilon}$ and $\underline{\epsilon}'$ have the same length k, they differ only in one sign, and $w := n_+(T) - n_-(T)$ is the writhe of T (this quantity is an invariant of framed tangles). Hence, up to degree shifts, $[\![T^{\underline{\epsilon}}]\!]$ is isomorphic to $[\![T^k]\!]$ for any sequence $\underline{\epsilon}$ of length k.

Choose an oriented (m,n)–tangle T' with orientations of bottom and top endpoints encoded by sequences $\underline{\epsilon}$ and $\underline{\epsilon}'$. Let $S_T(T')$ be the cobordism drawn by the isotopy $T'T^{\underline{\epsilon}} \simeq T^{\underline{\epsilon}'}T'$ that slides T' along a ribbon carrying T. The functoriality of the formal bracket implies the existence of a chain map

$$S_T(T')_*\colon C_{CK}(T') \underset{A^m}{\otimes} C_{CK}(T^{\underline{\epsilon}}) \longrightarrow C_{CK}(T^{\underline{\epsilon}'}) \underset{A^n}{\otimes} C_{CK}(T') \tag{5.24}$$

functorial in T' up to a sign. In particular, there is a morphism

$$\begin{tikzcd} A^m \arrow[r, "C_{CK}(T^{\underline{\epsilon}})"] \arrow[d, "C_{CK}(T')"'] \arrow[dr, phantom, "\Swarrow"] & A^m \arrow[d, "C_{CK}(T')"] \\ A^n \arrow[r, "C_{CK}(T^{\underline{\epsilon}'})"'] & A^n \end{tikzcd} \tag{5.25}$$

in $\mathrm{hTr}_q(\mathfrak{D}(\mathbf{DB}))$, the quantum horizontal trace of the derived category of diagrammatic bimodules (bounded, because arc algebras have finite global dimensions). We can then apply the quantum Hochschild homology and use (5.23) to get a homomorphism $Kh_{\mathbb{A}_q}(K^m) \longrightarrow Kh_{\mathbb{A}_q}(K^n)$, denoted also by $S_T(T')_*$.

Theorem D. *The homomorphism $S_T(T')_*$ depends only on the annular closure K of T. It is functorial in T' up to a sign, intertwines the action of $\mathcal{U}_q(\mathfrak{sl}_2)$, and factors through the Jones skein relation:*

$$q^2 S_T(\overcrossing)_* - q^{-2} S_T(\undercrossing)_* = (q - q^{-1}) S_T(\asymp)_*. \tag{5.26}$$

In particular, the n–strand braid group acts on the quantum annular homology $Kh_{\mathbb{A}_q}(\widehat{T}^n)$ of the n–cabling of a framed $(1,1)$–tangle T.

We need one property of the quantum Hochschild homology to prove the theorem. Recall that a chain complex C of (B, A)–bimodules admits a left dual if and only if it is bounded with each $C_i \in \mathscr{R}ep(B, A)$.

Lemma 5.11. *Choose algebras A and B, and chain complexes M and N of (A, A)– and (B, B)–bimodules respectively. Assume there is an exact sequence*

$$0 \longrightarrow C' \xrightarrow{\iota} C \xrightarrow{\pi} C'' \longrightarrow 0, \tag{5.27}$$

of left dualizable chain complexes of (B, A)–bimodules and a commuting diagram

$$\begin{tikzcd} C' \otimes_A M \arrow[r, "\iota\otimes\mathrm{id}"] \arrow[d, "\sigma'"'] & C \otimes_A M \arrow[r, "\pi\otimes\mathrm{id}"] \arrow[d, "\sigma"] & C'' \otimes_A M \arrow[d, "\sigma''"] \\ N \otimes_B C' \arrow[r, "\mathrm{id}\otimes\iota"] & N \otimes_B C \arrow[r, "\mathrm{id}\otimes\pi"] & N \otimes_B C'' \end{tikzcd} \tag{5.28}$$

Then $\sigma_* = \sigma'_* + \sigma''_*$, where $\sigma'_*, \sigma_*, \sigma''_* \colon qHH(M, A) \longrightarrow qHH(N, B)$ are homomorphisms induced by the vertical chain maps.

Proof. Choose B–linear generators $\{c'_1, \dots c'_r\}$ for C' and $\{c''_1, \dots, c''_s\}$ for C''. The sequence (5.27) splits in each degree when viewed as a sequence of chain complexes of B–modules, so that there is a set of B–linear generators $\{c_1, \dots, c_{r+s}\}$ of C satisfying $\iota(c'_i) = c_i$ for $i \leqslant r$ and $\pi(c_{r+i}) = c''_i$ for $i \leqslant s$.

Recall that $\sigma_* \colon qHH(M; A) \longrightarrow qHH(N; B)$ is defined by the sequence of maps

$$\begin{aligned} qHH(M; A) \xrightarrow{coev_*} qHH(M \otimes {}^*C \otimes C; A) \xrightarrow{\theta} qHH(C \otimes M \otimes {}^*C; B) \\ \xrightarrow{\sigma} qHH(N \otimes C \otimes {}^*C; B) \xrightarrow{ev_*} qHH(N; B) \end{aligned} \tag{5.29}$$

where the dual chain complex ${}^*C$ is constructed by taking ${}^*C_i := \mathrm{Hom}_B(C_{-i}, B)$ with generators ${}^*c_i$ dual to c_i. Explicitly,

$$\sigma_*([m]) = \left[\sum_{i=1}^{r+s} q^{-|c_i|} (\mathrm{id} \otimes {}^*c_i)(\sigma(c_i \otimes m)) \right] \tag{5.30}$$

for any cycle $m \in M$. The maps σ'_* and σ''_* are defined likewise. In particular, ${}^*c'_i = {}^*c_i \circ \iota$ for $i \leqslant r$ and ${}^*c_{i+r} = {}^*c''_i \circ \pi$ for $i \leqslant s$. Because the diagram (5.28) commutes, it follows that

$$\begin{aligned} (\mathrm{id} \otimes {}^*c_i)(\sigma(c_i \otimes m)) &= (\mathrm{id} \otimes {}^*c_i)(\sigma(\iota(c'_i) \otimes m)) \\ &= (\mathrm{id} \otimes ({}^*c_i \circ \iota))(\sigma'(c'_i \otimes m)) = (\mathrm{id} \otimes {}^*c'_i)(\sigma'(c'_i \otimes m)) \end{aligned}$$

for $i \leqslant r$, and

$$\begin{aligned} (\mathrm{id} \otimes {}^*c_{i+r})(\sigma(c_{i+r} \otimes m)) &= (\mathrm{id} \otimes ({}^*c''_i \circ \pi))(\sigma(c_{i+r} \otimes m)) \\ &= (\mathrm{id} \otimes {}^*c''_i)(\sigma''(\pi(c_{i+r}) \otimes m)) = (\mathrm{id} \otimes {}^*c''_i)(\sigma''(c''_i \otimes m)) \end{aligned}$$

for $i \leqslant s$. This proves that $\sigma_*([m]) = \sigma'_*([m]) + \sigma''_*([m])$. ∎

Proof of Theorem D. It is clear from construction that $S_T(-)_*$ intertwines the action of $\mathcal{U}_q(\mathfrak{sl}_2)$, and functoriality follows from functoriality of $C_{CK}(-)$: sliding $T''T'$ through $T^{\underline{\epsilon}}$ can be either performed at once, resulting in $S_T(-)$, or one can slide T' and T'' separately, obtaining $S_T(T'')_* \circ S_T(T')_*$. The Jones relation follows from Lemma 5.11 applied to Viro exact sequences from Proposition 2.2. What remains is to show that $S_T(T')_*$ does not depend on how the knot K is cut open. For that assume $T = T_1T_0$ is a composition of two framed $(1,1)$–tangles and consider the following diagram in $\mathrm{hTr}_q(\mathfrak{D}(\mathbf{DB}))$

$$\begin{tikzcd}
A^n \arrow[r, "C_{CK}(T_0^n)"] \arrow[d, "C_{CK}(T_0^n)"'] \arrow[dr, phantom, "\boxed{\mathrm{id}}"] & A^n \arrow[r, "C_{CK}(T_1^n)"] \arrow[dl, equal] \arrow[d, phantom, "\boxed{\mathrm{id}}"] & A^n \arrow[d, "C_{CK}(T_0^n)", red] \arrow[dl, equal, red] \arrow[dl, phantom, "\boxed{\mathrm{id}}"'] \\
A^n \arrow[r, "C_{CK}(T_1^n)"] \arrow[d, "C_{CK}(T')"'] \arrow[dr, phantom, "\boxed{S_{T_1}(T')}"] & A^n \arrow[r, "C_{CK}(T_0^n)"] \arrow[d, dashed, red, "C_{CK}(T')"] \arrow[dr, phantom, "\boxed{S_{T_0}(T')}"] & A^n \arrow[d, "C_{CK}(T')", red] \\
A^n \arrow[r, "C_{CK}(T_1^n)"] \arrow[d, "C_{CK}(T_0^n)^*"'] \arrow[dr, equal] \arrow[d, phantom, "\boxed{coev}"] & A^n \arrow[r, "C_{CK}(T_0^n)"] \arrow[dr, equal, red] \arrow[d, phantom, "\boxed{\mathrm{id}}"] & A^n \arrow[d, "C_{CK}(T_0^n)^*", red] \arrow[dl, phantom, "\boxed{ev}"'] \\
A^n \arrow[r, "C_{CK}(T_0^n)"] & A^n \arrow[r, "C_{CK}(T_1^n)"] & A^n
\end{tikzcd} \tag{5.31}$$

where a box in each region contains a 2–morphism that goes from the top right to the bottom left corner. The top part of the diagram represents the canonical isomorphism in $\mathrm{hTr}_q(\mathfrak{D}(\mathbf{DB}))$ between $C_{CK}(T_1^nT_0^n)$ and $C_{CK}(T_0^nT_1^n)$, and the bottom part represents the inverse to that isomorphism. The two middle squares encode the map $S_{T_0T_1}(T')_*$.

Consider the trapezoid at the right edge of the diagram, bounded by the 1–morphisms drawn in red. The trace relation allows us to move it to the left, which results in a diagram that encodes $S_{T_1T_0}(T')_*$. Thence, it is enough to show that the 2–morphism in the trapezoid has degree zero. Indeed, it is a composition of three degree zero 2–morphisms: the identity on $C_{CK}(T_0^n)$ followed by $S_{T_0}(T')$ and the evaluation map $C_{CK}(T_0^n)^* \otimes C_{CK}(T_0^n) \longrightarrow A^n$. $\square$

The map $S_T(T')_* \colon Kh_{\mathbb{A}_q}(\hat{T}^n) \longrightarrow Kh_{\mathbb{A}_q}(\hat{T}^m)$ can be computed without referring to traces. Let us identify the two copies of T' at both endpoints of $S_T(T')$. This results in a surface $\hat{S}_T(T')$ in $V \times I$, where V is a solid torus, with closures $\widehat{T}^n$ and $\widehat{T}^m$ as its boundary. Let $h \colon V \longrightarrow V$ be an embedding that takes the core of V to $\widehat{T} \subset V$. Then $\hat{S}_T(T')$ coincides with the image of $\mathbb{S}^1 \times T' \subset V \times I$ under $h \times \mathrm{id} \colon V \times I \longrightarrow V \times I$. In particular, $S_T(T')_*$ can be computed from the chain map assigned to $\hat{S}_T(T')$ in the annular homology theory.

Let us finish with a remark about signs. The construction due to Chen and Khovanov is functorial only up to an overall sign, so is the map $S_T(T')_*$ and the action. Therefore, the action is well-defined only if the ground ring $\Bbbk$ has characteristic two (so that $-1 = +1$). To have the action defined over other rings, such as $\mathbb{Z}[q^{\pm 1}]$, one needs to start with a strictly functorial version of the Chen–Khovanov construction, or fix the signs explicitly in the annular homology. It seems the latter approach is plausible: there is a version of a Lee spectral sequence in the non-quantized setting, which can be used to find canonical generators. This approach was used successfully in [GLW15].

However, the powers of q are well-defined. In particular, we can assign to any surface in $\mathbb{S}^1 \times \mathbb{R}^3$ a polynomial in q defined up to an overall sign. It follows from Theorem D that the polynomial associated to $\mathbb{S}^1 \times L$, where L is a link, is the Jones polynomial of L. In particular, the invariant do not collapse.

5.6 Quantum homology of links in a thickened Möbius band

We can quantize the APS homology of links in a twisted line bundle over a Möbius band similarly to that for annular links. We have quantized the Bar-Natan skein category of a Möbius band by defining $\mathscr{BN}_q(\mathbb{M}) := \mathrm{hTr}_q(\mathbf{BN}(\mathbb{R}\times I), \rho_*)$, the quantum horizontal trace of $\mathbf{BN}(\mathbb{R}\times I)$ twisted with respect to the reflection $\rho(s,t) := (-s,t)$ of the stripe $\mathbb{R}\times I$. The reflection ρ induces also an automorphism of A^n, which we shall denote with the same symbol, and a bifunctor on the bicategory of diagrammatic bimodules. The following result is immediate from the definition of $\mathbf{F}_{CK}$.

Lemma 5.12. *Choose an (m,n)–tangle T and write T^{flip} for its reflection along the vertical axis. Then*

$$\rho_*\mathbf{F}_{CK}(T) \cong \mathbf{F}_{CK}(T^{flip}) \tag{5.32}$$

as (A^m, A^n)–bimodules.

Thence, we can define a TQFT functor $\mathscr{F}_{\mathbb{M}_q}\colon \mathscr{BN}_q(\mathbb{M}) \longrightarrow \mathscr{M}od_{\Bbbk}$ by composing $\mathbf{F}_{CK}$ with the quantum Hochschild homology twisted by ρ. It follows immediately from Proposition 4.11 that

$$qHH_{>0}(A^n,\rho) = 0 \qquad \text{and} \qquad qHH_0(A^n,\rho) \cong \mathrm{coInv}({}_{\rho}E^n), \tag{5.33}$$

where the latter is generated by idempotents corresponding to symmetric cup diagrams (i.e. those fixed by ρ). In particular, if we write c_n for the collection of n parallel separating curves, each wrapping $\mathbb{M}$ twice, and γ for the nonseparating curve, then

$$\mathscr{F}_{\mathbb{M}_q}(c_n) \cong qHH(A^{2n},\rho) \cong E^n \cong V^{\otimes n} \qquad \text{and} \qquad \mathscr{F}_{\mathbb{M}_q}(\gamma\cup c_n) = 0 \tag{5.34}$$

where $E^n \cong qHH(A^{2n},\rho)$ takes an idempotent $e\in E^n$ to $e\otimes\rho(e)\in E^{2n}$. In particular, we shall be interested only in the subcategory $\mathscr{BN}_q^{ev}(\mathbb{M})$ generated by collections that do not contain γ. These collections are characterized by the following property: if $\mu\subset\mathbb{M}$ cuts the band into a square, then each object from $\mathscr{BN}_q^{ev}(\mathbb{M})$ intersects μ in an even number of points.

As in the annular case, we can restrict the category $\mathscr{BN}_q^{ev}(\mathbb{M})$ further to $\mathscr{E}$ by taking only collections of nontrivial curves with minimal intersection with μ (i.e. each curve intersects μ exactly twice). The argument from Lemma 5.5 adapted to this case shows that the inclusion $\mathscr{E} \longrightarrow \mathscr{BN}_q^{ev}(\mathbb{R}\times I)$ is an equivalence of categories. The following result is straightforward.

Lemma 5.13. *Let $\gamma\subset\mathbb{M}$ be a nonseparating curve. Then there is a commutative diagram of functors*

$$\begin{tikzcd}
\mathscr{BN}_{q^2}(\mathbb{A}) \arrow[rr, "\Phi"] \arrow[dr, "\mathscr{F}_{\mathbb{A}_q}"'] & & \mathscr{BN}_q^{ev}(\mathbb{M}) \arrow[dl, "\mathscr{F}_{\mathbb{M}_q}"] \\
& \mathscr{M}od_{\Bbbk} &
\end{tikzcd} \tag{5.35}$$

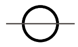
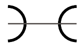
where $\Phi\colon \mathscr{BN}_{q^2}(\mathbb{A}) \longrightarrow \mathscr{BN}_q(\mathbb{M})$ is induced by the diffeomorphism $\mathbb{A}\approx\mathbb{M}-\gamma$.

Proof. The thesis follows by comparing Φ with the functor $\mathrm{hTr}_{q^2}(\mathbf{BN}) \longrightarrow \mathrm{hTr}_q(\mathbf{BN},\rho_*)$ that adds n vertical lines next to an (n,n)–tangle T. The difference in powers of q appears, because going once through a membrane in $\mathrm{hTr}_{q^2}(\mathbf{BN}) \cong \mathscr{BN}_{q^2}(\mathbb{A})$ corresponds to going twice through the membrane in $\mathrm{hTr}_q(\mathbf{BN},\rho_*) \cong \mathscr{BN}_q(\mathbb{M})$. ☐

Each cobordism in $\mathscr{E}$ can be decomposed into a composition of cobordisms from the image of Φ and projective planes with a disk removed, each with a nontrivial curve in $\mathbb{M}$ as its boundary. These can be seen in turn as the saddle cobordisms between a trivial and a nontrivial curve, with the trivial curve capped off. They correspond under the quotient map $(\mathbb{R}\times I)\times I \longrightarrow \mathbb{M}\,\widetilde{\times}\, I$ to surfaces $\cup\times I$ and $\cap\times I$, and we denote them by $S_\cup$ and $S_\cap$ respectively.

Lemma 5.14. *$\mathscr{F}_{\mathbb{M}_q}(S_\cap)\colon V \longrightarrow \Bbbk$ evaluates v_+ and v_- to q and 1 respectively, whereas $\mathscr{F}_{\mathbb{M}_q}(S_\cup)\colon \Bbbk \longrightarrow V$ takes 1 to $v_+ + q^{-1}v_-$.*

Proof. The module $V := qHH(A^2;\rho)$ assigned by $\mathscr{F}_{\mathbb{M}_q}$ to a nontrivial curve has canonical generators

$b_0 :=$  and $b_1 :=$  (5.36)

which corresponds under (5.34) to $v_+ + q^{-1}v_-$ and v_- respectively. The thesis now follows from direct computations. Indeed, the map $\mathscr{F}_{\mathbb{M}_q}(S_\cup)$ is given by the sequence

$$\begin{aligned}\Bbbk &\longrightarrow qHH({}^*\mathbf{F}_{CK}(\cup)\otimes \mathbf{F}_{CK}(\cup);\Bbbk,\mathrm{id})\\ &\longrightarrow qHH(\mathbf{F}_{CK}(\cup)\otimes {}^*\mathbf{F}_{CK}(\cup);A^2,\rho)\longrightarrow qHH(A^2,\rho)\end{aligned}$$

which takes $1\in\Bbbk$ into b_0. Dually, $S_\cap$ is assigned the sequence

$$\begin{aligned}qHH(A^2,\rho) &\longrightarrow qHH({}^*\mathbf{F}_{CK}(\cap)\otimes \mathbf{F}_{CK}(\cap);A^2,\rho)\\ &\longrightarrow qHH(\mathbf{F}_{CK}(\cap)\otimes {}^*\mathbf{F}_{CK}(\cap);\Bbbk,\mathrm{id})\longrightarrow \Bbbk\end{aligned}$$

which takes b_0 and b_1 to $q+q^{-1}$ and 1 respectively. ☐

The above is enough to compute $\mathscr{F}_{\mathbb{M}_q}$ on all morphisms in $\mathscr{BN}_q(\mathbb{M})$. A quick comparison with the formulas from Section 2 proves that $\mathscr{F}_{\mathbb{M}_q}$ is a deformation of the APS construction.

Theorem 5.15. *The the following diagram of functors*

$$\begin{tikzcd}\mathrm{hTr}(\mathbf{BN}^{ev},\rho_*) \arrow[r, "\widehat{(-)}"] \arrow[d, "\mathrm{hTr}(\mathbf{F}_{CK})"'] & \mathscr{BN}(\mathbb{M}) \arrow[d, "\mathscr{F}_{\mathbb{M}}"] \\ \mathrm{hTr}(\mathbf{DB},\rho_*) \arrow[r, "HH"] & \mathscr{M}od_{\Bbbk}\end{tikzcd} \tag{5.37}$$

commutes up to a natural isomorphism. In particular, for any $(2n,2n)$–tangle T and its annular closure $\widehat{T}$ there is an isomorphism of chain complexes

$$CKh_{\mathbb{M}}(\hat{T}) \cong HH(C_{CK}(T);A^{2n},\rho) \tag{5.38}$$

natural with respect to the chain maps associated to tangle cobordisms.

6 Odds and ends

6.1 Twisting

We used a quasi-isomorphism $C_{CK}(T') \otimes_{A^n} C_{CK}(T^{\underline{\epsilon}}) \cong C_{CK}(T^{\underline{\epsilon}'}) \otimes_{A^n} C_{CK}(T')$ of complexes of diagrammatic bimodules in Section 5.5 to construct the action. The existence of such an isomorphism implies that the family of chain complexes $C_{CK}(T') \otimes (-)$ constitute a natural transformation of the identity functor on $\mathscr{D}(\mathbf{DB})$, so that it can be also used to modify the annular homology as in Proposition 3.18.

Definition 6.1. We call a family $\underline{M} := \{M_n\}$ of chain complexes of (A^n, A^n)–bimodules an *annular twistor* if for every (m,n)-tangle there is a natural quasi-isomorphism

$$\phi_T \colon M_n \underset{A^n}{\otimes} C_{CK}(T) \xrightarrow{\cong} C_{CK}(T) \underset{A^m}{\otimes} M_m. \tag{6.1}$$

The following statement is a specialization of Proposition 3.18 and it follows immediately from the above definition.

Proposition 6.2. *Choose tangles $T \in \mathbf{Tan}(n,n)$ and $T' \in \mathbf{Tan}(m,m)$ such that $\hat{T} = \hat{T}'$. Then complexes $qCH(M_n \otimes_{A^n} C_{CK}(T); A^n)$ and $qCH(M_m \otimes_{A^m} C_{CK}(T'); A^m)$ are quasi-isomorphic for any annular twistor $\underline{M}$.*

Thence, $qHH(M_n \otimes_{A^n} C_{CK}(T); A^n)$ depends only on the annular closure $\widehat{T}$ of an (n,n)–tangle T. We call it the *annular homology twisted by $\underline{M}$* and use the notation $Kh_{\mathbb{A}_q}(\widehat{T}; \underline{M})$.

Example 6.3. Choose a framed $(1,1)$-tangle T and denote by T^n its n-cabling. Then $M_n := C_{CK}(T^n)$ constitute an annular twistor. The homology $Kh_{\mathbb{A}_q}(\widehat{T}'; \underline{M})$ is the quantum annular homology of $\hat{T}'$ computed for the annulus embedded into $\mathbb{R}^3$ along T. In particular, for $q = 1$ there is a spectral sequence from $Kh_{\mathbb{A}}(\widehat{T}'; \underline{M})$ to the Khovanov homology of the satellite link with companion $\widehat{T}$.

6.2 Generalized annular homology

There is another generalization of the annular homology, which is very close to twisting. In this section we assume that $\Bbbk$ is a ring of characteristic two, so that the Chen–Khovanov construction is strictly functorial.

Let us fix a $(1,1)$–tangle T and denote by $T^!$ its mirror image. Write $V_T := CKh_{\mathbb{A}_q}(\widehat{T})$ and $V_T^* := CKh_{\mathbb{A}_q}(\widehat{T}^!)$ for the quantized annular chain complexes of the annular closures of T and $T^!$. They form a dual pair, with evaluation and coevalution maps induced by cobordisms $T \times \cap$ and $T \times \cup$ in $(\mathbb{A} \times I) \times I$ respectively, see Example A.5 and Figure 8 for a picture of the evaluation cobordism.

We generalize $\mathscr{F}_{\mathbb{A}_q} \colon \mathscr{BN}(\mathbb{A}) \longrightarrow \mathrm{g}\mathscr{R}ep(\mathscr{U}_q(\mathfrak{sl}_2))$ by assigning to essential circles alternatively V_T and V_T^*. An annulus with essential boundary is then assigned one of the evaluation of coevaluation maps, and merging a trivial circle to an essential one is determined by the cobordisms merging an unknot to $\widehat{T}$ or $\widehat{T}'$. It follows from the functoriality that this produces a well defined TQFT functor $\mathscr{F}^T_{\mathbb{A}_q} \colon \mathscr{BN}(\mathbb{A}) \longrightarrow \mathscr{D}(\mathscr{U}_q(\mathfrak{sl}_2))$, valued in the derived category of representations of $\mathscr{U}_q(\mathfrak{sl}_2)$.

Definition 6.4. Let T be a $(1,1)$–tangle and choose an annular link L with diagram D. The *T–annular homology* $Kh^T_{\mathbb{A}_q}(L)$ of L is the homology of the chain complex $\mathscr{F}^T_{\mathbb{A}_q}[\![D]\!]$.

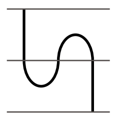
It is tempting to express the above construction using twistors. Indeed, one can define M_n as the alternating tensor product $V_T \otimes V_T^* \otimes V_T \otimes \dots$ of n factors. Them the duality between V_T and V_T^* can be used to construct a chain map

$$C_{CK}(T) \underset{A^m}{\otimes} M_m \longrightarrow M_n \underset{A^n}{\otimes} C_{CK}(T)$$

for every (m,n)–tangle T. However, it is not a quasi-isomorphism. In fact, when $m = n$ then it induces a nontrivial map on Hochschild homology

$$qHH(C_{CK}(T) \underset{A^n}{\otimes} M_n; A^n) \xrightarrow{\cong} qHH(M_n \underset{A^n}{\otimes} C_{CK}(T); A^n) \longrightarrow qHH(C_{CK}(T) \underset{A^n}{\otimes} M_n; A^n)$$

which, in the simplest case of $T = \bigcirc$ seen as a composition of a cup and a cap, is multiplication by the quantum dimension of $V_T \otimes V_T^*$ equal to the Jones polynomial of $\widehat{T}\#\widehat{T}'$. This seams to be the only issue, because for the zigzag



the induced map is a quasi-isomorphism due to the relation between the evaluation and coevaluation.

A Review of some categorical constructions

The material presented here is widely known, and the main goal of this section is to fix the notation. More details on bicategories can be found in the excellent paper [Be67], but the reader is also referred to [EGNO09], because many results about monoidal categories immediately translates to bicategories. For an introduction to homological algebra and derived categories see [Wei95]. Grothendieck groups are treated in [Mi, Wei13].

A.1 Bicategories

Recall that a *category* $\mathscr{C}$ consists of a class of objects $\mathrm{Ob}(\mathscr{C})$ and a class of morphisms $\mathscr{C}(x,y)$ for each pair of objects x and y. We say that $\mathscr{C}$ is *locally small* when each $\mathscr{C}(x,y)$ is a set, and $\mathscr{C}$ is *small* when objects form a set as well. A *bicategory* $\mathbf{C}$ is a "higher level" analogue of a category. It consists of

- a class of objects $\mathrm{Ob}(\mathbf{C})$,
- a category $\mathbf{C}(x,y)$ for each pair of objects (x,y), whose objects and morphisms are called *1–* and *2–morphisms* respectively, and which are represented by single and double arrows,
- a unit $\mathrm{id}_x \in \mathbf{C}(x,x)$ for each object x,
- a functor $\circ : \mathbf{C}(y,z) \times \mathbf{C}(x,y) \longrightarrow \mathbf{C}(x,z)$ for each triple of objects (x,y,z), and
- natural isomorphisms

$$\mathfrak{a}\colon f\circ(g\circ h) \overset{\cong}{\Longrightarrow} (f\circ g)\circ h \qquad \mathfrak{l}\colon \mathrm{id}_y \circ f \overset{\cong}{\Longrightarrow} f \qquad \mathfrak{r}\colon f\circ \mathrm{id}_x \overset{\cong}{\Longrightarrow} f \tag{A.1}$$

  called *associators* and *unitors*, which satisfy the pentagon and triangle axioms of a monoidal category [ML98].

A bicategory is called *strict* or a *2–category* if the natural isomorphisms are identities. Likewise for categories we call $\mathbf{C}$ a *locally small* bicategory when each category $\mathbf{C}(x, y)$ is small, and $\mathbf{C}$ is *small* when also $\mathrm{Ob}(\mathbf{C})$ is a set.

Associators and unitors are often omitted for clarity. According to the MacLane's Coherence Theorem [ML98, Chapter VII.2] there is the only way how to insert these isomorphisms back when necessary.

Notation. In this paper we denote categories with calligraphic letters $\mathscr{C}$, $\mathscr{D}$, etc., whereas bold letters $\mathbf{C}$, $\mathbf{D}$, etc. are reserved for bicategories. Identity morphisms are written as id_x, and identity 2–morphisms as $\mathbf{1}_f$. If $\mathbf{C}$ is a bicategory, then the composition in $\mathbf{C}(x, y)$ is denoted by $*$ and called *vertical*, whereas $\circ$ is the *horizontal composition.* These come from the common convention to draw 1–morphisms horizontally and 2–morphisms vertically, see the diagram to the right of a 2–morphism $\alpha\colon f \Rightarrow g$.

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$$

Example A.1. Tangles in a thickened surface $F \times I$ constitute a bicategory $\mathbf{Tan}(F)$, which objects are finite collections of points in F, 1–morphisms are tangles with endpoints on $F \times \partial I$, and 2–morphisms are tangle cobordisms. Horizontal composition is given by stacking, i.e. by the homeomorphism $(F \times I) \cup (F \times I) \approx F \times I$ where the gluing identifies $F \times \{1\}$ from the right copy of $F \times I$ with $F \times \{0\}$ with from the left one.

Example A.2. The bicategory $\mathbf{Cob}$ is a less dimensional analogue of the previous example. Its objects are points on a line, 1–morphisms are flat tangles in a stripe $\mathbb{R} \times I$, and 2–morphisms are surfaces in $(\mathbb{R} \times I) \times I$. Again, the horizontal composition is induced by the diffeomorphism $(\mathbb{R} \times I) \cup (\mathbb{R} \times I) \approx \mathbb{R} \times I$.

Example A.3. Rings, bimodules, and bimodule maps form a bicategory $\mathbf{Bimod}$, where $\mathbf{Bimod}(A, B) := \mathscr{B}imod(B, A)$ is the category of (B, A)–bimodules and the horizontal composition is given by the tensor product: $M \circ N := M \otimes_B N$ for an (C, B)–bimodule M and a (B, A)–bimodule N. The somewhat unorthodox choice for the category $\mathbf{Bimod}(A, B)$ is motivated by the interpretation of an (B, A)–bimodule M as a functor $M \otimes_A (-)\colon \mathscr{M}od(A) \longrightarrow \mathscr{M}od(B)$. In fact, every right exact functor between module categories is of this form.

Example A.4. Let $\mathscr{R}ep(A, B) \subset \mathscr{B}imod(A, B)$ be the subcategory formed by those bimodules that are finitely generated and projective as left A–modules. Each bimodule $M \in \mathscr{R}ep(B, A)$ can be viewed as a functor $M \otimes_A (-)\colon \mathscr{R}ep(A) \longrightarrow \mathscr{R}ep(B)$ between categories of projective modules. In particular, they are closed under tensor products, constituting a bicategory $\mathbf{Rep} \subset \mathbf{Bimod}$. As in the previous example, we have $\mathbf{Rep}(A, B) = \mathscr{R}ep(B, A)$.

Choose bicategories $\mathbf{C}$ and $\mathbf{D}$. A *bifunctor* $\mathbf{F}\colon \mathbf{C} \longrightarrow \mathbf{D}$ maps objects, 1–morphisms, and 2–morphisms of $\mathbf{C}$ into objects, 1–morphisms, and 2–morphisms of $\mathbf{D}$ respectively. It is equipped with a natural isomorphism

$$\mathfrak{m}\colon \mathbf{F}(g) \circ \mathbf{F}(f) \overset{\cong}{\Longrightarrow} \mathbf{F}(g \circ f) \tag{A.2}$$

satisfying certain coherence axiom, which allows us to forget that some 1–morphisms are not equal: if two 1–morphisms f and g in $\mathbf{D}$ are connected by a 2–morphism formed by $\mathfrak{m}$, associators and unitors (from both $\mathbf{C}$ and $\mathbf{D}$), such a 2–morphism is unique. In particular, there exists a unique canonical isomorphism $\mathrm{id}_{\mathbf{F}(x)} \overset{\cong}{\Longrightarrow} \mathbf{F}(\mathrm{id}_x)$.

Finally, a *natural transformation* $\eta\colon \mathbf{F} \Longrightarrow \mathbf{G}$ between two bifunctors is a collection of 1–morphisms $\eta_x\colon \mathbf{F}(x) \longrightarrow \mathbf{G}(x)$, one per object $x \in \mathbf{C}$, and invertible 2–morphisms $\eta_f\colon \mathbf{G}(f) \circ \eta_x \Longrightarrow \eta_y \circ \mathbf{F}(f)$, one per 1–morphisms $f \in \mathbf{C}(x, y)$, such that the equality

$$\begin{tikzcd} \mathbf{F}(x) \arrow[r, "\eta_x"] \arrow[d, bend right=60, "\mathbf{F}(g)"'] \arrow[d, bend left=60, "\mathbf{F}(f)"] \arrow[d, phantom, "\overset{\mathbf{F}(\alpha)}{\Longleftarrow}"] \arrow[dr, phantom, "\swarrow\eta_f"] & \mathbf{G}(x) \arrow[d, "\mathbf{G}(f)"] \\ \mathbf{F}(y) \arrow[r, "\eta_y"'] & \mathbf{G}(y) \end{tikzcd} = \begin{tikzcd} \mathbf{F}(x) \arrow[r, "\eta_x"] \arrow[d, "\mathbf{F}(g)"'] \arrow[dr, phantom, "\eta_g\swarrow"] & \mathbf{G}(x) \arrow[d, bend right=60, "\mathbf{G}(g)"'] \arrow[d, bend left=60, "\mathbf{G}(f)"] \arrow[d, phantom, "\overset{\mathbf{G}(\alpha)}{\Longleftarrow}"] \\ \mathbf{F}(y) \arrow[r, "\eta_y"'] & \mathbf{G}(y) \end{tikzcd} \tag{A.3}$$

holds for every 2–morphism $\alpha\colon f \Longrightarrow g$. Moreover, η_f must be coherent with all the other canonical 2–isomorphisms (associators, unitors, the structure 2–isomorphisms of $\mathbf{F}$ and $\mathbf{G}$), but we shall not write the conditions explicitly. These axioms allow us to safely forget about η_f in most computations.

A bicategory $\mathbf{C}$ has *left duals* if each $f \in \mathbf{C}(x, y)$ admits ${}^*\!f \in \mathbf{C}(y, x)$ together with coevaluation and evaluation 2–morphisms

$$\mathrm{id}_y \xRightarrow{coev} {}^*\!f \circ f \qquad\qquad f \circ {}^*\!f \xRightarrow{ev} \mathrm{id}_x \tag{A.4}$$

fitting into commuting triangles

$$\begin{tikzcd} f \arrow[r, Rightarrow, "\mathbf{1}_f \circ coev"] \arrow[dr, Rightarrow, "\mathbf{1}_f"'] & f \circ {}^*\!f \circ f \arrow[d, Rightarrow, "ev \circ \mathbf{1}_f"] \\ & f \end{tikzcd} \qquad \begin{tikzcd} {}^*\!f \arrow[r, Rightarrow, "coev \circ \mathbf{1}_{{}^*\!f}"] \arrow[dr, Rightarrow, "\mathbf{1}_{{}^*\!f}"'] & {}^*\!f \circ f \circ {}^*\!f \arrow[d, Rightarrow, "\mathbf{1}_{{}^*\!f} \circ ev"] \\ & {}^*\!f \end{tikzcd} \tag{A.5}$$

where for clarity associators and unitors are omitted. The morphism ${}^*\!f$ is called the *left dual* to f. We define the *right dual* f^* of f by reversing the order of the horizontal composition in (A.4) and (A.5). If a dual 1–morphism exists, then it is unique up to an isomorphism. In particular, dual pairs are preserved by bifunctors.

Example A.5. The mirror image $T^!$ of a tangle T, obtained from the flip of $F \times I$, is both the left and right dual of T. The evaluation 2-morphism is obtained by revolving T in four dimensions along the input surface $F \times \{0\}$, i.e. it is the image of the map $(p, t, s) \longmapsto (p, t\cos(s\pi), t\sin(s\pi))$ with $(p, t) \in T$ and $s \in I$, suitable normalized (see Fig. 8). The coevaluation is defined dually by a rotation along the output surface $F \times \{1\}$.

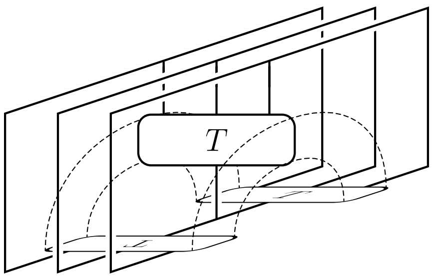


Figure 8: The evaluation cobordism $TT^! \Longrightarrow \mathbf{1}_3$ for a tangle $T \in \mathbf{Tan}(1, 3)$.

Example A.6. An (A,B)–bimodule M has a left dual if and only if it is finitely generated and projective as a left A–module. Explicitly, ${}^*M := \mathrm{Hom}_A(M,A)$ is the bimodule of left A–linear functions on M with $(b\cdot f)(m) := f(mb)$ and $(f\cdot a)(m) := f(m)a$. The evaluation and coevalution maps are defined as follows

$$\begin{aligned} ev\colon M \underset{B}{\otimes} {}^*M &\longrightarrow A & m\otimes\alpha &\longmapsto \alpha(m),\\ coev\colon B &\longrightarrow {}^*M \underset{A}{\otimes} M & 1 &\longmapsto \sum_i {}^*m_i\otimes m_i \end{aligned}$$

where $\{m_i\}$ are A–linear generators of M. Likewise, M has a right dual, the bimodule of right B–linear functions $M^* := \mathrm{Hom}_B(M,B)$, if and only if M is finitely generated and projective as a right B–modules.

It follows that **Rep** is formed by those bimodules that have left duals in **Bimod**. However, a left dual to a bimodule from **Rep** may not be projective as a left module. In particular, **Rep** does not have left duals.

A.2 Elements of homological algebra

Choose an algebra A and recall that by A–modules we understand finitely generated left A–modules. We write $Com(A)$ for the category of bounded complexes of A–modules and $Com^-(A)$ for the category of complexes with *bounded head*, where the *head of C* is the subcomplex starting at C_0 and following the direction of the differential. Hence, the head of C is generated by $C_{\leqslant 0}$ in the homological convention, with the differential $\partial\colon C_i \longrightarrow C_{i-1}$, while by $C_{\geqslant 0}$ in the cohomological convention, with the differential going the other way.

Remark. Because both homological and cohomological conventions are used in this paper, we prefer the notion of "having bounded head" instead of the ambiguous term "bounded from below." In particular, a projective resolution of an A–module lives in $Com^-(A)$ independently of the chosen convention.

We construct the *homotopy categories* $Com_{/h}(A)$ and $Com^-_{/h}(A)$ by identifying homotopic chain maps between complexes. A chain map $f\colon C \longrightarrow C'$ is a *quasi-isomorphism* if it induces an isomorphism on homology. A homotopy equivalence is a quasi-isomorphism, but not every quasi-isomorphism has a homotopy inverse. The *derived categories* $\mathcal{D}(A)$ and $\mathcal{D}^-(A)$ are obtained from $Com_{/h}(A)$ and $Com^-_{/h}(A)$ respectively by adding formal inverses to all quasi-isomorphisms. The following is the main result of basic homological algebra.

Theorem A.7. *For every complex $C\in Com^-(A)$ there is a complex $P(C)\in Com^-(A)$ with each C_i a projective module and a quasi-isomorphism $P(C)\longrightarrow C$. The complex $P(C)$ is unique up to a homotopy equivalence, and every chain map $f\colon C\longrightarrow D$ admits a lift to a chain map $P(f)\colon P(C)\longrightarrow P(D)$, unique up to a chain homotopy.*

The complex $P(C)$ is called the *projective resolution* of C. It follows that a quasi-isomorphism between complexes of projective modules admits a homotopy inverse. Hence, $\mathcal{D}^-(A)$ is identified with the homotopy category of chain complexes of projective modules with bounded head. The last statement does not hold for bounded complexes, unless we require that each A–module has finite projective resolution. In particular, we can require that A has *finite global dimension*, which means that there is a fixed number N such that each module has a resolution of length at most N. Clearly, it is enough to check the condition for indecomposable modules only. If each module has finite length, such as when A is Artinian, then one has to check the condition only for simple modules.

Corollary A.8. *$\mathscr{D}(A)$ is equivalent to the homotopy category of bounded complexes of projective A–modules if A is an Artinian algebra of finite global dimension.*

More generally, Theorem A.7 holds for every abelian category $\mathscr{A}$, in which every object is a quotient of a projective one (we say then that $\mathscr{A}$ *has enough projectives*). In particular, we can start with the category $\mathscr{B}imod(A, B)$ of (A, B)–bimodules, construct the categories of complexes $Com(A, B)$ and $Com^-(A, B)$, and then the derived categories $\mathscr{D}(A, B)$ and $\mathscr{D}^-(A, B)$.

A functor $\mathscr{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ between abelian categories preserves quasi-isomorphisms if it is exact, i.e. when it preserves exactness of sequences. If $\mathscr{F}$ is only right exact, that is for every short exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ the sequence

$$\mathscr{F}(X') \longrightarrow \mathscr{F}(X) \longrightarrow \mathscr{F}(X'') \longrightarrow 0$$

is still exact, then it is exact when restricted to sequences of projective objects. Assume $\mathscr{A}$ has enough projectives. Then there is a unique functor $\mathscr{D}^-\mathscr{F}\colon \mathscr{D}^-(\mathscr{A}) \longrightarrow \mathscr{D}^-(\mathscr{B})$ that agrees with $\mathscr{F}$ on complexes of projective objects. The functor $\mathscr{D}^-\mathscr{F}$ is called the *right derived functor of* $\mathscr{F}$ and it is constructed by applying $\mathscr{F}$ to the projective resolution of its argument:

$$\mathscr{D}^-\mathscr{F}(C) := \mathscr{F}(P(C)). \tag{A.6}$$

Clearly, the same holds for bounded derived categories if each A–module has a finite projective resolution.

Example A.9. The tensor functor $M \otimes_A (-)\colon \mathscr{M}od_A \longrightarrow \mathscr{M}od_B$ with $M \in \mathscr{B}imod(B, A)$ is right exact. Its derived functor can be computed with a help of the *bar resolution* of the algebra A, the chain complex $R_n(A) := A^{\otimes(n+2)}$ with the differential

$$d_n := \sum_{i=0}^{n} (-1)^i d_i^n, \qquad d_i^n(a_0 \otimes \cdots \otimes a_{n+1}) := a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}. \tag{A.7}$$

It is a projective resolution of A regarded as an (A, A)–bimodule, so that $M \otimes_A R_\bullet(A)$ is a chain complex of right projective A–modules.

Example A.10. The Hochschild homology $HH(M; A)$ of an (A, A)–bimodule M is defined as

$$HH(M; A) := H(\mathrm{coInv}(M \underset{A}{\otimes} R_\bullet(A)), \tag{A.8}$$

where $\mathrm{coInv}(N) := N/\mathrm{span}_{\Bbbk}\{an - na \mid a \in A, n \in N\}$ is the $\Bbbk$–module of coinvariants of an (A, A)–bimodule N. The functor $\mathrm{coInv}(-)$ is right exact in the category of (A, A)–bimodules, and it is exact when restricted to $\mathscr{R}ep(A, A)$. Because $M \otimes_A R_\bullet(A)$ is a resolution of M in $\mathscr{R}ep(A, A)$, the Hochschild homology can be interpreted as the derived functor of $\mathrm{coInv}(-)$. Alternatively, one can use the isomorphism $\mathrm{coInv}(N) \cong N \otimes_{A^e} A$, where $A^e := A \otimes A^{op}$, and interpret HH as the derived tensor product with A.

In this paper we occasionally use derived categories of subcategories of $\mathscr{B}imod(A, B)$, like $\mathscr{R}ep(A, B)$ or the category of diagrammatic bimodules (the bimodules assigned by

Chen and Khovanov to flat tangles, see Section 2.6). These are examples of exact categories, where a category $\mathscr{A}$ is *exact* if it is a subcategory of a certain abelian category $\hat{\mathscr{A}}$, called the *abelian completion of* $\mathscr{A}$, closed under extensions and kernels of those morphisms that are epimorphisms in $\hat{\mathscr{A}}$. Given an exact category $\mathscr{A}$ one can still construct categories of complexes and their homotopy categories, but quasi-isomorphisms are defined by computing homology of complexes in the completion $\hat{\mathscr{A}}$. This means there are functors between derived categories

$$\mathscr{D}(\mathscr{A}) \longrightarrow \mathscr{D}(\hat{\mathscr{A}}) \qquad \text{and} \qquad \mathscr{D}^-(\mathscr{A}) \longrightarrow \mathscr{D}^-(\hat{\mathscr{A}}) \tag{A.9}$$

induced by inclusions, which in general are not equivalences. When $\mathscr{A} = \mathscr{R}ep(A)$ then $\mathscr{D}(\mathscr{A})$ and $\mathscr{D}^-(\mathscr{A})$ are homotopy categories of complexes of projective modules. The second isomorphism in (A.9) is then an equivalence for any algebra A due to Theorem A.7, whereas for the first one we have to assume that A–modules admit finite projective resolutions. A similar discussion applies to $\mathscr{A} = \mathscr{R}ep(A, B)$ and to the category of diagrammatic bimodules.

Finally, if $\mathbf{C}$ is a bicategory with each $\mathbf{C}(x, y)$ abelian or exact, we define $\mathscr{D}(\mathbf{C})$ and $\mathscr{D}^-(\mathbf{C})$ by replacing morphism categories with the suitable versions of their derived categories. Both are bicategories if the horizontal composition functor is exact or right exact, in the latter case replacing it with its derived functor (in case of the bounded version it requires each object to have a finite projective resolution). The examples used in this paper are the bimodule bicategories **Bimod** and **Rep**, as well as the bicategory **DB** of diagrammatic bimodules, see Section 2.6.

A.3 Grothendieck groups

We shall use in this paper a few kinds of Grothendieck groups, definitions of which we remind in this section. All categories here are assumed to be essentially small, i.e. isomorphism classes of objects form a set.

Given a category $\mathscr{C}$, let $\Pi\mathscr{C}$ be the set of isomorphism classes of objects of $\mathscr{C}$. If $\mathscr{C}$ is additive, we can define a semigroup structure on $\Pi\mathscr{C}$ by setting $[x] + [y] := [x \oplus y]$. In this case, the *split Grothendieck group* $K_0^{sp}(\mathscr{C})$ of $\mathscr{C}$ is defined by enlarging $\Pi\mathscr{C}$ with formal inverses. In other words,

$$K_0^{sp}(\mathscr{C}) = {}^{\mathbb{Z}(\Pi\mathscr{C})}\big/_{[x]+[y]=[x\oplus y]} \tag{A.10}$$

is the abelian group generated by isomorphism classes of objects subject to the relation $[x] + [y] = [x \oplus y]$.

Let $\mathscr{C}$ be an exact category with an abelian completion $\mathscr{A}$ as defined in the previous section. We say that a sequence of morphisms in $\mathscr{C}$

$$0 \longrightarrow x' \longrightarrow x \longrightarrow x'' \longrightarrow 0 \tag{A.11}$$

is exact if it is exact in $\mathscr{A}$. Main examples are abelian categories, the category $\mathscr{R}ep(A)$ of projective A–modules, and the category $\mathscr{R}ep(A, B)$. The *(exact) Grothendieck group* $K_0(\mathscr{C})$ of $\mathscr{C}$ is defined as a quotient of the free abelian group $\mathbb{Z}(\Pi\mathscr{C})$ by the relation $[x'] + [x''] = [x]$ for every short exact sequence (A.11). Notice that $K_0(\mathscr{C}) = K_0^{sp}(\mathscr{C})$ when $\mathscr{C} = \mathscr{R}ep(A)$, because every short exact sequence in $\mathscr{R}ep(A)$ splits.

Notation. Choose an algebra A. It is common to write $K_0(A)$ for $K_0^{sp}(\mathscr{R}ep(A))$ and $G_0(A)$ for $K_0(\mathscr{M}od(A))$.

The Grothendieck group is also defined for a derived category $\mathscr{A}$. Recall that for every chain map $f\colon C \longrightarrow D$ there is the *cone complex* $C(f)$ with

$$C(f)_n := C_{n+1} \oplus D_n \qquad \text{and} \qquad d = \begin{pmatrix} -d_C & 0 \\ f & d_D \end{pmatrix} \tag{A.12}$$

where we use the cohomological convention (i.e. the differential increases the degree). There is a short exact sequence of complexes

$$0 \longrightarrow D \longrightarrow C(f) \longrightarrow C[1] \longrightarrow 0 \tag{A.13}$$

where $C[1]_n = C_{n+1}$ with differential $-d_C$. Exact sequences of this sort are called *distinguished triangles*, and they are usually written as

$$C \xrightarrow{f} D \longrightarrow C(f) \longrightarrow C[1]. \tag{A.14}$$

The Grothendieck groups of $\mathscr{D}(\mathscr{A})$ and $\mathscr{D}^-(\mathscr{A})$ are defined by imposing $[x'] + [x''] = [x]$ for any distinguished triangle $x' \longrightarrow x \longrightarrow x'' \longrightarrow x'[1]$. Notice that $C(f)$ has trivial homology when f is a quasi-isomorphism. In particular, it follows that $[x[1]] = -[x]$ (take $f = \mathrm{id}_x$).

There is a functor $I\colon \mathscr{A} \longrightarrow \mathscr{D}(\mathscr{A})$ that takes $x \in \mathscr{A}$ to a complex with x in degree 0 and vanishing otherwise. The functor is full, faithful, and takes exact sequences to distinguished triangles (the cone complex of a monomorphism $f\colon x' \longrightarrow x$ is quasi-isomorphic to the cokernel of f), descending to a homomorphism of Grothendieck groups.

Theorem A.11. *The functor $I\colon \mathscr{A} \longrightarrow \mathscr{D}(\mathscr{A})$ induces an isomorphism on Grothendieck groups. In particular, there are equalities*

$$[C] = \sum_i (-1)^i [C_i] = \sum_i (-1)^i [H_i] \tag{A.15}$$

for a chain complex $C \in \mathit{Com}(\mathscr{A})$ with homology $H_\bullet$.

Finally, assume that $\mathscr{C}$ is graded, i.e. it admits an autoequivalence $(-)\{1\}\colon \mathscr{C} \longrightarrow \mathscr{C}$. Then all the Grothendieck groups are modules over $\mathbb{Z}[q, q^{-1}]$ with $q[x] := [x\{1\}]$.

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