

SKEIN CONSTRUCTION OF IDEMPOTENTS IN BIRMAN-MURAKAMI-WENZL ALGEBRAS

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ABSTRACT. We give skein theoretic formulas for minimal idempotents in the Birman-Murakami-Wenzl algebras. These formulas are then applied to derive various known results needed in the construction of quantum invariants and modular categories. In particular, an elementary proof of the Wenzl formula for quantum dimensions is given. This proof does not use the representation theory of quantum groups and the character formulas.

INTRODUCTION

The Birman-Murakami-Wenzl algebras are deformations of the Brauer centralizer algebras [4, 13]. They are quotients of the Artin braid groups algebras, and have appeared in connection with the Kauffman link invariant and the quantum groups of types B, C and D. The Birman-Murakami-Wenzl algebras are generically semi-simple, and their structure was given by Wenzl [20]. They play a key role in the construction of quantum invariants, modular categories and Topological Quantum Field Theories, as was shown by Turaev and Wenzl [18, 19]. Our purpose here is to study the structure of these algebras without using their representation theory. In a separate article, we will pursue Turaev and Wenzl's program and construct four series of modular categories. Together with the present paper this construction will be reasonably self-contained.

Our main results are the following.

- We give explicit formulas for minimal idempotents in the Birman-Murakami-Wenzl algebras K_n . These are then used to obtain the semi-simple decomposition of K_n , together with a basis of *matrix units*. Similar results were obtained by Ram and Wenzl [15] using Jones *basic construction*.
- We give a skein theoretic proof of the Wenzl formula for the quantum dimensions of these minimal idempotents. The key point is here the proof of the

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recursive formula (8). This formula is further used to derive versions of the Wenzl formula corresponding to the quantum group specializations and to discuss the existence of idempotents in the non generic case.

Conventions. Throughout this paper, the manifolds are compact, smooth and oriented. By a *link* we mean an isotopy class of an unoriented framed link. Here, a framing is a non-singular normal vector field up to homotopy. By a *tangle* in a 3-manifold M we mean an isotopy class of a framed tangle relative to the boundary. Here the boundary of the tangle is a finite set of points in ∂M , together with a non zero vector tangent to ∂M at each point. Note that a framing together with an orientation is equivalent to a trivialization of the normal bundle up to homotopy. By an *oriented link* we mean an isotopy class of a link together with a trivialization of the normal bundle up to homotopy. By an *oriented tangle* we mean an isotopy class of a tangle together with a trivialization of the normal bundle, up to homotopy relative to the boundary. Here the boundary of the tangle is a finite set of points in ∂M , together with a trivialization of the tangent space to ∂M at each point. In the figures, a preferred convention using the plane gives the framing (blackboard framing).

1. THE BIRMAN-MURAKAMI-WENZL RIBBON CATEGORY

1.1. **Kauffman skein relations.** Let M be a 3-manifold (possibly with a given finite set l of framed points on the boundary). We denote by $\mathcal{K}(M)$ (resp. $\mathcal{K}(M, l)$) the k -module freely generated by links in M (and tangles in M that meet ∂M in l) modulo (the relative isotopy and) the Kauffman skein relations in Figure 1.

$$\begin{aligned} \begin{array}{c} \diagdown \\ \diagup \end{array} - \begin{array}{c} \diagup \\ \diagdown \end{array} &= (s - s^{-1}) \left(\begin{array}{c} | \\ | \end{array} - \begin{array}{c} \smile \\ \frown \end{array} \right) \\ \begin{array}{c} | \\ \circ \end{array} &= \alpha \begin{array}{c} | \\ | \end{array}, \quad \begin{array}{c} \circ \\ | \end{array} = \alpha^{-1} \begin{array}{c} | \\ | \end{array} \\ L \amalg \bigcirc &= \left(\frac{\alpha - \alpha^{-1}}{s - s^{-1}} + 1 \right) L \end{aligned}$$

FIGURE 1. Kauffman skein relations

We suppose that k is an integral domain containing the invertible elements α , s and that $\frac{\alpha - \alpha^{-1}}{s - s^{-1}}$ lives in k . We call $\mathcal{K}(M)$ the skein module of M .

For example, $\mathcal{K}(S^3) \cong k$. The isomorphism sends any link L in S^3 to its Kauffman polynomial $\langle L \rangle$, normalized by $\langle \emptyset \rangle = 1$.

1.2. The Birman-Murakami-Wenzl category. The Birman-Murakami-Wenzl category K is defined as follows. An object of K is a disc D^2 equipped with a finite set of points and a non zero tangent vector at each point. Unless otherwise specified, we will use the second vector of the standard basis (the vector $\sqrt{-1}$ in complex notation). If $\beta = (D^2, l_0)$ and $\gamma = (D^2, l_1)$ are two objects, the module $Hom_K(\beta, \gamma)$ is $\mathcal{K}(D^2 \times [0, 1], l_0 \times 0 \amalg l_1 \times 1)$. The notation $K(\beta, \gamma)$ and K_β will be used for $Hom_K(\beta, \gamma)$ and $End_K(\beta)$, respectively. For composition, we use the *covariant* notation:

$$\begin{aligned} K(\beta, \gamma) \times K(\gamma, \delta) &\rightarrow K(\beta, \delta) \\ (f, g) &\mapsto fg \end{aligned}$$

In our figures the time parameter goes upwards, so that the morphism fg is depicted with g lying above f , and the normal vector field is orthogonal to the plane and points ‘inside the blackboard’.

The Birman-Murakami-Wenzl category is a ribbon Ab-category (see [16, Ch II]). Ribbon categories admit a theory of traces of morphisms and dimensions of objects for which we will use the terminology *quantum trace* and *quantum dimension*. In the case of the category of finite dimensional vector spaces, equipped with trivial braiding and twist, these traces and dimensions coincide with the usual ones [16, Section I.1.7 and Lemma II.4.3.1]. We will use the notation $\langle f \rangle$ for the quantum trace of $f \in K_\beta$. This quantum trace is equal to the value of the closure of f in $\mathcal{K}(S^3) \cong k$ obtained by gluing of $D^2 \times \{0\}$ and $D^2 \times \{1\}$ in K_β along the identity map.

We denote by n the object formed by the n points $\{(2j-1)/n - 1 : j = 1, \dots, n\}$ (0 is the trivial object). If we consider only these standard objects n , $n \geq 0$, we obtain a full subcategory equivalent to K , which was named the Kauffman category by Turaev who first introduced it in [17, Section 7.7].

The algebra $K_n = End_K(n)$ is isomorphic to the Birman-Murakami-Wenzl algebra which is the quotient of the braid group algebra $k[B_n]$ by the Kauffman skein relations [7, 13]. For a proof of the above isomorphism, see [12] or [17]. This algebra is a deformation of the Brauer algebra (i.e. the centralizer algebra of the semi-simple Lie algebras of types B,C and D). If k is a field, then the algebra K_n is known to be semi-simple [20], except possibly if s is a root of unity, or $\alpha = \pm s^n$ for some $n \in \mathbb{Z}$. Its simple components correspond to the partitions $\lambda = (\lambda_1, \dots, \lambda_p)$ with $|\lambda| = \sum_i \lambda_i = n - 2r$, $r = 0, 1, \dots, [n/2]$.

The Birman-Murakami-Wenzl algebra K_n is generated by the identity $\mathbf{1}_n$, positive transpositions e_1, \dots, e_{n-1} and hooks h_1, \dots, h_{n-1} drawn in Figure 2.

$$\begin{aligned}
e_i &= \left| \cdots \right| \left| \begin{array}{c} i \quad i+1 \\ \diagdown \quad / \\ \diagup \quad \diagdown \end{array} \right| \left| \cdots \right| \\
h_i &= \left| \cdots \right| \left| \begin{array}{c} i \quad i+1 \\ \frown \\ \smile \end{array} \right| \left| \cdots \right|
\end{aligned}$$

FIGURE 2. Generators of K_n

If the e_i are supposed to be invertible, then a complete system of relations [20] is given by

$$\begin{aligned}
(B1) \quad e_i e_{i+1} e_i &= e_i e_{i+1} e_i \\
(B2) \quad e_i e_j &= e_j e_i, \text{ for } |i - j| \geq 2 \\
(R1) \quad h_i e_i &= \alpha^{-1} h_i \\
(R2) \quad h_i e_{i-1}^{\pm 1} h_i &= \alpha^{\pm 1} h_i \\
(K) \quad e_i - e_i^{-1} &= (s - s^{-1})(\mathbb{1}_n - h_i).
\end{aligned}$$

The quotient of K_n by the ideal I_n generated by h_{n-1} is isomorphic to Hecke algebra H_n . We will use the knowledge of this Hecke algebra H_n to study K_n . Note that $I_n = \{(a \otimes \mathbb{1}_1) h_{n-1} (b \otimes \mathbb{1}_1) : a, b \in K_{n-1}\}$.

2. HECKE ALGEBRAS

The Hecke category H is defined similarly as above, using the Homfly skein theory. An object in this category is a disc D^2 equipped with a set of points with a trivialization of the tangent space at each point. If $\beta = (D^2, l_0)$ and $\gamma = (D^2, l_1)$ are two objects, the module $Hom_H(\beta, \gamma) = H(\beta, \gamma)$ is the Homfly skein module $\mathcal{H}(D^2 \times [0; 1], l_0 \times 0 \amalg l_1 \times 1)$. Here the Homfly skein module in M is freely generated by oriented framed tangles in M modulo (the relative isotopy and) the Homfly relations given in Figure 3. Note that there we have specialized the three variable Homfly skein theory for framed links.

We also simply denote by n the object formed by the n points $\{(2j-1)/n - 1 : j = 1, \dots, n\}$, equipped with the standard trivialization.

The positive permutation braids w_π represent a basis of the module H_n , indexed by permutations π . The symmetrizers and antisymmetrizers in H_n are represented respectively by the following elements f_n and g_n of the braid group algebra.

$$f_n = \frac{1}{[n]!} s^{-\frac{n(n-1)}{2}} \sum_{\pi \in \mathcal{S}_n} s^{l(\pi)} w_\pi,$$

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \nearrow \\ \searrow \end{array} & - & \begin{array}{c} \nwarrow \\ \swarrow \end{array} & = & (s - s^{-1}) & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \\
 \begin{array}{c} \circlearrowleft \\ \uparrow \end{array} & = & \alpha & \begin{array}{c} \uparrow \\ \uparrow \end{array} & & \begin{array}{c} \circlearrowright \\ \uparrow \end{array} & = & \alpha^{-1} & \begin{array}{c} \uparrow \\ \uparrow \end{array} \\
 L \cup \bigcirc & = & \frac{\alpha - \alpha^{-1}}{s - s^{-1}} L & & & & & &
 \end{array}
 \end{array}$$

FIGURE 3. Homfly skein relations

$$g_n = \frac{1}{[n]!} s^{\frac{n(n-1)}{2}} \sum_{\pi \in \mathcal{S}_n} (-s)^{-l(\pi)} w_\pi .$$

Here $l(\pi)$ is the length of the permutation π . We work in the generic case. This means that, in the domain k , the quantum integers $[j] = \frac{s^j - s^{-j}}{s - s^{-1}}$ are asked to be invertible for every $j > 0$.

For a Young diagram λ of size n , we denote by \square_λ the object of the category H formed with one point for each cell c of λ , equipped with the standard trivialization. If c has index (i, j) (i -th row, and j -th column), then the corresponding point in D^2 is $\frac{j+i\sqrt{-1}}{n+1}$. Following Aiston and Morton [1], we can define in H_{\square_λ} a minimal idempotent y_λ which is a version of the corresponding Young idempotent of the symmetric group algebra. The idea of the construction is to insert symmetrizers along rows and antisymmetrizers along columns, and then to normalize. A skein computation of the normalizing coefficient (of slightly different idempotents) appeared in [24]. Details about the construction of these idempotents can also be found in [5].

A standard tableau t with shape a Young diagram $\lambda = \lambda(t)$ is a labeling of the cells, with the integers 1 to n , which is increasing along rows and columns. We denote by t' the tableau obtained by removing the cell numbered by n . We define $\alpha_t \in H(n, \square_\lambda)$ and $\beta_t \in H(\square_\lambda, n)$ by

$$\alpha_1 = \beta_1 = \mathbf{1}_1 ,$$

$$\alpha_t = (\alpha_{t'} \otimes \mathbf{1}_1) \varrho_t y_\lambda ,$$

$$\beta_t = y_\lambda \varrho_t^{-1} (\beta_{t'} \otimes \mathbf{1}_1) .$$

Here $\varrho_t \in H(\square_{\lambda(t')} \otimes 1, \square_\lambda)$ is a standard isomorphism.

The following theorems are shown in [5].

Theorem 2.1. *The family $\alpha_t\beta_\tau$ for all standard tableaux t, τ such that $\lambda(t) = \lambda(\tau)$ forms a basis for H_n , and (here $\delta_{\tau s}$ is the Kronecker delta)*

$$\alpha_t\beta_\tau\alpha_s\beta_\sigma = \delta_{\tau s}\alpha_t\beta_\sigma .$$

This gives explicitly an algebra isomorphism

$$\bigoplus_{|\lambda|=n} \mathcal{M}_{d_\lambda}(k) \approx H_n ,$$

where d_λ is the number of standard tableaux with shape λ , and $\mathcal{M}_{d_\lambda}(k)$ is the algebra of $d_\lambda \times d_\lambda$ matrices with coefficients in k . The diagonal elements $p_t = \alpha_t\beta_t$ are the *path idempotents* described in [21]. The *minimal central idempotent* corresponding to the partition λ is

$$z_\lambda = \sum_{\lambda(t)=\lambda} p_t .$$

The minimal idempotents y_λ and the path idempotents p_t satisfy the following branching formula.

Theorem 2.2 (Branching formula).

$$\begin{aligned} y_\lambda \otimes \mathbf{1}_1 &= \sum_{\substack{\lambda \subset \mu \\ |\mu| = |\lambda| + 1}} (y_\lambda \otimes \mathbf{1}_1) y_\mu (y_\lambda \otimes \mathbf{1}_1) , \\ p_t \otimes \mathbf{1}_1 &= \sum_{\tau' = t} p_{\tau'} . \end{aligned}$$

We have omitted in these formulas the standard isomorphisms respectively between $\square_\lambda \otimes 1$ and \square_μ , and between $(n-1) \otimes 1$ and n .

The result for the quantum dimensions is given in the following theorem [21]. Here we denote by $\langle \rangle^h$ the quantum trace in the ribbon Hecke category.

Theorem 2.3 (Quantum dimension).

$$\langle y_\lambda \rangle^h = \prod_{\text{cells}} \frac{\alpha s^{cn(c)} - \alpha^{-1} s^{-cn(c)}}{s^{hl(c)} - s^{-hl(c)}}$$

The assertion above can be proven by a skein calculation (see [24, Prop.2.4] or [2]). Here is a sketch of the proof. We first check the formula for columns 1^n , by using the recursive formula for the antisymmetrizers y_{1^n} . We note [11, Ch.1] that the right hand side in Theorem 2.3 is the Schur polynomial in the $\langle y_{1^n} \rangle^h$. We then proceed recursively on the number of cells.

If λ contains two distinct sub-diagrams μ and ν with $|\mu| = |\nu| = |\lambda| - 1$, then we get the result by considering $(y_\mu \otimes \mathbf{1}_1)(y_\nu \otimes \mathbf{1}_1)$ (considered as a composition of morphisms in H_{\square_λ}). This defines a quasi-idempotent which can be normalized. Whence we get

a minimal idempotent which belongs to the simple component indexed by λ , whose quantum trace gives the required formula.

We obtain the remaining cases, namely the rectangular diagrams, $\lambda = (j, \dots, j)$ by using the branching formula for $\mu = (j, \dots, j, j - 1)$.

3. IDEMPOTENTS IN BIRMAN-MURAKAMI-WENZL ALGEBRAS

The quotient of K_n by the ideal I_n generated by h_{n-1} is isomorphic to the Hecke algebra H_n . We denote by π_n the canonical projection map

$$\pi_n : K_n \longrightarrow H_n .$$

The main idea of our construction is to define a multiplicative section $s_n : H_n \rightarrow K_n$ and to use it for the transport of the idempotents from the Hecke algebra to the Birman-Murakami-Wenzl one. In this section, we suppose that the ground ring k is the field $\mathbb{Q}(\alpha, s)$. The computation of the quantum dimensions in Section 7 will permit to discuss the non generic case.

Theorem 3.1. *There exists a unique multiplicative homomorphism $s_n : H_n \rightarrow K_n$, such that*

$$\begin{aligned} \pi_n \circ s_n &= \text{id}_{H_n} \text{ and} \\ s_n(x)y &= ys_n(x) = 0 \text{ for } \forall x \in H_n \text{ and } \forall y \in I_n \end{aligned}$$

Corollary 3.2. $K_n \cong I_n \oplus H_n \cong I_n \oplus \left(\bigoplus_{|\lambda|=n} \mathcal{M}_{d_\lambda}(k) \right)$.

The theorem above gives minimal central idempotents in K_n ,

$$\tilde{z}_\lambda = s_n(z_\lambda) ,$$

and also minimal path idempotents,

$$\tilde{p}_t = s_n(p_t) .$$

The quantum trace of \tilde{p}_t depends only on $\lambda = \lambda(t)$; it is denoted by $\langle \lambda \rangle$. As we did in Section 2, for each Young diagram λ , we consider an object \square_λ whose points correspond to the cells of λ . From Theorem 3.1, we get the section $s_{\square_\lambda} : H_{\square_\lambda} \rightarrow K_{\square_\lambda}$, and we obtain a minimal idempotent in K_{\square_λ} ,

$$\tilde{y}_\lambda = s_{\square_\lambda}(y_\lambda) \in K_{\square_\lambda} .$$

Lemma 3.3. *If λ and μ are two distinct Young diagrams with the same size $|\lambda| = |\mu| = n$, then for every $x \in K(\square_\lambda, \square_\mu)$ one has $\tilde{y}_\lambda x \tilde{y}_\mu = 0$.*

Proof. We have $x = \tilde{h} + y$ with $\tilde{h} = s_n(h)$, $h \in H_n$ and $y \in I_n$. We omit here the isomorphisms between $K(n, \square_\lambda)$ and $K(n, \square_\mu)$. Then

$$s_n(y_\lambda)x s_n(y_\mu) = s_n(y_\lambda)\tilde{h}s_n(y_\mu) = s_n(y_\lambda h y_\mu).$$

The result follows from the corresponding property in the category H . \square

We will need the following absorbing property which also results from computation in the Hecke algebra [5, Cor.1.10].

Lemma 3.4. *If ν is a Young diagram obtained from the Young diagram λ by removing one cell, then one has:*

$$\tilde{y}_\lambda(\tilde{y}_\nu \otimes \mathbf{1}_1)\tilde{y}_\lambda = \tilde{y}_\lambda .$$

Proof of Theorem 3.1. If the section s_n exists, then it is unique. This can be seen as follows. Let U_n be the central idempotent corresponding to the factor I_n . If s'_n is another section, then we have for every $x \in H_n$

$$s_n(x) - s'_n(x) = (s_n(x) - s'_n(x))U_n = 0 .$$

We will construct the section s_n by induction on n . The result is certainly true for $n = 1$, since we have that $K_1 \approx H_1 \approx k$.

Let us assume that we have constructed s_m satisfying the conditions of the theorem, for every $m < n$, so that we have minimal idempotents, \tilde{p}_t , for every t such that $|\lambda(t)| < n$.

Let λ be a Young diagram whose size is $|\lambda| = n - 1$, then we have a minimal idempotent $\tilde{y}_\lambda \in K_{\square_\lambda}$. If ν is a Young diagram included in λ , such that $|\nu| = n - 2$, then we define $\tilde{y}_{(\lambda, \nu)} \in K_{\square_\lambda \otimes 1}$, by

$$(1) \quad \tilde{y}_{(\lambda, \nu)} = \frac{\langle \nu \rangle}{\langle \lambda \rangle} (\tilde{y}_\lambda \otimes \mathbf{1}_1)(\tilde{y}_\nu \otimes h_1)(\tilde{y}_\lambda \otimes \mathbf{1}_1) .$$

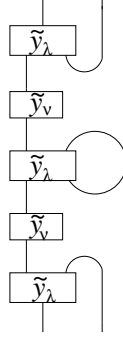
Here, the standard isomorphisms between $\square_\lambda \otimes 1$ and $\square_\nu \otimes 2$ are omitted. We need that the quantum dimensions $\langle \lambda \rangle$, with $|\lambda| = n - 1$ are not zero. This result follows from [7, Theorem 3.7], and will also be proved, by considering the specializations corresponding to Brauer algebras in the next section.

Lemma 3.5. *a) If ν and μ are two distinct Young diagrams of size $n - 2$, included in λ , then $\tilde{y}_{(\lambda, \nu)}\tilde{y}_{(\lambda, \mu)} = 0$.*

b) If ν is a Young diagram of size $n - 2$, included in λ , then $\tilde{y}_{(\lambda, \nu)}$ is an idempotent.

Proof. By the induction hypothesis we can apply Lemmas 3.3 and 3.4 to Young diagrams of size $m < n$. The statement a) follows then from Lemma 3.3 applied to ν and μ with $|\nu| = |\mu| = n - 2$.

The square of \tilde{y}_ν is equal to $\left(\frac{\langle \nu \rangle}{\langle \lambda \rangle}\right)^2$ times the skein element represented by the following tangle.



Let us consider the intermediate morphism

$$(\tilde{y}_\nu \otimes \cup_{\lambda/\nu})(\tilde{y}_\lambda \otimes \mathbf{1}_1)(\tilde{y}_\nu \otimes \cap_{\lambda/\nu})$$

(the subscript in $\cup_{\lambda/\nu}$ and $\cap_{\lambda/\nu}$ indicate which isomorphism in $K(\square_\lambda \otimes 1, \square_\nu \otimes 2)$ is used). The minimality of the idempotent \tilde{y}_ν implies that this morphism is equal to \tilde{y}_ν , up to a coefficient which is obtained by considering the trace (we use the absorbing property 3.4).

$$(2) \quad (\tilde{y}_\nu \otimes \cup_{\lambda/\nu})(\tilde{y}_\lambda \otimes \mathbf{1}_1)(\tilde{y}_\nu \otimes \cap_{\lambda/\nu}) = \frac{\langle \lambda \rangle}{\langle \nu \rangle} \tilde{y}_\nu .$$

Statement b) follows. \square

Let t be a standard tableau whose size is $n - 1$, with shape $\lambda(t) = \lambda$. We define $\mathbf{a}_t \in K(n - 1, \square_\lambda)$ and $\mathbf{b}_t \in K(\square_\lambda, n - 1)$ by lifting to the category K the elements α_t and β_t defined in Section 2. If α_t and β_t are represented by linear combinations of ‘braids’ (elements of the braid groupoid), then \mathbf{a}_t and \mathbf{b}_t are given by the following formulas.

$$\begin{aligned} \mathbf{a}_t &= s_{n-1}(\mathbf{1}_{n-1})\alpha_t = \alpha_t \tilde{y}_\lambda , \\ \mathbf{b}_t &= \beta_t s_{n-1}(\mathbf{1}_{n-1}) = \tilde{y}_\lambda \beta_t . \end{aligned}$$

We then have the formula

$$\tilde{p}_t = s_{n-1}(p_t) = \mathbf{a}_t \mathbf{b}_t .$$

For a Young diagram ν with $n - 2$ cells, included in $\lambda = \lambda(t)$, we define $\tilde{p}_{(t,\nu)}$ and $\tilde{p}_t^+ \in K_n$, by

$$\begin{aligned} \tilde{p}_{(t,\nu)} &= (\mathbf{a}_t \otimes \mathbf{1}_1) \tilde{y}_{(\lambda,\nu)} (\mathbf{b}_t \otimes \mathbf{1}_1) , \\ \tilde{p}_t^+ &= \tilde{p}_t \otimes \mathbf{1}_1 - \sum_{\substack{\nu \subset \lambda(t) \\ |\nu|=n-2}} \tilde{p}_{(t,\nu)} . \end{aligned}$$

Using Lemma 3.5, we get the following.

Lemma 3.6. *i) $\tilde{p}_{(t,\nu)}\tilde{p}_{(\tau,\mu)} = \delta_{t\tau}\delta_{\nu\mu}\tilde{p}_{(t,\nu)}$; ii) $\tilde{p}_t^+\tilde{p}_\tau^+ = \delta_{t\tau}\tilde{p}_t^+$.*

We define a linear homomorphism s'_n from the braid group algebra $k[B_n]$ to K_n by

$$(3) \quad \forall x \in k[B_n] \quad s'_n(x) = \sum_{|\lambda(t)|=|\lambda(\tau)|=n-1} \tilde{p}_t^+ x \tilde{p}_\tau^+ .$$

We will show that this homomorphism induces a well defined linear map $s_n : H_n \rightarrow K_n$ which is a section of π_n , and prove multiplicativity. The proof will be complete with the two following lemmas. \square

Lemma 3.7. *$\tilde{p}_t^+ y = y\tilde{p}_t^+ = 0$ for $\forall y \in I_n$.*

Proof. We want to show that $\tilde{p}_t^+ y = 0$ for any $y \in I_n$. We write

$$y = (a \otimes \mathbf{1}_1)h_{n-1}(b \otimes \mathbf{1}_1) .$$

By the induction hypothesis, we have the result if a is in I_{n-1} . So it is enough to consider the case where $a = s_{n-1}(x)$ for x an element of the matrix units basis described in Section 2, i.e. for $a = \mathbf{a}_\sigma \mathbf{b}_\tau$, where τ and σ are standard tableaux with the same shape, whose size is $n-1$. If $\sigma \neq t$, then we have $\tilde{p}_t \mathbf{a}_\sigma = 0$ and the result follows. It remains to check the case where $\sigma = t$.

$$\begin{aligned} \tilde{p}_t^+(\mathbf{a}_t \mathbf{b}_\tau \otimes \mathbf{1}_1)h_{n-1} &= \tilde{p}_t^+(\mathbf{a}_t \otimes \mathbf{1}_1)(\mathbf{b}_{\tau'} \otimes h_1) \\ &= (\tilde{p}_t \otimes \mathbf{1}_1)(\mathbf{a}_t \otimes \mathbf{1}_1)(\mathbf{b}_{\tau'} \otimes h_1) \\ &\quad - \sum_{\substack{\nu \subset \lambda \\ |\nu|=n-2}} \frac{\langle \nu \rangle}{\langle \lambda(t) \rangle} (\mathbf{a}_t \otimes \mathbf{1}_1)(\tilde{y}_\nu \otimes h_1)(\mathbf{b}_t \otimes \mathbf{1}_1)(\mathbf{a}_t \otimes \mathbf{1}_1)(\mathbf{b}_{\tau'} \otimes h_1) \\ &= (\mathbf{a}_t \otimes \mathbf{1}_1)(\mathbf{b}_{\tau'} \otimes h_1) \\ &\quad - \frac{\langle \lambda(\tau') \rangle}{\langle \lambda(t) \rangle} (\mathbf{a}_t \otimes \mathbf{1}_1)(\tilde{y}_{\lambda(\tau')} \otimes h_1)(\tilde{y}_{\lambda(t)} \otimes \mathbf{1}_1)(\mathbf{b}_{\tau'} \otimes h_1) \\ &= 0 \end{aligned}$$

The result $\tilde{p}_t^+ y = 0$ follows; $y\tilde{p}_t^+ = 0$ can be obtained similarly. \square

Lemma 3.8. *The map s'_n induces a well defined multiplicative homomorphism*

$$s_n : H_n \rightarrow K_n$$

such that $\pi_n \circ s_n = \text{id}_{H_n}$.

Proof. From Lemma 3.7, we can see that the Homfly skein relation is respected, whence we have that s_n is well defined. We have that $\pi_n(\tilde{p}_t^+ \otimes \mathbf{1}_1) = p_t \otimes \mathbf{1}_1$. This implies that $\pi_n \circ s_n = \text{id}_{H_n}$. The computation below shows the multiplicativity.

$$\begin{aligned} s_n(x)s_n(y) &= \sum_{t,\sigma,\tau} \tilde{p}_t^+ x \tilde{p}_\sigma^+ y \tilde{p}_\tau^+ \\ &= \sum_{t,\sigma,\tau} \tilde{p}_t^+ x (\tilde{p}_\sigma \otimes \mathbf{1}_1) y \tilde{p}_\tau^+ \\ &= s_n \left(\sum_{\sigma} x (p_\sigma \otimes \mathbf{1}_1) y \right) \\ &= s_n(xy) . \end{aligned}$$

□

4. BRAUER ALGEBRAS

Brauer centralizer algebras were introduced in [8] in relation with the representation theory of the orthogonal and symplectic groups (see also [23]). Their structure was obtained by Wenzl in [22]. We emphasize also Nazarov’s computations of the action of generators on their irreducible representations in [14]. His work includes the dimension formulas and inspired our computation of the quantum dimensions in Section 7.

We have defined the Birman-Murakami-Wenzl algebras K_n by using Kauffman skein theory. Brauer algebras can be defined in a similar way by using the classical version of Kauffman relations given in Figure 4.

$$\begin{array}{c} \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \\ \begin{array}{c} | \\ \circ \end{array} = \begin{array}{c} | \\ \circ \end{array} \\ L \quad \text{II} \quad \circ = N \quad L \end{array}$$

FIGURE 4. Classical Kauffman skein relations

Here any coefficient ring is allowed, and N could be an indeterminate. If N is a natural number, then we obtain Brauer algebras with complex coefficients as a specialization of Birman-Murakami-Wenzl algebras, with coefficient ring $\mathbb{C}[s^{\pm 1}]$ and $\alpha = s^{N-1}$, by setting $s = 1$. It is a classical fact [23, Ch.5], that there exists an algebra homomorphism Φ_n from this Brauer algebra, denoted by $D_n(N)$, to the centralizer algebra $\text{End}_{O(N)}(V^{\otimes n})$, where $V = \mathbb{C}^N$ is the fundamental representation

of the orthogonal group $O(N)$. If we denote by (u_1, \dots, u_N) the canonical basis of V , then Φ_n is defined on the generators by

$$\begin{aligned} \Phi_n(e_i).u_{j_1} \otimes \dots \otimes u_{j_i} \otimes u_{j_{i+1}} \otimes \dots \otimes u_{j_n} &= u_{j_1} \otimes \dots \otimes u_{j_{i+1}} \otimes u_{j_i} \otimes \dots \otimes u_{j_n} , \\ \Phi_n(h_i).u_{j_1} \otimes \dots \otimes u_{j_i} \otimes u_{j_{i+1}} \otimes \dots \otimes u_{j_n} &= \delta_{j_i j_{i+1}} \sum_{\nu=1}^n u_{j_1} \otimes \dots \otimes u_{j_\nu} \otimes u_{j_\nu} \otimes \dots \otimes u_{j_n} . \end{aligned}$$

In the above, δ is the Kronecker delta. For $N \geq n$, this homomorphism is injective. In fact the above assignment extends to a monoidal functor from the specialized BMW category, to the linear category. This functor is compatible with the (trivial) ribbon structures on these categories, and so it respects the ‘quantum’ traces. In the case of the linear category the quantum trace coincides with the usual one. We get that, with the given specializations,

$$\forall x \in D_n(N) \quad \langle x \rangle = \text{trace}(\Phi_n(x)) .$$

In particular, if x is a non trivial idempotent, then its quantum dimension is a non zero natural number.

The quotient of $D_n(N)$ by the ideal I_n generated by h_{n-1} is isomorphic to the symmetric group algebra $\mathbb{C}[\mathcal{S}_n]$, the classical counterpart of the Hecke algebra. We denote by π_n the canonical projection map

$$\pi_n : D_n(N) \longrightarrow \mathbb{C}[\mathcal{S}_n] .$$

Theorem 4.1. *If N is an integer greater or equal to n , then, there exists a multiplicative homomorphism $s_n : \mathbb{C}[\mathcal{S}_n] \rightarrow D_n(N)$, such that*

$$\pi_n \circ s_n = \text{id}_{\mathbb{C}[\mathcal{S}_n]} \text{ and}$$

$$\forall x \in \mathbb{C}[\mathcal{S}_n] \quad \forall y \in I_n \quad s_n(x)y = ys_n(x) = 0 .$$

Remark 4.2. As a corollary we get a non trivial minimal idempotent $\tilde{p}_t = s_n(p_t)$ for every standard tableau of size n , whose quantum trace is a non zero natural number. Here p_t is the minimal path idempotent in the symmetric group algebra.

Proof. The recursive construction of the preceding section can be done. The only point to check is that at each step the quantum dimensions $\langle \lambda \rangle$ are not zero. By the induction hypothesis, we have non trivial minimal idempotents $\tilde{p}_t = s_n(p_t)$ for every standard tableau of size $n - 1$. If the shape of t is λ , then we have that $\langle \lambda \rangle = \langle \tilde{p}_t \rangle = \text{trace}(\Phi(\tilde{p}_t))$ is a non zero integer. \square

Remark 4.3. The above completes the proof of Theorem 3.1. At each step of the recursive construction, we needed that the quantum dimensions $\langle \lambda \rangle$ with $|\lambda| = n - 1$ are non zero. This is the case, because they become non zero integers if we apply the rank $N \geq n$ Brauer specialization. The same remark shows that Theorem 4.1 holds if N is generic (e.g. for the Brauer algebra with ground field $\mathbb{Q}(N)$).

5. MATRIX UNITS IN BIRMAN-MURAKAMI-WENZL ALGEBRAS

In this section we will describe a matrix units basis in K_n (compare with [15]), and show the branching formula. Recall that in the categories H and K , for each Young diagram λ , we have defined an object \square_λ whose points correspond to the cells of λ . In the proof of Theorem 3.1, we have used the section $s_{\square_\lambda} : H_{\square_\lambda} \rightarrow K_{\square_\lambda}$ to define a minimal idempotent $\tilde{y}_\lambda = s_{\square_\lambda}(y_\lambda) \in K_{\square_\lambda}$.

A sequence $\Lambda = (\Lambda_1, \dots, \Lambda_n)$ of Young diagrams, in which two consecutive diagrams Λ_i and Λ_{i+1} differ by exactly one cell will be called an up and down tableau of length n , and shape Λ_n . Those up and down tableaux of length n such that $n = |\Lambda_n|$ (up tableaux) correspond bijectively with standard tableaux as described in Section 2.

For an up and down tableau Λ of length n , we denote by Λ' the tableau of length $n - 1$ obtained by removing the last Young diagram in the sequence Λ . We define $\mathbf{a}_\Lambda \in K(n, \square_\Lambda)$ and $\mathbf{b}_\Lambda \in K(\square_\Lambda, n)$ by

$$\mathbf{a}_1 = \mathbf{b}_1 = \mathbf{1}_1 ,$$

if $|\Lambda_n| = |\Lambda_{n-1}| + 1$, then

$$\begin{aligned} \mathbf{a}_\Lambda &= (\mathbf{a}_{\Lambda'} \otimes \mathbf{1}_1) \tilde{y}_{\Lambda_n} \\ \mathbf{b}_\Lambda &= \tilde{y}_{\Lambda_n} (\mathbf{b}_{\Lambda'} \otimes \mathbf{1}_1) , \end{aligned}$$

if $|\Lambda_n| = |\Lambda_{n-1}| - 1$, then

$$\begin{aligned} \mathbf{a}_\Lambda &= \frac{\langle \Lambda_n \rangle}{\langle \Lambda_{n-1} \rangle} (\mathbf{a}_{\Lambda'} \otimes \mathbf{1}_1) (\tilde{y}_{\Lambda_n} \otimes \cap) \\ \mathbf{b}_\Lambda &= (\tilde{y}_{\Lambda_n} \otimes \cup) (\mathbf{b}_{\Lambda'} \otimes \mathbf{1}_1) . \end{aligned}$$

Here we have omitted the standard isomorphism in $K(\square_{\Lambda_{n-1}} \otimes 1, \square_{\Lambda_n})$ and in $K(\square_{\Lambda_{n-1}} \otimes 1, \square_{\Lambda_n} \otimes 2)$. Note that for an up tableau, the definition is coherent with the one given for the corresponding standard tableau in the proof of Theorem 3.1.

Theorem 5.1. *a) The family $\mathbf{a}_\Lambda \mathbf{b}_\Xi$ for all up and down tableaux Λ, Ξ of length n , such that $\Lambda_n = \Xi_n$ forms a basis for K_n , and*

$$\mathbf{a}_\Lambda \mathbf{b}_\Xi \mathbf{a}_L \mathbf{b}_X = \delta_{\Xi L} \mathbf{a}_\Lambda \mathbf{b}_X .$$

b) *There exists an algebra isomorphism*

$$\bigoplus_{|\lambda|=n, n-2, \dots} \mathcal{M}_{d_\lambda^{(n)}}(k) \approx K_n ,$$

where $d_\lambda^{(n)}$ is the number of the up and down tableaux of length n with shape λ , and $\mathcal{M}_d(k)$ is the algebra of $d \times d$ matrices with coefficients in k .

The diagonal elements $q_\Lambda = \mathbf{a}_\Lambda \mathbf{b}_\Lambda$ are the path idempotents associated with the inclusions $K_i \subset K_{i+1}$ the minimal central idempotent corresponding to the partition λ is

$$z_\lambda^{(n)} = \sum_{\Lambda_n = \lambda} q_\Lambda .$$

If Λ corresponds to a standard tableau t (Hecke part), then one has $q_\Lambda = \tilde{p}_t$, and if $|\lambda| = n$ then $z_\lambda^{(n)} = \tilde{z}_\lambda$.

Proof. From Lemma 3.3 if $|\lambda| = |\mu|$, and from Lemma 3.6 if $|\lambda| \neq |\mu|$, we get

$$(4) \quad \tilde{y}_\lambda K(\square_\lambda, \square_\mu) \tilde{y}_\mu = 0 .$$

This implies that $\mathbf{b}_\Lambda \mathbf{a}_\Xi = 0$ if $\Lambda \neq \Xi$.

Using the properties 3.4 and 2, we show that, if n is the length of Λ , we have

$$\mathbf{b}_\Lambda \mathbf{a}_\Lambda = \tilde{y}_{\Lambda_n} .$$

Hence we have that

$$\mathbf{a}_\Lambda \mathbf{b}_{\Xi} \mathbf{a}_L \mathbf{b}_X = \delta_{\Xi L} \mathbf{a}_\Lambda \mathbf{b}_X .$$

The independence follows.

To show that the family $\mathbf{a}_\Lambda \mathbf{b}_\Xi$ generate K_n , we proceed recursively on n . Using the map s_n , we see from the known result in H_n that the $\mathbf{a}_\Lambda \mathbf{b}_\Xi$, where Λ_n and Ξ_n is the same Young diagram with n cells, generate the Hecke part $\tilde{H}_n = s_n(H_n)$ of K_n . It remains to consider I_n . From the induction hypothesis, we get that I_n is generated by the following elements

$$(\mathbf{a}_\Lambda \otimes \mathbf{1}_1)(\mathbf{b}_\Xi \otimes \mathbf{1}_1) h_{n-1}(\mathbf{a}_L \otimes \mathbf{1}_1)(\mathbf{b}_X \otimes \mathbf{1}_1) .$$

These are zero if $\Xi' \neq L'$, and else are equal to

$$\frac{\langle \Lambda_{n-1} \rangle}{\langle \lambda \rangle} \mathbf{a}_{(\Lambda, \lambda)} \mathbf{b}_{(X, \lambda)} ,$$

where $\lambda = \Xi'_{n-2} = L'_{n-2}$. □

The path idempotents q_Λ satisfy the following branching formula.

Theorem 5.2 (Branching formula).

$$q_\Lambda \otimes \mathbf{1}_1 = \sum_{\Xi'=\Lambda} q_{\Xi'} .$$

We recall that in the above formula the shape of Ξ is either one cell more, either one cell less than the shape of Λ .

Proof. The coordinates of $q_\Lambda \otimes \mathbf{1}_1$ in the standard basis are obtained by computing $\mathbf{b}_\Xi(q_\Lambda \otimes \mathbf{1}_1)\mathbf{a}_L$. The result is zero unless Ξ and L have the same shape μ and $L' = \Xi' = \Lambda$; and in the latter case the result is \tilde{y}_μ . \square

As a corollary we also have a branching formula for the minimal idempotents \tilde{y}_λ (the obvious isomorphisms are omitted).

Corollary 5.3.

$$\tilde{y}_\lambda \otimes \mathbf{1}_1 = \sum_{\substack{\lambda \subset \mu \\ |\mu| = |\lambda| + 1}} (\tilde{y}_\lambda \otimes \mathbf{1}_1) \tilde{y}_\mu (\tilde{y}_\lambda \otimes \mathbf{1}_1) + \sum_{\substack{\nu \subset \lambda \\ |\nu| = n - 1}} \tilde{y}_{(\lambda, \nu)} .$$

Proof. We can decompose $\tilde{y}_\lambda \otimes 1$ by using the minimal central idempotents in $K_{|\lambda|+1}$. We get

$$\tilde{y}_\lambda \otimes 1 = \sum_{\Xi} (\tilde{y}_\lambda \otimes \mathbf{1}_1) q_{\Xi} (\tilde{y}_\lambda \otimes \mathbf{1}_1) .$$

In the above, only those up and down tableaux Ξ with $\Xi_{|\lambda|} = \lambda$ contribute. \square

6. BRAIDING AND TWIST COEFFICIENTS

Proposition 6.1. *i) (Twist coefficient) Let $\tilde{y}_\mu \in K_{\square_\mu}$ be the minimal idempotent, then*

$$(5) \quad \begin{array}{c} | \\ \circlearrowleft \\ | \\ \boxed{\tilde{y}_\mu} \end{array} = \alpha^{|\mu|} s^{2 \sum_{c \subset \mu} cn(c)} \tilde{y}_\mu .$$

Here the content of a cell c in the i -th row and the j -th column of μ is defined by $cn(c) := j - i$.

ii) (Braiding coefficient) Suppose that $\lambda \subset \mu$ and $\mu - \lambda$ contains only one cell c . Then

$$(6) \quad \begin{array}{c} \tilde{y}_\mu \\ \text{---} \\ \text{---} \\ \tilde{y}_\lambda \\ \text{---} \\ \tilde{y}_\mu \end{array} = s^{2cn(c)} \tilde{y}_\mu .$$

Suppose that $\mu \subset \lambda$ and $\lambda - \mu$ contains only one cell c , then

$$(7) \quad \begin{array}{c} \tilde{y}_{(\lambda, \mu)} \\ \text{---} \\ \text{---} \\ \tilde{y}_\lambda \\ \text{---} \\ \tilde{y}_{(\lambda, \mu)} \end{array} = \alpha^{-2} s^{-2cn(c)} \tilde{y}_{(\lambda, \mu)} .$$

Proof. The statements (5) and (6) follow from the corresponding in the Hecke algebra (see [5, Prop.1.11]). Using the definition of the idempotent $\tilde{y}_{(\lambda, \mu)}$, we can bring (7) to the form

$$\left(\frac{\langle \mu \rangle}{\langle \lambda \rangle} \right)^2 (\tilde{y}_\lambda \otimes \mathbf{1}_1) ((\tilde{y}_\mu x \tilde{y}_\mu) \otimes h_{\lambda/\mu}) (\tilde{y}_\lambda \otimes \mathbf{1}_1) ,$$

where $x \in K_{\square_\mu}$ is depicted below.



By the Schur lemma, $\tilde{y}_\mu x \tilde{y}_\mu = c \tilde{y}_\mu$ with $c \in k$. Taking the quantum trace of this morphism, we get

$$\langle \tilde{y}_\mu x \tilde{y}_\mu \rangle = \alpha^{-2} s^{-2cn(c)} \langle \lambda \rangle = c \langle \mu \rangle .$$

The first equality is due to the Kauffman skein relations and the statement (6) above. □

7. QUANTUM DIMENSION

The formula for the quantum dimension $\langle \lambda \rangle$ was obtained by Wenzl [20, Theorem 5.5]. The proof there rests on the representation theory of the quantum group $U_q so(2n + 1)$. We give below an alternative proof for this formula. Our method is inspired by the Nazarov computation of the matrix elements of the action of the hook generators on the canonical basis of the irreducible representations of Brauer algebras [14]. Here we work with scalar field $k = \mathbb{Q}(s, \alpha)$, hence we have that all the quantum dimensions $\langle \lambda \rangle$ are invertible.

Suppose that $\lambda = (\lambda_1, \dots, \lambda_m)$ is obtained from μ by adding one cell in the i th row. Let l be the number of pairwise distinct rows in the diagram μ . Then one can obtain $l + 1$ diagrams by adding a cell to μ , and l diagrams by removing a cell from μ . Let c_1, \dots, c_{l+1} and d_1, \dots, d_l be the contents (defined in Prop.6.1) of these cells respectively. Denote by b_1, \dots, b_{2l+1} the scalars

$$\alpha s^{2c_1}, \dots, \alpha s^{2c_{l+1}},$$

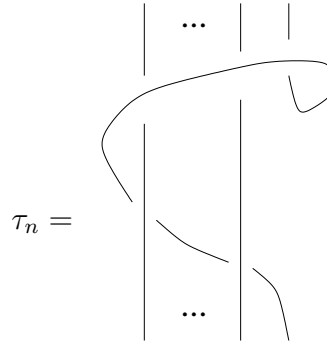
$$\alpha^{-1} s^{-2d_1} \dots, \alpha^{-1} s^{-2d_l},$$

and by b the value among b_1, \dots, b_{l+1} corresponding to the diagram λ .

Theorem 7.1. *One has*

$$(8) \quad \frac{\langle \lambda \rangle}{\langle \mu \rangle} = \alpha b^{-1} \left(\frac{b - b^{-1}}{s - s^{-1}} + 1 \right) \prod_{b_j \neq b} \frac{b - b_j^{-1}}{b - b_j} .$$

Proof. Let τ_n be the element of the algebra K_n defined below.



For $i > 0$ the equation in K_{n+1} ,

$$h_n(\tau_n^i \otimes \mathbf{1}_1)h_n = Z_n^{(i)} \otimes h_1 ,$$

defines a central element $Z_n^{(i)}$ in K_{n-1} . We consider the formal power series in u^{-1} ,

$$Z_n(u) = \sum_{i \geq 0} Z_n^{(i)} u^{-i} .$$

We have that

$$Z_n(u) \otimes h_1 = h_n \left(\frac{u}{u - \tau_n} \otimes \mathbf{1}_1 \right) h_n .$$

We denote by $Z_n(\mu, u)$ the series given by the action of $Z_n(u)$ on the simple component of K_{n-1} indexed by μ .

A canonical basis of K_n is given in Theorem 5.1. Let Λ be an up and down tableau, whose length is $n + 1$, and such that

$$\Lambda_{n-1} = \Lambda_{n+1} = \mu \text{ and } \Lambda_n = \lambda .$$

Write the products $h_n q_\Lambda$ and $q_\Lambda h_n$ in the canonical basis.

$$h_n q_\Lambda = \sum_{\Xi_{n+1} = \mu} h_n(\Xi, \Lambda) \mathbf{a}_\Xi \mathbf{b}_\Lambda \quad \text{and} \quad q_\Lambda h_n = \sum_{\Xi_{n+1} = \mu} h'_n(\Lambda, \Xi) \mathbf{a}_\Lambda \mathbf{b}_\Xi .$$

Let J_Λ be the set of up and down tableaux $\Xi = (\Xi_1, \dots, \Xi_{n+1})$ such that $\Xi_m = \Lambda_m$ for every $m \neq n$. By considering $q_\Xi h_n q_\Lambda$ (resp. $q_\Lambda h_n q_\Xi$), we get

$$h_n q_\Lambda = \sum_{\Xi \in J_\Lambda} h_n(\Xi, \Lambda) \mathbf{a}_\Xi \mathbf{b}_\Lambda \quad \text{and} \quad q_\Lambda h_n = \sum_{\Xi \in J_\Lambda} h'_n(\Lambda, \Xi) \mathbf{a}_\Lambda \mathbf{b}_\Xi .$$

Using the three lemmas below the proof can be accomplished as follows. From Lemma 7.2, Lemma 7.3 and (10) we have

$$\frac{\langle \lambda \rangle}{\langle \mu \rangle} = \text{res}_{u=b} \frac{Z(\mu, u)}{u} = \text{res}_{u=b} \frac{Q(\mu, u)}{u}$$

The required formula follows now from (12). □

Lemma 7.2. *One has*

$$h_n(\Lambda, \Lambda) = h'_n(\Lambda, \Lambda) = \frac{\langle \lambda \rangle}{\langle \mu \rangle} .$$

Proof. Let Λ' be as usual obtained by removing the last term in the sequence Λ . We have

$$q_\Lambda = \mathbf{a}_\Lambda \mathbf{b}_\Lambda = \frac{\langle \mu \rangle}{\langle \lambda \rangle} (\mathbf{a}_{\Lambda'} \otimes 1) (\tilde{y}_\mu \otimes h_{\lambda/\mu}) (\mathbf{b}_{\Lambda'} \otimes 1)$$

We will obtain the diagonal term $h_n(\Lambda, \Lambda)$ by computing the quantum trace of $h_n q_\Lambda$.

$$h_n(\Lambda, \Lambda) \langle \mu \rangle = \langle h_n q_\Lambda \rangle = \frac{\langle \mu \rangle}{\langle \lambda \rangle} \langle T \rangle ,$$

where T is represented in the picture below.

$$T = \begin{array}{c} \text{---} \\ \boxed{b_{\Lambda'}} \\ \text{---} \\ \boxed{a_{\Lambda'}} \\ \text{---} \end{array} \begin{array}{c} \circ \\ \circ \end{array}$$

Clearly, $\langle T \rangle = \langle \lambda \rangle^2 \langle \mu \rangle^{-1}$ and we get the required result for $h_n(\Lambda, \Lambda)$; $h'_n(\Lambda, \Lambda)$ is calculated similarly. \square

Let us denote by $\text{res}_{u=b} \frac{Z(\mu, u)}{u}$ the residue of $\frac{Z(\mu, u)}{u}$ at $u = b$, i.e. the coefficient by $(u - b)^{-1}$ in the Laurent expansion of this function in the neighborhood of the point $u = b$.

Lemma 7.3.

$$h_n(\Lambda, \Lambda) = \text{res}_{u=b} \frac{Z(\mu, u)}{u}$$

Proof. We have that

$$h_n(Z_n(u) \otimes \mathbf{1}_2)u^{-1} = h_n((u - \tau_n)^{-1} \otimes \mathbf{1}_1)h_n.$$

Multiplying on the left by q_Λ , we can express the above formula in the canonical basis. We denote by ζ_Λ the diagonal term of index Λ . From the left hand side, we get

$$\zeta_\Lambda = h_n(\Lambda, \Lambda)Z(\mu, u)u^{-1}.$$

Let us compute the coefficient from the right hand side.

$$\begin{aligned} \zeta_\Lambda q_\Lambda &= q_\Lambda h_n((u - \tau_n)^{-1} \otimes \mathbf{1}_1)h_n q_\Lambda \\ &= \sum_{L \in J_\Lambda} \sum_{\Xi \in J_\Lambda} h'_n(\Lambda, L)h_n(\Xi, \Lambda) \mathbf{a}_\Lambda \mathbf{b}_L ((u - \tau_n)^{-1} \otimes \mathbf{1}_1) \mathbf{a}_\Xi \mathbf{b}_\Lambda. \end{aligned}$$

In the the above sum, the term of indices L and Ξ is zero unless $L = \Xi$, and in this case the action of τ_n multiplies by the coefficient among b_1, \dots, b_{2l+1} corresponding to the Young diagram Ξ_n . These coefficients are distinct, and we know that $h_n(\Lambda, \Lambda) = \frac{\langle \lambda \rangle}{\langle \mu \rangle}$ is not zero. This implies that $Z(\mu, u)u^{-1}$ is a rational function in u , whose residue at $u = b$ is equal to $h_n(\Lambda, \Lambda)$. \square

The problem is now to compute the series $Z(\mu, u)$. We set

$$(9) \quad Q_n(u) = Z_n(u) + \frac{\alpha^{-1}}{s - s^{-1}} - \frac{u^2}{u^2 - 1},$$

$$(10) \quad Q(\mu, u) = Z(\mu, u) + \frac{\alpha^{-1}}{s - s^{-1}} - \frac{u^2}{u^2 - 1}.$$

Lemma 7.4.

(11)

$$Q_{n+1}(u) \left(\frac{1}{u} - (s - s^{-1})^2 \frac{\tau_n}{(u - \tau_n)^2} \right) = (Q_n(u) \otimes \mathbf{1}_1) \left(\frac{1}{u} - (s - s^{-1})^2 \frac{\tau_n^{-1}}{(u - \tau_n^{-1})^2} \right)$$

(12)

$$Q(\mu, u) = \left(\frac{\alpha}{s - s^{-1}} + \frac{u\alpha}{u^2 - 1} \right) \prod_j \frac{u - b_j^{-1}}{u - b_j}$$

Proof. In the following computations, we will drop some $\mathbf{1}_1$. This means that by drawing the figures corresponding to these computation, one may have to add a vertical string on the right in order to get coherent equalities. For example, we write

$$(13) \quad e_n^{-1} \tau_{n+1} = \tau_n e_n .$$

From the above, using the skein relation, we obtain

$$(14) \quad e_n^{-1}(u - \tau_{n+1}) = (u - \tau_n)e_n^{-1} - (s - s^{-1})\tau_n(\mathbf{1}_{n+1} - h_n) ,$$

$$(15) \quad (u - \tau_n)e_n = e_n(u - \tau_{n+1}) + (s - s^{-1})(\mathbf{1}_{n+1} - h_n)\tau_{n+1} .$$

This implies the following equalities for the formal series

$$(16) \quad \frac{1}{u - \tau_n} e_n^{-1} = e_n^{-1} \frac{1}{u - \tau_{n+1}} - (s - s^{-1}) \frac{\tau_n}{u - \tau_n} (\mathbf{1}_{n+1} - h_n) \frac{1}{u - \tau_{n+1}} ,$$

$$(17) \quad e_n \frac{1}{u - \tau_{n+1}} = \frac{1}{u - \tau_n} e_n + (s - s^{-1}) \frac{1}{u - \tau_n} (\mathbf{1}_{n+1} - h_n) \frac{\tau_{n+1}}{u - \tau_{n+1}} .$$

We also have the formula symmetric to (17)

$$(18) \quad \frac{1}{u - \tau_{n+1}} e_n = e_n \frac{1}{u - \tau_n} + (s - s^{-1}) \frac{\tau_{n+1}}{u - \tau_{n+1}} (\mathbf{1}_{n+1} - h_n) \frac{1}{u - \tau_n} .$$

We note that τ_n and τ_{n+1} commute, and that

$$h_n \tau_{n+1}^i = h_n \tau_n^{-i} .$$

Multiplying (16) on the left by e_n , we get

$$e_n \frac{1}{u - \tau_n} e_n^{-1} = \frac{1}{u - \tau_{n+1}} - (s - s^{-1}) e_n \frac{1}{u - \tau_{n+1}} \frac{\tau_n}{u - \tau_n} + (s - s^{-1}) e_n \frac{u}{u - \tau_n} h_n \frac{1}{u - \tau_n^{-1}} - (s - s^{-1}) e_n h_n \frac{1}{u - \tau_n^{-1}} .$$

Using (17), (18) and the skein relations, we get

$$\begin{aligned} e_n \frac{1}{u-\tau_n} e_n^{-1} &= \frac{1}{u-\tau_{n+1}} - (s-s^{-1})\alpha^{-1}h_n \frac{1}{u-\tau_n^{-1}} \\ &\quad - (s-s^{-1})\frac{1}{u-\tau_n} e_n \frac{\tau_n}{u-\tau_n} - (s-s^{-1})^2 \frac{1}{u-\tau_n} \frac{\tau_{n+1}}{u-\tau_{n+1}} \frac{\tau_n}{u-\tau_n} \\ &\quad + (s-s^{-1})^2 \frac{1}{u-\tau_n} h_n \frac{\tau_n^{-1}}{u-\tau_n^{-1}} \frac{\tau_n}{u-\tau_n} + (s-s^{-1})\frac{u}{u-\tau_{n+1}} \alpha^{-1} h_n \frac{1}{u-\tau_n^{-1}} \\ &\quad - (s-s^{-1})^2 u \frac{\tau_{n+1}}{u-\tau_{n+1}} \frac{1}{u-\tau_n} h_n \frac{1}{u-\tau_n^{-1}} + (s-s^{-1})^2 u \frac{\tau_n^{-1}}{u-\tau_n^{-1}} h_n \frac{1}{u-\tau_n} h_n \frac{1}{u-\tau_n^{-1}}. \end{aligned}$$

We multiply on each side by h_{n+1} , and use the relations

$$h_{n+1}h_n h_{n+1} = h_{n+1}, \quad h_{n+1}e_n h_{n+1} = \alpha h_{n+1} \text{ and } \tau_n h_{n+1} = h_{n+1}\tau_n.$$

$$\begin{aligned} \frac{Z_n(u)}{u} h_{n+1} &= \left(\frac{Z_{n+1}(u)}{u} - (s-s^{-1})\alpha^{-1} \frac{1}{u-\tau_n^{-1}} \right. \\ &\quad - (s-s^{-1})\alpha \frac{\tau_n}{(u-\tau_n)^2} - (s-s^{-1})^2 \frac{\tau_n}{(u-\tau_n)^2} (Z_{n+1}(u) - \frac{\alpha-\alpha^{-1}}{s-s^{-1}} - 1) \\ &\quad + (s-s^{-1})^2 \frac{1}{(u-\tau_n)^2 (u-\tau_n^{-1})} + (s-s^{-1}) \frac{u\alpha^{-1}}{(u-\tau_n^{-1})^2} \\ &\quad \left. - (s-s^{-1})^2 u \frac{\tau_n^{-1}}{(u-\tau_n)(u-\tau_n^{-1})^2} + (s-s^{-1})^2 \frac{\tau_n^{-1}}{(u-\tau_n^{-1})^2} Z_n(u) \right) h_{n+1}. \end{aligned}$$

The recursive formula (11) can be deduced. It can be written

$$Q_{n+1}(u) = Q_n(u) \left(\frac{(u-\tau_n)^2}{(u-\tau_n^{-1})^2} \frac{(u-s^{-2}\tau_n^{-1})(u-s^2\tau_n^{-1})}{(u-s^{-2}\tau_n)(u-s^2\tau_n)} \right).$$

Hence we have that

$$Q(\lambda, u) = Q(\mu, u) \left(\frac{(u-b)^2}{(u-b^{-1})^2} \frac{(u-s^{-2}b^{-1})(u-s^2b^{-1})}{(u-s^{-2}b)(u-s^2b)} \right).$$

Recall that $b = \alpha s^{2cn(\lambda/\mu)}$ is the eigenvalue of τ_n corresponding to $q_{\lambda'}$. The formula (12) is then established recursively. Note that

$$Z(1, u) = \frac{u}{u-\alpha} \left(\frac{\alpha-\alpha^{-1}}{s-s^{-1}} + 1 \right).$$

Whence we have the formula for $Q(1, u)$. □

Using Theorem 7.1 we can deduce Wenzl's dimension formula [20, Theorem 5.5]. Here λ^\vee denote the transposed Young diagram, so that λ_j^\vee is the length of the j th column of λ . Let $n \in \mathbb{N}$ and $d \in \mathbb{Z}$, we set

$$[y+d] = \frac{\alpha s^d - \alpha^{-1} s^{-d}}{s-s^{-1}}, \quad [n] = \frac{s^n - s^{-n}}{s-s^{-1}}.$$

Theorem 7.5 (Wenzl's formula).

$$(19) \quad \langle \lambda \rangle = \prod_{(j,j) \in \lambda} \frac{[y + \lambda_j - \lambda_j^\vee] + [hl(j, j)]}{[hl(j, j)]} \prod_{\substack{(i,j) \in \lambda \\ i \neq j}} \frac{[y + d_\lambda(i, j)]}{[hl(i, j)]}$$

Here, $hl(i, j)$ denote the hook-length of the cell (i, j) , i.e. $hl(i, j) = \lambda_i + \lambda_j^\vee - i - j + 1$, and $d_\lambda(i, j)$ is defined by

$$d_\lambda(i, j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1 & \text{if } i \leq j \\ -\lambda_i^\vee - \lambda_j^\vee + i + j - 1 & \text{if } i > j. \end{cases}$$

If we define $d'_\lambda(i, j)$ by

$$d'_\lambda(i, j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1 & \text{if } i < j \\ -\lambda_i^\vee - \lambda_j^\vee + i + j - 1 & \text{if } i \geq j, \end{cases}$$

then we can write Wenzl's formula as follows.

$$(20) \quad \langle \lambda \rangle = \prod_{(i,j) \in \lambda} \frac{\alpha^{\frac{1}{2}} s^{\frac{1}{2}} d_\lambda(i,j) - \alpha^{-\frac{1}{2}} s^{-\frac{1}{2}} d_\lambda(i,j)}{s^{\frac{1}{2}} hl(i,j) - s^{-\frac{1}{2}} hl(i,j)} \prod_{(i,j) \in \lambda} \frac{\alpha^{\frac{1}{2}} s^{\frac{1}{2}} d'_\lambda(i,j) + \alpha^{-\frac{1}{2}} s^{-\frac{1}{2}} d'_\lambda(i,j)}{s^{\frac{1}{2}} hl(i,j) + s^{-\frac{1}{2}} hl(i,j)}$$

Note. It would be nice to interpret the above formula by decomposing λ as a tensor product of two objects in some bigger category.

Proof. We will prove the formula (20). We first write the recursive formula (8) in a more convenient form. We denote by $(i, \lambda_i) = (\lambda_{i'}^\vee, i')$ the unique cell in the skew diagram λ/μ .

$$(21) \quad \frac{\langle \lambda \rangle}{\langle \mu \rangle} = \left(\frac{\alpha s^{2\lambda_i - 2i} - \alpha^{-1} s^{-2\lambda_i + 2i}}{s - s^{-1}} + 1 \right) \\ \times \prod_{j < i} \frac{(\alpha s^{\lambda_i - i + \lambda_j - j + 1} - \alpha^{-1} s^{-\lambda_i + i - \lambda_j + j - 1})(s^{\lambda_i - i - \lambda_j + j} - s^{-\lambda_i + i + \lambda_j - j})}{(\alpha s^{\lambda_i - i + \lambda_j - j} - \alpha^{-1} s^{-\lambda_i + i - \lambda_j + j})(s^{\lambda_i - i - \lambda_j + j - 1} - s^{-\lambda_i + i + \lambda_j - j + 1})} \\ \times \prod_{j' < i'} \frac{(\alpha s^{-\lambda_{i'}^\vee + i' - \lambda_{j'}^\vee + j' - 1} - \alpha^{-1} s^{\lambda_{i'}^\vee - i' + \lambda_{j'}^\vee - j' + 1})(s^{-\lambda_{i'}^\vee + i' + \lambda_{j'}^\vee - j'} - s^{\lambda_{i'}^\vee - i' - \lambda_{j'}^\vee + j'})}{(\alpha s^{-\lambda_{i'}^\vee + i' - \lambda_{j'}^\vee + j'} - \alpha^{-1} s^{\lambda_{i'}^\vee - i' + \lambda_{j'}^\vee - j' - 1})(s^{-\lambda_{i'}^\vee + i' + \lambda_{j'}^\vee - j' + 1} - s^{\lambda_{i'}^\vee - i' - \lambda_{j'}^\vee + j' - 1})}$$

Here the first big product gives the contribution of the coefficients b_ξ corresponding to cells in the rows 1 to $i - 1$. Note that some factors cancel if two among these rows have equal length.

We can write $\langle \lambda \rangle = \psi_\lambda(\alpha^{\frac{1}{2}}, s^{\frac{1}{2}}) \psi'_\lambda(\alpha^{\frac{1}{2}}, s^{\frac{1}{2}})$, where ψ_λ and ψ'_λ satisfy the following recursive formulas.

$$\frac{\psi_\lambda(\beta, t)}{\psi_\mu(\beta, t)} = \frac{\beta t^{2\lambda_i - 2i + 1} - \beta^{-1} t^{-2\lambda_i + 2i - 1}}{t - t^{-1}} \\ \times \prod_{j < i} \frac{(\beta t^{\lambda_i - i + \lambda_j - j + 1} - \beta^{-1} t^{-\lambda_i + i - \lambda_j + j - 1})(t^{\lambda_i - i - \lambda_j + j} - t^{-\lambda_i + i + \lambda_j - j})}{(\beta t^{\lambda_i - i + \lambda_j - j} - \beta^{-1} t^{-\lambda_i + i - \lambda_j + j})(t^{\lambda_i - i - \lambda_j + j - 1} - t^{-\lambda_i + i + \lambda_j - j + 1})} \\ \times \prod_{j' < i'} \frac{(\beta t^{-\lambda_{i'}^\vee + i' - \lambda_{j'}^\vee + j' - 1} - \beta^{-1} t^{\lambda_{i'}^\vee - i' + \lambda_{j'}^\vee - j' + 1})(t^{-\lambda_{i'}^\vee + i' + \lambda_{j'}^\vee - j'} - t^{\lambda_{i'}^\vee - i' - \lambda_{j'}^\vee + j'})}{(\beta t^{-\lambda_{i'}^\vee + i' - \lambda_{j'}^\vee + j'} - \beta^{-1} t^{\lambda_{i'}^\vee - i' + \lambda_{j'}^\vee - j' - 1})(t^{-\lambda_{i'}^\vee + i' + \lambda_{j'}^\vee - j' + 1} - t^{\lambda_{i'}^\vee - i' - \lambda_{j'}^\vee + j' - 1})} \\ \frac{\psi'_\lambda(\beta, t)}{\psi'_\mu(\beta, t)} = \frac{\beta t^{-2\lambda_{i'}^\vee + 2i' - 1} + \beta^{-1} t^{2\lambda_{i'}^\vee - 2i' + 1}}{t + t^{-1}} \\ \times \prod_{j < i} \frac{(\beta t^{\lambda_i - i + \lambda_j - j + 1} + \beta^{-1} t^{-\lambda_i + i - \lambda_j + j - 1})(t^{\lambda_i - i - \lambda_j + j} + t^{-\lambda_i + i + \lambda_j - j})}{(\beta t^{\lambda_i - i + \lambda_j - j} + \beta^{-1} t^{-\lambda_i + i - \lambda_j + j})(t^{\lambda_i - i - \lambda_j + j - 1} + t^{-\lambda_i + i + \lambda_j - j + 1})} \\ \times \prod_{j' < i'} \frac{(\beta t^{-\lambda_{i'}^\vee + i' - \lambda_{j'}^\vee + j' - 1} + \beta^{-1} t^{\lambda_{i'}^\vee - i' + \lambda_{j'}^\vee - j' + 1})(t^{-\lambda_{i'}^\vee + i' + \lambda_{j'}^\vee - j'} + t^{\lambda_{i'}^\vee - i' - \lambda_{j'}^\vee + j'})}{(\beta t^{-\lambda_{i'}^\vee + i' - \lambda_{j'}^\vee + j'} + \beta^{-1} t^{\lambda_{i'}^\vee - i' + \lambda_{j'}^\vee - j' - 1})(t^{-\lambda_{i'}^\vee + i' + \lambda_{j'}^\vee - j' + 1} + t^{\lambda_{i'}^\vee - i' - \lambda_{j'}^\vee + j' - 1})}$$

By induction, we can obtain the general formulas for $\psi_\lambda(\beta, t)$ and $\psi'_\lambda(\beta, t)$.

$$\psi_\lambda(\beta, t) = \prod_{(i,j) \in \lambda} \frac{\beta t^{d_\lambda(i,j)} - \beta^{-1} t^{-d_\lambda(i,j)}}{t^{hl(i,j)} - t^{-hl(i,j)}}$$

$$\psi'_\lambda(\beta, t) = \prod_{(i,j) \in \lambda} \frac{\beta t^{d'_\lambda(i,j)} + \beta^{-1} t^{-d'_\lambda(i,j)}}{t^{hl(i,j)} + t^{-hl(i,j)}}$$

Whence we get (20). □

The following proposition gives the quantum dimension formulas for the specializations corresponding to the quantum groups of B,C,D series (compare [9]). We consider here only partitions with at most n rows.

Proposition 7.6. *a) For $\alpha = s^{2n}$ (B_n specialization), one has for a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ (which may have zero coefficients),*

$$\langle \lambda \rangle = \prod_{j=1}^n \frac{[n + \lambda_j - j + 1/2]}{[n - j + 1/2]} \prod_{1 \leq i < j \leq n} \frac{[2n + \lambda_i - i + \lambda_j - j + 1][\lambda_i - i - \lambda_j + j]}{[2n - i - j + 1][j - i]}$$

b) For $\alpha = s^{2n-1}$ (D_n specialization), one has for a partition $\lambda = (\lambda_1, \dots, \lambda_n)$,

$$\langle \lambda \rangle = \prod_{1 \leq i < j \leq n} \frac{[2n + \lambda_i - i + \lambda_j - j][\lambda_i - i - \lambda_j + j]}{[2n - i - j][j - i]} \text{ if } \lambda_n = 0;$$

$$\langle \lambda \rangle = 2 \prod_{1 \leq i < j \leq n} \frac{[2n + \lambda_i - i + \lambda_j - j][\lambda_i - i - \lambda_j + j]}{[2n - i - j][j - i]} \text{ if } \lambda_n \neq 0;$$

c) For $\alpha = -s^{2n+1}$ (C_n specialization), one has for a partition $\lambda = (\lambda_1, \dots, \lambda_n)$,

$$\langle \lambda \rangle = (-1)^{|\lambda|} \prod_{j=1}^n \frac{[2n + 2 + 2\lambda_j - 2j]}{[2n + 2 - 2j]} \prod_{1 \leq i < j \leq n} \frac{[2n + 2 + \lambda_i - i + \lambda_j - j][\lambda_i - i - \lambda_j + j]}{[2n + 2 - i - j][j - i]}$$

Note. Observing that $\langle \lambda \rangle_{\alpha, s} = \langle \lambda^\vee \rangle_{\alpha, -s^{-1}} = \langle \lambda^\vee \rangle_{-\alpha^{-1}, s}$, we get formulas for the specializations which are *symmetric* to the above ones (compare with [9]). For example, for $\alpha = s^{-2n-1}$, one has for a partition λ whose first part λ_1 is at most n , and whose transposed partition is $\lambda^\vee = (\lambda_1^\vee, \dots, \lambda_n^\vee)$,

$$\langle \lambda \rangle = (-1)^{|\lambda|} \prod_{j=1}^n \frac{[2n + 2 + 2\lambda_j^\vee - 2j]}{[2n + 2 - 2j]} \prod_{1 \leq i < j \leq n} \frac{[2n + 2 + \lambda_i^\vee - i + \lambda_j^\vee - j][\lambda_i^\vee - i - \lambda_j^\vee + j]}{[2n + 2 - i - j][j - i]}$$

Proof. Suppose that $\lambda = (\lambda_1, \dots, \lambda_n)$, and that μ is obtained from the Young diagram λ by removing one cell from the i th row, then we can write the recursive formula (21) as follows.

If $i = n$ and $\lambda_n = 1$, then

$$(22) \quad \frac{\langle \lambda \rangle}{\langle \mu \rangle} = \left(\frac{\alpha s^{2-2n} - \alpha^{-1} s^{-2+2n}}{s-s^{-1}} + 1 \right) \times \prod_{j < n} \frac{(\alpha s^{\lambda_n - n + \lambda_j - j + 1} - \alpha^{-1} s^{-\lambda_n + n - \lambda_j + j - 1})(s^{\lambda_n - n - \lambda_j + j} - s^{-\lambda_n + n + \lambda_j - j})}{(\alpha s^{\lambda_n - n + \lambda_j - j} - \alpha^{-1} s^{-\lambda_n + n - \lambda_j + j})(s^{\lambda_n - n - \lambda_j + j - 1} - s^{-\lambda_n + n + \lambda_j - j + 1})},$$

else

$$(23) \quad \frac{\langle \lambda \rangle}{\langle \mu \rangle} = \left(\frac{\alpha s^{2\lambda_i - 2i} - \alpha^{-1} s^{-2\lambda_i + 2i}}{s-s^{-1}} + 1 \right) \times \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{(\alpha s^{\lambda_i - i + \lambda_j - j + 1} - \alpha^{-1} s^{-\lambda_i + i - \lambda_j + j - 1})(s^{\lambda_i - i - \lambda_j + j} - s^{-\lambda_i + i + \lambda_j - j})}{(\alpha s^{\lambda_i - i + \lambda_j - j} - \alpha^{-1} s^{-\lambda_i + i - \lambda_j + j})(s^{\lambda_i - i - \lambda_j + j - 1} - s^{-\lambda_i + i + \lambda_j - j + 1})} \times \frac{(s-s^{-1})}{(\alpha s^{2\lambda_i - 2i - 1} - \alpha^{-1} s^{-2\lambda_i + 2i + 1})} \times \frac{(\alpha s^{-n + \lambda_i - i} - \alpha^{-1} s^{n - \lambda_i + i})}{(s^{n + \lambda_i - i} - s^{-n - \lambda_i + i})}$$

In the a) case, $\alpha = s^{2n}$, if $i = n$ and $\lambda_n = 1$, we get

$$\frac{\langle \lambda \rangle}{\langle \mu \rangle} = (s + 1 + s^{-1}) \prod_{1 \leq i < n} \frac{[n + \lambda_j - j + 2][1 - n - \lambda_j + j]}{[n + \lambda_j - j + 1][-n - \lambda_j + j]},$$

else

$$\frac{\langle \lambda \rangle}{\langle \mu \rangle} = \frac{[n + \lambda_i - i + 1/2]}{[n + \lambda_i - i - 1/2]} \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{[2n + \lambda_i - i + \lambda_j - j + 1][\lambda_i - i - \lambda_j + j]}{[2n + \lambda_i - i + \lambda_j - j][\lambda_i - i - \lambda_j + j - 1]}.$$

The announced result follows. In the b) case, $\alpha = s^{2n-1}$, if $i = n$ and $\lambda_n = 1$, we get

$$\frac{\langle \lambda \rangle}{\langle \mu \rangle} = 2 \prod_{1 \leq i < n} \frac{[n + \lambda_j - j + 1][1 - n - \lambda_j + j]}{[n + \lambda_j - j][-n - \lambda_j + j]},$$

else

$$\frac{\langle \lambda \rangle}{\langle \mu \rangle} = \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{[2n + \lambda_i - i + \lambda_j - j][\lambda_i - i - \lambda_j + j]}{[2n + \lambda_i - i + \lambda_j - j - 1][\lambda_i - i - \lambda_j + j - 1]}.$$

The formulas in b) follow. In the c) case, $\alpha = -s^{2n+1}$, if $i = n$ and $\lambda_n = 1$, we get

$$\frac{\langle \lambda \rangle}{\langle \mu \rangle} = -(s^2 + s^{-2}) \prod_{1 \leq i < n} \frac{[n + 3 + \lambda_j - j][1 - n - \lambda_j + j]}{[n + 2 + \lambda_j - j][-n - \lambda_j + j]},$$

else

$$\frac{\langle \lambda \rangle}{\langle \mu \rangle} = -\frac{[2n + 2 + 2\lambda_i - 2i]}{[2n + 2\lambda_i - 2i]} \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{[2n + 2 + \lambda_i - i + \lambda_j - j][\lambda_i - i - \lambda_j + j]}{[2n + 1 + \lambda_i - i + \lambda_j - j][\lambda_i - i - \lambda_j + j - 1]}.$$

The formula follows. □

8. FORMULAS FOR THE IDEMPOTENTS AND THE NON GENERIC CASE

We conclude this paper with a summary of conditions needed to define our minimal idempotents \tilde{p}_t and \tilde{y}_λ . This is of importance in the non generic case where α and s are roots of unity and some of quantum integers $[m]$ are non invertible.

Let μ is a Young diagram obtained from λ by removing one cell. From Corollary 5.3 we get a formula for the idempotent \tilde{y}_λ . Here we denote by $\hat{y}_\lambda \in K_{\square_\lambda}$ any lifting of $y_\lambda \in H_{\square_\lambda}$. We have

$$\hat{y}_\lambda(\tilde{y}_\mu \otimes \mathbf{1}_1)\hat{y}_\lambda = \sum_{\substack{\mu \subset \nu \\ |\nu|=|\lambda|}} \hat{y}_\lambda(\tilde{y}_\mu \otimes \mathbf{1}_1)\tilde{y}_\nu(\tilde{y}_\mu \otimes \mathbf{1}_1)\hat{y}_\lambda + \sum_{\substack{\nu \subset \mu \\ |\nu|=|\lambda|-2}} \hat{y}_\lambda\tilde{y}_{(\mu,\nu)}\hat{y}_\lambda .$$

Note that the first sum on the right hand side lives in the Hecke summand of K_{\square_λ} , hence in each term we can replace \hat{y}_λ by \tilde{y}_λ . By using Lemmas 3.3 and 3.4 we get:

$$(24) \quad \tilde{y}_\lambda = \hat{y}_\lambda(\tilde{y}_\mu \otimes \mathbf{1}_1)\hat{y}_\lambda - \sum_{\substack{\nu \subset \mu \\ |\nu|=|\lambda|-2}} \hat{y}_\lambda\tilde{y}_{(\mu,\nu)}\hat{y}_\lambda$$

From the above formula we obtain a minimal idempotent \tilde{y}_λ if the following three conditions are satisfied:

- the quantum integer $[m]$ is non zero for any $m < \lambda_1 + \lambda_1^\vee$;
- the idempotent \tilde{y}_μ is defined for *some* $\mu \subset \lambda$, $|\mu| = |\lambda| - 1$;
- the coefficient $\frac{\langle \mu \rangle}{\langle \nu \rangle}$ given in Theorem 7.1 is nonzero for any $\nu \subset \mu \subset \lambda$, $|\nu| = |\mu| - 1 = |\lambda| - 2$.

Let t be a standard tableau, with shapes $\lambda(t) = \lambda$, and $\lambda(t') = \mu$. Then from the general formula 3 for the section s_n ($n = |\lambda|$), we get

$$(25) \quad \tilde{p}_t = \tilde{p}_{t'}^+ \hat{p}_t \tilde{p}_{t'}^+ .$$

Here \hat{p}_t is any lifting in K_n of the path idempotent $p_t \in H_n$. We obtain a minimal idempotent \tilde{p}_t if the following three conditions are satisfied:

- the quantum integer $[m]$ is non zero for any $m < \lambda_1 + \lambda_1^\vee$;
- the idempotent \tilde{y}_μ is defined;
- the coefficient $\frac{\langle \mu \rangle}{\langle \nu \rangle}$ given in Theorem 7.1 is nonzero for any $\nu \subset \mu \subset \lambda$, $|\nu| = |\mu| - 1 = |\lambda| - 2$.

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