A COHOMOLOGICAL STABILITY RESULT FOR PROJECTIVE SCHEMES OVER SURFACES

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Abstract. Let $\pi : X \to X_0$ be a projective morphism of schemes such that X_0 is noetherian and essentially of finite type over a field K. Let $i \in \mathbb{N}_0$, let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $\mathcal{L}$ be an ample invertible sheaf over X. We show that the set $\mathrm{Ass}_{X_0}(\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))$ of associated points of the higher direct image sheaf $\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F})$ ultimately becomes constant if n tends to $-\infty$, provided X_0 has dimension ≤ 2. If $X_0 = \mathbb{A}^3_K$, this stability result need not hold any more.

To prove this, we show that the set $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ of associated primes of the n-th graded component of the i-th local cohomology module of a finitely generated graded module M over a homogeneous noetherian ring $R = \oplus_{n\geq 0} R_n$ which is essentially of finite type over a field becomes ultimately constant in codimension 2 if n tends to $-\infty$.

1. Introduction

Let $\mathfrak{a}$ be an ideal of the noetherian ring A and let M be a finitely generated A-module. Moreover, let i be a non-negative integer and let $H^i_{\mathfrak{a}}(M)$ be the i-th local cohomology module of M with respect to $\mathfrak{a}$. In 1992, Huneke [11] asked, whether the set $\mathrm{Ass}_A(H^i_{\mathfrak{a}}(M))$ of associated primes of the A-module $H^i_{\mathfrak{a}}(M)$ is finite.

If $M = A$ and A is a regular local ring which contains a field K, the previous finiteness question finds an affirmative answer. This was shown by Huneke-Sharp [12] for $\mathrm{char}(K) > 0$ and by Lyubeznik [16] for $\mathrm{char}(K) = 0$. For further and more detailed statements see also [17]. For certain local but non-regular rings, the finiteness of the sets $\mathrm{Ass}_A(H^i_{\mathfrak{a}}(A))$ is known, too (cf [10] or [18], for example). Moreover, without any restriction on A and M, the set $\mathrm{Ass}_A(H^i_{\mathfrak{a}}(M))$ is finite, provided the A-modules $H^j_{\mathfrak{a}}(M)$ are finitely generated for all $j < i$ (cf [6], [13]). This latter result has found various nice extensions (cf [15] for example).

On the other hand Singh ([18]) gave the following surprisingly simple example, for which the above finiteness question has a negative answer:

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Let $A = \mathbb{Z}[x,y,z,u,v,w]/(xu+yv+zw)$ (with indeterminates x,y,z,u,v,w) and let $\mathfrak{a} := (u,v,w)A$. Then $\mathrm{Ass}_{\mathbb{Z}}(H^3_{\mathfrak{a}}(A))$ is infinite – and hence so is $\mathrm{Ass}_A(H^3_{\mathfrak{a}}(A))$. Katzman [14] gave a similar example of a ring containing a field and later Singh-Swanson [20] developed a method which allows to construct a great variety of such examples.

From the point of view of projective schemes, the above finiteness question finds a natural refinement. Namely, let $R = \oplus_{n\geq 0} R_n$ be a homogeneous noetherian ring, so that the base ring R_0 is noetherian and R is generated over R_0 by finitely many elements of R_1. Moreover, let $R_+ := \oplus_{n>0} R_n$ denote the irrelevant ideal of R, let $i \geq 0$ and let M be a finitely generated graded R-module. For each $n \in \mathbb{Z}$ let $H^i_{R_+}(M)_n$ be the n-th graded component of the local cohomology module $H^i_{R_+}(M)$ of M with respect to R_+. Then the R_0-module $H^i_{R_+}(M)_n$ is finitely generated for all $n \in \mathbb{Z}$ and vanishes for all $n >> 0$. Now, instead of our previous finiteness question, we ask:

(1.0) *Does the (finite) set* $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ *ultimately become constant, if* $n \to -\infty$*?*

This question of the *Asymptotic Stability of Associated Primes* obviously plays a crucial rôle in the study of the *Asymptotic Behaviour of Cohomology* (cf [1]), that is of the R_0-modules $H^i_{R_+}(M)_n$ for $n << 0$. It is easy to see, that an affirmative answer to (1.0) yields the finiteness of the set $\mathrm{Ass}_R(H^i_{R_+}(M))$.

On the other hand we may furnish the ring A of Singh with the grading for which x,y,z have degree 0 and u,v,w have degree one. Then $A_0 = \mathbb{Z}[x,y,z]$, and if we choose a prime number p and localize at $(x,y,z,p)A_0$, we get a homogeneous noetherian domain $R = \oplus_{n\geq 0} R_n$ with $R_0 = \mathbb{Z}[x,y,z]_{(p,x,y,z)}$, and such that $\mathrm{Ass}_R(H^3_{R_+}(R))$ is finite, but $\mathrm{Ass}_{R_0}(H^3_{R_+}(R)_n)$ is not asymptotically stable for $n \to -\infty$ (cf [3], [5]). This shows that (1.0) cannot be answered affirmatively in general, even if the set $\mathrm{Ass}_R(H^i_{R_+}(M))$ is finite. Also, whenever R_0 is a polynomial ring in at least three indeterminates over a field, there is a homogeneous noetherian domain $R = R_0 \oplus R_1 \oplus \cdots$ such that the set $\mathrm{Ass}_{R_0}(H^2_{R_+}(R)_n)$ is not asymtotically stable for $n \to -\infty$, (cf [1], [2]).

Nevertheless, (1.0) is known to have an affirmative answer in the following cases:

a) $H^j_{R_+}(M)_n = 0$ for all $j < i$ and all $n << 0$ (cf [4]);
b) R_0 is essentially of finite type over a field and of dimension ≤ 1 (cf [2]).

In the present paper we show that (1.0) also has an affirmative answer if the base ring R_0 is of dimension 2 and essentially of finite type over a field. Observe that in view of the examples of Singh and Swanson [20] this is the best possible result one may expect, if only a bound on the dimension of the base ring R_0 is imposed.

In fact, we shall prove our result in a more general, geometric setting.

Namely, let X_0 be a noetherian scheme and let $\pi : X \to X_0$ be a projective scheme over X_0. Moreover, let $\mathcal{L}$ be an ample invertible sheaf over X and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, for each $i > 0$ and each $n \in \mathbb{Z}$, the higher direct image sheaf $\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F})$ is coherent over $\mathcal{O}_{X_0}$ and vanishes for all $n >> 0$.

We shall prove the following result on the asymptotic behaviour of the sheaves $\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F})$ in which $i \geq 0$ is fixed and n tends to $-\infty$ (cf Corollary 5.4):

(1.1) *Assume that X_0 is essentially of finite type over a field and of dimension ≤ 2. Then, for each $i \geq 0$ the set*

$$\mathrm{Ass}_{X_0}(\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))$$

of points $x_0 \in X_0$ associated to the higher direct image sheaf $\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F})$, ultimately becomes constant if $n \to -\infty$.

The restriction on the dimension of the base scheme X_0 may seem surprising, but in accordance with our introductory observations we can say (cf Example 5.7):

(1.2) *If X_0 is an affine* 3*-space over a field, the conclusion of* (1.1) *need not hold any more.*

The above statement (1.1) is a conclusion of more general stability result (cf Theorem 5.5):

(1.3) *Assume that X_0 is essentially of finite type over a field and let $i \in \mathbb{N}_0$. Then, the set*

$$\mathrm{Ass}_{X_0}(\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes \mathcal{F}))^{\leq 2}$$

of points $x_0 \in X_0$ which are associated to $\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes \mathcal{F})$ and of codimension ≤ 2 in X_0, ultimately becomes constant if $n \to -\infty$.

This latter result is a consequence of the following asymptotic stability result which holds for affine base schemes (cf Theorem 5.3):

(1.4) *Assume that X_0 is affine and essentially of finite type over a field. Let $i \geq 0$. Then, the set*

$$\mathrm{Ass}_{\mathcal{O}(X_0)}(H^i(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))^{\leq 2}$$

of primes of $\mathcal{O}(X_0)$ which are of height ≤ 2 and associated to the cohomology module $H^i(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F})$, ultimately becomes constant if $n \to -\infty$.

The proof of (1.4) is easy, if once the case of a very ample sheaf $\mathcal{L}$ is established (cf Proposition 5.2). This latter case follows from an asymptotic stability result on graded components of local cohomology modules.

Namely, let $R = \oplus_{n \geq 0} R_n$ and $R_+ := \oplus_{n>0} R_n$ be as above let $i \geq 0$ and let M be a finitely generated graded R-module. Then, using our previous notation we have the following stability result (cf Theorem 4.1):

(1.5) *Assume that R_0 is essentially of finite type over a field.*

Then, the set $\mathrm{Ass}_{R_0}\,(H^i_{R_+}(M)_n)^{\leq 2}$ *of primes* $\mathfrak{p}_0 \subset R_0$ *which are of height* ≤ 2 *and associated to the* R_0*-module* $H^i_{R_+}(M)_n$*, ultimately becomes constant if* $n \to -\infty$*.*

We shall prove this result in two steps: Before we establish the general case, we show that the requested asymptotic stability of associated primes holds if R_0 is in addition a domain (cf Proposition 3.5). This is an essential generalization of a similar but much weaker stability result given in [1].

Our paper is organized as follows: In Section 2 we give a few preliminaries. In Section 3 we prove our main result (1.5) under the hypothesis that R_0 is in addition a domain. In Section 4 we show that this latter hypotheses can be omitted. There we also shall explicitly give an example, essentially due to Singh-Swanson [20], which shows that (1.5) need not hold if R_0 is a three-variate polynomial ring over a field. In Section 5 we apply the results of Section 4 to projective schemes and establish our statements (1.1) - (1.4). For basic notions and notations from Algebraic Geometry and Commutative Algebra we refer to [9] resp. [8].

2. Preliminaries

By $\mathbb{N}$ we denote the set of positive integers, by $\mathbb{N}_0$ the set of non-negative integers.

2.1. **Notations and Conventions.** A) Throughout this paper let $R = R_0 \oplus R_1 \oplus \cdots$ be a *homogeneous noetherian ring*. So R is $\mathbb{N}_0$-graded, R_0 is noetherian and there are finitely many elements $a_1, \cdots, a_k \in R_1$ such that $R = R_0[a_1, \cdots, a_k]$.

By R_+ we denote the *irrelevant ideal* of R, thus $R_+ = R_1 \oplus R_2 \oplus \cdots$.

Polynomial rings over R_0 are always furnished with their standard grading, so that they are homogeneous.

B) If $i \in \mathbb{N}_0$ and M is a graded R-module, we write $H^i_{R_+}(M)$ for the *i-th local cohomology module of M with respect to R_+*, and we always furnish this module with its natural grading.

For $n \in \mathbb{Z}$ we denote by $H^i_{R_+}(M)_n$ *the n-th graded component* of $H^i_{R_+}(M)$.

Keep in mind that if the graded R-module M is finitely generated, the R_0-module $H^i_{R_+}(M)_n$ is finitely generated for all $n \in \mathbb{Z}$ and vanishes for all $n >> 0$. •

2.2. **Definition.** Let $\mathcal{S}$ be a set and let $(\mathcal{S}_n)_{n\in\mathbb{Z}}$ be a family of subsets of $\mathcal{S}$. We say that (the set) $\mathcal{S}_n$ is *asymptotically stable for* $n \to -\infty$ if there is some $n_0 \in \mathbb{Z}$ such that $\mathcal{S}_n = \mathcal{S}_{n_0}$ for all $n \leq n_0$. •

2.3. **Notation.** A) Let A be (commutative unitary) ring. If $\mathfrak{a} \subseteq A$ is an ideal, we denote the variety $\{\mathfrak{p} \in \operatorname{Spec}(A) \,|\, \mathfrak{a} \subseteq \mathfrak{p}\}$ of $\mathfrak{a}$ by $\operatorname{Var}(\mathfrak{a})$. In this situation we write $\min(\mathfrak{a})$ for the set of minimal members of $\operatorname{Var}(\mathfrak{a})$. $\operatorname{Max}(A)$ is used to denote the set of maximal ideals in A, whereas $\operatorname{Min}(A)$ is used to denote the set of minimal primes of A.

B) If $\mathcal{S} \subseteq \operatorname{Spec}(A)$ and $\ell \in \mathbb{N}_0$, we write

$$\mathcal{S}^{=\ell} := \{\mathfrak{p} \in \mathcal{S} \,|\, \operatorname{height}(\mathfrak{p}) = \ell\} \text{ and } \mathcal{S}^{\leq \ell} := \{\mathfrak{p} \in \mathcal{S} \,|\, \operatorname{height}(\mathfrak{p}) \leq \ell\}.$$

C) If T is an A-module we write $\operatorname{Ass}_A(T)$ for the set of associated primes of T in A.

Quite often we shall have to use the following two results on asymptotic stability of sets of associated primes:

2.4. **Proposition.** *Assume that R_0 is essentially of finite type over a field K (so that $R_0 = S^{-1}R_0'$, where $R_0' = K[a_1, \ldots, a_r]$ is a finitely generated K-algebra and $S \subseteq R_0'$ is multiplicatively closed). Let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module. Then, the set $Ass_{R_0}(H^i_{R_+}(M)_n)^{\leq 1}$ is asymptotically stable for $n \to -\infty$.*

Proof. See [2, Theorem 3.7]. ∎

2.5. **Proposition.** *Assume that R_0 is semilocal, essentially of finite type over a field and of dimension ≤ 2. Let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module.*

Then, the set $\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)$ is asymptotically stable for $n \to -\infty$.

Proof. See [2, Corollary 4.8]. ∎

In addition, the following two results will be used as technical tools.

2.6. **Proposition.** *Assume that R_0 is a domain and let M be a finitely generated graded R-module. Then, there is an element $x \in R_0 \setminus \{0\}$ such that the $(R_0)_x$-module $H^i_{R_+}(M)_x$ is torsion free (or vanishes) for all $i \in \mathbb{N}_0$.*

Proof. See [2, Theorem 2.5]. ∎

2.7. **Proposition.** *Assume that $(R_0, \mathfrak{m}_0)$ is local and of dimension ≤ 1. Then, for each $i \in \mathbb{N}_0$ and each finitely generated graded R-module M, the R-module $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ is artinian.*

Proof. See [3, Theorem 2.5 b)]. ∎

3. Integral Base Rings

3.1. **Lemma.** *Let $r \in \mathbb{N}$, let $(A, \mathfrak{m})$ be a noetherian local ring and let $x \in \mathfrak{m}$. Let T be a finitely generated A-module such that $\dim_A(T) \leq 2$ and $\dim_A(T/xT) \leq 1$. Assume that $x^r\Gamma_{\mathfrak{p}A_\mathfrak{p}}(T_\mathfrak{p}) = 0$ for each $\mathfrak{p} \in \operatorname{Ass}_A(T)$ with $\dim(A/\mathfrak{p}) = 1$. Assume in addition, that $\mathfrak{m} \in Ass_A(T)$. Then $\mathfrak{m} \in \operatorname{Ass}_A(T/x^{r+1}T)$.*

Proof. It suffices to show that $\Gamma_\mathfrak{m}(T) \nsubseteq x^{r+1}T$. As $\mathfrak{m} \in \operatorname{Ass}_A(T)$ we have $\Gamma_\mathfrak{m}(T) \neq 0$. Set

$$\mathcal{S} := \{\mathfrak{p} \in \operatorname{Ass}_A(T) \,\Big|\, \dim(A/\mathfrak{p}) = 1\}.$$

Assume first, that $\mathcal{S} = \emptyset$. As $\dim_A(T/xT) \leq 1$ and $\dim(T) \leq 2$ it follows that x avoids all members of $\operatorname{Ass}_A(T)\backslash\{\mathfrak{m}\}$, so that x is a non-zero divisor with respect to $T/\Gamma_\mathfrak{m}(T)$. We obtain $xT \cap \Gamma_\mathfrak{m}(T) = x(\Gamma_\mathfrak{m}(T) \underset{T}{:} x) = x\Gamma_\mathfrak{m}(T)$, hence by Nakayama $xT \cap \Gamma_\mathfrak{m}(T) \subsetneq \Gamma_\mathfrak{m}(T)$, thus $\Gamma_\mathfrak{m}(T) \nsubseteq xT$. Therefore $\Gamma_\mathfrak{m}(T) \nsubseteq x^{r+1}T$.

So, let $\mathcal{S} \neq \emptyset$. Choose $\mathfrak{p} \in \mathcal{S}$. Then $0 \underset{A}{:} T \subseteq \mathfrak{p}$ and $\Gamma_{\mathfrak{p}A_\mathfrak{p}}(T_\mathfrak{p}) \neq 0$. As $x^r\Gamma_{\mathfrak{p}A_\mathfrak{p}}(T_\mathfrak{p}) = 0$ it follows $x \in \mathfrak{p}$, thus $\mathfrak{p} \in \operatorname{Var}(xA + (0 \underset{A}{:} T))\backslash\{\mathfrak{m}\} = \operatorname{Supp}_A(T/xT)\backslash\{\mathfrak{m}\}$. Therefore $\mathcal{S} \subseteq \operatorname{Supp}_A(T/xT)\backslash\{\mathfrak{m}\}$.

Now, let $\mathfrak{p} \in \operatorname{Supp}_A(T/xT)\backslash\{\mathfrak{m}\}$. As $\dim_A(T/xT) \leq 1$ it follows that $\mathfrak{p}$ is generic in $\operatorname{Supp}_A(T/xT) = \operatorname{Var}(xA + (0 \underset{A}{:} T))$. Therefore $xA_\mathfrak{p} + (0 \underset{A\mathfrak{p}}{:} T_\mathfrak{p}) = (xA + (0 \underset{A}{:} T))_\mathfrak{p}$ is a $\mathfrak{p}A_\mathfrak{p}$-primary ideal of $A_\mathfrak{p}$, hence $\Gamma_{xA}(T)_\mathfrak{p} \cong \Gamma_{xA_\mathfrak{p}}(T_\mathfrak{p}) = \Gamma_{(xA_\mathfrak{p} + (0 \underset{A_\mathfrak{p}}{:} T_\mathfrak{p}))}(T_\mathfrak{p}) = \Gamma_{\mathfrak{p}A_\mathfrak{p}}(T_\mathfrak{p})$.

Assume first that $\mathfrak{p} \notin \operatorname{Ass}_R(T)$. Then $\Gamma_{\mathfrak{p}A_\mathfrak{p}}(T_\mathfrak{p}) = 0$ and hence $x^r\Gamma_{\mathfrak{p}A_\mathfrak{p}}(T_\mathfrak{p}) = 0$. If $\mathfrak{p} \in \operatorname{Ass}_R(T)$, then $\mathfrak{p} \in \mathcal{S}$, hence $x^r\Gamma_{\mathfrak{p}A_\mathfrak{p}}(T_\mathfrak{p})$. It follows that $x^r\Gamma_{xA}(T)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Supp}_A(T/xT)\backslash\{\mathfrak{m}\}$, hence $\operatorname{Supp}_A(x^r\Gamma_{xA}(T)_\mathfrak{p}) \subseteq \{\mathfrak{m}\}$. So, there is some $n \in \mathbb{N}$ with $\mathfrak{m}^n x^r \Gamma_{xA}(T) = 0$.

Assume now, that $\Gamma_\mathfrak{m}(T) \subseteq x^{r+1}T$. Let $t \in \Gamma_\mathfrak{m}(T)$. Then, there is some $s \in T$ with $t = x^{r+1}s$. Moreover $\Gamma_\mathfrak{m}(T) \subseteq \Gamma_{xA}(T)$ shows that $x^{r+1}s = t \in \Gamma_{xA}(T)$, so that $s \in \Gamma_{xA}(T)$. Consequently $\mathfrak{m}^n x^r s = 0$, whence $x^r s \in \Gamma_\mathfrak{m}(T)$, thus $t = x(x^r s) \in x\Gamma_\mathfrak{m}(T)$. This yields $\Gamma_\mathfrak{m}(T) \subseteq x\Gamma_\mathfrak{m}(T)$ and hence the contradiction that $\Gamma_\mathfrak{m}(T) = 0$. ■

3.2. **Lemma.** *Let $m \in \mathbb{Z}$, let R_0 be essentially of finite type over a field, let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module. Assume that*

$$\sharp \bigcup_{n \leq m} Ass_{R_0}(H^i_{R_+}(M)_n)^{=2} < \infty.$$

Then $Ass_{R_0}(H^i_{R_+}(M)_n)^{\leq 2}$ is asymptotically stable for $n \to -\infty$.

Proof. Observe that $\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2} = \operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 1} \dot\cup \operatorname{Ass}_{R_+}(H^i_{R_+}(M)_n)^{=2}$ for all $n \in \mathbb{Z}$. According to Proposition 2.4 the set $\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 1}$ is asymptotically stable for $n \to -\infty$. Therefore, the set $\mathcal{S} := \bigcup_{n \leq m} \operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2}$ is finite.

According to Proposition 2.5 the set $\mathcal{T}_n^{(\mathfrak{p}_0)} := \mathrm{Ass}_{(R_0)_{\mathfrak{p}_0}}(H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n)$ is asymptotically stable for $n \to -\infty$ and all $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)^{\leq 2}$. As $H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n \cong (H^i_{R_+}(M)_n)_{\mathfrak{p}_0}$, we have $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n) = \bigcup_{\mathfrak{p}_0 \in \mathcal{S}}\{\mathfrak{q}_0 \cap R_0 \,|\, \mathfrak{q}_0 \in \mathcal{T}_n^{(\mathfrak{p}_0)}\}$ and this gives our claim. ■

3.3. **Lemma.** *Let $B = \oplus_{n\geq 0} B_n$ be a positively graded ring such that $(B_0, \mathfrak{n}_0)$ is noetherian local and let $G = \oplus_{n\in\mathbb{Z}} G_n$ be a graded B-module which is artinian and such that the B_0-module G_n is finitely generated for all $n \in \mathbb{Z}$. Then, there is some $r \in \mathbb{N}$ such that $\mathfrak{n}_0^r G = 0$.*

Proof. For each $n \in \mathbb{Z}$, the B_0-module $G_n \cong G_{\geq n}/G_{\geq(n+1)}$ is artinian and finitely generated. As G is artinian, there is some $r \in \mathbb{N}$ with $\mathfrak{n}_0^r G = \mathfrak{n}_0^{r+1} G$. So, for each $n \in \mathbb{Z}$ we have $\mathfrak{n}_0^r(G_n) = (\mathfrak{n}_0^r G)_n = (\mathfrak{n}_0^{r+1} G)_n = \mathfrak{n}_0^{r+1}(G_n) = \mathfrak{n}_0(\mathfrak{n}_0^r(G_n))$, thus $\mathfrak{n}_0^r G_n = 0$ by Nakayama. ■

3.4. **Lemma.** *Let $k, \ell, m \in \mathbb{Z}$ be such that $\ell \geq 0$ and $k > 0$. Let $x_1, \cdots, x_k$ be indeterminates and assume that there is a surjective homomorphism of graded rings*

$$R_0[\underline{x}] := R_0[x_1, \cdots, x_k] \twoheadrightarrow R.$$

Assume that

$$\sharp \bigcup_{n\leq m} \mathrm{Ass}_{R_0}(H^k_{R_0]\underline{x}]_+}(N)_n)^{\leq \ell} < \infty$$

for all finitely generated graded $R_0[\underline{x}]$-modules N.

Then, for all finitely generated graded R-modules M and all $i \in \mathbb{N}_0$ we have

$$\sharp \bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq \ell} < \infty.$$

Proof. By the graded base ring independence property of local cohomology we may assume that $R = R_0[\underline{x}]$. For $i > k$ we have $H^i_{R_+}(M) = 0$ and there is nothing to show. For $i = k$ our claim follows as $H^i_{R_+}(M)_n = 0$ for all $n >> 0$. So, let $i < k$. There is an exact sequence of finitely generated graded R-modules

$$0 \to N \to F \to M \to 0$$

in which $F = \oplus_{j=1}^r R(a_j)$ is a graded free R-module with $a_j \in \mathbb{Z}$ for $j = 1, \cdots, r$. As $R = R_0[\underline{x}]$ we have $H^i_{R_+}(F) = 0$, so that for each $n \in \mathbb{Z}$ there is a monomorphism of R_0-modules $0 \to H^i_{R_+}(M)_n \to H^{i+1}_{R_+}(N)_n$ and hence $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq \ell} \subseteq \mathrm{Ass}_{R_0}(H^{i+1}_{R_+}(N)_n)^{\leq \ell}$. Now, we may conclude by descending induction on i. ■

3.5. **Proposition.** *Let R_0 be a domain which is essentially of finite type over a field, let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module.*

Then $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2}$ is asymptotically stable for $n \to -\infty$.

Proof. According to Lemma 3.2 it suffices to show that the set

$$\bigcup_{n\leq m} \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{=2} =: \mathcal{W}^i_m$$

is finite for some $m \in \mathbb{Z}$. To prove this, we use Lemma 3.4 and restrict ourselves to the case where $R = R_0[\underline{x}] = R_0[x_1, \cdots, x_k]$ is a polynomial ring and where $i = k$. According to Proposition 2.6 there is an element $x \in R_0\setminus\{0\}$ such that the $(R_0)_x$-module $H^k_{R_+}(M)_x$ is torsion-free (or vanishes). Therefore

$$\mathrm{Ass}_{R_0}(H^k_{R_+}(M)_n)\setminus\{0\} \subseteq \mathrm{Var}(xR_0),\ \forall n \in \mathbb{Z}.$$

According to Proposition 2.4 the set $\mathrm{Ass}_{R_0}(H^k_{R_+}(M)_n)^{\leq 1}$ is asymptotically stable for $n \to -\infty$. So, there is an integer m and a finite set $U \subseteq \mathrm{Var}(xR_0)$ such that

$$\mathrm{Ass}_{R_0}(H^k_{R_+}(M)_n)^{=1} = U,\ \forall n \leq m.$$

Now, let $\mathfrak{p}_0 \in U$. Then the graded $R_{\mathfrak{p}_0}$-module $\Gamma_{\mathfrak{p}_0 R}(H^k_{R_+}(M))_{\mathfrak{p}_0} \cong \Gamma_{\mathfrak{p}_0 R_{\mathfrak{p}_0}}(H^k_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0}))$ is artinian by Proposition 2.7. Moreover, for each $n \in \mathbb{Z}$ the n-th graded component $\Gamma_{\mathfrak{p}_0}(H^k_{R_+}(M)_n)_{\mathfrak{p}_0}$ of this module is finitely generated over $(R_{\mathfrak{p}_0})_0 = (R_0)_{\mathfrak{p}_0}$. As $x \in \mathfrak{p}_0$ we thus find a positive integer $r(\mathfrak{p}_0)$ such that $x^{r(\mathfrak{p}_0)}\Gamma_{\mathfrak{p}_0}(H^k_{R_+}(M))_{\mathfrak{p}_0} = 0$ (cf Lemma 3.3). Thus

$$x^{r(\mathfrak{p}_0)}\Gamma_{\mathfrak{p}_0}(H^k_{R_+}(M)_n)_{\mathfrak{p}_0} = 0,\ \forall n \in \mathbb{Z}.$$

Now, set $r := 1$ if $U = \emptyset$, and $r := \max\{r(\mathfrak{p}_0) \,|\, \mathfrak{p}_0 \in U\}$ otherwise.

By the base ring independence of local cohomology we have isomorphisms of R_0-modules

$$H^k_{R_+}(M/x^{r+1}M)_n \cong H^k_{(R/x^{r+1}R)_+}(M/x^{r+1}M)_n$$

which yield that

$$\begin{aligned}&\mathrm{Ass}_{R_0}(H^k_{R_+}(M/x^{r+1}M)_n)^{\leq 2}\\ &= \{\mathfrak{p}_0 \in \mathrm{Var}(xR_0) \,|\, \mathfrak{p}_0/x^{r+1}R_0 \in \mathrm{Ass}_{R_0/x^{r+1}R_0}(H^k_{(R/x^{r+1}R)_+}(M/x^{r+1}M)_n)^{\leq 1}\}.\end{aligned}$$

As the set $\mathrm{Ass}_{R_0/xR_0}(H^k_{(R/x^{r+1}R)_+}(M/x^{r+1}M)_n)^{\leq 1}$ is asymptotically stable for $n \to -\infty$ (cf Proposition 2.4), it follows that the set

$$\mathcal{T} := \bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{R_0}(H^k_{R_+}(M/x^{r+1}M)_n)^{\leq 2}$$

is finite.

Now, let $n \leq m$ and let $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(H^k_{R_+}(M)_n)^{=2}$. Then $(A, \mathfrak{m}) := ((R_0)_{\mathfrak{p}_0}, \mathfrak{p}_0(R_0)_{\mathfrak{p}_0})$ is a noetherian local domain of dimension 2 with $x \in \mathfrak{m}\setminus\{0\}$ and $\mathfrak{m}$ is associated to $T := (H^k_{R_+}(M)_n)_{\mathfrak{p}_0}$. Moreover, $\dim_A(T/xT) \leq \dim(A/xA) = 1$. In addition, if $\mathfrak{p} \in \mathrm{Ass}_A(T)$ with $\dim(A/\mathfrak{p}) = 1$, we may write $\mathfrak{p} = \mathfrak{q}_0 A$ with $\mathfrak{q}_0 := \mathfrak{p} \cap R_0 \in \mathrm{Ass}_{R_0}(H^k_{R_+}(M)_n)^{=1} \subseteq U$. Therefore

$$x^r\Gamma_{\mathfrak{p}A_{\mathfrak{p}}}(T_{\mathfrak{p}}) = x^r\Gamma_{\mathfrak{q}_0(R_0)_{\mathfrak{q}_0}}(H^k_{R_+}(M)_n)_{\mathfrak{q}_0} = 0.$$

According to Lemma 3.1 we thus get $\mathfrak{m} \in \operatorname{Ass}_A(T/x^{r+1}T)$, whence

$$\mathfrak{p}_0 \in \operatorname{Ass}_{R_0}(H^k_{R_+}(M)_n/x^{r+1}H^k_{R_+}(M)_n)^{=2}.$$

Now, we consider the following commutative diagram of graded R-modules with exact first row in which π is the canonical epimorphism.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M/(0 \underset{M}{:} x^{r+1}) & \longrightarrow & M & \longrightarrow & M/x^{r+1}M & \longrightarrow & 0 \\
 & & {\scriptstyle\pi}\uparrow & & \| & & & & \\
 & & M & \xrightarrow{x^{r+1}} & M & & & &
\end{array}$$

As $H^{k+1}_{R_+}(\operatorname{Ker}(\pi)) = 0$ we thus get a commutative diagram of R_0-modules with exact first row and an epimorphism $\psi := H^k_{R_+}(\pi)$

$$\begin{array}{ccccc}
H^k_{R_+}(M/(0 \underset{M}{:} x^{r+1}))_n & \xrightarrow{\varphi} & H^k_{R_+}(M)_n & \longrightarrow & H^k_{R_+}(M/x^{r+1}M)_n \\
{\scriptstyle\psi}\uparrow & & \| & & \\
H^k_{R_+}(M)_n & \xrightarrow{x^{r+1}} & H^k_{R_+}(M)_n & &
\end{array}$$

It follows that $\operatorname{Im}(\varphi) = x^{r+1}H^k_{R_+}(M)_n$, so that there is a monomorphism

$$0 \to H^k_{R_+}(M)_n/x^{r+1}H^k_{R_+}(M)_n \to H^k_{R_+}(M/x^{r+1}M)_n.$$

Therefore $\mathfrak{p}_0 \in \operatorname{Ass}_{R_0}(H^k_{R_+}(M/x^{r+1}M)_n)^{=2} \subseteq \mathcal{T}$. So, $\mathcal{W}^k_m \subseteq \mathcal{T}$. As $\mathcal{T}$ is finite, this proves our claim. ∎

4. General Base Rings

4.1. **Theorem.** *Let R_0 be essentially of finite type over a field. Let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module.*

Then, the set $\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2}$ is asymptotically stable for $n \to -\infty$.

Proof. By the graded base ring independence property of local cohomology we may assume that $R = R_0[\underline{x}] = R_0[x_1, \cdots, x_k]$ is a polynomial ring. Moreover, we may write $R_0 = S_0^{-1}A_0$, where A_0 is a subring of R_0 which is of finite type over a field K, and where $S_0^{-1} \subseteq A_0$ is multiplicatively closed. Let $m_1, \dots, m_s \in M$ be homogeneous elements such that $M = \sum_{j=1}^s Rm_i$. As $A_0[\underline{x}]$ is a subring of $R_0[\underline{x}]$ we may consider the finitely generated graded $A_0[\underline{x}]$-module $N := \sum_{j=1}^s A_0[\underline{x}]m_j$. As $R = S_0^{-1}A_0[\underline{x}]$

and $M = S_0^{-1}N$, the graded flat base change property of local cohomology yields an isomorphism of R_0-modules

$$H^i_{R_+}(M)_n = H^i_{(S_0^{-1}A_0[\underline{x}])_+}(S_0^{-1}N)_n \cong S_0^{-1}H^i_{A_0[\underline{x}]_+}(N)_n$$

for each $n \in \mathbb{Z}$. So, for each $n \in \mathbb{Z}$ we have

$$\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2} = \{\mathfrak{q}_0 R_0 \,\big|\, \mathfrak{q}_0 \in \mathrm{Ass}_{A_0}(H^i_{A_0[\underline{x}]_+}(N)_n)^{\leq 2} \text{ and } \mathfrak{q}_0 \cap S_0 = \emptyset\}.$$

Thus it suffices to show that $\mathrm{Ass}_{A_0}(H^i_{A_0[\underline{x}]_+}(N)_n)^{\leq 2}$ is asymptotically stable for $n \to -\infty$. Therefore we may assume that R_0 is of finite type over the field K.

We do this by induction on the dimension deficiency

$$\delta = \delta(R_0) := \dim(R_0) - \min\{\dim(R_0/\mathfrak{q}_0) \,\big|\, \mathfrak{q}_0 \in \mathrm{Min}(R_0)\}.$$

Assume first that $\delta = 0$, so that $\dim(R_0/\mathfrak{q}_0) = \dim(R_0)$ for each $\mathfrak{q}_0 \in \mathrm{Min}(R_0)$. Let $d := \dim(R_0)$. By the Normalization Lemma there is a polynomial ring $B_0 := K[y_1, \cdots, y_d] \subseteq R_0$ such that R_0 is a finite integral extension of B_0. Moreover $\mathfrak{q}_0 \cap B_0 = 0$ for each $\mathfrak{q}_0 \in \mathrm{Min}(R_0)$ so that

$$\mathrm{height}(\mathfrak{p}_0 \cap B_0) = \mathrm{height}(\mathfrak{p}_0) \text{ for all } \mathfrak{p}_0 \in \mathrm{Spec}(R_0).$$

As $R = R_0[\underline{x}]$ is a finite integral extension of $B := B_0[\underline{x}]$, we see that M is a finitely generated graded B-module. So, according to Proposition 3.5 the set $\mathrm{Ass}_{B_0}(H^i_{B_+}(M)_n)^{\leq 2}$ is asymptotically stable for $n \to -\infty$. In particular, the set

$$\mathcal{W}' := \bigcup_{n \in \mathbb{Z}} \mathrm{Ass}_{B_0}(H^i_{B_+}(M)_n)^{=2}$$

is finite. As R_0 is a finite integral extension of B_0, it follows that the set

$$\mathcal{W} := \{\mathfrak{p}_0 \in \mathrm{Spec}(R_0) \,\big|\, \mathfrak{p}_0 \cap B_0 \in \mathcal{W}'\}$$

is finite.

Now, let $n \in \mathbb{Z}$ and let $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{=2}$. By the graded base ring independence property of local cohomology there is an isomorphism of B_0-modules $H^i_{R_+}(M)_n \cong H^i_{B_+}(M)_n$. Therefore $\mathfrak{p}_0 \cap B_0 \in \mathrm{Ass}_{B_0}(H^i_{B_+}(M)_n)$. As $\mathrm{height}(\mathfrak{p}_0 \cap B_0) = \mathrm{height}(\mathfrak{p}_0) = 2$ it follows that $\mathfrak{p}_0 \in \mathcal{W}$. So, the set $\bigcup_{n \in \mathbb{Z}} \mathrm{Ass}_{R_0}(H^i_{R_+}(M))^{=2}$ is finite and hence by Lemma 3.2 the set $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2}$ is asymptotically stable for $n \to -\infty$.

Next, let $\delta = \delta(R_0) > 0$. We write $\mathfrak{q}_0^{(1)}, \mathfrak{q}_0^{(2)}, \cdots, \mathfrak{q}_0^{(r)}$ for the different minimal primes of R_0, assuming that $\dim(R_0/\mathfrak{q}_0^{(j)}) = \dim(R_0)$ for $j = 1, \cdots, s$ and $\dim(R_0/\mathfrak{q}_0^{(\ell)}) < \dim(R_0)$

for $\ell = s+1, \cdots, r$ for some $s \in \{1, \cdots, r-1\}$. By prime avoidance we find elements

$$f_1, \cdots, f_p \in \bigcap_{j=1}^{s} \mathfrak{q}_0^{(j)} \setminus \bigcup_{\ell=s+1}^{r} \mathfrak{q}_0^{(\ell)}$$
$$g_1, \cdots, g_q \in \bigcap_{\ell=s+1}^{r} \mathfrak{q}_0^{(\ell)} \setminus \bigcup_{j=1}^{s} \mathfrak{q}_0^{(j)}$$

such that

$$\mathfrak{a}_0 := \bigcap_{j=1}^{s} \mathfrak{q}_0^{(j)} = (f_1, \cdots, f_p) \text{ and } \mathfrak{b}_0 := \bigcap_{\ell=s+1}^{r} \mathfrak{q}_0^{(\ell)} = (g_1, \cdots, g_q).$$

It follows

$$\bigcup_{\mu=1}^{p} \operatorname{Spec}(R_0)_{f_\mu} \cup \bigcup_{\nu=1}^{q} \operatorname{Spec}(R_0)_{f_\nu} = \operatorname{Spec}(R_0) \setminus \operatorname{Var}(\mathfrak{a}_0 + \mathfrak{b}_0).$$

Let $\mathfrak{p}_0 \in \operatorname{Var}(\mathfrak{a}_0 + \mathfrak{b}_0)$. Then, there are indices $j \in \{1, \cdots, s\}$ and $\ell \in \{s+1, \cdots, r\}$ such that $\mathfrak{q}_0^{(j)} \subseteq \mathfrak{p}_0$ and $\mathfrak{q}_0^{(\ell)} \subseteq \mathfrak{p}_0$. As R_0 is of finite type over a field, it follows $\operatorname{height}(\mathfrak{p}_0) = \operatorname{height}(\mathfrak{p}_0/\mathfrak{q}_0^{(j)}) = \dim(R_0/\mathfrak{q}_0^{(j)}) - \dim(R_0/\mathfrak{p}_0) > \dim(R_0/\mathfrak{q}_0^{(\ell)}) - \dim(R_0/\mathfrak{p}_0) = \operatorname{height}(\mathfrak{p}_0/\mathfrak{q}_0^{(\ell)}) > 0$, so that $\operatorname{height}(\mathfrak{p}_0) \geq 2$. This shows that the set $\operatorname{Var}(\mathfrak{a}_0 + \mathfrak{b}_0)^{=2}$ consists of minimal primes of $\mathfrak{a}_0 + \mathfrak{b}_0$ and hence is finite. Consequently the set

$$\mathcal{S} := \operatorname{Spec}(R_0)^{=2} \setminus \Big[\bigcup_{\mu=1}^{p} \operatorname{Spec}(R_0)_{f\mu} \cup \bigcup_{\nu=1}^{q} \operatorname{Spec}(R_0)_{f\nu}\Big]$$

is finite.

Now, observe that

$$\operatorname{Min}((R_0)_{f_\mu}) = \{(\mathfrak{q}_0^{(\ell)})_{f_\mu} \,\big|\, \ell = s+1, \cdots, r\}, \quad (\mu = 1, \cdots, p);$$
$$\operatorname{Min}((R_0)_{g_\nu}) = \{(\mathfrak{q}_0^{(j)})_{g_\nu} \,\big|\, j = 1, \cdots, s\}, \qquad (\nu = 1, \cdots, q).$$

Using again that R_0 is of finite type over a field we thus get

$$\delta((R_0)_{f_\mu}) = \max_{\ell=s+1}^{r}\{\dim(R_0/\mathfrak{q}_0^{(\ell)})\} - \min_{\ell=s+1}^{r}\{\dim(R_0/\mathfrak{q}_0^{(\ell)})\}$$
$$< \dim(R_0) - \min_{t=1}^{r}\{\dim(R/\mathfrak{q}_0^{(t)})\} = \delta(R_0)$$

for $\mu = 1, \cdots, p$. Moreover it follows that $\delta((R_0)_{g_\nu}) = 0$ for $\nu = 1, \cdots, q$.

So by induction, for all $\mu \in \{1, \cdots, p\}$ and all $\nu \in \{1, \cdots, q\}$, the sets

$$\operatorname{Ass}_{(R_0)_{f_\mu}}(H^i_{(R_{f_\mu})_+}(M_{f_\mu})_n)^{\leq 2} \text{ and } \operatorname{Ass}_{(R_0)_{g_\nu}}(H^i_{(R_{g_\nu})_+}(M_{g_\nu})_n)^{\leq 2}$$

are asymptotically stable for $n \to -\infty$. By the graded flat base change property of local cohomology it follows easily that the sets

$$\mathcal{W}_\mu := \bigcup_{n\in\mathbb{Z}} \big[\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{=2} \cap \ \mathrm{Spec}(R_0)_{f_\mu}\big]$$

$$\mathcal{V}_\nu := \bigcup_{n\in\mathbb{Z}} \big[\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{=2} \cap \ \mathrm{Spec}(R_0)_{g_\nu}\big]$$

are finite for all $\mu \in \{1, \cdots, p\}$ and all $\nu \in \{1, \cdots, q\}$.

As $\bigcup_{n\in\mathbb{Z}} \ \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{=2} \subseteq \bigcup_{\mu=1}^p \mathcal{W}_\mu \cup \bigcup_{\nu=1}^q \mathcal{V}_\nu \cup \mathcal{S}$ our claim follows by Lemma 3.2. ■

4.2**. Corollary.** *Let R_0, i and M as in Theorem 4.1, and let* $\dim(R_0) \leq 2$.
Then, the set $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ *is asymptotically stable for* $n \to -\infty$.

4.3**. Example.** (cf [20, Remark 4.2], [1, Example (7.5)]) Let K be a field and let x, y, z, u, v be indeterminates. Let $R_0 = K[x, y, z]$ and consider the homogeneous noetherian R_0-algebra

$$R := R_0[u, v]/(y^2u^2 + xyzuv + z^2v^2).$$

Then, the set $\bigcup_{n\leq 0} \ \mathrm{Ass}_{R_0}(H^2_{R_+}(R)_n)$ is infinite, so that the set $\mathrm{Ass}_{R_0}(H^2_{R_+}(R)_n)$ is not asymptotically stable for $n \to -\infty$.

This shows, that the conclusion of Corollary 4.2 need not hold if $\dim(R_0) \geq 3$. •

5. Applications to Ample Divisors

5.1**. Notations and Conventions.** A) Throughout this section let X_0 denote a noetherian scheme and let $\pi : X \to X_0$ be a projective scheme over X with very ample sheaf $\mathcal{O}_X(1)$.

B) Let $\ell \in \mathbb{N}_0$. If Y is a noetherian scheme and if $Z \subseteq Y$, we set

$$Z^{\leq \ell} := \{z \in Z \,\big|\, \ \mathrm{codim}_Y(z) \leq \ell\},$$

where $\mathrm{codim}_Y(z) = \dim(\mathcal{O}_{Y,z})$ denotes the codimension of the (closure of the) point $z \in Y$.

C) If Y is a scheme and $\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-modules, we write $\mathrm{Ass}_Y(\mathcal{F})$ for the set $\{y \in Y \,\big|\, \mathfrak{m}_{Y,y} \in \mathrm{Ass}_{\mathcal{O}_{Y,y}}(\mathcal{F}_y)\}$ of points in Y which are associated to $\mathcal{F}$.

D) The symbol $\tilde{\cdot}$ is used to denote induced sheaves. •

5.2. **Proposition.** *Assume that X_0 is affine and essentially of finite type over a field. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$.*

Then, the set $\mathrm{Ass}_{\mathcal{O}(X_0)}(H^i(X, \mathcal{F}(n)))^{\leq 2}$ is asymptotically stable for $n \to -\infty$.

Proof. Let $R_0 := \mathcal{O}(X_0)$. Then there is a homogeneous noetherian R_0-algebra $R = \oplus_{n \geq 0} R_n$ such that $X_0 = \mathrm{Spec}(R_0), X = \mathrm{Proj}(R)$ and $\mathcal{O}_X(1) = R(1)^{\sim}$. Moreover there is a finitely generated graded R-module M such that $\mathcal{F} = \tilde{M}$. For each $n \in \mathbb{Z}$ the Serre-Grothendieck correspondence yields a short exact sequence of R_0-modules

$$0 \to H^0_{R_+}(M)_n \to M_n \to H^0(X, \mathcal{F}(n)) \to H^1_{R_+}(M)_n \to 0$$

and isomorphisms of R_0-modules

$$H^j(X, \mathcal{F}(n)) \cong H^{j+1}_{R_+}(M)_n \text{ for all } j > 0.$$

Now, we conclude by Theorem 4.1. ■

5.3. **Theorem.** *Assume that X_0 is affine and essentially of finite type over a field. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules, let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$.*

Then, the set $\mathrm{Ass}_{\mathcal{O}(X_0)}(H^i(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))^{\leq 2}$ is asymptotically stable for $n \to -\infty$.

Proof. There is an integer $r_0 \in \mathbb{N}$ such that $\mathcal{L}^r$ is very ample for each $r > r_0$. Fix such an r and let $t \in \{0, \cdots, r-1\}$. If we apply Proposition 5.2 with $\mathcal{L}^r$ instead of $\mathcal{O}_X(1)$ and $\mathcal{L}^t \otimes_{\mathcal{O}_X} \mathcal{F}$ instead of $\mathcal{F}$ we find an integer n_t and a set $\mathcal{S}_t \subseteq \mathrm{Spec}(\mathcal{O}(X_0))$ such that

$$\mathrm{Ass}_{\mathcal{O}(X_0)}(H^i(X, \mathcal{L}^{nr+t} \otimes_{\mathcal{O}_X} \mathcal{F}))^{\leq 2} = \mathcal{S}_t \text{ for all } n \leq n_t.$$

Choosing a second integer $s > r_0$ we find some $m \in \mathbb{Z}$ and some set $\mathcal{T} \subseteq \mathrm{Spec}(\mathcal{O}(X_0))$ such that

$$\mathrm{Ass}_{\mathcal{O}(X_0)}(H^i(X, \mathcal{L}^{ns} \otimes \mathcal{F}))^{\leq 2} = \mathcal{T} \text{ for all } n \leq m.$$

Choosing s such that it has no common divisor with r, we find integers $n \leq n_t$ and $n' \leq m$ such that $nr + t = n's$. This shows that $\mathcal{S}_t = \mathcal{T}$ for $t = 0, \cdots, r-1$. From this our claim follows immediately. ■

5.4. **Corollary.** *Let $X_0, \mathcal{F}, \mathcal{L}$ and i be as in Theorem 5.3, and let* $\dim(X_0) \leq 2$.

Then, the set $\mathrm{Ass}_{\mathcal{O}(X_0)}(H^i(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F})$ is asymptotically stable for $n \to -\infty$. ■

5.5. **Theorem.** *Let X_0 be essentially of finite type over a field. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$.*

Then, the set $\mathrm{Ass}_{X_0}(\mathcal{R}^i \pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))^{\leq 2}$ is asymptotically stable for $n \to -\infty$.

Proof. The result is local in the base. So we may assume that X_0 is affine. Now we conclude by Theorem 5.3 as $\mathcal{R}^i \pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}) \cong H^i(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F})^{\sim}$.

5.6. **Corollary.** *Let $X_0, \mathcal{F}, \mathcal{L}$ and i be as in Theorem 5.5, and let* $\dim(X_0) \leq 2$. *Then, the set* $\mathrm{Ass}_{X_0}(\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))$ *is asymptotically stable for* $n \to -\infty$. ■

5.7. **Example.** Let R_0 and R be as in Example 4.3, so that $X_0 := \mathrm{Spec}(R_0)$ is the affine 3-space $\mathbb{A}^3_K$ and $X = \mathrm{Proj}(R) \subseteq \mathbb{P}^1_{X_0}$ is the projective line over $\mathbb{A}^3_K$. Then the Serre-Grothendieck Correspondence (cf Proof of Proposition 5.2) yields, that the set

$$\bigcup_{n \leq 0} \mathrm{Ass}_{\mathcal{O}(X_0)}(H^1(X, \mathcal{O}_X(n))) \text{ is not finite .}$$

Therefore the set

$$\mathrm{Ass}_{\mathcal{O}(X_0)}(H^1(X, \mathcal{O}_X(n))) = \mathrm{Ass}_{X_0}(\mathcal{R}^1\pi_*(\mathcal{O}_X(1)^n))$$

is not asymptotically stable for $n \to -\infty$. This shows, that in a surprisingly simple situation the conclusions of Corollaries 5.4 and 5.6 need not hold over a base scheme X_0 of dimension ≥ 3. •

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