FOUR LECTURES ON LOCAL COHOMOLOGY

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Third Lecture: Finiteness Results

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Introduction

These notes grew out of four introductory lectures on Local Cohomology, held at the

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The aim of these lectures was to give a first introduction to Local Cohomology, encouraging the audience to penetrate further in the subjects along the lines of the lecture notes [B] and [B-F] and the textbook [B-S].

In particular we suggest to the reader, which is not familiar yet with the subject, to consult in a next step the lecture notes [B-F], which are available as PDF. For those readers, who have already a more extended background in Commutative Algebra, we suggest to go on directly with [B-S].

Concerning Commutative Algebra we tried to use as far as possible only things which were treated at the Workshop by introductory lectures, notably: Basics of noetherian rings and modules, associated primes, Krull dimension, polynomial rings, localization, completion, graded rings and modules.

We also did use a number of basic results from Algebraic Geometry without giving their proves. For these results we recommend as a reference [H] or [R]. Moreover, we did state and use a number of results on Local Cohomology whose proves are found in [B-F] or in [B-S].

We also added a number of examples (which in full detail are treated in [B-S]) in order to illustrate the main results in concrete situations.

As basic references in Commutative Algebra we suggest [E], [M] or [S]. Moreover we added a few basic references to each of the four lectures in our final bibliography.

Finally, we express our very best thanks to the organizers of the Workshop and the Sisters of the Congregation of the Holy Family for their kind and generous hospitality during and after the workshop.

First Lecture: Torsion and Local Cohomology

1 Torsion Functors

Notation 1.1 Let R be a noetherian ring and let $\mathfrak{a} \subseteq R$ be an ideal. For an R-module M and a submodule $N \subseteq M$ let

$$\big(N \underset{M}{:} \mathfrak{a}\big) := \big\{m \in M \,\big|\, am \in M,\ \forall a \in \mathfrak{a}\big\}.$$

Observe, that $N \underset{M}{:} \mathfrak{a}$ is a submodule of M and that $N \subseteq N \underset{M}{:} \mathfrak{a}$. •

Definition 1.2 *The $\mathfrak{a}$-torsion submodule of an R-module M is defined by*

$$\Gamma_{\mathfrak{a}}(M) := \bigcup_{n \in \mathbb{N}} \big(0 \underset{M}{:} \mathfrak{a}^n\big) = \big\{m \in M \,\big|\, \exists n \in \mathbb{N} : \mathfrak{a}^n m = 0\big\}.$$

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Remark and Exercises 1.3 A) Let $\mathfrak{a}, \mathfrak{b} \subseteq R$ ideals and let M be an R-module. Then:

a) $\Gamma_0(M) = M, \Gamma_R(M) = 0$;

b) $\mathfrak{a} \subseteq \mathfrak{b} \Longrightarrow \Gamma_{\mathfrak{b}}(M) \subseteq \Gamma_{\mathfrak{a}}(M)$;

c) $\Gamma_{\sqrt{\mathfrak{a}}}(M) = \Gamma_{\mathfrak{a}}(M)$;

d) $\Gamma_{\mathfrak{a}+\mathfrak{b}}(M) = \Gamma_{\mathfrak{b}}(M) \cap \Gamma_{\mathfrak{b}}(M)$.

B) Moreover

a) If $h : M \to N$ is a homomorphism of R-modules, then $h(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$;

b) $\Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$.

C) Finally, if the R-module M is finitely generated we can say:

a) $\exists n \in \mathbb{N} : \mathfrak{a}^n \Gamma_{\mathfrak{a}}(M) = 0$;

b) $\exists m \in \mathbb{N} : \mathfrak{a}^m M \cap \Gamma_{\mathfrak{a}}(M) = 0$. •

Notation 1.4 A) Let M be an R-module. The set of *zero divisiors* of R with respect to M is denoted by $\mathrm{ZD}_R(M)$, whereas the set of *non-zero divisors* of R with respect to M is denoted by $\mathrm{NZD}_R(M)$, thus:

$$\mathrm{ZD}_R(M) := \{x \in R \,|\, \exists m \in M\backslash\{0\} : xm = 0\};\ \mathrm{NZD}_R(M) := R\backslash \mathrm{ZD}_R(M).$$

B) The set of prime ideals of R is called the *spectrum of* R and denoted by $\mathrm{Spec}(R)$. If $\mathfrak{a} \subseteq R$ is an ideal, the *variety of* $\mathfrak{a}$ in $\mathrm{Spec}(R)$ is denoted by $\mathrm{Var}(\mathfrak{a})$, thus:

$$\mathrm{Var}(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(R) \,|\, \mathfrak{a} \subseteq \mathfrak{p}\}.$$

Keep in mind, that the set of all varieties $\{\mathrm{Var}(\mathfrak{a}) | \mathfrak{a} \subseteq R \text{ an ideal}\}$ is the family of closed sets of a topology on $\mathrm{Spec}(R)$: the *Zariski topology.*

C) If M is an R-module we denote by $\mathrm{Ass}_R(M)$ the set of *associated primes of* M, thus

$$\mathrm{Ass}_R(M) := \big\{\mathfrak{p} \in \mathrm{Spec}(R) \,\big|\, \exists m \in M : \big(0 \underset{R}{:} m\big) = \mathfrak{p}\big\}.$$

Keep in mind that

$$\mathrm{NZ}_R(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}$$

and that $\mathrm{Ass}_R(M)$ is finite, if M is finitely generated. •

Proposition 1.5 *Let M be a finitely generated R-module. Then*

a) $\mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(M) \cap \mathrm{Var}(\mathfrak{a})$.

b) $\mathrm{Ass}_R(M/\Gamma_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(M)\backslash\mathrm{Var}(\mathfrak{a})$.

Proof: [B-F, 1.8]. ■

Remark and Exercise 1.6 A) Fix the ideal $\mathfrak{a}$ of the noetherian ring R. Let $h : M \to N$ be a homomorphism of R-modules. Then $h(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$ (cf 1.3 B) a)), so that we may define a homomorphism of R-modules

$$\Gamma_{\mathfrak{a}}(h) : \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(N),\ m \mapsto h(m) =: \Gamma_{\mathfrak{a}}(h)(m).$$

B) Using the notation of part A), one easily verifies:

a) $\Gamma_{\mathfrak{a}}(id_M) = \mathrm{id}_{\Gamma_{\mathfrak{a}}(M)}$, where id_U denotes the identity map $U \to U$ of a set U.

b) $\Gamma_{\mathfrak{a}}(h \circ \ell) = \Gamma_{\mathfrak{a}}(h) \circ \Gamma_{\mathfrak{a}}(\ell)$, where $\ell : L \to M$ and $h : M \to N$ are homomorphisms of R-modules.

c) $\Gamma_{\mathfrak{a}}(xh) = x\Gamma_{\mathfrak{a}}(h)$, where $x \in R$ and $h : M \to N$ is a homomorphism of R-modules.

C) Moreover it is easy to check that for any exact sequence of R-modules $0 \to L \xrightarrow{\ell} M \xrightarrow{h} N$, the induced sequence of R-modules

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(L) \xrightarrow{\Gamma_{\mathfrak{a}}(\ell)} \Gamma_{\mathfrak{a}}(M) \xrightarrow{\Gamma_{\mathfrak{a}}(h)} \Gamma_{\mathfrak{a}}(N)$$

is exact.

D) Statements a) - d) of part B) tell us, that the assignment

$$\Gamma_{\mathfrak{a}}(\bullet) = \Gamma_{\mathfrak{a}} : (M \xrightarrow{h} N) \mapsto\!\!\leadsto (\Gamma_{\mathfrak{a}}(M) \xrightarrow{\Gamma_{\mathfrak{a}}(h)} \Gamma_{\mathfrak{a}}(N))$$

defines a *covariant linear functor in the category of R-modules* or – for short – a *covariant functor of R-modules* $\Gamma_{\mathfrak{a}}(\bullet) = \Gamma_{\mathfrak{a}}$. By part C), this functor is *left exact.* $\bullet$

Definition 1.7 The left exact covariant functor $\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{a}}(\bullet)$ of 1.6 D) is called *the $\mathfrak{a}$-torsion functor.* $\bullet$

Remark and Exercise 1.8 Consider the exact sequence of $\mathbb{Z}$-modules

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and show that the 2-torsion functor $\Gamma_{2\mathbb{Z}}$ is not an exact functor. $\bullet$

2 Local Cohomology Functors

Again, let R be a noetherian ring and let $\mathfrak{a} \subseteq R$ be an ideal. Let $i \in \mathbb{N}_0$. We define the i-th local cohomology functor with respect to $\mathfrak{a}$ as the i-th right derived functor of the torsion functor $\Gamma_{\mathfrak{a}}$. We briefly recall this construction:

Reminders 2.1 A) An R-module I is said to be *injective*, if for each injective homomorphism of R-modules $M \overset{h}{\rightarrowtail} N$ and each homomorphism of R-modules $\ell : M \to I$ there is a homomorphism of R-modules $\tilde{\ell} : N \to I$ such that $\tilde{\ell} \circ h = \ell$.

B) The *Lemma of Eckmann-Schopf* says, that each R-module M is a submodule of an injective R-module. $\bullet$

Reminder 2.2 A) A *cocomplex of R-modules* $(M^\bullet, d^\bullet)$ is a sequence of R-modules

$$\cdots \longrightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} M^{i+2} \longrightarrow \cdots$$

such that $\operatorname{Im}(d^{i-1}) \subseteq \operatorname{Ker}(d^i)$ for all $i \in \mathbb{Z}$. A cocomplex of the form

$$\cdots 0 \longrightarrow 0 \longrightarrow 0 \cdots \longrightarrow 0 \longrightarrow M^i \xrightarrow{d^i} M^{i+1} \longrightarrow \cdots$$

shall be written as $0 \longrightarrow M^i \xrightarrow{d^i} M^{i+1} \longrightarrow \cdots$.

B) Let $(M^\bullet, d^\bullet)$ and $(N^\bullet, e^\bullet)$ be cocomplexes of R-modules. By a *homomorphism of cocomplexes* (of R-modules)

$$h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$$

we mean a family $(h^i)_{i\in\mathbb{Z}}$ of homomorphism of R-modules which give rise to the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & M^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "h^{i-1}"] & M^i \arrow[r, "d^i"] \arrow[d, "h^i"] & M^{i+1} \arrow[r] \arrow[d, "h^{i+1}"] & \cdots . \\
\cdots \arrow[r] & N^{i-1} \arrow[r, "e^{i-1}"] & N^i \arrow[r, "e^i"] & N^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Observe, that we have the *identity homomorphism*

$$(\mathrm{id}_{M^n})_{n\in\mathbb{Z}} =: \mathrm{id}_{(M^\bullet, d^\bullet)} : (M^\bullet, d^\bullet) \to (M^\bullet, d^\bullet)$$

of cocomplexes and the *composition*

$$h^\bullet \circ \ell^\bullet := (h^n \circ \ell^n) : (L^\bullet, f^\bullet) \to (N^\bullet, e^\bullet)$$

of two homomorphisms of cocomplexes $\ell^\bullet : (L^\bullet, f^\bullet) \to (M^\bullet, d^\bullet)$ and $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$. Moreover, we define the *sum*

$$h^\bullet + \ell^\bullet := (h^n + \ell^n)_{n\in\mathbb{Z}} : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$$

of two homomorphisms of cocomplexes

$$h^\bullet, \ell^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, d^\bullet)$$

and the *product*

$$xh^\bullet = (xh^n)_{n\in\mathbb{Z}} : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$$

of the homomorphism of cocomplexes $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ with $x \in R$.

It turns out that the cocomplexes of R-modules form a category. Moreover, if $(M^\bullet, d^\bullet)$ and $(N^\bullet, e^\bullet)$ are cocomplexes of R-modules, the set

$$\mathrm{Hom}_R((M^\bullet, d^\bullet), (N^\bullet, e^\bullet)) \;=\; \left\{ h^\bullet \;\middle|\; \begin{array}{c} h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, d^\bullet) \text{ is a} \\ \text{homomorphism of cocomplexes} \end{array} \right\}$$

carries a natural structure of R-module. This structure is compatible with composition in the obvious sense. So, the category of cocomplexes of R-modules is an *R-linear category* ...

C) Let $(M^\bullet, d^\bullet)$ be a cocomplex of R-modules and let $n \in \mathbb{Z}$. The n-th *cohomology of* $(M^\bullet, d^\bullet)$ is defined by

$$H^n(M^\bullet, d^\bullet) := \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1}).$$

If $(N^\bullet, e^\bullet)$ is a second cocomplex of R-modules and $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ is a homomorphism of cocomplexes, there is an *induced homomorphism*

$$\begin{array}{ccc}
H^n(M^\bullet, d^\bullet) & \xrightarrow{\quad H^n(h^\bullet) \quad} & H^n(N^\bullet, e^\bullet) \\
\| & & \| \\
\mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1}) & & \mathrm{Ker}(e^n)/\mathrm{Im}(e^{n-1}) \\
\cup & & \cup \\
m + \mathrm{Im}(d^{n-1}) & \longmapsto & h^n(m) + \mathrm{Im}(e^{n-1})
\end{array}$$

It is easy to verify, that induced homomorphisms behave well under taking compositions, sums and products with elements of R:

a) $H^n(\mathrm{id}_{(M^\bullet, d^\bullet)}) = \mathrm{id}_{H^n(M^\bullet, d^\bullet)}$;

b) $H^n(\ell^\bullet \circ h^\bullet) = H^n(\ell^\bullet) \circ H^n(h^\bullet)$;

c) $H^n(\ell^\bullet + h^\bullet) = H^n(\ell^\bullet) + H^n(h^\bullet)$;

d) $H^n(x\ell^\bullet) = xH^n(\ell^\bullet), (x \in R)$.

So, for fixed $n \in \mathbb{Z}$, the assignment

$$H^n(\bullet) = H^n : ((M^\bullet, d^\bullet) \xrightarrow{h^\bullet} (N^\bullet, e^\bullet)) \mapsto \big(H^n(M^\bullet, d^\bullet) \xrightarrow{H^n(h^\bullet)} H^n(N^\bullet, e^\bullet)\big)$$

defines a (covariant linear) functor from (the category of) cocomplexes of R-modules to (the category of) R-modules: the n-th *cohomology functor*. $\bullet$

Reminder and Exercise 2.3 A) Let $h^\bullet, \ell^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ be two homomorphisms of cocomplexes. A *homotopy from* $h^\bullet$ *to* $\ell^\bullet$ is a family of homomorphisms of R-modules $t_i : M^i \to N^{i-1}$ such that

$$h^i - \ell^i = t_{i+1} \circ d^i + e^{i-1} \circ t_i, \ (\forall i \in \mathbb{Z}).$$

If there is such a homotopy from $h^\bullet$ to $\ell^\bullet$, we say that $h^\bullet$ is *homotopic to* $\ell^\bullet$ and write $h^\bullet \sim \ell^\bullet$. This defines an equivalence relation on the R-module $\mathrm{Hom}_R((M^\bullet, d^\bullet), (N^\bullet, e^\bullet))$.

B) It is most important for us, that *"homotopic homomorphisms of cocomplexes are cohomologeous"*

$$h^\bullet \sim \ell^\bullet \Longrightarrow H^n(h^\bullet) \sim H^n(\ell^\bullet), \ \begin{pmatrix} h^\bullet, \ell^\bullet \in \mathrm{Hom}_R((M^\bullet, d^\bullet), (N^\bullet, e^\bullet)) \\ n \in \mathbb{Z} \end{pmatrix}.$$

C) Let F be a covariant linear functor of R-modules. Then, for each cocomplex of R-modules $(M^\bullet, d^\bullet) : \cdots \longrightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \longrightarrow \cdots$ we get an induced cocomplex

$$(F(M^\bullet), F(d^\bullet)) : \cdots \longrightarrow F(M^{i-1}) \xrightarrow{F(d^{i-1})} F(M^i) \xrightarrow{F(d^i)} F(M^{i+1}) \longrightarrow \cdots.$$

Moreover, if $h^\bullet : (M^\bullet, d^\bullet) \to (N^\bullet, e^\bullet)$ is a homomorphism of cocomplexes, there is an induced homomorphism of cocomplexes

$$F(h^\bullet) = (F(h^n))_{n \in \mathbb{Z}} : (F(M^\bullet), F(d^\bullet)) \to (F(N^\bullet), F(e^\bullet)).$$

Now, let $h^\bullet, \ell^\bullet \in \mathrm{Hom}_R((M^\bullet, d^\bullet), (N^\bullet, e^\bullet))$. Then:

a) $h^\bullet \sim \ell^\bullet \Longrightarrow F(h^\bullet) \sim F(\ell^\bullet)$;

b) $h^\bullet \sim \ell^\bullet \Longrightarrow H^n(F(h^\bullet)) = H^n(F(\ell^\bullet)), \ (\forall n \in \mathbb{Z})$. $\bullet$

Reminder and Exercise 2.4 A) Let M be an R-module. A *right resolution* $((E^\bullet, e^\bullet); b)$ of M consists of a cocomplex of R-modules $(E^\bullet, e^\bullet)$ for which $E^i = 0$ for all $i < 0$, and a homomorphism $b : M \to E^0$ such that the sequence $0 \longrightarrow M \xrightarrow{b} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$ is exact. $(E^\bullet, e^\bullet)$ is called a *resolving complex* for M and b is called a *coaugmentation*.

B) Let $h : M \to N$ be a homomorphism of R-modules, let $((D^\bullet, d^\bullet); a)$ be a right resolution of M and let $(E^\bullet, e^\bullet); b)$ be a right resolution of N. Then, a *right resolution of* h between $(D^\bullet, d^\bullet)$ and $(E^\bullet, e^\bullet)$ is a homomorphism of cocomplexes $h^\bullet : D^\bullet \to E^\bullet$ such that $h^0 \circ a = b \circ h$.

C) An *injective resolution* of the R-module M is a right resolution $((I^\bullet, d^\bullet); a)$ of M such that all the R-modules I^i are injective. It follows from the Lemma of Eckmann-Schopf (cf 2.1 B)):

a) Each R-module M has an injective resolution $((I^\bullet, d^\bullet); a)$.

Using the defining property of injective modules, we also may prove:

b) Let $M \xrightarrow{h} N$ be a homomorphism of R-modules, let $((E^\bullet, e^\bullet); b)$ be a right resolution of M and let $((I^\bullet, d^\bullet); a)$ be an injective resolution of N. Then, h has a resolution $h^\bullet : (E^\bullet, e^\bullet) \to (I^\bullet, d^\bullet)$. Moreover, if $\ell^\bullet : (E^\bullet, e^\bullet) \to (I^\bullet, d^\bullet)$ is a second resolution of h, then $h^\bullet \sim \ell^\bullet$.

D) Now, let $F = F(\bullet)$ be a covariant functor of R-modules. It then follows easily by statement b) of part C) and by 2.3 C) b):

a) Let $h : M \to N$, $((E^\bullet, e^\bullet); b)$ and $((I^\bullet, d^\bullet); a)$ be as above. Let $h^\bullet, \ell^\bullet : (E^\bullet, e^\bullet) \to (I^\bullet, d^\bullet)$ be the right resolutions of h. Then $H^n(F(h^\bullet)) = H^n(F(\ell^\bullet))$ for all $n \in \mathbb{Z}$.

From this we may deduce:

b) Let $((I^\bullet, d^\bullet); a)$ and $((J^\bullet, e^\bullet); b)$ be two injective resolutions of the R-mlodule M and let $i^\bullet : (I^\bullet, d^\bullet) \to (J^\bullet, e^\bullet)$ be a resolution of $\mathrm{id}_M : M \to M$ (which exists by C) a)). Then, for each n we have isomorphisms of R-modules

$$H^n(F(i^\bullet)) : H^n(F(I^\bullet), F(d^\bullet)) \xrightarrow{\cong} H^n(F(J^\bullet), F(e^\bullet)).$$

Moreover, if $j^\bullet : (I^\bullet, d^\bullet) \to (J^\bullet, e^\bullet)$ is a second resolution of id_M, then

$$H^n(F(i^\bullet)) = H^n(F(j^\bullet)) \text{ for all } n \in \mathbb{Z}.$$

$\bullet$

Construction and Exercise 2.5 A) By a *choice of injective resolutions of* R-modules $\mathbb{I}_\star$ we mean an assignment $M \rightsquigarrow \mathbb{I}_M = ((I_M^\bullet, d_M^\bullet); a_M)$ which, to each R-module M assigns an injective resolution of M. (Such assignments exist by 2.4 C).)

B) Fix a choice of injective resolutions of Rmodules $\mathbb{I}_\star$. Let F be a covariant functor of R-modules. For each $n \in \mathbb{Z}$ set

$$\mathcal{R}^n_{\mathbb{I}_\star} F(M) := H^n(F(I_M^\bullet), F(d_M^\bullet)).$$

Let $\mathbb{J}_*$ be a second choice of injective resolutions. For each R-module M let $i_M^\bullet : (I_M^\bullet, d_M^\bullet) \to (J_M^\bullet, e_M^\bullet)$ be a resolution of $\mathrm{id}_M : M \to M$ between $\mathbb{I}_M = ((I_M^\bullet, d_M^\bullet); a_M)$ and $\mathbb{J}_M = ((J_M^\bullet, e_M^\bullet); b_M)$. Then, according to 2.4 D) b) we have isomorphisms

a) $\quad H^n(F(i_M^\bullet)) : \mathcal{R}^n_{\mathbb{I}_*} F(M) \xrightarrow{\cong} \mathcal{R}^n_{\mathbb{J}_*} F(M), \ (\forall n \in \mathbb{Z})$

which in addition depend only on $\mathbb{I}_\star$ and $\mathbb{J}_\star$. So, up to the isomorphisms of a), the module $\mathcal{R}^n_{\mathbb{I}_\star} F(M)$ is independent of the choice of injective resolution $\mathbb{I}_\star$. Therefore we write

$$\mathcal{R}^n F(M) := \mathcal{R}^n_{\mathbb{I}_\star} F(M).$$

C) Let $\mathbb{I}_\star$ be as above and let $h : M \to N$ be a homomorphism of R-modules. Let $h^\bullet : (I_M^\bullet, d_M^\bullet) \to (I_N^\bullet, d_N^\bullet)$ be a right resolution of h. By 2.4 D) a), the induced homomorphisms of R-modules

$$H^n(F(h^\bullet)) : \left\{ \begin{array}{ccc} H^n(F(I_M^\bullet), F(d_M^\bullet)) & \longrightarrow & H^n(F(I_N^\bullet), F(d_N^\bullet)) \\ \| & & \| \\ \mathcal{R}^n F(M) & & \mathcal{R}^n F(N) \end{array} \right.$$

depend only on h and not on the chosen resolution of h. We therefore set

$$\mathcal{R}^n F(h) := H^n(F(h^\bullet)), \ (\forall n \in \mathbb{Z}).$$

It is not hard to verify, that the assignment

$$\mathcal{R}^n F(\bullet) = \mathcal{R}^n F : (M \xrightarrow{h} N) \rightsquigarrow (\mathcal{R}^n F(M) \xrightarrow{\mathcal{R}^n F(h)} \mathcal{R}^n F(N))$$

defines a covariant functor of R-modules: the n-th *right derived functor* $\mathcal{R}^n F$ *of* $F, (n \in \mathbb{Z})$.

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Definition 2.6 Let $\mathfrak{a}$ be an ideal of the noetherian ring R. Let $n \in \mathbb{Z}$. *The* n*-th local cohomology functor* $H^n_\mathfrak{a}(\bullet) = H^n_\mathfrak{a}$ *with respect to* $\mathfrak{a}$ is defined as the n-th right derived functor of $\Gamma_\mathfrak{a}$:

$$H^n_\mathfrak{a}(\bullet) = H^n_\mathfrak{a} := \mathcal{R}^n \Gamma_\mathfrak{a}(\bullet) = \mathcal{R}^n \Gamma_\mathfrak{a}.$$

If M is an R-module, $H^n_\mathfrak{a}(M)$ is called *the* n*-th local cohomology module of* M *with respect to* $\mathfrak{a}$. •

3 Basic Properties

Remark and Exercise 3.1 A) Let F be a functor of R-modules. Then one has:

a) $\mathcal{R}^n F(M) = 0$ for all $n < 0$ and all R-modules M.

b) If I is an injective R-module, then $\mathcal{R}^n F(I) = 0$ for all $n > 0$.

c) If F is an exact functor, then $\mathcal{R}^n F(M) = 0$ for all $n > 0$ and all R-modules M.

d) If F is left exact, for each R-module M, there is an isomorphism

$$\alpha_F^M : F(M) \xrightarrow{\cong} \mathcal{R}^0 F(M).$$

B) Let $\mathfrak{a}$ be an ideal of the noetherian ring R. Then, by translation we get from the above statements:

a) $H^n_{\mathfrak{a}}(M) = 0$ for all $n < 0$ and all R-modules M.

b) If I is an injective R-module, then $H^n_{\mathfrak{a}}(I) = 0$ for all $n > 0$.

c) For each R-module M, there is an isomorphism of R-modules $\alpha_{\mathfrak{a}}^M : \Gamma_{\mathfrak{a}}(M) \xrightarrow{\cong} H^0_{\mathfrak{a}}(M)$. •

C) By 1.3 A) c) we clearly have

a) $H^n_{\mathfrak{a}}(M) = H^n_{\sqrt{\mathfrak{a}}}(M)$ for all $n \in \mathbb{Z}$ and all R-modules M.

Remark and Construction 3.2 A) Let F be a covariant functor of R-modules. Moreover, let

$$\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{\ell} P \to 0$$

be a short exact sequence. Then, one may construct a family of homomorphisms of R-modules

$$\delta_{\mathbb{S}}^{n,F} : \big(\mathcal{R}^n F(P) \to \mathcal{R}^{n+1} F(N)\big)_{n \in \mathbb{N}_0}$$

such that the sequence

a)
$$\left\{ \begin{array}{l} 0 \longrightarrow \mathcal{R}^0 F(N) \xrightarrow{\mathcal{R}^0 F(h)} \mathcal{R}^0 F(M) \xrightarrow{\mathcal{R}^0 F(\ell)} \mathcal{R}^0 F(P) \\ \xrightarrow{\delta_{\mathbb{S}}^{0,F}} \mathcal{R}^1 F(N) \xrightarrow{\mathcal{R}^1 F(h)} \mathcal{R}^1 F(M) \xrightarrow{\mathcal{R}^0 F(\ell)} \mathcal{R}^1 F(P) \\ \xrightarrow{\delta_{\mathbb{S}}^{1,F}} \mathcal{R}^2 F(N) \longrightarrow \quad \cdots \end{array} \right.$$

is exact. The homomorphism $\delta^{n,F}_{\mathbb{S}}$ is called the n-th *connecting homomorphism with respect to F associated to* $\mathbb{S}$. The exact sequence a) is called the *right derived sequence with respect to F associated to* $\mathbb{S}$.

Moreover, the construction of the connecting homomorphisms $\delta^{n,F}_{\mathbb{S}}$ is *natural*, that is, it has the following property:

a) For each commutative diagram of R-modules

$$\begin{array}{ccccccccc}
\mathbb{S}: 0 & \longrightarrow & N & \xrightarrow{h} & M & \xrightarrow{\ell} & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\
\mathbb{S}': 0 & \longrightarrow & N' & \xrightarrow{h'} & M' & \xrightarrow{\ell'} & P' & \longrightarrow & 0
\end{array}$$

with exact rows $\mathbb{S}$ and $\mathbb{S}'$ and for all $n \in \mathbb{N}_0$ we have the commutative diagram

$$\begin{array}{ccc}
\mathcal{R}^n F(P) & \xrightarrow{\delta^{n,F}_{\mathbb{S}}} & \mathcal{R}^{n+1} F(N) \\
{\scriptstyle \mathcal{R}^n F(w)}\downarrow & & \downarrow{\scriptstyle \mathcal{R}^n F(u)} \\
\mathcal{R}^n F(P') & \xrightarrow{\delta^{n,F}_{\mathbb{S}'}} & \mathcal{R}^{n+1} F(N')
\end{array} .$$

For the construction and th proves of the stated properties of the connected homomorphisms, we recommend to consult [B-F, 3.5, 3.6 and 3.7].

B) Let $\mathfrak{a}$ be an ideal of the noetherian ring R and let $\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{\ell} P \to 0$ be a short exact sequence. Then, having in mind the definition 2.6 we have the connecting homomorphisms with respect to $\Gamma_{\mathfrak{a}}$ associated to $\mathbb{S}$:

$$\delta^{n,\mathfrak{a}}_{\mathbb{S}} := \delta^{n,\Gamma_{\mathfrak{a}}}_{\mathbb{S}} : H^n_{\mathfrak{a}}(P) \to H^{n+1}_{\mathfrak{a}}(N).$$

We call these *connecting homomorphisms with respect to* $\mathfrak{a}$ *associated to* $\mathbb{S}$. These now occur in the exact sequence

$$\text{a)} \qquad \left\{ \begin{array}{l}
0 \longrightarrow H^0_{\mathfrak{a}}(N) \xrightarrow{H^0_{\mathfrak{a}}(h)} H^0_{\mathfrak{a}}(M) \xrightarrow{H^0_{\mathfrak{a}}(\ell)} H^0_{\mathfrak{a}}(P) \\
\xrightarrow{\delta^{0,\mathfrak{a}}_{\mathbb{S}}} H^1_{\mathfrak{a}}(N) \xrightarrow{H^1_{\mathfrak{a}}(h)} H^1_{\mathfrak{a}}(M) \xrightarrow{H^1_{\mathfrak{a}}(\ell)} H^1_{\mathfrak{a}}(P) \\
\xrightarrow{\delta^{1,\mathfrak{a}}_{\mathbb{S}}} H^2_{\mathfrak{a}}(N) \longrightarrow \cdots
\end{array} \right.$$

which is called the *cohomology sequence with respect to* $\mathfrak{a}$ *associated to* $\mathbb{S}$. This sequence is *natural* as was made clear already above. •

C) Let $\mathfrak{a} \subseteq R$ be as above and let M be an R-module. Let $x \in \mathrm{NZD}_R(M)$. Then, we have a short exact sequence of R-modules

$$\mathbb{S} : 0 \to M \xrightarrow{x\cdot} M \xrightarrow{p} M/xM \to 0,$$

in which $x\cdot$ denotes the *multiplication map* $m \mapsto xm$ and p denotes the canonical map $m \mapsto m + xM$. As each of the functors $H^n_{\mathfrak{a}}$ is linear, we have $H^n_{\mathfrak{a}}(x\cdot) = H^n_{\mathfrak{a}}(x\ \mathrm{id}_M) = xH^n_{\mathfrak{a}}(\ \mathrm{id}_M) = x\ \mathrm{id}_{H^n_{\mathfrak{a}}}(M) = x\cdot : H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M)$. So, the cohomology sequence with respect to $\mathfrak{a}$ associated to $\mathbb{S}$ takes the form:

$$\text{a)} \qquad \begin{cases} 0 \longrightarrow H^0_{\mathfrak{a}}(N) \xrightarrow{x\cdot} H^0_{\mathfrak{a}}(M) \xrightarrow{H^0_{\mathfrak{a}}(p)} H^0_{\mathfrak{a}}(M/xM) \\ \xrightarrow{\delta^{0,\mathfrak{a}}_{\mathbb{S}}} H^1_{\mathfrak{a}}(M) \xrightarrow{x\cdot} H^1_{\mathfrak{a}}(M) \xrightarrow{H^1_{\mathfrak{a}}(p)} H^1_{\mathfrak{a}}(M/xM) \\ \xrightarrow{\delta^{1,\mathfrak{a}}_{\mathbb{S}}} H^2_{\mathfrak{a}}(M) \longrightarrow \quad \cdots \quad . \end{cases}$$

This is an exact sequence, which will be used often. •

Definition 3.3 Let $\mathfrak{a}$ be an ideal of the noetherian ring R. An R-module M is said to be $\mathfrak{a}$-*torsion* if $M = \Gamma_{\mathfrak{a}}(M)$. •

Remark and Exercise 3.4 Let $\mathfrak{a}$ be an ideal of the noetherian ring R. Then:

a) If M is an R-module, $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion.

b) Submodules and homomorphic images of $\mathfrak{a}$-torsion modules are $\mathfrak{a}$-torsion.

c) A finitely generated R-module M is $\mathfrak{a}$-torsion if and only if there is some $n \in \mathbb{N}$ with $\mathfrak{a}^n M = 0$. •

Proposition 3.5 *Let $\mathfrak{a}$ be an ideal of the noetherian ring R, let $n \in \mathbb{N}_0$ and let M be an R-module. Then the module $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion.*

Proof. This follows from the construction of $H^n_{\mathfrak{a}}(M) = \mathcal{R}^n\Gamma_{\mathfrak{a}}(M) = H^n(\Gamma_{\mathfrak{a}}(I^\bullet), \Gamma_{\mathfrak{a}}(d^\bullet)) = \mathrm{Ker}(\Gamma_{\mathfrak{a}}(d^n))/\mathrm{Im}(\Gamma_{\mathfrak{a}}(d^{n-1})) \subseteq \Gamma_{\mathfrak{a}}(I^n)/\mathrm{Im}(\Gamma_{\mathfrak{a}}(d^{n-1}))$ on use of 3.4 a), b). ■

Proposition 3.6 *Let $\mathfrak{a}$ be an ideal of the noetherian ring R and let I be an injective R-module. Then $\Gamma_{\mathfrak{a}}(I)$ is an injective R-module, too.*

Proof. [B-F, 3.13] or [B-S]. ■

Corollary 3.7 *Let $\mathfrak{a}$ be an ideal of the noetherian ring R and let M be an $\mathfrak{a}$-torsion R-module. Then, M has an injective resolution $((I^\bullet, d^\bullet); a)$ in which all the injective modules I^n are $\mathfrak{a}$-torsion.*

Proof. [B-F, 3.14, 3.15]. ■

Theorem 3.8 *Let $\mathfrak{a}$ be an ideal of the noetherian ring R and let M be an $\mathfrak{a}$-torsion R-module. Then $H^n_{\mathfrak{a}}(M) = 0$ for all $n > 0$.*

Proof. By 3.7 the module M has an injective resolution $((I^\bullet, d^\bullet); a)$ such that I^n is $\mathfrak{a}$-torsion for all $n \in \mathbb{N}_0$. Let $n > 0$. It follows $H^n_{\mathfrak{a}}(M) = \mathcal{R}^n\Gamma_{\mathfrak{a}}(M) = H^n(\Gamma_{\mathfrak{a}}(I^\bullet), \Gamma(d^\bullet)) = H^n(I^\bullet, d^\bullet) = \operatorname{Ker}(d^n)/\operatorname{Im}(d^{n-1}) = 0$. ■

Corollary 3.9 *Let $\mathfrak{a}$ be an ideal of the noetherian ring R. Let M be an R-module and let $N \subseteq M$ be a submodule which is $\mathfrak{a}$-torsion. Let $M \xrightarrow{p} M/N$ be the canonical map. Then*

a) $H^0_{\mathfrak{a}}(p) : H^0_{\mathfrak{a}}(M) \to H^0_{\mathfrak{a}}(M/N)$ *is surjective.*

b) $H^n_{\mathfrak{a}}(p) : H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M/N)$ *is an isomorphism for all $n > 0$.*

Proof. The cohomology sequence with respect to $\mathfrak{a}$ and associated to $0 \longrightarrow N \xrightarrow{\text{incl.}} M \xrightarrow{p} M/N \longrightarrow 0$ has the shape

$$\begin{array}{l} 0 \longrightarrow H^0_{\mathfrak{a}}(N) \longrightarrow H^0_{\mathfrak{a}}(M) \xrightarrow{H^0_{\mathfrak{a}}(p)} H^0_{\mathfrak{a}}(M/N) \\ \xrightarrow{\delta^0} H^1_{\mathfrak{a}}(N) \longrightarrow H^1_{\mathfrak{a}}(M) \xrightarrow{H^1_{\mathfrak{a}}(p)} H^1_{\mathfrak{a}}(M/N) \longrightarrow \cdots H^{n-1}_{\mathfrak{a}}(M/N) \\ \xrightarrow{\delta^{n-1}} H^n_{\mathfrak{a}}(N) \longrightarrow H^n_{\mathfrak{a}}(M) \xrightarrow{H^n_{\mathfrak{a}}(p)} H^n_{\mathfrak{a}}(M/N) \longrightarrow H^{n+1}_{\mathfrak{a}}(N) \longrightarrow \cdots \end{array}$$

By 3.8 we have $H^n_{\mathfrak{a}}(N) = 0$ for all $n > 0$. ■

Second Lecture: Vanishing Results

4 Grade and Depth

Throughout this section, let R be a noetherian ring and let $\mathfrak{a} \subseteq R$ be an ideal. If $\mathcal{S} \subseteq \mathbb{R}$ is a set of real numbers, we form $\inf(\mathcal{S})$ and $\sup(\mathcal{S})$ in $\mathbb{R} \cup \{-\infty, \infty\}$, with the convention that $\inf(\emptyset) = \infty$ and $\sup(\emptyset) = -\infty$.

Definition 4.1 The $\mathfrak{a}$-*depth of a finitely generated R-module M* is defined as

$$t_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

•

Our goal is to characterize the $\mathfrak{a}$-depth of a finitely generated R-module in "non-cohomological terms".

Reminder and Exercise 4.2 A) Let M be an R-module. A sequence $x_1, \cdots, x_r \in R$ is called an *M-sequence* if

$$x_i \in \mathrm{NZD}_R(M/\sum_{j=1}^{i-1} x_j M), \quad \text{for } i = 1, \cdots, r.$$

B) Let M be as above and let $x_1, \cdots, x_r \in R$. Then:

a) $x_1, \cdots, x_r$ is an M-sequence if and only if $x_1 \in \mathrm{NZD}_R(M)$ and $x_2, \cdots, x_r$ is an M/x_1M-sequence. •

Definition 4.3 The $\mathfrak{a}$-*grade of* an R-module M is defined by

$$\mathrm{grade}_M(\mathfrak{a}) := \begin{cases} 0, \text{ if } \mathfrak{a} \subseteq \mathrm{ZD}_R(M) \\ \sup\left\{ r \in \mathbb{N} \,\middle|\, \exists x_1, \cdots, x_r \in \mathfrak{a} : x_1, \cdots, x_r \text{ is an } M\text{-sequence} \right\}. \end{cases}$$

If $x_1, \cdots, x_r$ is a sequence of elements of R, we say that r is the *length* of the sequence. So, $\mathrm{grade}_M(\mathfrak{a})$ is 0, if there is no M-sequence consisting of elements of $\mathfrak{a}$. Otherwise, $\mathrm{grade}_M(\mathfrak{a})$ is the supremum of the lengths of all M-sequence which consist of elements of $\mathfrak{a}$. •

Proposition 4.4 *Let M be a finitely generated R-module. Let $r \in \mathbb{N}$. The following statements are equivalent:*

(i) *There is an M-sequence $x_1, \cdots, x_r \in \mathfrak{a}$.*

(ii) *$H^i_{\mathfrak{a}}(M) = 0$ for all $i < r$.*

Proof. "(i) $\Longrightarrow$ (ii)": (*Induction on r*). As $x_1 \in \mathfrak{a} \cap \mathrm{NZD}_R(M)$ we may conclude by 3.1 B) c) that $H^0_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(M) \subseteq \Gamma_{x_1R}(M) = \bigcup_{n\in\mathbb{N}}(0 \underset{M}{:} x_1^n R) = 0$, and this proves the case $r = 1$.

Let $r > 1$. Then $x_1, \cdots, x_{r-1} \in \mathfrak{a}$ form an M-sequence. So, by induction $H^i_{\mathfrak{a}}(M) = 0$ for all $i < r-1$. It remains to be shown that $H^{r-1}_{\mathfrak{a}}(M) = 0$. According to 3.2 C) a) we have an exact sequence

$$H^{r-2}_{\mathfrak{a}}(M/x_1M) \to H^{r-1}_{\mathfrak{a}}(M) \xrightarrow{x\cdot} H^{r-1}_{\mathfrak{a}}(M).$$

According to 4.2 B) $x_2, \cdots, x_r$ is an M/x_1M-sequence. In particular by induction we get $H^i_{\mathfrak{a}}(M/x_1M) = 0$ for all $i < r-1$. It follows that the map $x\cdot : H^{r-1}_{\mathfrak{a}}(M) \to H^{r-1}_{\mathfrak{a}}(M)$ is injective, hence, that $x \in \mathrm{NZD}_R(H^{r-1}_{\mathfrak{a}}(M))$. Consequently $x^n \in \mathrm{NZR}_R(H^{r-1}_{\mathfrak{a}}(M))$ for all $n \in \mathbb{N}$. As $H^{r-1}_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion (cf 3.5) and as $x \in \mathfrak{a}$ it follows $H^{r-1}_{\mathfrak{a}}(M) = 0$.

"(ii) $\Longrightarrow$ (i)": Assume that $H^i_{\mathfrak{a}}(M) = 0$ for all $i \in \{0, \cdots, r-1\}$. We have to find an M-sequence $x_1, x_2, \cdots, x_r \in \mathfrak{a}$. In view of 3.1 B) c) we get $\Gamma_{\mathfrak{a}}(M) \cong H^0_{\mathfrak{a}}(M) = 0$. So, 1.5 a) implies $\mathrm{Ass}_R(M) \cap \mathrm{Var}(\mathfrak{a}) = \mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(0) = \emptyset$ so that $\mathfrak{a} \subseteq \mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Ass}_R(M)$. As $\mathrm{Ass}_R(M)$ is finite it follows from the Prime Avoidance Principle, that $\mathfrak{a} \nsubseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p} = \mathrm{ZD}_R(M)$ (cf 1.4 C)), hence that $\mathfrak{a} \cap \mathrm{NZD}_R(M) \neq \emptyset$. So, there is an element $x_1 \in \mathfrak{a} \cap \mathrm{NZD}_R(M)$. This proves the case $r = 1$. So, let $r > 1$. By 3.2 C) a) we have exact sequences

$$H^{i-1}_{\mathfrak{a}}(M) \to H^{i-1}_{\mathfrak{a}}(M/x_1M) \to H^i_{\mathfrak{a}}(M), \ (i \in \mathbb{N}).$$

These show that $H^j_{\mathfrak{a}}(M/x_1M) = 0$ for all $j < r-1$. By induction, there is an M/x_1M-sequence $x_2, \cdots, x_r$ consisting of elements $x_i \in \mathfrak{a}$. By 4.2 B) $x_1, \cdots, x_r$ becomes an M-sequence. ■

Theorem 4.5 *Let M be a finitely generated R-module. Then $t_{\mathfrak{a}}(M) = \mathrm{grade}_M(\mathfrak{a})$.*

Proof. Easy from 4.4. ■

5 Dimension and Cohomological Dimension

Let R be a noetherian ring and let $\mathfrak{a} \subseteq R$ be an ideal.

Definition 5.1 The *cohomological dimension of the R-module M with respect to* $\mathfrak{a}$ is defined as:

$$\mathrm{cd}_{\mathfrak{a}}(M) := \sup\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

•

Reminder and Exercise 5.2 A) Let M be a finitely generated R-module. The *dimension of M* is defined as the supremum of lengths of chains of primes in the variety of the annilator $0 \underset{R}{:} M \subseteq R$ of M.

$$\dim(M) := \sup\left\{\ell \in \mathbb{N}_0 \mid \exists \mathfrak{p}_0, \mathfrak{p}_1, \cdots, \mathfrak{p}_\ell \in \mathrm{Var}(0 \underset{R}{:} M) : \mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \cdots \subsetneqq \mathfrak{p}_\ell\right\}.$$

B) Keep in mind the following facts:

a) $\dim(M) = -\infty \Longleftrightarrow M = 0 \Longleftrightarrow \dim(M) < 0$.

b) $N \subseteq M$ submodule $\Longrightarrow \dim(N), \dim(M/N) \leq \dim(M)$.

c) $x \in \mathrm{NZD}_R(M) \Longrightarrow \dim(M/xM) \leq \dim(M) - 1$. •

Theorem 5.3 *(Vanishing Theorem of Grothendieck): If M is a finitely generated R-module, then $\mathrm{cd}_{\mathfrak{a}}(M) \leq \dim(M)$.*

Proof. Let $d := \dim(M)$. If $d = \infty$, there is nothing to prove. If $d = -\infty$, we have $M = 0$ and hence $H^i_{\mathfrak{a}}(M) = 0$ for all $i \in \mathbb{Z}$, and our claim is clear. So, let $d \in \mathbb{N}_0$. We have to show that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > d$. Let $\overline{M} := M/\Gamma_{\mathfrak{a}}(M)$. According to 3.9 b) we have $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(\overline{M})$ for all $i > 0$. By 5.2 B) b) we have $\overline{d} := \dim(\overline{M}) \leq d$. It thus suffices to show that $H^i_{\mathfrak{a}}(\overline{M}) = 0$ for all $i > \overline{d}$. As $\Gamma_{\mathfrak{a}}(\overline{M}) = 0$ (cf 3.1 B) c)), we may replace M by $\overline{M}$ and thus assume that $\Gamma_{\mathfrak{a}}(M) = 0$. So, by 4.4 there is an $x \in \mathfrak{a} \cap \mathrm{NZD}_R(M)$. According to 3.2 C) a) there are exact sequences

$$H^{i-1}_{\mathfrak{a}}(M/xM) \to H^i_{\mathfrak{a}}(M) \xrightarrow{x} H^i_{\mathfrak{a}}(M), \ (\forall i > 0).$$

As $x \in \mathfrak{a}$ and as $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion, it suffices to show that $H^{i-1}_{\mathfrak{a}}(M/xM) = 0$ for all $i > d$ (cf proof of 4.4, "(i) $\Longrightarrow$ (ii)"). Assume first that $d = 0$. Then, by 5.2 B) c) $\dim(M/xM) \leq -1$, hence $M/xM = 0$ (cf 5.2 B) a)). It follows that $H^{i-1}_{\mathfrak{a}}(M/xM) = 0$ for all $i > 0 = d$. So, let $d > 0$. By 5.2 B) c) it follows $\dim(M/xM) \leq d - 1$. Now, by induction $H^{i-1}_{\mathfrak{a}}(M/xM) = 0$ for all $i > d$. ■

6 Arithmetic Rank and Cohomological Dimension

Again, let $\mathfrak{a}$ be an ideal of the noetherian ring R.

Definition 6.1 The *arithmetic rank* of $\mathfrak{a}$ is defined as

$$\mathrm{ara}(\mathfrak{a}) := \inf \Big\{ r \in \mathbb{N}_0 \,\Big|\, \exists x_1, \cdots, x_r \in R : \sqrt{\sum_{i=1}^{r} Rx_i} = \sqrt{\mathfrak{a}} \Big\}.$$

Remark 6.2 If $\mathfrak{a}$ is generated by r elements $x_1, \cdots, x_r$, then clearly $\mathrm{ara}(\mathfrak{a}) \leq r$. So:

a) $\mathrm{ara}(\mathfrak{a}) < \infty$;

b) $\mathrm{ara}(\mathfrak{a}) = 0 \Longleftrightarrow \sqrt{\mathfrak{a}} = \sqrt{0} \Longleftrightarrow \mathfrak{a} \subseteq \sqrt{0}$. •

Remark 6.3 Let $\mathfrak{b} \subseteq R$ be a second ideal. Then, for each R-module M there is an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0_{\mathfrak{a}+\mathfrak{b}}(M) & \longrightarrow & H^0_{\mathfrak{a}}(M) \oplus H^0_{\mathfrak{b}}(M) & \longrightarrow & H^0_{\mathfrak{a}\cap\mathfrak{b}}(M) & \\ \longrightarrow & H^1_{\mathfrak{a}+\mathfrak{b}}(M) & \longrightarrow & H^1_{\mathfrak{a}}(M) \oplus H^1_{\mathfrak{b}}(M) & \longrightarrow & H^1_{\mathfrak{a}\cap\mathfrak{b}}(M) & \\ \longrightarrow & H^2_{\mathfrak{a}+\mathfrak{b}}(M) & \longrightarrow & H^2_{\mathfrak{a}}(M) \oplus H^2_{\mathfrak{b}}(M) & \longrightarrow & \cdots & , \end{array}$$

the *Mayer-Vietoris sequence* with respect to $\mathfrak{a}$ and $\mathfrak{b}$ associated to M. For the construction of this sequence see [B-F, (4.11), (4.12), (4.13), (4.14), (4.15)]. For a different approach see [B-S, Chap. 3]. •

Lemma 6.4 *Let $x \in R$ and let M be an R-module. Then $H^i_{Rx}(M) = 0$ for all $i > 1$.*

Proof. Let $\eta : M \to M_x := \{x^n | n \in \mathbb{N}_0\}^{-1}M$ be the natural homomorphism of R-modules defined by $m \mapsto \frac{m}{1}$ for all $m \in M$. Then $\mathrm{Ker}(\eta) = \Gamma_{Rx}(M)$. Let $\overline{M} := M/\Gamma_{Rx}(M)$; we get an exact sequence

$$0 \to \overline{M} \xrightarrow{\overline{\eta}} M_x \to M_x/\overline{\eta}(\overline{M}) \to 0,$$

where $\overline{\eta}$ is defined by $m + \Gamma_{Rx}(M) \mapsto \eta(m)$.

It follows from 3.2 B) that there are exact sequences

$$H^{i-1}_{Rx}(M_x/\overline{\eta}(\overline{M})) \to H^i_{Rx}(\overline{M}) \to H^i(M_x) \to H^i_{Rx}(M_x/\overline{\eta}(\overline{M})) \text{ for all } i > 0.$$

It is easy to verify that $M_x/\overline{\eta}(\overline{M})$ is Rx-torsion. It follows by 3.8 that

$$H^{i-1}_{Rx}(M_x/\overline{\eta}(\overline{M})) = H^i_{Rx}(M_x/\overline{\eta}(\overline{M})) = 0 \text{ for all } i > 1.$$

Therefore $H^i_{Rx}(\overline{M}) \cong H^i_{Rx}(M_x)$ for all $i > 1$. Moreover, by 3.9 b) we have $H^i_{Rx}(M) \cong H^i_{Rx}(\overline{M})$ for all $i > 0$. It thus suffices to show that $H^i_{Rx}(M_x) = 0$ for all $i > 1$. Observe, that the multiplication map $x\cdot : M_x \to M_x$ is an isomorphism of R-modules. Therefore $x\cdot = H^i_{Rx}(x\cdot) : H^i_{Rx}(M_x) \to H^i_{Rx}(M_x)$ is an isomorphism, hence injective. As $H^i_{Rx}(M_x)$ is Rx-torsion, it follows $H^i_{Rx}(M_x) = 0$ for all $i \geq 0$. ■

Theorem 6.5 *(Vanishing Theorem of Hartshorne): If M is a finitely generated R-module, then $cd_{\mathfrak{a}}(M) \geq ara(\mathfrak{a})$.*

Proof. Let $r \in \mathbb{N}$ and let $\mathfrak{a} = \sum_{j=1}^{r} Rx_i$. According to 3.1 C) a) and 6.2 it suffices to show that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > r$. The case $r = 1$ is clear by 6.4. So, let $r > 1$. We write $\mathfrak{b} := \sum_{j=1}^{r-1} Rx_i$. As $\mathfrak{a} = \mathfrak{b} + Rx_r$, the Mayer-Vietoris sequence of the ideals $\mathfrak{b}$ and Rx_r associated to M yields exact sequences

$$H^{i-1}_{\mathfrak{b}\cap(Rx_r)}(M) \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{b}}(M) \oplus H^i_{Rx_r}(M), \ (\forall i > 0).$$

By induction $H^i_{\mathfrak{b}}(M) = 0$ for all $i \geq r$. By 6.4 $H^i_{Rx_r}(M) = 0$ for all $i > 1$. As $\sqrt{\mathfrak{b} \cap (Rx_r)} = \sqrt{\mathfrak{b} \cdot Rx_r} = \sqrt{\sum_{j=1}^{r-1} x_j x_r R}$ we have $H^{i-1}_{\mathfrak{b}\cap(Rx_r)}(M) = H^{i-1}_{\sum_{j=1}^{r-1} x_j x_r R}(M)$ for all $i > 0$ by 1.3 C) a). But by induction, the right hand side module vanishes for all $i > r$. On use of the above sequences we get $H^i_{\mathfrak{a}}(M) = 0$ for all $i > r$. ■

7 Affine Varieties: Numbers of Defining Equations

Reminders 7.1 A) Let $r \in \mathbb{N}$, let k be an algebraically closed field and consider the polynomial ring $k[x_1, \cdots, x_r]$. Let $\emptyset \neq \mathcal{S} \subseteq k[x_1, \cdots, x_r]$. The *algebraic set defined by* $\mathcal{S}$ is the set

$$V(\mathcal{S}) := \{(c_1, \cdots, c_r) = \underline{c} \in k^r \mid f(\underline{c}) = 0, \ \forall f \in \mathcal{S}\}.$$

We also convene that $V(\emptyset) = k^r$. Algebraic sets $V \subseteq k^r$ are often called *affine (algebraic)* varieties in k^r. If $f_1, \cdots, f_n \in k[x_1, \cdots, x_n]$ are finitely many polynomials, we write

$$V(f_1, \cdots, f_n) := V(\{f_1, \cdots, f_n\})$$

and we say that $V = V(f_1, \cdots, f_n)$ *is defined by the n-equations $f_i = 0, i = 1, \cdots, n$.* A basic question of algebraic geometry asks:

Which is the minimal number $n = n(V)$ such that a given affine variety $V \subseteq k^n$ may be defined by n equations?

B) It is immediate, that

a) $V(\mathcal{S}) = V(\sum_{f\in\mathcal{S}} fk[x_1, \cdots, x_n])$.

Therefore:

b) Each affine variety $V \subseteq k^n$ is of the form $V = V(\mathfrak{a})$, with an ideal $\mathfrak{a} \subseteq k[x_1, \cdots, x_r]$.

As each ideal is finitely generated we conclude from statement a)

c) Each affine variety $V \subseteq k^n$ is of the form $V = V(f_1, \cdots, f_n)$, with finitely many polynomials $f_1, \cdots, f_n \in k[x_1, \cdots, x_r]$.

C) Let $V \subseteq k^r$ be an affine variety. The *vanishing ideal* of V is defined as

$$I(V) := \{f \in k[x_1, \cdots, x_r] \,|\, f(V) = 0\}.$$

This is indeed an ideal of $k[x_1, \cdots, x_r]$ which moreover is radical, that is $I(V) = \sqrt{I(V)}$. It is easy to verify that

a) $V(I(V)) = V$ for each affine variety $V \subseteq k^r$.

Moreover, it is a consequence of *"Hilbert's Nullstellensatz"*, that

b) $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for each ideal $\mathfrak{a} \subseteq k[x_1, \cdots, x_r]$. •

Now, we can characterize the minimal number of equations needed to define an affine variety.

Theorem 7.2 *Let k be an algebraically closed field. Let $\mathfrak{a}$ be an ideal of the polynomial ring $k[x_1, \cdots, x_r]$. Let $V = V(\mathfrak{a})$. Then* $\operatorname{ara}(\mathfrak{a}) = \operatorname{ara}(I(V))$ *and this number is the smallest number n such that there are n-polynomials $f_1, \cdots, f_n \in k[x_1, \cdots, x_r]$ for which $V = V(f_1, \cdots, f_n)$.*

Proof. As $I(V) = I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ (cf 7.1 C) b)) we have $\operatorname{ara}(\mathfrak{a}) = \operatorname{ara}(I(V))$. Let $f_1, \cdots, f_n \in k[x_1, \cdots, x_r]$ such that $V = V(f_1, \cdots, f_n)$. It follows $\sqrt{\sum_{i=1}^{n} f_i k[x_1, \cdots, x_r]} = I(V(\sum_{i=1}^{r} f_i k[x_1, \cdots, x_r])) = I(V(f_1, \cdots, f_n)) = I(V) = \sqrt{\mathfrak{a}}$, hence $n \geq \operatorname{ara}(\mathfrak{a})$.

Let $m = \operatorname{ara}(\mathfrak{a})$ and let $f_1, \cdots, f_m \in k[x_1, \cdots, x_r]$ such that $\sqrt{\sum_{i=1}^{m} f_i k[x_1, \cdots, x_r]} = \sqrt{\mathfrak{a}}$. It follows $V(f_1, \cdots, f_m) = V(\sum_{i=1}^{m} f_i k[x_1, \cdots, x_r]) = V(I(V(\sum_{i=1}^{m} f_i k[x_1, \cdots, x_r])))$ $= V(\sqrt{\sum_{i=1}^{m} f_i k[x_1, \cdots, x_r]}) = V(\sqrt{\mathfrak{a}}) = V(I(V(\mathfrak{a}))) = V(I(V)) = V$ (cf 7.2 C) a), b)). ■

Corollary 7.3 *Let k be an algebraically closed field, let $\mathfrak{a}$ be an ideal of the polynomial ring $k[x_1, \cdots, x_r] =: R$ and let $c = \operatorname{cd}_{\mathfrak{a}}(R)$. Then, one needs at least c equations to define the affine variety $V = V(\mathfrak{a}) \subseteq k^n$.*

Proof. Clear by 7.2 and 6.5. ■

Exercise 7.4 Consider the polynomial ring $k[x_1, \cdots, x_r] =: R$ over the field k. Let $\mathfrak{m} := \sum_{i=1} x_i R$. Use 4.5 and 6.5 to show that $H^i_{\mathfrak{m}}(R) \neq 0$ if and only if $i = r$. •

Example 7.5 Let k be an algebraically closed field, let $V = V(x_1x_3, x_1x_4, x_2x_3, x_2x_4) \subseteq k^4$. As $(x_2x_3)^2 = (x_1x_4 + x_2x_3)x_2x_3 - x_1x_3x_2x_4$ and $(x_1x_4)^2 = (x_1x_4 + x_2x_3)x_1x_4 - x_1x_3x_2x_4$, we have $V = V(x_1x_3, x_1x_4 + x_2x_3, x_2x_4)$, so that V can be defined by 3 equations. Hence $\text{ara}(I(V)) \leq 3$.

As $I(V) = \sqrt{(x_1x_3, x_1x_4, x_2x_3x_2x_4)} = \sqrt{(x_1, x_3) \cdot (x_2, x_4)} = \sqrt{(x_1, x_2) \cap (x_3, x_4)} = (x_1, x_2) \cap (x_3, x_4)$ we have $H^3_{I(V)}(R) = H^3_{(x_1,x_2)\cap(x_3,x_4)}(R)$, where $R = k[x_1, x_2, x_3, x_4]$ (cf 3.1 C) a)). So, the Mayer-Vietoris sequence gives rise to an exact sequence

$$H^3_{I(V)}(R) \to H^4_{(x_1,\cdots,x_4)}(R) \to H^4_{(x_1,x_2)}(R) \oplus H^4_{(x_3,x_4)}(R),$$

By 7.4 we have $H^4_{(x_1,x_2,x_3,x_4)}(R) \neq 0$. By 6.5 we have $H^4_{(x_1,x_2)}(R) = H^4_{(x_3,x_4)}(R) = 0$. So $H^3_{I(V)}(R) \neq 0$ hence $\text{cd}_{I(V)}(R) \geq 3$, thus $\text{ara}(I(V)) \geq 3$, hence $\text{ara}(I(V)) = 3$. •

Comment and Exercise 7.6 In the previous example we have $V = V(x_1, x_2) \cup V(x_3, x_4)$. So V is the union of the two planes $V(x_1, x_2)$ and $V(x_2, x_3)$ in k^4 which intersect each other in the origin $(0, 0, 0, 0)$. On the other hand, the union of two planes which intersect each other in a line, can be defined by 2 equations. •

8 Affine Varieties: Extending Regular Functions

Reminder and Exercise 8.1 A) Let k be an algebraically closed field, let $r \in \mathbb{N}$ and let $V \subseteq k^r$ be an affine algebraic variety. Then V is said to be *irreducible* if $V \neq \emptyset$ and if V cannot be written as the union of two proper subsets $V_1, V_2 \subsetneqq V$ which are again affine varieties in k^r. It is easy to verify, that the following statements are equivalent:

(i) V is irreducible;

(ii) $I(V) \subseteq k[x_1, \cdots, x_r]$ is a prime ideal;

(iii) $V = V(\mathfrak{p})$, where $\mathfrak{p} \subseteq k[x_1, \cdots, x_r]$ is a prime ideal.

B) We assume from now on, that V is irreducible and furnish V with its *Zariski-topology*. So, the open sets of V are precisely the sets of the form $V \backslash W$, where $W \subseteq k^r$ is an affine variety. Equivalently: The closed sets of V are the affine varieties $W \subseteq k^r$ with $W \subseteq V$.

C) Let $U \subseteq V$ be a non-empty open set. A function $f : U \to k$ is said to be *regular* if it is locally presented by rational functions, more precisely:

For each $p \in U$, there are polynomials $h_p, g_p \in k[x_1, \cdots, x_r]$ and an open neighborhood $W_p \subseteq U$ of p such that:

$$\forall q \in W_p : g_p(q) \neq 0 \text{ and } f(q) = \frac{h_p(q)}{g_p(q)}.$$

We set

$$\mathcal{O}(U) := \{f : U \to k \,\big|\, f \text{ is a regular function }\}.$$

It is easy to see, that $\mathcal{O}(U)$ is a subring of the ring of all functions $U \to k$. We thus call $\mathcal{O}(U)$ the *ring of regular functions on* U. Let us note two important facts, for which we refer to [B-F, (7.1), (7.4)]

a) $\mathcal{O}(U)$ is a domain.

b) The restriction map $k[x_1, \cdots, x_r] \xrightarrow{\pi} \mathcal{O}(V)$ given by $f \mapsto f\restriction_V$ is a surjective homomorphism of rings with $\mathrm{Ker}(\pi) = I(V)$.

D) Now, let $Z \subseteq V$ be a closed subset. Then

$$I_V(Z) := \{f \in \mathcal{O}(V) \,\big|\, f(Z) = 0\}$$

is a radical ideal of $\mathcal{O}(V)$. We call this ideal the *vanishing ideal* of Z in $\mathcal{O}(V)$. Keep in mind that $I_{k^n}(V) = I(V) \subseteq k[x_1, \cdots, x_n] = \mathcal{O}(k^n)$ (cf C) b)).

Theorem 8.2 *Let $V \subseteq k^r$ be an irreducible affine variety and let $U \subsetneqq V$ be a non-empty open subset. Then, there is an exact sequence of $\mathcal{O}(V)$-modules*

$$0 \longrightarrow \mathcal{O}(V) \xrightarrow{\mathrm{res}_{VU}} \mathcal{O}(U) \longrightarrow H^1_{I_V(V\setminus U)}(\mathcal{O}(V)) \longrightarrow 0$$

in which res_{VU} is the restriction map defined by $f \mapsto f\restriction_U$.

Proof. See [B-F, (7.8)]. ■

Corollary 8.3 *Let V and U be as above. Then, the following statements are equivalent:*

(i) Each regular function $f : U \to k$ may be extended to a regular function $\tilde{f} : V \to k$.

(ii) $H^1_{I_V(V\setminus U)}(\mathcal{O}(V)) = 0$

(iii) There are functions $f_1, f_2 \in I_V(V\setminus U)$ which form an $\mathcal{O}(V)$-sequence.

Proof. "(i) $\Longleftrightarrow$ (ii)": Statement (i) is equivalent to the surjectivity of the restriction map $\mathrm{res}_{VU} : \mathcal{O}(V) \to \mathcal{O}(U)$ and hence implies statement (ii) by 8.2 – and conversely.

"(ii) $\Longrightarrow$ (iii)": As $V\backslash U \subseteq V$ we have $I(V) \subseteq I(V\backslash U)$. As $V\backslash U \neq V$ we have $V(I(V\backslash U)) = V\backslash U \neq V = V(I(V))$ (cf 7.1 C) a)) and hence $I(V) \subsetneqq I(V\backslash U)$. So, there is some $f \in I(V\backslash U)\backslash I(V)$. Therefore $f\restriction_V (V\backslash U) = f(V\backslash U) = 0$ and $f\restriction_V \neq 0$ (cf 8.1 C) b)). This shows, that $I_V(V\backslash U) \neq 0$. As $\mathcal{O}(V)$ is a domain (cf 8.1 C) a)), this implies $H^0_{I_V(V\backslash U)}(\mathcal{O}(V)) = 0$ (cf 4.4). So, statement (ii) is equivalent to $t_{I_V(V\backslash U)}(\mathcal{O}(V)) \geq 2$. By 4.4 this is equivalent to statement (iii). ∎

Example and Exercise 8.4 A) *(Hartshorne)* Let k be an algebraically closed field and consider the homomorphism of polynomial rings

$$h : k[x_1, x_2, x_3, x_4] \to k[x, y]$$

given by $x_1 \mapsto x,\ x_2 \mapsto xy,\ x_3 \mapsto y(y-1),\ x_4 \mapsto y^2(y-1)$.

Then, clearly

$$\mathrm{Im}(h) = k[x, xy, y(y-1), y^2(y-1)].$$

Moreover, as $k[x, y]$ is a domain, $\mathrm{Ker}(h) \subseteq k[x_1, x_2, x_3, x_4]$ is a prime ideal. So by 8.1 A) a)

$$V := V(\mathrm{Ker}(h)) \subseteq k^4$$

is an irreducible affine variety.

Keeping in mind the Homomorphism Theorem, the Nullstellensatz (cf 7.1 C) b)) and 8.1 C) b) we get isomorphisms of k-algebras

$$k[x, xy, y(y-1), y^2(y-1)] \cong k[x_1, x_2, x_3, x_4]/\mathrm{Ker}(h) = k[x_1, x_2, x_3, x_4]/I(V) \cong \mathcal{O}(V).$$

We thus identify

$$\begin{aligned}&\mathcal{O}(V) = k[x, xy, y(y-1), y^2(y-1)] \text{ and}\\ &x_1\restriction_V = x,\ x_2\restriction_V = xy,\ x_3\restriction_V = y(y-1),\ x_4\restriction_V = y^2(y-1).\end{aligned}$$

B) Clearly $\mathrm{Ker}(h) \subseteq (x_1, x_2, x_3, x_4)$, hence $\{\underline{0}\} = V(x_1, x_2, x_3, x_4) \subseteq V(\mathrm{Ker}(h)) = V$. We set

$$U := V\backslash\{\underline{0}\}.$$

Then $V\backslash U = \{\underline{0}\}$ and hence

$$I_V(V\backslash U) = (x_1\restriction_V, x_2\restriction_V, x_3\restriction_V, x_4\restriction_V) = (x, xy, y(y-1), y^2(y-1)).$$

Now, it is easy to check that

$$k[x, y] = \mathcal{O}(V) + y\mathcal{O}(V),\ y \notin \mathcal{O}(V) \text{ and } I_V(V\backslash U)k[x, y] \subseteq \mathcal{O}(V).$$

So, $k[x,y]/\mathcal{O}(V)$ is a non-zero cyclic $\mathcal{O}(V)$-module annihilated by $I_V(V\backslash U)$. Therefore, $k[x,y]/\mathcal{O}(V)$ is a non-zero homomorphic image of $\mathcal{O}(V)/I_V(V\backslash U) \cong k$, hence $k[x,y]/\mathcal{O}(V) \cong k$. So, we get an exact sequence of finitely generated $\mathcal{O}(V)$-modules.

$$0 \to \mathcal{O}(V) \to k[x,y] \to k \to 0.$$

Now, x and $y(y-1) \in I_V(V\backslash U)$ form a $k[x,y]$-sequence, so that $H^i_{I_V(V\backslash U)}(k[u,v]) = 0$ for $i = 0,1$ (cf 4.4). Thus, applying local cohomology to the above sequence, we get

$$H^1_{I_V(U\backslash U)}(\mathcal{O}(V)) \cong k.$$

Therefore, by 8.3 we must have a regular function on U, which cannot be extended to a regular function on V.

C) To make the latter statement more explicit we consider the map $\alpha : k^2 \to k^4$ given by $(x,y) \mapsto (x, xy, y(y-1), y^2(y-1))$. If $f \in \mathrm{Ker}(h)$ we have $f(\alpha(x,y)) = f(x, xy, y(y-1), y^2(y-1)) = f(h(x_1), h(x_2), h(x_3), h(x_4))) = 0$. This shows that $\mathrm{Im}(\alpha) \subseteq V(\mathrm{Ker}(h)) = V$. So, we may write

$$\alpha : k^2 \to V;\ (x,y) \mapsto (x, xy, y(y-1), y^2(y-1)).$$

As the coordinates of α are given by regular (actually polynomial) functions, we can say that α is a morphism of algebraic varieties.

Now, consider the open sets

$$U_1 := V\backslash V(x_1),\ U_3 := V\backslash V(x_3).$$

Then $x = x_1\restriction_V$ has no zero in U_1 and $y(y-1) = x_3\restriction_V$ has no zero in U_3. Therefore

$$U = U_1 \cup U_3.$$

Now, we can define a map

$$\beta : U \to k^2;\ \underbrace{(x_1, x_2, x_3, x_4)}_{=:p} \mapsto \begin{cases} (x_1, \frac{x_2}{x_1}), & \text{if } p \in U_1 \\ (x_1, \frac{x_4}{x_3}), & \text{if } p \in U_3 \end{cases}.$$

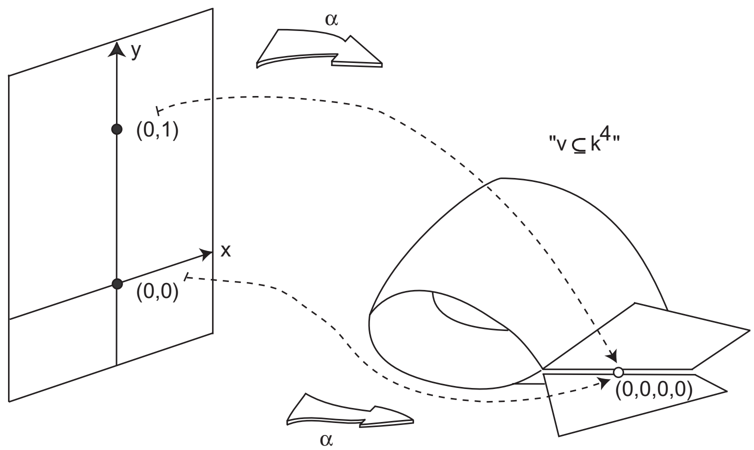
This map has regular coordinates and hence is a morphism.

Clearly we have

$$\alpha^{-1}(0,0,0,0) = \{(0,0),(0,1)\},$$

and moreover β is inverse to $\alpha\restriction_{k^2\backslash\{(0,0),(0,1)\}}$ so that

$$\alpha\restriction: \underbrace{k^2\backslash\{(0,0),(0,1)\}}_{=:W} \xrightarrow{\cong} U = V\backslash\{(0,0,0,0)\}.$$



In particular we have $\mathcal{O}(W) \cong \mathcal{O}(U)$. So, α maps k^2 onto V by just mapping $(0,0)$ and $(0,1)$ to the same point $(0,0,0,0)$

D) Now, we can give explicitly a non-extendable regular function on U:

Namely, let $\beta_2 \in \mathcal{O}(U)$ be the second component of β, so that

$$\beta_2((x_1,x_2,x_3,x_4)=p) = \begin{cases} \frac{x_2}{x_1}, & \text{if } p \in U_1 \\ \frac{x_4}{x_3}, & \text{if } p \in U_3 \end{cases}.$$

Then, β_2 cannot be extended to a regular function on V: Namely, for all $p \in U$ we have $\beta_2(p) = y(\beta(p))$. Choosing $(x,y) \in k^2 \backslash \{(0,0),(0,1)\}$ we thus get $\beta_2(\alpha(x,y)) = y(\beta(\alpha(x,y))) = y(x,y) = y$.

Assume now, that β_2 can be extended to a regular function γ on V. Consider the regular function

$$\sigma : k \to k;\ (y \xrightarrow{\sigma} \gamma(\alpha(0,y))).$$

Then, for all $y \neq 0,1$ we have

$$\sigma(y) = \gamma(\alpha(0,y)) = \beta_2(\alpha(0,y)) = y.$$

This means, that σ is given by the polynomial $y \in k[y] = \mathcal{O}(k)$. On the other hand

$$\sigma(0) = \gamma(\alpha(0,0)) = \gamma(0,0,0,0) = \gamma(\alpha(0,1)) = \sigma(1),$$

which yields the contradiction $0 = 1$. Therefore, $\beta_2 : U \to k$ cannot be extended regularly to V. •

As a preparation for later arguments we suggest the following exercise.

Exercise 8.5 A) Let k be an algebraically closed field. Show that the proper closed non-empty subsets of the *line* $L = k$ are precisely the finite subsets of L.

B) Let $r \in \mathbb{N}$ and let $V \subseteq k^r$ be a *curve*, that is an irreducible affine variety of dimension 1, so that $\dim(\mathcal{O}(V)) = 1$. Show that the conclusion of part A) holds, if we replace L by V. $\bullet$

Third Lecture: Finiteness Results

9 Localization and Local Cohomology

Proposition 9.1 *Let R be a noetherian ring, let $S \subseteq R$ be a non-empty multiplicately closed set and let I be an injective R-module. Then $S^{-1}I$ is an injective $S^{-1}R$-module.*

Proof. [B-F, (5.1)]. ■

Theorem 9.2 *Let R be a noetherian ring, let $\mathfrak{a} \subseteq R$ be an ideal, let $S \subseteq R$ be a non-empty multiplicatively closed set and let M be an R-module. Then, for each $n \in \mathbb{N}_0$ there is an isomorphism of $S^{-1}R$-modules*

$$\varrho^n_{\mathfrak{a},M} : S^{-1}H^n_{\mathfrak{a}}(M) \xrightarrow{\cong} H^n_{\mathfrak{a}S^{-1}R}(S^{-1}M).$$

Proof. (Sketch; for details see [B-F, (5.2) - (5.6)]) Let $0 \longrightarrow M \xrightarrow{a} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots$ be an injective resolution of M. Then, by 9.1 and by the exactness of the localization functor

$$S^{-1}\bullet : (M \xrightarrow{f} N) \rightsquigarrow (S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N)$$

(from R-modules to $S^{-1}R$-modules) we see that

$$0 \longrightarrow S^{-1}M \xrightarrow{S^{-1}a} S^{-1}I^0 \xrightarrow{S^{-1}d^0} S^{-1}I^1 \xrightarrow{S^{-1}d^1} S^{-1}I^2 \longrightarrow \cdots$$

is an injective resolution of the $S^{-1}R$-module $S^{-1}M$. Therefore

$$H^n_{\mathfrak{a}S^{-1}R}(S^{-1}M) = H^n\big(\Gamma_{\mathfrak{a}s^{-1}}(S^{-1}I^\bullet), \Gamma_{\mathfrak{a}S^{-1}}(S^{-1}d^\bullet)\big).$$

It is easy to verify, that for each R-module N one has $\Gamma_{\mathfrak{a}S^{-1}}(S^{-1}N) = S^{-1}\Gamma_{\mathfrak{a}}(N)$. Therefore $H^n_{\mathfrak{a}S^{-1}}(S^{-1}M) = H^n\big(\Gamma_{\mathfrak{a}S^{-1}}(S^{-1}I^\bullet), \Gamma_{\mathfrak{a}S^{-1}}(S^{-1}d^\bullet)\big)$ $(= H^n(S^{-1}\Gamma_{\mathfrak{a}}(I^\bullet), S^{-1}\Gamma_{\mathfrak{a}}(d^\bullet))$. As the functor $S^{-1}\bullet$ is exact, it commutes with cohomology, so that

$$H^n_{\mathfrak{a}S^{-1}R}(S^{-1}M) = S^{-1}H^n(\Gamma_{\mathfrak{a}}(I^\bullet), \Gamma_{\mathfrak{a}}(d^\bullet)) = S^{-1}H^n_{\mathfrak{a}}(M).$$

■

Remark 9.3 Theorem 9.2 may be expressed in the form:

Local Cohomology commutes with Localization. •

Proposition 9.4 *Let $\mathfrak{a}$ be an ideal of the noetherian ring R and let M be a finitely generated R module such that $\mathrm{cd}_{\mathfrak{a}}(M) > 0$. Then, there is a $j \in \mathbb{N}$ such that the R-module $H^j_{\mathfrak{a}}(M)$ is not finitely generated.*

Proof. We have to show that if $H^c_{\mathfrak{a}}(M) \neq 0$ for some $c > 0$, then $H^j_{\mathfrak{a}}(M)$ is not finitely generated for some $j \in \mathbb{N}$.

There is some $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $H^c_{\mathfrak{a}}(M)_{\mathfrak{p}} \neq 0$. We have to find some $j \in \mathbb{N}$ such that the $R_{\mathfrak{p}}$-module $H^j_{\mathfrak{a}}(M)_{\mathfrak{p}}$ is not finitely generated. By 9.2 we may replace $R, \mathfrak{a}, M$ respectively by $R_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}}, M_{\mathfrak{p}}$ and hence assume that $(R, \mathfrak{m})$ is local. As $H^c_{\mathfrak{a}}(M) \neq 0$ and as $H^c_R = \mathcal{R}^c\Gamma_R = \mathcal{R}^c = 0$, we must have $\mathfrak{a} \subseteq \mathfrak{m}$ Let $\overline{M} = M/\Gamma_{\mathfrak{a}}(M)$. As $H^i_{\mathfrak{a}}(\overline{M}) \cong H^i_{\mathfrak{a}}(M)$ for all $i > 0$, we may replace M by $\overline{M}$ and hence assume that $H^0_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(M) = 0$, so that there is some $x \in \mathfrak{a} \cap \mathrm{NZD}_R(M)$, (cf 4.4).

In particular, there are exact sequences

$$H^{i-1}_{\mathfrak{a}}(M) \to H^{i-1}_{\mathfrak{a}}(M/xM) \to H^i_{\mathfrak{a}}(M) \xrightarrow{x\cdot} H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M/xM) \text{ for all } i \in \mathbb{N}.$$

Assume first, that $\dim(M) = 1$. Then, by Grothendieck's Vanishing Theorem 5.3 we must have $c = 1$, hence $H^1_{\mathfrak{a}}(M) \neq 0$. Also by this same vanishing theorem (and as $\dim(M/xM) \leq \dim(M) - 1$, cf 5.2 B) c)) we have $H^1_{\mathfrak{a}}(M/xM) = 0$. Applying the above sequence with $i = 1$ we thus get an epimorphism $H^1_{\mathfrak{a}}(M) \xrightarrow{x\cdot} H^1_{\mathfrak{a}}(M)$, so that $xH^1_{\mathfrak{a}}(M) = H^1_{\mathfrak{a}}(M) \neq 0$. By Nakayama, $H^1_{\mathfrak{a}}(M)$ cannot be finitely generated.

So, let $\dim(M) > 1$. If $H^c_{\mathfrak{a}}(M)$ is not finitely generated, we choose $j = c$. So, assume that $H^c_{\mathfrak{a}}(M)$ is finitely generated. Then, by Nakayama the map $H^c_{\mathfrak{a}}(M) \xrightarrow{x\cdot} H^c_{\mathfrak{a}}(M)$ is not surjective. Applying the above sequence with $i = c$, we get $H^c_{\mathfrak{a}}(M/xM) \neq 0$. As $\dim(M/xM) \leq \dim(M) - 1$ (cf 5.2 B) c)) it follows by induction that $H^\ell_{\mathfrak{a}}(M/xM)$ is not finitely generated for some $\ell \in \mathbb{N}$. Applying the above sequence with $i = \ell + 1$ we thus may choose $j \in \{\ell, \ell + 1\}$. ■

Corollary 9.5 *Let $\mathfrak{a}$ be an ideal of the noetherian ring R and let M be a finitely generated R-module. If $H^i_{\mathfrak{a}}(M) \neq 0$ for some $i > 0$, then there is a $j > 0$ such that $H^j_{\mathfrak{a}}(M)$ is not finitely generated.* •

10 Associated Primes of Local Cohomology Modules

Theorem 10.1 *Let $\mathfrak{a}$ be an ideal of the noetherian ring R and let M be a finitely generated R-module. Let $i \in \mathbb{N}_0$ be such that $H^j_{\mathfrak{a}}(M)$ is finitely generated for all $j < i$. Let $N \subseteq H^i_{\mathfrak{a}}(M)$ be a finitely generated submodule. Then the set $\mathrm{Ass}_R(H^i_{\mathfrak{a}}(M)/N)$ is finite.*

Proof. (Induction on i). The case $i = 0$ is clear as $H^0_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(M) \subseteq M$ is finitely generated. So, let $i > 0$ and set $\overline{M} = M/\Gamma_{\mathfrak{a}}(M)$. Then $H^0_{\mathfrak{a}}(\overline{M}) \cong \Gamma_{\mathfrak{a}}(\overline{M}) = 0$ and $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(\overline{M})$ for all $i > 0$. In particular, $H^j_{\mathfrak{a}}(M) = 0$ for all $j < i$. We thus may replace M by $\overline{M}$ and hence assume that $H^0_{\mathfrak{a}}(M) = 0$. By 4.4 we thus find an element $y \in \mathfrak{a} \cap \mathrm{NZD}(M)$.

Moreover, as N is finitely generated and $\mathfrak{a}$-torsion, there is some $n \in \mathbb{N}$ such that $y^n N = 0$. Let $x := y^n$. Then $x \in \mathfrak{a} \cap \mathrm{NZD}(M)$ and $xN = 0$.

On use of the cohomology sequence with respect to $\mathfrak{a}$ and associated to the exact sequence $\mathbb{S} : 0 \to M \xrightarrow{x\cdot} M \xrightarrow{p} M/xM \to 0$ we now get a commutative diagram with exact rows and columns

$$\begin{tikzcd}
H^{n-1}_{\mathfrak{a}}(M) \arrow[r, "\varepsilon"] & H^{i-1}_{\mathfrak{a}}(M/xM) \arrow[r, "\delta"] \arrow[d, "\pi"] & H^i_{\mathfrak{a}}(M) \arrow[r, "x\cdot"] \arrow[d, "\varrho"] & H^i_{\mathfrak{a}}(M) \arrow[d, "\|"] \\
 & T := H^{i-1}_{\mathfrak{a}}(M/xM)/\delta^{-1}(N) \arrow[r, "\overline{\delta}"] \arrow[d] & H^i_{\mathfrak{a}}(M)/N \arrow[r, "\overline{x\cdot}"] \arrow[d] & H^i_{\mathfrak{a}}(M) \\
 & 0 & 0 &
\end{tikzcd}$$

in which $\varepsilon = H^{i-1}_{\mathfrak{a}}(p)$ is induced by the canonical map $p : M \to M/xM$, in which δ is the connecting homomorphism $\delta^i_{\mathbb{S},\mathfrak{a}}$, in which π and ϱ are the canonical maps given by $m + \delta^{-1}(N) \overset{\overline{\delta}}{\mapsto} \delta(m)$ and $m + N \overset{\overline{x\cdot}}{\mapsto} xm$. Observe that $\mathrm{Ker}(\delta) = \varepsilon(H^{i-1}_{\mathfrak{a}}(M))$ is finitely generated. As R is noetherian and N is finitely generated, it follows that $\delta^{-1}(N)$ is finitely generated.

By the exact sequences resulting from $\mathbb{S}$

$$H^{j-1}_{\mathfrak{a}}(M) \to H^{j-1}_{\mathfrak{a}}(M/xM) \to H^j_{\mathfrak{a}}(M) \quad (j > 0),$$

we see that $H^k_{\mathfrak{a}}(M/xM)$ is finitely generated for all $k < i - 1$. So, by induction we have $\sharp\mathrm{Ass}_R(T) < \infty$. As N is finitely generated, we also have $\sharp\mathrm{Ass}_R(N) < \infty$. It therefore suffices to show that

$$\mathrm{Ass}_R(H^i_{\mathfrak{a}}(M)/N) \subseteq \mathrm{Ass}_R(T) \cup \mathrm{Ass}_R(N).$$

So, let $\mathfrak{p} \in \mathrm{Ass}_R(H^i_{\mathfrak{a}}(M)/N) \setminus \mathrm{Ass}_R(T)$. It suffices to show that $\mathfrak{p} \in \mathrm{Ass}_R(N)$. With an appropriate element $h \in H^i_{\mathfrak{a}}(M)$ we may write $\mathfrak{p} = (0 \underset{R}{:} \varrho(h))$. Consider the submodule $U := \overline{\delta}^{-1}(R\varrho(h)) \subseteq T$. The second row of the above diagram gives rise to an exact sequence

$$0 \longrightarrow U \xrightarrow{\overline{\delta}\restriction} R\varrho(h) \xrightarrow{\overline{x\cdot}\restriction} \overline{x} \cdot R\varrho(h) \longrightarrow 0,$$

where the maps $\overline{\delta}\restriction$ and $\overline{x\cdot}\restriction$ are obtained by restriction of $\overline{\delta}$ respectively of $\overline{x\cdot}$. As $U \subseteq T$ we have $\mathrm{Ass}_R(U) \subseteq \mathrm{Ass}_R(T)$ and hence $\mathfrak{p} \notin \mathrm{Ass}_R(U)$. As $\mathfrak{p} = (0 \underset{R}{:} \varrho(h)) \in \mathrm{Ass}_R(R\varrho(h))$, the above exact sequence yields $\mathfrak{p} \in \mathrm{Ass}_R(xR\varrho(h))$. As $\overline{x} \cdot R\varrho(h) = R\overline{x}\varrho(h) = Rxh$ we get $\mathfrak{p} \in \mathrm{Ass}_R(Rxh)$. So, there is some $s \in R$ such that $\mathfrak{p} = (0 \underset{R}{:} sxh)$.

As $x \in \mathfrak{a}$ and $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion, there is some $m \in \mathbb{N}$ with $x^m(xsh) = 0$ Therefore we have $x^m \in (0 \underset{R}{:} xsh) = \mathfrak{p}$, hence $x \in \mathfrak{p} = (0 \underset{R}{:} \varrho(h))$.

From this we obtain $xh + N = \varrho(xh) = x\varrho(h) = 0$, hence $xh \in N$ and thus $xsh \in N$. As $\mathfrak{p} = (0 \underset{R}{:} sxh)$, we get indeed $\mathfrak{p} \in \mathrm{Ass}_R(N)$. ■

As an immediate application (namely by taking $N = 0$) we get:

Corollary 10.2 *Let $\mathfrak{a}$ be an ideal of the noetherian ring R and let M be a finitely generated R-module. Let $i \in \mathbb{N}_0$ be such that $H^j_{\mathfrak{a}}(M)$ is finitely generated for all $j < i$. Then the set $\mathrm{Ass}_R(H^i_{\mathfrak{a}}(M))$ is finite.*

Example and Exercise 10.3 *(A.K. Singh)* Let x, y, z, u, v, w be inderminates. We set

$$R := \mathbb{Z}[x, y, z, u, v, w]/(xu + yv + zw) \text{ and } \mathfrak{a} := (x, y, z)R.$$

Then according to [Si] we have $\sharp\mathrm{Ass}_R(H^3_{\mathfrak{a}}(R)) = \infty$.

Show that for this choice of R and $\mathfrak{a}$ we have

$$H^i_{\mathfrak{a}}(R) \begin{cases} = 0, \text{ if } i \neq 2, 3, \\ \text{not finitely generated if } i = 2, 3. \end{cases}$$

•

11 The Cohomological Finiteness Dimension

Definition 11.1 Let $\mathfrak{a}$ be an ideal of the noetherian ring R and let M be a finitely generated R-module. The *(cohomological) $\mathfrak{a}$-finiteness dimension of M with respect to $\mathfrak{a}$* is defined by

$$f_{\mathfrak{a}}(M) := \inf\{r \in \mathbb{N}_0 \mid H^r_{\mathfrak{a}}(M) \text{ not finitely generated }\}.$$

•

Remark 11.2 A) Let $R, \mathfrak{a}$ and M be as above. Then:

a) $f_{\mathfrak{a}}(M) \in \mathbb{N} \cup \{\infty\}$ with $f_{\mathfrak{a}}(M) = \infty$ if and only if $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i \in \mathbb{N}_0$, hence if and only if $\mathrm{cd}_{\mathfrak{a}}(M) \leq 0$ (cf 9.4).

b) $t_{\mathfrak{a}}(M) \leq f_{\mathfrak{a}}(M)$.

c) If $f_{\mathfrak{a}}(M) < \infty$, then $f_{\mathfrak{a}}(M) \leq \mathrm{cd}_{\mathfrak{a}}(M)$.

We now may formulate 9.4 in the form:

d) $\mathrm{cd}_{\mathfrak{a}}(M) > 0 \Longrightarrow f_{\mathfrak{a}}(M) \leq \mathrm{cd}_{\mathfrak{a}}(M)$.

B) Keeping the notation of part A) we may reformulate 10.2 in the form

a) $i \leq f_{\mathfrak{a}}(M) \Longrightarrow \sharp \mathrm{Ass}_R(H^i_{\mathfrak{a}}(M)) < \infty$. •

Proposition 11.3 *Let $\mathfrak{a}$ be an ideal of the noetherian ring R and let M be a finitely generated R-module. Then*

$$f_{\mathfrak{a}}(M) = \inf\{r \in \mathbb{N}_0 \,\big|\, \mathfrak{a} \nsubseteq \sqrt{0 \underset{R}{:} H^r_{\mathfrak{a}}(M)}\}.$$

Proof. Let $s_{\mathfrak{a}}(M) := \inf\{r \in \mathbb{N}_0 | \mathfrak{a} \nsubseteq \sqrt{0 \underset{R}{:} H^r_{\mathfrak{a}}(M)}\}$. First, let $i < f_{\mathfrak{a}}(M)$. Then $H^i_{\mathfrak{a}}(M)$ is a finitely generated $\mathfrak{a}$-torsion module. It follows that there is some $n \in \mathbb{N}$ with $\mathfrak{a}^n H^i_{\mathfrak{a}}(M) = 0$, (cf 3.4 c)). So, we get $\mathfrak{a} \subseteq \sqrt{0 \underset{R}{:} H^i_{\mathfrak{a}}(M)}$ for all $i < f_{\mathfrak{a}}(M)$. Consequently $f_{\mathfrak{a}}(M) \leq s_{\mathfrak{a}}(M)$.

We now prove the converse inequality. To do so, it suffices to show that for each $s \in \mathbb{N}$ we have the implication

$$\mathfrak{a} \subseteq \sqrt{(0 \underset{R}{:} H^i_{\mathfrak{a}}(M))} \text{ for all } i < s \Longrightarrow s \leq f_{\mathfrak{a}}(M).$$

We prove this by induction on s. If $s = 1$ there is nothing to show (cf 11.2 A) a)). So, let $s > 1$. It suffices to show that the modules $H^i_{\mathfrak{a}}(M)$ are finitely generated for all $i \in \{1, \cdots, s-1\}$.

Let $\overline{M} := M/\Gamma_{\mathfrak{a}}(M)$. Then $H^0_{\mathfrak{a}}(\overline{M}) \cong \Gamma_{\mathfrak{a}}(\overline{M}) = 0$ and $H^i_{\mathfrak{a}}(\overline{M}) \cong H^i_{\mathfrak{a}}(M)$ for all $i > 0$. Therefore $\mathfrak{a} \subseteq \sqrt{0 : H^i_{\mathfrak{a}}(\overline{M})}$ for all $i < s$ and it suffices to show the R-modules $H^i_{\mathfrak{a}}(\overline{M})$ are finitely generated for all $i \in \{1, \cdots, s-1\}$. This allows to replace M by $\overline{M}$ and hence to assume that $H^0_{\mathfrak{a}}(M) = 0$.

We thus find an element $x \in \mathfrak{a} \cap \mathrm{NZD}_R(M)$. Now, for each $i < s$ there is some $n_i \in \mathbb{N}$ with $\mathfrak{a}^{n_i} \subseteq 0 \underset{R}{:} H^i_{\mathfrak{a}}(M)$. Let $n = \max\{n_i | i < s\}$. Then $x^n \in \mathfrak{a}^n \cap \mathrm{NZD}_R(M)$. In particular we have

$x^n H^i_{\mathfrak{a}}(M) = 0$ for all $i < s$. Applying cohomology to the exact sequence $0 \to M \xrightarrow{x^n\cdot} M \to M/x^nM \to 0$ we thus get exact sequences

$$0 \to H^{i-1}_{\mathfrak{a}}(M) \to H^{i-1}_{\mathfrak{a}}(M/x^nM) \to H^i_{\mathfrak{a}}(M) \to 0$$

for all $i \in \{1, \cdots, s-1\}$. As $\mathfrak{a}^n H^i_{\mathfrak{a}}(M) = 0$ for all these i it follows $\mathfrak{a}^{2n} H^{i-1}_{\mathfrak{a}}(M/x^nM) = 0$ and hence $\mathfrak{a} \subseteq \sqrt{0 : H^{i-1}_{\mathfrak{a}}(M/x^nM)}$ for all $i \in \{1, \cdots, s-1\}$. So, by induction the modules $H^{i-1}_{\mathfrak{a}}(M/x^nM)$ are finitely generated for all $i \in \{1, \cdots, s-1\}$. Now, the above exact sequences show that $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < s$. ■

Lemma 11.4 *Let $\mathfrak{a}$ be an ideal of the noetherian ring R. Let L be an R-module such that $\sharp Ass_R(L) < \infty$. Assume that for each $\mathfrak{p} \in Ass_R(L)$ there is some $n_{\mathfrak{p}} \in \mathbb{N}$ such that $(\mathfrak{a}^{n_{\mathfrak{p}}} L)_{\mathfrak{p}} = 0$. Then, with $n := \max\{n_{\mathfrak{p}} | \mathfrak{p} \in Ass_R(L)\}$ we have $\mathfrak{a}^n L = 0$.*

Proof. Let $x \in L$ and let $t_1, \cdots, t_r \in L$ be such that $\mathfrak{a}^n x = \sum_{i=1}^r Rt_i$. Let $\mathfrak{p} \in \mathrm{Ass}_R(L)$. Then $(\mathfrak{a}^n x)_{\mathfrak{p}} \subseteq (\mathfrak{a}^n L)_{\mathfrak{p}} \subseteq (\mathfrak{a}^{n_{\mathfrak{p}}} L)_{\mathfrak{p}} = 0$, hence $(\sum_{i=1}^r Rt_i)_{\mathfrak{p}} = 0$.

So, for each $i \in \{1, \cdots, r\}$ there is some $s_{i,\mathfrak{p}} \in R \backslash \mathfrak{p}$ such that $s_{i,\mathfrak{p}} t_i = 0$. Let $s_{\mathfrak{p}} := \prod_{i=1}^r s_{i,\mathfrak{p}}$. Then $s_{\mathfrak{p}} \in R\backslash\mathfrak{p}$ and $s_{\mathfrak{p}} t_i = 0$ for $i = 1, \cdots, r$, hence $s_{\mathfrak{p}} \mathfrak{a}^n x = 0$.

Let $\mathfrak{b} := \sum_{\mathfrak{p} \in \mathrm{Ass}_R(L)} Rs_{\mathfrak{p}}$. Then clearly $\mathfrak{b}\mathfrak{a}^n x = 0$. As $s_{\mathfrak{p}} \notin \mathfrak{p}$ we must have $\mathfrak{b} \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_R(L)$. As $\sharp \mathrm{Ass}_R(L) < \infty$ we get by the Prime Avoidance Principle that $\mathfrak{b} \nsubseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(L)} \mathfrak{p} = \mathrm{ZD}_R(L)$. So, there is an element $z \in \mathfrak{b} \cap \mathrm{NZD}_R(L)$. But now, $z\mathfrak{a}^n x \subseteq \mathfrak{b}\mathfrak{a}^n x = 0$ implies $\mathfrak{a}^n x = 0$. As $x \in L$ was arbitrarily chosen, we get $\mathfrak{a}^n L = 0$. ■

Theorem 11.5 *(Local-Global Principle of Faltings) Let $\mathfrak{a}$ be an ideal of the noetherian ring R, let $r \in \mathbb{N}$ and let M be a finitely generated R-module. Then, the following statements are equivalent:*

(i) $H^i_{\mathfrak{a}}(M)$ *is finitely generated for all $i < r$.*

(ii) The $R_{\mathfrak{p}}$-module $H^i_{\mathfrak{a}}(M)_{\mathfrak{p}}$ is finitely generated for all $i < r$ and for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

(iii) The $R_{\mathfrak{p}}$-module $H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M)_{\mathfrak{p}}$ is finitely generated for all $i < r$ and all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. "(i) $\Longrightarrow$ (ii)": Clear by the basic properties of localization.

"(ii) $\Longleftrightarrow$ (iii)": Clear by the fact that local cohomology commutes with localization (cf 9.2).

"(ii) $\Longrightarrow$ (i)": (Induction on r). The case $r = 1$ is clear as $H^0_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(M)$ is finitely generated. So, let $r > 1$. By induction we know already that $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < r-1$. It remains to be shown that $L := H^{r-1}_{\mathfrak{a}}(M)$ is finitely generated. By 11.3 it suffices to show that $\mathfrak{a} \subseteq \sqrt{0 \underset{R}{:} L}$, hence to find an $n \in \mathbb{N}$ with $\mathfrak{a}^n L = 0$.

By 10.2 we have $\mathrm{Ass}_R(L) < \infty$. Let $\mathfrak{p} \in \mathrm{Ass}_R(L)$. By our hypothesis $L_\mathfrak{p}$ is finitely generated over $R_\mathfrak{p}$. As L is $\mathfrak{a}$-torsion, $L_\mathfrak{p}$ is $\mathfrak{a}R_\mathfrak{p}$-torsion. So, there is some $n_\mathfrak{p} \in \mathbb{N}$ with $(\mathfrak{a}^{n_\mathfrak{p}} L)_\mathfrak{p} = \mathfrak{a}^{n_\mathfrak{p}} R_\mathfrak{p} L_\mathfrak{p} = (\mathfrak{a}R_\mathfrak{p})^{n_\mathfrak{p}} L_\mathfrak{p} = 0$, (cf 3.4 c)). Now, we conclude by 11.4. ■

Corollary 11.6 *Let $R, \mathfrak{a}$ and M be as in 11.5. Then*

$$f_\mathfrak{a}(M) = \min\{f_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}) \,\big|\, \mathfrak{p} \in \mathrm{Spec}(R)\} = \min\{f_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}) \,\big|\, \mathfrak{p} \in \mathrm{Var}(\mathfrak{a}) \cap \mathrm{Supp}(M)\}.$$

Proof. The first equation is immediate by 11.5. The second equation is easy from 9.2. ■

12 The Finiteness Theorem

Definition 12.1 A) Let $(R, \mathfrak{m})$ be a noetherian local ring. Then, the *depth* of a finitely generated R-module M is defined by (cf 4.3)

$$\mathrm{depth}_R(M) := \mathrm{grade}_M(\mathfrak{m}).$$

So, by 4.5 we may write

$$\mathrm{depth}_R(M) = t_\mathfrak{m}(M) = \inf\{i \in \mathbb{N}_0 \,\big|\, H^i_\mathfrak{m}(M) \neq 0\}.$$

B) Let $\mathfrak{a}$ be an ideal of the noetherian ring R and let M be a finitely generated R-module. We define a $\mathfrak{a}$-*adjusted depth of M at* a prime $\mathfrak{p} \in \mathrm{Spec}(R)$ by

$$\mathrm{adj}_\mathfrak{a}\mathrm{depth}(M_\mathfrak{p}) := \mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \ \mathrm{height}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p}),$$

where $\mathrm{height}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p})$ is understood to be the height of the ideal $\overline{\mathfrak{a}} := (\mathfrak{a}+\mathfrak{p})/\mathfrak{p} \subseteq R/\mathfrak{p} =: \overline{R}$, hence by the definition of height:

$$\mathrm{height}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p}) = \min\{\dim(\overline{R}_{\overline{\mathfrak{p}}} \,\big|\, \overline{\mathfrak{p}} \in \mathrm{Var}(\overline{\mathfrak{a}})\}$$

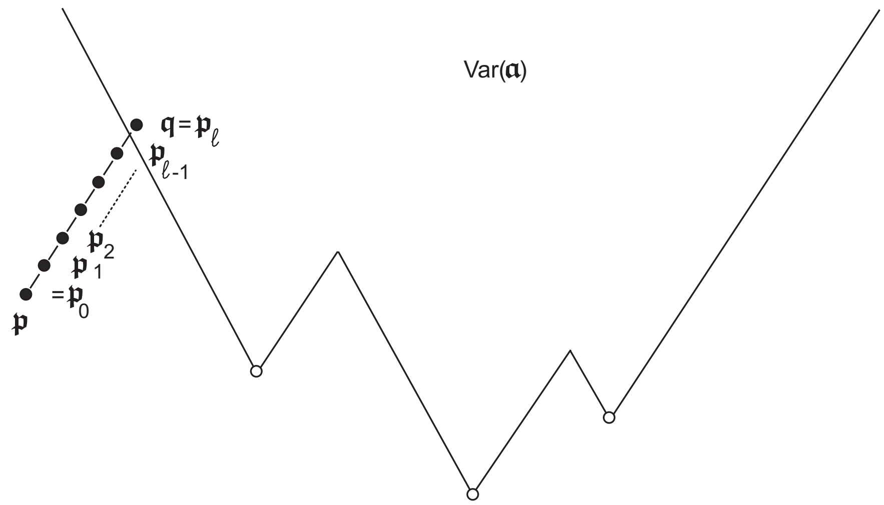
•

Remark 12.2 Keep the notation and hypotheses of 12.1. Then, the number

$\mathrm{height}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p})$ corresponds to the *"distance of* $\mathfrak{p}$ *from the variety* $\mathrm{Var}(\mathfrak{a})$*"*.

More precisely, if $\mathfrak{q} \in \mathrm{Spec}(R)$ with $\mathfrak{p} \subseteq \mathfrak{q}$ one can consider $\mathrm{height}(\mathfrak{q}/\mathfrak{p})$ as "the distance between $\mathfrak{p}$ and $\mathfrak{q}$" measured in terms of lengths of chains of primes which connect $\mathfrak{p}$ and $\mathfrak{q}$:

$$\mathrm{height}(\mathfrak{q}/\mathfrak{p}) = \max\left\{\ell \in \mathbb{N}_0 \,\big|\, \exists \mathfrak{p}_0, \cdots, \mathfrak{p}_\ell \in \mathrm{Spec}(R) : \mathfrak{p} \subseteq \mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_\ell \subseteq \mathfrak{q}\right\}.$$



Therefore we can say

$$\text{height}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p}) = \inf\{\ \text{height}(\mathfrak{q}/\mathfrak{p}) \,\big|\, \mathfrak{q} \subseteq \mathfrak{q} \ \text{ and } \ \mathfrak{q} \in \text{Var}(\mathfrak{a})\} := \text{“distance } (\mathfrak{a}, \text{Var}(\mathfrak{a}))\text{”}.$$

So, the adjusted depth measures the usual depth of M at $\mathfrak{p}$, that is $\text{depth}_{R_\mathfrak{p}}(M_\mathfrak{p})$, and adds to it the distance $\mathfrak{p}$ has from $\text{Var}(\mathfrak{a})$.

Having large depth at $\mathfrak{p}$, means that M “behaves well at $\mathfrak{p}$”. So, the $\mathfrak{a}$-adjusted depth measures the “well-behaviour of M at $\mathfrak{p}$” giving a bonus to points $\mathfrak{p}$ which are far away from $\text{Var}(\mathfrak{a})$. Therefore, the $\mathfrak{a}$-adjusted depth tells us, how well behaved M is at points $\mathfrak{p} \in \text{Spec}(R)$ “near the variety $\text{Var}(\mathfrak{a})$ of $\mathfrak{a}$”.

Reminder 12.3 By Krull’s Principal Ideal Theorem the maximal ideal $\mathfrak{m}$ of a local noetherian ring R cannot be generated be less than $\text{height}(\mathfrak{m}) = \dim(R)$ elements. A noetherian local ring $(R, \mathfrak{m})$ whose maximal ideal $\mathfrak{m}$ can be generated by $\dim(R)$ element is called a *regular local ring.*

A noetherian ring R is said to be *regular*, if $R_\mathfrak{p}$ is a regular local ring for each $\mathfrak{p} \in \text{Spec}(R)$.

•

Theorem 12.4 *(Finiteness Theorem of Grothendieck) Assume that the noetherian ring R is a homomorphic image of a regular ring. Then*

$$f_\mathfrak{a}(M) = \inf\{\text{adj}_\mathfrak{a}\text{depth}(M_\mathfrak{p}) \,\big|\, \mathfrak{p} \in \text{Spec}(R) \setminus \text{Var}(\mathfrak{a})\}.$$

Proof. See [B-S, 9.5.2]. ∎

Remarks 12.5 A) The hypothesis that R is a homomorphic image of a regular ring is not at all restrictive for most applications. It is satisfied for example whenever R is "essentially of finite" over a field (or over $\mathbb{Z}$). Let us recall, that a ring R is *essentially of finite type over a ring* R_0 if R is a ring of fractions of a finitely generated R_0-algebra.

B) The hypothesis that R is a homomorphic image of a regular ring can be replaced by the weaker condition, that R is a so-called *tolerable* ring, that is a ring which is *universally catenary* and whose *formal fibres are all Cohen-Macaulay rings* (cf [B-S, 9.6.7]). Let us recall here that a noetherian ring R is called a *Cohen-Macaulay ring* if all its localizations $R_{\mathfrak{p}}$, with $\mathfrak{p} \in \mathrm{Spec}(R)$ are local Cohen-Macaulay rings. A local noetherian ring $(R, \mathfrak{m})$ is said to be a Cohen-Macaulay ring if its depth is maximal, that is $\mathrm{depth}_R(R) = \dim(R)$.

In particular, all homomorphic images of Cohen-Macaulay rings are tolerable. Moreover all regular rings are Cohen-Macaulay rings. So, in 12.4 one can replace the condition "regular" by the weaker condition "Cohen-Macaulay". •

Exercises 12.6 A) Let k be an algebraically closed field, let $r \in \mathbb{N}$ and let $V \subseteq k^r$ be an irreducible affine variety. Let $U \subsetneqq V$ be a non-empty open subset and let $Z = V \setminus U$. Prove that $\mathcal{O}(U)$ is a finitely generated $\mathcal{O}(V)$-module if and only if Z is of "codimension ≥ 2 in V", that is if and only if $\mathrm{height}(I_V(Z)) \geq 2$.

B) Let $(R, \mathfrak{m})$ be a local domain which is a homomorphic image of local (noetherian) Cohen-Macaulay ring. Show that the R-modules $H^i_{\mathfrak{m}}(R)$ are finitely generated for all $i < \dim(R)$ if and only if "R is Cohen-Macaulay on its punctured spectrum", that is if and only if $R_{\mathfrak{p}}$ is Cohen-Macaulay ring for all $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$. •

Fourth Lecture: Connectivity in Algebraic Varieties

13 Analytically Irreducible Rings

Definition 13.1 A local noetherian domain $(R, \mathfrak{m})$ is said to be *analytically irreducible* if its completion $(\hat{R}, \mathfrak{m}\hat{R})$ with respect to the $\mathfrak{m}$-adic topology is an integral domain. •

Theorem 13.2 *(Vanishing Theorem of Hartshorne-Lichtenbaum) Let $(R, \mathfrak{m})$ be an analytically irreducible domain and let $\mathfrak{a} \subseteq \mathfrak{m}$ be an ideal of R such that $\dim(R/\mathfrak{a}) > 0$. Then $\mathrm{cd}_{\mathfrak{a}}(R) < \dim(R)$.*

Proof. [B-S, 8.2.10]. ■

Theorem 13.3 *(Non-Vanishing Theorem of Grothendieck) Let* $(R, \mathfrak{m})$ *be a local noetherian ring and let* M *be a finitely generated* R*-module. Then* $H_{\mathfrak{m}}^{\dim(M)}(M) \neq 0$.

Proof. [B-S, 6.1.4]. ∎

Proposition 13.4 *Assume that* $(R, \mathfrak{m})$ *is a local analytically irreducible domain and let* $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{m}$ *be two ideals such that* $\dim(R/\mathfrak{a}), \dim(R/\mathfrak{b}) > 0 = \dim(R/(\mathfrak{a}+\mathfrak{b}))$. *Then*

$$ara(\mathfrak{a} \cap \mathfrak{b}) \geq \dim(R) - 1.$$

Proof. Let $d := \dim(R)$. We have to show that $\operatorname{ara}(\mathfrak{a} \cap \mathfrak{b}) \geq d-1$. The Mayer-Vietoris sequence gives an exact sequence

$$H_{\mathfrak{a}\cap\mathfrak{b}}^{d-1}(R) \to H_{\mathfrak{a}+\mathfrak{b}}^{d}(R) \to H_{\mathfrak{a}}^{d}(R) \oplus H_{\mathfrak{b}}^{d}(R).$$

By 13.2 we have $H_{\mathfrak{a}}^{d}(R) = H_{\mathfrak{b}}^{d}(R) = 0$. As $\dim(R/(\mathfrak{a}+\mathfrak{b})) = 0$ we have $\sqrt{\mathfrak{a}+\mathfrak{b}} = \mathfrak{m}$, hence $H_{\mathfrak{a}+\mathfrak{b}}^{d}(R) = H_{\sqrt{\mathfrak{a}+\mathfrak{b}}}^{d}(R) = H_{\mathfrak{m}}^{d}(R) \neq 0$ (cf 3.1 C) a) and 13.3).

It follows $H_{\mathfrak{a}\cap\mathfrak{b}}^{d-1}(R) \neq 0$, hence $\operatorname{cd}_{\mathfrak{a}\cap\mathfrak{b}}(R) \geq d-1$, thus $\operatorname{ara}(\mathfrak{a} \cap \mathfrak{b}) \geq d-1$ (cf 6.5). ∎

14 Affine Algebraic Cones

Reminders 14.1 Let k be an algebraically closed field and let $r \in \mathbb{N}$. An *affine (algebraic) cone (with vertex* $\underline{0} \in k^{r+1}$*)* is an affine variety $V \subseteq k^{r+1}$ such that $\underline{0} \in V$ and such that for each $p \in V \backslash \{\underline{0}\}$ the straight line joining p and $\underline{0}$ is contained in V. As an exercise one can prove that for an affine variety $V \subseteq k^{r+1}$ the following are equivalent:

(i) V is an affine cone;

(ii) $V = V(\mathfrak{a})$, where $\mathfrak{a} \subseteq k[x_0, \cdots, x_r]$ is a graded ideal;

(iii) $I(V) \subseteq k[x_0, \cdots, x_r]$ is a graded ideal;

(iv) $V = V(f_1, \cdots, f_t)$ with homogeneous polynomials $f_i \in k[x_0, \cdots, x_r]$.

B) Let $V \subseteq k^{r+1}$ b an irreducible affine cone. Then $\mathcal{O}(V) \cong k[x_0, \cdots, x_r]/I(V)$ is a domain (cf 8.1 C)). As the ideal $I(V) \subseteq k[x_0, \cdots, x_r]$ is graded, the ring $k[x_0, \cdots, x_r]/I(V)$ carries a natural grading, given by

$$(k[x_0, \cdots, x_r]/I(V))_n = (k[x_0, \cdots, x_r]_n + I(V))/I(V) \cong k[x_0, \cdots, x_r]_n/I(V)_n;$$

(prove this as an exercise). Correspondingly, the domain $\mathcal{O}(V)$ carries a natural grading

a) $\mathcal{O}(V) = \oplus_{n\geq 0}\mathcal{O}(V)_n$, with $\mathcal{O}(V)_n \cong k[x_0, \cdots, x_r]_n/I(V)_n$, $(\forall n \in \mathbb{N}_0)$.

In particular

b) $\mathcal{O}(V) = k[x_0\restriction_V, \cdots, x_r\restriction_V]$ with $x_i\restriction_V \in \mathcal{O}(V)_1$ for $i = 0, \cdots, r$.

C) Keep the notations of part B). Let $I_V(\{\underline{0}\}) \subseteq \mathcal{O}(V)$ be the vanishing ideal of $\underline{0}$ in $\mathcal{O}(V)$. It is easy to check that

a) $I_V(\{\underline{0}\}) = \oplus_{n>0}\mathcal{O}(V)_n =: \mathcal{O}(V)_+$.

We now consider the *local ring of V at its vertex*:

$$\mathcal{O}_{V,\underline{0}} := \mathcal{O}(V)_{I_V(\{\underline{0}\})} = \mathcal{O}(V)_{\mathcal{O}(V)_+} \text{ and the ideal } \mathfrak{m}_{V,\underline{0}} := I_V(\{\underline{0}\})\mathcal{O}(V)_{I_V(\{\underline{0}\})}$$

Then:

b) $(\mathcal{O}_{V,\underline{0}}, \mathfrak{m}_{V,0})$ is a local noetherian domain.

c) $\dim(\mathcal{O}_{V,\underline{0}}) = \dim(\mathcal{O}(V)) := \dim(V)$. •

Reminders and Exercises 14.2 A) Let k be a field and let $R = \oplus_{n\in\mathbb{N}_0}R_0$ be a *noetherian homogeneous k-algebra*. So, we have $R_0 = k$ and $R = k[f_0, \cdots, f_r]$ with finitely many elements $f_0, \cdots, f_r \in R_1$. Consider the *irrelevant ideal* $R_+ := \oplus_{n>0}R_n$ of R. As $R/R_+ \cong k$, we see that R_+ is a maximal ideal of R. Moreover, any graded ideal of R is contained in R_+. So R_+ is the *graded* (or homogeneous) maximal ideal of R.

B) Keep the previous notations. We consider the local ring $(R_{R_+}, R_+R_{R_+})$. As $R_+ \subseteq R$ is a maximal ideal it follows easily, that for each $n \in \mathbb{N}$, the natural homomorphism

$$\varepsilon_n : R/(R_+)^n \to R_{R_+}/(R_+R_{R_+})^n,\ \left(x + (R_+)^n \mapsto \frac{x}{1} + (R_+R_{R_+})^n\right)$$

is an isomorphism of rings. So, we get an isomorphism of rings

a) $\varepsilon := \varprojlim_n \varepsilon_n : (R, R_+)^\wedge = R/(R_+)^n \xrightarrow[\cong]{} \varprojlim_n R_{R_+}/(R_+R_{R_+})^n = (R_{R_+}, R_+R_{R_+})^\wedge$

between the R_+-adic completion of the homogeneous k-algebra R and the $R_+R_{R_+}$-adic completion of the local ring R_{R_+}.

C) Next, consider the direct product of k-vector spaces $\Pi_{n\in\mathbb{N}_0}R_n =: \hat{R}$. On $\hat{R}$ we may introduce a binary operation

$$\cdot : \hat{R}\times\hat{R}\to\hat{R};\ (x_n)_{n\in\mathbb{N}}\cdot(y_n)_{n\in\mathbb{N}} := (\sum_{i+j=n} x_iy_j)_{n\in\mathbb{N}}.$$

Then, it is not hard to verify:

a) $\hat{R}$ furnished with its standard addition and the previous multiplication "$\cdot$" is a commutative local ring with unit element $1_{\hat{R}} = (1,0,0,\cdots)$ and maximal ideal $\Pi_{n>0}R_n =: \widehat{R_+}$.

Moreover, it is immediate to see

b) The inclusion map $R = \oplus_{n\in\mathbb{N}_0}R_n \overset{i}{\rightarrowtail} \Pi_{n\in\mathbb{N}_0}R_n = \hat{R}$ is a homomorphism of rings and $R_+\hat{R} = \widehat{R_+}$.

Finally, we can say

c) R is a domain if and only if $\hat{R}$ is.

D) As R is homogeneous, we have

a) $(R_+)^n = R_{\geq n} = \oplus_{m\geq n}R_m,$ for all $n\in\mathbb{N}$.

In particular, for each $n\in\mathbb{N}$ and each $\overline{x}\in R/(R_+)^n$ there is a unique element $\nu^{(n)}(\overline{x})\in R_0\oplus\cdots\oplus R_{n-1}$ such that $\nu^{(n)}(\overline{x}) + (R_+)^n = \overline{x}$. We write $\nu^{(n)}(\overline{x}) = \sum_{i=0}^{n-1}\nu_i^{(n)}(\overline{x})$ with $\nu_i^{(n)}(\overline{x})\in R_i$. Keep in mind that

$$(R,R_+)^\wedge = \varprojlim_n (R/(R_+)^n) = \{(\overline{x}^{(n)})_{n\in\mathbb{N}}\in\Pi_{n\in\mathbb{N}}(R/(R_+)^n)\,\big|\,\overline{x}^{(n+1)}\overset{\text{can}}{\mapsto}\overline{x}^{(n)},\ \forall n\in\mathbb{N}\}$$
$$= \{(\overline{x}^{(n)})_{n\in\mathbb{N}}\in\Pi_{n\in\mathbb{N}}(R/(R_+)^n)\,\big|\,\nu_i^{(n+1)}(\overline{x}^{(n+1)}) = \nu_i^{(n)}(\overline{x}^{(n)}) \text{ for all } n\in\mathbb{N} \text{ and all } i<n\}.$$

So, we have a bijective map

b) $\psi : (R,R_+)^\wedge \to \hat{R} = \Pi_{n\geq 0}R_n$, given by $(\overline{x}^{(n)})_{n\in\mathbb{N}} \mapsto (\nu_n^{(n+1)}(\overline{x}^{(n+1)}))_{n\in\mathbb{N}_0}$.

It is not too hard to calculate that ψ is a homomorphism of rings. So, we get isomorphisms of rings (cf B) a)):

c) $\psi : (R, R_+)^\wedge \xrightarrow{\cong} \hat{R} = \Pi_{n \geq 0} R_n;\ \psi \circ \varepsilon^{-1} : (R_+, R_+ R_{R_+})^\wedge \xrightarrow{\cong} \Pi_{n \geq 0} R_n.$ •

Lemma 14.3 *Let k be an algebraically closed field, let $r \in \mathbb{N}$ and let $V \subseteq k^{r+1}$ be an irreducible affine cone. Then, the local ring $\mathcal{O}_{V,\underline{0}}$ of V at its vertex $\underline{0}$ is an analytically irreducible noetherian local domain with* $\dim(\mathcal{O}_{V,0}) = \dim(V)$.

Proof. By 14.1 C) we already know that $\mathcal{O}_{V,0}$ is a local noetherian domain of dimension $\dim(V)$ and with maximal ideal $\mathfrak{m}_{V,\underline{0}}$. We set $R := \mathcal{O}(V) = \oplus_{n \geq 0} \mathcal{O}(V)_n$. According to 14.1 B) we know that R is a noetherian homogeneous k-algebra and a domain. Now, on use of 14.2 D) c) we get $\widehat{\mathcal{O}_{V.0}} = (R_{R_+})^\wedge = (R_{R_+}, R_+ R_{R_+})^\wedge \cong \Pi_{n \geq 0} R_n$. So, by 14.2 C) c) we see that $\widehat{\mathcal{O}_{V,0}}$ is a domain. ∎

Lemma 14.4 *Let k be an algebraically closed field, let $r \in \mathbb{N}$ and let $V \subseteq k^{r+1}$ be an irreducible affine cone. Let $Z \subseteq V$ be another affine cone and let $Z_1, Z_2 \subseteq Z$ be two closed subsets such that $Z_1 \cup Z_2 = Z, Z_1 \cap Z_2 = \{\underline{0}\}$ and $Z_1, Z_2 \supsetneqq \{\underline{0}\}$. Then $I_V(\{\underline{0}\}) \subseteq \mathcal{O}(V)$ is not a minimal prime of $I_V(Z_i)$ for $i = 1, 2$.*

Proof. Assume to the contrary, that $I_V(\{\underline{0}\})$ is a minimal prime of $I_V(Z_i)$ for $i = 1$ or for $i = 2$. Without loss of generality, we may assume that $I_V(\{\underline{0}\})$ is a minimal prime of $I_V(Z_1)$.

We find some $\tilde{f} \in \mathcal{O}(V) \backslash I_V(\{\underline{0}\})$ which is contained in the intersection of all (the finitely many) minimal primes of $I_V(Z_1)$ different from $I_V(\{\underline{0}\})$. According to 8.1 C) b) there is a polynomial $f \in k[x_0, \cdots, x_r]$ with $\tilde{f} = f\restriction_V$. We consider the open set

$$U := V \backslash V(f) = \{p \in V \mid \tilde{f}(p) \neq 0\}.$$

Clearly, $\underline{0} \in U$. Next, let $q \in Z_1 \backslash \{\underline{0}\}$. Then $I_V(\{\underline{0}\}) \neq I_V(\{q\}) \supseteq I_V(Z_1)$ shows that the maximal ideal $I_V(\{\underline{0}\}) \subseteq \mathcal{O}(V)$ must contain a minimal prime $\mathfrak{p} \neq I_V(\{\underline{0}\})$ of $I_V(Z_1)$. In particular we get $\tilde{f} \in \mathfrak{p} \subseteq I_V(\{q\})$, hence $\tilde{f}(q) = 0$, thus $q \notin U$. This first shows that $U \cap Z_1 = \{\underline{0}\}$.

Now, let $L \subseteq k^{r+1}$ be the straight line through $\underline{0}$ and q. As Z is a cone we have $L \subseteq Z$. Moreover, $L = (L \cap Z_1) \cup (L \cap Z_2)$, where $L \cap Z_i \subseteq L$ is closed. As $q \in Z_1 \backslash \{\underline{0}\}$ it follows from $Z_1 \cap Z_2 = \{\underline{0}\}$, that $q \notin Z_2$. Therefore $L \cap Z_2 \subsetneqq L$ and so $L \cap Z_2$ is finite, (cf 8.5 A).

Moreover $U \cap L$ is an open neighborhood of $\underline{0}$ in L. In particular, there is a point $p \in (L \cap U) \backslash \{\underline{0}\}$. As $U \cap Z_1 = \{\underline{0}\}$ it follows $p \notin Z_1$, hence $L \cap Z_1 \subsetneqq L$. So $L \cap Z_1$ is a finite set, too. So, the infinite set L is the union of the two finite sets $L \cap Z_1$, and $L \cap Z_2$, a contradiction. ∎

Proposition 14.5 *Let k be an algebraically closed field, let $r \in \mathbb{N}$, let $V \subseteq k^{r+1}$ be an irreducible affine cone of dimension $d+1$. Let $s < d$ and let $f_1, \cdots, f_s \in k[x_0, \cdots, x_r]$ be homogeneous polynomials.*

Then $[V \cap V(f_1, \cdots, f_s)] \backslash \{\underline{0}\}$ is connected.

Proof. Clearly $Z := V \cap V(f_1, \cdots, f_s) = V(I(V) + (f_1, \cdots, f_s))$ is an affine cone with $I(Z) = \sqrt{I(V) + (f_1, \cdots, f_s)}$. Writing $\bullet \restriction$ for the restriction map $k[x_0, \cdots, x_r] \twoheadrightarrow \mathcal{O}(V)$ we get $I_V(Z) = \bullet \restriction (I(Z)) = \sqrt{(f_1 \restriction, \cdots, f_s \restriction)}$. In particular we have $\operatorname{ara}(I_V(Z)) \leq s$. Assume that $Z \backslash \{\underline{0}\}$ is disconnected. Then, $Z \backslash \{\underline{0}\}$ is the disjoint union of two non-empty relatively closed subsets. We thus may write $Z = Z_1 \cup Z_2$ with closed sets $Z_1, Z_2 \subseteq V$ with $Z_1, Z_2 \supsetneqq \{\underline{0}\}$ and $Z_1 \cap Z_2 = \{\underline{0}\}$.

It follows $I_V(Z) = I_V(Z_1 \cup Z_2) = I_V(Z_1) \cap I_V(Z_2)$ and $\sqrt{I_V(Z_1) + I_V(Z_2)} = I_V(\{\underline{0}\})$. Moreover 14.4 implies that $I_V(\{\underline{0}\})$ is not a minimal prime of $I_V(Z_i)$ for $i = 1, 2$.

Now, consider the analytically irreducible local noetherian domain $R := \mathcal{O}_{V,\underline{0}}$ of dimension $d+1$ (cf 14.3) and the ideals $\mathfrak{a} := I_V(Z_1)\mathcal{O}_{V,\underline{0}}$ and $\mathfrak{b} := I_V(Z_2)\mathcal{O}_{V,\underline{0}}$.

It follows $\mathfrak{a} \cap \mathfrak{b} = I_V(Z)\mathcal{O}_{V,\underline{0}}$ and hence $\operatorname{ara}(\mathfrak{a} \cap \mathfrak{b}) \leq \operatorname{ara}(I_V(Z)) \leq s$. Moreover by 14.4 $I_V(\{0\})$ is not a minimal prime of $I_V(Z_i)$ for $i = 1, 2$. Therefore $\dim(R/\mathfrak{a}), \dim(R/\mathfrak{b}) > 0 = \dim(R/(\mathfrak{a} + \mathfrak{b}))$. In addition $\sqrt{I_V(Z_1) + I_V(Z_2)} = I_V(\{0\}$.

By 13.4 it follows $s \geq \operatorname{ara}(\mathfrak{a} \cap \mathfrak{b}) \geq d + 1 - 1 = d$, a contradiction. ■

Proposition 14.6 *Let k be an algebraically closed field, let $r \in \mathbb{N}$ and let $V, W \subseteq k^{r+1}$ be two irreducible affine cones such that $\dim(V) + \dim(W) > r + 2$. Then, the set $V \cap W \backslash \{\underline{0}\}$ is connected.*

Proof. Consider the diagonal embedding $\delta : k^{r+1} \to k^{r+1} \times k^{r+1}$; $(\underline{c} \mapsto (\underline{c}, \underline{c}))$ and the diagonal $\Delta = \operatorname{Im}(\underline{\delta})$. Writing $\mathcal{O}(k^{r+1} \times k^{r+1}) = k[x_0, \cdots, x_r, y_0, \cdots, y_r]$, we have

$$\Delta = V(x_0 - y_0, x_1 - y_1, \cdots, x_r - y_r).$$

Moreover, the diagonal embedding δ yields an isomorphism of algebraic varieties

$$\delta \restriction : V \cap W \xrightarrow{\cong} (V \times W) \cap \Delta$$

and hence a homeomorphism. Observe that $\delta(\underline{0}) = (\underline{0}, \underline{0})$. It thus suffices to show that $[(V \times W) \cap \Delta] \backslash \{(\underline{0}, \underline{0})\}$ is connected. But $V \times W \subseteq k^{r+1} \times k^{r+1} = k^{2r+2}$ is an irreducible affine variety with $\dim(V \times W) = \dim(V) + \dim(W)$. Clearly $V \times W$ is also a cone with vertex $(\underline{0}, \underline{0})$. By 14.5 and as $r + 1 < \dim(V \times W) - 1$, it follows that

$$[(V \times W) \cap \Delta] \backslash \{(\underline{0}, \underline{0})\} = [(V \times W) \cap V(x_0 - y_0, x_1 - y_1, \cdots, x_r - y_r)] \backslash \{(\underline{0}, \underline{0})\}$$

is connected. ■

15 Projective Varieties

Reminder 15.1 A) Let k be an algebraically closed field and let $r \in \mathbb{N}$. We define the *projective r-space* $\mathbb{P}^r_k$ as the space of all lines $L \subseteq k^{r+1}$ through $\underline{0}$. For $\underline{c} = (c_0, \cdots, c_r) \in k^{r+1} \backslash \{\underline{0}\}$, we write

$$\pi(\underline{c}) = (c_0 : \cdots : c_r) := k\underline{c} \subseteq k^{r+1}$$

for the line running through $\underline{0}$ and $\underline{c}$. We thus may write

$$\mathbb{P}^r_k = \{(c_0 : \cdots : c_r) \,\big|\, (c_0, \cdots, c_r) \in k^{r+1} \backslash \{\underline{0}\}\}$$

and get a surjective map

$$\pi : k^{r+1} \backslash \{\underline{0}\} \twoheadrightarrow \mathbb{P}^r_k, \ (\underline{c} \mapsto (c_0 : c_1 : \cdots : c_r)),$$

the *natural projection.*

Observe in particular that

a) $(c_0 : \cdots : c_r) = (b_0 : \cdots : b_r) \Longleftrightarrow \exists \lambda \in k \backslash \{0\} : \lambda \underline{c} = \underline{b}$.

B) A *projective (algebraic) variety* $V \subseteq \mathbb{P}^r_k$ is a set of the form $V := \pi(\tilde{V} \backslash \{\underline{0}\})$ with $\tilde{V} \subseteq k^{r+1}$ an affine cone. In this situation we also write

$$V = \mathbb{P}(\tilde{V})$$

and call V the *projectivization* of $\tilde{V}$. Observe that in this case $\tilde{V}$ is uniquely determined by V and has the form $\tilde{V} = \pi^{-1}(V) \cup \{\underline{0}\}$. We call $\tilde{V}$ the *affine cone* over V and denote it by $\mathrm{cone}(V)$. Thus:

$$\mathrm{cone}(V) := \pi^{-1}(V) \cup \{\underline{0}\}.$$

C) Keeping the above notations we thus have two bijections which are inverse to each other:

$$\left\{ \tilde{V} \subseteq k^{r+1} \,\middle|\, \begin{array}{c} \tilde{V} = \text{ affine} \\ \text{cone} \end{array} \right\} \underset{\mathrm{cone}(\bullet)}{\overset{\mathbb{P}(\bullet)}{\rightleftarrows}} \left\{ V \subseteq \mathbb{P}^r_k \,\middle|\, \begin{array}{c} V = \text{ projective} \\ \text{variety} \end{array} \right\}.$$

In view of this the following statements seem not very surprising for a projective variety $V \subseteq \mathbb{P}^r_k$

a) $\dim(\mathrm{cone}(V)) = \dim(V) + 1$;

b) $\mathrm{cone}(V)$ irreducible $\Longleftrightarrow V$ irreducible ;

c) $\mathrm{cone}(V) \backslash \{\underline{0}\}$ connected $\Longleftrightarrow V$ connected .

Moreover:

d) The assignments $\mathbb{P}(\bullet)$ and $\mathrm{cone}(\bullet)$ commute with finite unions and intersections. $\bullet$

Theorem 15.2 *(Connectedness Theorem of Bertini-Grothendieck) Let k be an algebraically closed field, let $r \in \mathbb{N}$ and let $V \subseteq \mathbb{P}^r_k$ be an irreducible projective variety of dimension $d > 1$. Let $s < d$ and let $f_1, \cdots, f_s \in k[x_0, \cdots, x_r]$ be homogeneous polynomials.*

Then $V \cap \mathbb{P}(V(f_1, \cdots, f_s))$ is connected.

Proof. Clear from 14.5 and 15.1 C). ∎

Theorem 15.3 *(Connectedness Theorem of Fulton-Hansen and Faltings) Let k be an algebraically closed field, let $r \in \mathbb{N}$ and let $V, W \subseteq \mathbb{P}^r_k$ be two irreducible projective varieties such that $\dim(V) + \dim(W) > r$. Then $V \cap W$ is connected.*

Proof. Easy from 14.6 and 15.1 C). ∎

Example 15.4 A) We write $\mathcal{O}(k^3) = k[x_1, x_2, x_3]$, where k is an algebraically closed field. We consider the two surfaces in k^3 given by

$$\overset{\circ}{V} := V(x_1 - x_2x_3),\ \overset{\circ}{W} := V(x_1 + x_2^2 - x_2x_3 - 1),$$

which are indeed both irreducible as their defining polynomials are irreducible. Then

$$\begin{aligned}\overset{\circ}{V} \cap \overset{\circ}{W} &= V(x_1 - x_2x_3, x_1 + x_2^2 - x_2x_3 - 1)\\ &= V(x_1 - x_2x_3, x_2^2 - 1)\\ &= V(x_1 - x_2x_3, x_2 + 1) \cup V(x_1 - x_2x_3, x_2 - 1)\\ &= \underbrace{V(x_1 + x_3, x_2 + 1)}_{\mathbb{L}_1^0 :=} \cup \underbrace{V(x_1 - x_3, x_2 - 1)}_{\mathbb{L}_2^0 :=}.\end{aligned}$$

So, $\overset{\circ}{V} \cap \overset{\circ}{W}$ is the union of the two skew lines $\mathbb{L}_1^0$ and $\mathbb{L}_2^0$ and hence is disconnected. But, on the other hand

$$\dim(\overset{\circ}{V}) + \dim(\overset{\circ}{W}) = 2 + 2 > 3.$$

This means that the analogue of 15.3 need not hold in the affine setting. Now, let us write $\mathcal{O}(k^4) = k[x_0, x_1, x_2, x_3]$ and consider the projective varieties

$$\mathbb{P}(V(x_0x_1 - x_2x_3)) =: V \subseteq \mathbb{P}^3_k \quad \text{and} \qquad \mathbb{P}(V(x_0x_1 + x_2^2 - x_2x_3 - x_0^2)) =: W \subseteq \mathbb{P}^3_k$$

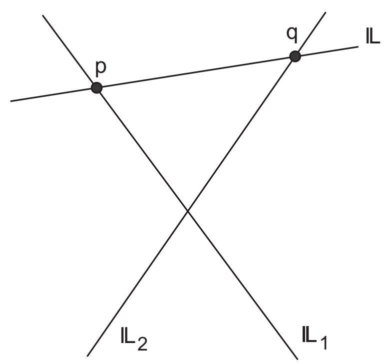
whose affine cones are respectively

$$\operatorname{cone}(V) = V(x_0x_1 - x_2x_3) \subseteq k^4 \text{ and } \operatorname{cone}(W) = V(x_0x_1 + x_2^2 - x_2x_3 - x_0^2) \subseteq k^4$$

In particular, these cones are both irreducible (as their defining polynomials are) and of dimension $4-1=3$. So, V and W are both irreducible and of dimension 2. As $\dim(V) + \dim(W) = 4 > 3$ we may expect by 15.3 that $V \cap W$ is connected. Indeed it is easy to check that $V(x_0x_1 - x_2x_3, x_0x_1 + x_2^2 - x_2x_3) = V(x_1 + x_3, x_2 + x_0) \cup V(y_1 - x_1, x_2 - x_0) \cup V(x_0, x_2)$. Therefore we have

$$\begin{aligned} V \cap W &= \mathbb{P}(\operatorname{cone}(V)) \cap \mathbb{P}(\operatorname{cone}(W)) \\ &= \mathbb{P}(\operatorname{cone}(V) \cap \operatorname{cone}(W)) \\ &= \mathbb{P}(V(x_0x_1 - x_2x_3, x_0x_1 + x_2^2 - x_2x_3 - x_0^2)) \\ &= \mathbb{P}(V(x_1 + x_3, x_2 + x_0) \cup V(x_1 - x_3, x_2 - x_0) \cup V(x_0, x_2)) \\ &= \underbrace{\mathbb{P}(V(x_1 + x_3, x_2 + x_0))}_{\mathbb{L}_1 :=} \cup \underbrace{\mathbb{P}(V(x_1 - x_3, x_2 - x_0)}_{\mathbb{L}_2 :=} \cup \underbrace{\mathbb{P}(V(x_0, x_2))}_{\mathbb{L} :=}. \end{aligned}$$

Now, $\mathbb{L}_1, \mathbb{L}_2 \subseteq \mathbb{P}_3^k$ form a pair of skew lines and $\mathbb{L} \subseteq \mathbb{P}_3^k$ is a line which intersects $\mathbb{L}_1$ at $p := (0:1:0:-1)$ and $\mathbb{L}_2$ at $q := (0:1:0:1)$.



In particular $V \cap W = \mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}$ is connected, as predicted by the connectedness theorem 15.3.

B) To get a slightly better insight, we consider the canonical embedding

$$\sigma_0 : k^2 \hookrightarrow \mathbb{P}_k^3,\ ((c_1, c_2, c_3) \mapsto (1 : c_1 : c_2 : c_3))$$

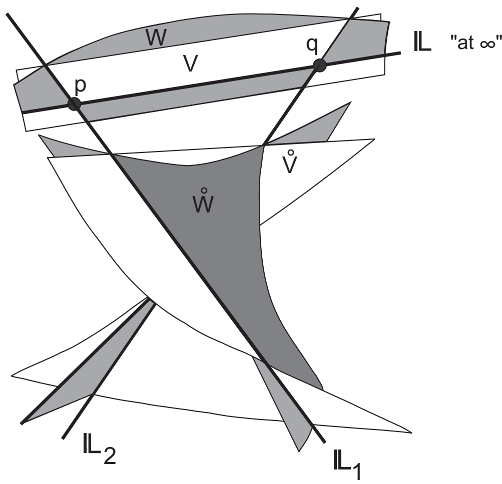
and identify $k^3 := \operatorname{Im}(\sigma_0) = \{(1 : c_1 : c_2 : c_3) | c_i \in k\}$. So, $\mathbb{P}_k^3 = k^3 \mathring{\cup} \underbrace{\mathbb{P}(V(x_0))}_{\mathbb{H}:=}$, where $\mathbb{H} \subseteq \mathbb{P}_k^3$ is "the plane at infinite". Now we have the situation:

$$\mathbb{L}_1 = \mathbb{L}_1^0 \mathring{\cup} \{p\},\ \mathbb{L}_2 = \mathbb{L}_2^0 \cup \{p\},\ \text{and } \mathbb{L} \subseteq \mathbb{H},\ \text{thus } \mathbb{L} \cap k^3 = \emptyset.$$

Moreover

$$V = \mathring{V} \mathring{\cup} \mathbb{L} \text{ and } W = \mathring{W} \mathring{\cup} \mathbb{L}.$$

In other words: $\mathbb{L}_i$ is the projective closure of $\mathbb{L}_i^0$, V is the projective closure of $\overset{\circ}{V}$, W is the projective closure of $\overset{\circ}{W}$ and moreover the closures V and W intersect at the line $\mathbb{L}$ at infinity and $V \cap W$ becomes connected.



C) We can look at previous the example in yet another way: We have a surface $\overset{\circ}{V} \subseteq k^3$ which is irreducible. We have intersected $\overset{\circ}{V}$ with the irreducible (hypersurface-) variety

$$\overset{\circ}{W} = V(x_1 + x_2^2 - x_2 x_1 - 1)$$

and did get a non-connected intersection! This shows that the Bertini-Grothendieck connectedness theorem need not hold in the affine setting.

On the other hand $V \cap \mathbb{P}(V(x_0 x_1 + x_2^2 - x_2 x_3 - x_0^2)) = V \cap W$ is connected, in accordance with the Bertini-Grothendieck connectedness theorem 15.2. •

Remark 15.5 *For a complete treatment of the theme of this lecture, for sharper results and further extensions, see Chapter 19 of [B-S].*

Bibliography

Basic references for our lectures

[B] BRODMANN, M.: *Lectures on Local Cohomology*, Autumn School held at the University of Quinhon (Vietnam) in September 1999, Institute of Mathematics, Hanoi (2001).

[B-F] BRODMANN, M. and FUMASOLI, S.: *First lectures on local cohomology*; (Revised and extended preliminary version), Preprint, PDF, load down: http://www.math.unizh.ch/brodmann (click: "Publikationen") (2004).

[B-S] BRODMANN, M. and SHARP, R.Y.: *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics 60, Cambridge University Press (1998).

Basic references in Commutative Algebra

[E] EISENBUD, D.: *Commutative algebra with a view towards algebraic geometry*, Springer, New York (1996).

[M] MATSUMURA, H.: *Commutative ring theory*, Cambridge Studies in Advanced Mathematics 8, Cambridge University Press (1980).

[S] SHARP, R.Y.: *Steps in commutative Algebra*, Second Edition, LMS Student Texts 51, Cambridge University Press (2000).

Basic references in Algebraic Geometry

[H] HARTSHORNE, R.: *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer (1977).

[R] REID, M.: *Undergraduate Algebraic Geometry*, LMS Student Texts 12, Cambridge University Press (1990).

Further references

First Lecture:

[G1] GROTHENDIECK, A.: *Local cohomology*, Lecture Notes in Mathematics 41, Springer (1967).

[H1] HARTSHORNE, R.: *Residues and duality*, Lecture Notes in Mathematics 20, Springer (1966).

[N] NORTHCOTT, D.G.: *An introduction to homological algebra*, Cambridge University Press (1960).

[R] ROTMAN, J.J.: *An introduction to homological algebra*, Academic Press, Orlando (1979).

Second Lecture:

[Br-He] BRUNS, W., HERZOG, J.: *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press (1993).

[G2] GROTHENDIECK, A.: *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux*, (SGA2), Séminaire de Géométrie Algébrique du Bois Marie 1962, North Holland (1968).

[Hu-Ly] HUNEKE, C., LYUBEZNIK, G.: *On the vanishing of local cohomology modules*, Inventiones Mathematicae 102 (1990) 72 - 93.

Third Lecture:

[B-L] BRODMANN, M., LASHGARI, F.A.: *A finiteness result for associated primes of local cohomology modules*, Proceedings of the AMS 128 (2000) 2851 - 2853.

[B-R-S] BRODMANN, M., ROTTHAUS, C., SHARP, R.Y.: *On annihilation and associated primes of local cohomology modules*, Journal of Pure and Applied Algebra 153 (2000) 197 - 227.

[F1] FALTINGS, G.: *Der Endlichkeitssatz in der lokalen Kohomologie*, Mathematische Annalen 255 (1981) 45 - 56.

[Hu] HUNEKE, C.: *Problems on local cohomology*, Research Notes in Mathematics 2 (1992) 93 - 108.

[Si] SINGH, A.K.: *p-torsion elements in local cohomology modules*, Mathematics Research Letters 7 (2000) 165 - 176.

Fourth Lecture:

[B-Ru] BRODMANN, M., RUNG, J.: *Local cohomology and the connectedness dimension in algebraic varieties*, Commentarii Mathematici Helvetici 61 (1986) 481 - 490.

[F2] FALTINGS, G.: *Algebraization of some formal vector bundles*, Annals of Mathematics 110 (1979) 501 - 504.

[Fu-Ha] FULTON, W., HANSEN, J.: *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings*, Annals of Mathematics 110 (1979) 159 - 166.

[H3] HARTSHORNE, R.: *Complete intersections and connectedness*, American Journal of Mathematics 84 (1962) 403 - 450.