Intrinsic Finite Element Methods for the Computation of Fluxes for Poisson's Equation.

P. G. Ciarlet* P. Ciarlet, Jr.† S. A. Sauter‡ C. Simian§

Abstract

In this paper we consider an intrinsic approach for the direct computation of the fluxes for problems in potential theory. We develop a general method for the derivation of intrinsic conforming and non-conforming finite element spaces and appropriate lifting operators for the evaluation of the right-hand side from abstract theoretical principles related to the second Strang Lemma. The convergence of this intrinsic finite element method is proved.

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1 Introduction

In this paper our goal is to develop a general method for the *derivation* of *intrinsic* conforming and non-conforming finite elements from theoretical principles for the discretization of elliptic partial differential equations. More precisely, we employ the stability and convergence theory for non-conforming finite elements based on the second Strang lemma and derive from these principles weak compatibility conditions for non-conforming finite elements. In other words, we show that local polynomial finite element spaces for elliptic problems in divergence form *must* satisfy those compatibility conditions in order to estimate the perturbation in the second Strang lemma in a consistent way.

As a simple model problem for the introduction of our method, we consider Poisson's equation but emphasize that this method is applicable also for much

*(mapgc@cityu.edu.hk), Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong

†(patrick.ciarlet@ensta-paristech.fr), Laboratoire POEMS, UMR 7231, ENSTA ParisTech, 828, boulevard des Maréchaux, 91762 Palaiseau Cedex, France

‡(stas@math.uzh.ch), Institut für Mathematik, Universität Zürich, Winterthurerstr 190, CH-8057 Zürich, Switzerland

§(corina.simian@math.uzh.ch), Institut für Mathematik, Universität Zürich, Winterthurerstr 190, CH-8057 Zürich, Switzerland

more general (systems of) elliptic equations. We consider the intrinsic formulation of Poisson's equation, i.e., the minimization of the energy functional in the space of *admissible* energies which will be defined below. The goal is to construct piecewise polynomial finite element spaces for the *direct* approximation of the physical quantity of interest, i.e., the flux, the electrostatic field, the velocity field, etc. depending on the underlying application. To take into account essential boundary conditions we have to construct a *lifting operator* as the left inverse of the elementwise gradient operator, that is, an operator defined element by element – whose realization turns out to be quite simple.

There is a vast literature on various conforming and non-conforming, primal, dual, mixed formulations of elliptic differential equations and conforming as well as non-conforming discretization. Since our main focus is the development of a *concept* for deriving conforming and non-conforming intrinsic finite elements from theoretical principles and not the presentation of a specific new finite element space we omit an extensive list of references on the analysis of specific families of finite elements spaces but refer to the classical monographs [4], [16], and [3], and the references therein.

Intrinsic formulations of the Lamé equations modelling linear three-dimensional elasticity have been first derived in [5]. An intrinsic finite element space has been developed in [6] and [7] by modifying the lowest order Nédélec finite elements (cf. [13], [14]) such that the compatibility conditions which arise from the intrinsic formulation are satisfied.

The approach we propose allows us to recover the non-conforming Crouzeix-Raviart element [9], the Fortin-Soulie element [10], the Crouzeix-Falk element [8], and the Gauss-Legendre elements [2], [18] as well as the standard conforming *hp*-finite elements.

The paper is organized as follows.

In Section 2 we introduce our model problem and the relevant function spaces for the intrinsic formulation of the continuous problem as an energy minimization problem.

In Section 3 we derive weak continuity conditions for the characterization of the admissible energy space. Based on these conditions we derive conforming intrinsic polynomial finite element spaces and show that they are (necessarily) the gradients of the well-known Lagrange *hp*-finite element spaces.

In Section 4 we infer from the proof of the second Strang lemma appropriate compatibility conditions at the interfaces between elements of the mesh so that the non-conforming perturbation of the original bilinear form can be estimated in a consistent way. We derive *all* types of piecewise polynomial finite element that satisfy this condition and also derive a local basis for these spaces.

Finally, in Section 5 we summarize the main results and give some conclusions.

2 Model Problem

We consider the model problem of finding, for a given electric charge density $\rho \in L^2(\Omega)$, an electrostatic field $\mathbf{e}$ in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, which satisfies

$$-\operatorname{div}(\varepsilon \mathbf{e}) = \rho \qquad \text{in } \Omega, \tag{1}$$

where ε denotes the electrostatic permeability. In the electrostatic case, one may further write $\mathbf{e} = \nabla \phi$, where ϕ is the electrostatic potential, known up to a constant. We consider that the potential ϕ is constant on each connected component of the boundary $\Gamma := \partial\Omega$. Classically, this amounts to saying that (1) is complemented with a perfect conductor boundary condition, namely[1], $\mathbf{e} \times \mathbf{n}_{|\partial\Omega} = 0$, where $\mathbf{n}$ is the unit outward normal vector field to $\partial\Omega$.

Throughout the paper we assume that

$$\Omega \subset \mathbb{R}^d \text{ is a bounded Lipschitz domain with connected boundary } \Gamma. \tag{2}$$

As a consequence of this assumption, $\phi_{|\partial\Omega}$ is constant. Since ϕ is known up to a constant, we may choose an electrostatic potential such that $\phi_{|\partial\Omega} = 0$. Hence, the variational formulation of (1) restricted to the domain Ω is based on the space

$$\mathbf{E}(\Omega) := \nabla H_0^1(\Omega),$$

where $H^1(\Omega)$ is the usual Sobolev space which contains $L^2(\Omega)$ functions with weak first derivatives in $L^2(\Omega)$ and $H_0^1(\Omega) \subset H^1(\Omega)$ is the subspace containing only those functions in $H^1(\Omega)$ with zero traces at the boundary Γ.

Remark 1 If $\partial\Omega$ consists of disjoint connected components Γ_k, $0 \le k \le q$, i.e., $\partial\Omega = \bigcup_{k=0}^{q} \Gamma_k$, with $\overline{\Gamma_k} \cap \overline{\Gamma_{k'}} = \emptyset$ for $k \neq k'$, then the space $E(\Omega)$ is given by

$$\mathbf{E}(\Omega) = \left\{ \nabla v \mid v \in H^1(\Omega),\ v\big|_{\Gamma_0} = 0 \quad \text{and, for all } 1 \le k \le q, \quad v\big|_{\Gamma_k} = c_k \right\}$$

for arbitrary constants $c_k \in R$, $1 \le k \le q$. To reduce technicalities in this paper, we will only consider domains that satisfy (2).

Given a scalar field v, we define its (weak) vector curl by: $\mathbf{curl}v := (-\partial_2 v, \partial_1 v)^{\mathrm{T}}$. Likewise, given a vector field $\mathbf{e}$, we define its (weak) scalar curl by: $\operatorname{curl}\mathbf{e} = \partial_2 e_1 - \partial_1 e_2$. Finally, we let $\mathbf{a} \cdot \mathbf{b}$ denote the Euclidean scalar product for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$.

We recall a well-known result below. The proof can be found in [12].

Proposition 2 *Let $\Omega \subset \mathbb{R}^d$ satisfy (2). The operator $\nabla : H_0^1(\Omega) \to \mathbf{E}(\Omega)$ is an isomorphism and thus its inverse operator $\Lambda : \mathbf{E}(\Omega) \to H_0^1(\Omega)$ is continuous.*

[1] For $d = 3$, $\mathbf{a} \times \mathbf{b}$ is the usual vector product and in two dimensions we use $\mathbf{a} \times \mathbf{b} := a_2 b_1 - a_1 b_2$.

Let $d = 2$ and $\mathbf{L}^2(\Omega) := L^2(\Omega) \times L^2(\Omega)$. It holds

$$\mathbf{E}(\Omega) = \left\{ \mathbf{e} \in \mathbf{L}^2(\Omega) \mid \int_\Omega \mathbf{e} \cdot \mathbf{curl} v = 0 \quad \forall v \in H^1(\Omega) \right\} \tag{3}$$
$$= \left\{ \mathbf{e} \in \mathbf{L}^2(\Omega) \mid \operatorname{curl} \mathbf{e} = 0 \text{ in } H^{-1}(\Omega) \text{ and } \mathbf{e} \times \mathbf{n} = 0 \text{ in } H^{-1/2}(\Gamma) \right\}.$$

In order to ensure existence and uniqueness of the variational formulation and convergence estimates for the finite element discretization we impose the following assumptions on the electrostatic permeability.

Assumption 3 *The electrostatic permeability ε in (1) satisfies $\varepsilon \in L^\infty(\Omega)$ and*

$$0 < \varepsilon_{\min} := \operatorname*{ess\,inf}_{x \in \Omega} \varepsilon(x) \leq \operatorname*{ess\,sup}_{x \in \Omega} \varepsilon(x) =: \varepsilon_{\max} < \infty. \tag{4}$$

There exists a partition $\mathcal{P} := (\Omega_j)_{j=1}^J$ of Ω into J (possibly curved) polygons such that, for all $r \in \mathbb{N}$, it holds

$$\|\varepsilon\|_{PW^{r,\infty}(\Omega)} := \max_{1 \leq j \leq J} \left\| \varepsilon|_{\Omega_j} \right\|_{W^{r,\infty}(\Omega_j)} < \infty.$$

The variational problem reads: Find $\mathbf{e} \in \mathbf{E}(\Omega)$ such that

$$\int_\Omega \varepsilon \mathbf{e} \cdot \tilde{\mathbf{e}} = \int_\Omega \rho \Lambda \tilde{\mathbf{e}} \qquad \forall \tilde{\mathbf{e}} \in \mathbf{E}(\Omega). \tag{5}$$

Equivalently the solution $\mathbf{e}$ can be characterized as the minimizer on $\mathbf{E}(\Omega)$ of the functional

$$j : \mathbf{E}(\Omega) \to \mathbb{R} \quad j(\tilde{\mathbf{e}}) := \frac{1}{2} \int_\Omega \varepsilon \tilde{\mathbf{e}} \cdot \tilde{\mathbf{e}} - \int_\Omega \rho \Lambda \tilde{\mathbf{e}}.$$

In most physical applications the quantity $\mathbf{e}$, or the flux $\varepsilon \mathbf{e}$, is the physical quantity of interest rather than the potential $u = \Lambda \mathbf{e}$ and our goal is to *derive* conforming and non-conforming finite element spaces for the direct approximation of $\mathbf{e}$ in (5) from conditions which arise from the abstract convergence theory.

3 Conforming Intrinsic Finite Element Spaces

In this paper we restrict our studies to two-dimensional, bounded, polygonal domains $\Omega \subset \mathbb{R}^2$ and simplicial triangulations. As a convention we assume that a triangle is a closed set and the edges are also closed sets. The interior of a triangle τ is denoted by $\overset{\circ}{\tau}$ and we write $\overset{\circ}{E}$ for the relative interior of an edge E. The finite element method is based on triangulations, or meshes, $\mathcal{T}$ of Ω which are regular in the sense of [4]: a) For each $\mathcal{T}$, the triangles form a partition of Ω, i.e., $\overline{\Omega} = \cup_{\tau \in \mathcal{T}} \tau$, b) for each $\mathcal{T}$, the intersection of the interiors of any two non-identical triangles is either empty, a common vertex, or a common edge, and c) the family of meshes is shape-regular, i.e., the minimal and maximal angles of the triangles are uniformly bounded away from 0 and π. In a mesh $\mathcal{T}$, the set of all interior edges is denoted by $\mathcal{E}$ and the set of edges lying on $\partial\Omega$ is $\mathcal{E}_{\partial\Omega}$. The set of interior vertices is $\mathcal{V}$ and the set of vertices lying on $\partial\Omega$ is $\mathcal{V}_{\partial\Omega}$. Finally, we denote by h the meshsize of a mesh $\mathcal{T}$, namely $h := \max_{\tau\in\mathcal{T}} h_\tau$, where h_τ is the diameter of τ.

For $p \in \mathbb{N}_0$ let $\mathbb{P}_p$ denote the space of polynomials of degree $\leq p$, i.e., consisting of the functions $\sum_{i=0}^{p}\sum_{j=0}^{p-i} a_{i,j}x_1^i x_2^j$ for some real coefficients $a_{i,j}$. For $\omega \subset \Omega$, we write $\mathbb{P}_p(\omega)$ for polynomials of degree $\leq p$ defined on ω. Given $\mathcal{T}$, we define the finite element spaces

$$\left.\begin{array}{l} S_{\mathcal{T}}^{p,m} := \left\{u \in H^{m+1}(\Omega) \mid \forall\tau\in\mathcal{T} : u|_{\overset{\circ}{\tau}} \in \mathbb{P}_p\right\}, \\ \mathbf{S}_{\mathcal{T}}^{p,m} := S_{\mathcal{T}}^{p,m} \times S_{\mathcal{T}}^{p,m}, \end{array}\right\} \quad \text{for } m = -1, 0,$$

$$S_{\mathcal{T},0}^{p,0} := S_{\mathcal{T}}^{p,0} \cap H_0^1(\Omega),$$

and

$$\mathbf{E}_{\mathcal{T}}^{p} := \left\{\mathbf{e} \in \mathbf{S}_{\mathcal{T}}^{p,-1} \mid \int_\Omega \mathbf{e}\cdot\mathbf{curl}v = 0 \quad \forall v \in H^1(\Omega)\right\}. \tag{6}$$

From (3) we conclude that $\mathbf{E}_{\mathcal{T}}^p \subset \mathbf{E}(\Omega)$ is a piecewise polynomial finite element space which gives rise to the conforming Galerkin discretization of (5) by these *intrinsic* finite elements: Find $\mathbf{e}_{\mathcal{T}} \in \mathbf{E}_{\mathcal{T}}^p$ such that

$$\int_\Omega \varepsilon\mathbf{e}_{\mathcal{T}}\cdot\tilde{\mathbf{e}}_{\mathcal{T}} = \int_\Omega \rho\Lambda\tilde{\mathbf{e}}_{\mathcal{T}} \qquad \forall\tilde{\mathbf{e}}_{\mathcal{T}} \in \mathbf{E}_{\mathcal{T}}^p. \tag{7}$$

In the rest of Section 3, we will derive a local basis for $\mathbf{E}_{\mathcal{T}}^p$ and a realization of the lifting operator Λ. We define for later purpose the **piecewise curl** and the **piecewise gradient operators** by

$$\left.\begin{array}{r} (\operatorname{curl}_{\mathcal{T}}\mathbf{e})(\mathbf{x}) := \partial_2 e_1(\mathbf{x}) - \partial_1 e_2(\mathbf{x}) \\ \nabla_{\mathcal{T}} u(\mathbf{x}) = (\partial_1 u(\mathbf{x}), \partial_2 u(\mathbf{x}))^{\mathrm{T}} \end{array}\right\} \qquad \forall\mathbf{x} \in \Omega\backslash\left(\bigcup_{E\in\mathcal{E}} E\right).$$

3.1 Local Characterization of Conforming Intrinsic Finite Elements

In this section, we will develop a local characterization of conforming intrinsic finite elements. This approach generalizes that of [6], where such finite element approximations were considered for the first time (for the system of two-dimensional linearized elasticity).

For an edge $E \in \mathcal{E}\cup\mathcal{E}_{\partial\Omega}$ let $\mathbf{n}_E$ denote a unit vector which is orthogonal to E. The orientation for the inner edges is arbitrary but fixed while the orientation for the boundary edges is such that $\mathbf{n}_E$ points toward the exterior of Ω. Let $\mathbf{t}_E$ denote an oriented unit vector along E, which obeys the convention that $\det[\mathbf{t}_E, \mathbf{n}_E] = 1$.

For the inner edges $E \in \mathcal{E}$, we define the pointwise tangential jumps $[\mathbf{e} \cdot \mathbf{t}_E]_E : E \to \mathbb{R}$ for $\mathbf{x} \in \overset{\circ}{E}$ by

$$[\mathbf{e} \cdot \mathbf{t}_E]_E (\mathbf{x}) = \lim_{\varepsilon \searrow 0} (\mathbf{e} (\mathbf{x} + \varepsilon \mathbf{n}_E) \cdot \mathbf{t}_E - \mathbf{e} (\mathbf{x} - \varepsilon \mathbf{n}_E) \cdot \mathbf{t}_E) .$$

Lemma 4 *Let the boundary of Ω be connected. The space $\mathbf{E}_{\mathcal{T}}^p$ can be characterized by local conditions according to*

$$\begin{aligned} \mathbf{E}_{\mathcal{T}}^p = \Big\{ \mathbf{e} \in \mathbf{S}_{\mathcal{T}}^{p,-1} \mid & \operatorname{curl}_{\mathcal{T}} \mathbf{e} = 0 \\ & \text{and for all } E \in \mathcal{E} \quad [\mathbf{e} \cdot \mathbf{t}_E]_E = 0 \\ & \text{and for all } E \in \mathcal{E}_{\partial\Omega} \quad \mathbf{e} \cdot \mathbf{t}_E|_E = 0 \Big\} . \end{aligned} \tag{8}$$

Proof. We denote the right-hand side in (8) by $\tilde{\mathbf{E}}_{\mathcal{T}}^p$ and prove $\mathbf{E}_{\mathcal{T}}^p = \tilde{\mathbf{E}}_{\mathcal{T}}^p$. Let $\mathbf{e} \in \mathbf{E}_{\mathcal{T}}^p$. Consider the curl-condition (6) with test-fields v.

Part a: For $\tau \in \mathcal{T}$, let $v \in \mathcal{D}(\tau) := \{u \in C^\infty(\tau) \mid \operatorname{supp} u \subset\subset \tau\}$. Then,

$$\int_\tau (\operatorname{curl} \mathbf{e})\, v = \int_\tau \mathbf{e} \cdot \mathbf{curl} v = 0.$$

Since $\tau \in \mathcal{T}$ and $v \in \mathcal{D}(\tau)$ are arbitrary, we conclude that $\operatorname{curl}_{\mathcal{T}} \mathbf{e} = 0$ holds.

Part b: For $E \in \mathcal{E}$, let $\tau_1, \tau_2 \in \mathcal{T}$ be such that $E = \tau_1 \cap \tau_2$. We set $\omega_E := \tau_1 \cup \tau_2$. We choose $v \in \mathcal{D}\left(\overset{\circ}{\omega}_E\right)$. Then

$$\int_{\tau_1} \mathbf{e} \cdot \mathbf{curl} v + \int_{\tau_2} \mathbf{e} \cdot \mathbf{curl} v = 0.$$

For $i = 1, 2$, denote by $\mathbf{n}^i = \left(n_1^i, n_2^i\right)^{\mathrm{T}}$ the exterior normal for τ_i. Trianglewise partial integration yields (by taking into account $v = 0$ on $\partial\omega_E$)

$$\begin{aligned} 0 &= \int_{\partial\tau_1} \left(-e_1 n_2^1 + e_2 n_1^1\right) v + \int_{\partial\tau_2} \left(-e_1 n_2^2 + e_2 n_1^2\right) v + \int_{\omega_E} (\operatorname{curl}_{\mathcal{T}} \mathbf{e})\, v \\ &= \int_E \left(-e_1 n_2^1 + e_2 n_1^1\right) v + \int_E \left(-e_1 n_2^2 + e_2 n_1^2\right) v + \int_{\omega_E} (\operatorname{curl}_{\mathcal{T}} \mathbf{e})\, v. \end{aligned}$$

We already proved $\operatorname{curl}_{\mathcal{T}} \mathbf{e} = 0$. Note that $\left(-n_2^1, n_1^1\right)^{\mathrm{T}} = -\left(-n_2^2, n_1^2\right)^{\mathrm{T}}$ is tangential to E so that

$$0 = \int_E [\mathbf{e} \cdot \mathbf{t}_E]_E\, v.$$

Since $v \in \mathcal{D}\left(\overset{\circ}{\omega}_E\right)$ is arbitrary, we conclude $[\mathbf{e} \cdot \mathbf{t}_E]_E = 0$.

Part c: Let $E \in \mathcal{E}_{\partial\Omega}$ and $\tau \in \mathcal{T}$ such that $E \subset \partial\tau$. Let

$$\mathcal{D}_E(\tau) := \left\{ v|_\tau : v \in \mathcal{D}\left(\mathbb{R}^2\right) \text{ and } v = 0 \text{ in some neighborhood of } \Omega \backslash \tau \right\}.$$

Repeating the argument as in b) by taking into account that $v \in \mathcal{D}_E(\tau)$ in general does not vanish on E leads to $\mathbf{e} \cdot \mathbf{t}_E = 0$ in this case.

Thus, we have proved $\mathbf{E}_{\mathcal{T}}^{p} \subset \tilde{\mathbf{E}}_{\mathcal{T}}^{p}$.

Part d: To prove the opposite inclusion we consider $\mathbf{e} \in \tilde{\mathbf{E}}_{\mathcal{T}}^{p}$. Then, for all $v \in H^1(\Omega)$ it holds

$$\begin{aligned}
{}_{(H^1(\Omega))'}\langle \operatorname{curl} \mathbf{e}, v \rangle_{H^1(\Omega)} &= \int_\Omega \mathbf{e} \cdot \mathbf{curl} v = \sum_{\tau \in \mathcal{T}} \int_\tau \mathbf{e} \cdot \mathbf{curl} v \\
&= \sum_{\tau \in \mathcal{T}} \int_\tau \left(\operatorname{curl}_{\mathcal{T}} \mathbf{e} \right) v + \sum_{\tau \in \mathcal{T}} \int_{\partial \tau} \left(-e_1 n_2^\tau + e_2 n_1^\tau \right) v \\
&= \sum_{\tau \in \mathcal{T}} \int_\tau \left(\operatorname{curl}_{\mathcal{T}} \mathbf{e} \right) v + (-1)^{\sigma_E} \sum_{E \in \mathcal{E}} \int_E \left[\mathbf{e} \cdot \mathbf{t}_E \right]_E v \\
&\qquad + \sum_{E \in \mathcal{E}_{\partial\Omega}} \int_E \left(\mathbf{e} \cdot \mathbf{t}_E \right) v \\
&= 0.
\end{aligned}$$

Above, $\sigma_E \in \{0, 1\}$, depending on the orientation of $\mathbf{t}_E$.
Hence, $\tilde{\mathbf{E}}_{\mathcal{T}}^{p} \subset \mathbf{E}_{\mathcal{T}}^{p}$ and the assertion follows. ■

Next we define triangle-, edge-, and vertex-oriented local subspaces of $\mathbf{E}_{\mathcal{T}}^{p}$:
For any $\tau \in \mathcal{T}$, we define

$$\mathbf{B}_\tau^p := \left\{ \mathbf{e} \in \mathbf{E}_{\mathcal{T}}^{p} \mid \operatorname{supp} \mathbf{e} \subset \tau \right\}. \tag{9}$$

For any $E \in \mathcal{E}$, we set

$$\mathcal{T}_E := \{ \tau \in \mathcal{T} : E \subset \partial \tau \} \qquad \omega_E := \bigcup_{\tau \in \mathcal{T}_E} \tau, \tag{10}$$

and define $\mathbf{B}_E^p$ implicitly by the direct sum decomposition

$$\mathbf{B}_E^p \oplus \left(\bigoplus_{\tau \in \mathcal{T}_E} \mathbf{B}_\tau^p \right) := \left\{ \mathbf{e} \in \mathbf{E}_{\mathcal{T}}^{p} \mid \operatorname{supp} \mathbf{e} \subset \omega_E \right\}. \tag{11}$$

For any $V \in \mathcal{V}$, we set

$$\mathcal{E}_V := \{ E \in \mathcal{E} : V \in \partial E \}, \ \mathcal{T}_V := \{ \tau \in \mathcal{T} : V \in \tau \}, \ \omega_V := \bigcup_{\tau \in \mathcal{T}_V} \tau. \tag{12}$$

Then $\mathbf{B}_V^p$ is implicitly defined by the condition

$$\mathbf{B}_V^p \oplus \left(\bigoplus_{E \in \mathcal{E}_V} \mathbf{B}_E^p \right) \oplus \left(\bigoplus_{\tau \in \mathcal{T}_V} B_\tau^p \right) := \left\{ \mathbf{e} \in \mathbf{E}_{\mathcal{T}}^{p} \mid \operatorname{supp} \mathbf{e} \subset \omega_V \right\}. \tag{13}$$

Proposition 5 *Let the boundary of Ω be connected. The space $\mathbf{E}_{\mathcal{T}}^{p}$ can be decomposed as the direct sum*

$$\mathbf{E}_{\mathcal{T}}^{p} = \left(\bigoplus_{V \in \mathcal{V}} \mathbf{B}_V^p \right) \oplus \left(\bigoplus_{E \in \mathcal{E}} \mathbf{B}_E^p \right) \oplus \left(\bigoplus_{\tau \in \mathcal{T}} \mathbf{B}_\tau^p \right). \tag{14}$$

The proof is a direct consequence of Proposition 8 which will be proved in Section 3.3.

3.2 Integration

We start with a lemma on integration of curl-free polynomials. Let

$$\mathbf{P}_{\mathrm{curl}}^{p} := \{\mathbf{e} \in \mathbb{P}_p \times \mathbb{P}_p : \operatorname{curl} \mathbf{e} = 0\} \tag{15}$$

and, for $\tau \in \mathcal{T}$, we write $\mathbf{P}_{\mathrm{curl}}^{p}(\tau) := \{\mathbf{e}|_{\tau} : \mathbf{e} \in \mathbf{P}_{\mathrm{curl}}^{p}\}$ to indicate the domain of the functions explicitly.

Lemma 6 *For any $\tau \in \mathcal{T}$ and any $\mathbf{e} \in \mathbf{P}_{\mathrm{curl}}^{p}(\tau)$, it holds*

$$\emptyset \neq \left\{u \in H^1(\tau) \mid \nabla u = \mathbf{e}\right\} \subset \mathbb{P}_{p+1}(\tau). \tag{16}$$

Proof. Let $\tau \in \mathcal{T}$ and $\mathbf{e} \in \mathbf{P}_{\mathrm{curl}}^{p}(\tau)$. In [12, 1] it is proved that there exists $u \in H^1(\tau)$, unique up to a constant, such that $\nabla u = \mathbf{e}$ and, hence, the left-hand side in (16) is proved. Let $\mathbf{m}_\tau$ be the center of mass for τ. Then Poincaré's theorem yields that the path integral

$$U(\mathbf{x}) := \int_{\gamma_{\mathbf{x}}} \mathbf{e} \quad \text{with} \quad \gamma_{\mathbf{x}} \text{ denoting the straight path } \overline{\mathbf{m}_\tau \mathbf{x}} \tag{17}$$

defines some U such that $\nabla U = \mathbf{e}$. Since $\mathbf{e} \in \mathbf{P}_{\mathrm{curl}}^{p}(\tau)$, there are coefficients $\mathbf{a}_\mu \in \mathbb{R}^2$ such that

$$\mathbf{e}(\mathbf{x}) = \sum_{|\mu| \leq p} \mathbf{a}_\mu (\mathbf{x} - \mathbf{m}_\tau)^\mu$$

with the usual multiindex notation $\mu \in \mathbb{N}_0^2$, $|\mu| := \mu_1 + \mu_2$, $\mathbf{w}^\mu := w_1^{\mu_1} w_2^{\mu_2}$. To evaluate the integral in (17) we employ the affine pullback $\chi_{\mathbf{x}} : [0,1] \to \overline{\mathbf{m}_\tau \mathbf{x}}$, $\chi_{\mathbf{x}} := \mathbf{m}_\tau + t(\mathbf{x} - \mathbf{m}_\tau)$ and obtain

$$\begin{aligned}
U(\mathbf{x}) &= \int_0^1 \mathbf{e} \circ \chi_{\mathbf{x}}(t) \cdot \chi_{\mathbf{x}}'(t)\, dt \\
&= \sum_{|\mu| \leq p} \mathbf{a}_\mu \cdot (\mathbf{x} - \mathbf{m}_\tau) \int_0^1 (t(\mathbf{x} - \mathbf{m}_\tau))^\mu\, dt \\
&= \sum_{|\mu| \leq p} (\mathbf{a}_\mu \cdot (\mathbf{x} - \mathbf{m}_\tau)) (\mathbf{x} - \mathbf{m}_\tau)^\mu \int_0^1 t^{|\mu|} dt \\
&= \sum_{|\mu| \leq p} \mathbf{a}_\mu \cdot (\mathbf{x} - \mathbf{m}_\tau) \frac{(\mathbf{x} - \mathbf{m}_\tau)^\mu}{|\mu| + 1} \in \mathbb{P}_{p+1}.
\end{aligned}$$

Since the functions in the set $\{\ldots\}$ in (16) differ only by a constant we have proved the second inclusion in (16). ■

Lemma 6 motivates the definition of the local lifting $\lambda_\tau^c : \mathbf{P}_{\mathrm{curl}}^{p}(\tau) \to \mathbb{P}^{p+1}(\tau)$ for $\tau \in \mathcal{T}$, $\mathbf{e} \in \mathbf{P}_{\mathrm{curl}}^{p}(\tau)$, and $c \in \mathbb{R}$ by

$$\lambda_\tau^c(\mathbf{e}) := U + c \quad \text{with} \quad U \text{ as in (17)}. \tag{18}$$

Note that the space in (16) satisfies

$$\left\{u \in H^1(\tau) \mid \nabla u = \mathbf{e}\right\} = \left\{\lambda_\tau^c(\mathbf{e}) : c \in \mathbb{R}\right\}.$$

Corollary 7 *Let the boundary of* Ω *be connected.* $\Lambda : \mathbf{E}_{\mathcal{T}}^{p} \to S_{\mathcal{T},0}^{p+1,0}$ *is an isomorphism with inverse* $\nabla : S_{\mathcal{T},0}^{p+1,0} \to \mathbf{E}_{\mathcal{T}}^{p}$.

Proof. From Lemma 6 we conclude that

$$\Lambda \mathbf{E}_{\mathcal{T}}^{p} \subset S_{\mathcal{T}}^{p+1,-1}$$

holds. Since $\mathbf{E}_{\mathcal{T}}^{p} \subset \mathbf{E}$, the mapping properties of the lifting Λ imply

$$\Lambda \mathbf{E}_{\mathcal{T}}^{p} \subset H_0^1(\Omega).$$

Hence

$$\Lambda \mathbf{E}_{\mathcal{T}}^{p} \subset S_{\mathcal{T}}^{p+1,-1} \cap H_0^1(\Omega) = S_{\mathcal{T},0}^{p+1,0}.$$

On the other hand, we have $S_{\mathcal{T},0}^{p+1,0} \subset H_0^1(\Omega)$ and hence $\nabla S_{\mathcal{T},0}^{p+1,0} \subset \mathbf{E}$. Furthermore, it is clear that

$$\nabla S_{\mathcal{T},0}^{p+1,0} \subset \mathbf{S}_{\mathcal{T}}^{p,-1}.$$

Hence,

$$\nabla S_{\mathcal{T},0}^{p+1,0} \subset \mathbf{S}_{\mathcal{T}}^{p,-1} \cap \mathbf{E} = \mathbf{E}_{\mathcal{T}}^{p}$$

from which we finally conclude that

$$S_{\mathcal{T},0}^{p+1,0} \subset \Lambda \mathbf{E}_{\mathcal{T}}^{p}$$

holds. ■

3.3 A Local Basis for Conforming Intrinsic Finite Elements

Corollary 7 shows that a basis for the spaces $\mathbf{B}_V^p$, $\mathbf{B}_E^p$, $\mathbf{B}_\tau^p$ can easily be constructed by using the standard basis functions for hp-finite element spaces (cf. [16]). We recall briefly their definition. Let

$$\widehat{\mathcal{N}}^p := \left\{ \frac{(i,j)^{\mathrm{T}}}{p} : (i,j) \in \mathbb{N}_0^2 \text{ with } i+j \leq p \right\}$$

denote the equispaced unisolvent set of nodal points on the unit triangle $\hat{\tau}$ with vertices $(0,0)^{\mathrm{T}}$, $(1,0)^{\mathrm{T}}$, $(0,1)^{\mathrm{T}}$. For a triangle $\tau \in \mathcal{T}$ with vertices $\mathbf{A}^\tau$, $\mathbf{B}^\tau$, $\mathbf{C}^\tau$, let $\chi_\tau : \hat{\tau} \to \tau$ denote the affine mapping $\chi_\tau(\hat{\mathbf{x}}) := \mathbf{A}^\tau + (\mathbf{B}^\tau - \mathbf{A}^\tau)\hat{x}_1 + (\mathbf{C}^\tau - \mathbf{A}^\tau)\hat{x}_2$. Then, the set of interior nodal points are given by

$$\mathcal{N}^p := \left\{ \chi_\tau\left(\hat{N}\right) \mid \hat{N} \in \widehat{\mathcal{N}}^p, \tau \in \mathcal{T} \right\} \backslash \partial\Omega. \tag{19}$$

The Lagrange basis for $S_{\mathcal{T},0}^{p,0}$ can be indexed by the nodal points $N \in \mathcal{N}^p$ and is characterized by

$$b_{p,N}^{\mathcal{T}} \in S_{\mathcal{T},0}^{p,0} \quad \text{and} \quad \forall N' \in \mathcal{N}^p \quad b_{p,N}^{\mathcal{T}}(N') = \begin{cases} 1 & N = N', \\ 0 & N \neq N'. \end{cases} \tag{20}$$

Recall that the triangles in $\mathcal{T}$ are by convention closed sets and the edges in $\mathcal{E}$ are closed.

Proposition 8 *Let the boundary of* Ω *be connected. Let* $\mathbf{B}_\tau^p$, $\mathbf{B}_E^p$, $\mathbf{B}_V^p$ *be defined by (9), (11), (13). A basis*

for the space $\mathbf{B}_\tau^p$ *is given by* $\left\{\nabla b_{p+1,N}^{\mathcal{T}} \mid N \in \overset{\circ}{\tau} \cap \mathcal{N}^{p+1}\right\}$ *for all* $\tau \in \mathcal{T}$,

for the space $\mathbf{B}_E^p$ *is given by* $\left\{\nabla b_{p+1,N}^{\mathcal{T}} \mid N \in \overset{\circ}{E} \cap \mathcal{N}^{p+1}\right\}$ *for all* $E \in \mathcal{E}$,

for the space $\mathbf{B}_V^p$ *is given by* $\left\{\nabla b_{p+1,V}^{\mathcal{T}}\right\}$ *for all* $V \in \mathcal{V}$.

Proof. Corollary 7 implies that $(\nabla b_{p+1,N}^{\mathcal{T}})_{N\in\mathcal{N}^{p+1}}$ is a basis of $\mathbf{E}_{\mathcal{T}}^p$. The assertion follows simply by sorting these basis functions, according as to whether they are associated with a single triangle, with two triangles with a side in common, and with triangles with a vertex in common. ■

Remark 9 Proposition 8 shows that (7) is equivalent to the standard Galerkin finite element formulation of (1): Find $u_{\mathcal{T}} \in S_{\mathcal{T},0}^{p+1,0}$ such that

$$\int_\Omega \varepsilon \nabla u_{\mathcal{T}} \cdot \nabla v_{\mathcal{T}} = \int_\Omega \rho v_{\mathcal{T}} \qquad \forall v_{\mathcal{T}} \in S_{\mathcal{T},0}^{p+1,0}$$

via $e_{\mathcal{T}} = \nabla u_{\mathcal{T}}$. However, the derivation via the intrinsic variational formulation has the advantage of providing insights on how to design non-conforming intrinsic finite element.

4 Non-Conforming Intrinsic Finite Elements

4.1 (Implicit) Definition of Non-Conforming Intrinsic Finite Elements

In this section, we will define non-conforming intrinsic finite element spaces to approximate the solution of (5). As a minimal requirement we assume that the non-conforming finite element space $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p$ satisfies

$$\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p \subset \mathbf{L}^2(\Omega) \quad \text{and} \quad \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p \not\subset \mathbf{E}(\Omega) \quad \text{and} \quad \dim \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p < \infty. \tag{21}$$

We further require that $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p$ is a piecewise polynomial, trianglewise curl-free finite element space and that the conforming space $\mathbf{E}_{\mathcal{T}}^p$ is a subspace of $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p$:

$$\mathbf{E}_{\mathcal{T}}^p \subset \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p \subset \left\{\mathbf{e} \in \mathbf{S}_{\mathcal{T}}^{p,-1} \mid \mathrm{curl}_{\mathcal{T}}\,\mathbf{e} = 0\right\}. \tag{22}$$

For the definition of a variational formulation we have to extend the lifting operator Λ to an operator $\Lambda_{\mathcal{T}}$ which satisfies

$$\Lambda_{\mathcal{T}} : \left(\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p + \mathbf{E}(\Omega)\right) \to L^2(\Omega) \tag{23}$$

$$\Lambda_{\mathcal{T}} : \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p \to S_{\mathcal{T}}^{p+1,-1} \tag{24}$$

as well as the consistency condition

$$\Lambda_{\mathcal{T}}\mathbf{e} = \Lambda\mathbf{e} \qquad \forall \mathbf{e} \in \mathbf{E}(\Omega). \tag{25}$$

The complete definitions of $\mathbf{E}^{p}_{\mathcal{T},\mathrm{nc}}$ and $\Lambda_{\mathcal{T}}$ will be based on the convergence theory for non-conforming finite elements according to the second Strang lemma (cf. [4, Th. 4.2.2]): this lemma will specify how to define them and obtain in the end an optimal order of convergence (see Theorem 14 hereafter).

In the same spirit as in Section 3, we first define the operator $\Lambda_{\mathcal{T}}$ elementwise by the local lifting operators λ^{c}_{τ} as in (18):

$$(\Lambda_{\mathcal{T}}\mathbf{e})|_{\overset{\circ}{\tau}} := \lambda^{c_\tau}_{\tau}\left(\mathbf{e}|_{\overset{\circ}{\tau}}\right) \qquad \forall \tau \in \mathcal{T} \quad \forall \mathbf{e} \in \mathbf{E}^{p}_{\mathcal{T},\mathrm{nc}}. \tag{26}$$

Note that the coefficients $(c_\tau)_{\tau\in\mathcal{T}}$ are at our disposal.

From (26) we conclude that $\nabla_{\mathcal{T}}$ is a left-inverse to $\Lambda_{\mathcal{T}}$, i.e.,

$$\forall \mathbf{e} \in \mathbf{E}^{p}_{\mathcal{T},\mathrm{nc}} : \nabla_{\mathcal{T}}\Lambda_{\mathcal{T}}\mathbf{e} = \mathbf{e}. \tag{27}$$

A compatibility assumption on $\mathbf{E}^{p}_{\mathcal{T},\mathrm{nc}}$ concerning the jumps of functions across edges is formulated next. For an edge E with endpoints A^E, B^E the affine mapping $\chi_E : [-1,1] \to E$ is given by $\chi_E(\xi) = A^E + \frac{\xi+1}{2}\left(B^E - A^E\right)$. The space of univariate polynomials of degree $\leq p$ along the edge E is given by

$$\mathbb{P}_p(E) := \left\{ q \circ \chi_E^{-1} \mid q \text{ is a polynomial of degree } \leq p \text{ on } [-1,1] \right\}. \tag{28}$$

On the one hand, given $\mathbf{e} \in \mathbf{E}^{p}_{\mathcal{T}}$, one has $[\Lambda_{\mathcal{T}}\mathbf{e}]_E = 0$ for all $E \in \mathcal{E}$, and $\Lambda_{\mathcal{T}}\mathbf{e} = 0$ on $\partial\Omega$. On the other hand, for elements of the non-conforming finite element space $\mathbf{E}^{p}_{\mathcal{T},\mathrm{nc}}$, we require that these conditions are *weakly* enforced. Given $\tilde{\mathbf{e}} \in \mathbf{E}^{p}_{\mathcal{T},\mathrm{nc}}$, keeping in mind that, along every edge E, the jump $[\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}]_E$ is a polynomial of degree $\leq (p+1)$, we conclude that the chosen edge compatibility condition reads:

$$\begin{aligned} &\int_E [\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}]_E\, q = 0 \quad \forall q \in \mathbb{P}_p(E), \ \forall E \in \mathcal{E} \quad \text{and} \\ &\int_E \Lambda_{\mathcal{T}}\tilde{\mathbf{e}}\, q = 0 \quad \forall q \in \mathbb{P}_p(E), \ \forall E \in \mathcal{E}_{\partial\Omega}. \end{aligned} \tag{29}$$

Remark 10 One could choose a priori the degree of the polynomials q between 0 and $p+1$. Indeed, a degree equal to $p+1$ defines conforming finite elements, because (29) then implies $[\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}]_E = 0$ across all interior edges E, and $\Lambda_{\mathcal{T}}\tilde{e} = 0$ on $\partial\Omega$, and Lemma 4 leads to $\tilde{e} \in E^{p}_{\mathcal{T}}$. On the other hand, a degree strictly lower than $p+1$ in the implicit definition (29) of $E^{p}_{\mathcal{T},\mathrm{nc}}$ leads to a non-conforming finite element space, such that $E^{p}_{\mathcal{T}}$ is a strict subset of $E^{p}_{\mathcal{T},\mathrm{nc}}$. The degree p of the polynomials q, which is chosen here, yields an optimal order of convergence (see Theorem 14), whereas a degree strictly lower than p yields a sub-optimal order of convergence.

For any inner edge $E \in \mathcal{T}$, we may choose $q = 1$ in the left condition of (29) to obtain $\int_E [\Lambda_{\mathcal{T}} \tilde{\mathbf{e}}]_E = 0$. Let h_E denote the length of E. The combination of a Poincaré inequality with a trace inequality then yields

$$\begin{aligned} \|[\Lambda_{\mathcal{T}} \tilde{\mathbf{e}}]_E\|_{L^2(E)} &\leq C h_E \|[\mathbf{t}_E \cdot \nabla_{\mathcal{T}} \Lambda_{\mathcal{T}} \tilde{\mathbf{e}}]_E\|_{L^2(E)} \\ &\overset{(27)}{=} C h_E \|[\mathbf{t}_E \cdot \tilde{\mathbf{e}}]_E\|_{L^2(E)} \leq \tilde{C} h_E^{1/2} \|\tilde{\mathbf{e}}\|_{L^2(\omega_E)} . \end{aligned} \tag{30}$$

In a similar fashion we obtain for all boundary edges $E \in \mathcal{E}_{\partial\Omega}$ and all $\mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ the estimate

$$\|\Lambda_{\mathcal{T}} \tilde{\mathbf{e}}\|_{L^2(E)} \leq \tilde{C} h_E^{1/2} \|\tilde{\mathbf{e}}\|_{L^2(\omega_E)} . \tag{31}$$

These considerations are summarized in the following definition.

Definition 11 *Let the boundary of Ω be connected. The non-conforming intrinsic finite element space $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ is given by*

$$\mathbf{E}^p_{\mathcal{T},\mathrm{nc}} := \left\{ \mathbf{e} \in \mathbf{S}^{p,-1}_{\mathcal{T}} \mid \operatorname{curl}_{\mathcal{T}} \mathbf{e} = 0 \quad \textit{and} \quad \textit{(29) is satisfied} \right\}.$$

This definition directly implies that condition (22), i.e., $\mathbf{E}^p_{\mathcal{T}} \subset \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ holds.
In Section 4.2 we will prove the following direct sum decomposition

$$\mathbf{E}^p_{\mathcal{T},\mathrm{nc}} = \mathbf{E}^p_{\mathcal{T}} \oplus \begin{cases} \bigoplus_{E \in \mathcal{E}} \operatorname{span}\left\{\nabla_{\mathcal{T}} U^E_{p+1}\right\} & p \text{ even,} \\ \bigoplus_{\tau \in \mathcal{T}} \operatorname{span}\left\{\nabla_{\mathcal{T}} U^\tau_{p+1}\right\} & p \text{ odd} \end{cases} \tag{32}$$

with functions U^E_{p+1} and U^τ_{p+1} defined in respectively (42) and (47). As a consequence, one deduces the following definition of the extended lifting operator.

Definition 12 *Let the boundary of Ω be connected. For a function $\mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ with*

$$\mathbf{e} = \mathbf{e}_1 + \begin{cases} \sum_{E \in \mathcal{E}} \alpha_E \nabla_{\mathcal{T}} U^E_{p+1} & \textit{if } p \textit{ is even,} \\ \sum_{\tau \in \mathcal{T}} \alpha_\tau \nabla_{\mathcal{T}} U^\tau_{p+1} & \textit{if } p \textit{ is odd} \end{cases} \tag{33}$$

for some $\mathbf{e}_1 \in \mathbf{E}^p_{\mathcal{T}}$ and real coefficients α_E resp. α_τ, the extended lifting operator $\Lambda_{\mathcal{T}}$ is given by

$$\Lambda_{\mathcal{T}} \mathbf{e} := \Lambda \mathbf{e}_1 + \begin{cases} \sum_{E \in \mathcal{E}} \alpha_E U^E_{p+1} & \textit{if } p \textit{ is even,} \\ \sum_{\tau \in \mathcal{T}} \alpha_\tau U^\tau_{p+1} & \textit{if } p \textit{ is odd.} \end{cases}$$

Proposition 13 *Let the boundary of Ω be connected. For any $\mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ with simply connected support $\omega_{\mathbf{e}} := \operatorname{supp} \mathbf{e}$, it holds*

$$\operatorname{supp} \Lambda_{\mathcal{T}} \mathbf{e} \subset \omega_{\mathbf{e}}.$$

Proof. We split $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$ according to (33) with $\mathbf{e}_1 \in \mathbf{E}$. Since the sum, in (32), is direct we conclude[2] that $\operatorname{supp} \mathbf{e}_i \subset \omega_{\mathbf{e}}$ for $i = 1, 2$. From Proposition 2

[2]Here, we also used the property that for a polynomial $q \in \mathbb{P}_p(\omega)$, $\omega \subset \Omega$ with positive area measure, it holds either $q|_\omega = 0$ or $\operatorname{supp} q = \omega$. In our application we choose $q = \mathbf{e}_1 + \mathbf{e}_2$ and apply the argument trianglewise.

we obtain $\Lambda_{\mathcal{T}}\mathbf{e}_1 = \Lambda\mathbf{e}_1 \in H_0^1(\Omega)$. Since $\mathbf{e}_1|_{\Omega\backslash\omega_{\mathbf{e}}} = 0$ Poincaré's theorem implies that $\Lambda\mathbf{e}_1|_{\omega_i} = c_i$, i.e., is constant on each disjoint connected component ω_i of $\Omega\backslash\omega_{\mathbf{e}}$. Since $\omega_{\mathbf{e}}$ is simply connected, each component ω_i has an intersection $\overline{\omega_i}\cap\partial\Omega$ with positive length. The property $\Lambda\mathbf{e}_1 \in H_0^1(\Omega)$ implies that $\Lambda\mathbf{e}_1|_{\omega_i} = 0$. This proves $\operatorname{supp}\Lambda_{\mathcal{T}}\mathbf{e}_1 \subset \omega_{\mathbf{e}}$.

For even p, the definition of $\Lambda_{\mathcal{T}}$ for the non-conforming part $\mathbf{e}_2$ (in particular $\Lambda_{\mathcal{T}}\left(\nabla_{\mathcal{T}}U_{p+1}^E\right) = U_{p+1}^E$) implies that $\operatorname{supp}\nabla_{\mathcal{T}}U_{p+1}^E = \operatorname{supp}U_{p+1}^E$ so that $\operatorname{supp}\Lambda_{\mathcal{T}}\mathbf{e}_2 \subset \omega_{\mathbf{e}}$. The proof for odd p is by an analogous argument. ■

Equipped with $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p$ and $\Lambda_{\mathcal{T}}$, the non-conforming Galerkin discretization of (5) reads: Find $\mathbf{e}_{\mathcal{T}} \in \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p$ such that

$$\int_\Omega \varepsilon\mathbf{e}_{\mathcal{T}}\cdot\tilde{\mathbf{e}} = \int_\Omega \rho\Lambda_{\mathcal{T}}\tilde{\mathbf{e}} \qquad \forall\tilde{\mathbf{e}} \in \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p. \tag{34}$$

We say that the exact solution $\mathbf{e} \in L^2(\Omega)\times L^2(\Omega)$ is piecewise smooth over a partition $\mathcal{P} = (\Omega_j)_{j=1}^J$ of Ω into J (possibly curved) polygons, if there exists some positive integer s such that

$$\mathbf{e}_{|\Omega_j} \in H^s(\Omega_j)\times H^s(\Omega_j) \quad \text{for } j = 1,2,\ldots,J.$$

We write $\mathbf{e} \in PH^s(\Omega)\times PH^s(\Omega)$ and refer for further properties and generalizations to non-integer values of s, e.g., to [15, Sec. 4.1.9].

For the approximation results, the finite element meshes $\mathcal{T}$ are assumed to be compatible with the partition $\mathcal{P}$ in the following sense: for all $\tau \in \mathcal{T}$, there exists a single index j such that $\mathring{\tau}\cap\Omega_j \neq \emptyset$.

Theorem 14 *Let the boundary of Ω be connected. Let the electrostatic permeability ε satisfy Assumption 3 and let $\rho \in L^2(\Omega)$. As an additional assumption on the regularity of the exact solution, we require that the exact solution of (5) satisfies $\mathbf{e} \in PH^s(\Omega)\times PH^s(\Omega)$ for some positive integer s. Assume that the non-conforming finite element space $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p$ and the extended lifting operator $\Lambda_{\mathcal{T}}$ are defined on a compatible mesh $\mathcal{T}$, as in Definitions 11 and 12. Then, the non-conforming Galerkin discretization (34) has a unique solution which satisfies*

$$\|\mathbf{e}-\mathbf{e}_{\mathcal{T}}\|_{\mathbf{L}^2(\Omega)} \leq Ch^r\|\mathbf{e}\|_{PH^r(\Omega)}.$$

with $r := \min\{p+1,s\}$. The constant C only depends on $\varepsilon_{\min}$, $\varepsilon_{\max}$, $\|\varepsilon\|_{PW^{r,\infty}(\Omega)}$, p, and the shape regularity of the mesh.

Proof. The second Strang lemma applied to the non-conforming Galerkin discretization (34) implies the existence of a unique solution which satisfies the error estimate

$$\|\mathbf{e}-\mathbf{e}_{\mathcal{T}}\|_{\mathbf{L}^2(\Omega)} \leq \left(1+\frac{\varepsilon_{\max}}{\varepsilon_{\min}}\right)\inf_{\tilde{\mathbf{e}}\in\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p}\|\mathbf{e}-\tilde{\mathbf{e}}\|_{\mathbf{L}^2(\Omega)} + \frac{1}{\varepsilon_{\min}}\sup_{\tilde{\mathbf{e}}\in\mathbf{E}_{\mathcal{T},\mathrm{nc}}^p\backslash\{0\}}\frac{|\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}})|}{\|\tilde{\mathbf{e}}\|_{\mathbf{L}^2(\Omega)}},$$

where

$$\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}}) := \int_\Omega \varepsilon\mathbf{e}\cdot\tilde{\mathbf{e}} - \int_\Omega \rho\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}.$$

The approximation properties of $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ are inherited from the approximation properties of $\mathbf{E}^p_{\mathcal{T}}$ in the first infimum because of the inclusion $\mathbf{E}^p_{\mathcal{T}} \subset \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ in (22). For the second term we obtain

$$\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}}) = \int_\Omega \varepsilon (\nabla \Lambda \mathbf{e}) \cdot \tilde{\mathbf{e}} - \int_\Omega \rho \Lambda_{\mathcal{T}} \tilde{\mathbf{e}}. \tag{35}$$

Note that $\rho \in L^2(\Omega)$ implies that $\operatorname{div}(\varepsilon \nabla u) \in L^2(\Omega)$ and, in turn, that the jump $[\varepsilon \mathbf{e} \cdot \mathbf{n}_E]_E$ equals zero and the restriction $(\varepsilon \mathbf{e} \cdot \mathbf{n}_E)|_E$ is well defined. We may apply trianglewise integration by parts to (35) to obtain

$$\begin{aligned}\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}}) &= \int_\Omega (\varepsilon \mathbf{e} \cdot \nabla_{\mathcal{T}} \Lambda_{\mathcal{T}} \tilde{\mathbf{e}} - \rho \Lambda_{\mathcal{T}} \tilde{\mathbf{e}}) \\ &= -\sum_{E \in \mathcal{E}} \int_E \varepsilon (\mathbf{e} \cdot \mathbf{n}_E) [\Lambda_{\mathcal{T}} \tilde{\mathbf{e}}]_E + \sum_{E \in \mathcal{E}_{\partial\Omega}} \int_E \varepsilon (\mathbf{e} \cdot \mathbf{n}_E) \Lambda_{\mathcal{T}} \tilde{\mathbf{e}}.\end{aligned}$$

Let $q_E \in \mathbb{P}_p(E)$ denote the best approximation of $\varepsilon \mathbf{e} \cdot \mathbf{n}_E|_E$ with respect to the $L^2(E)$ norm. Then, the combination of (29) with standard approximation properties and a trace inequality leads to

$$\begin{aligned}|\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}})| &= \left| -\sum_{E \in \mathcal{E}} \int_E \left(\varepsilon \frac{\partial u}{\partial \mathbf{n}_E} - q_E \right) [\Lambda_{\mathcal{T}} \tilde{\mathbf{e}}]_E + \sum_{E \in \mathcal{E}_{\partial\Omega}} \int_E \left(\varepsilon \frac{\partial u}{\partial \mathbf{n}_E} - q_E \right) \Lambda_{\mathcal{T}} \tilde{\mathbf{e}} \right| \\ &\leq \sum_{E \in \mathcal{E}} \left\| \varepsilon \frac{\partial u}{\partial \mathbf{n}_E} - q_E \right\|_{L^2(E)} \| [\Lambda_{\mathcal{T}} \tilde{\mathbf{e}}]_E \|_{L^2(E)} \\ &\qquad + \sum_{E \in \mathcal{E}_{\partial\Omega}} \left\| \varepsilon \frac{\partial u}{\partial \mathbf{n}_E} - q_E \right\|_{L^2(E)} \| \Lambda_{\mathcal{T}} \tilde{\mathbf{e}} \|_{L^2(E)} \\ &\leq C \left(\sum_{E \in \mathcal{E}} h_E^{r-1/2} \| \mathbf{e} \|_{H^r(\tau_E)} \| [\Lambda_{\mathcal{T}} \tilde{\mathbf{e}}]_E \|_{L^2(E)} \right. \\ &\qquad \left. + \sum_{E \in \mathcal{E}_{\partial\Omega}} h_E^{r-1/2} \| \mathbf{e} \|_{H^r(\tau_E)} \| \Lambda_{\mathcal{T}} \tilde{\mathbf{e}} \|_{L^2(E)} \right),\end{aligned}$$

where C depends only on p, $\|\varepsilon\|_{W^r(\tau_E)}$, and the shape regularity of the mesh, and τ_E is one triangle of ω_E. The estimates (30) - (31) along with the shape regularity of the mesh lead to the consistency estimate

$$\begin{aligned}|\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}})| &\leq C \left(\sum_{E \in \mathcal{E}} h_E^r \| \mathbf{e} \|_{H^r(\tau_E)} \| \tilde{\mathbf{e}} \|_{L^2(\omega_E)} + \sum_{E \in \mathcal{E}_{\partial\Omega}} h_E^r \| \mathbf{e} \|_{H^r(\tau_E)} \| \tilde{\mathbf{e}} \|_{L^2(\omega_E)} \right) \\ &\leq \tilde{C} h^r \| \mathbf{e} \|_{PH^r(\Omega)} \| \tilde{\mathbf{e}} \|_{L^2(\Omega)},\end{aligned}$$

which completes the proof. ■

Remark 15 If one chooses in (29) a degree $p' < p$ for the test-polynomials q, then the order of convergence behaves like $h^{r'} \|\mathbf{e}\|_{H^{r'}(\Omega)}$, with $r' := \min\{p'+1, s\}$, because the best approximation q_E now belongs to $P_{p'}(E)$.

4.2 A Local Basis for Non-Conforming Intrinsic Finite Elements

Like in Proposition 5, we construct the space $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ by defining basis functions whose supports are given by a single triangle $\tau \in \mathcal{T}$, edge-oriented basis functions whose supports are given by ω_E, for $E \in \mathcal{E}$, and vertex-oriented basis functions whose supports are given by ω_V, $V \in \mathcal{V}$. The corresponding spaces are denoted by $\mathbf{B}^p_{\tau,\mathrm{nc}}$, $\mathbf{B}^p_{E,\mathrm{nc}}$, $\mathbf{B}^p_{V,\mathrm{nc}}$ and defined as follows. The triangle supported subspaces are given by

$$\mathbf{B}^p_{\tau,\mathrm{nc}} := \left\{ \mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \mid \operatorname{supp} \mathbf{e} \subset \tau \right\} \qquad \forall \tau \in \mathcal{T}. \tag{36}$$

The definitions of $\mathcal{T}_E$, ω_E, $\mathcal{E}_V$, $\mathcal{T}_V$, ω_V are given in (10) and (12). The edge- and vertex-oriented subspaces are given implicitly by the following direct sum decompositions

$$\mathbf{B}^p_{E,\mathrm{nc}} \oplus \bigoplus_{\tau \in \mathcal{T}_E} \mathbf{B}^p_{\tau,\mathrm{nc}} = \left\{ \mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \mid \operatorname{supp} \mathbf{e} \subset \omega_E \right\} \ \forall E \in \mathcal{E}, \tag{37}$$

$$\mathbf{B}^p_{V,\mathrm{nc}} \oplus \bigoplus_{E \in \mathcal{E}_V} \mathbf{B}^p_{E,\mathrm{nc}} \oplus \bigoplus_{\tau \in \mathcal{T}_V} \mathbf{B}^p_{\tau,\mathrm{nc}} = \left\{ \mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \mid \operatorname{supp} \mathbf{e} \subset \omega_V \right\} \ \forall V \in \mathcal{V}. \tag{38}$$

In Theorem 21, we will prove that $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ can be decomposed into a direct sum of these local subspaces.

4.2.1 Triangle Supported Basis Functions

In this section, let $\tau \in \mathcal{T}$ denote any fixed triangle in the mesh. The Lagrange basis of $\mathbb{P}_p(\tau)$ with respect to $\mathcal{N}^p \cap \tau$ is denoted by $b^\tau_{p,N}$, $N \in \mathcal{N}^p \cap \tau$, and is characterized by

$$b^\tau_{N,p} \in \mathbb{P}_p(\tau) \quad \text{and} \quad \forall N' \in \mathcal{N}^p \cap \tau \quad b^\tau_{N,p}(N') = \begin{cases} 1 & \text{if } N = N', \\ 0 & \text{if } N \neq N'. \end{cases}$$

We denote the (discontinuous in general) extension by zero of $b^\tau_{p,N}$ to $\Omega \backslash \tau$ again by $b^\tau_{p,N}$. From Lemma 6 and Conditions (22), (29), we deduce

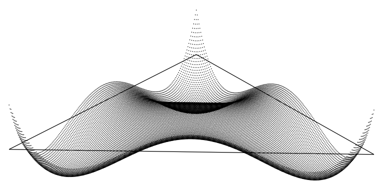
$$\mathbf{B}^p_{\tau,\mathrm{nc}} = \Big\{ \left.\mathbf{e}\right|_\tau \in \nabla \mathbb{P}_{p+1}(\tau) \mid \operatorname{supp} \mathbf{e} \subset \tau \quad \text{and} \\ \forall E \subset \partial\tau, \ \forall q \in \mathbb{P}_p(E) : \int_E q \Lambda_{\mathcal{T}} \mathbf{e} = 0 \Big\}. \tag{39}$$

According to (39), it is clear that $\mathbf{B}^p_\tau \subset \mathbf{B}^p_{\tau,\mathrm{nc}}$. In the next step, we use the compatibility conditions in (39) for the explicit characterization of $\mathbf{B}^p_{\tau,\mathrm{nc}}$.

Lemma 16 *Let the boundary of Ω be connected. For $\tau \in \mathcal{T}$, the non-conforming finite element space $\mathbf{B}^p_{\tau,\mathrm{nc}}$ is given by*

$$\mathbf{B}^p_{\tau,\mathrm{nc}} = \begin{cases} \mathbf{B}^p_\tau & \text{if } p \text{ is even,} \\ \mathbf{B}^p_\tau + \operatorname{span}\left\{ \nabla_{\mathcal{T}} U^\tau_{p+1} \right\} & \text{if } p \text{ is odd,} \end{cases} \tag{40}$$

where U^τ_{p+1} is defined in (42).



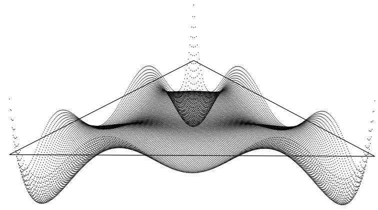


Figure 1: Representation of U_{p+1} for $p = 3$ (left) and $p = 5$ (right)

Proof. Pick some $\mathbf{e} \in \mathbf{B}^p_{\tau,\mathrm{nc}}$, let $u := \Lambda_{\mathcal{T}}\mathbf{e}$ and denote the restrictions to τ by $\mathbf{e}_\tau$ and u_τ. For $E \in \mathcal{E} \cup \mathcal{E}_{\partial\Omega}$, let χ_E be as in (28) the affine pullback to $[-1,1]$. Let $L_p : [-1,1] \to \mathbb{R}$ denote the Legendre polynomials of degree p with the normalization convention that $L_p(1) = 1$. In turn, this implies $L_p(-1) = (-1)^p$. We lift them to the edge E via $L_p^E := L_p \circ \chi_E^{-1}$. It is well known that L_{p+1}^E satisfies the orthogonality condition

$$(L_{p+1}^E, q)_{L^2(E)} = 0 \qquad \forall q \in \mathbb{P}_p(E).$$

The compatibility condition in (39) therefore implies, for all $E \subset \partial\tau$, that

$$u_\tau|_E = c_E \cdot L_{p+1}^E \quad \text{for some } c_E \in \mathbb{R}. \tag{41}$$

The relation $u_\tau \in \mathbb{P}_{p+1}(\tau)$ implies that $u_\tau|_{\partial\tau}$ is continuous so that u_τ is continuous at every vertex of τ. We distinguish two cases.

Let p be even. In this case we have $L_{p+1}(1) = -L_{p+1}(-1)$ so that the continuity at the vertices of τ implies $c_E = 0$. Thus $u_\tau|_{\partial\tau} = 0$ and we have proved (40) for even p.

Let p be odd. Now we have $L_{p+1}(1) = L_{p+1}(-1)$ so that $c_E = c_\tau$ for all $E \subset \partial\tau$ and some fixed c_τ. For any $N \in \mathcal{N}^{p+1} \cap \partial\tau$, we denote by $E_N \subset \partial\tau$ a fixed, but arbitrary, edge such that $N \in E_N$. We define the function (cf. Figure 1))

$$U_{p+1}^\tau := \sum_{N \in \mathcal{N}^{p+1} \cap \partial\tau} L_{p+1}^{E_N}(N)\, b_{p+1,N}^\tau \tag{42}$$

whose gradient $\nabla_{\mathcal{T}} U_{p+1}^\tau$ satisfies the compatibility condition across the edges. This leads to the assertion for odd p.

■

Remark 17 The space $B^p_{\tau,\mathrm{nc}}$ satisfies the compatibility conditions (29). A basis of $B^p_{\tau,\mathrm{nc}}$ for even p is given by $\left\{\nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{\tau}\right\}$, while a basis for odd p is given by $\left\{\nabla_{\mathcal{T}} b^{\mathcal{T}}_{p+1,N} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{\tau}\right\} \cup \left\{\nabla_{\mathcal{T}} U^{\tau}_{p+1}\right\}$.

4.2.2 Edge-oriented Basis Functions

Lemma 18 *Let the boundary of Ω be connected. For $E \in \mathcal{E}$, the non-conforming finite element space $\mathbf{B}^p_{E,\mathrm{nc}}$ as defined in (37) is explicitly given by*

$$\mathbf{B}^p_{E,\mathrm{nc}} = \begin{cases} \mathbf{B}^p_E + \operatorname{span}\left\{\nabla_{\mathcal{T}} U^E_{p+1}\right\} & \text{if } p \text{ is even,} \\ \mathbf{B}^p_E & \text{if } p \text{ is odd,} \end{cases} \tag{43}$$

where U^E_{p+1} is defined in (47).

Proof. Given $\mathbf{e} \in \mathbf{B}^p_E$, it follows from (11) that $\operatorname{supp} \mathbf{e} \subset \omega_E$, without being restricted to a single triangle (otherwise, $\mathbf{e} \in \mathbf{B}^p_\tau$ for some $\mathcal{T}_E$). Then it follows from the implicit Definitions (36) and (37) that $\mathbf{e} \in \mathbf{B}^p_{E,\mathrm{nc}}$. Hence, $\mathbf{B}^p_E \subset \mathbf{B}^p_{E,\mathrm{nc}}$.

For $E \in \mathcal{E}$, the space $\mathbf{B}^p_{E,\mathrm{nc}}$ was defined implicitly by (37). Since any $\mathbf{e} \in \mathbf{B}^p_{E,\mathrm{nc}}$ can be expressed locally on $\tau \in \mathcal{T}_E$ by $\left.\mathbf{e}\right|_\tau = \nabla v_\tau$ for some $v_\tau \in \mathbb{P}_{p+1}(\tau)$ (cf. Lemma 6)) we have

$$\mathbf{B}^p_{E,\mathrm{nc}} \subset \bigoplus_{\tau \in \mathcal{T}_E} \operatorname{span}\left\{\nabla_{\mathcal{T}} b^\tau_{N,p+1} \mid N \in \mathcal{N}^{p+1} \cap \tau\right\},$$

where we recall that $b^\tau_{N,p+1}$ are the Lagrange basis functions on τ and vanish on $\Omega \backslash \tau$. Since the functions $b^\tau_{N,p+1}$ for the inner nodes $N \in \mathcal{N}^{p+1} \cap \overset{\circ}{\tau}$ belong to the space $\mathbf{B}^p_{\tau,\mathrm{nc}}$, we obtain from (37)

$$\mathbf{B}^p_{E,\mathrm{nc}} \subset \bigoplus_{\tau \in \mathcal{T}_E} \operatorname{span}\left\{\nabla_{\mathcal{T}} b^\tau_{N,p+1} \mid N \in \mathcal{N}^{p+1} \cap \partial\tau\right\}.$$

For $\mathbf{e} \in \mathbf{B}^p_{E,\mathrm{nc}}$, let $u := \Lambda_{\mathcal{T}} \mathbf{e}$ and $u_\tau := \left.u\right|_\tau$, $\tau \in \mathcal{T}_E$. By arguing as in the case of triangle-supported basis functions, we derive from the compatibility conditions (29)

$$[u]_E = c_E L^E_{p+1} \quad \text{and} \quad \forall E' \subset \partial\omega_E \quad \left.u\right|_{E'} = c_{E'} L^{E'}_{p+1}. \tag{44}$$

Again, the relation $u_\tau \in \mathbb{P}_{p+1}(\tau)$ implies the continuity of u_τ at the vertices of τ.

Let p **be even.** The continuity of u_τ along $\partial\tau$ and the endpoint properties of $L^{E'}_{p+1}$ imply that $u_\tau\left(A^E\right) = u_\tau\left(B^E\right)$ for $\tau \in \mathcal{T}_E$, where A^E, B^E denote the endpoints of E (cf. Figure 2). Hence, $[u]_E\left(A^E\right) = [u]_E\left(B^E\right)$. Since $L^E_{p+1}\left(A^E\right) = -L^E_{p+1}\left(B^E\right)$ we conclude from the first condition in (44) that $c_E = 0$ holds so that u is continuous across E. Recall that the edges are closed and define

$$b^E_{p+1,N} := \begin{cases} \left.b^{\mathcal{T}}_{p+1,N}\right|_{\omega_E} & \text{on } \omega_E, \\ 0 & \text{on } \Omega \backslash \omega_E, \end{cases} \tag{45}$$

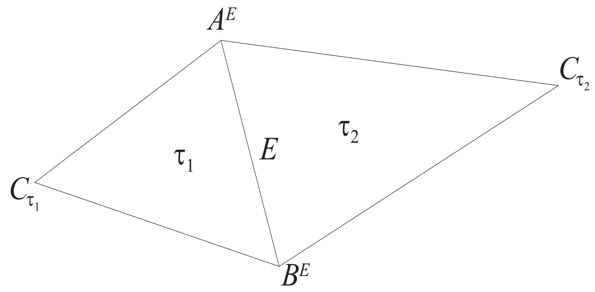


Figure 2: Edge $E \in \mathcal{E}$ with endpoints A^E, B^E and two neighboring triangles τ_1, τ_2,

where $b^{\mathcal{T}}_{p+1,N}$ are as in (20). The space $\mathbf{R}^p_{E,\mathrm{nc}}$ is given implicitly by the decomposition

$$\mathbf{B}^p_{E,\mathrm{nc}} = \mathbf{B}^p_E \oplus \mathbf{R}^p_{E,\mathrm{nc}}. \tag{46}$$

Note that then

$$\mathbf{R}^p_{E,\mathrm{nc}} \subset \operatorname{span}\left\{\nabla_{\mathcal{T}} b^E_{p+1,N} \mid N \in \mathcal{N}^{p+1} \cap \partial\omega_E\right\}.$$

Pick $\mathbf{e} \in \mathbf{R}^p_{E,\mathrm{nc}}$ and set $u := \Lambda_{\mathcal{T}}\mathbf{e}$. The continuity property $[u]_E = 0$ which we already derived implies that $c_{E'} = c$ for all $E' \subset \partial\omega_E$. This leads to $u = cU^E_{p+1}$ with

$$U^E_{p+1} := \sum_{N \in \mathcal{N}^{p+1} \cap \partial\omega_E} L^{E_N}_{p+1}(N)\, b^E_{p+1,N} \quad \text{and} \quad b^E_{p+1,N} \text{ as in (45)}, \tag{47}$$

where, again, for $N \in \mathcal{N}^{p+1} \cap \partial\omega_E$ we assign some edge $E_N \subset \partial\omega_E$ such that $N \in E_N$. Hence $\mathbf{R}^p_{E,\mathrm{nc}} = \operatorname{span}\left\{\nabla_{\mathcal{T}} U^E_{p+1}\right\}$ and the assertion follows for even p.

Let p be odd. We have

$$\mathbf{B}^p_{E,\mathrm{nc}} = \mathbf{B}^p_E \oplus \mathbf{R}^p_{E,\mathrm{nc}}, \tag{48}$$

Pick $\mathbf{e} \in \mathbf{R}^p_{E,\mathrm{nc}}$ and set $u := \Lambda_{\mathcal{T}}\mathbf{e}$. For any edge $E' \subset \partial\omega_E \cap \partial\tau$, the restriction of u_τ to E' must be a multiple of a Legendre polynomial. The continuity of u_τ along $\partial\tau$ implies in particular the continuity at C_τ (cf. Figure 2). Hence, $u_\tau|_{\partial\omega_E \cap \partial\tau} = c_\tau\, U^\tau_{p+1}\big|_{\partial\omega_E \cap \partial\tau}$ for some c_τ and U^τ_{p+1} as defined in (42), and

$$\tilde{u} = u - \sum_{\tau \in \mathcal{T}_E} c_\tau U^\tau_{p+1}$$

vanishes at $\partial\omega_E$. Since the jump of $\tilde{u}$ across E vanishes in A^E and B^E the first condition in (44) implies that $\tilde{u}$ is continuous in ω_E and vanishes on $\partial\omega_E$. From this we conclude that $\tilde{u} \in \mathbf{B}^p_E$. The characterization of $\mathbf{R}^p_{E,\mathrm{nc}}$ as a direct sum in (48) shows that $u = 0$ and thus $\mathbf{R}^p_{E,\mathrm{nc}} = \{0\}$. ■

Remark 19 The space $B^p_{E,\mathrm{nc}}$ satisfies the compatibility conditions (29). A basis of $B^p_{E,\mathrm{nc}}$ for odd p is given by $\left\{\nabla_{\mathcal{T}} b^{\mathcal{T}}_{p+1,N} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{E}\right\}$ while for even p we may choose $\left\{\nabla_{\mathcal{T}} b^{\mathcal{T}}_{p+1,N} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{E}\right\} \cup \left\{\nabla_{\mathcal{T}} U^E_{p+1}\right\}$.

4.2.3 Vertex-oriented Basis Functions

In this section we will find an explicit representation of the vertex-oriented subspace $\mathbf{B}^p_{V,\mathrm{nc}}$ defined by (37).

Lemma 20 *Let the boundary of Ω be connected. It holds*

$$\mathbf{B}^p_{V,\mathrm{nc}} = \begin{cases} \{0\} & \text{if } p \text{ is even}, \\ \mathbf{B}^p_V & \text{if } p \text{ is odd}. \end{cases} \tag{49}$$

Proof. In a first step, we will prove that the subspace $\mathbf{R}^{\mathcal{T}}_{p+1,V}$, which is implicitly defined by

$$\mathbf{R}^{\mathcal{T}}_{p+1,V} \oplus \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}^p_{E,\mathrm{nc}} \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}^p_{\tau,\mathrm{nc}} = \left\{\mathbf{e}' \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \middle| \operatorname{supp} \mathbf{e}' \subset \omega_V\right\}, \tag{50}$$

satisfies

$$\mathbf{R}^{\mathcal{T}}_{p+1,V} \subset \mathbf{B}^p_V. \tag{51}$$

In the second step, we will show that for even p the inclusion

$$\mathbf{B}^p_V \subset \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}^p_{E,\mathrm{nc}} \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}^p_{\tau,\mathrm{nc}} \tag{52}$$

holds so that the first case in (49) follows. In the case of odd p we first note that $\mathbf{B}^p_V = \operatorname{span}\left\{\nabla b^{\mathcal{T}}_{p+1,V}\right\}$. We will prove that, for all $V \in \mathcal{V}$ (cf. (43)),

$$\nabla b^{\mathcal{T}}_{p+1,V} \notin \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}^p_{E,\mathrm{nc}} \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}^p_{\tau,\mathrm{nc}}. \tag{53}$$

From (38) and (51), we conclude that $\mathbf{R}^{\mathcal{T}}_{p+1,V} = \mathbf{B}^p_V$.

1st Step. Choose any

$$\mathbf{e} \in \left\{\mathbf{e}' \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \middle| \operatorname{supp} \mathbf{e}' \subset \omega_V\right\} \tag{54}$$

and set $u := \Lambda_{\mathcal{T}} \mathbf{e}$.

Let p be odd. For $\tau \in \mathcal{T}_V$, the edge E^τ is given by the condition $E^\tau \subset \partial\tau \cap \partial\omega_V$ (cf. Figure 3). Since $L^{E^\tau}_{p+1}$ has even degree the values at the endpoints A^τ, B^τ of E^τ equal one. We set $u_\tau := u|_\tau$ and define

$$\tilde{u} := u - \sum_{\tau\in\mathcal{T}_V} u_\tau\left(A^\tau\right) U^\tau_{p+1}.$$

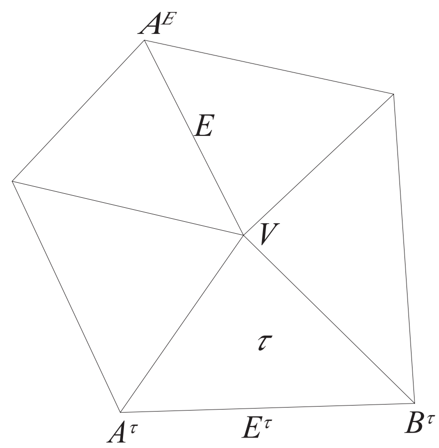


Figure 3: A vertex $V \in \mathcal{V}$, neighboring triangle $\tau \in \mathcal{T}_V$, and neighboring edge $E \in \mathcal{T}_V$.

Hence, $\tilde{u} = 0$ on $\partial\omega_V$. Any edge $E \in \mathcal{E}_V$ has V as one endpoint; denote the other one by A^E. We employ the condition $[\tilde{u}]_E = c_E L_{p+1}^E$ at the point A^E to obtain $c_E = 0$. Hence $\tilde{u}$ is continuous and vanishes on $\partial\omega_V$. Consequently, $\tilde{u}$ is a conforming function, i.e.,

$$\begin{aligned}\nabla\left(u - \sum_{\tau\in\mathcal{T}_V} u_\tau\left(A^\tau\right) U_{p+1}^\tau\right) &\in \mathbf{B}_V^p \oplus \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}_E^p \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}_\tau^p \\ &\subset \mathbf{B}_V^p \oplus \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}_{E,\mathrm{nc}}^p \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}_{\tau,\mathrm{nc}}^p.\end{aligned}$$

Hence, (50) implies $\mathbf{R}_{p+1,V}^{\mathcal{T}} \subset \mathbf{B}_V^p$.

Let p be even. We number the edges in $\mathcal{E}_V$ counter-clockwise $\mathcal{E}_V = \{E_1, \ldots, E_q\}$ (see Figure 4) for some q and, to simplify the notation, we set $E_0 := E_q$ and $E_{q+1} := E_1$. The triangle which has E_{i-1} and E_i as edges and V as a vertex is denoted by τ_i. Each edge E_i has V as an endpoint; denote by A_i the other one. We further set $E_i^{\mathrm{out}} := \partial\tau_i \cap \partial\omega_V$. We define recursively $u_0 := u$ and, for $k = 1, 2, \ldots, q$,

$$u_k = u_{k-1} - \frac{\left(u_{k-1}\right)_{\tau_k}\left(A_k\right)}{U_{p+1}^{E_k}\left(A_k\right)} U_{p+1}^{E_k}.$$

Note that $u_q = 0$ on $\partial\omega_V \backslash E_1^{\mathrm{out}}$. By arguing as for the case of odd p we deduce that u_q is continuous on $\omega_V \backslash E_1$. Since $u_q|_{E_1^{\mathrm{out}}} = c_1 L_{p+1}^{E_1^{\mathrm{out}}}$ for some $c_1 \in \mathbb{R}$, the property $u_q\left(A_q\right) = 0$ and $L_{p+1}^{E_1^{\mathrm{out}}}\left(A_q\right) \neq 0$ implies $c_1 = 0$. Hence, $u_q|_{\partial\omega_V} = 0$. Arguing as in the case of odd p finally yields that u_q is continuous on ω_V and the assertion follows as in the case of odd p.

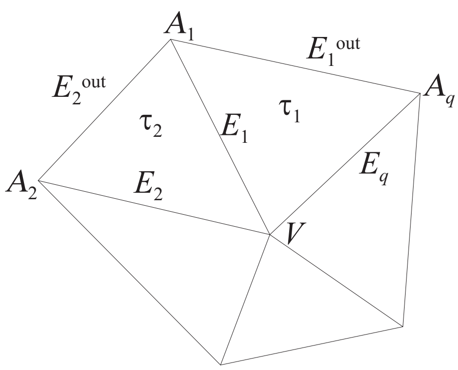


Figure 4: Vertex $V \in \mathcal{V}$ and outgoing edges – numbered counterclockwise. The triangles $\tau_i \in \mathcal{T}_V$ contain E_{i-1}, E_i as edges and V as a vertex.

This finishes the proof of (51).

2nd Step: To prove (52) we again distinguish between even and odd values of p.

Let p **be even.** Then, by using U_{p+1}^{E} as in (47), we define a function

$$w_1 := b_{p+1,V}^{\mathcal{T}} - \frac{1}{q} \sum_{E \in \mathcal{E}_V} U_{p+1}^{E}(V)\, U_{p+1}^{E} \tag{55}$$

which is continuous in ω_V and vanishes at V and at all inner nodes $\mathcal{N}^{p+1} \cap \overset{\circ}{\tau}$, $\tau \in \mathcal{T}_V$. Two consecutive terms in the sum in (55) define the function

$$\left(U_{p+1}^{E_{i-1}}(V)\, U_{p+1}^{E_{i-1}} + U_{p+1}^{E_i}(V)\, U_{p+1}^{E_i} \right)\Big|_{E_i^{\text{out}}},$$

which is a multiple of the Legendre polynomial $L_{p+1}^{E_i^{\text{out}}}$. From (55) we conclude that this function has values 0 at both endpoints of E_i^{out} so that $w_1 = 0$ on $\partial\omega_V$.

Next, the function

$$w_2 = w_1 - \sum_{E \in \mathcal{E}_V} \sum_{N \in \mathcal{N}^{p+1} \cap \overset{\circ}{E}} w_1(N)\, b_{p+1,N}^{\mathcal{T}} \tag{56}$$

vanishes at all nodal points $\mathcal{N}^{p+1} \cap \left(\overset{\circ}{\omega}_E \right)$ and the jumps across $E \in \mathcal{E}_V$ have to vanish due to the compatibility condition. Since w_1 as well as the basis functions in the sum (56) vanish along $\partial\omega_E$, we conclude that w_2 vanishes also on $\partial\omega_E$ and thus $w_2 = 0$ in Ω. Hence, we have established (52), or, more precisely, that

$$\nabla b_{p+1,V}^{\mathcal{T}} \in \bigoplus_{E \in \mathcal{E}_V} \mathbf{B}_{E,\text{nc}}^{p}.$$

Let p be odd. We will prove (53) by contradiction and assume that

$$\nabla b_{p+1,V}^{\mathcal{T}} \in \bigoplus_{E\in\mathcal{E}_V} \mathbf{B}_{E,\mathrm{nc}}^{p} \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}_{\tau,\mathrm{nc}}^{p}.$$

We then infer from Remark 17 and Remark 19 that

$$b_{p+1,V}^{\mathcal{T}} = \underbrace{\sum_{N\in\mathcal{N}^{p+1}\backslash\mathcal{V}} \alpha_N b_{p+1,N}^{\mathcal{T}}}_{=:v_{\mathrm{c}}} + \underbrace{\sum_{\tau\in\mathcal{T}} \alpha_\tau U_{p+1}^{\tau}}_{v_{\mathrm{nc}}} \tag{57}$$

for some real coefficients α_N and α_τ. Since $b_{p+1,N}^{\mathcal{T}}$ and v_{c} are continuous in Ω, the function v_{nc} must also be continuous. By contradiction it is easy to prove that

$$C^0(\Omega) \cap \bigoplus_{\tau\in\mathcal{T}} \operatorname{span}\left\{U_{p+1}^{\tau}\right\} = \operatorname{span}\left\{U_{p+1}\right\} \text{ with } U_{p+1} := \sum_{\tau\in\mathcal{T}} U_{p+1}^{\tau},$$

so that $v_{\mathrm{nc}} \in \operatorname{span}\{U_{p+1}\}$. Since $v_{\mathrm{c}}(V) = 0$ and $b_{p+1,V}^{\mathcal{T}}(V) = 1$, we obtain from (57) that $v_{\mathrm{nc}}(V) = 1$. The restriction of U_{p+1} to any edge $E \in \mathcal{E} \cup \mathcal{E}_{\partial\Omega}$ is a Legendre polynomial of even degree, which implies $v_{\mathrm{nc}}(V') = 1$, for every $V' \in \mathcal{V} \cup \mathcal{V}_{\partial\Omega}$. But the functions $b_{p+1,V}^{\mathcal{T}}$ and v_{c} vanish on $\partial\Omega$. This contradicts $v_{\mathrm{nc}}(V') = 1$ for the boundary points $V' \in \mathcal{V}_{\partial\Omega}$. ■

4.2.4 Properties of the Non-Conforming Intrinsic Basis functions

Theorem 21 *Let the boundary of Ω be connected. A basis of $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^{p}$ is given by*

$$\left\{\nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1}\backslash\mathcal{V}\right\} \cup \bigcup_{E\in\mathcal{T}} \left\{\nabla_{\mathcal{T}} U_{p+1}^{E}\right\} \quad \text{if } p \text{ is even,} \tag{58}$$

and by

$$\left\{\nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1}\right\} \cup \bigcup_{\tau\in\mathcal{T}} \left\{\nabla_{\mathcal{T}} U_{p+1}^{\tau}\right\} \quad \text{if } p \text{ is odd.} \tag{59}$$

Remark 22 At first glance, it seems that $B_V^p \not\subset E_{\mathcal{T},\mathrm{nc}}^p$ for even p. Actually, this subspace of $E_{\mathcal{T}}^p$ has already been taken into account; see (52).

Proof. By construction, the space $\widetilde{\mathbf{E}_{\mathcal{T},\mathrm{nc}}^{p}}$ of the functions found in (58) as in (59) is a subspace of $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^{p}$. So, it remains to prove $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^{p} \subset \widetilde{\mathbf{E}_{\mathcal{T},\mathrm{nc}}^{p}}$.

Let p be odd. The arguments in the following are very similar to those in the proof of Lemma 20 for odd p. Let $u := \Lambda_{\mathcal{T}}\mathbf{e}$. Pick some $\tau \in \mathcal{T}$ having at least one edge on $\partial\Omega$. Condition (29) implies that for all edges $E \subset \partial\tau \cap \partial\Omega$, the restriction $\left.u\right|_E$ is a multiple of the Legendre polynomial L_{p+1}^{E}. The continuity of $\left.u\right|_\tau$ on τ implies that there exists a function $\tilde{u} := cU_{p+1}^{\tau}$ with $\nabla\tilde{u} \in \mathbf{B}_{\tau,\mathrm{nc}}^{p}$ for some c such that $u_1 := u - \tilde{u}$ satisfies $\left.u_1\right|_{\partial\tau\cap\partial\Omega} = 0$. Since u_1 vanishes at the endpoints of all such edges $E \in \mathcal{E}_{\partial\Omega}$, the function u_1 is also continuous across the other edges $E \subset \partial\tau \cap \Omega$. Let

$$\tilde{u}_1 = \sum_{N \in \mathcal{N}^{p+1} \cap \overset{\circ}{\tau}} u_1(N)\, b_{p+1,N}^{\mathcal{T}} + \sum_{E \subset \partial\tau \cap \Omega} \sum_{N \in \mathcal{N}^{p+1} \cap \overset{\circ}{E}} u_1(N)\, b_{p+1,N}^{\mathcal{T}} + \sum_{V \in \partial\tau \cap \Omega} u_1(V)\, b_{p+1,V}^{\mathcal{T}}$$

and note that $\tilde{u}_1 \in \widetilde{\mathbf{E}_{\mathcal{T},\mathrm{nc}}^{p}}$. In particular Lemma 20 implies that $b_{p+1,V}^{\mathcal{T}} \in \widetilde{\mathbf{E}_{\mathcal{T},\mathrm{nc}}^{p}}$. Note that $u_2 := u_1 - \tilde{u}_1$ vanishes on τ. Iterating this construction for the remaining triangles finally results in a function that vanishes on Ω. Thus we have found a linear representation of u by functions in $\widetilde{\mathbf{E}_{\mathcal{T},\mathrm{nc}}^{p}}$.

Let p be even. Again the arguments are very similar to those in the proof of Lemma 20 for even p. We omit the details here. ■

Proposition 23 *Let the boundary of Ω be connected. The lowest order non-conforming intrinsic finite elements are given by*

$$\mathbf{E}_{\mathcal{T},\mathrm{nc}}^{0} = \operatorname{span}\left\{ \nabla_{\mathcal{T}} U_1^E : E \in \mathcal{E} \right\},$$

where the functions U_1^E are the standard non-conforming Crouzeix-Raviart basis functions (cf. [9]).

Proof. Choosing $p = 0$ and taking into account that $N^1 = \mathcal{V}$ we conclude from (58) that a basis for $\mathbf{E}_{\mathcal{T},\mathrm{nc}}^{0}$ is given by $\bigcup_{E \in \mathcal{T}} \left\{ \nabla_{\mathcal{T}} U_1^E \right\}$.

To show the connection to the Crouzeix-Raviart basis functions, we consider an edge $E \in \mathcal{E}$ with neighboring triangles τ_1 and τ_2. From (47), we deduce that U_1^E is affine on each of the triangles τ_1, τ_2 with value 1 at the endpoints of E and value -1 at the vertices of τ_1, τ_2 that are opposite to E. Hence, U_1^E coincides with the standard Crouzeix-Raviart basis functions; see again [9]. ■

5 Conclusions

In this article we developed a general method for constructing of finite element spaces from intrinsic conforming and non-conforming conditions. As a model problem we have considered the Poisson equation, but this approach is by no means limited to this model problem. Using theoretical conditions in the spirit of the second Strang lemma, we have derived conforming and non-conforming finite element spaces of arbitrary order for the fluxes. For these spaces, we also derived sets of local basis functions.

It turns out that the lowest order non-conforming space is spanned by the trianglewise gradients of the standard non-conforming Crouzeix-Raviart basis functions. In general, all polynomial non-conforming spaces are spanned by the gradients of standard *hp*-finite element basis functions *enriched* by some edge oriented non-conforming basis functions for even polynomial degree and by some triangle-supported non-conforming basis functions for odd polynomial degree. As a by-product, this methodology allowed us to recover the well-known **non-conforming Crouzeix-Raviart element** [9] (cf. Proposition 23). By using a similar but more technical reasoning (cf. [17]), it can be shown that our intrinsic derivation of non-conforming finite elements also allows to recover the **second order non-conforming Fortin-Soulie element** [10, 11], the **third order Crouzeix-Falk element** [8], and the family of **Gauss-Legendre elements** [2], [18].

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