ON VARIETIES OF ALMOST MINIMAL DEGREE IN SMALL CODIMENSION

MARKUS BRODMANN, PETER SCHENZEL

ABSTRACT. The present research grew out of the authors' joint work [2]. It continues the study of the structure of projective varieties of almost minimal degree, focussing to the case of small codimension. In particular, we give a complete list of all occuring Betti diagrams in the cases where $\operatorname{codim} X \leq 4$.

1. INTRODUCTION

Let $X \subset \mathbb{P}^r_K$ denote an irreducible and reduced projective variety over an algebraically closed field K. We always assume that X is non-degenerate, that is not contained in a hyperplane. Then, the degree and the codimension of X satisfy the inequality $\deg X \geq \operatorname{codim} X + 1$ (cf. for instance [9]). Varieties for which equality holds are called called *varieties of minimal degree*. These varieties are completely classified (cf. for instance [7] and [9]). In particular they are arithmetically Cohen-Macaulay and have a linear minimal free resolution. In particular, their Betti numbers are explicitly known.

In case $\deg X = \operatorname{codim} X + 2$, the variety X is called a *variety of almost minimal degree*. Here one has a much greater variety of possible Betti numbers. The investigation of homological properties of varieties of almost minimal degree was initiated by Hoa, Stückrad, and Vogel (cf. [10]). We refer also to [2] and [12] for certain improvements of their results. In particular the Castelnuovo-Mumford regularity of a variety X of almost minimal degree satisfies $\operatorname{reg} X \leq 2$ (cf. [6] for the definition of the Castelnuovo-Mumford regularity).

In the framework of polarized varieties of Δ-genus 1, Fujita (cf. [3] and [4]) provides a satisfactory description of varieties of almost minimal degree. The study of varieties of almost minimal degree is pursued by the authors (cf. [2]) from the arithmetic point of view. It turns out that $X \subset \mathbb{P}^r_K$ is either an arithmetically normal Del Pezzo variety or a proper projection of a variety of minimal degree. By a proper projection of a variety $Z \subset \mathbb{P}^{r+1}_K$ we always mean a projection from a point $p \in \mathbb{P}^{r+1}_K \setminus Z$. See also 3.1 for the precise statement.

The aim of the present paper is to investigate varieties of almost minimal degree and of low codimension, in particular their Betti diagrams. More precisely, we describe the structure of the minimal free resolution of a variety X of almost minimal degree of $\operatorname{codim} X \leq 4$ by listing all possible Betti diagrams. Let us recall that the structure of arithmetically Cohen-Macaulay resp. Gorenstein varieties in codimension 2 resp. 3 is known by the Theorems of Hilbert-Burch resp. Buchsbaum-Eisenbud (cf. [6]). So, we

The second named author was partially supported by Swiss National Science Foundation (Project No. 20 - 103491/1).

need not to discuss these cases in detail any more. As the Betti diagram, the degree and the codimension are not affected if X is replaced by a cone over X, we shall assume that X is not a cone. The most surprising fact is, that the dimension of X is always ≤ 6 (cf. Section 2 for the precise statements). Our main technical tool is a result shown by the authors in [2], which says that apart from an exceptional case, (that is the generic projection of the Veronese surface in $\mathbb{P}^5_K$) any non-arithmetically normal (and in particular non-arithmetically Cohen-Macaulay) variety of almost minimal degree $X \subset \mathbb{P}^r_K$ (which is not a cone) is contained in a variety of minimal degree $Y \subset \mathbb{P}^r_K$ such that $\operatorname{codim}(X, Y) = 1$.

Acknowledgement . The authors thank Uwe Nagel for making his preprint [12] available. They also thank for the referee's very valuable suggestions.

2. MAIN RESULTS

Let $X \subset \mathbb{P}^r_K$ denote a non-degenerate variety of almost minimal degree, hence an integral closed subscheme with $\deg X = \operatorname{codim} X + 2$ not contained in a hyperplane $\mathbb{P}^{r-1}_K \subset \mathbb{P}^r_K$. We use the abbreviations $\dim X = d$ and $\operatorname{codim} X = c$. Let $S = K[x_0, \ldots, x_r]$ denote the polynomial ring in $r + 1$ variables, so that $\mathbb{P}^r_K = \operatorname{Proj}(S)$. Let $A_X = S/I_X$ denote the homogeneous coordinate ring of X, where $I_X \subset S$ is the defining ideal of X. The codepth of A_X is defined as the difference $\operatorname{codepth} A_X := \dim A_X - \operatorname{depth} A_X$, where $\operatorname{depth} A_X$ denotes the depth of A_X.

For the notion of Betti diagrams we follow the suggestion of Eisenbud (cf. [6]). That is, in a diagram the number in the i-th column and the j-th row is

$$\dim_K \operatorname{Tor}^S_i(K, I_X)_{i+j}.$$

Outside the range of the diagram all the entries are understood to be zero. Our main results are.

Theorem 2.1. $(\operatorname{codim} X = 2)$ *Let $X \subset \mathbb{P}^r_K$ be a non-degenerate variety of degree 4 and codimension 2 which is not a cone. Then X is of one of the following types:*

(a) *X is a complete intersection cut out by two quadrics.*
(b) *$\dim X \leq 4$ and the Betti diagram of I_X has the form*

	1	2	3
1	1	0	0
2	3	4	1

.

(c) *(The exceptional case) X is a generic projection of the Veronese surface $F \subset \mathbb{P}^5_K$ and the Betti diagram of I_X has the shape*

	1	2	3	4
1	0	0	0	0
2	7	10	5	1

.

Moreover for any $1 \leq d \leq 4$ there are examples as mentioned in (b) such that $\dim X = d$.

In the case where X is a Cohen-Macaulay variety, Theorem 2.1 (b) has been shown by Nagel (cf. [12]). Under this additional assumption one has $\dim X \leq 2$. Theorem 2.1 grew out of our aim to understand Nagel's arguments. Our approach enables us to investigate the cases of codimension three and four as well.

Theorem 2.2. $(\operatorname{codim} X = 3)$ *Let $X \subset \mathbb{P}^r_K$ denote a non-degenerate variety of degree 5 and codimension 3 which is not a cone. Then the following cases may occur:*

(a) $\dim X \leq 6$ *and X is the Pfaffian variety defined by the five Pfaffians of a skew symmetric 5×5 matrix of linear forms. Its minimal free resolution is given by the Buchsbaum-Eisenbud complex.*

(b) $\dim X \leq 4$ *and* $\operatorname{codepth} A_X = 1$. *The Betti diagram of the defining ideal I_X has the following form*

	1	2	3	4
1	4	2	0	0
2	1	6	5	1

.

(c) $\dim X \leq 5$ *and* $\operatorname{codepth} A_X = 2$. *The Betti diagram of I_X has the following shape*

	1	2	3	4	5
1	3	2	0	0	0
2	6	16	15	6	1

.

For any of the dimensions and codepths admitted in (a), (b) and (c) resp. there are examples of varieties of almost minimal degree.

The next result concerns the case where X is of codimension 4. A new phenomenon occurs in this situation. Namely, for the same codepth two different Betti diagrams may occur.

Theorem 2.3. $(\operatorname{codim} X = 4)$ *Let $X \subset \mathbb{P}^r_K$ denote a non-degenerate variety of degree 6 and codimension 4 which is not a cone. Then the following four cases may occur:*

(a) $\dim X \leq 4$ *and X is arithmetically Gorenstein. Its minimal free resolution has the following form*

$$0 \to S(-6) \to S^9(-4) \to S^{16}(-3) \to S^9(-2) \to I_X \to 0.$$

(b) $\dim X \leq 4$ *and* $\operatorname{codepth} A_X = 1$. *The Betti diagram of the defining ideal I_X has one of the following two forms:*

	1	2	3	4	5
1	8	12	3	0	0
2	1	4	10	6	1

resp.

	1	2	3	4	5
1	8	11	3	0	0
2	0	4	10	6	1

.

(c) $\dim X \leq 5$ *and* $\operatorname{codepth} A_X = 2$. *The Betti diagram of I_X is of the form:*

	1	2	3	4	5	6
1	7	8	3	0	0	0
2	3	19	30	21	7	1

.

(d) $\dim X \leq 6$ *and* $\operatorname{codepth} A_X = 3$. *The Betti diagram of I_X has the following shape:*

	1	2	3	4	5	6	7
1	6	8	3	0	0	0	0
2	10	40	65	56	28	8	1

.

For any of the dimensions, codepths and Betti diagrams admitted in (a), (b), (c) and (d) resp. there are examples of varieties of almost minimal degree.

The final result of this note concerns varieties of almost minimal degree and of codimension less than half the embedding dimension. First of all note (cf. Example 4.8) that in codimension 6 there are varieties of almost minimal degree having the same $\operatorname{codepth}$ but with rather different Betti diagrams. That is, the corresponding statements to Theorem 2.1, 2.2 and 2.3 are not true in hihgher codimension.

Corollary 2.4. *Let $X \subset \mathbb{P}_K^r$ be a non-degenerate variety of almost minimal degree which is not a cone. Suppose that $\dim X > \operatorname{codim} X + 2$ and $\operatorname{codim} X \geq 3$. Then $\operatorname{codim} X = 3$ and X is arithmetically Gorenstein. Therefore $\dim X \leq 6$ and X is defined by the five Pfaffians of a skew symmetric 5×5 matrix of linear forms.*

Moreover, in the case where X is a variety of almost minimal degree and $\dim X \leq \operatorname{codim} X + 2$, that is when X is not necessary a Del Pezzo variety there are estimates for the Betti numbers and their vanishing (cf. 3.3 for the details).

3. Outline of the Proofs

Let $X \subset \mathbb{P}_K^r$ denote a non-degenerate reduced irreducible variety of almost minimal degree. By the work of the authors (cf. [2]) it follows that X is either a normal Del Pezzo variety – in this case X is arithmetically Gorenstein – or a projection of a variety of minimal degree.

First, we consider the case of non-arithmetically normal varieties of almost minimal degree. In this situation we have the following characterization in which $\operatorname{Sec}_P(Z)$ denotes the secant cone of a projective variety Z with respect to a point P in the ambient space.

Lemma 3.1. *Let $X \subset \mathbb{P}_K^r$ denote a non-degenerate reduced irreducible variety which is not a cone. Let $1 \leq t \leq \dim X + 1 =: d + 1$. Then the following conditions are equivalent:*

(i) *X is not arithmetically normal, $\deg X = \operatorname{codim} X + 2$ and $\operatorname{depth} A_X = t$.*
(ii) *X is the projection of a variety $Z \subset \mathbb{P}_K^{r+1}$ of minimal degree from a point $P \in \mathbb{P}_K^{r+1} \setminus Z$ such that $\dim \operatorname{Sec}_P(Z) = t - 1$.*

Moreover, $1 \leq \operatorname{depth} A_X \leq 4$.

Proof. Cf. [2, Theorem 1.1 and Corollary 7.6]. □

In view of Lemma 3.1 there is some need for information about varieties of minimal degree in order to understand varieties of almost minimal degree. A variety of minimal degree $Z \subset \mathbb{P}_K^s$ is either

- a quadric hypersurface,
- a (cone over a) Veronese surface in $\mathbb{P}_K^5$, or
- a (cone over a) smooth rational normal scroll

(cf. [7] and [9] for the details and the history of this classification).

Next we recall a few basic facts about rational normal scrolls (cf. also [9]). Let

$$T = K[x_{10}, \ldots, x_{1a_1}, x_{20}, \ldots, x_{2a_2}, \ldots, x_{k0}, \ldots, x_{ka_k}]$$

be the polynomial ring. A (cone over a) rational normal scroll $S(a_1, \ldots, a_k)$ is defined as the "rank two subscheme in $\mathbb{P}^s_K = \mathrm{Proj}(T)$" defined by the matrix

$$M = \begin{pmatrix} x_{10} & \ldots & x_{1a_1-1} & \vdots & x_{20} & \ldots & x_{2a_2-1} & \vdots & \ldots & \vdots & x_{k0} & \ldots & x_{ka_k-1} \\ x_{11} & \ldots & x_{1a_1} & \vdots & x_{21} & \ldots & x_{2a_2} & \vdots & \ldots & \vdots & x_{k1} & \ldots & x_{ka_k} \end{pmatrix}$$

with $s = k - 1 + \sum_{i=1}^k a_i$. It is well known that $\dim S(a_1, \ldots, a_k) = k$ and therefore

$$\deg S(a_1, \ldots, a_k) = \operatorname{codim} S(a_1, \ldots, a_k) + 1 = \sum_{i=1}^k a_i,$$

so that $S(a_1, \ldots, a_k)$ is a variety of minimal degree. Keep in mind that $S(a_1, \ldots, a_k)$ is a proper cone if and only if $a_i = 0$ for some $i \in \{1, \ldots, k\}$, that is, if and only if there are indeterminates which do not occur in M.

Moreover we need some information about the Hilbert series $F(\lambda, A_X)$ of the graded K-algebra A_X. Let us recall that the Hilbert series of a graded K-algebra A is the formal power series defined by

$$F(\lambda, A) = \sum_{i \geq 0} (\dim_K A_i) \lambda^i.$$

The Hilbert series of a variety of almost minimal degree may be described as follows.

Lemma 3.2. *Let $X \subset \mathbb{P}^r_K$ denote a variety of almost minimal degree. Put $q = \operatorname{codepth} A_X$. Then*

$$F(\lambda, A_X) = \frac{1}{(1-\lambda)^{d+1}} (1 + (c+1)\lambda - \lambda(1-\lambda)^{q+1}),$$

where $c = \operatorname{codim} X$ and $d = \dim X$. Furthermore $\dim_K (I_X)_2 = \binom{c+1}{2} - q - 1$.

Proof. Cf. [2, Corollary 4.4]. □

As a consequence of Lemma 3.1 the authors (cf. [2]) derived some information about the Betti numbers of I_X for certain varieties $X \subset \mathbb{P}^r_K$ of almost minimal degree.

Lemma 3.3. *Let $X \subset \mathbb{P}^r_K$ be a variety of almost minimal degree which is not arithmetically Cohen-Macaulay. Suppose that X is not a generic projection of (a cone over) the Veronese surface $F \subset \mathbb{P}^5_K$. Then there exists a variety of minimal degree $Y \subset \mathbb{P}^r_K$ such that $X \subset Y$ and $\operatorname{codim}(X, Y) = 1$. Moreover*

$$\mathrm{Tor}_i^S(k, A_X) \simeq k^{u_i}(-i-1) \oplus k^{v_i}(-i-2) \text{ for } 1 \leq i \leq c+q,$$

where $c = \operatorname{codim} X, q = \operatorname{codepth} A_X$ and

(a) $u_1 = \binom{c+1}{2} - q - 1,$
$i\binom{c}{i+1} \leq u_i \leq (c+1)\binom{c}{i} - \binom{c}{i+1}$, *if* $1 < i < c - q,$
$u_i = i\binom{c}{i+1}$, *if* $c - q \leq i < c,$
$u_i = 0$, *if* $c \leq i \leq c + q.$

(b) $\max\{0, \binom{c+q-1}{i+1} - (i+2)\binom{c}{i+1}\} \leq v_i \leq \binom{c+q+1}{i+1}$, *if* $1 \leq i < c - q - 1,$
$v_i = \binom{c+q+1}{i+1} - (i+2)\binom{c}{i+1}$, *if* $\max\{1, c - q - 1\} \leq i < c,$
$v_i = \binom{c+q+1}{i+1}$, *if* $c \leq i \leq c + q.$

In addition $v_i - u_{i+1} = \binom{c+q+1}{i+1} - (c+1)\binom{c}{i+1} + \binom{c}{i+2}$ *for all* $1 \leq i < c$.

Proof. Cf. [2, Theorem 1.1 and Theorem 8.3]. □

In the particular case where $\dim X = 1$ the statement about the Betti numbers has been shown independently by Nagel (cf. [12]).

In the following remark, we add a comment concerning the ”exceptional cases” of a generic projection of (a cone over) the Veronese surface $F \subset \mathbb{P}^5_K$ and of an arithmetically Cohen-Macaulay variety.

Remark 3.4. A) (The exceptional case) Let $F \subset \mathbb{P}^5_K$ be the Veronese surface defined by the 2×2-minors of the symmetric matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}.$$

Let $P \in \mathbb{P}^5_K \setminus F$ denote a point. Suppose that $\operatorname{rank} M|_P = 3$, that is the case of a generic point. Remember that $\det M = 0$ defines the secant variety of F. Then the projection of F from P defines a surface $X \subset \mathbb{P}^4_K$ of almost minimal degree and $\operatorname{depth} A_X = 1$. The surface X is cut out by seven cubics (cf. 3.2), so that it is not contained in a variety of minimal degree.
B) (The arithmetically normal case) Let $X \subset \mathbb{P}^r_K$ denote an arithmetically normal variety of almost minimal degree. Then X is not a birational projection of a scroll and hence a maximal Del Pezzo variety (cf. [2, Theorem 1.2]). In particular X is arithmetically Gorenstein. If in addition $\operatorname{codim} X \geq 3$, Fujita's classification of normal maximal Del Pezzo varieties yields $\dim X \leq 4$ (cf. [5, (8.11), (9.17)]). If $\operatorname{codim} X \geq 4$, the same classification shows that $\dim X \leq 4$.
C) If X is an arithmetical normal Del Pezzo variety it is in general not a one codimensional subvariety of a variety of minimal degree. Namely, let $X \subset \mathbb{P}^9_K$ be the smooth codimension three variety cut out by the 4×4-Pfaffians of a generic skew-symmetric 5×5-matrix of linear forms. Then X is not contained in a variety Y of minimal degree such that $\operatorname{codim}(X, Y) = 1$ (cf. [2] for the details).

Proofs. After these preparations, we now come to the proofs of our Theorems. The proof of statement 2.1 (a) is easy. Now let us consider the statements 2.2 (a) and 2.3 (a). In both cases X is arithmetically Gorenstein. If X is a birational projection of a scroll, we have $\dim A_X = \operatorname{depth} A_X \leq 4$ (cf. Lemma 3.1). Otherwise X is arithmetically normal (cf. [2, Theorem 1.2]). So, by 3.4 B) we have $\dim X \leq 6$ if $\operatorname{codim} X \geq 3$ and $\dim X \leq 4$ if $\operatorname{codim} X \geq 4$. This gives us the dimension estimates. The statements on the minimal free resolutions now follow from well known results. In fact the structure of the minimal free resolution of I_X given in Theorem 2.3 (a) is a consequence of [13, Theorem B].

Next we prove statement (c) of Theorem 2.1. To this end let $X \subset \mathbb{P}^4_K$ be a generic projection of the Veronese surface $F \subset \mathbb{P}^5_K$. Then $\dim X = 2, \operatorname{codim} X = 2$ and $\operatorname{depth} A_X = 1$. As seen above, I_X does not contain any quadric. Therefore I_X has a linear resolution. Remember that $\operatorname{reg} I_X = 3$ (cf. [2]). A computation with the aid of the Hilbert series (cf. 3.2) gives the structure of the Betti diagram of 2.1 (c).

For all other statements of Theorems 2.1, 2.2 and 2.3 we may assume that the variety of almost minimal degree $X \subset \mathbb{P}^r_K$ is not arithmetically Cohen-Macaulay and not a projection of the Veronese surface $F \subset \mathbb{P}^5_K$. So, X is contained in a variety of minimal degree Y of codimension $c-1$ (cf. 3.3). That is, Y is defined as the zero locus of $\binom{c}{2}$ quadrics. On the other hand the defining ideal I_X is generated by $\binom{c+1}{2} - q - 1$ quadrics (cf. 3.2). This implies that

$$\dim_K(I_X/I_Y)_2 = c - q - 1 \geq 0,$$

where $c = \operatorname{codim} X, q = \operatorname{codepth} A_X \geq 1$. Considering all possibilities that arise for $c = 2, 3, 4$ it follows that, with the exception of the case in which $c = 4, q = 1$, Lemma 3.3 furnishes the corresponding Betti diagrams.

The particular case where $c = 4, q = 1$, yields the following shape of the Betti diagram

	1	2	3	4	5
1	8	u_2	3	0	0
2	v_1	4	10	6	1

with $v_1 - u_2 = -11$. In order to finish the proof we observe that $v_1 \leq 1$ (cf. Lemma 3.5).

To complete the proof of Theorems 2.1, 2.2 and 2.3 we have to prove the stated constraints on the occuring dimensions and codepths. To do so, we may assume that X is not arithmetically Cohen-Macaulay. Therefore by Lemma 3.1 X is the projection of a variety of minimal degree $Z \subset \mathbb{P}^{r+1}_K$ from a point $P \in \mathbb{P}^{r+1}_K \setminus Z$ such that $\dim \operatorname{Sec}_P(Z) = t - 1$, where $t = \operatorname{depth} A_X$.

Next let us analyze this situation in more detail. To this end let $Z = S(a_1, \ldots, a_k)$ for certain integers $a_i, i = 1, \ldots, k$. Then it follows that

$$r = k - 2 + \sum_{i=1}^{k} a_i \text{ and } c + 2 = \sum_{i=1}^{k} a_i,$$

where $c = \operatorname{codim} X$.

As X is not a cone, Z cannot be a cone over a rational normal scroll. Therefore $\min\{a_i : i = 1, \ldots, k\} \geq 1$. So, for a given codimension c we have to investigate all the possible partitions

$$c + 2 = \sum_{i=1}^{k} a_i, \text{ with } k \geq 1 \text{ and } a_1 \geq a_2 \geq \ldots \geq a_k \geq 1.$$

For $c = 2$ we thus get the following possible types for the rational normal scroll Z :

k	a_1	a_2	a_3	a_4	$r+1$	$\dim X$
1	4				4	1
2	3	1			5	2
2	2	2			5	2
3	2	1	1		6	3
4	1	1	1	1	7	4

This proves already that $\dim X \leq 4$.

Next, we discuss the case in which the codimension equals 3. Here, there are the following possibilities for the type of the scroll Z:

k	a_1	a_2	a_3	a_4	a_5	$r+1$	$\dim X$
1	5					5	1
2	4	1				6	2
2	3	2				6	2
3	3	1	1			7	3
3	2	2	1			7	3
4	2	1	1	1		8	4
5	1	1	1	1	1	9	5

Therefore $\dim X \leq 5$. We already know that $u_1 \geq 3$. Now, on use of Lemma 3.3 we easily get the requested constraints in Theorem 2.2.

Finally, let $\operatorname{codim} X = 4$. Then, as above there is the following list of possible types for the scroll Z :

k	a_1	a_2	a_3	a_4	a_5	a_6	$r+1$	$\dim X$
1	6						6	1
2	5	1					7	2
2	4	2					7	2
2	3	3					7	2
3	4	1	1				8	3
3	3	2	1				8	3
3	2	2	2				8	3
4	3	1	1	1			9	4
4	2	2	1	1			9	4
5	2	1	1	1	1		10	5
6	1	1	1	1	1	1	11	6

As above it follows that $\dim X \leq 6$. We know that $u_1 \geq 6$. So, by Lemma 3.3 we get the requested constraints in Theorem 2.3.

Finally we prove Corollary 2.4. Assume that X is not arithmetically normal. Then, by Theorem 1.2 of [2] we know that X is a birational projection of a rational normal scroll $Z \subset \mathbb{P}_K^{r+1}$ from a point $p \in \mathbb{P}_K^{r+1} \setminus Z$. As X is not a cone, Z is not a cone either and therefore $\operatorname{codim} X + 2 = \sum_{i=1}^k a_i \geq k = \dim X$. This contradicts the assumption of Corollary 2.4. Therefore X is arithmetically normal and so $\dim X \leq 6$ by Remark 3.4. That is, $\operatorname{codim} X = 3$. So our claim follows by Theorem 2.2.

For the existence of the samples described in Theorems 2.1 and 2.2 we refer to the next section.

We close this section with a result on the number of cubics in a minimal generating set of the defining ideal of a certain varieties of almost minimal degree.

Lemma 3.5. *Let $X \subset \mathbb{P}_K^r$ be a variety of almost minimal degree with $\operatorname{codepth} A_X = 1$ and $c := \operatorname{codim} X \geq 4$. Then the defining ideal I_X of X is generated by $\binom{c+1}{2} - 2$ quadrics and at most one cubic.*

Proof. First we reduce the problem to the case in which $\dim X = 1$. Let $d = \dim X > 1$. By an argument of Bertini type (cf. [11]) we may find generic linear forms $l_1, \dots, l_{d-1} \in S_1$ such that $W = X \cap \mathbb{P}^{c+1}_K \subset \mathbb{P}^{c+1}_K := \mathrm{Proj}(S/(l_0, \dots, l_{d-1})S)$ is a non-degenerate integral variety of almost minimal degree. As $l_0, \dots, l_{d-1}$ are chosen generically and $\mathrm{depth}\, A_X = d$ they form an A_X-regular sequence. Therefore the Betti diagrams of I_X and I_W are the same. In particular $\mathrm{codepth}\, A_W = 1$.

So we assume $X \subset \mathbb{P}^s_K$ with $\dim X = 1$ and $s = c + 1$. The statement about the number of quadrics is a consequence of Lemma 3.2. Since X is of almost minimal degree we know that $I = I_X$ is 3-regular (cf. 3.3). Write $I = (J, LS)$ with $J = I_2 S$ and with a K-vector space $L \subset S_3$ such that $I_3 = J_3 \oplus L$. Our aim is to show that $\dim_K L \leq 1$.

After an appropriate linear coordinate change we may assume that $x_s \in S_1$ is generic. Let $T := S/x_s S = K[x_0, \dots, x_{s-1}]$. Then $R := T/(J, L)T \simeq S/(I, x_s S)$ defines a scheme Z of $s + 1$ points in semi uniform position in $\mathbb{P}^{s-1}_K$. The short exact sequence $0 \to A_X(-1) \xrightarrow{x_s} A_X \to R \to 0$ induces an isomorphism $H^0_{R_+}(R) \simeq K(-2)$. But this means that the vanishing ideal of Z in T has the form $(J, L, q)T$ with an appropriate quadric $q \in S_2$. Since $s \geq 3$ the minimal free resolution of this ideal has the form

$$T^{a_2}(-3) \xrightarrow{\phi} T^{a_1}(-2) \xrightarrow{\pi} (J, L, q)T \to 0.$$

This allows us to write $(J, L, q)T = (J, q)T$ and to assume that the first $a_1 - 1$ generators of $T^{a_1}(-2)$ are mapped by π onto a K-basis of $(JT)_2$ and the last generator is mapped by π to $q \cdot 1_T$. Clearly, ϕ is given by a matrix with linear entries. This shows that $M := JT :_T q \subset T$ is a proper ideal generated by linear forms.

As $JT \subseteq M$ and as $(J, qT) = IT$ is of height $s-1$ we must have $s-2 \leq \mathrm{height}\, M \leq s$. As M is generated by linear forms, $(T/M)_1$ is a K-vector space of dimension $t \in \{0, 1, 2\}$. So the graded short exact sequence

$$0 \to T/M(-2) \to T/JT \to T/(J, q)T \to 0$$

shows that

$$\dim_K(IT)_3 = \dim_K((J, q)T)_3 = \dim_K(JT)_3 + t.$$

Therefore, we may write $(I, x_s) = (J, L', x_s)$ where $L' \subseteq L \subset S_3$ is a K-vector space of dimension $\leq t$. As I is a prime ideal and as $x_s \in S_1 \setminus I$ it follows $I = (J, L')$, hence $L' = L$. So, if $\dim_K L' \leq 1$, we are done.

Otherwise, $\dim_K L' = \dim_K L = 2 = t$ and we may write $I = (J, k_1, k_2)$ with $k_1, k_2 \in S_3$. As $\mathrm{height}\, I = s - 1$ it follows $\mathrm{height}\, J \geq s - 3$. As $JT \subseteq M$ and as $\mathrm{height}\, M = s - 2$ we have $\mathrm{height}\, JT \leq s - 2$. As x_s is a generic linear form, this means that $\mathrm{height}\, J \leq s - 3$ and hence $\mathrm{height}\, J = s - 3$. As $I = (J, k_1, k_2)$ is a prime ideal of height $s - 1 = \mathrm{height}\, J + 2$, the ideal J must be prime too.

As x_s is generic and $\mathrm{height}\, J \leq s - 3$, we may conclude by Bertini's theorem that $JT \subset T$ defines an integral subscheme of $\mathbb{P}^{s-1}_K = \mathrm{Proj}(T)$. So, the saturation

$$JT :_T \langle T_+ \rangle \subset T \text{ of } JT \text{ in } T$$

is a prime ideal of height $s - 2$. As $JT \subseteq M \subset T_+$ and as M is a prime ideal we get $JT :_T \langle T_+ \rangle = M$. Therefore

$$\mathrm{Proj}(T/IT) = \mathrm{Proj}(T/(J, q)T) = \mathrm{Proj}(T/(M, qT))$$

consists of two points, so that $s+1=2$, a contradiction. So, the case $\dim_K L' \geq 2$ does not occur at all. $\square$

The previous Lemma 3.5 is inspired by a corresponding statement for curves of degree $r+2$ in $\mathbb{P}^r_K$ shown by the authors (cf. [1, Lemma (6.4)]).

Moreover, if $\operatorname{codepth} A_X \geq 2$ the number of cubics needed to define X is not bounded by 1 (cf. the examples in [2, Section 9]).

4. EXAMPLES

In this section we want to confirm the existence of all types of varieties of almost minimal degree $X \subset P^r_K$ which are described in the Theorems 2.1, 2.2 and 2.3.

First of all we want to show the existence of a Del Pezzo variety as required by Theorem 2.3 (a).

Example 4.1. Let $X = \mathbb{P}^2_K \times \mathbb{P}^2_K \subset \mathbb{P}^8_K$ be the Segre product of two projective planes. Its defining ideal I_X is generated by the 2×2-minors of the the following generic 3×3-matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 \end{pmatrix}.$$

It is easy to see that $\dim X = 4, \operatorname{codim} X = 4$ and $\deg X = 6.$ Therefore, X is a variety of almost minimal degree. Moreover, A_X is a Cohen-Macaulay and therefore a Gorenstein ring. An example of dimension 3 is the Segre product $\mathbb{P}^1_K \times \mathbb{P}^1_K \times \mathbb{P}^1_K \subset \mathbb{P}^7_K$ (cf. [5, (8.11), 6)]). Examples of smaller dimensions are obtained by taking generic linear sections.

In the next examples let us show that the two different Betti diagrams of statement (b) in 2.3 indeed occur. Note that they require $\operatorname{codepth} A_X = 1$ and $\operatorname{codim} X = 4.$

Example 4.2. Consider the rational normal surface scroll $Z = S(3,3) \subset \mathbb{P}^7_K$. Let $P_1 = (0:0:0:0:1:0:0:1)$ and $P_2 = (1:0:0:0:0:0:1)$ in $\mathbb{P}^7_K$. Then $P_i \in \mathbb{P}^7_K \setminus Z, i = 1,2,$ as it is easily seen. Define X_i to be the projection of Z from $P_i, i = 1,2.$ Then $\dim X_i = 2$ and $\operatorname{codepth} A_{X_i} = 1$ for $i = 1,2.$ The Betti diagrams of $I_{X_i}, i = 1,2,$ are those of Theorem 2.3 (b).

Now we construct non-arithmetically Cohen-Macaulay varieties of almost minimal degree of the type mentioned in Theorem 2.2. To this end we use the possible rational normal scrolls $Z = S(a_1,\ldots,a_k)$ of the proof of Theorem 2.2 which after appropriate projection furnish the varieties we are looking for. The construction of the examples corresponding to Theorems 2.1 and 2.3 follows similarly, and so we skip the details in these cases.

Example 4.3. Let $Z = S(5) \subset \mathbb{P}^5_K$ denote the rational normal curve of degree 5. Choose $P \in \mathbb{P}^5_K \setminus Z$ a generic point. Then, the projection $X \subset \mathbb{P}^4_K$ of Z from P is an example of a variety of almost minimal degree with $\dim X = 1$ and $\operatorname{codepth} A_X = 1.$

Next we want to investigate the case of surfaces.

Example 4.4. Let $Z = S(4,1) \subset \mathbb{P}^6_K$. Consider the two points $P_1 = (0:1:0:0:0:0:0)$ and $P_2 = (0:0:1:0:0:0:0)$. Then $P_i \in \mathbb{P}^6_K \setminus Z$, for $i = 1,2$. Let $X_i, i = 1,2,$

denote the projection of Z from P_i. Then $\operatorname{codepth} A_{X_1} = 1$ and $\operatorname{codepth} A_{X_2} = 2$. The same type of examples may be produced by projections of the scroll $S(3,2)$.

Our next examples are of dimension 3.

Example 4.5. Let $Z = S(3,1,1) \subset \mathbb{P}^7_K$. Consider the points $P_1 = (0:1:0:0:0:0:0:0)$ and $P_2 = (0:0:0:1:1:0:0:0)$. Then it is easy to see that $P_i \in \mathbb{P}^7_K \setminus Z, i = 1,2$. Let X_i denote the projection of Z from P_i. Then $\operatorname{codepth} A_{X_1} = 2$ and $\operatorname{codepth} A_{X_1} = 1$. The same type of examples may be produced by projections from the scroll $S(2,2,1)$.

Now, let us consider the situation of fourfolds.

Example 4.6. Consider the scroll $Z = S(2,1,1,1) \subset \mathbb{P}^8_K$. Let $P_1 = (0:1:0:0:0:0:0:0:0)$ and $P_2 = (0:0:0:0:0:0:1:1:0)$. Then $P_i \in \mathbb{P}^8_K \setminus Z, i = 1,2$. Let $X_i \subset \mathbb{P}^7$ denote the projection of Z from $P_i, i = 1,2$. Then $\operatorname{codepth} A_{X_1} = 2$, while $\operatorname{codepth} A_{X_2} = 1$.

Finally let us consider the case where $\dim X = 5$.

Example 4.7. Let $Z = S(1,1,1,1,1) \subset \mathbb{P}^9_K$ be the Segre variety. Then $P \in \mathbb{P}^9_K \setminus Z$ for the point $P = (0:1:1:0:0:0:0:0:0:0)$. Let $X \subset \mathbb{P}^8_K$ denote the projection of Z from P. Then $\operatorname{depth} A_X = 4$, and therefore $\operatorname{codepth} A_X = 2$. Finally observe that $\operatorname{codepth} A_X = 1$ is impossible if $\dim X = 5$, as $\operatorname{depth} A_X \leq 4$ (cf. Lemma 3.1).

The Examples 4.3 – 4.7 provide the existence of the samples claimed by Theorem 2.2. Similar constructions provide varieties as mentioned in Theorem 2.1 and 2.3.

In the final examples, we will show that in higher codimension, the shape of the Betti diagram of I_X for a variety X of minimal degree may vary in a much stronger way: In fact the "beginning of the Betti diagrams" may be rather different from each other.

Example 4.8. Let $Z = S(8) \subset \mathbb{P}^8_K$ denote the rational normal curve of degree 8. Let $P_1 = (0:0:0:0:0:0:1:0:0), P_2 = (0:0:0:0:0:1:0:0:0)$ and $P_3 = (0:0:0:0:1:0:0:0:0)$. Then $P_i \in \mathbb{P}^8_K \setminus Z$ for $i = 1,2,3$. Let $X_i \subset \mathbb{P}^7_K, i = 1,2,3$, denote the projection of Z from P_i. Then the Betti diagrams of $I_{X_i}, i = 1,2,3$, resp. have the form:

i		1	2	3	4	5	6	7
1	1	19	58	75	44	5	0	0
	2	1	6	15	20	21	8	1
2	1	19	57	70	34	5	0	0
	2	0	1	5	20	21	8	1
3	1	19	57	69	34	5	0	0
	2	0	0	5	20	21	8	1

In all three cases $\operatorname{codim} X_i = 4$ and $\operatorname{codepth} A_{X_i} = 1$. Remember that the number of cubics in the defining ideals is bounded by 1 (cf. Lemma 3.5). It follows that X_1 is contained in the scroll $S(5,1)$, while X_2 is contained in the scroll $S(4,2)$ and X_3 is contained in the scroll $S(3,3)$.

In view of the Example 4.8 and corresponding examples in higher dimensions one might expect that the type of the rational normal scroll Y, that contains the variety X of almost minimal degree as a one codimensional subvariety, determines the Betti diagram "near the beginning of the resolution". In small codimensions, the different types of these scrolls Y are much more limited than imposed by Theorems 2.1, 2.2 and 2.3. It seems rather challenging to understand the rôle of the scrolls Y for the beginning of the minimal free resolution of I_X.

Moreover the examples in 4.8 show that the estimates for the Betti numbers given in Lemma 3.3 near the beginning of the Betti diagram are fairly weak.

Remark 4.9. To compute the Betti diagrams and hence the arithmetic depths of the above examples, we have made use of the computer algebra system SINGULAR (cf. [8]). Moreover, there is in preparation a conceptual approach for the computation of the $\operatorname{depth} A_X$ in terms of the center of the projection and the secant variety of $S(a_1, \ldots, a_k) \subset \mathbb{P}_K^{r+1}$.

REFERENCES

[1] BRODMANN, M., SCHENZEL, P.: *Curves of degree $r+2$ in $\mathbb{P}^r$: Cohomological, geometric and homological aspects*, J. Algebra 242 (2001), 577 - 623.

[2] BRODMANN, M., SCHENZEL, P.: *Arithmetic properties of projective varieties of almost minimal degree*, to appear in J. Algebraic Geom.

[3] FUJITA, T.: *Classification of projective varieties of Δ-genus one*. Proceedings of the Japan Academy of Science, Ser. A Math. Sci 58 (1982), 113 - 116.

[4] FUJITA, T.: *Projective varieties of Δ-genus one*. In Algebraic and Topological Theories – To the Memory of Dr. Takehiko Miyata (Kinokuniya, Tokyo, 1985), 149 - 175.

[5] FUJITA, T.: *Classification theories of polarized varieties*, London Mathematical Society Lecture Notes Series 155, Cambridge University Press, 1990.

[6] EISENBUD, D.: *The Geometry of Syzygies – A second course in Commutative Algebra and Algebraic Geometry*, Graduate Texts in Mathematics, Vol. 229, Springer-Verlag, New York, 2005.

[7] EISENBUD, D., HARRIS, J.: *On varieties of minimal degree (a centennial account)*. In Proceedings of Symposion of Pure Mathematics Vol 46, pp 3 - 13, American Mathematical Society, Providence, 1987.

[8] GREUEL, G.M., PFISTER, G. ET AL: *Singular* 3.0, *A Computer Algebra System for Polynomial Computations*. Center for Computer Algebra, University of Kaiserslautern (2005) (http://www.singular.uni-kl.de).

[9] HARRIS, J.: *Algebraic Geometry: A First Course*. Graduate Texts in Mathematics, Vol. 133, Springer-Verlag, New York, 1992.

[10] HOA, T., STÜCKRAD, J., VOGEL, W.: *Towards a structure theory of projective varieties of degree $=$ codimension $+2$*. J. of Pure and Appl. Algebra 71 (1991), 203 - 231.

[11] JOUANOLOU, J.P.: *Théorèmes de Bertini et applications*. Progress in Mathematics, Vol. 42, Birkhäuser, Basel, 1983.

[12] NAGEL,U: *Minimal free resolutions of projective subschemes of small degree*, Preprint, 2005, (http://arxiv.org/math.AG/0505347).

[13] SCHENZEL, P.: *Über die freien Auflösungen extremaler Cohen-Macaulay-Ringe*, J. Algebra 64, 93-101 (1980).

UNIVERSITÄT ZÜRICH, INSTITUT FÜR MATHEMATIK, WINTERTHURER STR. 190, CH – 8057 ZÜRICH, SWITZERLAND

E-mail address: `brodmann@math.unizh.ch`

MARTIN-LUTHER-UNIVERSITÄT HALLE-WITTENBERG, FACHBEREICH MATHEMATIK UND INFORMATIK, D – 06 099 HALLE (SAALE), GERMANY
E-mail address: schenzel@mathematik.uni-halle.de