A Posteriori Error Estimation for Highly Indefinite Helmholtz Problems

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Abstract

We develop a new analysis for residual-type a posteriori error estimation for a class of highly indefinite elliptic boundary value problems by considering the Helmholtz equation at high wavenumber $k > 0$ as our model problem. We employ classical conforming Galerkin discretization by using *hp*-finite elements.

The key role in the analysis is played by a new estimate of the L^2-error by the error in the H^1-norm which allows to absorb the critical, wavenumber-depending part of the error in the elliptic part. The estimate for our posteriori error estimator then becomes *independent* of the, possibly, high wavenumber $k > 0$ while, in contrast, the constant in the estimate by using the classical theory is amplified by a factor k.

It turns out that the optimal choice of the polynomial degree p is $O(\log(k))$ and, hence, all estimates in this paper are explicit in the mesh width h and the degree p.

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1 Introduction

In this paper we will introduce a new analysis for residual-based a posteriori error estimation. We consider the conforming Galerkin method with *hp*-finite elements applied to a class of highly indefinite boundary value problems, which arise, e.g., when electromagnetic or acoustic scattering problems are modelled in the frequency domain. As our model problem we consider a highly indefinite Helmholtz equation with oscillatory solutions.

Residual-based a posteriori error estimates for elliptic problems have been introduced in [4], [5] and their theory for elliptic problems is now fairly completely established (cf. [18], [1]). To sketch the principal idea and to explain the

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goal of this paper let u denote the (unknown) solution of the weak formulation of an elliptic second order PDE with appropriate boundary conditions. Typically the solution belongs to some infinite-dimensional Sobolev space H. Let u_S denote a computed Galerkin solution based on a finite dimensional subspace $S \subset H$. A (reliable) a posteriori error estimator is a *computable* functional η which depends on u_S and the given data such that an estimate of the form

$$\|u - u_S\|_H \leq C\eta\,(u_S) \tag{1}$$

holds for a (minimal) constant C which either is known explicitly or sharp upper bounds are available. We emphasize that in the literature various refinements of this concept of a posteriori error estimation exist while for the purpose of our introduction this simple definition is sufficient.

In the classical theory the constant C depends linearly on the norm of the solution operator of the PDE in some appropriate function spaces, more precisely, it depends reciprocally on the *inf-sup constant* γ. In [12] it was proved for the Helmholtz problem with Robin boundary conditions that for certain classes of physical domains the reciprocal inf-sup constant $1/\gamma$ (and, hence, also the constant C in (1)) grows linearly with the wavenumber. See also [10] for further estimates of the inf-sup constant for the Helmholtz problem. However, this implies that for large wavenumbers the classical a posteriori estimation becomes useless because the error then typically is highly overestimated. Additional difficulties arise for the a posteriori error estimation for highly indefinite problems because the existence and uniqueness of the classical Galerkin solution is ensured only if the mesh width is sufficiently small.

In contrast to definite elliptic problems, there exist only relatively few publications in the literature on a posteriori estimation for highly indefinite problems (cf. [2], [3]).

In [14] and [15] a new a priori convergence theory for Galerkin discretizations of highly indefinite boundary value problems has been developed which is based on new regularity estimates (the *splitting lemmas* as in [14] and [15]) where the solution is split into a "rough part" with wavenumber-independent regularity constant and a "smooth" part with high-order regularity in (weighted) Sobolev spaces but more critical dependence on the regularity constant on the wavenumber. This theory allows in the a priori convergence theory to "absorb" the L^2-error of the PDE which depends critically on the wavenumber in the wavenumber-independent part of the equation.

In this paper, we will develop a new a posteriori analysis based on similar ideas: The L^2-part of the a posteriori error will be estimated by the H^1-error and then can be compensated by an appropriate choice of the *hp*-finite element space.

The paper is structured as follows. In Section 2, we will consider as our model problem the high frequency, time harmonic scattering of an acoustic wave at some bounded domain in an unbounded exterior domain and transform it to a finite domain by using a Dirichlet-to-Neumann boundary operator resp.

some approximation to it. We define a conforming Galerkin *hp*-finite element discretization for its numerical approximation.

In Section 3, we summarize the a priori analysis as in [14] and [15] which will be needed a) to determine the minimal *hp*-finite element space for a stable Galerkin discretization and b) to estimate the *adjoint approximation property* which will appear as weights in our a posteriori error estimation.

In Section 4, we will present the a posteriori error estimator and prove its reliability and efficiency. It will turn out that the optimal polynomial degree p will depend logarithmically on the wavenumber and, hence, the finite element interpolation theory has to be explicit with respect to the mesh width h and the polynomial degree p.

In a forthcoming paper, we will focus on numerical experiments and also on the definition of an *hp*-refinement strategy in order to obtain a convergent adaptive algorithm

2 Model Helmholtz Problems and their Discretization

2.1 Model Problems

The Helmholtz equation describes wave phenomena in the frequency domain which, e.g., arises if electromagnetic or acoustic waves are scattered from or emitted by bounded physical objects. In this light, the computational domain for such wave problems, typically, is the unbounded complement of a bounded domain $\Omega^{\text{in}} \subset \mathbb{R}^d$, $d = 1, 2, 3$, i.e., $\Omega^{\text{out}} := \mathbb{R}^d \backslash \overline{\Omega^{\text{in}}}$. Throughout this paper, we assume that Ω^{in} has a Lipschitz boundary $\Gamma^{\text{in}} := \partial\Omega^{\text{in}}$.

The Helmholtz problem depends on the wavenumber k. In most parts of the paper (exceptions: Remarks 15, 17 and Corollaries 27, 28) we allow for variable wavenumber $k : \Omega^{\text{out}} \to \mathbb{R}$ but always assume that k is real-valued, nonnegative, and a positive constant outside a sufficiently large ball (cf. (10)).

For a given right-hand side $f \in L^2(\Omega^{\text{out}})$, the *Helmholtz problem* is to seek $U \in H^1_{\text{loc}}(\Omega^{\text{out}})$ such that

$$\left(-\Delta - k^2\right) U = f \qquad \text{in } \Omega^{\text{out}} \tag{2a}$$

is satisfied. Towards infinity, *Sommerfeld's radiation condition* is imposed

$$\left|\partial_r U - \mathrm{i}\, kU\right| = o\left(|x|^{\frac{1-d}{2}}\right) \qquad \text{for } |x| \to \infty, \tag{2b}$$

where ∂_r denotes differentiation in radial direction and $|\cdot|$ the Euclidian vector norm. For simplicity we restrict here to *homogeneous Dirichlet boundary condition* on Γ^{in}

$$\left.U\right|_{\Gamma^{\text{in}}} = 0. \tag{2c}$$

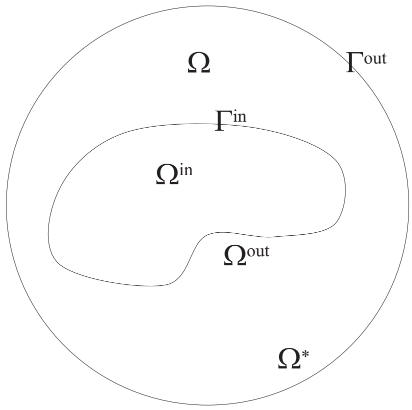

Figure 1: Scatterer Ω^{in} with boundary Γ^{in} and exterior domain Ω^{out}. The support of f is assumed to be contained in the bounded region $\Omega^{\star}$. The domain for the weak variational formulation is $\Omega = \Omega^{\star} \backslash \Omega^{\text{in}}$.

We assume that f is local in the sense that there exists some bounded, simply connected Lipschitz domain[1] $\Omega^{\star}$ such that a) $\overline{\Omega^{\text{in}}} \subset \Omega^{\star}$, b) $\operatorname{supp}(f) \subset \Omega^{\star}$, and c) k is constant in a neighborhood of $\partial\Omega^{\star}$. The *computational domain* (cf. Figure 1) will be

$$\Omega := \Omega^{\star} \backslash \overline{\Omega^{\text{in}}} \tag{3}$$

and, next, we will derive appropriate boundary conditions at the outer boundary $\Gamma^{\text{out}} := \partial\Omega^{\star}$. Problem (2) can be reformulated in an equivalent way as a *transmission problem* by seeking functions $u \in H^1(\Omega)$ and $u^{\text{out}} \in H^1_{\text{loc}}(\mathbb{R}^d \backslash \overline{\Omega^{\star}})$ such that

$$\begin{aligned}
\left(-\Delta - k^2\right) u = f & \qquad \text{in } \Omega, \\
\left(-\Delta - k^2\right) u^{\text{out}} = 0 & \qquad \text{in } \mathbb{R}^d \backslash \overline{\Omega^{\star}}, \\
u = 0 & \qquad \text{on } \Gamma^{\text{in}}, \\
u = u^{\text{out}} \quad \text{and} \quad \partial_n u = \partial_n u^{\text{out}} & \qquad \text{on } \Gamma^{\text{out}}, \\
\left|\partial_r u^{\text{out}} - \mathrm{i}\, k u^{\text{out}}\right| = o\left(|x|^{\frac{1-d}{2}}\right) & \qquad \text{for } |x| \to \infty.
\end{aligned} \tag{4}$$

Here, n denotes the normal vector pointing into the *exterior domain* $\mathbb{R}^d \backslash \overline{\Omega^{\star}}$ and ∂_n denotes differentiation in normal direction.

It can be shown that, for given $g \in H^{1/2}(\Gamma^{\text{out}})$, the problem: Find $w \in H^1_{\text{loc}}\left(\mathbb{R}^d \backslash \overline{\Omega^{\star}}\right)$ such that

$$\begin{aligned}
\left(-\Delta - k^2\right) w = 0 & \qquad \text{in } \mathbb{R}^d \backslash \overline{\Omega^{\star}}, \\
w = g & \qquad \text{on } \Gamma^{\text{out}}, \\
\left|\partial_r w - \mathrm{i}\, k w\right| = o\left(|x|^{\frac{1-d}{2}}\right) & \qquad \text{for } |x| \to \infty
\end{aligned} \tag{5}$$

[1] Since Ω^{in} is bounded, $\Omega^{\star}$ always can be chosen as a ball. Other choices of $\Omega^{\star}$ might be preferable in certain situations.

has a unique solution. The mapping $g \mapsto w$ is called the *Steklov–Poincaré operator* and denoted by $S_P : H^{1/2}(\Gamma^{\text{out}}) \rightarrow H^1_{\text{loc}}(\mathbb{R}^d \backslash \overline{\Omega^\star})$. The *Dirichlet-to-Neumann* (DtN) map is given by $T_k := \gamma_1 S_P : H^{1/2}(\Gamma^{\text{out}}) \rightarrow H^{-1/2}(\Gamma^{\text{out}})$, where $\gamma_1 := \partial_n$ is the normal derivative operator at Γ^{out}. Hence, problem (4) can be reformulated as: Find $u \in H^1(\Omega)$ such that

$$\begin{aligned} \left(-\Delta - k^2\right) u = f \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \Gamma^{\text{in}}, \\ \partial_n u = T_k u \qquad & \text{on } \Gamma^{\text{out}}. \end{aligned} \tag{6}$$

The previous problems are posed in the *weak formulation* given by: Find

$$u \in \mathcal{H} := \left\{ u \in H^1(\Omega) : u|_{\Gamma^{\text{in}}} = 0 \right\}$$

such that

$$A_{\text{DtN}}(u,v) := \int_\Omega \left(\langle \nabla u, \nabla \bar{v} \rangle - k^2 u \bar{v} \right) - \int_{\Gamma^{\text{out}}} (T_k u)\, \bar{v} = \int_\Omega f \bar{v} \qquad \text{for all } v \in \mathcal{H}. \tag{7}$$

Since the numerical realization of the nonlocal DtN map T_k is costly, various approaches exist in the literature to approximate this operator by a local operator. The most simple one is the use of Robin boundary conditions leading to

$$\begin{aligned} \left(-\Delta - k^2\right) u = f \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \Gamma^{\text{in}}, \\ \partial_n u = \mathrm{i}\, k u \qquad & \text{on } \Gamma^{\text{out}}. \end{aligned} \tag{8}$$

The weak formulation of this equation is given by: Find $u \in \mathcal{H}$ such that

$$A_{\text{Robin}}(u,v) := \int_\Omega \left(\langle \nabla u, \nabla \bar{v} \rangle - k^2 u \bar{v} \right) - \int_{\Gamma^{\text{out}}} \mathrm{i}\, k u \bar{v} = \int_\Omega f \bar{v} \qquad \text{for all } v \in \mathcal{H}. \tag{9}$$

In most parts of this paper we allow indeed that k is a function varying in Ω, while the following conditions are always assumed to be satisfied:

$$\begin{aligned} k \in L^\infty\left(\mathbb{R}^d, \mathbb{R}\right), \qquad & 0 \leq \operatorname{essinf}_{x\in\Omega} k(x) \leq \operatorname{esssup}_{x\in\Omega} k(x) =: k_{\max} < \infty, \\ & k = k_{\text{const}} \text{ outside a large ball}, \\ & k = k_{\text{const}} \text{ in an neighborhood } \mathcal{U}^\star_{\text{const}} \text{ of } \Gamma^{\text{out}}. \end{aligned} \tag{10}$$

Let $\mathcal{U}_{\text{const}} := \mathcal{U}^\star_{\text{const}} \cap \overline{\Omega}$. The constants in the estimates in this paper will depend on $k_{\max}$, and $\mathcal{U}_{\text{const}}$ (through a trace inequality as in Lemma 2) but hold uniformly for all functions k satisfying (10).

2.2 Abstract Variational Formulation

Notation 1 *For a Lebesgue-measurable set $\omega \subset \mathbb{R}^d$ and $p \in [1,\infty]$, $m \in \mathbb{N}$, we denote by $L^p(\omega)$ the usual Lebesgue space with norm $\|\cdot\|_{L^p(\omega)}$ and by $H^m(\omega)$ the usual Sobolev spaces with norm $\|\cdot\|_{H^m(\omega)}$. The seminorm which contains* *only the derivatives of highest order is denoted by* $|\cdot|_{H^m(\omega)}$. *We equip the space* $\mathcal{H}$ *with the norm*

$$\|v\|_{\mathcal{H};\Omega} := \left(\|\nabla v\|^2_{L^2(\Omega)} + \|k_+ v\|^2_{L^2(\Omega)} \right)^{1/2} \quad \text{with} \quad k_+ := \max\{1, k\} \tag{11}$$

which is obviously equivalent to the $H^1(\Omega)$*-norm.*

Since Γ^{out} is a Lipschitz manifold and $\mathcal{U}_{\text{const}}$ is a Lipschitz domain, it is well known that the following trace estimates hold (see [7, (1.6.6) Theorem]).

Lemma 2 *There exists a constant* C_{tr} *depending only on* $\mathcal{U}_{\text{const}}$ *such that*

$$\forall u \in H^1(\Omega): \quad \|u\|_{H^{1/2}(\Gamma^{\text{out}})} \leq C_{\text{tr}} \|u\|_{\mathcal{H};\mathcal{U}_{\text{const}}} \tag{12a}$$

and

$$\forall u \in H^1(\Omega): \quad \|u\|_{L^2(\Gamma^{\text{out}})} \leq C_{\text{tr}} \|u\|^{1/2}_{L^2(\mathcal{U}_{\text{const}})} \|u\|^{1/2}_{H^1(\mathcal{U}_{\text{const}})}. \tag{12b}$$

Corollary 3 *For* $u \in H^1(\Omega)$, *we have*

$$\left\| \sqrt{k} u \right\|_{L^2(\Gamma^{\text{out}})} \leq C_{\text{tr}} \|u\|_{\mathcal{H};\mathcal{U}_{\text{const}}} \leq C_{\text{tr}} \|u\|_{\mathcal{H};\Omega}.$$

Proof. Since $k = k_{\text{const}}$ on $\mathcal{U}_{\text{const}}$, there holds

$$\begin{aligned} k_{\text{const}} \|u\|^2_{L^2(\Gamma^{\text{out}})} &\leq C^2_{\text{tr}} k_{\text{const}} \|u\|_{L^2(\mathcal{U}_{\text{const}})} \|u\|_{H^1(\mathcal{U}_{\text{const}})} \\ &\leq \frac{C^2_{\text{tr}}}{2} \left(k^2_{\text{const}} \|u\|^2_{L^2(\mathcal{U}_{\text{const}})} + \|u\|^2_{H^1(\mathcal{U}_{\text{const}})} \right) \\ &= \frac{C^2_{\text{tr}}}{2} \left(\left(1 + k^2_{\text{const}}\right) \|u\|^2_{L^2(\mathcal{U}_{\text{const}})} + |u|^2_{H^1(\mathcal{U}_{\text{const}})} \right) \\ &\leq C^2_{\text{tr}} \left(\|k_+ u\|^2_{L^2(\mathcal{U}_{\text{const}})} + |u|^2_{H^1(\mathcal{U}_{\text{const}})} \right). \end{aligned} \tag{13}$$

■

Both sesquilinear forms A_{DtN} (7) and A_{Robin} (9) belong to the following class of forms (see Proposition 6).

Assumption 4 (Variational formulation) *Let* $\Omega \subset \mathbb{R}^d$, *for* $d \in \{2, 3\}$, *be a bounded Lipschitz domain. Then* $\mathcal{H}$, *equipped with the norm* $\|\cdot\|_{\mathcal{H};\Omega}$, *is a closed subspace of* $H^1(\Omega)$. *We consider a sesquilinear form* $A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ *that can be decomposed into* $A = a - b$, *where*

$$a(v, w) := \int_\Omega \left(\langle \nabla v, \nabla \bar{w} \rangle - k^2 v \bar{w} \right)$$

and the sesquilinear form b *satisfies the following properties:*

(a) $b : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ *is a* continuous *sesquilinear form with*

$$|b(v, w)| \leq C_b \|v\|_{\mathcal{H};\Omega} \|w\|_{\mathcal{H};\Omega} \qquad \text{for all } v, w \in \mathcal{H}, \tag{14}$$

for some positive constant C_b.

(b) There exist $\theta \geq 0$ and $\gamma_{\text{ell}} > 0$ such that the following Gårding inequality *holds:*

$$\operatorname{Re}\left(a(v,v) - b(v,v)\right) + \theta\|k_+ v\|^2_{L^2(\Omega)} \geq \gamma_{\text{ell}}\|v\|^2_{\mathcal{H};\Omega} \qquad \text{for all } v \in \mathcal{H}. \tag{15}$$

(c) The adjoint problem*: Find $z \in \mathcal{H}$ such that*

$$a(v,z) - b(v,z) = (v,f)_{L^2(\Omega)} \qquad \text{for all } v \in \mathcal{H} \tag{16}$$

is uniquely solvable for every $f \in L^2(\Omega)$ with bounded solution operator $Q_k^\star : L^2(\Omega) \to \mathcal{H}$, $f \mapsto z$, more precisely, the (k-dependent) constant

$$C_k^{\text{adj}} := \sup_{f \in L^2(\Omega)\setminus\{0\}} \frac{\|Q_k^\star\left(k_+^2 f\right)\|_{\mathcal{H};\Omega}}{\|k_+ f\|_{L^2(\Omega)}} \tag{17}$$

is finite.

Problem 5 *Let A be a sesquilinear form as in Assumption 4. For given $f \in L^2(\Omega)$, we seek $u \in \mathcal{H}$ such that*

$$a(u,v) - b(u,v) = \int_\Omega f\bar{v} \qquad \text{for all } v \in \mathcal{H}. \tag{18}$$

Proposition 6 *Both sesquilinear forms A_{Robin} (9) and A_{DtN} (7) (under the additional condition that Γ^{out} is a sufficiently large sphere) satisfy Assumption 4.*

Proof. The proof is a slight modification of the corresponding proofs for constant wavenumber k in [14] and [12]. Condition (a) for A_{Robin} follows from Corollary 3. For A_{DtN} we employ that k is constant in $\mathcal{U}_{\text{const}}$ and Γ^{out} is a sphere of a large radius $R > 0$. Hence, from the proof of [14, Lemma 3.3] it follows that

$$\left|\int_{\Gamma^{\text{out}}} (T_k u)\,\bar{v}\right| \\ \leq C\left(R^{-1}\|u\|_{H^{1/2}(\Gamma^{\text{out}})}\|v\|_{H^{1/2}(\Gamma^{\text{out}})} + k_{\text{const}}\|u\|_{L^2(\Gamma^{\text{out}})}\|v\|_{L^2(\Gamma^{\text{out}})}\right).$$

By using Corollary 3 we obtain

$$\left|\int_{\Gamma^{\text{out}}} (T_k u)\,\bar{v}\right| \leq C\left(1 + \frac{1}{R}\right) C_{\text{tr}}^2 \|u\|_{\mathcal{H};\Omega}\|v\|_{\mathcal{H};\Omega}$$

and the continuity of A_{DtN} follows.

For condition (b) and Robin boundary conditions, we employ

$$\begin{aligned}\operatorname{Re}\left(A_{\text{Robin}}(v,v)\right) + 2\|k_+ v\|^2_{L^2(\Omega)} &\geq \int_\Omega \left(|\nabla v|^2 + k_+^2|v|^2 + \left(k_+^2 - k^2\right)|v|^2\right) \\ &\geq \|v\|^2_{\mathcal{H};\Omega}\end{aligned}$$

and (15) holds with $\theta = 2$ and $\gamma_{\text{ell}} = 1$.

For the sesquilinear form A_{DtN} we employ [14, Lemma 3.3 (2)] to obtain

$$\begin{aligned}
&\operatorname{Re}\left(A_{\text{DtN}}(v,v)\right) + 2\|k_+ v\|^2_{L^2(\Omega)} \\
&\geq \left(\int_\Omega \left(|\nabla v|^2 + k_+^2 |v|^2 + \left(k_+^2 - k^2\right) |v|^2 \right) - \operatorname{Re}\left(\int_{\Gamma^{\text{out}}} T_k v \, \bar{v} \right) \right) \\
&\geq \|v\|^2_{\mathcal{H};\Omega}
\end{aligned}$$

and (15) again holds with $\theta = 2$ and $\gamma_{\text{ell}} = 1$.

For condition (c) we may apply Fredholm's theory and, hence, it suffices to prove that

$$a(u,v) - b(u,v) = 0 \qquad \text{for all } v \in \mathcal{H} \tag{19}$$

implies $u = 0$. For Robin boundary conditions we argue as in [12, (8.1.2)] and for DtN boundary conditions as in [14, Proof of Theorem 3.8] to see that (19) implies $u \in H_0^1(\Omega)$. Hence, u solves

$$\int_\Omega \left(\langle \nabla u, \nabla \bar{v} \rangle - k^2 u \bar{v} \right) = 0 \qquad \text{for all } v \in \mathcal{H}. \tag{20}$$

Let $\Omega^{\star\star}$ be a bounded domain such that $\Omega \subset \Omega^{\star\star} \subset \mathbb{R}^d \backslash \overline{\Omega^{\text{in}}}$ and $\Gamma^{\text{out}} \subset \Omega^{\star\star}$. The extension of u by zero to $\Omega^{\star\star}$ is denoted by u_0. It satisfies $u \in \mathcal{H}(\Omega^{\star\star}) := \left\{ u \in H^1(\Omega^{\star\star}) \mid u|_{\Gamma^{\text{in}}} = 0 \right\}$ and

$$\int_\Omega \left(\langle \nabla u_0, \nabla \bar{v} \rangle - k^2 u_0 \bar{v} \right) = 0 \qquad \text{for all } v \in \mathcal{H}(\Omega^{\star\star}).$$

Elliptic regularity theory implies that $u_0 \in H^2(Q)$ for any compact subset $Q \subset \Omega^{\star\star}$, in particular, in an open $\Omega^{\star\star}$ neighborhood of Γ^{out}. The unique continuation principle (cf. [11, Ch. 4.3]) implies that $u_0 = 0$ in $\Omega^{\star\star}$ so that $u = 0$ in Ω.

■

2.3 Discretization

2.3.1 Conforming Galerkin Discretization

A *conforming Galerkin discretization* of Problem 5 is based on the definition of a finite dimensional subspace $S \subset \mathcal{H}$ and is given by: Find $u_S \in S$ such that

$$a(u_S, v) - b(u_S, v) = \int_\Omega f \bar{v} \qquad \text{for all } v \in S. \tag{21}$$

2.3.2 *hp*-Finite Elements

As an example for S as above, we will define *hp*-finite elements on a finite element mesh $\mathcal{T}$ consisting of simplices with maximal mesh width h and local polynomial degree p. Before formulating the conditions on the mesh in an abstract way, we give an example of a typical construction.

Example 7 (Patchwise construction of FE mesh.) *Let Ω denote a bounded domain.*

(a) We assume that a polyhedral (polygonal in 2D) domain $\widetilde{\Omega}$ along with a bi-Lipschitz mapping $\chi : \widetilde{\Omega} \to \Omega$ is given. Let $\widetilde{\mathcal{T}}^{\mathrm{macro}} = \{\widetilde{K}_i^{\mathrm{macro}} : 1 \le i \le q\}$ denote a conforming finite element mesh for $\widetilde{\Omega}$ consisting of open simplices which are regular in the sense of [9]. $\widetilde{\mathcal{T}}^{\mathrm{macro}}$ is considered as a coarse partition of $\widetilde{\Omega}$, i.e., the diameters of the elements in $\widetilde{\mathcal{T}}^{\mathrm{macro}}$ are of order 1. We assume that the restrictions $\chi_i := \chi|_{\widetilde{K}_i^{\mathrm{macro}}}$ are analytic *for all $1 \le i \le q$.*

(b) The finite element mesh with step size h is generated by refining the mesh $\widetilde{\mathcal{T}}^{\mathrm{macro}}$ in some standard (conforming) way and denoted by $\widetilde{\mathcal{T}} = \{\widetilde{K}_i : 1 \le i \le N\}$. The corresponding finite element mesh for Ω then is defined by $\mathcal{T} = \{K = \chi(\widetilde{K}) : \widetilde{K} \in \widetilde{\mathcal{T}}\}$.

Note that, for any $K = \chi(\widetilde{K}) \in \mathcal{T}$, there exists an affine bijection $A_K : \widehat{K} \to \widetilde{K}$ which maps the reference element *$\widehat{K} := \{x \in \mathbb{R}^d_{>0} : \sum_{i=1}^d x_i < 1\}$ to the simplex $\widetilde{K}$. A parametrization $F_K : \widehat{K} \to K$ can be chosen by $F_K := R_K \circ A_K$, where $R_K := \chi|_{\widetilde{K}}$ is independent of the mesh width $h := \max\{h_K : K \in \mathcal{T}\}$, where $h_K := \operatorname{diam}(K)$.*

Concerning the polynomial degree distribution, it will be convenient (cf. [16, (10)]) to assume that the polynomial degrees of neighboring elements are comparable: There exists a constant $c_{\mathrm{p}} > 0$ such that

$$c_{\mathrm{p}}^{-1}(p_K + 1) \le p_{K'} + 1 \le c_{\mathrm{p}}(p_K + 1) \qquad \text{for all } K, K' \in \mathcal{T} \text{ with } \overline{K} \cap \overline{K'} = \emptyset. \tag{22}$$

To formulate the smoothness and scaling assumptions on R_K and A_K in an abstract way we have to introduce some notation first. For a function $v : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$, we write

$$|\nabla^n v(x)|^2 = \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = n} \frac{n!}{\alpha!} |\partial^\alpha v(x)|^2. \tag{23}$$

Assumption 8 *Each element map F_K can be written as $F_K = R_K \circ A_K$, where A_K is an affine map and the maps R_K and A_K satisfy for constants C_{affine}, C_{metric}, $\gamma > 0$ independent of h_K:*

$$\|A_K'\|_{L^\infty(\widehat{K})} \le C_{\mathrm{affine}} h_K, \qquad \|(A_K')^{-1}\|_{L^\infty(\widehat{K})} \le C_{\mathrm{affine}} h_K^{-1}$$
$$\|(R_K')^{-1}\|_{L^\infty(\widetilde{K})} \le C_{\mathrm{metric}}, \qquad \|\nabla^n R_K\|_{L^\infty(\widetilde{K})} \le C_{\mathrm{metric}} \gamma^n n! \qquad \text{for all } n \in \mathbb{N}_0.$$

Here, $\widetilde{K} = A_K(\widehat{K})$.

Remark 9 *Assumption 8 will be used in Section 3 for the a priori analysis and the derivation of the minimal hp-finite element space which leads to a stable discretization of the Helmholtz problem. It will turn out that the a posteriori estimate contains a weight which requires an a priori estimate. Since higher polynomial orders p are relevant for this, Assumption 8 also contains bounds on higher order derivatives of the element maps. The constants* C_{affine}, C_{metric} *describe the* shape-regularity *of the finite element mesh, i.e., they are a measure for possible distortions of the elements. The constants in the following estimates depend on the constants* C_{affine}, C_{metric} *and are moderately bounded if the shape regularity of the mesh is reasonably small.*

Definition 10 (*hp*-finite element space) *For meshes* $\mathcal{T}$ *with element maps* F_K *as in Assumption 8 the hp-finite element space of piecewise (mapped) polynomials is given by*

$$S^{p,1}(\mathcal{T}) := \{v \in \mathcal{H} : v|_K \circ F_K \in \mathbb{P}_p \text{ for all } K \in \mathcal{T}\}, \tag{24}$$

where $\mathbb{P}_p$ *denotes the space of polynomials of degree* p*. For chosen* $\mathcal{T}$ *and* p*, we may let* $S = S^{p,1}(\mathcal{T})$*.*

3 A Priori Analysis

In this section, we collect those results on existence, uniqueness, stability, and regularity for the Helmholtz problem (5), which later will be used for the analysis of the a posteriori error estimator.

3.1 Well-posedness

Proposition 11 *Let* $\Omega^{\text{in}} \subset \mathbb{R}^d$*,* $d = 2, 3$*, in* (2a) *be a bounded Lipschitz domain which is star-shaped with respect to the origin. Let* $\Gamma^{\text{out}} := \partial B_R$ *for some* $R > 0$*. Then,* (7) *admits a unique solution* $u \in \mathcal{H}$ *for all* $f \in \mathcal{H}'$ *which depends continuously on the data.*

Proposition 12 *Let* Ω *be a bounded Lipschitz domain. For all* $f \in \left(H^1(\Omega)\right)'$*, a unique solution* u *of problem* (9) *exists and depends continuously on the data.*

For the proofs of these propositions for constant k we refer, e.g., to [12, Prop. 8.1.3] and [8, Lemma 3.3], while for variable k one may argue as in Proposition 6.

3.2 Discrete Stability and Convergence

An essential role for the stability and convergence of the Galerkin discretization is played by the adjoint approximability which has been introduced in [15]; see also [17], [6].

Definition 13 (Adjoint approximability) *For a finite dimensional subspace $S \subset \mathcal{H}$, we define the* adjoint approximability *of Problem 5 by*

$$\eta_k^\star(S) := \sup_{f \in L^2(\Omega)\backslash\{0\}} \frac{\inf_{v\in S} \|Q_k^\star\left(k_+^2 f\right) - v\|_{\mathcal{H};\Omega}}{\|k_+ f\|_{L^2(\Omega)}}, \tag{25}$$

where $Q_k^\star$ is as in (17).

Theorem 14 (Stability and convergence) *Let $\gamma_{\mathrm{ell}}, \theta, C_b, C_k^{\mathrm{adj}}$ be as in Assumption 4 and S as in Section 2.3.1. Then the condition*

$$\eta_k^\star\left(S\right) \leq \frac{\gamma_{\mathrm{ell}}}{2\theta\left(1+C_b\right)}, \tag{26}$$

implies the following statements:

(a) The discrete inf-sup condition is satisfied:

$$\inf_{v\in S\backslash\{0\}} \sup_{w \in S\backslash\{0\}} \frac{|a(v,w)-b(v,w)|}{\|v\|_{\mathcal{H};\Omega}\|w\|_{\mathcal{H};\Omega}} \geq \frac{\gamma_{\mathrm{ell}}}{2+\gamma_{\mathrm{ell}}/(1+C_b)+2\theta C_k^{\mathrm{adj}}} > 0. \tag{27}$$

(b) Let S satisfy (26). Then, the Galerkin method based on S is quasi-optimal, i.e., for every $u \in \mathcal{H}$ there exists a unique $u_S \in S$ with $a(u-u_S,v)-b(u-u_S,v) = 0$ for all $v \in S$, and there holds

$$\|u-u_S\|_{\mathcal{H};\Omega} \leq \frac{2}{\gamma_{\mathrm{ell}}}(1+C_b) \inf_{v\in S} \|u-v\|_{\mathcal{H};\Omega}, \tag{28}$$

$$\|k_+\left(u-u_S\right)\|_{L^2(\Omega)} \leq \frac{2}{\gamma_{\mathrm{ell}}}(1+C_b)^2 \eta_k^\star(S) \inf_{v\in S} \|u-v\|_{\mathcal{H};\Omega}. \tag{29}$$

The proof follows very closely the proofs of [14, Thms. 4.2, 4.3].

Remark 15 *In [14], [15], it is proved for the case of constant wave number k, that for S as in Section 2.3.2, i.e., hp-finite elements, the conditions*

$$p = O\left(\log(k)\right) \qquad \text{and} \qquad \frac{kh}{p} = O\left(1\right) \tag{30}$$

imply (26) and lead to the "minimal" finite element space for discretization of the Helmholtz equations. In this light, terms in the a-posteriori error estimates which grow polynomially in p are expected to grow, at most, logarithmically with respect to k and, hence, are moderately bounded, also for large wavenumbers.

4 A Posteriori Error Estimation

The following Assumption collects the requirements for the a posteriori error estimation.

Assumption 16

(a) The continuous Helmholtz problem satisfies Assumption 4.

(b) S is a hp-finite element space as explained in Section 2.3.2 and satisfies Assumption 8 and (22).

(c) $u_S \in S$ is the computed solution satisfying the Galerkin equation.

Remark 17

(a) Assumption 16 does not require the stability condition (26) to be satisfied which is only sufficient *for existence and uniqueness of the discrete problem. We only assume that u_S exists, is computed, and solves the Galerkin equation for the specific problem. To be on the safe side in the case of constant wave number k, one can start the discretization process with the a priori choice (30) of p and h which implies (26) and, in turn, the existence and uniqueness of a Galerkin solution for any right-hand side in $L^2(\Omega)$.*

(b) The constant in the a posteriori error estimate will contain the term $\eta_k^\star(S)$ as a factor. In order to get an explicit upper bound, an a priori estimate of the quantity is required which can be found for constant wavenumber in [14, Theorem 5.5] and [15, Prop. 5.3, Prop. 5.6].

For a simplicial finite element mesh $\mathcal{T}$, the boundary of any element $K \in \mathcal{T}$ consists of $(d-1)$-dimensional simplices. We call (the relatively open interior of) these lower dimensional simplices the *edges* of K, although this terminology is related to the case $d = 2$. The set of all edges of all elements in $\mathcal{T}$ is denoted by $\mathcal{E}^\star$. The subset $\mathcal{E}^\partial \subset \mathcal{E}^\star$ consists of all edges which are contained in Γ^{out} while the subset $\mathcal{E}^\Omega \subset \mathcal{E}^\star$ consists of all edges that are contained in Ω. Finally, we set $\mathcal{E} := \mathcal{E}^\Omega \cup \mathcal{E}^\partial$, the set of all edges that are not in $\overline{\Gamma^{\text{in}}}$. The set of simplex vertices that are not contained in $\overline{\Gamma^{\text{in}}}$ is denoted by $\mathcal{N}$ and, for the cardinality of a discrete set, we write $|\mathcal{N}|$, $|\mathcal{E}|$, etc. For a subset $\mathcal{M} \subset \Omega$ we define simplex neighborhoods about $\mathcal{M}$ by

$$\begin{aligned}
\omega^0_{\mathcal{M}} &:= \left\{\overline{\mathcal{M}}\right\}, \\
\omega^j_{\mathcal{M}} &:= \bigcup \left\{\overline{K} \mid K \in \mathcal{T} \text{ and } \overline{K} \cap \omega^{j-1}_{\mathcal{M}} \neq \emptyset\right\}, \quad j \geq 1, \\
h_{\mathcal{M}} &:= \max\left\{h_K \mid \overline{\mathcal{M}} \cap \overline{K} \neq \emptyset\right\}, \\
p_{\mathcal{M}} &:= \max\left\{p_K + 1 \mid \overline{\mathcal{M}} \cap \overline{K} \neq \emptyset\right\}, \\
\mathcal{E}_{\mathcal{M}} &:= \left\{E \in \mathcal{E}^\star \mid \overline{\mathcal{M}} \cap \overline{E} \neq \emptyset\right\}.
\end{aligned} \tag{31}$$

Definition 18 (Residual) *For $v \in S$ we define the* volume residual $\operatorname{res}(v) \in L^2(\Omega)$ *and the* edge residual $\operatorname{Res}(v) \in L^2(\cup_{E\in\mathcal{E}} E)$ *by*

$$\operatorname{res}(v) := f + \Delta v + k^2 v \qquad \text{on } K \in \mathcal{T},$$

$$\operatorname{Res}(v) := \begin{cases} [\partial_n v]_E & \text{on } E \in \mathcal{E}^\Omega, \\ -\partial_n v + \mathrm{i}\, k v & \text{on } E \in \mathcal{E}^\partial. \end{cases}$$

Here $[v]_E$ is the jump *of the given function v on the edge E, i.e., the difference of the limits in points $x \in E$ from both sides.*

In the definitions above we used exact data f, k. We will later, Section 4.2, replace these by approximations.

The residual $\operatorname{Res}(v)$ is defined for the Robin boundary condition (8) for simplicity. With an obvious modification of this definition, we could also insert a term $T_k v$ here, instead of $\mathrm{i}\, kv$, for the DtN boundary condition (6).

Definition 19 (Error estimator) *Given a set of* weights $\alpha = \{\alpha_K, \alpha_E : K \in \mathcal{T},\ E \in \mathcal{E}\}$*, we define for $v \in S$ the* error estimator

$$\eta(v,\alpha) := \left(\sum_{K\in\mathcal{T}} \alpha_K^2 \left\| \operatorname{res}(v) \right\|_{L^2(K)}^2 + \sum_{E\in\mathcal{E}} \alpha_E^2 \left\| \operatorname{Res}(v) \right\|_{L^2(E)}^2 \right)^{1/2}. \tag{32}$$

The choice of the weights α_K, α_E are related to an interpolation estimate which we explain next.

Assumption 20 (Interpolation operator) *Let $I_S : \mathcal{H} \to S$ denote a continuous linear operator that satisfies the* local relative approximation property: *There are constants $\alpha_K > 0$ for all $K \in \mathcal{T}$ and $\alpha_E > 0$ for all $E \in \mathcal{E}$ such that*

$$\left\| v - I_S v \right\|_{L^2(K)} \leq \alpha_K \left\| v \right\|_{\mathcal{H};\omega_K^m}, \tag{33a}$$

$$\left\| v - I_S v \right\|_{L^2(E)} \leq \alpha_E \left\| v \right\|_{\mathcal{H};\omega_E^m}, \tag{33b}$$

for some $m = O(1)$ independent of h_K, p_K.

The weights in (32) can be chosen as the minimal constants in (33) for any given operator I_S that satisfies the above mentioned properties. In [16, Thms 2.1, 2.2], a Clément-type *hp*-interpolation operator has been constructed which leads to specific choices of α_K, α_E.

Theorem 21 *Let $\Omega \subset \mathbb{R}^2$ and let $p = (p_K)_{K\in\mathcal{T}}$ denote a polynomial degree distribution satisfying (22). Let Assumption 16(a), (b) be satisfied. Then there exist $C > 0$, that depends only on the shape-regularity of the grid (cf. Remark 9), and a linear operator $I_S : H^1_{\mathrm{loc}}(\mathbb{R}^2) \to S$ such that for all simplices $K \in \mathcal{T}$ and all edges $E \in \mathcal{E}_K$ we have*

$$\left\| u - I_S u \right\|_{L^2(K)} + \frac{h_K}{p_K} \left\| \nabla I_S u \right\|_{L^2(K)} + \sqrt{\frac{h_K}{p_K}} \left\| u - I_S u \right\|_{L^2(E)} \leq C_0 \frac{h_K}{p_K} \left\| \nabla u \right\|_{L^2(\omega_K^4)}.$$

Proof. This result has been proven in [16] in a vertex oriented setting, but is easily reformulated as stated above using shape uniformity and quasi-uniformity in the polynomial degree (22). ■

Corollary 22 *Let the Assumptions of Theorem 21 be satisfied. The constants α_K, α_E in Assumption 20 can be chosen according to*

$$\alpha_K := C_0 \frac{h_K}{p_K}, \qquad \alpha_E := C_0 \left(\frac{h_K}{p_K}\right)^{1/2}.$$

Theorem 25 will show that this $\eta(u_S, \alpha)$ can be used for a posteriori error estimation. That it estimates the error from above is called *reliability*, that it estimates the error from below is called *efficiency*.

4.1 Reliability

According to Assumption 16 the exact solution $u \in \mathcal{H}$ and the Galerkin solution $u_S \in S$ of (18) and (21), respectively, exist. In view of inequality (15), we estimate the error $e = u - u_S$, $\operatorname{Re}(a(e,e) - b(e,e))$, and $\|k_+ e\|_{L^2(\Omega)}$ separately in terms of $\eta(u_S, \alpha)$.

Lemma 23 *Let Assumption 16 be satisfied. Assume that there exists a linear and bounded linear operator $I_S : \mathcal{H} \to S$ as in Assumption 20. Then there is a constant $C_1 > 0$, that depends only on the shape-regularity of the grid (cf. Remark 9), such that*

$$|\operatorname{Re}(a(e,e) - b(e,e))| \leq C_1 \eta(u_S, \alpha) \|e\|_{\mathcal{H};\Omega}$$

with α as in (33).

Proof. Using the solution properties and integration by parts yields the error representation

$$\begin{aligned} a(e,e) - b(e,e) &= a(e, e - I_S e) - b(e, e - I_S e) \\ &= \sum_{K \in \mathcal{T}} \int_K \operatorname{res}(u_S)(e - I_S e) + \sum_{E \in \mathcal{E}} \int_E \operatorname{Res}(u_S)(e - I_S e). \end{aligned}$$

We use the assumed interpolation estimates (33) and get with the Cauchy–Schwarz inequality

$$\begin{aligned} &|\operatorname{Re}(a(e,e) - b(e,e))| \\ &\quad \leq \left(\sum_{K \in \mathcal{T}} \alpha_K^2 \|\operatorname{res}(u_S)\|_{L^2(K)}^2 + \sum_{E \in \mathcal{E}} \alpha_E^2 \|\operatorname{Res}(u_S)\|_{L^2(E)}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}} \|e\|_{\mathcal{H};\omega_K^4}^2 \right)^{1/2} \\ &\quad \leq C_1 \eta(u_S, \alpha) \|e\|_{\mathcal{H};\Omega}. \end{aligned}$$

■

Lemma 24 *Let Assumptions 16 and 20 be satisfied. Then, with C_1 from Lemma 23 and $\eta_k^\star(S)$ as in* (25)

$$\|k_+ e\|_{L^2(\Omega)} \leq C_1 \eta_k^\star(S) \eta(u_S, \alpha). \tag{34}$$

Proof. We define z by (16) with $f := k_+^2 e$. Let $z_S \in S$ denote the best approximation of z with respect to the $\|\cdot\|_{\mathcal{H};\Omega}$-norm. We have, by using Galerkin's orthogonality and the arguments as in the proof of Lemma 23,

$$\begin{aligned}\|k_+ e\|_{L^2(\Omega)}^2 &= a(e,z) - b(e,z) = a(e, z - z_S) - b(e, z - z_S) \\ &= \sum_{K\in\mathcal{T}} \int_K \operatorname{res}(u_S)(z - z_S) + \sum_{E\in\mathcal{E}} \int_E \operatorname{Res}(u_S)(z - z_S).\end{aligned}$$

We further follow the arguments of the mentioned proof and, by using the definition of $\eta_k^\star(S)$, we get

$$\|k_+ e\|_{L^2(\Omega)}^2 \le C_1 \eta(u_S, \alpha) \|z - z_S\|_{\mathcal{H};\Omega} \le C_1 \eta(u_S,\alpha)\eta_k^\star(S) \|k_+ e\|_{L^2(\Omega)}$$

and this gives (34). ■

Theorem 25 (Reliability estimate) *Let Assumptions 16 and 20 be satisfied. Then, with C_1 from Lemma 23,*

$$\|e\|_{\mathcal{H};\Omega} \le \frac{1}{\gamma_{\text{ell}}} C_1 \left(1 + (\gamma_{\text{ell}}\theta)^{1/2} \eta_k^\star(S)\right) \eta(u_S, \alpha).$$

Proof. The combination of (15), (34) with the bounds obtained in Lemma 23 and 24 yields

$$\begin{aligned}\gamma_{\text{ell}} \|e\|_{\mathcal{H};\Omega}^2 &\le \operatorname{Re}\left(a(e,e) - b(e,e)\right) + \theta \|k_+ e\|_{L^2(\Omega)}^2 \\ &\le C_1 \eta(u_S,\alpha) \|e\|_{\mathcal{H};\Omega} + \theta C_1^2 \eta_k^\star(S)^2 \eta(u_S,\alpha)^2\end{aligned}$$

so that

$$\begin{aligned}\|e\|_{\mathcal{H};\Omega} &\le \frac{1}{\gamma_{\text{ell}}} C_1 \eta(u_S,\alpha) + \left(\frac{\theta}{\gamma_{\text{ell}}}\right)^{1/2} C_1 \eta_k^\star(S) \eta(u_S,\alpha) \\ &\le \frac{1}{\gamma_{\text{ell}}} C_1 \left(1 + (\gamma_{\text{ell}}\theta)^{1/2} \eta_k^\star(S)\right) \eta(u_S,\alpha).\end{aligned}$$

■

In the previous arguments res and Res were defined with exact data functions f, k. If we define $\widetilde{\eta}$ in terms of $\widetilde{\operatorname{res}}$ and $\widetilde{\operatorname{Res}}$, where f, k have been replaced by polynomial approximations $\widetilde{f}, \widetilde{k}$ the results holds with the following modification.

Corollary 26 *Let $\widetilde{f}, \widetilde{k}$ be approximations to f, k. Then*

$$\begin{aligned}\eta(u_S,\alpha) \le \sqrt{3}\Bigg(\widetilde{\eta}(u_S,\alpha) &+ \Big(\sum_{K\in\mathcal{K}} \alpha_K^2 \|f - \widetilde{f}\|_{L^2(K)}^2\Big)^{1/2} \\ &+ \Big(\sum_{K\in\mathcal{K}} \alpha_K^2 \|(k^2 - \widetilde{k}^2) u_S\|_{L^2(K)}^2\Big)^{1/2}\Bigg).\end{aligned}$$

Proof. We notice

$$\operatorname{res}(u_S) = f + k^2 u_S + \Delta u_S = \widetilde{f} + \widetilde{k}^2 u_S + \Delta u_S + f - \widetilde{f} + (k^2 - \widetilde{k}^2) u_S$$
$$= \widetilde{\operatorname{res}}(u_S) + f - \widetilde{f} + (k^2 - \widetilde{k}^2) u_S$$

and, on Γ^{out},

$$\operatorname{Res}(u_S) = -\partial_n u_S + \mathrm{i}\, k u_S = \widetilde{\operatorname{Res}}(u_S)$$

since k is constant on Γ^{out}. We thus obtain

$$\eta(u_S, \alpha)^2 \le 3\widetilde{\eta}(u_S, \alpha)^2 + 3 \sum_{K \in \mathcal{K}} \alpha_K^2 \|f - \widetilde{f}\|_{L^2(K)}^2 + 3 \sum_{K \in \mathcal{K}} \alpha_K^2 \|(k^2 - \widetilde{k}^2) u_S\|_{L^2(K)}^2 .$$

■

An explicit estimate of the error by the error estimator requires an upper bound for the adjoint approximation property $\eta_k^\star(S)$. Such estimates for *hp*-finite elements spaces for constant wavenumbers k are derived in [14] and [15] for problem (7) and (9). We summarize the results as the following corollaries.

Corollary 27 (Robin boundary conditions) *Consider problem (9) with constant wavenumber k, where $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded Lipschitz domain.* ***Either*** *Ω has an analytic boundary* ***or*** *it is a convex polygon in $\mathbb{R}^2$ with vertices A_j, $j = 1, \ldots, J$. We use the approximation space S described in Section 2.3.2. If Ω is a polygon, then the hp-finite element space S is employed where, in addition, $L = O(p)$ geometric mesh grading steps are performed towards the vertices — for the details we refer to [15]. Let $f \in L^2(\Omega)$ and $k \ge k_0 > 1$ and assume that $\Gamma^{\mathrm{in}} = \emptyset$, i.e., we consider the pure Robin problem. Let Assumption 16 (a) and (b) as well as Assumption 20 be satisfied. Then there exist constants δ, $\tilde{c} > 0$ that are independent of h, p, and k such that the conditions*

$$\frac{kh}{p} \le \delta \quad \text{and} \quad p \ge 1 + \tilde{c} \log(k)$$

imply the k-independent a posteriori error estimate

$$\|e\|_{\mathcal{H};\Omega} \le \frac{1}{\gamma_{\mathrm{ell}}} C_1 \left(1 + (\gamma_{\mathrm{ell}}\theta)^{1/2} \check{C}\right) \eta(u_S, \alpha),$$

where $\check{C}$ only depends on δ and $\tilde{c}$.

Corollary 28 (DtN boundary conditions) *Consider problem (7) for constant wavenumber k, where Ω has an analytic boundary. Let Assumption 16 (a) and (b) as well as Assumption 20 be satisfied and assume that the constant C_k^{adj} in (17) grows at most polynomially in k, i.e., there exists some $\beta \ge 0$ such that*[2] *$C_k^{\mathrm{adj}} \le C k^\beta$. Let $f \in L^2(\Omega)$ and $k \ge k_0 > 1$. Then there exist constants*

[2] See [10] for sufficient conditions on the domain which implies this growth condition.

δ, $\tilde{c} > 0$ *that are independent of* h, p, *and* k *such that the conditions*

$$\frac{kh}{p} \leq \delta \quad \text{and} \quad p \geq 1 + \tilde{c}\log(k)$$

imply the k*-independent a posteriori error estimate*

$$\|e\|_{\mathcal{H};\Omega} \leq \frac{1}{\gamma_{\mathrm{ell}}} C_1 \left(1 + (\gamma_{\mathrm{ell}}\theta)^{1/2}\check{C}\right) \eta(u_S, \alpha)$$

where $\check{C}$ *only depends on* δ *and* $\tilde{c}$.

4.2 Efficiency

The localized version of the error estimator is given by

$$\eta_K(v, \alpha) := \left(\alpha_K^2 \left\|\operatorname{res}(v)\right\|_{L^2(K)}^2 + \frac{1}{2}\sum_{E\in\mathcal{E}(K)} \alpha_E^2 \left\|\operatorname{Res}(v)\right\|_{L^2(E)}^2\right)^{1/2},$$

where $\mathcal{E}(K) := \{E \in \mathcal{E} : E \subset \partial K\}$. Note that $\eta(v, \alpha) = \sqrt{\sum_{K\in\mathcal{T}} \eta_K^2(v, \alpha)}$.

In view of Corollary 26 let us define approximations $\widetilde{f}, \widetilde{k}$ to f, k, respectively, as local $L^2(K)$-projections onto a polynomial of degree p_K (or some $q_K \sim p_K$). In this case we use the notation $\widetilde{\operatorname{res}}$ and $\widetilde{\eta}$ accordingly. Also we set

$$k_{K,+} := \max\{\|k\|_{L^\infty(K)}, 1\}$$

and, for any subset $\omega \subset \Omega$,

$$\delta_\omega^2 := \left\|f - \widetilde{f}\right\|_{L^2(\omega)}^2 + \left\|\left(k^2 - \widetilde{k}^2\right) u_S\right\|_{L^2(\omega)}^2.$$

Theorem 29 *Let Assumptions 16 and (10) be satisfied and let the mesh be shape regular (cf. Remark 9). We assume that* Ω *is either an interval* ($d = 1$)*, or a polygonal domain* ($d = 2$)*, or a Lipschitz polyhedron* ($d = 3$)*, and that the element maps* F_K *are affine. We assume the resolution condition:*

$$\frac{k_{K,+}h_K}{p_K} \lesssim 1 \qquad \text{for all } K \in \mathcal{T}. \tag{35}$$

Then, there exists a constant C *depending only on the constants in Assumption 8 and 4 — and in particular, is independent of* k, p_K, h_K *and* u, u_S *— so that*

$$\widetilde{\eta}_K(u_S, \alpha) \leq C p_K^{3/2} \left(\alpha_K \frac{p_K}{h_K} + \alpha_E \left(\frac{p_K}{h_K}\right)^{1/2}\right) \left(\|u - u_S\|_{\mathcal{H};\omega_K} + \frac{\delta_{\omega_K}}{k_{K,+}}\right), \tag{36}$$

where α_K, α_E *are weights in (32) such that (33a) and (33b) hold*[3]*. For* $d = 2$*, the choices as in Corollary 22 lead to*

$$\widetilde{\eta}_K(u_S, \alpha) \leq C p_K^{3/2} \left(\|u - u_S\|_{\mathcal{H};\omega_K} + \frac{\delta_{\omega_K}}{k_{K,+}}\right). \tag{37}$$

[3] Recall that in general α_K depends on h_K (cf. Corollary 22 for $d = 2$).

Proof. We apply the results [16, Lem. 3.4, 3.5]. There, the proofs are given for two space dimensions, i.e., $d = 2$. They carry over to the case $d = 1$ simply by using [16, Lem. 2.4] instead of [16, Thm. 2.5]. For the case $d = 3$, are careful inspection of the proofs in [16, Thm. 2.5] (which is given in [13, Thm. D2]) and [16, Lem. 2.6] shows that these lemmata also hold for $d = 3$. Hence, the proof of [16, Lem. 3.4, 3.5] can be used verbatim for the cases $d = 1$ and $d = 3$. We choose $\alpha = 0$ in [16, Lem. 3.4, 3.5]. Following these lines of arguments we get for any $\varepsilon > 0$, $K \in \mathcal{T}$, $E \in \mathcal{E}(K)$,

$$\frac{h_K^2}{p_K^2} \|\widetilde{\operatorname{res}}(u_S)\|_{L^2(K)}^2 \leq C(\varepsilon)\left(p_K^2 \|\nabla(u-u_S)\|_{L^2(K)}^2 + p_K^{1+2\varepsilon}\frac{h_K^2}{p_K^2}\left(\left\|k^2(u-u_S)\right\|_{L^2(K)}^2 + \delta_K^2\right)\right)$$

and

$$\frac{h_K}{p_K}\left\|\widetilde{\operatorname{Res}}(u_S)\right\|_{L^2(E)}^2 \leq C(\varepsilon)p_K^{2\varepsilon}\left(p_K^2\|\nabla(u-u_S)\|_{L^2(\omega_K)}^2 + p_K^{1+2\varepsilon}\frac{h_K^2}{p_K^2}\left(\left\|k^2(u-u_S)\right\|_{L^2(\omega_K)}^2 + \delta_{\omega_K}^2\right)\right).$$

Hence,

$$\begin{aligned}
&\alpha_K^2\|\widetilde{\operatorname{res}}(u_S)\|_{L^2(K)}^2 + \alpha_E^2\left\|\widetilde{\operatorname{Res}}(u_S)\right\|_{L^2(E)}^2 \qquad (38)\\
&\leq \left(\alpha_K\frac{p_K}{h_K}\right)^2\frac{h_K^2}{p_K^2}\|\widetilde{\operatorname{res}}(u_S)\|_{L^2(K)}^2 + \left(\alpha_E^2\frac{p_K}{h_K}\right)\frac{h_K}{p_K}\left\|\widetilde{\operatorname{Res}}(u_S)\right\|_{L^2(E)}^2\\
&\leq C(\varepsilon)p_K^2\left(\alpha_K^2\frac{p_K^2}{h_K^2} + \alpha_E^2\frac{p_K^{1+2\varepsilon}}{h_K}\right)\\
&\quad\left(\|\nabla(u-u_S)\|_{L^2(\omega_K)}^2 + 4p_K^{2\varepsilon}\frac{k_{K,+}^2h_K^2}{p_K^3}\|k(u-u_S)\|_{L^2(\omega_K)}^2 + p_K^{2\varepsilon}\frac{h_K^2}{p_K^3}\delta_{\omega_K}^2\right).
\end{aligned}$$

For the special choice $\varepsilon = 1/2$ and with condition (35) we finally get

$$\widetilde{\eta}_K^2(u_S,\alpha) \leq Cp_K^3\left(\alpha_K^2\frac{p_K^2}{h_K^2} + \alpha_E^2\frac{p_K}{h_K}\right)\left(\|u-u_S\|_{\mathcal{H};\omega_K}^2 + k_{K,+}^{-2}\delta_{\omega_K}^2\right).$$

■

Remark 30

(a) *It is possible to choose any* $\varepsilon > 0$ *in (38) (with* $C(\varepsilon) \sim 1/\varepsilon$*). The factor* $p_K^{3/2}$ *in the estimates (36), (37) then can be replaced by* $p^{1+\varepsilon}$*, while condition* (35) *has the weaker form* $k_{K,+}h_K/p_K \leq p_K^{1/2-\varepsilon}$ *(for* $\varepsilon \leq 1/2$*). However, in view of* $p_K \sim \log(k)$ *we think that this is of minor importance.*

(b) *Theorem 29 could be completed by the data saturation condition, say in case of* (37)*,* $C\delta_{\omega_K}p_K^{3/2}k_{K,+} \leq 1/2$*, which would then allow to bound* $\widetilde{\eta}_K(u_S,\alpha)$ *directly by the error.*

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