Error Estimates for Finite Element Discretizations of Elliptic Problems with Oscillatory Coefficients.

D. Peterseim*† S. Sauter‡

December 15, 2010

Abstract

In this paper we will consider elliptic boundary value problems with oscillatory diffusion coefficient, say A. We will derive regularity estimates in Sobolev norms which are weighted by certain derivatives of A. The constants in the regularity estimates then turn out to be independent of the variations in A.

These regularity results will be employed for the derivation of error estimates for hp-finite element discretizations which are explicit with respect to the local variations of the diffusion coefficient.

Keywords weighted regularity, elliptic problem, oscillatory diffusion, hp finite elements

AMS subject classifications 65N30, 35B65 , 35J57.

1 Introduction

The numerical solution of elliptic boundary value problems by the Galerkin finite element method consists of the construction of an "appropriate" finite element mesh and the choice of the (local) polynomial degrees of approximation. An optimal construction should be adapted to the local behavior of the exact solution and, hence, should take into account

a) local singularities of the solution (e.g. singularities at re-entrant corners or at non-smooth interfaces),

*Institut für Mathematik, Humboldt-Universitaet zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany, e-mail: peterseim@math.hu-berlin.de

†The work of the first author was supported by the DFG Research Center Matheon Berlin through project C33.

‡Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland, e-mail: stas@math.uzh.ch

b) effects of, possibly, singular perturbations in the solutions (e.g. indefiniteness, boundary layers, etc.),

c) oscillatory coefficients.

Early publications on local singularities for elliptic problems are [12], [8] – in the meantime a fairly complete theory for this types of irregularities in the solution is available in the literature. In addition, optimal mesh grading for the finite element method has been developed in an a priori way so that the optimal convergence rates are preserved in the presence of singularities.

The same holds true for some classes of singular perturbations. For second order elliptic problems with analytic coefficients the (global) high-order regularity in weighted Sobolev spaces has been derived explicitly in terms of the (global) growth of the derivatives of the coefficients (see [17, Theorem 5.3.10]). In [16] and [15], a wave-number explicit regularity theory for highly-indefinite Helmholtz-type problems has been developed and optimal finite element spaces have been constructed.

In this paper, we are interested in Case c), i.e., coefficients in the elliptic PDE which are oscillatory. The main focus is on coefficients which are smooth but, possibly, highly oscillatory in some parts of the domain. Emphasis is on the case that these parts are *not* distributed uniformly or periodically over the domain. In this light, this paper can be regarded as a generalization of the regularity theory for elliptic PDEs with periodic coefficients (see, e.g., [18], [14] which is based on Fourier transform with special kernel functions).

Based on the local oscillatory behavior of the *coefficients*, we will construct finite element spaces which are optimally adapted to the regularity of the *solution*. The theory is based on the local regularity results derived in [17, Chap. 5] – the main difference is that we use this local regularity to derive a weight function for the definition of weighted Sobolev norms so that the constants in the regularity estimates becomes independent of the local variations of the coefficients.

Nowadays, a posteriori error estimation is commonly used for the control of adaptive mesh refinement or, in general, of the adaptive enrichment of finite element spaces. However, for many singularly perturbed or parameter dependent problems such as, e.g., convection dominated problems, highly indefinite scattering problems, high-frequency eigenvalue problems, etc., the condition "the mesh width has to be *sufficiently* small" typically arises (also for discretizations with a posteriori error control). For singularly perturbed problems or high frequency scattering problems, this condition is often so restrictive that the initial mesh must be chosen very fine and further refinement exceeds computer capacity. Thus, the generation of optimal initial meshes is of utmost importance and our goal is to present a new concept for this purpose.

We further emphasize that our focus in this paper is not in the study of the regularity of problems with piecewise smooth coefficients being discontinuous at "sharp" interfaces (because such type of problems are already studied in the literature by using broken norms (if the interface is smooth) or by Sobolev spaces which are weighted by certain singular functions). In the case of smooth interfaces, this allows the use of standard finite element spaces which have to resolve the (smooth) interface and the resulting convergence estimates are asymptotically optimal. However, if the coefficient does not jump over a *sharp* interface but changes rapidly its values over an interface "zone", the regularity of the solution is polluted in a neighborhood of this zone and the broken regularity estimates become useless. However, we emphasize that a generalization of our regularity estimates to sharp interfaces with discontinuous diffusion coefficients is possible (but technical) by using [17, Lemma 5.5.8].

In the literature, regularity results for problems with highly oscillatory coefficients in combination with error estimates for finite element discretizations exist for *periodic* settings, such as full space problems or problems on tori where the diffusion coefficient is of the form $A^{\mathrm{per}}\left(\frac{x}{\varepsilon}\right)$, i.e., oscillates on a small scale ε, and is periodic. If, in this case, the coefficient is also smooth, one can prove the error estimate, e.g., for a *hp*-finite element discretization

$$\|u-u_h\|_{H^1(\Omega)} \leq C_f \min\left\{1, \left(\frac{h}{\varepsilon}\right)^p\right\}$$

(cf. [14, p. 539]).

Our results are in the same spirit but the analysis is not based on Fourier techniques but on local regularity estimates so that our theory covers also non-periodic settings and quite general non-uniform oscillatory diffusion coefficients.

In [7], diffusion problems with even more general L^∞ coefficients are considered. It is proved that also in such cases there exists a (local) *generalized* finite element basis with the following property: For any shape regular finite element mesh of step size h there exist $O\left(\log\frac{1}{h}\right)$ *local* basis functions per nodal point such that the corresponding Galerkin solution u_h satisfies the error estimate

$$\|u-u_h\|_{H^1(\Omega)} \leq C_f h,$$

where C_f depends on the right-hand side f and the global bounds of the diffusion coefficients (cf. (2.1)) but not on its variations. On one hand, this result is more general than the ones stated in Theorem 5.2 and Corollary 5.3. On the other hand, the definition of the basis functions in [7] is not constructive while the results in this paper apply to standard *hp* finite element spaces.

2 Setting

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let the diffusion matrix $A \in L^\infty\left(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}}\right)$ be uniformly elliptic:

$$\begin{aligned} 0 < \alpha(A,\Omega) &:= \operatorname*{ess\,inf}_{x\in\Omega} \inf_{v\in\mathbb{R}^d\setminus\{0\}} \frac{\langle A(x)v, v\rangle}{\langle v, v\rangle}, \\ \infty > \beta(A,\Omega) &:= \operatorname*{ess\,sup}_{x\in\Omega} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \frac{\langle A(x)v, v\rangle}{\langle v, v\rangle}. \end{aligned} \tag{2.1}$$

For $m \in \mathbb{N}_0$, let $H^m(\Omega)$ denote the usual Sobolev spaces with norm $\|\cdot\|_{H^m(\Omega)}$ and let $H_0^m(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^m(\Omega)}$. The dual space of $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega)$.

For given $f \in L^2(\Omega)$, we are seeking $u \in H_0^1(\Omega)$ such that

$$a\left(u,v\right) := \int_\Omega \langle A\nabla u, \nabla v\rangle = \int_\Omega fv =: F(v) \quad \text{for all} v \in H_0^1(\Omega). \tag{2.2}$$

The elliptic regularity theory tells us that for smooth data (domain Ω, diffusion coefficient A) the condition $f \in H^m(\Omega)$ implies $u \in H^{m+2}(\Omega)$ and there is a constant C (depending on the data and m) such that

$$\|u\|_{H^{m+2}(\Omega)} \leq C \|f\|_{H^m(\Omega)}.$$

In case that the domain Ω is, e.g., a polygonal domain and/or A is only piecewise smooth and discontinuous along polygonal interfaces it is well known that high order regularity can be preserved in weighted Sobolev spaces.

In this paper, we will study the effect of a smooth but oscillatory diffusion coefficient and introduce new types of weighted Sobolev spaces where the regularity constants are independent of such oscillations. This regularity estimates are then interlinked with the Galerkin discretization of (2.2) by *hp* finite elements, because it allows to balance the local estimates of the interpolation error on the single finite element simplices: The weight function used in the definition of the oscillation adapted Sobolev norms encodes the strength of the oscillations of the diffusion coefficients on the scale of the finite element mesh.

3 Oscillation Adapted Sobolev Norms

We assume that A – besides (2.1) – satisfies $A \in C^p(\overline{\Omega}, \mathbb{R}^{d\times d}_{\text{sym}})$ for some smoothness parameter $p \in \mathbb{N}_{\geq 1}$. In the subsequent definition we quantify the smoothness of the coefficient relative to subdomains of Ω.

Definition 3.1 (Oscillation condition) *Let $A \in C^p(\overline{\Omega}, \mathbb{R}^{d\times d}_{\text{sym}})$ for some $p \in \mathbb{N}_{\geq 1}$. A subset $\omega \subset \Omega$ fulfills the* oscillation condition of order p *if*

$$\operatorname{osc}(A,\omega,p) := \max_{1\leq q\leq p} \left\{ \frac{1}{q!} \left(\operatorname{diam}\omega\right)^q \|\nabla^q A\|_{L^\infty(\omega)} \right\} \leq 1. \tag{3.1}$$

Note that the oscillation condition is fulfilled if and only if

$$\operatorname{diam}\omega \max_{1\leq q\leq p} \left\{ \left(\frac{1}{q!} \|\nabla^q A\|_{L^\infty(\omega)} \right)^{1/q} \right\} \leq 1. \tag{3.2}$$

Correspondingly, we define a function $H_{p,A} : \Omega \to \mathbb{R}_{>0}$ which turns out to measure the "variation" of the regularity for problem (2.2) from a standard Poisson problem. This function will depend on the smoothness parameter p.

The construction is as follows. We subdivide some bounding box $Q_0 \supset \bar{\Omega}$ into hypercubes such that the oscillation condition is fulfilled for every such cube. In the following, a cube $Q := \{x \in \mathbb{R}^d : \|x - c_Q\|_\infty \le R_Q\}$ is represented by its center c_Q and its radius R_Q (its halved width). For any parameter $\rho > 0$

$$B_\rho(Q) := \{x \in \mathbb{R}^d : \|x - c_Q\|_\infty \le \rho R_Q\}$$

defines a ρ-scaled version of the cube Q. Clearly, $B_1(Q) = Q$.

Algorithm 3.2 (Oscillation adapted covering) *Let $Q_0 \supset \bar{\Omega}$ be some closed bounding box of Ω. For $p \in \mathbb{N}_{\ge 1}$, a subdivision $\mathcal{Q} = \mathcal{Q}_p(A)$ of Q_0 into closed cubes is defined by:*

$\mathcal{Q} = \{Q_0\}$, $\mathcal{Q}^* := \emptyset$
while $\mathcal{Q}^* \neq \mathcal{Q}$ ***do***
  $\mathcal{Q}^* := \mathcal{Q}$
  for $Q \in \mathcal{Q}^*$ ***do***
    if $\operatorname{osc}(A, B_2(Q) \cap \Omega, p) > 1$ ***then***
      Q is subdivided into 2^d disjoint, congruent cubes $q_1, \dots, q_{2^d}$ and
      $\mathcal{Q} = \mathcal{Q} \setminus Q \cup \{q_1, \dots, q_{2^d}\}$
    end if
  end for
end while

Remark 3.3 *Since $A \in C^p(\overline{\Omega}, \mathbb{R}^{d\times d}_{\text{sym}})$, Algorithm 3.2 terminates because, under this assumption, the oscillations of A are bounded globally over Ω, i.e.,*

$$\operatorname{osc}(A, \Omega, p) \le C_{A,\Omega}, \text{ and, hence, } \operatorname{osc}(A, Q, p) \le 2^{-\ell} \left(\frac{\operatorname{diam} Q_0}{\operatorname{diam} \Omega}\right) C_{A,\Omega}$$

if $Q \in \mathcal{Q}_p(A)$ is the result of ℓ refinement steps in Algorithm 3.2. Hence, the maximal number of refinement steps is bounded from above by $\left\lceil \log_2(1 + C_{A,\Omega} \frac{\operatorname{diam} Q_0}{\operatorname{diam} \Omega}) \right\rceil$.

We shall now make a few observations concerning the local-quasi uniformity of the subdivisions $\mathcal{Q}_p(A)$ generated by Algorithm 3.2. We call two cubes $Q_1, Q_2 \in \mathcal{Q}$ *neighbored* if their boundaries have a common point. The set of neighbors of some cube Q will be denoted by $\mathcal{N}(Q)$.

Proposition 3.4 *If $P, Q \in \mathcal{Q}_p(A)$ are neighbored then $\frac{1}{4} R_Q \le R_P \le 4 R_Q$.*

Proof. $P \in \mathcal{Q} := \mathcal{Q}_p(A)$ implies that the father $\tilde{P}$ of P in the hierarchical construction of $\mathcal{Q}$ does not fulfill the oscillation condition. Necessarily, $B_2(\tilde{P}) \not\subset B_2(Q)$. Since $\tilde{P} \cap Q \neq \emptyset$, we get $\|c_Q - c_{\tilde{P}}\|_\infty \le R_Q + R_{\tilde{P}}$. The condition $B_2(\tilde{P}) \not\subset B_2(Q)$ can be rewritten as

$$R_Q + 3R_{\tilde{P}} > R_{B_2(Q)} = 2R_Q,$$

which yields $R_{\tilde{P}} > \frac{1}{3} R_Q$ and $R_P = \frac{1}{2} R_{\tilde{P}} > \frac{1}{6} R_Q$. Since radii are successively halved in Algorithm 3.2, we finally get $R_P \ge \frac{1}{4} R_Q$. ■

Proposition 3.5 *There exits* $C_{\mathrm{ol}} \in \mathbb{N}$ *depending only on* d *such that for all* $Q \in \mathcal{Q}_p(A)]$ *and for all* $\eta \in [0, 1[$ *it holds*

$$\#\{P \in \mathcal{Q}_p(A) : \ |P \cap B_{1+\eta}(Q)| > 0\} \leq C_{\mathrm{ol}} M_d(\eta),$$

where $M_1(\eta) = \log(1-\eta)$ *and* $M_d(\eta) = (1-\eta)^{1-d}$ *if* $d \geq 2$.

Proof. Let $\mathcal{Q} := \mathcal{Q}_p(A)$ and $\mathcal{P}_Q := \{P \in \mathcal{Q} : \ |P \cap B_{1+\eta}(Q)| > 0\}$. An upper bound for the number of elements of $\mathcal{P}_Q$ can be derived by considering the scenario where Q is surrounded by layers each of which consists of congruent cubes of minimal size. The layers are defined recursively: The first layer contains all neighbors of Q; for $k \geq 2$ the $(k)-$th-th layer contains all neighbors of elements of the $(k-1)-$th layer that are not contained in the layers $1, 2, \ldots, k-1$. By Proposition 3.4 all neighbors of Q have at least radius $\frac{1}{4}R_Q$. We will show later (see the end of this proof) that elements of the k-th layer have at least radius $2^{-(k+1)}R_Q$. Assuming the latter statement is true, we compute that the thickness of the first K layers is $(2(1-(1/2)^{K+1})-1)R_Q$. Thus there can be at most $K = \lceil -\log(1-\eta)/\log(2) \rceil$ layers within the η-neighborhood of Q and the proof is finished for $d = 1$. If $d \geq 2$ then the number of elements in $\mathcal{P}_Q$ can be bounded by $\sum_{k=1}^{K} 2^{(d-1)k} = \frac{2^{d-1}}{2^{d-1}-1} \frac{1-(1-\eta)^{d-1}}{(1-\eta)^{d-1}}$. This bound depends only on η and d but not on Q. It can be written in terms of $M_d(\eta)$ and the proof is finished.

The missing estimate on the minimal layer thickness is proved recursively. Assume that the radii of the k-th layer elements are bounded from below by $2^{-(k+1)}R_Q$ for all $k = 1, \ldots, L$ and that P is an element of layer $L+1$. Then

$$\|c_Q - c_P\|_\infty \geq R_Q + \sum_{k=1}^{L} 2(2^{-(k+1)}R_Q) + R_P = \left(\sum_{k=0}^{L} 2^{-k}\right) R_Q + R_P.$$

If $R_P \geq 2^{-(L+1)}R_Q$ nothing has to be shown. Otherwise, if $R_P \leq 2^{-(L+2)}R_Q$ the intersection of $\tilde{P}$ and elements of the layers $1, \ldots, L$ is of measure zero; $\tilde{P}$ being the father of P in the hierarchical construction of $\mathcal{Q}$. This yields $\|c_Q - c_{\tilde{P}}\|_\infty \geq \left(\sum_{k=0}^{L} 2^{-k}\right) R_Q + 2R_P$, and

$$\max_{y \in B_2(\tilde{P})} \|y - c_Q\|_\infty \geq \left(\sum_{k=0}^{L} 2^{-k}\right) R_Q + 6R_P.$$

As in the proof of Proposition 3.4 a necessary condition on R_P can be derived from the fact that $B_2(\tilde{P})$ does not fulfill the resolution condition while $B_2(Q)$ does, i.e. $B_2(\tilde{P}) \not\subset B_2(Q)$:

$$\left(\sum_{k=0}^{L} 2^{-k}\right) R_Q + 6R_P > 2R_Q.$$

Thus $R_P > \frac{1}{6}2^{-L}R_Q = \frac{2}{3}2^{-(L+2)}R_Q$. Since radii are successively halved in Algorithm 3.2, we get the desired result $R_P \geq 2^{-(L+2)}R_Q$. ∎

Density functions are now given by the local element size in $\mathcal{Q}_p(A)$.

Definition 3.6 (Oscillation adapted density) *Let $\mathcal{Q}_p(A)$, $p \in \mathbb{N}_{\geq 1}$, be a covering of Ω generated by Algorithm 3.2. Then $\mathcal{Q}_p(A)$-piecewise constant functions $H_{p,A} : \cup\mathcal{Q}_p(A) \to \mathbb{R}_{>0}$ are defined by*

$$H_{p,A}(x) := \min\{\operatorname{diam} Q : Q \in \mathcal{Q}_p(A) \text{ with } x \in Q\} \quad \text{for } x \in \cup\mathcal{Q}.$$

It will turn out that the function $H_{p,A}$ contains important information of the diffusion coefficient A for higher order regularity estimates. However, the construction of $H_{p,A}$ via subdivisions into (overlapping) cubes is not well suited for the representation of the geometry of Ω and for finite element discretizations thereon. In view of the fact that smooth domains, curvilinear polygons, and curved polyhedra are the relevant geometries for our theory we will construct a regular finite element mesh (cf. [3]) consisting of (possibly curved) simplices. The distribution of the simplices in this mesh is controlled by the oscillation adapted function $H_{p,A}$.

In a first step, we introduce an initial coarse mesh that resolves the geometry. In a second step, based on the function $H_{p,A}$, the initial mesh is refined according to the oscillations of the coefficient.

Definition 3.7 (Macro triangulation, refinement, parametrization)

a) We assume that there exists a polyhedral (polygonal in 2D) domain $\widetilde{\Omega}$ along with a bi-Lipschitz mapping $\chi : \widetilde{\Omega} \to \Omega$. Let $\widetilde{\mathcal{T}}^{\mathrm{macro}} = \{\widetilde{K}_i^{\mathrm{macro}} : 1 \leq i \leq q\}$ denote a conforming finite element mesh for $\widetilde{\Omega}$ consisting of simplices which are regular in the sense of [3]. $\widetilde{\mathcal{T}}^{\mathrm{macro}}$ is considered as a coarse partition of $\widetilde{\Omega}$, i.e., the diameters of the elements in $\widetilde{\mathcal{T}}^{\mathrm{macro}}$ are of order 1. We assume that the restrictions $\chi_i := \chi|_{\widetilde{K}_i^{\mathrm{macro}}}$ are analytic for all $1 \leq i \leq q$. The macro mesh for Ω is then given by

$$\mathcal{T}^{\mathrm{macro}} := \left\{ K = \chi(\widetilde{K}^{\mathrm{macro}}) : \widetilde{K}^{\mathrm{macro}} \in \widetilde{\mathcal{T}} \right\}.$$

b) Using the macro mesh as the initial mesh we introduce a recursive refinement procedure REFINE. *The input of* REFINE *is a finite element mesh $\mathcal{T}$, where some elements are marked for refinement, and the output is a new conforming finite element mesh $\mathcal{T}^{\mathrm{refine}}$ in in the sense of [3]. The output is derived by refining the corresponding simplicial mesh $\widetilde{\mathcal{T}}$ in a standard way (e.g., in 2D, by first connecting the midpoints of the marked triangle edges and second eliminating hanging nodes by some suitable closure algorithm). The resulting mesh is denoted by $\widetilde{\mathcal{T}}^{\mathrm{refine}} = \{\widetilde{K}_i : 1 \leq i \leq N\}$. The corresponding finite element mesh for Ω is denoted by $\mathcal{T}^{\mathrm{refine}} = \{K = \chi(\widetilde{K}) : \widetilde{K} \in \widetilde{\mathcal{T}}^{\mathrm{refine}}\right\}$. As a simplifying assumption on the refinement strategy we assume that the elimination of hanging nodes causes refinement of non-marked triangles only in the first layer around marked triangles. In certain* *cases this strategy generates meshes with some "flat" triangles, i.e., the constant measuring the shape regularity of the mesh is increased.*[1]

c) *Note that there exists an affine bijection* $A_K : \widehat{K} \to \widetilde{K}$ *which maps the reference element* $\widehat{K} := \{x \in (\mathbb{R}_{\geq 0})^d : \sum_{i=1}^d x_i \leq 1\}$ *to the simplex* $\widetilde{K}$ *for any* $K = \chi(\widetilde{K}) \in \mathcal{T}$, *where* $\mathcal{T}$ *is derived from* $\mathcal{T}^{\text{macro}}$ *by repeated application of* REFINE. *A parametrization* $F_K : \widehat{K} \to K$ *can be written as* $F_K = R_K \circ A_K$, *where* A_K *is an affine map and the maps* R_K *and* A_K *satisfy for constants* C_{affine}, C_{metric}, $\gamma > 0$*:*

$$\begin{aligned} \|A_K'\|_{L^\infty(\widehat{K})} &\leq C_{\text{affine}} \operatorname{diam}(K), \\ \|(A_K')^{-1}\|_{L^\infty(\widehat{K})} &\leq C_{\text{affine}} \operatorname{diam}(K)^{-1}, \\ \|(R_K')^{-1}\|_{L^\infty(\widetilde{K})} &\leq C_{\text{metric}}, \\ \|\nabla^n R_K\|_{L^\infty(\widetilde{K})} &\leq C_{\text{metric}} \gamma^n n! \quad \textit{for } n \in \mathbb{N}_0. \end{aligned} \tag{3.3}$$

Driven by the density function $H_{p,A}$, the actual oscillation adapted meshes are derived by successively refining the macro mesh as follows.

Algorithm 3.8 (Oscillation adapted finite element mesh) *Let* $\mathcal{T}^{\text{macro}}$ *be a subdivision of* $\overline{\Omega}$ *in the sense of Definition 3.7 and let* $p \in \mathbb{N}_{\geq 1}$. *A subdivision* $\mathcal{T}_p(A)$ *of* Ω *that (as we will prove) reflects the regularity of the coefficient is defined by:*

$\mathcal{T} := \mathcal{T}^{\text{macro}}$
for $q = 1, 2, \ldots, p$ ***do***
  $\mathcal{M} := \mathcal{T}$
  while $\mathcal{M} \neq \emptyset$ ***do***
    $\mathcal{M} := \{K \in \mathcal{T} : \operatorname{diam}(K) > \min_{x \in K} H_{q,A}(x)\}$
    $\mathcal{T} = \text{REFINE}(\mathcal{T}, \mathcal{M})$
  end while
end for

Remark 3.9

a) *The mesh* $\mathcal{T}_p(A)$ *serves as a starting mesh for further regular refinements. Then, the final mesh* $\mathcal{T}_h$ *is again a finite element mesh for* Ω *and satisfies: For all* $K \in \mathcal{T}_p(A)$, *there exists a set of sons* $\operatorname{sons}(K) \subset \mathcal{T}_h$ *such that* $K = \bigcup\{K' : K' \in \operatorname{sons}(t)\}$. *The diameter of* $K \in \mathcal{T}_h$ *is denoted by* h_K *and we are using the maximal mesh width* $h := \max\{h_K : K \in \mathcal{T}_h\}$ *as the index in* $\mathcal{T}_h$.

[1] One could avoid this by allowing that the closure algorithm spreads out by more than one triangle layer about the red refined triangles. The generalization of our theory to this version of the closure algorithm however would require further technicalities in our theory. To avoid this for sake of readability we impose our simplifying assumption on the closure algorithm.

b) For $\ell, \ell' \in \mathbb{N}_{\geq 1}$, $\ell < \ell'$, let $\mathcal{T}_\ell(A)$ and $\mathcal{T}_{\ell'}(A)$ denote the meshes generated by Algorithm 3.8 by using the same initial mesh $\mathcal{T}^{\text{macro}}$. Then, by construction, $\mathcal{T}_{\ell'}$ is a refinement of $\mathcal{T}_\ell$.

Our goal is to discretize (2.2) by the Galerkin finite element method. It will turn out that the ratio $\max\{h_{K'} : K' \in \operatorname{sons} K\}/h_K$, $K \in \mathcal{T}_p(A)$, plays the essential role for the error estimates.

We shall prove that the mesh $\mathcal{T}_p(A)$ has analogue properties as the mesh $\mathcal{Q}_p(A)$ – more precisely it satisfies Propositions 3.4 and 3.5. In addition, it is a simplicial finite element mesh. For $K \in \mathcal{T}$ and $\rho \geq 1$, some scaled neighborhood of K is defined by

$$K_\rho := \{x \in \mathbb{R}^d : \exists y \in K : \|y - x\|_2 \leq \tfrac{\rho}{2} \operatorname{diam}(K)\}. \tag{3.4}$$

Note that Definition 3.7 implies that $\rho_\mathcal{T} := \max\{\operatorname{diam}(T)^d/|T| : T \in \mathcal{T}\}$ is bounded from above by a constant which only depends on C_{affine} and C_{metric} from (3.3).

Lemma 3.10 *Let $\mathcal{Q} = \mathcal{Q}_p(A)$ and $\mathcal{T} = \mathcal{T}_p(A)$, $p \in \mathbb{N}_{\geq 1}$, be the subdivisions generated by Algorithm 3.2 resp. 3.8 which "resolve" the coefficient A. Then it holds:*

a) There exist $C_1(d, C_{\text{affine}}, C_{\text{metric}}), C_2(d) \in \mathbb{N}$ such that for all $Q \in \mathcal{Q}$ and all $K \in \mathcal{T}_p$, $p \in \mathbb{N}_{\geq 1}$, there holds

$$\#\{T \in \mathcal{T} : T \cap Q \neq \emptyset\} \leq C_1 \quad \text{and} \quad \#\{P \in \mathcal{Q} : P \cap K \neq \emptyset\} \leq C_2.$$

b) For all $\eta \in [0, 1[$, there exists $C'_{\text{ol}}(C_1, C_{\text{ol}}, C_2) \in \mathbb{N}$ such that for all $K \in \mathcal{T}$ there holds

$$\#\{T \in \mathcal{T}_p : |T \cap K_{1+\eta}| > 0\} \leq C'_{\text{ol}} \begin{cases} \log(1-\eta) & \text{if } d = 1, \\ (1-\eta)^{1-d} & \text{if } d \geq 2. \end{cases}$$

Proof. Let $K \in \mathcal{T}$ and $Q \in \mathcal{Q}$ be given such that $K \cap Q \neq \emptyset$. Then, depending on the actual realization of the procedure REFINE, there exist $\theta > 0$ such that

$$\theta \operatorname{diam}(Q) \leq \operatorname{diam}(K) \leq \operatorname{diam}(Q). \tag{3.5}$$

Let $\mathcal{N}(Q) := \bigcup\{P \in \mathcal{Q} : P \text{ and } Q \text{ are neighbored}\}$ denote a neighborhood of Q in $\mathcal{Q}$. Then C_1 can be estimated by

$$\begin{aligned} |\mathcal{N}(Q)| &= \sum_{T \in \mathcal{T} :\, T \cap \mathcal{N}(Q) \neq \emptyset} |T \cap \mathcal{N}(Q)| \geq \sum_{T \in \mathcal{T} :\, T \cap Q \neq \emptyset} |T \cap \mathcal{N}(Q)| \\ &\geq \sum_{T \in \mathcal{T} :\, T \cap Q \neq \emptyset} \tfrac{\theta}{4\rho_\mathcal{T}} \operatorname{diam}(Q)^d \geq C_1 \tfrac{\theta}{4\rho_\mathcal{T}} |Q|. \end{aligned}$$

This implies that C_1 is bounded in terms of θ, $\rho_\mathcal{T}$, and d. An analogue argument proves that C_2 is finite and therefore Part *a)*.

Part *b)* follows from Part *a)* as we will explain next. There are at most C_1 cubes which intersect K. Proposition 3.5 shows that in an η-neighborhood of every such cube there are at most $C_{\mathrm{ol}} M_d(\eta)$ elements of $\mathcal{Q}$. Therefore, $K_{1+\eta}$ is covered by at most $C_1 C_{\mathrm{ol}} M_d(\eta)$ cubes. Due to Part *a)* each of the latter cubes is again intersected by at most C_2 simplices. ■

We finally introduce weighted (mesh-dependent) Sobolev norms.

Definition 3.11 (Oscillation adapted Sobolev norms) *Let $\mathcal{T}_p(A)$, $p \in \mathbb{N}_{\geq 1}$, be the subdivision of Ω generated by Algorithm 3.8. A weighted seminorm $|\cdot|_{p+1,A}$ in $H^{p+1}(\Omega)$ is defined by*

$$|u|_{p+1,A} := \frac{1}{p!}\left(\sum_{K\in\mathcal{T}_p(A)} \operatorname{diam}(K)^{2p}\|\nabla^{p+1}u\|^2_{L^2(K)}\right)^{1/2},$$

while corresponding full norms are given by

$$\|u\|_{p+1,A} := \sqrt{\|u\|^2_{H^1(\Omega)} + \sum_{\ell=2}^{p+1}|u|^2_{\ell,A}}.$$

By construction the seminorms $|\cdot|_{p+1,A}$ are equivalent to the weighted seminorms $\frac{1}{p!}|H^p_{p,A}\nabla^{p+1}\cdot|_{L^2(\Omega)}$ and

$$\operatorname{osc}(A, K_2, p) < 1 \quad \text{for all } K \in \mathcal{T}_p(A). \tag{3.6a}$$

We omit the proof of equivalence and focus on a related property that will be used later on.

Lemma 3.12 *For all $K \in \mathcal{T}_p(A)$ the lower estimate*

$$h_K \geq c \max\left\{\tau, \left(\max_{1\leq q\leq p}\left\{\left(\frac{\|\nabla^q A\|_{L^\infty(K^\star)}}{q!}\right)^{1/q}\right\}\right)^{-1}\right\} \tag{3.6b}$$

holds with a constant τ representing the minimal mesh size in the initial macro mesh $\mathcal{T}^{\mathrm{macro}}$ (cf. Definition 3.7), and a constant $c > 0$ depending only on the shape parameters in $\mathcal{T}^{\mathrm{macro}}$ and, through (3.5)*, the procedure* REFINE*; $K^\star := K_C$ denotes the C-scaled version of K (cf. (3.4), where the constant C depends only on the shape parameters in $\mathcal{T}^{\mathrm{macro}}$ and the procedure* REFINE*.*

Proof. If K is an element of the initial macro triangulation $\mathcal{T}^{\mathrm{macro}}$ then, by choosing τ appropriately, the assertion can always be satisfied. If $K \in \mathcal{T}_p(A)$ originates from some father simplex $\tilde{K}$ through refinement, $\tilde{K}$ or one of its neighbors was marked in Algorithm 3.8. The marking of $\tilde{K}$ implies the existence of some $Q \in \mathcal{Q}_p(A)$ so that $\operatorname{diam}(\tilde{K}) > \operatorname{diam}(Q)$. If $Q \neq Q_0$ then the scaled version of its father $B_2(\tilde{Q})$ violates the oscillation condition (see (3.2)), i.e.,

$$\operatorname{diam}(B_2(\tilde{Q})) \max_{1 \le q \le p} \left\{ \left(\|\nabla^q A\|_{L^\infty(B_2(\tilde{Q}))} / q! \right)^{1/q} \right\} > 1.$$

This yields

$$\operatorname{diam}(K) \ge \theta \operatorname{diam}(\tilde{K}) > (\theta/4) \operatorname{diam}(B_2(\tilde{Q})) \ge \frac{\theta}{4} \left(\max_{1 \le q \le p} \left\{ \left(\|\nabla^q A\|_{L^\infty(\tilde{Q}_2)} / q! \right)^{1/q} \right\} \right)^{-1}. \quad (3.7)$$

Based on mesh regularity a similar estimate can be derived in the case where the refinement of $\tilde{K}$ is due to preservation of conformity. Therefore (3.6b) is proved. ■

4 Oscillation Adapted Regularity

We start by stating the main result concerning the regularity estimates in weighted Sobolev norms, where the regularity constants are independent of the derivatives of the diffusion coefficient A. The proof is based on local interior regularity estimates which will be proved in Sections 4.2 and 4.3.

4.1 Main Regularity Result

Theorem 4.1 *Let $A \in C^p(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ satisfy (2.1) for some $p \in \mathbb{N}_{\ge 1}$ and assume $f \in H^{p-1}(\Omega)$. The corresponding solution of (2.2) is denoted by u. Further assume that the mesh $\mathcal{T}_p(A)$ is generated by Algorithm 3.8. Let the boundary $\partial\Omega$ be of class C^p.*

Then, the solution satisfies $u \in H^{p+1}(\Omega)$ and

$$\|u\|_{p+1,A} \le C_{11} C_{12}^p \|f\|_{H^{p-1}(\Omega)}. \quad (4.1)$$

The constants C_{11} and C_{12} are independent of p and the variation of A but depend on α, β as in (2.1), on C_{ol} (cf. Proposition 3.5) and on the constants in Definition 3.7(c), on the spatial dimension d, and on the geometry of the domain Ω through its diameter and the constants describing the regularity of the boundary $\partial\Omega$.

Proof. For $K \in \mathcal{T}_p(A)$, let $H_K := \sup\{H_{p,A}(x) : x \in K\}$ and let $H_{\max} := \|H_{p,A}\|_{L^\infty(\Omega)}$. Then by choosing $0 < \eta < 1$ as in Lemma 3.10(b) and using Lemmata 4.5, 4.6, and 4.8 we obtain

$$\sum_{K \in \mathcal{T}_p(A)} \frac{H_K^{2p} \left\|\nabla^{p+1} u\right\|^2_{L^2(K)}}{(p!)^2} \le C_8^2 C_9^{2p} p \sum_{K \in \mathcal{T}_p(A)} \left(\|\nabla u\|^2_{L^2(K_{1+\eta})} + \sum_{i=0}^{p-1} \frac{H_K^{2+2i}}{(i+1)!^2} \left\|\nabla^i f\right\|^2_{L^2(K_{1+\eta})} \right)$$

$$\overset{(3.6a)}{\le} C_{10}^2 C_9^{2p} p \left(\|\nabla u\|^2_{L^2(\Omega)} + \sum_{i=0}^{p-1} \frac{H_{\max}^{2+2i}}{(i+1)!^2} \left\|\nabla^i f\right\|^2_{L^2(\Omega)} \right).$$

Since $0 < \alpha := \alpha(A, \Omega)$ is bounded from below and $H_{\max} \leq \operatorname{diam} \Omega$ we obtain

$$\sum_{K \in \mathcal{T}_p(A)} \frac{H_K^{2p} \left\| \nabla^{p+1} u \right\|_{L^2(K)}^2}{(p)!^2} \leq C_{11}^2 C_9^{2p} p \left\| f \right\|_{H^{p-1}(\Omega)}^2 \leq C_{11}^2 (C_9')^{2p} \left\| f \right\|_{H^{p-1}(\Omega)}^2, \tag{4.2}$$

where C_{11} depends also on $H_{\max}$. For the estimate of the full norm, we get

$$\begin{aligned} \|u\|_{p+1,A}^2 &= \|u\|_{H^1(\Omega)}^2 + \sum_{\ell=1}^{p} |u|_{\ell+1,A}^2 \\ &\leq C \|f\|_{L^2(\Omega)}^2 + \sum_{\ell=1}^{p} C_{11}^2 (C_9')^{2\ell} \|f\|_{H^{\ell-1}(\Omega)}^2 \\ &\leq C_{11}^2 C_{12}^{2p} \|f\|_{H^{p-1}(\Omega)}^2 . \end{aligned}$$

■

4.2 Interior Regularity

For the local high order interior regularity estimates we employ the framework which has been developed in [19], [17]. By some technical reasons which are related to the construction of the oscillation adaptive covering and the finite element method we replace the Euclidean balls in [19], [17] by simplices. For $R > 0$, let the d-dimensional scaled unit simplex (with barycenter at the origin) be denoted by

$$\widehat{T}_R := \left\{ x - \frac{1}{d+1} (R, R, \ldots, R)^\intercal \mid x \in (\mathbb{R}_{\geq 0})^d \wedge \|x\|_{\ell^1} \leq R \right\}.$$

Lemma 4.2 (H^2-regularity) *Let $f \in L^2(\widehat{T}_R)$, $A \in C^1(\widehat{T}_R, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that $0 < \alpha := \alpha(A, \widehat{T}_R)$ and $\beta(A, \widehat{T}_R) =: \beta < \infty$ (cf. (2.1)). Assume that $\operatorname{osc}(A, \widehat{T}_R, 1) \leq 1$. Then, there exists a constant C_{I} depending only on α and β such that the weak solution u of*

$$-\operatorname{div}(A\nabla u) = f \quad \text{in } \widehat{T}_R, \quad u = 0 \quad \text{on } \partial\widehat{T}_R$$

is in $H^2(\widehat{T}_R)$ and satisfies

$$\left\| \nabla^2 u \right\|_{L^2(\widehat{T}_R)} \leq C_{\mathrm{I}} \left\| f \right\|_{L^2(\widehat{T}_R)}. \tag{4.3}$$

The proof follows by scaling the problem to the unit simplex $\widehat{T}_1$ (as explained in [17, Lemma 5.5.5]) and by using standard regularity estimates (see, e.g., [5], [6]).

Lemma 4.3 (interior regularity) *Let the assumptions of Lemma 4.2 be satisfied. Then, there exists a constant $C_{\mathrm{I}}' > 0$ depending only on $\alpha(A, \widehat{T}_R)$ and $\beta(A, \widehat{T}_R)$ such that any solution of*

$$-\operatorname{div}(A\nabla u) = f \quad \text{in } \widehat{T}_R \tag{4.4}$$

satisfies

$$\left\|\nabla^2 u\right\|_{L^2(\widehat{T}_r)} \leq C_{\mathrm{I}}' \left(\|f\|_{L^2(\widehat{T}_{r+\delta})} + \delta^{-1} \|\nabla u\|_{L^2\left(\widehat{T}_{r+\delta}\right)} + \delta^{-2} \|u\|_{L^2\left(\widehat{T}_{r+\delta}\right)} \right)$$

for all $r, \delta > 0$ *with* $r + \delta < R$.

Proof. (See [19, Lemma 5.7.1], [17, Lemma 5.5.11] for ball-shaped domains.) We employ a cutoff function χ being identically one on $\widehat{T}_{R-\delta}$, vanishing on $\widehat{T}_R \backslash \widehat{T}_{R-\delta/2}$ and satisfying $\left\|\nabla^j \chi\right\|_{L^\infty\left(\widehat{T}_R\right)} \leq C\delta^{-j}$, $j = 0, 1, 2$. Then, $U = \chi u$ satisfies

$$-\operatorname{div}(A\nabla U) = \chi f - 2\langle \nabla u, A\nabla \chi\rangle - u \operatorname{div}(A\nabla \chi) \quad \text{in } \widehat{T}_R \quad \text{and} \quad U = 0 \quad \text{on } \partial\widehat{T}_R.$$

By using (4.3) and triangle inequalities in combination with Hölder inequalities, we get

$$\begin{aligned}
\|\nabla u\|_{L^2\left(\widehat{T}_{R-\delta}\right)} &\leq \left\|\nabla^2 U\right\|_{L^2\left(\widehat{T}_R\right)} \\
&\leq C_{\mathrm{I}} \Big(\|f\|_{L^2\left(\widehat{T}_R\right)} \|\chi\|_{L^\infty\left(\widehat{T}_R\right)} + 2 \|\nabla u\|_{L^2\left(\widehat{T}_R\right)} \|A\|_{L^\infty\left(\widehat{T}_R\right)} \|\nabla \chi\|_{L^\infty\left(\widehat{T}_R\right)} \\
&+ \|u\|_{L^2\left(\widehat{T}_R\right)} \|\operatorname{div}(A\nabla\chi)\|_{L^\infty\left(\widehat{T}_R\right)} \Big). \qquad (4.5)
\end{aligned}$$

The assumptions on the cutoff function and A imply $\|\chi\|_{L^\infty\left(\widehat{T}_R\right)} \leq C$ and

$$\|A\|_{L^\infty\left(\widehat{T}_R\right)} \|\nabla \chi\|_{L^\infty\left(\widehat{T}_R\right)} \leq C\beta\left(A, \widehat{T}_R\right)\delta^{-1} \qquad (4.6\text{a})$$

$$\begin{aligned}
\|\operatorname{div}(A\nabla\chi)\|_{L^\infty\left(\widehat{T}_R\right)} &\leq C\left(R\,\|A'\|_{L^\infty\left(\widehat{T}_R\right)} (R\delta)^{-1} + \beta\left(A, \widehat{T}_R\right)\delta^{-2}\right) \\
&\leq C\delta^{-2}\left(1 + \beta\left(A, \widehat{T}_R\right)\right). \qquad (4.6\text{b})
\end{aligned}$$

The combination of (4.5) and (4.6) leads to the assertion. ■

For the estimate of the higher order derivatives we need some further notation. Let

$$N_{R,\ell}(v) := \frac{1}{[\ell]!} \sup_{R/2 \leq r < R} (R-r)^{2+\ell} \left\|\nabla^{\ell+2} v\right\|_{L^2\left(\widehat{T}_r\right)} \quad \ell \in \mathbb{N}_0 \cup \{-2, -1\}, \qquad (4.7\text{a})$$

$$M_{R,\ell}(v) := \frac{1}{\ell!} \sup_{R/2 \leq r < R} (R-r)^{2+\ell} \left\|\nabla^{\ell} v\right\|_{L^2\left(\widehat{T}_r\right)} \quad \ell \in \mathbb{N}_0, \qquad (4.7\text{b})$$

where $[\ell] := \max\{1, \ell\}$. Note that for any $1/2 \leq \eta < 1$:

$$\left\|\nabla^{\ell+2} v\right\|_{L^2\left(\widehat{T}_{\eta R}\right)} \leq \frac{[\ell]!}{((1-\eta)R)^{2+\ell}} N_{R,\ell}(v). \qquad (4.8\text{a})$$

Lemma 4.4 (interior higher order regularity) *For $\ell \in \mathbb{N}_0$, we assume (cf. (2.1)) that*

$$A \in C^{\ell+1}(\widehat{T}_R, \mathbb{R}^{d\times d}_{\text{sym}}), \quad \operatorname{osc}\left(A, \widehat{T}_R, \ell+1\right) \leq \kappa \quad \textit{for some } \kappa > 0,$$
$$0 < \alpha := \alpha\left(A, \widehat{T}_R\right), \quad \beta\left(A, \widehat{T}_R\right) =: \beta < \infty.$$

Then there exists $C_{\mathrm{I}}' > 0$ depending only on α, β, and d such that

$$N_{R,\ell}(u) \leq C_{\mathrm{I}}' \left(M_{R,\ell}(f) + (1+\kappa) \sum_{q=1}^{\ell+1} \frac{\ell+1}{2^q [\ell+1-q]} N_{R,\ell-q}(u) \right) \tag{4.9}$$

for any $f \in H^\ell\left(\widehat{T}_R\right)$ and any solution u of

$$-\operatorname{div}(A\nabla u) = f \quad \textit{on } \widehat{T}_R.$$

Proof. One easily checks that the proof of [17, Lem. 5.5.12] carries over to the scaled unit simplex (instead of balls) so that

$$N_{R,\ell}(u) \leq C_{\mathrm{I}}' \left(M_{R,\ell}(f) + \sum_{q=1}^{\ell+1} \binom{\ell+1}{q} \left(\frac{R}{2}\right)^q \|\nabla^q A\|_{L^\infty(\widehat{T}_R)} \frac{[\ell-q]!}{\ell!} N_{R,\ell-q}(u) \right. \tag{4.10}$$
$$\left. + N_{R,\ell-1}(u) + N_{R,\ell-2}(u) \right).$$

From $\operatorname{osc}\left(A, \widehat{T}_R, \ell+1\right) \leq \kappa$ we conclude that

$$N_{R,\ell}(u) \leq C_{\mathrm{I}}' \left(M_{R,\ell}(f) + \kappa \sum_{q=1}^{\ell+1} \frac{\ell+1}{2^q [\ell+1-q]} N_{R,\ell-q}(u) + N_{R,\ell-1}(u) + N_{R,\ell-2}(u) \right). \tag{4.11}$$

Since the factors in front of $N_{R,\ell-1}(u)$ and $N_{R,\ell-2}(u)$ above are 1, the assertion follows. ■

Lemma 4.5 *For $p \in \mathbb{N}_{\geq 1}$, we assume that*

$$A \in C^p(\widehat{T}_R, \mathbb{R}^{d\times d}_{\text{sym}}), \quad \operatorname{osc}\left(A, \widehat{T}_R, p\right) \leq \kappa \quad \textit{for some } \kappa > 0,$$
$$0 < \alpha := \alpha\left(A, \widehat{T}_R\right), \quad \beta\left(A, \widehat{T}_R\right) =: \beta < \infty.$$

Then, for any $f \in H^{p-1}\left(\widehat{T}_R\right)$ and any solution u of

$$-\operatorname{div}(A\nabla u) = f \quad \textit{on } \widehat{T}_R.$$

it holds

$$\frac{R^p}{p!}\left\|\nabla^{p+1}u\right\|_{L^2(\widehat{T}_{\eta R})} \leq C_1 C_2^p \left(\|\nabla u\|_{L^2(\widehat{T}_R)} + \sum_{i=0}^{p-1} \frac{R^{1+i}}{(i+1)!}\left\|\nabla^i f\right\|_{L^2(\widehat{T}_R)}\right)$$

for all $1/2 \leq \eta < 1$ *with*

$$C_1 := \frac{\lambda+1+C_{\mathrm{I}}'}{(1-\eta)(1+\lambda)}, \quad C_2 := \frac{\lambda+1}{2(1-\eta)}, \quad \lambda := 2C_{\mathrm{I}}'(1+\kappa). \tag{4.12}$$

Proof. The estimate

$$N_{R,-1}(u) \leq \frac{R}{2}\|\nabla u\|_{L^2(\widehat{T}_R)} \tag{4.13}$$

directly follows from (4.7a). Definition (4.7a) implies

$$N_{R,0}(u) = \sup_{R/2 \leq r < R} (R-r)^2 \left\|\nabla^2 u\right\|_{L^2(\widehat{T}_r)}.$$

Next, we estimate the recursion (4.9) and define

$$N_{-1} := N_{R,-1}(u) \quad \text{and, for } \ell = 0,1,2\ldots, \quad N_\ell := C_\ell + \lambda \sum_{q=1}^{\ell+1} \frac{\ell+2}{2^q(\ell+2-q)} N_{\ell-q}, \tag{4.14}$$

where $C_\ell := C_{\mathrm{I}}' M_{R,\ell}(f)$ and λ as in (4.12). It follows directly by comparing (4.9) with (4.14) that $N_{R,\ell}(u) \leq N_\ell$. We set $\tilde{N}_\ell := 2^\ell N_\ell/(\ell+2)$ and $\tilde{C}_\ell := C_\ell 2^\ell/(\ell+2)$ to obtain

$$\tilde{N}_{-1} = N_{-1}/2 \quad \text{and, for } \ell = 0,1,2\ldots, \quad \tilde{N}_\ell = \tilde{C}_\ell + \lambda \sum_{q=1}^{\ell+1} \tilde{N}_{\ell-q}.$$

This recursion can be resolved and we get, for all $\ell \geq 0$,

$$\begin{aligned} \tilde{N}_\ell &\leq \tilde{C}_\ell + \lambda(\lambda+1)^\ell \tilde{N}_{-1} + \lambda \sum_{i=0}^{\ell-1} (\lambda+1)^{\ell-1-i} \tilde{C}_i \\ &\leq (\lambda+1)^{\ell+1} \tilde{N}_{-1} + \sum_{i=0}^{\ell} (\lambda+1)^{\ell-i} \tilde{C}_i. \end{aligned}$$

By substituting back the original quantities we derive

$$\frac{N_\ell}{\ell+2} \leq \left(\frac{\lambda+1}{2}\right)^{\ell+1} N_{-1} + C_{\mathrm{I}}' \sum_{i=0}^{\ell} \left(\frac{\lambda+1}{2}\right)^{\ell-i} \frac{M_{R,i}(f)}{(i+2)}. \tag{4.15}$$

The combination of (4.8a), (4.13), (4.15), and $M_{R,i}(f) \leq \frac{1}{i!}\left(\frac{R}{2}\right)^{2+i}\left\|\nabla^i f\right\|_{L^2(\widehat{T}_R)}$ with some elementary estimates leads to the assertion. ∎

4.3 Regularity at the Boundary

For $R > 0$, let the d-dimensional scaled unit simplex be denoted by

$$\widehat{T}_R^+ := \left\{ x \mid x \in (\mathbb{R}_{\geq 0})^d \wedge \|x\|_{\ell^1} \leq R \right\}$$

and let $\Gamma_R^+ := \left\{ x \in \overline{\widehat{T}_R^+} \mid x_d = 0 \right\}$ be its horizontal facet.

For the estimate of the solution in $\widehat{T}_r^+$ by quantities in a certain neighborhood, we will proceed along the lines of [17, Sect. 5.5.3] and derive estimates for the normal and tangential derivatives at the boundary separately.

4.3.1 Control of Tangential Derivatives

Let $x = (x_1, x_2 \ldots, x_{d-1})$ denote the tangential variables with respect to Γ_R^+. The derivatives with respect to x are denoted by ∇_x. We will need the following notation

$$N_{R,\ell}^+(v) := \begin{cases} \dfrac{1}{\ell!} \sup\limits_{R/2 \leq r < R} (R-r)^{\ell+2} \left\|\nabla^2 \nabla_x^\ell v\right\|_{L^2(\widehat{T}_r^+)} & \text{if } \ell \geq 0, \\ \sup\limits_{R/2 \leq r < R} (R-r)^{\ell+2} \left\|\nabla^{2+\ell} v\right\|_{L^2(\widehat{T}_r^+)} & \text{if } \ell = -2, -1. \end{cases} \tag{4.16a}$$

$$M_{R,\ell}^+(v) := \frac{1}{\ell!} \sup_{R/2 \leq r < R} (R-r)^{\ell+2} \left\|\nabla_x^\ell v\right\|_{L^2(\widehat{T}_r^+)}. \tag{4.16b}$$

Lemma 4.6 *For $p \in \mathbb{N}_{\geq 1}$, we assume (cf. (2.1)) that*

$$A \in C^p(\widehat{T}_R^+, \mathbb{R}_{\text{sym}}^{d \times d}), \qquad \operatorname{osc}\left(A, \widehat{T}_R^+, p\right) \leq \kappa \quad \textit{for some } \kappa > 0,$$
$$0 < \alpha := \alpha\left(A, \widehat{T}_R^+\right), \quad \beta\left(A, \widehat{T}_R^+\right) =: \beta < \infty.$$

Then there exists $C_{\mathrm{B}}' > 0$ depending only on α, β, and d such that for all $f \in H^{p-1}\left(\widehat{T}_R^+\right)$ and any solution u of

$$-\operatorname{div}(A \nabla u) = f \quad \textit{in } \widehat{T}_R^+, \quad u = 0 \quad \textit{on } \Gamma_R^+ \tag{4.17}$$

we have

$$\frac{R^p}{p!} \left\|\nabla_x^{p-1} \nabla^2 u\right\|_{L^2(\widehat{T}_{\eta R}^+)} \leq C_1 C_2^p \left(\|\nabla u\|_{L^2(\widehat{T}_R^+)} + \sum_{i=0}^{p-1} \frac{R^{1+i}}{(i+1)!} \left\|\nabla^i f\right\|_{L^2(\widehat{T}_R^+)} \right)$$

for all $1/2 \leq \eta < 1$ with

$$C_1 := \frac{\lambda_{\mathrm{B}} + 1 + C_{\mathrm{B}}'}{(1-\eta)(1+\lambda)}, \quad C_2 := \frac{\lambda_{\mathrm{B}} + 1}{2(1-\eta)}, \quad \lambda_{\mathrm{B}} := 2C_{\mathrm{B}}'(1+\kappa).$$

Proof. Once again, one checks that the proof for [17, Lem. 5.5.15] carries over to the scaled unit simplex so that

$$N_{R,\ell}^{+}(u) \leq C_{\mathrm{B}} \left(M_{R,\ell}^{+}(f) + \sum_{q=1}^{\ell+1} \binom{\ell+1}{q} \left(\frac{R}{2}\right)^{q} \|\nabla^{q} A\|_{L^{\infty}(\widehat{T}_{R}^{+})} \frac{[\ell-q]!}{\ell!} N_{R,\ell-q}^{+}(u) \right.$$
$$\left. + N_{R,\ell-1}^{+}(u) + N_{R,\ell-2}^{+}(u) \right).$$

This estimate has the same form as (4.10) so that we may conclude

$$\frac{N_{R,\ell}^{+}(u)}{\ell+2} \leq \left(\frac{\lambda_{\mathrm{B}}+1}{2}\right)^{\ell+1} N_{R,-1}^{+}(u) + C_{\mathrm{B}}' \sum_{i=0}^{\ell} \left(\frac{\lambda_{\mathrm{B}}+1}{2}\right)^{\ell-i} \frac{M_{R,i}^{+}(f)}{(i+2)} \qquad (4.18)$$

and, finally, the assertion follows in the same way as in the proof of Lemma 4.5. ■

4.3.2 Control of Normal Derivatives

For the control of the normal derivatives, we introduce the quantity

$$N_{R,\ell,q}^{+}(v) := \frac{1}{[\ell+q]!} \sup_{R/2 \leq r < R} (R-r)^{\ell+q+2} \left\| \nabla_{x}^{\ell} \partial_{y}^{q+2} v \right\|_{L^{2}(\widehat{T}_{r}^{+})}, \qquad (4.19)$$

where, again, ∇_x denote the gradient with respect to the tangential variables x_i with respect to Γ_R^+, $1 \leq i \leq d-1$, and $\partial_y = \partial_{x_n}$ denote the derivative with respect to the normal direction.

Lemma 4.7 *For $t \in \mathbb{N}_0$, we assume that*

$$\begin{array}{lll} A \in C^{t+1}(\widehat{T}_R^+, \mathbb{R}_{\mathrm{sym}}^{d\times d}), & \operatorname{osc}(A, \widehat{T}_R^+, t+1) \leq \kappa & \textit{for some} \quad \kappa > 0, \\ 0 < \alpha := \alpha(A, \widehat{T}_R^+), & \dfrac{R^{\ell+m}}{\ell! m!} \left|\nabla_x^{\ell} \partial_y^{m} A\right| \leq \kappa & \textit{for all} \quad 1 \leq \ell + m \leq t+1, \\ \infty > \beta := \beta(A, \widehat{T}_R^+). & & \end{array}$$

Then, for all $f \in H^t(\widehat{T}_R^+)$ and corresponding solutions u of (4.17) we have

$$\left| N_{R,\ell,q}^{+}(u) \right| \leq C_5 K_1^{\ell} K_2^{q} \left(N_{R,-1}^{+}(u) + \sum_{s=0}^{\ell+q} M_{R,s}(f) \right)$$

for all $\ell \in \mathbb{N}_0$ and $q \in \mathbb{Z}_{\geq -2}$ with $\ell + q \leq t$. The constants C_5, K_1, K_2 only depend on α, β, d, λ_{B}, C_{B}', and κ.

Proof. We assume that $f \in C^t(\overline{\widehat{T}_R^+})$ and obtain the result for general $f \in H^t(\widehat{T}_R^+)$ by a standard density argument.

In the following, ℓ, q denote always integers which satisfy $\ell \in \mathbb{N}_0$, $q = -2, -1, 0, \ldots$, and $\ell + q \leq t$.

For $q=-2,-1$, the estimate $N^+_{R,\ell,q}(v) \leq N^+_{R,\ell+q}(v)$ directly follows from the definitions (4.16) and (4.19). This serves as the start of an induction. We assume that the assertion is proved for all

$$(\ell, q) \in \mathcal{I}_t(q') := \{(r,s) \mid 0 \leq r \leq t, -2 \leq s \leq \min\{q', t-r-1\}\}$$

for some $-1 \leq q' \leq t-1$. In the induction step, we will prove the result for all $(\ell,q) \in \mathcal{I}_t(q'+1)$. Taking into account the start of the induction, we may assume from now on that $\ell, q \geq 0$ and $\ell + q \leq t$.

Let $\hat{A}$ denote the $d \times d$ matrix with $\hat{A}_{i,j} := A_{i,j}$ for all $1 \leq i,j \leq d$ with $(i,j) \neq (d,d)$ and $\hat{A}_{d,d} := 0$. Then

$$-A_{d,d}\partial_y^2 u = f + \langle \operatorname{div} A, \nabla u\rangle + \hat{A} : \nabla^2 u$$

and

$$-\partial_y^2 u = \tilde{f} + \langle b, \nabla u\rangle + B : \nabla^2 u \quad \text{with} \quad \tilde{f} = f/A_{d,d},\ b = \frac{\operatorname{div} A}{A_{d,d}},\ B := A_{d,d}^{-1}\hat{A}.$$

With start with the contribution related to $\tilde{f}$. From Lemma A.2 we obtain

$$\frac{(R-r)^{\ell+q+2}}{(\ell+q)!}\left\|\nabla^{\ell+q}\tilde{f}\right\|_{L^2(\widehat{T}_r^+)} \leq \frac{2}{\alpha}\left(\frac{8}{3}\right)^{\frac{d-1}{2}}\gamma^{\ell+q}\sum_{s=0}^{\ell+q} M_{R,s}(f),$$

where $\gamma := \max\left\{2, \frac{8\kappa}{\alpha}\right\}$ and $\widehat{T}_r$ in the definition of $M_{R,s}(f)$ (cf. (4.7b)) has to be replaced by $\widehat{T}_r^+$.

Next, we will bound the term

$$M_{\ell,q}(b,u) := \frac{1}{(\ell+q)!}\sup_{R/2\leq r<R}(R-r)^{\ell+q+2}\left\|\nabla_x^\ell\partial_y^q\langle b,\nabla u\rangle\right\|_{L^2(\widehat{T}_r^+)}.$$

From [17, Lemma 5.5.18], we get

$$\begin{aligned} M_{\ell,q}(b,u) \leq{}& \frac{\ell!q!}{(\ell+q)!}\sum_{r=0}^{\ell}\sum_{s=0}^{q}\frac{\left\|\partial_y^s\nabla_x^r b\right\|_{L^\infty(\widehat{T}_R^+)}}{r!s!}\left(\frac{R}{2}\right)^{r+s+1}\times \\ &\times\frac{[\ell-r+q-s-1]!}{(\ell-r)!(q-s)!}\left(N^+_{R,\ell-r,q-1-s}(u) + N^+_{R,\ell+1-r,q-s-2}(u)\right). \end{aligned}$$

The bound

$$\frac{1}{r!s!}\left\|\partial_y^s\nabla_x^r b\right\|_{L^\infty(\widehat{T}_R^+)}\left(\frac{R}{2}\right)^{r+s+1} \leq \frac{C}{2}\left(\frac{\gamma}{2}\right)^{r+s} \tag{4.20}$$

is proved in Lemma A.3, where C depends on d, α, and κ. Thus

$$\begin{aligned} M_{\ell,q}(b,u) \leq{}& \frac{C\ell!q!}{2(\ell+q)!}\sum_{r=0}^{\ell}\sum_{s=0}^{q}\left(\frac{\gamma}{2}\right)^{r+s}\frac{[\ell-r+q-s-1]!}{(\ell-r)!(q-s)!}\times \\ &\times\left(N^+_{R,\ell-r,q-1-s}(u) + N^+_{R,\ell+1-r,q-s-2}(u)\right). \end{aligned}$$

Finally, we consider the term $B:\nabla^2 u$. From [17, Lemma 5.5.17] we derive the estimate

$$\begin{aligned}
&\frac{1}{[\ell+q]!}\sup_{R/2\le r<R}(R-r)^{\ell+q+2}\left\|\nabla_x^\ell\partial_y^q\left(B:\nabla^2u\right)\right\|_{L^2(\widehat{T}_r^+)}\\
&\le\sum_{r=0}^{\ell}\sum_{s=0}^{q}\binom{\ell}{r}\binom{q}{s}\left\|\partial_y^s\nabla_x^rB\right\|_{L^\infty(\widehat{T}_R^+)}\left(\frac{R}{2}\right)^{r+s}\times\\
&\times\frac{[\ell-r+q-s]!}{[\ell+q]!}\left(N^+_{R,\ell+1-r,q-1-s}(u)+N^+_{R,\ell+2-r,q-s-2}(u)\right).
\end{aligned}$$

Similarly as for the estimate (4.20) one shows

$$\frac{1}{r!s!}\left\|\partial_y^s\nabla_x^rB\right\|_{L^\infty(\widehat{T}_R^+)}\left(\frac{R}{2}\right)^{r+s}\le C\left(\frac{\gamma}{2}\right)^{r+s}\quad\text{with}\quad C:=\frac{2\kappa}{\alpha(\gamma-1)^2}\left(\frac{8}{3}\right)^{\frac{d-2}{2}}+\beta.$$

Thus

$$\begin{aligned}
&\frac{1}{[\ell+q]!}\sup_{R/2\le r<R}(R-r)^{\ell+q+2}\left\|\nabla_x^\ell\partial_y^q\left(B:\nabla^2u\right)\right\|_{L^2(\widehat{T}_R^+)}\\
&\le C\frac{\ell!q!}{[\ell+q]!}\sum_{r=0}^{\ell}\sum_{s=0}^{q}\left(\frac{\gamma}{2}\right)^{r+s}\frac{[\ell-r+q-s]!}{(\ell-r)!\,(q-s)!}\left(N^+_{R,\ell+1-r,q-1-s}(u)+N^+_{R,\ell+2-r,q-s-2}(u)\right).
\end{aligned}$$

In this way, we have proved

$$\begin{aligned}
N^+_{R,\ell,q}(u)&\le\frac{2}{\alpha}\left(\frac{8}{3}\right)^{\frac{d-1}{2}}\left(\frac{\gamma}{2}\right)^{\ell+q}\sum_{s=0}^{\ell+q}M_{R,s}(f)\\
&+\frac{\ell!q!}{[\ell+q]!}\frac{C}{2}\sum_{r=0}^{\ell}\sum_{s=0}^{q}\left(\frac{\gamma}{2}\right)^{r+s}\frac{[\ell-r+q-s-1]!}{(\ell-r)!\,(q-s)!}\left(N^+_{R,\ell-r,q-1-s}(u)+N^+_{R,\ell+1-r,q-s-2}(u)\right)\\
&+C\frac{\ell!q!}{[\ell+q]!}\sum_{r=0}^{\ell}\sum_{s=0}^{q}\left(\frac{\gamma}{2}\right)^{r+s}\frac{[\ell-r+q-s]!}{(\ell-r)!\,(q-s)!}\left(N^+_{R,\ell+1-r,q-1-s}(u)+N^+_{R,\ell+2-r,q-s-2}(u)\right).
\end{aligned}$$

To understand this recursion, we start by introducing

$$\begin{array}{ll}
C_1=\frac{2}{\alpha}\left(\frac{8}{3}\right)^{\frac{d-1}{2}}, & C_2=\gamma/2,\\
\tilde{N}^+_{\ell,q}:=N^+_{R,\ell,q}(u), & \hat{C}:=\max\{C_1,C\}
\end{array}$$

to obtain

$$\begin{aligned}
\tilde{N}^+_{\ell,q}\le\hat{C}C_2^{\ell+q}&\left(\sum_{s=0}^{\ell+q}M_{R,s}(f)\right.\\
&\left.+\frac{\ell!q!}{(\ell+q)!}\sum_{r=0}^{\ell}\sum_{s=0}^{q}C_2^{-r-s}\frac{(r+s)!}{r!s!}\left(\frac{\tilde{N}^+_{r,s-1}+\tilde{N}^+_{r+1,s-2}}{2[r+s]}+\tilde{N}^+_{r+1,s-1}+\tilde{N}^+_{r+2,s-2}\right)\right).
\end{aligned}$$

By using Stirling's formula and $\ell!/r! \leq \ell^{\ell-r}$ we get with $c := \mathrm{e}^{1/12}$

$$\frac{\ell!q!}{(\ell+q)!}\frac{(r+s)!}{r!s!} \leq c\left(\frac{\mathrm{e}}{\ell+q}\right)^{\ell+q-r-s} \ell^{\ell-r}q^{q-s} \leq c\left(\frac{\mathrm{e}\,\ell}{\ell+q}\right)^{\ell-r}\left(\frac{\mathrm{e}\,q}{\ell+q}\right)^{q-s} \leq c\,\mathrm{e}^{\ell-r+q-s}.$$

Thus,

$$\tilde{N}^{+}_{\ell,q} \leq \hat{C}C_2^{\ell+q}\sum_{s=0}^{\ell+q} M_{R,s}(f)$$
$$+ c\hat{C}\sum_{r=0}^{\ell}\sum_{s=0}^{q}(C_2\,\mathrm{e})^{\ell-r+q-s}\left(\frac{\tilde{N}^{+}_{r,s-1}+\tilde{N}^{+}_{r+1,s-2}}{2} + \tilde{N}^{+}_{r+1,s-1} + \tilde{N}^{+}_{r+2,s-2}\right).$$

By defining the quantities $N^{\mathrm{I}}_{\ell,q}$ via the recursion $N^{\mathrm{I}}_{\ell,q} := N^{+}_{R,\ell,q}(u)$ for $q = -2,-1$, and by

$$N^{\mathrm{I}}_{\ell,q} = \hat{C}C_2^{\ell+q}\sum_{s=0}^{\ell+q} M_{R,s}(f) + \frac{3c\hat{C}}{2}\sum_{r=0}^{\ell}\sum_{s=0}^{q}(C_2\,\mathrm{e})^{\ell-r+q-s}\left(\tilde{N}^{\mathrm{I}}_{r+1,s-1} + \tilde{N}^{\mathrm{I}}_{r+2,s-2}\right) \tag{4.21}$$

for $q \geq 0$ we conclude from obvious monotonicity considerations that $\tilde{N}^{+}_{\ell,q} \leq N^{\mathrm{I}}_{\ell,q}$.

Note that (4.18) implies after some simple estimates

$$N^{+}_{R,t}(u) \leq C_4 C_3^t\left(N^{+}_{R,-1}(u) + \sum_{i=0}^{t} M^{+}_{R,i}(f)\right). \tag{4.22}$$

with $C_3 = \mathrm{e}^{\frac{\lambda_{\mathrm{B}}+1}{2}}$ and $C_4 = C_3 + C'_{\mathrm{B}}$.

We will prove by induction that

$$N^{\mathrm{I}}_{\ell,q} \leq C_5 K_1^{\ell} K_2^{q}\left(N^{+}_{R,-1}(u) + \sum_{s=0}^{\ell+q} M_{R,s}(f)\right),$$

where

$$C_5 \geq \max\left\{C_4, 2\hat{C}\right\}, \quad K_1 \geq \max\left\{C_3, 2C_2\,\mathrm{e}\right\}, \quad K_2 \geq K_1\max\left\{1, 24c\hat{C}\right\}.$$

For $q = -2,-1$, we get from (4.22)

$$N^{\mathrm{I}}_{\ell,q} = N^{+}_{R,\ell,q}(u) \leq N^{+}_{R,\ell+q}(u) \leq C_4 C_3^{\ell+q}\left(N^{+}_{R,-1}(u) + \sum_{i=0}^{\ell+q} M^{+}_{R,i}(f)\right)$$

and the assumptions on C_5, K_1, K_2 imply the assertion.

For $q \geq 0$, we estimate the right-hand side (r.h.s.) in the recursion (4.21) by

$$
\begin{aligned}
r.h.s. &\leq \hat{C} C_2^{\ell+q} \sum_{s=0}^{\ell+q} M_{R,s}(f) \\
&+ \frac{3c\hat{C}}{2} K_1^{\ell+1} K_2^{q-1} C_5 \left(1 + \frac{K_1}{K_2}\right) \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \left(\frac{C_2\,\mathrm{e}}{K_1}\right)^r \left(\frac{C_2\,\mathrm{e}}{K_2}\right)^s \left(N_{R,-1}^{+}(u) + \sum_{m=0}^{\ell+q} M_{R,m}(f)\right) \\
&\leq C_5 K_1^{\ell} K_2^{q} \left(\left\{ 6c\hat{C} \frac{K_1}{K_2} \left(1 + \frac{K_1}{K_2}\right) \right\} N_{R,-1}^{+}(u) \right. \\
&+ \left. \left\{ \frac{\hat{C}}{C_5} \left(\frac{C_2}{K_1}\right)^{\ell} \left(\frac{C_2}{K_2}\right)^{q} + 6c\hat{C} \frac{K_1}{K_2} \left(1 + \frac{K_1}{K_2}\right) \right\} \sum_{m=0}^{\ell+q} M_{R,m}(f) \right).
\end{aligned}
$$

The assumptions on K_1, K_2 ensure that the expressions in the curly brackets are bounded by 1 so that the assertion follows. ■

Lemma 4.8 *For $p \in \mathbb{N}_{\geq 1}$, we assume that*

$$
\begin{aligned}
&A \in C^p(\widehat{T}_R^{+}, \mathbb{R}_{\mathrm{sym}}^{d\times d}), \qquad \operatorname{osc}\left(A, \widehat{T}_R^{+}, p\right) \leq \kappa \qquad \text{for some} \quad \kappa > 0, \\
&0 < \alpha := \alpha\left(A, \widehat{T}_R^{+}\right), \qquad \frac{R^{\ell+m}}{\ell! m!} \left|\nabla_x^{\ell} \partial_y^{m} A\right| \leq \kappa \qquad \text{for all} \quad 1 \leq \ell + m \leq p, \\
&\infty > \beta := \beta\left(A, \widehat{T}_R^{+}\right).
\end{aligned}
$$

Then for all $f \in H^{p-1}\left(\widehat{T}_R^{+}\right)$ and corresponding solutions u of (4.17) we have

$$
\frac{R^{\ell+q+1}}{[\ell+q]!} \left\|\nabla_x^{\ell} \partial_y^{q+2} u\right\|_{L^2(\widehat{T}_{\eta R}^{+})} \leq C_6 C_7^{\ell+q} \left(\|\nabla u\|_{L^2(\widehat{T}_R^{+})} + \sum_{i=0}^{\ell+q} \frac{1}{i!} \left(\frac{R}{2}\right)^{1+i} \left\|\nabla^i f\right\|_{L^2(\widehat{T}_R)} \right)
$$

for all $\ell \in \mathbb{N}_0$ and $q \in \mathbb{Z}_{\geq -2}$ with $\ell + q \leq p - 1$ and for all $1/2 \leq \eta < 1$, where $C_6 := \frac{C_5}{(1-\eta)^2}$ and $C_7 := \frac{\max\{K_1, K_2\}}{1-\eta}$.

4.3.3 Curved Boundaries

Next, we will lift the regularity estimates on the scaled unit simplex to possibly curved simplices of the finite element mesh. We explain the arguments only for the case of a simplex $K \in \mathcal{G}$ with one and only one edge, say E, on Γ. We denote the pullback to the (scaled) reference element by $F_K := R_K \circ A_K : \widehat{T}_{h_K}^{+} \rightarrow K$ which is chosen such that $F_K : \Gamma_{h_K}^{+} \rightarrow E$. The scaling of the reference triangle is chosen such that F_K and its derivatives are bounded independently of h.

From the invariance (up to multiplicative constants) of Sobolev norms under analytic coordinate transforms (see [17, Corollary 4.2.21]) we conclude that the estimates in Section 4.3.1 and 4.3.2 remain valid (with the substitutions $R \leftarrow h_K$ and $\widehat{T}_R^{+} \leftarrow K$) – now with multiplicative constants which depend in addition on bounds of derivatives of the pullback F_K.

5 Oscillation Adapted Finite Elements

As an application of the new regularity estimates we will derive error estimates for Galerkin hp-finite element discretizations of (2.2) We refer the reader to [1, 4, 9, 10, 21] for further details concerning hp methods.

Let $\mathcal{T}_p(A)$ be generated by Algorithm 3.8. We assume that the mesh $\mathcal{T}_h$ is a refinement of $\mathcal{T}_p(A)$ according to Definition 3.7 and satisfies (3.3) with moderate constants. Recall the definition of the subsets $\operatorname{sons}(K) \subset \mathcal{T}_h$ for $K \in \mathcal{T}_p(A)$ as in Remark 3.9a).

The hp-finite element space for the mesh $\mathcal{T}_h$ with polynomial degree p is given by

$$S_h^p := \left\{ u \in H_0^1(\Omega) \mid \forall K \in \mathcal{T}_h : u|_K \circ F_K \in \mathbb{P}_p \right\}, \tag{5.1a}$$

where the pullback is as in Definition 3.7(c).

The Galerkin discretization of (2.2) reads:

$$\text{Find } u_h \in S_h^p \quad \text{s.t.} \quad a(u_h, v) = F(v) \quad \forall v \in S_h^p. \tag{5.1b}$$

It is well known that the Galerkin solution exists, is unique, and satisfies the quasi-optimal error estimate in the form of Céa's lemma

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{1}{\alpha} \inf_{v \in S_h^p} \|u - v\|_{H^1(\Omega)}. \tag{5.2}$$

To obtain explicit convergence estimates in terms of h and p one has to construct an hp-interpolation operator and to use regularity estimates for the solution u in combination with approximation properties.

Theorem 5.1 *There exists an interpolation operator $\Pi_{h,p} : H^k(\Omega) \rightarrow S_h^p$ such that*

$$\|u - \Pi_{h,p} u\|_{H^1(K)} \leq C_{\text{apx}} \left(\frac{h_K}{p} \right)^p \|u\|_{H^{p+1}(K)}$$

holds for all $K \in \mathcal{T}_h$. The constant C_{apx} only depends on the constants in (3.3) and is independent of p, u, K, and the diameter $h_K := \operatorname{diam} K$.

A construction for the interpolation operator $\Pi_{h,p}$ and the proof of the theorem can be found, e.g., in [2, Lemma 4.5], [20, Lemma 17].

The combination of the local interpolation estimates as in Theorem 5.1 with the new regularity estimates (cf. Theorem 4.1) and Céa's lemma (5.2) gives us the error estimate for the Galerkin solution.

Theorem 5.2 *Let the assumption of Theorem 4.1 be satisfied. Let the hp-finite element discretization be as in (5.1). Then the Galerkin solution u_h exists, is unique, and satisfies the error estimate*

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{C_{11} C_{\text{apx}}}{c\alpha} \left(C_{13} h_{\text{eff}} \right)^p \|f\|_{H^{p-1}(\Omega)}, \tag{5.3a}$$

where

$$h_{\text{eff}} := \max_{K \in \mathcal{T}_p(A)} \left\{ \left(1 + \max_{1 \le q \le p} \left(\frac{\|\nabla^q A\|_{L^\infty(K^\star)}}{q!} \right)^{1/q} \right) \max_{K' \in \text{sons}(K)} h_{K'} \right\} \tag{5.3b}$$

with $K^\star$ as in Lemma 3.12.

Proof. We obtain

$$\begin{aligned}
\|u - u_h\|_{H^1(\Omega)}^2 &\le \frac{1}{\alpha^2} \sum_{K \in \mathcal{T}_h} \|u - \Pi_{h,p} u\|_{H^1(K)}^2 \le \frac{C_{\text{apx}}^2}{\alpha^2} \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p} \right)^{2p} \|u\|_{H^{p+1}(K)}^2 \\
&= \frac{C_{\text{apx}}^2}{\alpha^2} \sum_{K \in \mathcal{T}_p(A)} \sum_{K' \in \text{sons}(K)} \left(\frac{h_{K'}}{p} \right)^{2p} \|u\|_{H^{p+1}(K')}^2 \\
&\le \frac{C_{\text{apx}}^2}{\alpha^2} \sum_{K \in \mathcal{T}_p(A)} \left(\frac{\max_{K' \in \text{sons}(K)} h_{K'}}{h_K} \right)^{2p} \left(\frac{h_K}{p} \right)^{2p} \|u\|_{H^{p+1}(K)}^2 .
\end{aligned}$$

Remark 3.9(b) implies that

$$\sum_{K \in \mathcal{T}_p(A)} \left(\frac{h_K}{p} \right)^{2p} \|u\|_{H^{p+1}(K)}^2 \le \|u\|_{p+1,A}^2 .$$

From (3.1) and (3.6b) we conclude that

$$\|u - u_h\|_{H^1(\Omega)} \le \frac{C_{\text{apx}}}{c\alpha} (C h_{\text{eff}})^p \|u\|_{p+1,A} .$$

■

Corollary 5.3 *Let the assumption of Theorem 5.2 be satisfied. Assume that the coefficient A satisfies*

$$\frac{1}{q!} \|\nabla^q A\|_{L^\infty(\Omega)} \le C \varepsilon^{-q} \tag{5.4}$$

for some (small) $\varepsilon > 0$ and for all $1 \le q \le p$. Let p and h be chosen such that

$$p = \left\lceil \frac{\log h}{\log(C_9' h / \varepsilon)} \right\rceil \quad \text{and} \quad C_9' h < \varepsilon$$

holds. Then, the Galerkin discretization with the corresponding hp-finite element space S_h^p has a unique solution u_h which converges linearly

$$\|u - u_h\|_{H^1(\Omega)} \le C h \|f\|_{H^{p-1}(\Omega)} ,$$

where C is independent of ε, h, and f.

Proof. The combination of (5.3) and (5.4) yields

$$\|u - u_h\|_{H^1(\Omega)} \leq C \left(\frac{C_9' h}{\varepsilon}\right)^p \|f\|_{H^{p-1}(\Omega)}$$

with some constant C. By using $p = \left\lceil \frac{\log h}{\log(C_9' h/\varepsilon)} \right\rceil$ and the condition on h we obtain – after some straightforward manipulations – the assertion. ■

The above corollary shows that for problems with oscillations of characteristic length scale ε in the coefficient, standard hp-finite element basis functions on a uniform mesh of width $h \lesssim \varepsilon$ and polynomial degree $p \gtrsim \log 1/\varepsilon$ suffice to allow for an error bound proportional to ε. Note that in this case the dimension of the finite element space is of order $(\varepsilon \log(1/\varepsilon))^{-d}$. In contrast, a conventional $P1$ finite element method requires for the same error tolerance a mesh of width $h \lesssim \varepsilon^2$ so that the dimension of the $P1$ finite element space is much larger – more precisely of order $(\varepsilon)^{-2d}$.

Notice that the results of this paper apply to problems with coefficients of finite and infinite smoothness – analyticity is not required. In addition, there is no periodicity assumption: oscillations distributed in a non-uniform way are explicitly addressed.

Our theory does not cover problems with discontinuous coefficients. However, if the interfaces between smooth regions of the the diffusion coefficient are resolved by the initial macro finite element mesh, then, by exploiting [17, Lemma 5.5.8], the generalization of our regularity estimates to sharp interfaces with discontinuous diffusion matrix is possible.

The method shall not be regarded as a substitute for adaptive finite element methods driven by suitable a posteriori error estimators; the influence of the domain geometry and the right-hand side on the regularity of the solution are not handled by our theory. However, in many situations it is difficult to define an initial mesh (as coarse as possible) from which the adaptive algorithm successfully and efficiently applies. The a priori criterion for the design of minimal meshes (as it is used in Algorithms 3.2 and 3.8) might be useful in such situations. A comparison of our approach with a posteriori estimates regarding the resolution of oscillatory coefficients (by adaptive finite element meshes) is topic of future research.

A Derivatives of Composite Functions and of Products of Functions

Lemma A.1 *Let $\omega \subset \mathbb{R}^d$ be a domain and $a \in C^{t+1}(\omega)$ which satisfies* $\operatorname{osc}(a, \omega, t+1) \leq \kappa$ *and*

$$\forall x \in \omega \quad 0 < \alpha \leq a(x) \leq \beta < \infty.$$

Then $a^{-1} \in C^{t+1}(\omega)$ and satisfies, for $R := \operatorname{diam} \omega$,

$$\frac{R^{t+1}}{(t+1)!} \left|\nabla^{t+1} a^{-1}(x)\right| \leq \frac{2}{\alpha} \left(\frac{8}{3}\right)^{\frac{d-1}{2}} \gamma^{t+1} \qquad \text{(A.1)}$$

with $\gamma := \max\left\{2, \frac{8\kappa}{\alpha}\right\}$.

Proof. The existence of a^{-1} and $a \in C^{t+1}(\omega)$ follows readily from the generalized diBruno formula (cf. [13, Theorem 4.2]).

For $x \in \omega$ fixed and all $y \in \mathbb{C}^d$, let

$$\tilde{a}(y) := \sum_{\ell=0}^{t+1} \frac{1}{\ell!} \langle y-x, \nabla\rangle^{\ell} a(x) \quad \text{with} \quad \frac{1}{\ell!} \langle y-x, \nabla\rangle^{\ell} a := \sum_{\substack{\mu \in \mathbb{N}_0^d \\ |\mu| = \ell}} \frac{(y-x)^{\mu}}{\mu!} \partial^{\mu} a(x).$$

Also from [13, Theorem 4.2], it follows that $\partial^{\mu} a^{-1}(x)$ only depends on $\partial^{\nu} a(x)$ for $\nu_i \leq \mu_i$, $1 \leq i \leq d$. Hence, by choosing $y = x$, we obtain

$$\left(\tilde{a}^{-1}\right)^{(\mu)}(x) = \left(a^{-1}\right)^{(\mu)}(x) \quad \forall \mu \in \mathbb{N}_0^d \quad |\mu| = t+1.$$

Since $\tilde{a}$ is analytic we may apply Cauchy's integral formula to estimate the derivatives of $\tilde{a}$

$$\frac{1}{\mu!}\left(\tilde{a}^{-1}\right)^{(\mu)}(x) = \frac{1}{(2\pi i)^d} \oint_{C_r(x_1)} \oint_{C_r(x_2)} \cdots \oint_{C_r(x_d)} \frac{\tilde{a}^{-1}(v)}{(v-x)^{\mu+\mathbf{1}}} dv \tag{A.2}$$

with $\mathbf{1} = (1, 1, \ldots, 1)^{\intercal}$ and $C_r(x_i)$ is the circle in $\mathbb{C}$ about x_i with radius $r > 0$. The denominator satisfies $\left|(v-x)^{\mu+\mathbf{1}}\right| = r^{|\mu|+d}$ so that

$$\left|\frac{1}{\mu!}\left(\tilde{a}^{-1}\right)^{(\mu)}(x)\right| \leq r^{-|\mu|} \sup\left\{\left|\tilde{a}^{-1}(v)\right| : v \in \mathbb{C}^d \mid \forall 1 \leq i \leq d \quad v_i \in C_r(x_i)\right\}.$$

The assumptions on a imply

$$\left|\tilde{a}^{-1}(v)\right| = \frac{1}{a(x) + \langle v-x, \nabla\tilde{a}(\xi)\rangle}$$

for some $\xi \in \overline{vx}$. We set $e := \frac{v-x}{\|v-x\|}$ and obtain

$$\begin{aligned} |\langle v-x, \nabla\tilde{a}(\xi)\rangle| &= \left|\sum_{\ell=0}^{t} \frac{1}{\ell!} \langle \xi - x, \nabla\rangle^{\ell} \langle v-x, \nabla\rangle a(x)\right| \\ &\leq \sum_{\ell=0}^{t} \frac{r^{\ell+1}}{\ell!} \left|\langle e, \nabla\rangle^{\ell+1} a(x)\right|. \end{aligned}$$

Some tedious calculus leads to

$$\frac{1}{\ell!}\left|\langle e,\nabla\rangle^{\ell+1}a(x)\right| = (\ell+1)\sum_{\substack{\mu\in\mathbb{N}_0^d\\|\mu|=\ell+1}}\frac{e^{\mu}}{\mu!}\partial^{\mu}a(x)$$
$$= \frac{1}{\ell!}\sqrt{\sum_{\substack{\mu\in\mathbb{N}_0^d\\|\mu|=\ell+1}}\frac{(\ell+1)!}{\mu!}e^{2\mu}}\sqrt{\sum_{\substack{\mu\in\mathbb{N}_0^d\\|\mu|=\ell+1}}\frac{(\ell+1)!}{\mu!}\left|\partial^{\mu}a(x)\right|^2}$$
$$= \frac{1}{\ell!}\sqrt{\left(\sum_{i=1}^{d}e_i^2\right)^{\ell+1}}\left|\nabla^{\ell+1}a(x)\right| = \frac{1}{\ell!}\left|\nabla^{\ell+1}a(x)\right|.$$

Thus, with $R = \operatorname{diam}\omega$, by choosing $r = cR$ in (A.2) for some $0 < c < 1$, and by using the oscillation condition we obtain

$$|\langle v-x,\nabla\tilde{a}(\xi)\rangle| \leq \kappa\sum_{\ell=0}^{t}(\ell+1)c^{\ell+1} \leq \kappa c\sum_{\ell=0}^{\infty}(\ell+1)c^{\ell} = \frac{c\kappa}{(1-c)^2}.$$

By setting $c = \gamma^{-1}$ (cf. (A.1)) we get $|\langle v-x,\nabla\tilde{a}(\xi)\rangle| \leq \frac{\alpha}{2}$ so that $\left|\tilde{a}^{-1}(v)\right| \leq \frac{2}{\alpha}$. Hence,

$$\frac{R^{|\mu|}}{\mu!}\left|\left(\tilde{a}^{-1}\right)^{(\mu)}(x)\right| \leq \frac{2}{\alpha c^{|\mu|}}. \tag{A.3}$$

A summation over all $\mu\in\mathbb{N}_0^d$ with $|\mu| = t+1$ leads to

$$\frac{R^{t+1}}{(t+1)!}\left|\nabla^{t+1}a^{-1}(x)\right| = \frac{1}{(t+1)!}\sqrt{\sum_{\substack{\mu\in\mathbb{N}_0^d\\|\mu|=t+1}}\frac{(t+1)!}{\mu!}\left|R^{|\mu|}\partial^{\mu}\tilde{a}^{(\mu)}(x)\right|^2}$$
$$\leq \frac{2}{\alpha}\frac{1}{\sqrt{(t+1)!}c^{t+1}}\sqrt{\sum_{\substack{\mu\in\mathbb{N}_0^d\\|\mu|=t+1}}\mu!} \overset{\text{Lemma A.4}}{\leq} \frac{2}{\alpha}\left(\frac{8}{3}\right)^{\frac{d-1}{2}}c^{-t-1}.$$

■

Lemma A.2 *Let $\omega\subset\mathbb{R}^d$ be a domain and let $a\in C^{t+1}(\omega)$ satisfy the assumptions of Lemma A.1. Then, for $f\in C^{t+1}(\omega)$, it holds $\tilde{f} := f/a \in C^{t+1}(\omega)$ and $\tilde{f}$ satisfies, for $R := \operatorname{diam}\omega$ and $1\leq\ell\leq t+1$,*

$$\frac{R^{\ell}}{\ell!}\left|\nabla^{\ell}\tilde{f}(x)\right| \leq \frac{2}{\alpha}\left(\frac{8}{3}\right)^{\frac{d-1}{2}}\gamma^{\ell}\sum_{q=0}^{\ell}\frac{R^q\left|\nabla^q f(x)\right|}{q!}$$

with γ as in Lemma A.1.

Proof. From [17, Lemma A.1.3] we conclude that

$$\frac{R^\ell}{\ell!}\left|\nabla^\ell \tilde{f}(x)\right| \leq \sum_{q=0}^{\ell} \frac{R^q\left|\nabla^q f(x)\right|}{q!} \frac{R^{\ell-q}\left|\nabla^{\ell-q} a^{-1}(x)\right|}{(\ell-q)!}.$$

By using Lemma A.1 we get

$$\frac{R^\ell}{\ell!}\left|\nabla^\ell \tilde{f}(x)\right| \leq \frac{2}{\alpha}\left(\frac{8}{3}\right)^{\frac{d-1}{2}} \gamma^\ell \sum_{q=0}^{\ell} \frac{R^q\left|\nabla^q f(x)\right|}{q!}.$$

■

Lemma A.3 *Let $\omega \subset \mathbb{R}^d$ be a domain and let $A \in C^{t+1}(\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ be such that $0 < \alpha\left(A, \widehat{T}_R^+\right) =: \alpha$ and $\beta := \beta\left(A, \widehat{T}_R^+\right) < \infty$. For the oscillations we assume* $\operatorname{osc}(a, \omega, t+1) \leq \kappa$ *and*

$$\frac{R^{\ell+m}}{\ell! m!}\left|\nabla_x^\ell \partial_y^m A\right| \leq \kappa \quad \forall 1 \leq \ell+m \leq t+1.$$

Then, for $b = \frac{\operatorname{div} A}{A_{d,d}}$, it holds $b \in C^t(\omega, \mathbb{R}^d)$ and b satisfies, for $R := \operatorname{diam}\omega$ and $1 \leq \ell+m \leq t$,

$$\frac{R^{\ell+m+1}}{\ell! m!}\left|\nabla_x^\ell \partial_y^m b(x)\right| \leq C\gamma^{\ell+m} \quad \text{with} \quad C := 4\frac{\sqrt{d}\kappa}{\alpha}\left(\frac{8}{3}\right)^{\frac{d-2}{2}}\left(\frac{\gamma}{\gamma-1}\right)^3.$$

Proof. For $1 \leq \ell+m \leq t$, it holds

$$\frac{R^{\ell+m+1}}{\ell! m!}\left|\nabla_x^\ell \partial_y^m \frac{\operatorname{div} A}{A_{dd}}\right| \leq \sum_{r=0}^{\ell}\sum_{s=0}^{m} \frac{R^{r+s+1}\left|\nabla_x^r \partial_y^s \operatorname{div} A\right|}{r! s!} \frac{R^{\ell-r+m-s}\left|\nabla_x^{\ell-r}\partial_y^{m-s} A_{d,d}^{-1}\right|}{(\ell-r)!(m-s)!}. \tag{A.4}$$

Next, observe that

$$\begin{aligned}\frac{R^{r+s+1}}{r! s!}\left|\nabla_x^r \partial_y^s \operatorname{div} A\right| &\leq \sqrt{d}\left((r+1)\frac{R^{r+s+1}}{(r+1)! s!}\left|\nabla_x^{r+1}\partial_y^s A\right| + (s+1)\frac{R^{r+s+1}}{r!(s+1)!}\left|\nabla_x^r \partial_y^{s+1} A\right|\right)\\ &\leq \sqrt{d}(r+s+2)\kappa \end{aligned}\tag{A.5a}$$

and

$$\begin{aligned}\frac{R^{r+s}}{r! s!}\left|\nabla_x^r \partial_y^s A_{d,d}^{-1}\right| &= \frac{1}{\sqrt{r! s!}}\sqrt{\sum_{\substack{\mu\in\mathbb{N}_0^{d-1}\\ |\mu|=r}} \frac{1}{\mu! s!}\left|R^{r+s}\partial_x^\mu \partial_y^s A_{d,d}^{-1}\right|^2} \overset{(A.3)}{\leq} \frac{2\gamma^{r+s}}{\alpha}\frac{1}{\sqrt{r!}}\sqrt{\sum_{\substack{\mu\in\mathbb{N}_0^{d-1}\\ |\mu|=r}} \mu!}\\ &\leq \left(\frac{8}{3}\right)^{\frac{d-2}{2}}\frac{2\gamma^{r+s}}{\alpha}.\end{aligned}\tag{A.5b}$$

Inserting (A.5) into (A.4) leads to

$$\begin{aligned}\frac{R^{\ell+m+1}}{\ell!m!}\left|\nabla_x^\ell\partial_y^m\frac{\operatorname{div}A}{A_{dd}}\right| &\leq 2\frac{\sqrt{d}\kappa}{\alpha}\left(\frac{8}{3}\right)^{\frac{d-2}{2}}\sum_{r=0}^{\ell}\sum_{s=0}^{m}(r+s+2)\,\gamma^{\ell-r+m-s}\\ &\leq 2\frac{\sqrt{d}\kappa}{\alpha}\left(\frac{8}{3}\right)^{\frac{d-2}{2}}\gamma^{\ell+m}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(r+s+2)\,\gamma^{-r-s}\\ &\leq 4\frac{\sqrt{d}\kappa}{\alpha}\left(\frac{8}{3}\right)^{\frac{d-2}{2}}\frac{\gamma^{\ell+m+3}}{(\gamma-1)^3}.\end{aligned}$$

■

Lemma A.4 *It holds*

$$\sum_{\substack{\mu\in\mathbb{N}_0^d\\|\mu|=\ell}}\mu!\leq\ell!\left(\frac{8}{3}\right)^{d-1}. \tag{A.6}$$

Proof. Let

$$\sigma_d(\ell):=\begin{cases}\ell! & d=1,\\ \sum_{i=0}^{\ell}i!\sigma_{d-1}(\ell-i) & d>1\end{cases}$$

and observe that the left-hand side in (A.6) equals $\sigma_d(\ell)$. We prove the result by induction over d.

The case $d=1$ is trivial. For $d=2$, we employ the notation as in (cf. [11, (2.5)])

$$(-\ell)_i:=(-1)^i\frac{\Gamma(1+\ell)}{\Gamma(1+\ell-i)}\qquad\forall\ell\in\mathbb{N}_{\geq1}\text{ and }\forall0\leq i\leq\ell$$

to obtain

$$\begin{aligned}\frac{1}{\ell!}\sum_{i=0}^{\ell}i!(\ell-i)! &= \sum_{i=0}^{\ell}(-1)^i\frac{i!}{(-\ell)_i}\overset{[11,\,(7.2.4)]}{=}\frac{\ell+1}{2^{\ell+1}}\sum_{k=1}^{\ell+1}\frac{2^k}{k}\\ &=\frac{\ell+1}{2^{\ell+1}}\left(-\mathrm{i}\,\pi+B_2(2+\ell,0)\right)=:\psi(\ell),\end{aligned}$$

where $B_z(a,b)$ is the incomplete beta function. The function $\psi(x)$ is continuous for all $x\in\mathbb{R}_{\geq0}$ and satisfies

$$\psi(0)=1\quad\psi(\infty)=1.$$

Hence, there exists C such that, for all $\ell\in\mathbb{R}_{\geq0}$, it holds $\psi(\ell)\leq C$. Numerical tests show that the maximum of ψ is attained for $\psi(4)=8/3$ so that $\sigma_2(\ell)\leq\frac{8}{3}\ell!$.

Assume that the assertion holds for $d-1$. Then, the recursion formula gives us

$$\sigma_d(\ell)\leq\left(\frac{8}{3}\right)^{d-2}\sum_{i=0}^{\ell}i!(\ell-i)!\leq\left(\frac{8}{3}\right)^{d-1}\ell!.$$

■

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