On the Justification of Plate Models

Dietrich Braess* Stefan Sauter† Christoph Schwab‡

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Abstract

In this paper, we will consider the modelling of problems in linear elasticity on thin plates by the models of Kirchhoff–Love and Reissner–Mindlin. A fundamental investigation for the Kirchhoff plate goes back to Morgenstern [*Herleitung der Plattentheorie aus der dreidimensionalen Elastizitätstheorie.* Arch. Rational Mech. Anal. 4, 145–152 (1959)] and is based on the two-energies principle of Prager and Synge. This was half a centenium ago.

We will derive the Kirchhoff–Love model based on Morgenstern's ideas in a rigorous way (including the proper treatment of boundary conditions). It provides insights a) for the relation of the $(1, 1, 0)$-model with the $(1, 1, 2)$-model that differ by a quadratic term in the ansatz for the third component of the displacement field and b) for the rôle of the shear correction factor. A further advantage of the approach by the two-energy principle is that the extension to the Reissner–Mindlin plate model becomes very transparent and easy. Our study includes plates with reentrant corners.

1 Introduction

The plate models of Reissner–Mindlin and Kirchhoff–Love are usually applied to the solution of plate bending problems [15, 20, 17]. Their advantage is

*(Dietrich.Braess@rub.de), Fakultät für Mathematik, Ruhr-Universität Bochum, Germany.

†(stas@math.uzh.ch), Institut für Mathematik, Universität Zürich, Winterthurerstr 190, CH-8057 Zürich, Switzerland

‡(Schwab@sam.math.ethz.ch), ETH Zürich, Switzerland

not only the reduction of the dimension; one can also better control and avoid the locking phenomena which occur in finite element computations for thin elastic bodies. For this reason, the justification of plate models and the estimation of the *model error* is of interest and has a long history. Many papers are based on the so-called *asymptotic methods;* see [9, 10] and the references therein.

A fundamental investigation with a very different tool was done by Morgenstern in 1959 for the Kirchhoff plate [18]. His idea to use the two-energies principle of Prager and Synge [19] that is also denoted as *hypercircle method*, can now be found in a few papers [1, 3, 22, 26]. A second glance shows that the results in [18] depend on two conjectures. We will verify them before we deal with some conclusions for the Reissner–Mindlin plate and for finite element computations.

The results in [18] may be summarized as follows. Let $\Pi(v)$ be the internal stored energy of the plate for the three-dimensional displacement field v and $\Pi^c(\sigma)$ the complementary energy for a stress tensor field σ that satisfies the equilibrium condition

$$\operatorname{div}\sigma + f = 0. \tag{1.1}$$

Given a plate of thickness t, he constructed a solution u_t in the framework of the Kirchhoff model and an equilibrated stress tensor σ_t such that

$$\Pi_0 = \lim_{t\to 0} t^{-3}\Pi(u_t), \quad \Pi_0^c = \lim_{t\to 0} t^{-3}\Pi^c(\sigma_t)$$

exist. He reported that a student had proven

$$\Pi_0 = \Pi_0^c \tag{1.2}$$

by some tedious calculations. It follows from the two-energies principle and (1.2) that the plate model is correct for thin plates, i.e., for $t \to 0$. In [18], the effect of ignoring partially boundary conditions on the lateral boundary was not analyzed.

More precisely, the analysis was performed for a $(1, 1, 2)$-plate model, i.e., the ansatz for the transversal deflection contains a quadratic term in the x_3-variable

$$u_3(x) = w(x_1, x_2) + x_3^2\, W(x_1, x_2). \tag{1.3}$$

There is the curious situation that the quadratic term in (1.3) can be eventually neglected in numerical computations as a term of higher order provided that one is content with a relative error of order $O\left(t^{1/2}\right)$ and if the Poisson ratio in the material law is corrected. It is now folklore that a *shear correction factor* is required even in the limit $t \to 0$ if computations are performed without the second term on the right-hand side of (1.3). The magnitude of the factor, however, differs in the literature [4]. The results have been obtained by minimization arguments; we will justify the model and estimate the model error by completing the analysis with the two-energies principle. Although we start with the Kirchhoff plate, the extension to the Reissner–Mindlin plate is so transparent and easy that we consider it as easier than the analysis in [1, 3]. Our analysis covers plates with reentrant corners.

Although the quadratic term in (1.3) needs not to be computed, it is required for the correct design of the plate equations and the analysis. Roughly speaking, it is understood that the $(1,1,0)$-plate describes a *plain strain state* while the $(1,1,2)$-ansatz covers the *plain stress state* that is more appropriate here.

We also obtain some general hints for the finite element discretization. Of course, many facts are obvious, but surprisingly there are also counterintuitive consequences.

Section 2 provides a review of the displacement formulation of the Kirchhoff plate model in order to circumvent later some traps. Section 3 is concerned with the transition from the $(1,1,0)$-plate model to the $(1,1,2)$-plate. Since clamped plates are known to have boundary layers, we have to estimate them before the convergence analysis. This is one of the items not covered in [18]. The correctness of the model for thin plates is proven in Section 4 by the Theorem of Prager and Synge. In particular, a $t^{1/2}$ behavior shows that such a proof cannot be given by a power series in the thickness variable t. The extension to the Reissner–Mindlin plate is the topic of Section 5, and the remaining sections contain some aspects of the discretization and a posteriori error estimates.

2 Displacement formulation of the Kirchhoff plate

Let $\omega \subset \mathbb{R}^2$ be a smoothly bounded domain and $\Omega = \omega \times (-t/2, +t/2)$ be the reference configuration of the plate under consideration. The top and bottom surfaces are $\partial\Omega^{\pm} := \omega \times (\pm\frac{t}{2})$. The deformation of the plate under a body force f is given by the equations for the displacement field $u : \Omega \to \mathbb{R}^3$,

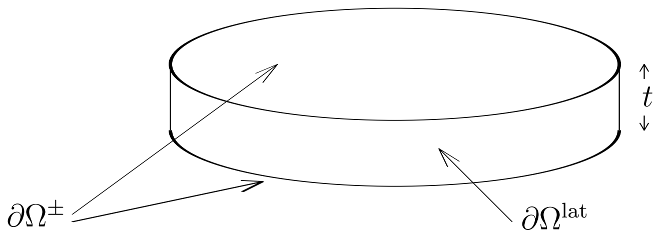

Figure 1: A plate and the three parts of its boundary

the strain tensor field $\varepsilon : \Omega \to \mathbb{R}^{3\times 3}_{\text{sym}}$, and the stress tensor field $\sigma : \Omega \to \mathbb{R}^{3\times 3}_{\text{sym}}$:

$$\begin{aligned} \varepsilon(u) &= \frac{1}{2}(\nabla u + (\nabla u)^T), \\ \sigma &= 2\mu\left(\varepsilon + \frac{\nu}{1-2\nu}(\operatorname{tr}\varepsilon)\delta_3\right), \\ \operatorname{div}\sigma &= -f. \end{aligned} \tag{2.1a}$$

Here, the Lamé constant μ and the Poisson ratio ν are material parameters, while δ_d is the $d\times d$ identity matrix. In addition, we have Neumann boundary conditions on the top and the bottom

$$\sigma \cdot n = g \quad \text{on } \partial\Omega^+ \cup \partial\Omega^- \tag{2.1b}$$

and Dirichlet conditions on the lateral boundary $\partial\Omega^{\text{lat}} := \partial\omega \times (-\frac{t}{2}, +\frac{t}{2})$ in the case of a hard clamped plate:

$$u = 0 \quad \text{on } \partial\Omega^{\text{lat}}. \tag{2.1c}$$

We restrict ourselves to plate bending and purely transversal loads:

$$f(x) = t^2\,(0,0,p(x_1,x_2)), \quad g(x_1,x_2,\pm t/2) = t^3\,(0,0,q^{\pm}(x_1,x_2)). \tag{2.2}$$

(See [3] for forces and tractions with non-zero components in the other directions.) Here the loads are scaled. Thus, (2.1) contains the classical equations associated to the variational form

$$\Pi(u) = \mu\left((\varepsilon(u),\varepsilon(u))_{0,\Omega} + \frac{\nu}{1-2\nu}(\operatorname{tr}\varepsilon(u),\operatorname{tr}\varepsilon(u))_{0,\Omega}\right) - \text{load}. \tag{2.3}$$

As usual, $(\cdot,\cdot)_{0,\Omega}$ denotes the inner product in $L_2(\Omega)$.

The $(1,1,2)$-Kirchhoff plate is described by the ansatz

$$\begin{aligned} u_i &= -x_3\,\partial_i w(x_1,x_2) \qquad \text{for } i=1,2,\\ u_3 &= w(x_1,x_2) + x_3^2\,W(x_1,x_2), \end{aligned} \tag{2.4}$$

and the strain tensor is obtained from the symmetric part of the gradient

$$\varepsilon = \begin{bmatrix} -x_3\,\partial_{11}w & & \mathit{symm.}\\ -x_3\,\partial_{12}w & -x_3\,\partial_{22}w & \\ \frac{1}{2}x_3^2\,\partial_1 W & \frac{1}{2}x_3^2\partial_2 W & 2x_3W \end{bmatrix}, \quad \operatorname{tr}\varepsilon = -x_3\Delta w + 2x_3W. \tag{2.5}$$

The associated stress tensor as given by the constitutive equation in (2.1a) is

$$\sigma_{\mathrm{KL}} = 2\mu \begin{bmatrix} -x_3\,\partial_{11}w & & \mathit{symm.}\\ -x_3\,\partial_{12}w & -x_3\,\partial_{22}w & \\ \frac{1}{2}x_3^2\,\partial_1 W & \frac{1}{2}x_3^2\partial_2 W & 2x_3W \end{bmatrix} + 2\mu\,\frac{\nu}{1-2\nu}(-x_3\Delta w + 2x_3W)\delta_3. \tag{2.6}$$

The integration over the thickness involves the integrals

$$\int_{-t/2}^{+t/2} x_3^2\mathrm{d}x_3 = \frac{1}{12}t^3 \quad\text{and}\quad \int_{-t/2}^{+t/2} x_3^4\mathrm{d}x_3 = \frac{1}{80}t^5. \tag{2.7}$$

Expressions like inner products for the middle surface ω are related to functions of two variables and to derivatives with respect to x_1 and x_2. Let

$$D^2w = \begin{bmatrix} \partial_{11}w & \partial_{12}w\\ \partial_{21}w & \partial_{22}w \end{bmatrix},$$

and we obtain with the ansatz (2.4)

$$\begin{aligned} \Pi(u) =& \frac{\mu}{12}t^3\Big((D^2w,D^2w)_{0,\omega} + 4\|W\|_{0,\omega}^2\\ &+\frac{\nu}{1-2\nu}\|\Delta w - 2W\|_{0,\omega}^2 + \frac{3}{40}t^2\|\nabla W\|_{0,\omega}^2\Big) - \text{load} \end{aligned}$$

with the load $t^3\int_\omega (p+q^+-q^-)w\,\mathrm{d}x_1\mathrm{d}x_2 =: t^3\int_\omega p_{\mathrm{total}}\,w\,\mathrm{d}x_1\mathrm{d}x_2$. Here the contribution of the quadratic term has been dropped, since it is of the order t^5. Next we note that

$$D^2w : D^2w = \sum_{i,k}(\partial_{ik}w)^2 = (\partial_{11}w + \partial_{22}w)^2 + 2\big((\partial_{12}w)^2 - \partial_{11}w\partial_{22}w\big)$$

and apply the identity $4W^2 + \frac{\nu}{1-2\nu}(z-2W)^2 = \frac{\nu}{1-\nu}z^2 + \frac{1-\nu}{1-2\nu}(2W - \frac{\nu}{1-\nu}z)^2$ with $z := \Delta w$, following [18]. Hence,

$$\begin{aligned}\Pi(u) =&\frac{\mu}{12}t^3\left(\|\Delta w\|_{0,\omega}^2 + \int_\omega 2[(\partial_{12}w)^2 - \partial_{11}w\partial_{22}w)]\mathrm{d}x_1\mathrm{d}x_2\right.\\ &\left.+\frac{\nu}{1-\nu}\|\Delta w\|_{0,\omega}^2 + \frac{1-\nu}{1-2\nu}\|2W - \frac{\nu}{1-\nu}\Delta w\|_{0,\omega}^2 + \frac{3}{40}t^2\|\nabla W\|_{0,\omega}^2\right)\\ &- t^3\int_\omega p_{\mathrm{total}}\, w\,\mathrm{d}x_1\mathrm{d}x_2\\ =&\frac{\mu}{12}t^3\left(\frac{1}{1-\nu}\|\Delta w\|_{0,\omega}^2 + \frac{1-\nu}{1-2\nu}\|2W - \frac{\nu}{1-\nu}\Delta w\|_{0,\omega}^2 + \frac{3}{40}t^2\|\nabla W\|_{0,\omega}^2\right.\\ &\left.+\int_\omega 2[(\partial_{12}w)^2 - \partial_{11}w\partial_{22}w)]\mathrm{d}x_1\mathrm{d}x_2\right) - t^3\int_\omega p_{\mathrm{total}}\, w\,\mathrm{d}x_1\mathrm{d}x_2. \qquad (2.8)\end{aligned}$$

The boundary conditions $u_1 = u_2 = 0$ on $\partial\Omega^{\mathrm{lat}}$ together with (2.4) imply $\nabla w = 0$ on $\partial\omega$. Since $w = 0$ on $\partial\omega$ means that the tangential component of the gradient vanishes, it suffices to have

$$\nabla w \cdot n = 0 \quad \text{on } \partial\omega.$$

Integration by parts yields $\int_\omega 2[(\partial_{12}w)^2 - \partial_{11}w\partial_{22}w)]\mathrm{d}x_1\mathrm{d}x_2 = 0$, and this integral can be dropped in (2.8). The minimization of Π leads to the so-called "plate equation" of Kirchhoff describing the bending of a thin plate occupying a plane domain ω which is clamped at its boundary $\partial\omega$:

$$\begin{aligned}\frac{\mu}{6(1-\nu)}\Delta^2 w &= p_{\mathrm{total}} && \text{in } \omega,\\ w &= \frac{\partial w}{\partial n} = 0 && \text{on } \partial\omega.\end{aligned} \qquad (2.9)$$

More precisely, the $(1,1,0)$-model, i.e. $W = 0$, yields (2.9) with a different coefficient in front of $\Delta^2 w$. The actual coefficient anticipates already some features of the $(1,1,2)$-model.

3 From the $(1,1,0)$-model to the $(1,1,2)$-model

We start with the $(1,1,0)$-model, i.e., we set $W = 0$. Since $1 + \frac{\nu}{1-\nu} + \frac{1-\nu}{1-2\nu}\left(\frac{\nu}{1-\nu}\right)^2 = \frac{1-\nu}{1-2\nu}$, it follows from (2.8) that here

$$t^{-3}\Pi(u) = \frac{\mu}{12}\frac{1-\nu}{1-2\nu}\|\Delta w\|_{0,\omega}^2 - (p_{\mathrm{total}}, w)_{0,\omega}\,. \qquad (3.1)$$

The shortcoming of (3.1) is obvious. The denominator of the coefficient in the first term tends to zero if $\nu \to 1/2$. The coercivity of the quadratic form becomes large for nearly incompressible material. Such a behavior is typical for a *plain strain state* and contradicts the fact that we have no Dirichlet boundary conditions on the top and on the bottom of the plate.

Turning to the $(1,1,2)$-model we consider both w and W as free. We insert a provisional step in which we ignore the Dirichlet boundary condition for W. If we ignore also the higher order term in (2.8) during the minimization, the minimum is attained for

$$2W = \frac{\nu}{1-\nu}\Delta w \tag{3.2}$$

and

$$t^{-3}\Pi(u) = \frac{\mu}{12}\,\frac{1}{1-\nu}\|\Delta w\|^2_{0,\omega} - (p_{\text{total}}, w)_{0,\omega} + \frac{3}{40}t^2\|\nabla W\|^2_{0,\omega}\,. \tag{3.3}$$

The main difference to (3.1) is the coefficient of the first term that describes the coercivity of the energy functional. It is consistent with a *plain stress state* that has a smaller stiffness whenever $\nu > 0$. The plate equation (2.9) is the Euler equation for the variational problem with the leading terms in (3.3).

Of course, the boundary condition $W = 0$ cannot be ignored. Morgenstern assumed that a suitable W can be obtained from the right-hand side of (3.2) by a cut-off next to the boundary [18], and a similar consideration can be found in [5]. A more precise treatment leads to a singularly perturbed problem. Fix w as the solution of the plate equation (2.9) and choose $W \in H^1_0(\omega)$ as the solution of the variational problem

$$\min_{W\in H^1_0(\omega)} \alpha\|W-\phi\|^2_{0,\omega} + t^2\|\nabla W\|^2_{0,\omega} \longrightarrow \min! \tag{3.4}$$

where $\phi := \frac{\nu}{2(1-\nu)}\Delta w \in L_2(\omega)$ and $\alpha := \frac{80}{3}\frac{1-\nu}{1-2\nu}$.

To obtain asymptotic error bounds of solutions of dimensionally reduced plate models with respect to the solution of the three dimensional problem, it is necessary to estimate the minimum of (3.4) in terms of t, as was proposed in [1]. There, asymptotic error bounds in terms of the plate thickness were found to depend on the *regularity of the solution w of* (2.9): specifically, in [1], $\phi \in H^1(\omega)$ was assumed. This is a realistic assumption if ω is either convex or smooth and $p_{total} \in H^{-1}(\omega)$ in (2.9). In case that ω has reentrant corners, or that $p_{total} \in H^{-2+s}(\omega)$ for some $0 < s < 1$, the arguments in [1] are not applicable, but the following result provides bounds on W.

Lemma 3.1 *Assume that ω is a Lipschitz polygon or a smooth domain and that $\phi \in H^s(\omega)$ for some $s \in [0,1]$, and that the Dirichlet problem for the Poisson equation in ω admits a shift theorem of order s. Then, for any $0 < t \leq 1$, the unique solution W of the variational problem*

$$\min_{W \in H_0^1(\omega)} \{t^2 \|\nabla W\|_{0,\omega}^2 + \|W - \phi\|_{0,\omega}^2\} \tag{3.5}$$

satisfies the following a priori estimates with constants independent of t:

a) if $\phi \in L^2(\omega)$, then

$$t^2 \|\nabla W\|_{0,\omega}^2 + \|W - \phi\|_{0,\omega}^2 \leq \|\phi\|_{0,\omega}^2 , \tag{3.6}$$

b) if $\phi \in H^s(\omega)$ with $0 < s < 1/2$, there exists a constant $c(s,\omega) > 0$ such that

$$t^2 \|\nabla W\|_{0,\omega}^2 + \|W - \phi\|_{0,\omega}^2 \leq c(s,\omega) t^{2s} \|\phi\|_{s,\omega}^2 , \tag{3.7}$$

c) if $\phi \in H^{1/2}(\omega)$, then for any $\varepsilon > 0$ there exists a constant $c(\varepsilon,\omega) > 0$ such that

$$t^2 \|\nabla W\|_{0,\omega}^2 + \|W - \phi\|_{0,\omega}^2 \leq c(\varepsilon,\omega) t^{1-\varepsilon} \|\phi\|_{\frac{1}{2},\omega}^2 ,$$

d) if $\phi \in H^1(\omega)$, there exists a constant $c(\omega) > 0$ such that

$$t^2 \|\nabla W\|_{0,\omega}^2 + \|W - \phi\|_{0,\omega}^2 \leq c(\omega)\, t \|\phi\|_{1,\omega}^2 .$$

Proof. Consider first the *case $s = 0$:* The minimum (3.5) is smaller than the value that is attained at the trivial candidate $W_0 = 0$. This proves (3.6) and thus assertion a).

The case d) was already addressed in [1]; we give a self-contained argument for completeness here. The unique minimizer $W \in H_0^1(\omega)$ of the variational problem (3.5) is the weak solution of the boundary value problem

$$-t^2 \Delta W + W = \phi \ \text{ in } \ \omega, \qquad W = 0 \ \text{ on } \ \partial\omega. \tag{3.8}$$

Multiplication of (3.8) by the test function $-\Delta W = -t^{-2}(W-\phi) \in L^2(\omega)$ and integration by parts yields, using $\phi \in H^1(\omega)$, that

$$t^2\|\Delta W\|_{0,\omega}^2 + \|\nabla W\|_{0,\omega}^2 = -\int_\omega \phi \Delta W dx \qquad (3.9)$$

$$= \int_\omega \nabla\phi \cdot \nabla W dx - (\gamma_0\phi, \gamma_1 W)_{0,\partial\omega} \qquad (3.10)$$

where γ_0 denotes the trace and γ_1 the normal derivative operator, respectively. Moreover, $(\cdot,\cdot)_{0,\partial\omega}$ denotes the $L^2(\partial\omega)$ inner product.

We focus on $\|\gamma_1 W\|_{0,\partial\omega}$. Since in this case $W \in H^2(\omega) \cap H_0^1(\omega)$, we find $\partial_i W \in H^1(\omega)$, $i = 1, 2$, and we recall the multiplicative trace inequality

$$\|\gamma_0\psi\|_{0,\partial\omega}^2 \le c(\omega)\left(\|\psi\|_{0,\omega}^2 + \|\psi\|_{0,\omega}\|\nabla\psi\|_{0,\omega}\right) \qquad \psi \in H^1(\omega). \qquad (3.11)$$

An elementary proof is provided, e.g., in [12]. [We note that there would be a faster proof if the trace operator γ_0 were continuous from $H^{1/2}(\omega) \to L^2(\partial\omega)$ which is, however, known to be false; see [14] for a counterexample.]

For applying (3.11) with $\psi = \partial_i W$ we note that $\gamma_0\nabla W = \nabla W|_{\partial\omega} \in L^2(\partial\omega)^2$. Since the exterior unit normal vector $n(x)$ on a Lipschitz boundary $\partial\omega$ belongs to $L^\infty(\partial\omega)^2$ by Rademacher's Theorem, we have $\gamma_1 W = \frac{\partial W}{\partial n}|_{\partial\omega} = n \cdot \gamma_0 \nabla W$ almost everywhere on $\partial\omega$. The $H^2(\omega)$ regularity of the Dirichlet problem for the Poisson equation on smooth or convex domains and the a priori estimate $\|W\|_{2,\omega} \le c\|\Delta W\|_{0,\omega}$ are used with (3.11) to estimate

$$\begin{aligned}
\|\gamma_1 W\|_{0,\partial\omega}^2 &= \|n \cdot \gamma_0\nabla W\|_{0,\partial\omega}^2 \le \|\gamma_0\nabla W\|_{0,\partial\omega}^2 \\
&\le C_1(\omega)\left(\|\nabla W\|_{0,\omega}^2 + \|\nabla W\|_{0,\omega}\|W\|_{2,\omega}\right) \\
&\le C_2(\omega)\left(\|\nabla W\|_{0,\omega}^2 + \|\nabla W\|_{0,\omega}\|\Delta W\|_{0,\omega}\right).
\end{aligned}$$

Inserting this bound into (3.10) and recalling $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for $a, b \ge 0$ we obtain

$$\begin{aligned}
t^2\|\Delta W\|_{0,\omega}^2 + \|\nabla W\|_{0,\omega}^2 &\le \|\nabla\phi\|_{0,\omega}\|\nabla W\|_{0,\omega} + \|\gamma_0\phi\|_{0,\partial\omega}\|\gamma_1 W\|_{0,\partial\omega} \\
&\le \frac{1}{4}\|\nabla W\|_{0,\omega}^2 + C_3(\omega)\left(\|\nabla\phi\|_{0,\omega}^2 + \|\gamma_0\phi\|_{0,\partial\omega}^2\right) \\
&+ C_4(\omega)\|\gamma_0\phi\|_{0,\partial\omega}\|\nabla W\|_{0,\omega}^{1/2}\|\Delta W\|_{0,\omega}^{1/2} \\
&\le \frac{1}{4}\|\nabla W\|_{0,\omega}^2 + C_3(\omega)\left(\|\nabla\phi\|_{0,\omega}^2 + \|\gamma_0\phi\|_{0,\partial\omega}^2\right) \\
&+ C_5(\omega)t^{-1}\|\gamma_0\phi\|_{0,\partial\omega}^2 + \frac{1}{4}\|\nabla W\|_{0,\omega}^2 + \frac{t^2}{2}\|\Delta W\|_{0,\omega}^2.
\end{aligned}$$

Absorbing the terms with W on the right-hand side into the left-hand side, multiplying the resulting inequality by t^2, and replacing $t^2\Delta W$ by $W - \phi$ results in

$$\|W-\phi\|_{0,\omega}^2 + t^2\|\nabla W\|_{0,\omega}^2 \leq C_6(\omega)t^2 \left(\|\nabla\phi\|_{0,\omega}^2 + \|\gamma_0\phi\|_{0,\partial\omega}^2\right) + C_7(\omega)t\|\gamma_0\phi\|_{0,\partial\omega}^2.$$

Recalling the assumption $\phi \in H^1(\omega)$ and referring to the multiplicative trace inequality (3.11) completes the proof in the case $s = 1$, i.e. d).

The proof of the intermediate case $0 < s \leq 1/2$, i.e., of assertions b) and c), cannot be obtained by simple interpolation and is shifted to the appendix. ∎

Remark 3.2 *(On the dependence of s on ω)*
We will show below that Lemma 3.1 implies estimates of the modeling error in plate models as the plate thickness t tends to zero. The rate s of convergence depends on which of the cases b), c) or d) is applicable. This, in turn, depends on the regularity parameters $s'(\omega)$ and $s^*(\omega)$ below and thus only on the geometry of ω.

Specifically, let ω be a bounded polygonal domain, and denote by $\theta^*(\omega) \in (0, 2\pi)$ the largest interior opening angle of ω at its vertices. Then, from the theory of singularities of elliptic problems (cf. e.g. [13] and the references there), the regularity of the Poisson equation $s^*(\omega)$ can be any number satisfying

$$0 < s^*(\omega) < \pi/\theta^*(\omega). \tag{3.12}$$

Analogously, $s'(\omega)$ is defined by the regularity of the Dirichlet problem (2.9) for the biharmonic equation in ω and therefore determined by the corner singularities. It can be any number satisfying

$$0 < s'(\omega) < \alpha'(\omega) \tag{3.13}$$

where $\alpha'(\omega)$ is the smallest positive real part of the roots of

$$\alpha \in \mathbb{C}: \qquad \alpha^2 \sin^2(\theta^*(\omega)) = \sin^2(\alpha\theta^*(\omega)). \tag{3.14}$$

We will use Lemma 3.1 with $s = \min\{s^*, s'\}$. In the case of a convex polygon ω we have $\theta^* < \pi$, and the choices $s' = s^* = 1$ are admissible. Under the regularity assumption $p_{total} \in H^{-1}(\omega)$ *the solution w of (2.9) satisfies $\Delta w \in H^1(\omega)$ and we are in case d) of Lemma 3.1. Below, we will show that* *then the modelling error in* (2.9) *for the pure bending of a hard clamped plate behaves* $O(t^{1/2})$ *as* $t \to 0$*, as asserted in [1].*

Nevertheless, Lemma 3.1 and our subsequent considerations yield the convergence order $O(t^{1/2})$ *in plates with polygonal midsurfaces* ω *that have reentrant corners provided that* $p_{total} \in H^{-1/2}(\omega)$*. Specifically, we obtain from* (3.12) *and* (3.14) *that* $s'^* > 1/2$ *for any* $\theta^* < 2\pi$*. Hence, for* $p_{total} \in H^{-3/2}(\omega)$*, we have the a priori estimate*

$$\|\Delta w\|_{H^s(\omega)} \le C(\omega)\|p_{total}\|_{H^{-2+s}(\omega)}$$

for all $0 \le s \le 1/2 < \min\{s'(\omega), s^*(\omega)\}$*. From cases b) and c) of Lemma 3.1 we find that for arbitrary small* $\varepsilon > 0$ *there exists* $c(s,\varepsilon,\omega) > 0$ *such that for all* $0 < t \le 1$ *holds*

$$t\,\|\nabla W\|_{0,\omega} + \|2W - \frac{\nu}{1-\nu}\Delta w\|_{0,\omega} \le c(s,\varepsilon,\omega)t^{\min\{s,1/2-\varepsilon\}}\|p_{total}\|_{H^{-2+s}(\omega)}\,. \tag{3.15}$$

Remark 3.3 *A proof of Lemma 3.1 by asymptotic expansions of* W *in* (3.9) *with respect to* t *seems elusive, since the data* ϕ *and* ω *lack even the regularity for defining the first nontrivial term of the asymptotic expansions in [2] in the cases which are of main interest to us.*

Consider the 3,3-component of the stress tensor

$$\left(\sigma_{\text{KL}}^{(1,1,2)}\right)_{33} = 2\mu\frac{1-\nu}{1-2\nu}x_3\left(2W - \frac{\nu}{1-\nu}\Delta w\right).$$

From (3.15) we conclude that $\left\|\left(\sigma_{\text{KL}}^{(1,1,2)}\right)_{33}\right\|_{0,\Omega} = O\left(t^{1/2}\right)\left\|\left(\sigma_{\text{KL}}^{(1,1,0)}\right)_{33}\right\|_{0,\Omega}$, i.e., the L_2 norm of σ_{33} is reduced at least by a factor of $t^{1/2}$. The change from the $(1,1,0)$-model to the $(1,1,2)$-model for small t induces a correction of the 3,3-component of the stress tensor that is of the same order as $\sigma_{33}^{(1,1,0)}$, more precisely,

$$\left\|\left(\sigma_{\text{KL}}^{(1,1,2)} - \sigma_{\text{KL}}^{(1,1,0)}\right)_{3,3}\right\|_{0,\Omega} \ge \left\|\left(\sigma_{\text{KL}}^{(1,1,0)}\right)_{3,3}\right\|_{0,\Omega}\left(1 - O(t^{1/2})\right).$$

This is an essential contribution when we move from the $(1,1,0)$-model to the $(1,1,2)$-model.

Note that the physical solution of (2.1) exhibits boundary layers (see [2]) at the lateral boundary with shrinking thickness $t \to 0$ which are incorporated into the two-dimensional model via the function W.

The scaling of the loads as in (2.2) makes the solution of the plate equation independent of the thickness, and (2.6) implies that

$$\|\sigma_{\rm KL}^{(1,1,2)}\|_{0,\Omega} = c\, t^{3/2}(1+o(1)). \tag{3.16}$$

We will see in the next section that the stress tensor for the three-dimensional problem and the equilibrated stress tensor in (4.3) below show the same behavior for thin plates. (Of course, statements on relative errors are independent of the scaling in (2.2)).

We compare (3.1) and (3.3); see also Table 1 in Section 6.2. The higher order term in (3.3) is estimated by Lemma 3.1, and it follows that

$$\Pi(u^{(1,1,0)}) = \frac{1-2\nu}{(1-\nu)^2}\,\Pi(u^{(1,1,2)})[1+O(t)]. \tag{3.17}$$

Note that the *correction factor*

$$\frac{1-2\nu}{(1-\nu)^2} \tag{3.18}$$

equals $0.4/0.49 \approx 0.82$ if we have a material with $\nu = 0.3$. This factor was already incorporated in the plate equation (2.9). In the literature a constant shear correction factor κ is often found, with the value $\kappa = 5/6$ going back to E. Reissner [20]. Without computing the function $x_3^2 W$ we obtain from Lemma 3.1 the following information on the resulting stress tensor in the $(1,1,2)$ model

$$\begin{aligned}
\sigma_{\rm KL} &= 2\mu \begin{bmatrix} -x_3\,\partial_{11}w & & \mathit{symm.} \\ -x_3\,\partial_{12}w & -x_3\,\partial_{22}w & \\ \frac12 x_3^2\,\partial_1 W & \frac12 x_3^2\partial_2 W & 0 \end{bmatrix} - 2\mu\frac{\nu}{1-\nu}\begin{bmatrix} x_3 & & \\ & x_3 & \\ & & 0\end{bmatrix}\Delta w \\
&\quad + 2\mu \begin{bmatrix} \frac{\nu}{1-2\nu}\,x_3 & & \mathit{symm.} \\ 0 & \frac{\nu}{1-2\nu}\,x_3 & \\ 0 & 0 & \frac{1-\nu}{1-2\nu}\,x_3 \end{bmatrix} \left(2W - \frac{\nu}{1-\nu}\Delta w\right) \\
&= 2\mu \begin{bmatrix} -x_3\,\partial_{11}w & & \mathit{symm.} \\ -x_3\,\partial_{12}w & -x_3\,\partial_{22}w & \\ 0 & 0 & 0 \end{bmatrix} - 2\mu\,\frac{\nu}{1-\nu}\begin{bmatrix} x_3 & & \\ & x_3 & \\ & & 0\end{bmatrix}\Delta w \\
&\quad + O(t^{1/2})\|\sigma_{\rm KL}\|_{0,\Omega}\,.
\end{aligned} \tag{3.19}$$

The relation is to be understood in the sense that the L_2 norm of the higher order terms that result from the contribution of W are $O(t^{1/2}\|\sigma_{\rm KL}\|_{0,\Omega})$.

4 Justification by the two-energies principle

The a priori assumptions in the preceding sections will now be justified. To this end, the internal stored energy will be determined in terms of strains or of stresses. The relation between those quantities is given by the elasticity tensor and its inverse, i.e., the *compliance tensor* $\mathcal{A}$; cf. (2.1a):

$$\mathcal{A}\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\nu}{1+\nu}(\mathrm{tr}\,\sigma)\delta_3\right), \qquad \mathcal{A}^{-1}\varepsilon = 2\mu\left(\varepsilon + \frac{\nu}{1-2\nu}(\mathrm{tr}\,\varepsilon)\delta_3\right).$$

The associated energy norms are

$$\|\sigma\|_{\mathcal{A}}^2 = \int_\Omega \mathcal{A}\sigma : \sigma\,\mathrm{d}x, \qquad \|\varepsilon\|_{\mathcal{A}^{-1}}^2 = \int_\Omega \mathcal{A}^{-1}\varepsilon : \varepsilon\,\mathrm{d}x.$$

The theorem of Prager and Synge [19] is applied to the differential equation with Dirichlet boundary conditions on $\partial\Omega^{\rm lat}$ and Neumann conditions on $\partial\Omega^+ \cup \partial\Omega^-$. The solutions of the $3D$ problem (2.1) are denoted by u^* and σ^*.

Theorem 4.1 *(Prager and Synge) Let $\sigma \in H(\mathrm{div},\Omega)$ satisfy the equilibrium condition and the Neumann boundary condition*

$$\begin{aligned} \mathrm{div}\,\sigma &= -f \quad \text{in } \Omega, \\ \sigma\cdot n &= g \quad \text{on } \partial\Omega^+\cup\partial\Omega^- \end{aligned}$$

and $u \in H^1(\Omega)$ satisfy the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega^{lat}.$$

Then

$$\|\varepsilon(u)-\varepsilon(u^*)\|_{\mathcal{A}^{-1}}^2 + \|\sigma-\sigma^*\|_{\mathcal{A}}^2 = \|\sigma - \mathcal{A}^{-1}\varepsilon(u)\|_{\mathcal{A}}^2\,. \tag{4.1}$$

The proof is based on the orthogonality relation $(\varepsilon(u)-\varepsilon(u^*),\sigma-\sigma^*)_{0,\Omega} = 0$ and can be found, e.g., in [19, 1, 6]. It reflects the fact that

$$[\Pi(u)-\Pi(u^*)] + [\Pi^c(\sigma^*)-\Pi^c(\sigma)] = \Pi(u)-\Pi^c(\sigma).$$

In this context the following corollary will be useful.

Corollary 4.2 *Let the assumptions of Theorem 4.1 prevail and $v \in H^1(\Omega)$ satisfy the boundary condition $v = 0$ on $\partial\Omega^{lat}$. If $\Pi(u) \le \Pi(v)$, then*

$$\|\varepsilon(u) - \varepsilon(u^*)\|^2_{\mathcal{A}^{-1}} + \|\sigma - \sigma^*\|^2_{\mathcal{A}} \le \|\sigma - \mathcal{A}^{-1}\varepsilon(v)\|^2_{\mathcal{A}}.$$

We start with the case of a body force as specified in (2.2) and zero tractions

$$\begin{array}{lll} \operatorname{div}\sigma & = -t^2\,(0,0,p(x_1,x_2)) & \text{in } \Omega, \\ \sigma\cdot n & = 0 & \text{on } \partial\Omega^+ \cup \partial\Omega^-. \end{array} \tag{4.2}$$

Following [18] we derive an appropriate equilibrated stress tensor from the solution of the plate equation (2.9). Set

$$\sigma_{\rm eq} = \begin{bmatrix} 12x_3M_{11} & 12x_3M_{12} & -(6x_3^2-\frac{3}{2}t^2)Q_1 \\ 12x_3M_{12} & 12x_3M_{22} & -(6x_3^2-\frac{3}{2}t^2)Q_2 \\ -(6x_3^2-\frac{3}{2}t^2)Q_1 & -(6x_3^2-\frac{3}{2}t^2)Q_2 & -(2x_3^3-\frac{1}{2}x_3t^2)\,p \end{bmatrix} \tag{4.3}$$

with $M : \omega \to \mathbb{R}^{2\times 2}_{\rm sym}$ and $Q : \omega \to \mathbb{R}^2$ given by

$$\begin{aligned} M_{ik} &:= -\frac{\mu}{6}\left(\partial_{ik}w + \frac{\nu}{1-\nu}\,\delta_{ik}\,\Delta w\right), \\ Q_i &:= (\operatorname{div} M)_i = -\frac{\mu}{6(1-\nu)}\partial_i\Delta w. \end{aligned} \tag{4.4}$$

It follows from (2.9) that $\operatorname{div} Q = -\frac{\mu}{6(1-\nu)}\Delta^2 w = -p$, and

$$\begin{aligned} (\operatorname{div}\sigma_{\rm eq})_3 &= -(6x_3^2 - \frac{3}{2}t^2)\operatorname{div} Q - (6x_3^2 - \frac{1}{2}t^2)\,p = -t^2 p, \\ (\operatorname{div}\sigma_{\rm eq})_i &= 12x_3(\partial_1 M_{i1} + \partial_2 M_{i2}) - 12x_3\,Q_i = 0 \qquad \text{for } i = 1,2. \end{aligned}$$

Thus the assumptions (4.2) are verified. These relations and (2.6) yield

$$\begin{aligned} &\sigma_{\rm eq} - \mathcal{A}^{-1}\varepsilon(u^{(1,1,2)}) \\ &= \frac{\mu}{6(1-\nu)} \begin{bmatrix} 0 & 0 & (6x_3^2-\frac{3}{2}t^2)\partial_1\Delta w - 6(1-\nu)x_3^2\partial_1 W \\ & 0 & (6x_3^2-\frac{3}{2}t^2)\partial_2\Delta w - 6(1-\nu)x_3^2\partial_2 W \\ \textit{symm.} & & -x_3(2x_3^2-\frac{1}{2}t^2)\Delta^2 w \end{bmatrix} \\ &- 2\mu\frac{\nu}{1-2\nu}x_3(2W - \frac{\nu}{1-\nu}\Delta w) \begin{bmatrix} \nu & & \\ & \nu & \\ & & 1-\nu \end{bmatrix}. \end{aligned}$$

Obviously, $\|\sigma_{\rm eq}\|_{0,\Omega} = O\left(t^{3/2}\right)$; cf. (3.16). Using Lemma 3.1, (2.7), and $\int_{-t/2}^{t/2}(6x_3^2 - \frac{3}{2}t^2)^2 {\rm d}x_3 = O\left(t^5\right)$ we end up with

$$\|\sigma_{\rm eq} - \mathcal{A}^{-1}\varepsilon(u^{(1,1,2)})\|_{0,\Omega} = O(t^{1/2})\,\|\sigma_{\rm eq}\|_{0,\Omega}. \tag{4.5}$$

Since $2\mu\|\cdot\|_{\mathcal{A}}^2 \le \|\cdot\|_{0,\Omega}^2 \le 2\mu(1+\frac{3\nu}{1-2\nu})\|\cdot\|_{\mathcal{A}}^2$, it follows that

$$\|\sigma_{\rm eq} - \mathcal{A}^{-1}\varepsilon(u^{(1,1,2)})\|_{\mathcal{A}} = O(t^{1/2})\|\sigma_{\rm eq}\|_{\mathcal{A}}.$$

Now the two-energies principle (Theorem 4.1) yields

$$\|\varepsilon(u_{\rm KL}^{(1,1,2)}) - \varepsilon(u^*)\|_{\mathcal{A}^{-1}}^2 + \|\sigma_{\rm eq} - \sigma^*\|_{\mathcal{A}}^2 = O(t)\,\|\sigma^*\|_{\mathcal{A}}^2, \tag{4.6}$$

and the model error becomes small for thin plates. This is summarized in the following theorem.

Theorem 4.3 *The model error of the* $(1,1,2)$*-Kirchhoff plate model becomes small for thin plates*

$$\|\varepsilon(u_{KL}^{(1,1,2)}) - \varepsilon(u^*)\|_{\mathcal{A}^{-1}} + \|\sigma_{eq} - \sigma^*\|_{\mathcal{A}} = O(t^{1/2})\|\sigma^*\|_{\mathcal{A}}, \tag{4.7}$$

Here we have implicitly assumed full regularity. From Remark 3.2 we know how the exponent has to be adapted in the case of plates with reentrant corners.

Remark 4.4 *We get a solution for the displacements, strains, and stresses with a relative model error of order $O(t^{1/2})$ without computing an (approximate) solution of (3.4) for the quadratic term $x_3^2 W$. Let $u^{(1,1,0*)}$ denote the displacement that we obtain from $u^{(1,1,2)}$ when we drop the quadratic term. It is obtained from the solution of the plate equation* (2.9)*. Obviously, the L_2 error of $u^{(1,1,0*)}$ differs from that of $u^{(1,1,2)}$ by a term of higher order. Next, we can set (see (2.5), (3.2))*

$$\varepsilon_{ij} = \begin{cases} x_3 \frac{\nu}{1-\nu}\Delta w & \textit{if } i,j = 3,3, \\ \varepsilon_{i,j}(u^{(1,1,0*)}) & \textit{otherwise.} \end{cases}$$

Finally the stresses may be taken from σ_{KL} in (3.19) *or from σ_{eq} in* (4.3)*.*

Note that the stress tensor is not derived from $\varepsilon(u^{(1,1,0*)})$ and the original material law. This may have consequences if the plate is connected to 3D-elements in finite element computations – and also for a posteriori error estimates.

The fractional power of t in the model error is also a hint that there may be complications with power series in the thickness variable.

Remark 4.5 *A consequence of* (3.17) *is also worth to be noted. We have* $\Pi\left(u^{(1,1,0*)}\right) = \Pi\left(u^{(1,1,2)}\right)(1+O(t))$. *The estimate shows that only a portion of order* $O(t^1)$ *of the energy can be absorbed by the boundary layer of the plate. It is known that the portion can be larger in shells.*

We turn to the pure traction problem

$$\begin{aligned} \operatorname{div}\sigma &= 0 && \text{in } \Omega, \\ \sigma\cdot n &= t^3\,(0,0,q^+(x_1,x_2)) && \text{on } \partial\Omega^+, \\ \sigma\cdot n &= 0 && \text{on } \partial\Omega^-. \end{aligned} \tag{4.8}$$

It can be handled similarly. Only the kinematical factor in $\sigma_{\mathrm{eq},33}$ has to be adapted:

$$\sigma_{\mathrm{eq},33} = (-2x_3^3 + \frac{3}{2}x_3t^2 + \frac{1}{2}t^3)\,q^+.$$

Obviously,

$$-2x_3^3 + \frac{3}{2}x_3t^2 + \frac{1}{2}t^3 = \begin{cases} t^3 & \text{for } x = +\frac{t}{2}, \\ 0 & \text{for } x = -\frac{t}{2}. \end{cases}$$

Therefore the boundary conditions from (4.8) are satisfied. The equations $(\operatorname{div}\sigma_{\mathrm{eq}})_i = 0$ for $i = 1,2$ are obtained as above. Finally,

$$(\operatorname{div}\sigma_{\mathrm{eq}})_3 = -(6x_3^2 - \frac{3}{2}t^2)\operatorname{div}Q + (-6x_3^2 + \frac{3}{2}t^2)\,q^+ = 0.$$

We have an equilibrated stress tensor again, and the relative error is of the order $O(t^{1/2})$ as before.

5 Extension to the Reissner–Mindlin plate

Recently, Alessandrini et al [1] provided a justification of the Reissner–Mindlin plate in the framework of mixed methods. We will see that we obtain the justification with the displacement formulation faster by an extension of the results from the preceding section, and we cover also the $(1,1,0)$-model that was not analyzed in [1]. The assumption (2.2) concerning vertical loads implies that the differences between the models are smaller than expected, provided that we measure them in terms of the energy norm.

Since there are several variants called by the same name, we have to be more precise. We consider the displacement formulation with the rotations θ_i no longer fixed by the Kirchhoff hypothesis:

$$\begin{aligned} u_i &= -x_3\,\theta_i(x_1,x_2) \qquad \text{for } i=1,2, \\ u_3 &= w(x_1,x_2) + x_3^2\,W(x_1,x_2) \end{aligned} \tag{5.1}$$

and the boundary conditions $w = W = \theta_i = 0$ on $\partial\omega$. Clearly, (5.1) covers the $(1,1,2)$-model, and we have the $(1,1,0)$-model if $W = 0$. The associated stress tensor is

$$\begin{aligned} \sigma_{\mathrm{RM}} = 2\mu &\begin{bmatrix} -x_3\,\partial_1\theta_1 & & symm. \\ -\frac{1}{2}x_3\,(\partial_1\theta_2+\partial_2\theta_1) & -x_3\,\partial_2\theta_2 & \\ \frac{1}{2}(\partial_1 w-\theta_1)+\frac{1}{2}x_3^2\,\partial_1 W & \frac{1}{2}(\partial_2 w-\theta_2)+\frac{1}{2}x_3^2\partial_2 W & 2x_3W \end{bmatrix} \\ &+ 2\mu\,\frac{\nu}{1-2\nu}\begin{bmatrix} x_3 & & \\ & x_3 & \\ & & x_3 \end{bmatrix}(-\operatorname{div}\theta + 2W). \end{aligned} \tag{5.2}$$

The minimization of the energy leads to a smaller value for the Reissner–Mindlin plate than for the Kirchhoff plate. It follows from Theorem 4.3 and Corollary 4.2 that also

$$\|\varepsilon(u_{\mathrm{RM}}^{(1,1,2)}) - \varepsilon(u^*)\|_{\mathcal{A}^{-1}} + \|\sigma_{\mathrm{eq}} - \sigma^*\|_{\mathcal{A}} = O(t^{1/2})\|\sigma^*\|_{\mathcal{A}}. \tag{5.3}$$

in accordance with the results in [3]. The computation of the quadratic term is not required if one proceeds also here in the spirit of Remark 4.4.

More interesting is the question whether the $(1,1,0)$-model of the Reissner–Mindlin plate has a substantial better behavior in the thin plate limit than the Kirchhoff plate of the same order. This was not discussed in [1]. We will conclude from the results in the preceding section that it is indeed not true again.

Proposition 5.1 *Let $0 < \nu < 1/2$. Then the $(1,1,0)$-model of the Reissner–Mindlin plate with non-zero load has a solution with*

$$\liminf_{t\to 0} \frac{\|\mathcal{A}^{-1}\varepsilon(u_{RM}^{(1,1,0)}) - \sigma^*\|_{\mathcal{A}}}{\|\sigma^*\|_{\mathcal{A}}} > 0. \tag{5.4}$$

Proof. Set $\sigma_{\rm RM} = \mathcal{A}^{-1}\varepsilon(u_{\rm RM}^{(1,1,0)})$ and suppose that

$$\|\sigma_{\rm RM} - \sigma^*\|_{\mathcal{A}} = o(1)\|\sigma^*\|_{\mathcal{A}} \tag{5.5}$$

holds for $t \to 0$. Theorem 4.3 and (3.16) yield

$$\|\sigma^*\|_{\mathcal{A}} = c\,t^{3/2}(1 + o(1)).$$

Let $\sigma_{\rm eq}$ be given by (4.3), where $w = w_{\rm pl}$ denotes the solution of the plate equation (2.9). In particular, the nonzero load implies $\Delta w_{\rm pl} \neq 0$. We know from Theorem 4.3 that $\|\sigma_{\rm eq} - \sigma^*\|_{\mathcal{A}} = O(t^{1/2})\|\sigma^*\|_{\mathcal{A}}$, and obtain from the Theorem of Prager and Synge the equality

$$\|\varepsilon(u_{\rm RM}^{(1,1,0)}) - \varepsilon(u^*)\|^2_{\mathcal{A}^{-1}} + \|\sigma_{\rm eq} - \sigma^*\|^2_{\mathcal{A}} = \|\sigma_{\rm eq} - \mathcal{A}^{-1}\varepsilon(u_{\rm RM}^{(1,1,0)})\|^2_{\mathcal{A}}\,.$$

Using again $\|\sigma_{\rm eq} - \sigma^*\|_{\mathcal{A}} = O(t^{1/2})\|\sigma^*\|_{\mathcal{A}}$, the hypothesis (5.5) implies that

$$\|\sigma_{\rm eq} - \sigma_{\rm RM}\|_{\mathcal{A}} = o(1)\|\sigma^*\|_{\mathcal{A}}.$$

The (3,3) component of $\sigma_{\rm eq}$ is a term of higher order in t, and we have from (5.2) (by setting $W = 0$ therein) that

$$2\mu\|x_3\,{\rm div}\,\theta\|_{0,\Omega} = O(t)\|\sigma^*\|_{\mathcal{A}}.$$

On the other hand, we conclude from (5.5) that the diagonal components for $i = 1, 2$ have to satisfy the conditions

$$2\mu\|x_3(\partial_i\theta_i - \partial_{ii}w_{\rm pl})\|_{0,\Omega} = o(1)\|\sigma^*\|_{\mathcal{A}}.$$

The triangle inequality yields $2\mu\|x_3\Delta w_{\rm pl}\|_{\mathcal{A}^{-1}} = o(1)\|\sigma^*\|_{\mathcal{A}}$ and eventually $\|\Delta w_{\rm pl}\|_{0,\omega} = o(1)$. Now there is a contradiction to $\Delta w_{\rm pl} \neq 0$. ∎

So it is not surprising that in the analysis of beams also the $(1, 2)$-model and not the $(1, 0)$-model is used [7].

6 Closing Remarks

6.1 A posteriori estimates of the model error

The two-energies principle and the theorem of Prager and Synge have been used for efficient *a posteriori error estimates* of other elliptic problems; see, e.g., [6, 8]. Since the principle was used successfully in the preceding sections for *a priori error estimates*, it is expected to be also a good candidate for deriving a posteriori error estimates of the model error. Remark 4.4, however, contains already a hint that special care is required.

Let u be an approximate solution of the displacement derived from w or the pair (w, θ) after solving the associated plate equations by a finite element method. We know from the a priori estimates that the relative error is of the order $O(t^{1/2})$. If we derive an error estimate for u via the two-energies principle directly, in principle, we compute a stress tensor σ from $\varepsilon(u)$ with the *original material law* in (2.1a) and compare it with an equilibrated stress tensor. This process, however, is implicitly performed within the framework of the $(1,1,0)$-model. The resulting bound cannot be better than the error of the $(1,1,0)$-model and cannot be efficient for $t \to 0$ since $\|\mathcal{A}^{-1}\varepsilon(u) - \sigma_{\mathrm{eq}}\|_{\mathcal{A}} / \|\sigma_{\mathrm{eq}}\|_{\mathcal{A}}$ does not converge to zero as $t \to 0$ (cf. Proposition 5.1).

The consequence is clear. We have to compute an approximate solution of the higher order term $x_3^2 W(x_1, x_2)$ if we want an efficient a posteriori error estimate by the two-energies principle. On the other hand, we will obtain then reliable estimates directly from the two-energy principle which are independent of a priori assumptions [21].

To compute W, the efficient numerical solution of the singularly perturbed problem (3.8) is necessary. The regularity and finite flement approximation of this problem is well understood; we refer to [16] for details on the design of FE approximations of W which converge exponentially, uniformly in the plate-thickness parameter t: appropriate finite element meshes are the *admissible boundary layer meshes* in the spirit of [16]. They have one layer of anisotropic, so-called "needle elements" of width $O(t)$ at the boundary.

6.2 Computational aspects

The $(1,1,0)$-models with appropriate shear correction factors and the $(1,1,2)$-models without correction lead to a relative error of $O(t^{1/2})$; cf. also [3] for the Reissner–Mindlin plate. Of course, a smaller error within the $O(t^{1/2})$ behavior is expected for computations with the $(1,1,2)$-ansatz.

Finite element approximations of w, θ, and W often serve for the discretization in the x_1, x_2 direction. Although the quadratic term $x_3^2 W$ may be considered only as a correction of the popular plate models, its finite element discretization requires more effort than it looks at first glance. It contains a big portion of the boundary layer, and we get only an improvement to the simplest models if the finite element solutions are able to resolve the layer.

Another short comment refers to computations when plates are connected with a body that is not considered as thin. More precisely, the total domain consists of two parts. The first part is modeled as a plate, while the second one is regarded as a 3-dimensional body. In order to avoid complications at the interface, a linear ansatz in the thickness direction, i.e., a $(1,1,1)$-model is used for the plate [24]. Since there is also the tendency to return to the full 3D models, the following question arises.

Table 1: Scaled coercivity constant of the plate model

model	factor of $\frac{\mu}{12}t^3\|\Delta w\|^2$ in the energy
$(1,1,0)$-model	$\frac{1-\nu}{1-2\nu}$
$(1,1,2)$-model	$\frac{1}{1-\nu}+O(t)$
m layers of $(1,1,1)$-model	$\frac{1}{1-\nu}+\frac{1}{m^2}\frac{\nu^2}{(1-\nu)(1-2\nu)}+O(t)$

Problem 6.1 *How stiff is the energy functional if the plate is represented by $m\geq 1$ layers of the $(1,1,1)$-model?*

We recall (2.8), but consider the energy before the integration over x_3 has been performed. The impact of the quadratic term is the fact that $\frac{\partial}{\partial_3}(x_3^2 W) - x_3\frac{\nu}{1-\nu}\Delta w$ is small. The model above with m layers implies that x_3^2 is approximated by a piecewise linear function $s(x_3)$. The $O(t)$ term in the energy is now augmented by the approximation error

$$\|\frac{\partial}{\partial_3}(x_3^2 - s(x_3))W\|_{0,\Omega}^2 = \|W\|_{0,\omega}^2 \int_{-t/2}^{+t/2}(2x_3 - s')^2 \mathrm{d}x_3.$$

We choose s as the interpolant of x_3^2 at the $m+1$ nodes of m subintervals. Elementary calculations yield

$$\int_{-t/2}^{+t/2}(2x_3 - s')^2 \mathrm{d}x_3 = \frac{1}{3m^2}t^3 = \frac{1}{m^2}\int_{-t/2}^{+t/2}(2x_3)^2 \mathrm{d}x_3.$$

Hence,

$$\|\frac{\partial}{\partial_3}(x_3^2 - s(x_3))W\|_{0,\Omega}^2 = \frac{1}{m^2}\|\frac{\partial}{\partial_3}(x_3^2)W\|_{0,\Omega}^2.$$

The resulting coercivity constant is listed in the last row of Table 1. In particular, the error is smaller than 2% if $\nu = 0.3$ and $m \geq 3$.

Remark 6.2 *Roughly speaking the shear correction factor in the* $(1, 1, 0)$*-models helps to compensate that there are no higher order terms in the* x_3*-direction. The correction factor* (3.18) *is not valid anymore in higher order models. The so-called* $(1, 1, 2)$ *model of plate bending also can show an increased consistency order with respect to the three-dimensional problems upon introduction of a suitable shear correction factor, whereas even higher order models will not exhibit improved asymptotic consistency upon introduction of a shear correction factor [4].*

In particular, this has to be taken into account when hierarchical a posteriori error estimates are used.

A Appendix: Completion of the Proof of Lemma 3.1

Here we prove the assertions b) and c) of Lemma 3.1 in order to complete the proof. To treat the intermediate cases $0 < s < 1$, the use of interpolation is suggestive, but the upper endpoint result for interpolation is not available, if the geometry of ω is such that $\phi \notin H^1(\omega)$. Other arguments that are based on fractional order Sobolev spaces are required. We start by recalling their definitions via interpolation and some basic properties.

Given $g \in L^2(\omega)$, the weak solution Z of the Dirichlet problem of the Poisson equation

$$-\Delta Z = g \quad \text{in} \quad \omega, \qquad Z = 0 \quad \text{on} \quad \partial\omega \tag{A.1}$$

belongs to $H^{1+s}(\omega) \cap H_0^1(\omega)$ for all $0 \leq s \leq s^*(\omega)$ for some $1/2 < s^*(\omega) \leq 1$ ($s^*(\omega) = 1$ for a smooth domain or a convex polygon); cf. Remark 3.2. Moreover, Z satisfies the a priori estimate

$$\|Z\|_{1+s,\omega} \leq C_s\|g\|_{-1+s,\omega} \quad \text{for all} \quad 0 \leq s \leq s^*(\omega), \tag{A.2}$$

and the Dirichlet Laplacean is an isomorphism

$$\Delta : H^{1+s}(\omega) \cap H_0^1(\omega) \to H^{-1+s}(\omega), \quad 0 \leq s \leq s^*(\omega),$$

where we define (cf. [25], Chapter 1) for $0 \le s \le 1$

$$H^{-1+s}(\omega) = \tilde{H}^{1-s}(\omega)' = ((L^2(\omega), H_0^1(\omega))_{1-s,2})' = (H^{-1}(\omega), L^2(\omega))_{s,2}$$

(real method of interpolation) with duality taken with respect to the "pivot" space $L^2(\omega) \simeq (L^2(\omega))'$. The spaces $\tilde{H}^\theta(\omega) := (L^2(\omega), H_0^1(\omega))_{\theta,2}$ satisfy

$$\begin{array}{rcllr} \tilde{H}^\theta(\omega) & \simeq & H^\theta(\omega) := (L^2(\omega), H^1(\omega))_{\theta,2} & \text{for} \quad 0 \le \theta < 1/2 & \\ \tilde{H}^\theta(\omega) & \subset & H^\theta(\omega) & \text{for} \quad 1/2 \le \theta \le 1. & \end{array} \tag{A.3}$$

Note that $\tilde{H}^{1/2}(\omega) \simeq H_{00}^{1/2}(\omega)$.

Now we are prepared to consider *case b):* $0 < s < 1/2$. We extend the $L^2(\omega)$-inner product in the right-hand side of (3.9) to $H^s(\omega) \times H^{-s}(\omega)$ which implies

$$t^2 \|\Delta W\|_{0,\omega}^2 + \|\nabla W\|_{0,\omega}^2 \;\; \le \;\; \|\phi\|_{s,\omega} \|\Delta W\|_{-s,\omega}, \quad 0 \le s < 1/2.$$

Since (A.1) constitutes the principal part of the problem (3.8), it follows that $W \in H_0^1(\omega) \cap H^{1+s}(\omega)$ with $0 < s \le s^*(\omega)$. We deduce from (A.2) and $H^{-s}(\omega) = (H^{-1}(\omega), L^2(\omega))_{1-s,2}$ that for $0 \le s < 1/2$

$$\begin{array}{rcl} t^2 \|\Delta W\|_{0,\omega}^2 + \|\nabla W\|_{0,\omega}^2 & \le & \|\phi\|_{s,\omega} \|\Delta W\|_{0,\omega}^{1-s} \|\Delta W\|_{-1,\omega}^{s} \\ & = & \|\phi\|_{s,\omega} \|\Delta W\|_{0,\omega}^{1-s} \|\nabla W\|_{0,\omega}^{s} \,. \end{array}$$

Using Young's inequality

$$|ab| \le \frac{1}{p} a^p + \frac{1}{q} b^q \quad \text{for} \quad 1/p + 1/q = 1, \quad 1 < p, q < \infty, \tag{A.4}$$

with $p^{-1} = 1 - s$, $q^{-1} = s$,

$$a = t^\alpha \|\Delta W\|_{0,\omega}^{1-s}, \quad b = t^{-\alpha} \|\nabla W\|_{0,\omega}^{s}\,,$$

and $\alpha = 1/[p+q]$, we find that there exists $C_8(s,\omega)$ such that for any $0 < t \le 1$ holds

$$\begin{array}{rcl} t^2 \|\Delta W\|_{0,\omega}^2 + \|\nabla W\|_{0,\omega}^2 & \le & C_8 \|\phi\|_{s,\omega} t^{s-1} \left(t \|\Delta W\|_{0,\omega} + \|\nabla W\|_{0,\omega} \right) \\ & \le & C_9 \|\phi\|_{s,\omega}^2 t^{2s-2} + \dfrac{t^2}{2} \|\Delta W\|_{0,\omega}^2 + \dfrac{1}{2} \|\nabla W\|_{0,\omega}^2 \,. \end{array}$$

We collect all terms with W on the left-hand side and proceed as in the proof of case d). Multiplying by t^2 and substituting $t^2 \Delta W = W - \phi$ implies that for all $0 < t \leq 1$

$$t^2 \|\nabla W\|_{0,\omega}^2 + \|W - \phi\|_{0,\omega}^2 \leq 2C_9 \|\phi\|_{s,\omega}^2 t^{2s}$$

which is b).

The proof of case c) is now immediate. Let $0 < \varepsilon < 1/2$. The estimate (3.7) with $s = 1/2 - \varepsilon/2$ yields the assertion. ∎

Remark A.1 *It is suggestive that, for $s = 1/2$, the rate* (3.15) *equals, in fact, $O(t^{1/2})$; a verification would require, however, different technical tools. The proof as in b) fails for $s = 1/2$ because ΔW in (3.9), in general, is* not *contained in the interpolation space $\left(H^{-1}(\omega), L^2(\omega)\right)_{\frac{1}{2},2}$. The characterization of $H^{-s}(\omega)$ by interpolation, however, was used in the proof of b) for the direct estimate of (3.9) (without integration by parts).*

To prove Lemma 3.1, part d, we have applied partial integration to (3.9) and then estimated the arising boundary integral by trace inequalities. One can generalize the proof of case d to $1/2 \leq s < 1$. However, it turns out that such a proof does not lead to a sharper estimate as in Lemma 3.1, part c.

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