ARITHMETIC PROPERTIES OF PROJECTIVE VARIETIES OF ALMOST MINIMAL DEGREE

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ABSTRACT. We study the arithmetic properties of projective varieties of almost minimal degree, that is of non-degenerate irreducible projective varieties whose degree exceeds the codimension by precisely 2. We notably show, that such a variety $X \subset \mathbb{P}^r$ is either arithmetically normal (and arithmetically Gorenstein) or a projection of a variety of minimal degree $\tilde{X} \subset \mathbb{P}^{r+1}$ from an appropriate point $p \in \mathbb{P}^{r+1} \setminus \tilde{X}$. We focus on the latter situation and study X by means of the projection $\tilde{X} \to X$.

If X is not arithmetically Cohen-Macaulay, the homogeneous coordinate ring B of the projecting variety $\tilde{X}$ is the endomorphism ring of the canonical module $K(A)$ of the homogeneous coordinate ring A of X. If X is non-normal and is maximally Del Pezzo, that is arithmetically Cohen-Macaulay but not arithmetically normal B is just the graded integral closure of A. It turns out, that the geometry of the projection $\tilde{X} \to X$ is governed by the arithmetic depth of X in any case.

We study in particular the case in which the projecting variety $\tilde{X} \subset \mathbb{P}^{r+1}$ is a (cone over a) rational normal scroll. In this case X is contained in a variety of minimal degree $Y \subset \mathbb{P}^r$ such that $\operatorname{codim}_Y(X) = 1$. We use this to approximate the Betti numbers of X.

In addition we present several examples to illustrate our results and we draw some of the links to Fujita's classification of polarized varieties of Δ-genus 1.

1. INTRODUCTION

Let $\mathbb{P}^r_k$ denote the projective r-space over an algebraically closed field k. Let $X \subset \mathbb{P}^r_k$ be an irreducible non-degenerate projective variety of dimension d. The degree $\deg X$ of X is defined as the number of points of $X \cap \mathbb{L}$, where $\mathbb{L}$ is a linear subspace defined by generically chosen linear forms $\ell_1, \ldots, \ell_d$. It is well known that

$$\deg X \geq \operatorname{codim} X + 1,$$

(cf e.g. [12]), where $\operatorname{codim} X = r - d$ is used to denote the codimension of X. In case equality holds, X is called a variety of minimal degree. Varieties of minimal degree are classified and well understood. A variety X of minimal degree is either a quadric hypersurface, a (cone over a) Veronese surface in $\mathbb{P}^5_k$, or a (cone over a smooth) rational normal scroll (cf [21, Theorem 19.9]). In particular these varieties are arithmetically Cohen-Macaulay and arithmetically normal.

The main subject of the present paper is to investigate varieties of almost minimal degree, that is irreducible, non-degenerate projective varieties $X \subset \mathbb{P}^r_k$ with $\deg X = \operatorname{codim} X + 2$. From the point of view of polarized varieties, Fujita [14], [15], [16] has studied extensively such

Date: Zürich and Halle, June 2005.

1991 *Mathematics Subject Classification.* Primary: 14H45, 13D02.

Key words and phrases. Betti numbers, variety of minimal degree.

The second author was partially supported by Swiss National Science Foundation (Projects No. 20 - 66980-01 & No. 20 - 103491 / 1).

varieties in the framework of varieties of Δ-genus 1. Nevertheless, in our investigation we take a purely arithmetic point of view and study our varieties together with a fixed embedding in a projective space.

A natural approach to understand a variety $X \subset \mathbb{P}^r_k$ of almost minimal degree is to view it (if possible) as a birational projection of a variety of minimal degree $\tilde{X} \subset \mathbb{P}^{r+1}_k$ from a point $p \in \mathbb{P}^{r+1}_k \setminus \tilde{X}$. If sufficiently many varieties of almost minimal degree can be obtained by such projections, we may apply to them the program of: "classifying by projections of classified varieties". It turns out, that this classification scheme can indeed be applied to an interesting class of varieties of almost minimal degree $X \subset \mathbb{P}^r_k$, namely those, which are not arithmetically normal or equivalently, to all those which are not simultaneously normal and arithmetically Gorenstein. More precisely, we shall prove the following result, in which $\mathrm{Sec}_p(\tilde{X})$ is used to denote secant cone of $\tilde{X}$ with respect to p:

Theorem 1.1. *Let $X \subset \mathbb{P}^r_k$ be a non-degenerate irreducible projective variety and let $t \in \{1, 2, \cdots, \dim X + 1\}$. Then, the following conditions are equivalent:*

(i) *X is of almost minimal degree, of arithmetic depth t and not arithmetically normal.*
(ii) *X is of almost minimal degree and of arithmetic depth t, where either $t \leq \dim X$ or else $t = \dim X + 1$and X is not normal.*
(iii) *X is of almost minimal degree and of arithmetic depth t, where either X is not normal and $t > 1$ or else X is normal and $t = 1$.*
(iv) *X is a (birational) projection of a variety $\tilde{X} \subset \mathbb{P}^{r+1}_k$ of minimal degree from a point $p \in \mathbb{P}^{r+1}_k \setminus \tilde{X}$ such that $\dim \mathrm{Sec}_p(\tilde{X}) = t - 1$.*

For the proof of this result see Theorem 5.6 (if $t \leq \dim X$) resp. Theorem 6.9 (if $t = \dim X + 1$). In the spirit of Fujita [14] we say that a variety of almost minimal degree is *maximally Del Pezzo* if it is arithmetically Cohen-Macaulay (or – equivalently – arithmetically Gorenstein). Then, as a consequence of Theorem 1.1 we have:

Theorem 1.2. *A variety $X \subset \mathbb{P}^r_k$ of almost minimal degree is either maximally Del Pezzo and normal or a (birational) projection of a variety $\tilde{X} \subset \mathbb{P}^{r+1}_k$ of minimal degree from a point $p \in \mathbb{P}^{r+1}_k \setminus \tilde{X}$.*

In this paper, our interest is focussed on those varieties $X \subset \mathbb{P}^r_k$ of almost minimal degree which are birational projections of varieties of minimal degree. As already indicated by Theorem 1.1 and in accordance with our arithmetic point of view, the arithmetic depth of X is the key invariant of our investigation. It turns out, that this arithmetic invariant is in fact closely related to the geometric nature of our varieties. Namely, the picture sketched in Theorem 1.1 can be completed as follows:

Theorem 1.3. *Let $X \subset \mathbb{P}^r_k$ be a variety of almost minimal degree and of arithmetic depth t, such that $X = \varrho(\tilde{X})$, where $\tilde{X} \subset \mathbb{P}^{r+1}_k$ is a variety of minimal degree and $\varrho : \mathbb{P}^{r+1}_k \setminus \{p\} \to \mathbb{P}^r_k$ is a birational projection from a point $p \in \mathbb{P}^{r+1}_k \setminus \tilde{X}$. Then:*

(a) *$\nu := \varrho \restriction: \tilde{X} \to X$ is the normalization of X.*
(b) *The secant cone $\mathrm{Sec}_p(\tilde{X}) \subset \mathbb{P}^{r+1}_k$ is a projective subspace $\mathbb{P}^{t-1}_k \subset \mathbb{P}^{r+1}_k$.*
(c) *The singular locus $\mathrm{Sing}(\nu) = \varrho(\mathrm{Sec}_p(\tilde{X}) \setminus \{p\}) \subset X$ of ν is a projective subspace $\mathbb{P}^{t-2}_k \subset \mathbb{P}^r_k$ and coincides with the non-normal locus of X.*

(d) *If* $t \leq \dim X, \mathrm{Sing}(\nu)$ *coincides with the non* S_2*-locus and the non-Cohen-Macaulay locus of* X *and the generic point of* $\mathrm{Sing}(\nu)$ *in* X *is of Goto-type.*
(e) *The singular fibre* $\nu^{-1}(\mathrm{Sing}(\nu)) = \mathrm{Sec}_p(\tilde{X}) \cap \tilde{X}$ *is a quadric in* $\mathbb{P}_k^{t-1} = \mathrm{Sec}_p(\tilde{X})$.

For the proves of these statements see Theorem 5.6 and Corollary 6.10.

Clearly, the projecting variety $\tilde{X} \subset \mathbb{P}_k^{r+1}$ of minimal degree plays a crucial rôle for X. We thus may distinguish the *exceptional case* in which X is a cone over the Veronese surface and the *general case* in which $\tilde{X}$ is a cone over a rational normal scroll. In this latter case, we have the following crucial result, in which we use the convention $\dim \emptyset = -1$:

Theorem 1.4. *Let* $X \subset \mathbb{P}_k^r$ *be a variety of almost minimal degree which is a birational projection of a (cone over a) rational normal scroll* $\tilde{X} \subset \mathbb{P}_k^{r+1}$ *from a point* $p \in \mathbb{P}_k^{r+1} \setminus \tilde{X}$. *Then, there is a (cone over a) rational normal scroll* $Y \subset \mathbb{P}_k^r$ *such that* $X \subset Y$ *and* $\mathrm{codim}_X(Y) = 1$. *Moreover, if the vertex of* $\tilde{X}$ *has dimension* h, *the dimension* l *of the vertex of* Y *satisfies* $h \leq l \leq h+3$. *In addition, the arithmetic depth* t *satisfies* $t \leq h+5$.

For a proof of this result see Theorem 7.3 and Corollary 7.5. It should be noticed, that there are varieties of almost minimal degree, which cannot occur as a 1-codimensional subvariety of a variety of minimal degree (cf Example 9.4 and Remark 6.5).

Our paper is built up following the idea, that the arithmetic depth $t := \operatorname{depth} A$ of a variety $X \subset \mathbb{P}_k^r = \mathrm{Proj}(S), S = k[x_0, \cdots, x_r]$, of almost minimal degree with homogeneous coordinate ring $A = A_X$ is a key invariant. In Section 2 we present a few preliminaries and discuss the special case where X is a curve.

In Section 3 we consider the case where $t = 1$. We show that the total ring of global sections $\oplus_{n\in\mathbb{Z}} H^0(X, \mathcal{O}_X(n))$ of X – that is the S_+-transform $D(A)$ of A – is the homogeneous coordinate ring of a variety $\tilde{X} \subset \mathbb{P}_k^{r+1}$ of minimal degree. In geometric terms: X is isomorphic to $\tilde{X}$ by means of a projection from a generic point $p \in \mathbb{P}_k^{r+1}$ and hence normal but not arithmetically normal (cf Propositions 3.1 and 3.4).

In Section 4 we begin to investigate the case $(1 <)\ t \leq \dim X$, that is the case in which X is not arithmetically Cohen-Macaulay. First, we prove some vanishing statements for the cohomology of X and describe the structure of the t-th deviation module $K^t(A)$ of A. Moreover we determine the Hilbert series of A and the number of defining quadrics of X (cf Theorem 4.2 and Corollary 4.4).

In Section 5 we aim to describe X as a projection if $t \leq \dim X$. As a substitute for the S_+-transform $D(A)$ of the homogeneous coordinate ring A (which turned out to be useful in the case $t = 1$) we now consider the endomorphism ring $B := \mathrm{End}_A(K(A))$ of the canonical module of A (cf Theorem 5.3). It turns out that B is the homogeneous coordinate ring of variety $\tilde{X} \subset \mathbb{P}_k^{r+1}$ of minimal degree, and this allows to describe X as a projection of $\tilde{X}$ (cf Theorem 5.6). Endomorphism rings of canonical modules have been studied extensively in a purely algebraic setting (cf [2], [28]). The striking point is the concrete geometric meaning of these rings in the case of varieties of almost minimal degree.

In Section 6 we study the case where $t = \dim X + 1$, that is the case where X is arithmetically Cohen-Macaulay. Now, X is a Del Pezzo variety in the sense of Fujita [16]. According to our arithmetic point of view we shall speak of maximal Del Pezzo varieties in order to distinguish them within the larger class of polarized Del Pezzo varieties. We shall give several equivalent characterizations of these varieties (cf Theorem 6.2). We shall in addition introduce the notion of Del Pezzo variety and show among other things that this notion coincides with Fujita's definition for the polarized pair $(X, \mathcal{O}_X(1))$ (cf Theorem 6.8). Finally we shall prove that the graded integral closure B of the homogeneous coordinate ring A of a non-normal maximal Del Pezzo variety $X \subset \mathbb{P}^r_k$ is the homogeneous coordinate ring of a variety of minimal degree $\tilde{X} \subset \mathbb{P}^{r+1}_k$ (cf Theorem 6.9) and describe X as a projection of $\tilde{X}$ (cf Corollary 6.10). Contrary to the case in which $t \leq \dim X$, we now cannot characterize B as the endomorphism ring of the canonical module $K(A)$, simply as A is a Gorenstein ring. We therefore study B by geometric arguments, which rely essentially on the fact that we know already that the non-normal locus of X is a linear subspace (cf Proposition 5.8). It should be noticed that on turn these geometric arguments seem to fail if $t \leq \dim X$.

In Section 7 we assume that X is a (birational) projection of a (cone over a) rational normal scroll $\tilde{X} \subset \mathbb{P}^{r+1}_k$. We then prove what is claimed by the previous Theorem 1.4. Here, we extensively use the determinantal description of rational normal scrolls (cf [21]). As an application we give some constraints on the arithmetic depth t of X (cf Corollary 7.5 and Corollary 7.6).

In Section 8 we study the Betti numbers of the homogeneous coordinate ring A of our variety of almost minimal degree $X \subset \mathbb{P}^r_k$. We focus on those cases, which after all merit a particular interest, that is the situation where $t \leq \dim X$ and X is a projection of a rational normal scroll. Using what has been shown in Section 7, we get a fairly good and detailed view on the behaviour of the requested Betti numbers.

Finally, in Section 9 we present various examples that illustrate the results proven in the previous sections. In several cases we calculated the Betti numbers of the vanishing ideal of the occuring varieties on use of the computer algebra system SINGULAR [20].

2. PRELIMINARIES

We first fix a few notation, which we use throughout this paper. By $\mathbb{N}_0$ (resp. $\mathbb{N}$) we denote the set of non-negative (resp. positive) integers.

Notation 2.1. A) Let k be an algebraically closed field, let $S := k[x_0, \cdots, x_r]$ be a polynomial ring, where $r \geq 2$ is an integer. Let $X \subset \mathbb{P}^r_k = \mathrm{Proj}(S)$ be a reduced irreducible projective variety of positive dimensions d. Moreover, let $\mathcal{J} = \mathcal{J}_X \subset \mathcal{O}_{\mathbb{P}^r_k}$ denote the sheaf of vanishing ideals of X, let $I = I_X = \oplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^r_k, \mathcal{J}(n)) \subset S$ denote the vanishing ideal of X and let $A = A_X := S/I$ denote the homogeneous coordinate ring of X.
B) If M is a finitely generated graded S-module and if $i \in \mathbb{Z}$, we use $H^i(M) = H^i_{S_+}(M)$ to denote the i-th *local cohomology module* of M with respect to the irrelevant ideal $S_+ = \oplus_{n\in\mathbb{N}} S_n$ of S. Let $D(M) = D_{S_+}(M)$ denote *the S_+-transform* $\varinjlim \mathrm{Hom}_S(S^n_+, M)$ of M. Moreover, let us introduce the i-th *deficiency module* of M:

$$K^i(M) = K^i_S(M) := \mathrm{Ext}^{r+1-i}_S(M, S(-r-1)).$$

The S-modules $D(M), H^i(M)$ and $K^i(M)$ are always furnished with their natural gradings.

Reminder 2.2. A) Let $i \in \mathbb{Z}$. If $U = \oplus_{n\in\mathbb{Z}} U_n$ is a graded S-module, we denote by ${}^*\mathrm{Hom}_k(U, k)$ the graded S-module $\oplus_{n\in\mathbb{Z}} \mathrm{Hom}_k(U_{-n}, k)$. If M is a finitely generated graded S-module, by graded local duality, we have isomorphisms of graded S-modules

$$K^i(M) \simeq {}^*\mathrm{Hom}_k(H^i(M), k) \text{ and} \tag{2.1}$$

$$H^i(M) \simeq {}^*\mathrm{Hom}_k(K^i(M), k) \simeq \mathrm{Hom}_S(K^i(M), E), \tag{2.2}$$

where E denotes the graded injective envelope of the S-module $k = S/S_+$.
B) By $\operatorname{depth} M$ we denote the depth of the finitely generated graded S-module M (with respect to the irrelevant ideal S_+ of S), so that

$$\begin{aligned} \operatorname{depth} M &= \inf\{i \in \mathbb{Z} \,\big|\, H^i(M) \neq 0\} \\ &= \inf\{i \in \mathbb{Z} \,\big|\, K^i(M) \neq 0\}, \end{aligned} \tag{2.3}$$

(with the usual convention that $\inf \emptyset = \infty$). Here $\operatorname{depth} A$ is called the *arithmetic depth of the variety* $X \subset \mathbb{P}^r_k$. If we denote the Krull dimension of M by $\dim M$, we have

$$\begin{aligned} \dim M &= \sup\{i \in \mathbb{Z} \,\big|\, H^i(M) \neq 0\} \\ &= \sup\{i \in \mathbb{Z} \,\big|\, K^i(M) \neq 0\}, \end{aligned} \tag{2.4}$$

(with the conventions that $\sup \emptyset = -\infty$ and $\dim 0 = -\infty$).
C) For a graded S-module $U = \oplus_{n \in \mathbb{Z}} U_n$, let $\operatorname{end} U := \sup\{n \in \mathbb{Z} \mid U_n \neq 0\}$ and $\operatorname{beg} U := \inf\{n \in \mathbb{Z} \mid U_n \neq 0\}$ denote the *end* resp. the *beginning* of U. In these notation, the *Castelnuovo-Mumford regularity* of the finitely generated graded S-module M is defined by

$$\operatorname{reg} M = \sup\{ \text{ end } H^i(M) + i \,\big|\, i \in \mathbb{Z}\} = \inf\{-\operatorname{beg} K^i(M)) + i \,\big|\, i \in \mathbb{Z}\}. \tag{2.5}$$

Keep in mind that the Castelnuovo-Mumford regularity of the variety $X \subset \mathbb{P}^r_k$ is defined as

$$\operatorname{reg} X = \operatorname{reg} I = \operatorname{reg} A + 1. \tag{2.6}$$

We are particularly interested in the *canonical module* of A, that is in the graded A-module

$$K(A) := K^{\dim(A)}(A) = K^{d+1}(A). \tag{2.7}$$

Remark 2.3. A) Let $0 < i < \dim(A) = d + 1$ and let $\mathfrak{p} \in \operatorname{Spec} S$ with $\dim S/\mathfrak{p} = i$. Then, the $S_\mathfrak{p}$-module $A_\mathfrak{p}$ has positive depth and hence vanishes or is of projective dimension $< \dim S_\mathfrak{p} = r + 1 - i$. Therefore $K^i(A)_\mathfrak{p} \simeq \operatorname{Ext}^{r+1-i}(A_\mathfrak{p}, S_\mathfrak{p}) = 0$. So

$$\dim K^i(A) < i \ \text{ for } \ 0 < i < \dim(A) = d + 1. \tag{2.8}$$

B) Let $n \in \mathbb{N}$ and let $f \in A_n \backslash \{0\}$. Then, f is A-regular and the short exact sequence $0 \to A(-n) \xrightarrow{f} A \to A/fA \to 0$ yields an epimorphism of graded A-modules $f : H^{d+1}(A)(-n) \twoheadrightarrow H^{d+1}(A)$. So, by the isomorphisms (2.1) of Reminder 2.2, the multiplication map $f : K^{d+1}(A) \to K^{d+1}(A)(n)$ is injective. Moreover, localizing at the prime ideal $I \subset S$ we get

$$K^{d+1}(A) \otimes_A \text{Quot}(A) \simeq K^{d+1}(A)_I \simeq \operatorname{Ext}^{r-d}_{S_I}(S_I/IS_I, S_I) \simeq S_I/IS_I = \text{Quot}\,(A).$$

So, we may resume:

(2.9) The canonical module $K(A)$ of A is torsion free and of rank 1.

C) Let $\ell \in S_1 \backslash \{0\}$ be a linear form. We write $T := S/\ell S$ and consider T as a polynomial ring in r indeterminates. For the T-deficiency modules $K^i_T(A/\ell A)$ of $A/\ell A$, the isomorphisms (2.1) of Reminder 2.2 together with the base ring independence of local cohomology furnish the following isomorphisms of graded $A/\ell A$-modules

$$K^i_T(A/\ell A) \simeq {}^*\operatorname{Hom}_k \left(H^i_{T_+}(A/\ell A), k\right) \simeq {}^*\operatorname{Hom}_k \left(K^i_{S_+}(A/\ell A), k\right) \simeq K^i_S(A/\ell A).$$

So for all $i \in \mathbb{Z}$ we obtain

$$K_T^i(A/\ell A) \simeq K_S^i(A/\ell A) = \mathrm{Ext}_S^{r+1-i}\left(A/\ell A, S(-r-1)\right). \tag{2.10}$$

D) Let ℓ be as above. If we apply $\mathrm{Ext}_S^{r+1-i}(\bullet, S(-r-1))$ to the short exact sequence $0 \to A(-1) \xrightarrow{\ell} A \to A/\ell A \to 0$ and keep in mind the isomorphisms (2.10), we get for each $i \in \mathbb{Z}$ an exact sequence of graded $A/\ell A$-modules

$$0 \to \left(K_S^{i+1}(A)/\ell K_S^{i+1}(A)\right)(1) \to K_T^i(A/\ell A) \to 0 :_{K_S^i(A)} \ell \to 0. \tag{2.11}$$

Correspondingly, applying local cohomology, we get for each $i \in \mathbb{Z}$ an exact sequence of graded $A/\ell A$-modules

$$0 \to H_{S_+}^i(A)/\ell H_{S_+}^i(A) \to H_{T_+}^i(A/\ell A) \to (0 :_{H_{S_+}^{i+1}(A)} \ell)(-1) \to 0. \tag{2.12}$$

E) We keep the above notation. In addition, we assume that $\ell \in S_1 \backslash \{0\}$ is chosen generically. Then, according to Bertini's Theorem (cf [24]) the hyperplane section

$$Y := X \cap \mathrm{Proj}(T) = \mathrm{Proj}(T/IT) \simeq \mathrm{Proj}(A/\ell A) \subset \mathrm{Proj}(T) = \mathbb{P}_k^{r-1}$$

is reduced and irreducible if $\dim A > 2$. The homogeneous coordinate ring of Y is

$$A' = A/(\ell A)^{\mathrm{sat}} \simeq T/(IT)^{\mathrm{sat}},$$

where $\bullet^{\mathrm{sat}}$ is used to denote the saturation of a graded ideal in a homogeneous k-algebra. Observe that we have the following isomorphisms of graded $A/\ell A$-modules (cf (2.1), (2.10)).

$$H_{S_+}^i(A/\ell A) \simeq H_{T_+}^i(A/\ell A) \simeq H_T^i(A') \text{ for all } i > 0; \tag{2.13}$$

$$K_S^i(A/\ell A) \simeq K_T^i(A/\ell A) \simeq K_T^i(A') \text{ for all } i > 0. \tag{2.14}$$

On use of (2.12) and (2.13) we now easily get

$$H_{T_+}^i(A')_{\geq m} = 0 \Rightarrow H_{S_+}^{i+1}(A)_{\geq m-1} = 0 \text{ for all } i > 0 \text{ and all } m \in \mathbb{Z}, \tag{2.15}$$

where, for a graded S-module $U = \oplus_{n \in \mathbb{Z}} U_n$, we use $U_{\geq n}$ to denote the m-th left truncation $\oplus_{n \geq m} U_n$ of U. Finally, if $\operatorname{depth} A > 1$, we have $A' = A/\ell A$. If $\operatorname{depth} A = 1$, we know that $H_{S_+}^1(A)$ is a finitely generated non-zero A-module so that, by Nakayama, $\ell H_{S_+}^1(A) \neq H_{S_+}^1(A)$ and hence $H_{T_+}^1(A') \neq 0$ (cf (2.12) and (2.13)). So, the arithmetic depth of Y behaves as follows

$$\operatorname{depth} A' = \begin{cases} \operatorname{depth} A - 1, & \text{if depth } A > 1, \\ 1, & \text{if depth } A = 1. \end{cases} \tag{2.16}$$

The aim of the present paper is to investigate the case in which the degree of X exceeds the codimension of X by 2. Keep in mind, that the degree of X always exceeds the codimension of X by 1. Therefore, we make the following convention.

Convention 2.4. We write $\dim X, \operatorname{codim} X$ and $\deg X$ for the dimension, the codimension and the degree of X respectively, so that $d = \dim X = \dim A - 1, \operatorname{codim} X = \operatorname{height} I = r - \dim X = r - d$. Keep in mind that

$$\deg X \geq \operatorname{codim} X + 1$$

(cf e.g. [12]). We say that X is of *almost minimal degree*, if $\deg X = \operatorname{codim} X + 2 = r - d + 2$. Note that X is called of *minimal degree* (cf [12]) whenever $\deg X = \operatorname{codim} X + 1$.

We now discuss the case in which X is a curve of almost minimal degree.

Remark 2.5. A) We keep the hypotheses and notations of Remark 2.3 and assume that $\dim X = 1$ and that $\deg X = \operatorname{codim} X + 2 = r + 1$. Then, for a generic linear form $\ell \in S_1 \setminus \{0\}$ and in the notation of part E) of Remark 2.3, the generic hyperplane section

$$Y := \operatorname{Proj}(T/IT) \simeq \operatorname{Proj}(A/\ell A) = \operatorname{Proj}(A') \subset \operatorname{Proj}(T) = \mathbb{P}_k^{r-1}$$

is a scheme of $r+1$ points in semi-uniform position in $\mathbb{P}_k^{r-1}$ (cf [3], [25]). Consequently, by (cf [19]) we can say that IT is generated by quadrics. Therefore we may conclude: The homogeneous T-module

$$H^0_{T_+}((IT)^{\text{sat}}/IT) \simeq H^0_T(A/\ell A) \text{ is generated in degree } 2. \tag{2.17}$$

Moreover, (cf [4, (2.4) a)])

$$\dim_k H^1_{T_+}(A/\ell A)_n = \dim_k H^1_{T_+}(A')_n = \begin{cases} r+1, & \text{if } n < 0 \\ r, & \text{if } n = 0 \\ \leq 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases} \tag{2.18}$$

So, by the exact sequences (2.12) and by statement (2.15) we get

$$H^1(A)/\ell H^1(A) \subset k(-1) \text{ and} \tag{2.19}$$

$$\text{end } H^2(A) \leq 0. \tag{2.20}$$

B) Assume first, that A is a Cohen-Macaulay ring. Then $H^1(A) = 0$ and so, by (2.20), the Hilbert polynomial $P_A(x) \in \mathbb{Q}[x]$ of A satisfies $P_A(n) = \dim_k A_n$ for all $n > 0$. As $\dim_k A_1 = r + 1$ it follows $P_A(x) = (r+1)x$ and hence $H^2(A)_{-n} \simeq A_n$ for all $n \in \mathbb{Z}$. So, by (2.1) $K(A)_n = K^2(A)_n \simeq A_n$ for all $n \in \mathbb{Z}$. As $K(A)$ is torsion-free of rank 1 (cf (2.9)), we get an isomorphism of graded A-modules $K(A) \simeq A(0)$. Therefore, A is a Gorenstein ring.

If A is normal, $X \subset \mathbb{P}_k^r$ is a smooth non-degenerate curve of genus $\dim_k K(A)_0 = 1$ and of degree $r+1$, hence an elliptic normal curve: we are in the case $\bar{I}$ of [4, (4.7) B)].

C) Yet assume that A is a Cohen-Macaulay (and hence a Gorenstein) ring. Assume that A is not normal. Let B denote the graded normalization of A. Then, there is a short exact sequence of graded S-modules $0 \to A \to B \xrightarrow{\pi} C \to 0$ with $\dim C = 1$. As $H^0_{S_+}(B) = H^1_{S_+}(B) = H^0_{S_+}(A) = H^1_{S_+}(A) = 0$ we get $H^0_{S_+}(C) = 0$ and an exact sequence of graded S-modules

$$0 \to H^1_{S_+}(C) \to H^2(A) \to H^2_{S_+}(B) \to 0.$$

As $\dim C = 1$ and $H^0_{S_+}(C) = 0$, there is some $c \in \mathbb{N}$ such that

$$\dim_k C_n + \dim_k H^1_{S_+}(C)_n = c \text{ for all } n \in \mathbb{Z}.$$

By (2.20) and the above sequence $\dim_k C_n = c$ for all $n > 0$. As $C_0 = 0$ and $\dim_k H^2(A)_0 = \dim_k A_0 = 1$ (cf part B)) it follows $c = 1$. As $H^0_{S_+}(C) = 0$, there exits a C-regular element

$h \in S_1 \setminus \{0\}$, and choosing $t \in C_1 \setminus \{0\}$ we get

(2.21) $$C = k[h]t \simeq k[h](-1).$$

Choose $\bar{y} \in B_1$ such that $\pi(\bar{y}) = t$. Then we get $B/A = (\bar{y}A + A)/A$ and hence $B = A[\bar{y}]$. So, if y is an indeterminate, there is a surjective homomorphism of homogeneous k-algebras

$$S[y] = k[x_0, \cdots, x_r, y] \overset{\beta}{\twoheadrightarrow} B,\ y \mapsto \bar{y},$$

which occurs in the commutative diagram

$$\begin{tikzcd} S[y] \arrow[r, "\beta"] & B \\ S \arrow[u, hook] \arrow[r, "\alpha"] & A \arrow[u, hook, "\alpha"'] \end{tikzcd}$$

where α is the natural map. Thus, the normalization $\tilde{X} := \mathrm{Proj}(B)$ of X is a curve of degree $r+1$ in $\mathrm{Proj}(S[y]) = \mathbb{P}^{r+1}_k$ – a rational normal curve – and the normalization morphism $\nu : \tilde{X} \to X$ is induced by a simple projection $\varrho : \mathbb{P}^{r+1} \setminus \{p\} \twoheadrightarrow \mathbb{P}^r_k$ with center $\{p\} = |\,\mathrm{Proj}(S[y]/S_+S[y])|$.

Moreover, by (2.21) we have $\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X \simeq \tilde{C} \simeq k$, so that $\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$ is supported in a single point $q \in X$, – the unique singularity of X – a double point. That is, we are in the case $\overline{\mathrm{III}}$ of [4, (4.7) B)].

D) We keep the notations and hypotheses of part A). But contrary to what we did in parts B) and C) we now assume that A is not Cohen-Macaulay, so that $H^1(A) \neq 0$. Then, by (2.19) and by Nakayama it follows $H^1(A)/\ell H^1(A) = k(-1)$. In particular $H^1(A)_1 \simeq k$ and the multiplication map $\ell : H^1(A)_n \to H^1(A)_{n+1}$ is surjective for all $n \geq 1$.

Now we claim that $H^1(A)_n = 0$ for all $n > 1$. Assuming the opposite, we would have an isomorphism $H^1(A)_1 \xrightarrow{\ell} H^1(A)_2$ and the exact sequence of graded S-modules $0 \to H^0_{S_+}(A/\ell A) \to H^1(A)(-1) \xrightarrow{\ell} H^2(A)$ would imply that $H^0_{T_+}(A/\ell A)_2 \simeq H^0_{S_+}(A/\ell A)_2 = 0$ and hence $H^0_{T_+}(A/\ell A) = 0$ (cf (2.17)). This would imply $\operatorname{depth} A > 1$, a contradiction. This proves our claim and shows (cf (2.1))

(2.22) $$H^1(A) \simeq k(-1) \text{ and } K^1(A) \simeq k(1).$$

By (2.18) (applied for $n = 1$) it follows that the natural map $H^1(A)_1 \to H^1(A/\ell A)_1$ is an isomorphism. So $H^2(A)_0 = 0$. In particular, we get

(2.23) $$P_A(x) = (r+1)x + 1, \text{ end } H^2(A) = -1,$$

where $P_A(x) \in \mathbb{Q}[x]$ is used to denote the Hilbert polynomial of A.

As $K(A)$ is torsion-free over the 2-dimensional domain A (cf (2.9)) and satisfies the second Serre property S_2 (cf [28, 3.1.1]), in view of the second statement of (2.23) we get :

(2.24) $$K(A) \text{ is a } CM\text{-module with beg } K(A) = 1.$$

According to (2.22), the S_+-transform $D(A)$ of A is a domain which appears in a short exact sequence $0 \to A \to D(A) \to k(-1) \to 0$. Choosing $\bar{y} \in D(A)_1 \setminus A_1$ we obtain $D(A) = A[\bar{y}]$. So, if y is an indeterminate, there is a surjective homomorphism of homogeneous k-algebras $S[y] = k[x_0, \cdots, x_r, y] \overset{\gamma}{\twoheadrightarrow} D(A)$, sending y to $\bar{y}$ and extending the natural map $\alpha : S \to A$ (cf part C)). In particular $\tilde{X} := \mathrm{Proj}(D(A))$ is a curve of degree $r+1$ in $\mathrm{Proj}(S[y]) = \mathbb{P}^{r+1}_k$ – a rational normal curve. Moreover, the natural morphism $\varepsilon : \tilde{X} \to X$ is an isomorphism induced by the simple projection $\varrho : \mathbb{P}_k^{r+1}\backslash\{p\} \twoheadrightarrow \mathbb{P}_k^r$ with $\{p\} = |\operatorname{Proj}(S[y]|S_+S[y])|$ $(\notin \tilde{X})$. That is, we are in the case $\bar{\text{II}}$ of [4, (4.7) B)].

3. The case "Arithmetic Depth $= 1$"

In this section we study varieties of almost minimal degree and arithmetic depth one. In particular, we shall extend the results of part D) of Remark 2.5 from curves to higher dimensions.

Proposition 3.1. *Let $X \subset \mathbb{P}_k^r$ be a projective variety of almost minimal degree such that* $\operatorname{depth} A = 1$ *and* $\dim X = d$*. Then*

(a) $H^i(A) = K^i(A) = 0$ *for all* $i \neq 1, d+1$*;*
(b) $\operatorname{end} H^{d+1}(A) = -\operatorname{beg} K(A) = -d$*;*
(c) $H^1(A) \simeq k(-1),\ K^1(A) \simeq k(1)$*;*
(d) $K(A)$ *is a torsion-free* CM*-module of rank one;*
(e) $D(A)$ *is a homogeneous* CM *integral domain with* $\operatorname{reg} D(A) = 1$ *and* $\dim_k D(A)_1 = r+2$*.*

Proof. (*Induction on* $d = \dim X$). For $d = 1$ all our claims are clear by the results of part D) of Remark 2.5.

So, let $d > 1$. Let $\ell \in S_1\backslash\{0\}$ be generic. Then in the notation of part E) of Remark 2.3 we have $\dim A' = d$ and $\operatorname{depth} A' = 1$ (cf (2.16)). By induction $H^i_{T_+}(A') = 0$ for all $i \neq 1, d$. So, by (2.15) we obtain

$$H^i(A) = 0, \text{ for all } i \neq 1, 2, d+1. \tag{3.1}$$

Moreover, by induction and in view of (2.13) we get $H^1_{T_+}(A/\ell A) \simeq k(-1)$. As $H^1(A)$ is a non-zero and finitely generated graded S-module, we have $\ell H^1(A) \neq H^1(A)$. So, by (2.12) we obtain

$$H^1(A)/\ell H^1(A) \simeq k(-1) \tag{3.2}$$

and

$$H^2(A) = 0. \tag{3.3}$$

Combining (3.1), (3.3) and (2.1), we get claim (a). By induction

$$\operatorname{end} H^d(A/\ell A) = \operatorname{end} H^d_{T_+}(A') = -d+1.$$

As $H^d(A) = 0$, (2.12) gives $\operatorname{end} H^{d+1}(A) = -d$. In view of (2.1) we get claim (b). Also, by induction $\operatorname{depth} K^d_T(A') = d$. As $d > 0$, we have $H^d_{T_+}(A') \simeq H^d_{T_+}(A/\ell A)$ and hence $K^d_T(A/\ell A) \simeq K^d_T(A')$, (cf (2.1)). As $K^d(A) = 0$, (2.9) and (2.11) prove statement (d). Moreover $D(A)$ is a positively graded finite integral extension domain of A such that $H^1_{S_+}(D(A)) = 0$ and $H^i_{S_+}(D(A)) \simeq H^i(A)$ for all $i > 1$, it follows from statements (a) and (b), that $D(A)$ is a CM-ring with $\operatorname{reg} D(A) = 1$. In view of (3.2) and the natural exact sequence

$$0 \to A \xrightarrow{\eta} D(A) \xrightarrow{\xi} H^1(A) \to 0 \tag{3.4}$$

there is some $\delta \in D(A)_1\backslash A$ such that $D(A) = A + \delta A$. In particular we have $D(A) = A[\delta]$ and $D(A)_1 \simeq A \oplus k$. Therefore statement (e) is proved.

It remains to show statement (c). In view of (2.1) it suffices to show that $H^1(A) \simeq k(-1)$. By (3.2) and as $H^1(A)_n = 0$ for all $n \gg 0$, there is an isomorphism of graded S-modules $H^1(A) \simeq S/\mathfrak{q}(-1)$, where $\mathfrak{q} \subset S$ is a graded S_+-primary ideal. We have to show that $\mathfrak{q} = S_+$. There is a minimal epimorphism of graded S-modules

$$\pi : S \oplus S(-1) \to D(A) \to 0$$

such that $\pi \restriction_S$ coincides with the natural map $\alpha : S \twoheadrightarrow A$ and $\pi(S(-1)) = \delta A = \delta S$. As $\operatorname{beg}(\text{Ker }(\alpha) = I) \geq 2$ and $\pi(S_1) \cap \pi(S(-1)_1) = A_1 \cap \delta k = 0$, it follows $\operatorname{beg}\text{Ker }(\pi) \geq 2$. Moreover, by statement (e) we have $\operatorname{reg} D(A) = 1$. Therefore a minimal free presentation of $D(A)$ has the form

$$S^\beta(-2) \to S \oplus S(-1) \xrightarrow{\pi} D(A) \to 0 \tag{3.5}$$

with $\beta \in \mathbb{N}_0$. It follows $\operatorname{Tor}_1^S(k, D(A)) \simeq k^\beta(-2)$. As $1 = \eta(1)$ is a minimal generator of the S-module $D(A)$, the sequence (3.4) induces an epimorphism of graded S-modules

$$\operatorname{Tor}_1^S(k, D(A)) \to \operatorname{Tor}_1^S(k, H^1(A)) \to 0.$$

Therefore

$$(\mathfrak{q}/S_+\mathfrak{q})(-1) \simeq \operatorname{Tor}_0^S(k, \mathfrak{q}(-1)) \simeq \operatorname{Tor}_1^S(k, (S/\mathfrak{q})(-1)) \simeq \operatorname{Tor}_1^S(k, H^1(A))$$

is concentrated in degree 2. So, by Nakayama, $\mathfrak{q}$ is generated in degree one, thus $\mathfrak{q} = S_+$. $\square$

Varieties of almost minimal degree and arithmetic depth one can be characterized as simple generic projections from varieties of minimal degree.

Reminder 3.2. A) Recall that an irreducible reduced non-degenerate projective variety $\tilde{X} \subset \mathbb{P}_k^s$ is said to be of *minimal degree* if deg $\tilde{X}$ = codim $\tilde{X} + 1$.
B) Projective varieties of minimal degree are rather well understood, namely (cf e.g. [21, Theorem 19.9]): A projective variety $\tilde{X} \subset \mathbb{P}_k^s$ of minimal degree is either

(3.6) a quadric hypersurface ,

(3.7) a (cone over a) Veronese surface in $\mathbb{P}_k^5$ or

(3.8) a (cone over a) rational normal scroll.

C) In particular, a variety $\tilde{X} \subset \mathbb{P}_k^S$ of minimal degree is arithmetically Cohen-Macaulay and arithmetically normal.

Remark 3.3. A) Let $\tilde{X} \subset \mathbb{P}_k^s$ be an irreducible reduced projective variety, let $p \in \mathbb{P}_k^s \backslash \tilde{X}$, let $\varrho : \mathbb{P}_k^s \backslash \{p\} \twoheadrightarrow \mathbb{P}_k^{s-1}$ be a projection with center p and let $X := \varrho(\tilde{X}) \subset \mathbb{P}_k^{s-1}$. Then, the induced morphism $\varrho \restriction : \tilde{X} \twoheadrightarrow X$ is finite. Moreover, we have $\deg X = \deg \tilde{X}$ if and only if $\varrho \restriction$ is birational, hence if and only if there is a line $\bar{\ell} \subset \mathbb{P}_k^s$ with $p \in \bar{\ell}$ and such that the scheme $\bar{\ell} \cap \tilde{X}$ is non-empty, reduced and irreducible. It is equivalent to say that there are lines $\bar{\ell} \subset \mathbb{P}_k^s$ which join p and $\tilde{X}$ and are not secant lines of $\tilde{X}$.

But this means precisely that the join Join$(p, \tilde{X})$ of p and $\tilde{X}$ is not contained in the secant cone $\operatorname{Sec}_p(\tilde{X})$ of $\tilde{X}$ of p. Observe that here $\operatorname{Sec}_p(\tilde{X})$ is understood as the union of all lines $\bar{\ell} \subset \mathbb{P}_k^s$ such that $\bar{\ell} \cap \tilde{X}$ is a scheme of dimension 0 and of degree ≥ 1.

Also, $\varrho \restriction$ is an isomorphism if and only if for any line $\bar{\ell} \subset \mathbb{P}_k^s$ with $p \in \bar{\ell}$, the scheme $\bar{\ell} \cap \tilde{X}$ is either empty or reduced and irreducible. It is equivalent to say that $p \notin \operatorname{Sec}(\tilde{X})$, where the secant variety $\operatorname{Sec}(\tilde{X})$ of $\tilde{X}$ is understood as the union of all lines $\bar{\ell} \subset \mathbb{P}_k^s$ such that $\bar{\ell} \cap \tilde{X}$ is a scheme of dimension 0 and of degree > 1, or else $\bar{\ell} \subset \tilde{X}$.

B) Assume now in addition that $\tilde{X} \subset \mathbb{P}_k^s$ is of minimal degree. Then by the above observation we can say:

$$X \subset \mathbb{P}_k^{s-1} \text{ is of almost minimal degree if and only if } \operatorname{Join}(p, \tilde{X}) \not\subset \operatorname{Sec}_p(\tilde{X}). \tag{3.9}$$

$$\begin{aligned} &X \subset \mathbb{P}_k^{s-1} \text{ is of almost minimal degree and } \varrho \restriction: \tilde{X} \to X \text{an isomorphism} \\ &\text{if and only if } p \notin \operatorname{Sec}(\tilde{X}), \text{ thus if and only if } \operatorname{Sec}_p(\tilde{X}) = \{p\}. \end{aligned} \tag{3.10}$$

Now, we can give the announced geometric characterization of varieties of almost minimal degree and arithmetic depth one.

Proposition 3.4. *The following statements are equivalent:*

(i) *X is of almost minimal degree and of arithmetic depth* 1.

(ii) *X is the projection $\varrho(\tilde{X})$ of a variety $\tilde{X} \subset \mathbb{P}_k^{r+1}$ of minimal degree from a point $p \in \mathbb{P}_k^{r+1} \setminus \operatorname{Sec}(\tilde{X})$.*

Proof. (i) $\Longrightarrow$ (ii): Assume that X is of almost minimal degree with $\operatorname{depth} A = 1$. Then, by statement e) Proposition 3.1, there is some $\bar{y} \in D(A)_1 \setminus A_1$ such that $D(A) = A[\bar{y}]$. Now, as in the last paragraph of part D) in Remark 2.5, we may view $\tilde{X} := \operatorname{Proj}(D(A))$ as a non-degenerate irreducible projective variety in $\mathbb{P}_k^{r+1}$ such that $\deg(\tilde{X}) = \deg(X)$ and such a projection $\varrho : \mathbb{P}_k^{r+1} \setminus \{p\} \twoheadrightarrow \mathbb{P}_k^r$ from an appropriate point $p \in \mathbb{P}_k^{r+1} \setminus \tilde{X}$ induces an isomorphism $\varrho \restriction: \tilde{X} \to X$. In view of (3.10) this proves statement (ii).

(ii) $\Longrightarrow$ (i): Assume that there is a variety $\tilde{X} \subset \mathbb{P}_k^{r+1}$ of minimal degree and a projection $\varrho : \mathbb{P}_k^{r+1} \setminus \{p\} \twoheadrightarrow \mathbb{P}_k^r$ from a point $p \notin \operatorname{Sec}(\tilde{X})$ with $X = \varrho(\tilde{X})$. Then, by (3.10), X is of almost minimal degree and $\varrho \restriction: \tilde{X} \to X$ is an isomorphism. It remains to show that $\operatorname{depth} A = 1$, hence that $H^1(A) \neq 0$. So, let B denote the homogeneous coordinate ring of $\tilde{X} \subset \mathbb{P}_k^{r+1}$. Then, the isomorphism $\varrho \restriction: \tilde{X} \to X$ leads to an injective homomorphism of graded integral domains $A \hookrightarrow B$ such that B/A is an A-module of finite length. Therefore $B \subset \bigcup_{n \in \mathbb{N}} (A :_B S_+^n) = D(A)$. As $\tilde{X} \subset \mathbb{P}_k^{r+1}$ is non-degenerate, we have $\dim_k B_1 = r + 2 > \dim_k A_1$, hence $A \subsetneqq B \subset D(A)$ so that $H^1(A) \simeq D(A)/A \neq 0$. $\square$

4. The non-arithmetically Cohen-Macaulay case

In this section we study projective varieties of almost minimal degree which are not arithmetically Cohen-Macaulay. So, we are interested in the case where $\deg X = \operatorname{codim} X + 2$ and $1 \leq \operatorname{depth} A \leq \dim X$.

Our first aim is to generalize Proposition 3.1. In order to do so, we prove the following auxiliary result, in which $\operatorname{NZD}_S(M)$ is used to denote the set of non-zero divisors in S with respect to the S-module M.

Lemma 4.1. *Let M be a finitely generated graded S-module, let $m \in \{0, \cdots, r\}$ and $n \in \mathbb{Z}$. Let $z_m, \cdots, z_r \in S_1$ be linearly independent over k such that $z_m \in \operatorname{NZD}_s(M)$ and such that there is an isomorphism of graded S-modules $M/z_m M \simeq (S/(z_m, \cdots, z_r))(n)$.*

Then, there are linearly independent elements $y_{m+1}, \cdots, y_r \in S_1$ such that there is an isomorphism of graded S-modules $M \simeq (S/(y_{m+1}, \cdots, y_r))(n)$.

Proof. By Nakayama there is an isomorphism of graded S-modules $M \simeq (S/\mathfrak{q})(n)$, where $\mathfrak{q} \subset S$ is a homogeneous ideal. In particular

$$z_m \in \mathrm{NZD}_S(S/\mathfrak{q}). \tag{4.1}$$

As $S/(z_m, \mathfrak{q}) \simeq (M/z_m M)(-n) \simeq S/(z_m, \cdots, z_r)$, we have

$$(z_m, \mathfrak{q}) = (z_m, z_{m+1}, \cdots, z_r).$$

Also, by (4.1), we have $z_m \notin \mathfrak{q}_1$, so that $\mathfrak{q}_1$ becomes a k-vector space of dimension $r - m$. Let $y_{m+1}, \cdots, y_r \in S_1$ form a k-basis of $\mathfrak{q}_1$. As

$$(y_{m+1}, \cdots, y_r) \subset \mathfrak{q} \subset (z_m, y_{m+1}, \cdots, y_r)$$

and in view of (4.1) we obtain

$$\mathfrak{q} = (y_{m+1}, \cdots, y_r) + \mathfrak{q} \cap z_m S = (y_{m+1}, \cdots, y_r) + z_m \mathfrak{q}.$$

So, by Nakayama $\mathfrak{q} = (y_{m+1}, \cdots, y_r)$. ☐

Now, we are ready to prove the first main result of this section, which recover results of [23] written down in the context of the modules of deficiency.

Theorem 4.2. *Assume that $X \subset \mathbb{P}^r_k$ is of almost minimal degree and that $t := \operatorname{depth} A \leq \dim X =: d$. Then*

(a) $H^i(A) = K^i(A) = 0$ *for all* $i \neq t, d+1$*;*
(b) $\operatorname{end} H^{d+1}(A) = -\operatorname{beg} K(A) = -d$*;*
(c) *There are linearly independent forms $y_{t-1}, \cdots, y_r \in S_1$ such that there is an isomorphism of graded S-modules*

$$K^t(A) \simeq (S/(y_{t-1}, \cdots, y_r))(2-t);$$

(d) $K(A)$ *is a torsion-free CM-module of rank one.*

Proof. (*Induction on t*). The case $t = 1$ is clear by Proposition 3.1.

So, let $t > 1$ and $\ell \in S_1 \setminus \{0\}$ be generic. Then, in the notation of Remark 2.3 E) we have $A' = A/\ell A$, $\dim A' = \dim Y + 1 = d$ and $\operatorname{depth} A' = t - 1 \leq d - 1 = \dim Y$ (cf (2.16)).

(a): By induction, $H^j_{T_+}(A') = 0$ for all $j \neq t-1, d$. So, (2.15) gives $H^i(A) = 0$ for all $i \neq 0, 1, t, d+1$. As $t > 1$, we have $H^0(A) = H^1(A) = 0$. In view of (2.1) this proves statement (a).

(b): By induction $\operatorname{end} H^d_{T_+}(A/\ell A) = -d + 1$. As $H^d(A) = 0$, (2.12) implies that $\operatorname{end} H^{d+1}(A) = -d$ and (2.1) gives our claim.

(c): By induction there are forms $z_{t-1}, \cdots, z_r \in S_1$ whose images $\bar{z}_{t-1}, \cdots, \bar{z}_r \in T_1$ are linearly independent over k and such that there is an isomorphism of graded T-modules $K^{t-1}(A/\ell A) = (T/(\bar{z}_{t-1}, \cdots, \bar{z}_r))(2 - (t-1))$. Let $z_{t-2} := \ell$. Then $z_{t-2}, \cdots, z_r \in S_1$ are linearly independent and

$$K^{t-1}_T(A/\ell A) \simeq (S/(z_{t-2}, z_{t-1}, \cdots, z_r))(3-t).$$

By statement (a) we have $K^{t-1}(A) = 0$. So, the sequence (2.11) gives an isomorphism of graded S-modules

$$K^t(A)/z_{t-2}K^t(A) \simeq (S/(z_{t-2}, \cdots, z_r))(2-t). \tag{4.2}$$

Assume first, that $t < d$. By induction $K_T^t(A/z_{t-2}A) = K_T^t(A')$ vanishes and hence (2.11) yields $0 :_{K^t(A)} z_{t-2} = 0$, thus $z_{t-2} \in \mathrm{NZD}_S(K^t(A))$. So, (4.2) and Lemma 4.1 imply statement (c).

Now, let $t = d$. Then (4.2) implies $\dim K^d(A)/z_{t-2}K^d(A) = d - 2$. Our first aim is to show that $\dim K^d(A) = d - 1$. If $d > 2$, this follows by the genericity of $z_{t-2} = \ell$. So, let $t = d = 2$. Then $A/\ell A$ is a domain of depth 1 which is the coordinate ring of a curve $Y \subset \mathbb{P}_k^{r-1}$ of almost minimal degree (cf Remark 2.5 A)). So, according to Remark 2.5 D) we have $H^1(A/\ell A) \simeq k(-1)$ and $H^2(A/\ell A)_n = 0$ for all $n \geq 0$. If we apply cohomology to the exact sequence $0 \to A(-1) \xrightarrow{\ell} A \to A/\ell A \to 0$ and keep in mind that $H^1(A) = 0$ we thus get $\ell : H^2(A)_{-1} \xrightarrow{\simeq} H^2(A)_0 \simeq k$. According to Remark 2.5 D) there is an isomorphism $\tilde{Y} \xrightarrow{\simeq} Y$, where $\tilde{Y} \subset \mathbb{P}_k^{r-1}$ is a rational normal curve, so that $Y \simeq \mathbb{P}_k^1$ is smooth. As Y is a hyperplane section of X it follows that the non-singular locus of X is finite. So, if we apply [1, Proposition 5.2] to the ample sheaf of $\mathcal{O}_X$-modules $\mathcal{L} := \mathcal{O}_X(1)$ and observe that $H^2(A)_n \simeq H^1(X, \mathcal{L}^{\otimes n})$ for all $n \in \mathbb{Z}$, we get that $H^2(A)_n \simeq k$ for all $n \leq 0$. Consequently, $K^2(A)_n \neq 0$ for all $n \geq 0$, hence $\dim K^2(A) > 0 = \dim K^2(A)/z_0K^2(A)$. Therefore $\dim K^2(A) = 1$, which concludes the case $t = d$.

According to (4.2) the S-module $K^d(A)/z_{t-2}K^d(A)$ is generated by a single homogeneous element of degree $d - 2$. By Nakayama, $K^d(A)$ has the same property. So, there is a graded ideal $\mathfrak{q} \subset S$ with $K^d(A) \simeq (S/\mathfrak{q})(2 - d)$. In particular we have $\dim S/\mathfrak{q} = d - 1$.

Now, another use of (4.2) yields

$$\begin{aligned} S/(\mathfrak{q}, z_{d-2}) \simeq (S/\mathfrak{q})/z_{d-2}(S/\mathfrak{q}) &\simeq K^t(A)(d-2)/z_{d-2}K^t(A)(d-2) \\ &\simeq (K^t(A)/z_{d-2}K^t(A))(d-2) \simeq S/(z_{t-2}, \cdots, z_r), \end{aligned}$$

so that $(\mathfrak{q}, z_{d-2}) = (z_{t-2}, \cdots, z_r)$ is a prime ideal. As

$$\dim S/\mathfrak{q} = d - 1 > \dim(S/(z_{t-2}, \cdots, z_r))$$

it follows, that $\mathfrak{q}$ is a prime ideal. Moreover, as $z_{d-2} \notin \mathfrak{q}$, we obtain $z_{d-2} \in \mathrm{NZD}_S(S/\mathfrak{q}) = \mathrm{NZD}_S(K^d(A))$.

Now, our claim follows from (4.2) and Lemma 4.1.

(d): In view (2.9) it remains to show that $\operatorname{depth} K(A) = d + 1$. By (2.9) and by induction we have

$$\ell \in \mathrm{NZD}_S(K(A)) \text{ and } \operatorname{depth} K_T^d(A/\ell A) = d. \tag{4.3}$$

So, by the sequence (2.11), applied with $i = d$, it suffices to show that $0 :_{K^d(A)} \ell = 0$. If $t < d$, this last equality follows from statement (a). If $t = d$, statement (c) yields $\operatorname{depth} K^d(A) = d - 1 > 0$ and by the genericity of ℓ we get $\ell \in \mathrm{NZD}_S(K^d(A))$. □

Remark 4.3. Keep the notations and hypotheses of Theorem 4.2. Then, by statement (c) of Theorem 4.2 and in view of (2.1) we get end $H^t(A) = 2 - t$. So, by statements (a) and (b) of Theorem 4.2 we obtain

$$\operatorname{reg}(A) = 2 \text{ and } \dim_k A_n = P_A(n), \text{ for all } n > 2 - t. \tag{4.4}$$

Corollary 4.4. *Let $X \subset \mathbb{P}_k^r$ be of almost minimal degree with $\dim X = d$ and $\operatorname{depth} A = t$. Then:*

(a) *The Hilbert series of A is given by*

$$F(\lambda, A) = \frac{1 + (r+1-d)\lambda}{(1-\lambda)^{d+1}} - \frac{\lambda}{(1-\lambda)^{t-1}}.$$

(b) *The Hilbert polynomial of A is given by*

$$P_A(n) = (r-d+2)\binom{n+d-1}{d} + \binom{n+d-1}{d-1} - \binom{n+t-2}{t-2}.$$

(c) *The number of independent quadrics in I is given by*

$$\dim_k(I_2) = t + \binom{r+1-d}{2} - d - 2.$$

Proof. (a): (*Induction on t*). If $t = 1$, $D(A)$ is a CM-module of regularity 1 (cf Proposition 3.1 (e)) and of multiplicity $\deg X = r - d + 2$. Therefore

$$F(\lambda, D(A)) = \frac{1 + (r+1-d)\lambda}{(1-\lambda)^{d+1}}. \tag{4.5}$$

In view of statement c) of Proposition 3.1 we thus get

$$F(\lambda, A) = F(\lambda, D(A)) - \lambda = \frac{1 + (r+1-d)\lambda}{(1-\lambda)^{d+1}} - \lambda$$

and hence our claim.

So, let $t > 1$. Then, as $t' = t - 1$ (cf (2.16)) and $A' = A/\ell A$ we get by induction

$$\begin{aligned} F(\lambda, A) = \frac{F(\lambda, A')}{1-\lambda} &= [\frac{1 + ((r-1) + 1 - (d-1))\lambda}{(1-\lambda)^{d}} - \frac{\lambda}{(1-\lambda)^{t-2}}](1-\lambda)^{-1} \\ &= \frac{1 + (r+1-d)\lambda}{(1-\lambda)^{d+1}} - \frac{\lambda}{(1-\lambda)^{t-1}}. \end{aligned}$$

(b), (c): These are purely arithmetical consequences of statement (a). □

Remark 4.5. Observe that Corollary 4.4 also holds if X is arithmetically Cohen-Macaulay. In this case, the shape of the Hilbert series $F(\lambda, A)$ (cf statement (a)) yields that A is a Gorenstein ring (cf [29]) which says that a projective variety of almost minimal degree which is arithmetically Cohen-Macaulay is already arithmetically Gorenstein. For $\dim X = 0$ this may be found in [23].

Finally, by Remark 2.5 B), by statement (2.9) and the exact sequence (2.12) it follows immediately by induction on $d = \dim X$ that $K(A) \cong A(1-d)$ if X is is arithmetically Cohen-Macaulay. This shows again that X is arithmetically Gorenstein.

5. Endomorphism Rings of Canonical Modules

Or next aim is to extend the geometric characterization of Proposition 3.4 to arbitrary non-arithmetically Cohen-Macaulay varieties of almost minimal degree.

We attack this problem via an analysis of the properties of the endomorphism ring of the canonical module $K(A)$ of A, which in the local case has been studied already in [2]. The crucial point is, that this ring has a geometric meaning in the context of varieties of almost minimal degree.

Notation 5.1. We write B for the endomorphism ring of the canonical module of A, thus

$$B := \mathrm{Hom}_S(K(A), K(A)).$$

Observe that B is a finitely generated graded A-module and that

$$B = \mathrm{Hom}_A(K(A), K(A)). \tag{5.1}$$

In addition we have a homomorphism of graded A-modules

$$\varepsilon : A \to B,\ a \mapsto a\, \mathrm{id}_{K(A)}. \tag{5.2}$$

Keep in mind, that B carries a natural structure of (not necessarily commutative) ring and that ε is a homomorphism of rings.

The homomorphism $\varepsilon : A \to B$ occurs to be of genuine interest for its own. So we give a few properties of it.

Proposition 5.2. *Let $d := \dim X \geq 1$. Then*

(a) $B = k \oplus B_1 \oplus B_2 \oplus \cdots$ *is a positively graded commutative integral domain of finite type over $B_0 = k$.*
(b) $\varepsilon : A \to B$ *is a finite injective birational homomorphism of graded rings.*
(c) *There is a (unique) injective homomorphism $\tilde{\varepsilon}$ of graded rings, which occurs in the commutative diagram*

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr, "\varepsilon"'] & & D(A) \arrow[dl, "\tilde{\varepsilon}"] \\ & B & \end{tikzcd}$$

(d) *If $\mathfrak{p} \in \mathrm{Spec}(A)$, the ring $A_{\mathfrak{p}}$ has the second Serre property S_2 if and only if the localized map $\varepsilon_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is an isomorphism.*
(e) $\varepsilon : A \to B$ *is an isomorphism if and only if A satisfies S_2.*
(f) $\tilde{\varepsilon} : D(A) \to B$ *is an isomorphism if and only if X satisfies S_2.*
(g) *B satisfies S_2 (as an A-module and as a ring).*
(h) *If the A-module $K(A)$ is Cohen-Macaulay, then B is Cohen-Macaulay (as an A-module and as a ring).*

Proof. (a), (b): By (2.9) (cf Remark 2.3) we know that $K(A)$ is torsion-free and of rank one. From this it follows easily that B is a commutative integral domain. Also the map $\varepsilon : A \to B$ is a homomorphism of A-modules, and so becomes injective by the torsion-freeness of the A-module $K(A)$. The intrinsic A-module structure on B and the A-module structure induced by ε are the same. As B is finitely generated as an A-module it follows that ε is a finite homomorphism of rings.

It is easy to verify that the natural grading of the A-module B respects the ring structure on B and thus turns B into a graded ring. In particular ε becomes a homomorphism of graded rings. As A is positively graded, ε is finite and B is a domain, it follows that B is finite. As k is algebraically closed and B_0 is a domain, we get $B_0 \simeq k$. As A is of finite type over k and ε is finite, B is of finite type over k, too.

(c): As $\dim A > 1$ we know that $H^1_{S_+}(A)$ is of finite length. Therefore

$$\mathrm{Ext}^j_S(H^1_{S_+}(A), S) = 0 \text{ for all } j \neq r+1.$$

So, the short exact sequence $0 \to A \to D(A) \to H^1(A) \to 0$ yields an isomorphism of graded A-modules

$$K(A) = \mathrm{Ext}_S^{r-d}(A, S(-r-1)) \simeq \mathrm{Ext}_S^{r-d}(D(A), S(-r-1)).$$

Therefore, $K(A)$ carries a natural structure of graded $D(A)$-module. As $D(A)$ is a birational extension ring of A, we can write

$$B = \mathrm{Hom}_A(K(A), K(A)) = \mathrm{Hom}_{D(A)}(K(A), K(A))$$

and hence consider B as a graded $D(A)$-module in a natural way. In particular, there is a homomorphism of rings

$$\tilde{\varepsilon} : D(A) \to B,\ c \mapsto c\,\mathrm{id}_{K(A)},$$

the unique homomorphism of rings $\tilde{\varepsilon}$ which appears in the commutative diagram

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr, "\varepsilon"'] & & D(A) \arrow[dl, "\tilde{\varepsilon}"] \\ & B & \end{tikzcd}$$

As $A, D(A)$ and B are domains and as ε is injective, $\tilde{\varepsilon}$ is injective, too. Clearly $\tilde{\varepsilon}$ is finite, and respects gradings.

(d): Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then, by the chain condition in $\mathrm{Spec}(S)$, $K(A)_{\mathfrak{p}}$ is nothing else than the canonical module $K_{A_{\mathfrak{p}}}$ of the local domain $A_{\mathfrak{p}}$. In particular we may identify $B_{\mathfrak{p}} = \mathrm{Hom}_A(K(A), K(A))_{\mathfrak{p}} \simeq \mathrm{Hom}_{A_{\mathfrak{p}}}(K_{A_{\mathfrak{p}}}, K_{A_{\mathfrak{p}}})$. Then, the natural map $\varepsilon_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ induced by ε coincides with the natural map

$$A_{\mathfrak{p}} \to \mathrm{Hom}_{A_{\mathfrak{p}}}(K_{A_{\mathfrak{p}}}, K_{A_{\mathfrak{p}}}), \quad b \mapsto b\,\mathrm{id}_{K_{A_{\mathfrak{p}}}}.$$

But this latter map is an isomorphism if and only if $A_{\mathfrak{p}}$ satisfies S_2 (cf [28, 3.5.2]).

(e): Is clear by statement (d).

(f): By statement (d), X satisfies S_2 if and only if $\varepsilon_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in \mathrm{Proj}(A)$. But this latter statement is equivalent to the fact that $B/\varepsilon(A)$ has finite length, thus to the fact that $B \subset \varepsilon(A) :_B A_+^n$ for some n, hence to $B \subset \tilde{\varepsilon}(D(A))$.

(g): Let $\mathfrak{p} \in \mathrm{Spec}(A)$ of $\mathrm{height} \geq 2$. Then, the canonical module $K_{A_{\mathfrak{p}}}$ is of $\mathrm{depth} \geq 2$ (cf [28, 3.1.1]). So, there is a $K_{A_{\mathfrak{p}}}$-regular sequence $x, y \in A_{\mathfrak{p}}$. By the left-exactness of the functor $\mathrm{Hom}_{A_{\mathfrak{p}}}(K_{A_{\mathfrak{p}}}, \cdot)$ it follows that x, y is a regular sequence with respect to $\mathrm{Hom}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}, A_{\mathfrak{p}}) = B_{\mathfrak{p}}$, so that $\mathrm{depth}_{A_{\mathfrak{p}}} B_{\mathfrak{p}} \geq 2$. This shows that the A-module B satisfies S_2. As B is finite over A, it satisfies S_2 as a ring.

(h): Assume that $K(A)$ is a Cohen-Macaulay module. Consider the exact sequence of graded A-modules $0 \to A \xrightarrow{\varepsilon} B \to B/A \to 0$. By statement (d) we have $(B/A)_{\mathfrak{p}} = 0$ as soon as $\mathfrak{p} \in \mathrm{Spec}(A)$ is of height ≤ 1. So $\dim B/A \leq d-1$ and ε induces an isomorphism of graded S-modules $H^{d+1}(A) \simeq H^{d+1}(B)$. By (2.1) we get an isomorphism of graded S-modules $K^{d+1}(B) \simeq K(A)$. Therefore, the A-module $K^{d+1}(B)$ is Cohen-Macaulay. In view of [28, 3.2.3] we thus get $H^i_{A_+}(B) = 0$ for $i = 2, \cdots, d$. By statement g) we have $H^i_A(B) = 0$ for $i = 0, 1$. So, the A-module B is Cohen-Macaulay. As B is finite over A, it becomes a Cohen-Macaulay ring. $\square$

We now apply the previous result in the case of varieties of almost minimal degree. We consider B as a graded extension ring of A by means of $\varepsilon : A \to B$.

Theorem 5.3. *Assume that $X \subset \mathbb{P}^r_k$ is of almost minimal degree and that $t := \operatorname{depth} A \leq \dim X =: d$. Then*

(a) *B is a finite graded birational integral extension domain of A and CM.*
(b) *There are linearly independent linear forms $y_{t-1}, \cdots, y_r \in S_1$ and an isomorphism of graded S-modules $B/A \simeq (S/(y_{t-1}, \cdots, y_r))(-1)$.*
(c) *The Hilbert polynomial of B is given by*

$$P_B(x) = (r-d+2)\binom{x+d-1}{d} + \binom{x+d-1}{d-1}.$$

(d) *If $t = 1$, then $B = D(A)$.*

Proof. (a): According to statement (d) of Theorem 4.2 the A-module $K(A)$ is CM. So our claim follows form statements (a), (b) and (h) of Proposition 5.2.

(d): Let $t = 1$. According to Proposition 3.4 and statement (3.10) of Remark 3.3, X is isomorphic to a variety $\tilde{X} \subset \mathbb{P}^{r+1}_k$ of minimal degree and thus CM (cf Reminder 3.2, part B)). So, by statement (f) of Proposition 5.2 we have $D(A) = B$.

(b): We proceed by induction on t. If $t = 1$, statement (d) gives $B/A \simeq H^1(A)$ and so we may conclude by statement (c) of Proposition 3.1. Let $t > 1$. We write $C = B/A$ and consider the exact sequence of graded S-modules

$$0 \to A \xrightarrow{\varepsilon} B \to C \to 0.$$

Let $\ell \in S_1 \backslash \{0\}$ be generic. As $t > 1$ and B is CM (as an A-module) we have $\operatorname{depth}_A(C) > 0$. Therefore $\ell \in \mathrm{NZD}(C)$. We write $A' = A/\ell A$ and $T = S/\ell S$. Then A' is a domain and $Y := \operatorname{Proj}(A') \subset \operatorname{Proj}(T) = \mathbb{P}^{r-1}_k$ is a variety of almost minimal degree (cf Remark 2.3).

Let $K(A') := K^d_T(A'), B' := \operatorname{Hom}_T(K(A'), K(A'))$ and $C' = B'/A'$. By induction there are linearly independent linear forms $\bar{z}_{t-1}, \cdots, \bar{z}_r \in T_1$ and an isomorphism of graded T-modules $B'/A' \simeq (T/(\bar{z}_{t-1}, \cdots, \bar{z}_r)T)(-1)$. We write $\ell = z_{t-2}$. Then, there are linear forms $z_{t-1}, \cdots, z_r \in S_1$ such that $z_{t-2}, z_{t-1}, \cdots, z_r$ are linearly independent and such that there is an isomorphism of graded S-modules $C' \simeq (S/(z_{t-2}, \cdots, z_r)S)(-1)$. As $z_{t-2} = \ell \in \mathrm{NZD}(C)$, it suffices to show that there is an isomorphism of graded S-modules $C/\ell C \simeq C'$ (cf Lemma 4.1). As $\ell \in \mathrm{NZD}(C)$ there is an exact sequence of graded S-modules

$$0 \to A' \xrightarrow{\alpha} B/\ell B \to C/\ell C \to 0$$

in which α is induced by ε. It thus suffices to construct an isomorphism of graded A-modules $\bar{\gamma}$, which occurs in the commutative diagram

$$\begin{tikzcd} & A' \arrow[dl, "\alpha"'] \arrow[dr, "\varepsilon'"] & \\ B/\ell B \arrow[rr, "\bar{\gamma}"'] & & B' \end{tikzcd}$$

where ε' is used to denote the natural map. As A' is a domain and as the A-module B is CM and torsion free of rank 1 (by statement (a)), the A'-module $B/\ell B$ is torsion-free and again of rank 1 (as $\ell \in S_1$ is generic). By statement (a) the A'-module B' is also torsion-free of rank 1.

So, it suffices to find an epimorphism $\gamma : B \to B'$, which occurs in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{can}_0"] \arrow[d, "\varepsilon"'] & A' \arrow[d, "\varepsilon"] & \\ B \arrow[r, "\gamma"] & B' \arrow[r] & 0 \end{tikzcd}$$

and hence such that $\gamma(1_B) = 1_{B'}$.

By our choice of ℓ and in view of statements (a), (c) and (d) of Theorem 4.2 we have $\ell \in \mathrm{NZD}(K^d(A)) \cap \mathrm{NZD}(K(A))$. So, by (2.11) of Remark 2.3 we get an exact sequence of graded S-modules

$$0 \to K(A)(-1) \xrightarrow{\ell} K(A) \xrightarrow{\pi} K(A')(1) \to 0.$$

Let $U := \mathrm{Ext}^1_S(K(A), K(A))(-1)$. If we apply the functors

$$\mathrm{Hom}_S(K(A), \cdot) \text{ and } \mathrm{Hom}_S(\cdot, K(A'))(1)$$

to the above exact sequence, we get the following diagram of graded S-modules with exact rows and columns

(5.3)
$$\begin{tikzcd} & 0 \arrow[d] & & \\ & B' = \mathrm{Hom}_S(K(A'), K(A')) \arrow[d, "\nu"] & & \\ \mathrm{Hom}_S(K(A), K(A)) \arrow[r, "\mu"] \arrow[d, equal] & \mathrm{Hom}_S(K(A), K(A'))(-1) \arrow[r] \arrow[d, "0"] & 0 :_U \ell \arrow[r] & 0 \\ B & \mathrm{Hom}_S(K(A), K(A')) & & \end{tikzcd}$$

where $\mu := \mathrm{Hom}_S(id_{K(A)}, \pi), \nu := \mathrm{Hom}_S(\pi, id_{K(A')})$. With $\gamma := \nu^{-1} \circ \mu$, it follows

$$\begin{aligned} \gamma(1_B) &= \gamma(id_{K(A)}) = \nu^{-1}(\mu(id_{K(A)})) = \nu^{-1}(\pi \circ id_{K(A)}) = \nu^{-1}(\pi) \\ &= \nu^{-1}(id_{K(A')(-1)} \circ \pi) = \nu^{-1}(\nu(id_{K(A')})) = id_{K(A')} = 1_{B'}. \end{aligned}$$

So, it remains to show that μ is surjective. It suffices to show that $(0 :_U \ell) = 0$. Assume to the contrary, that $0 :_U \ell \neq 0$. Then ℓ belongs to some associated prime ideal $\mathfrak{p} \in \mathrm{Ass}_S U$. As ℓ is generic, this means that $S_1 \subset \mathfrak{p}$ so that $S_+ \subset \mathfrak{p}$ and hence $0 :_U S_+ \neq 0$, thus $0 :_{(0:_U \ell)} S_+ \neq 0$. Therefore $\mathrm{depth}_S(0 :_U \ell) = 0$. In view of statement (a) we have $\mathrm{depth}_S(B/\ell B) = \mathrm{depth}_S(B') = d$. Moreover the above diagram (5.3) yields an exact sequence of graded S-modules

$$0 \to B/\ell B \to B' \to 0 :_U \ell \to 0,$$

which shows that $\mathrm{depth}_S(0 :_U \ell) \geq d - 1 > 0$, a contradiction.

(c): By statement (b) we have $P_{B/A}(x) = \binom{x+t-3}{t-2}$ if $t > 1$ and $P_{B/A}(x) = 0$ if $t = 1$. In view of statement (a) of Corollary 4.4 we get our claim. $\square$

Now, we are ready to draw a few conclusions about the geometric aspect.

Notation 5.4. A) We convene that $\mathbb{P}_k^{-1} = \emptyset$ and we use $\mathbf{CM}(X)$, $S_2(X)$ and $\mathrm{Nor}(X)$ to denote respectively the locus of Cohen-Macaulay points, S_2-points and normal points of X.

B) If $\nu : \tilde{X} \to X$ is a morphism of schemes, we denote by $\mathrm{Sing}(\nu)$ the set

$$\{x \in X \mid \nu_x^\sharp : \mathcal{O}_{X,x} \xrightarrow{\not\cong} (\nu_* \mathcal{O}_{\tilde{X}})_x\}$$

of all points $x \in X$ over which ν is singular.

Definition 5.5. We say that $x \in X$ is a *Goto* or *G-point*, if the local ring $\mathcal{O}_{X,x}$ is of "Goto type" (cf [17]) thus if $\dim \mathcal{O}_{X,x} > 1$ and

$$H^i_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x}) = \begin{cases} 0, & \text{if } i \neq 1, \dim \mathcal{O}_{X,x}, \\ \kappa(x), & \text{if } i = 1. \end{cases}$$

Theorem 5.6. *Assume that $X \subset \mathbb{P}_k^r$ is of almost minimal degree and that $t := \operatorname{depth} A \leq \dim X =: d$. Then*

(a) *B is the homogeneous coordinate ring of a d-dimensional variety $\tilde{X} \subset \mathbb{P}_k^{r+1}$ of minimal degree.*
(b) *B is the normalization of A.*
(c) *The normalization $\nu : \tilde{X} \to X$ given by the inclusion $\varepsilon : A \to B$ is induced by a projection $\varrho : \mathbb{P}_k^{r+1} \setminus \{p\} \to \mathbb{P}_k^r$ from a point $p \in \mathbb{P}_k^{r+1} \setminus \tilde{X}$.*
(d) *The secant cone $\mathrm{Sec}_p(\tilde{X}) \subset \mathbb{P}_k^{r+1}$ is a projective subspace of dimension $t-1$ and $\mathrm{Sing}(\nu) = \varrho(\mathrm{Sec}_p(\tilde{X}) \setminus \{p\}) \subset X$ is a projective subspace $\mathbb{P}_k^{t-2} \subset \mathbb{P}_k^r$.*
(e) *The generic point $x \in X$ of $\mathrm{Sing}(\nu)$ is a G-point.*
(f) $\mathrm{Nor}(X) = S_2(X) = \mathbf{CM}(X) = X \setminus \mathrm{Sing}\,(\nu)$.

Proof. (a): By Proposition 5.2 (a), (b) we see that B is an integral, positively graded k-algebra of finite type and with $\dim B = \dim A = d+1$. By Theorem 5.3 (b) and on use of Nakayama we have in addition $B = k[B_1]$ with $\dim_k B_1 = \dim_k A_1 + 1 = r+2$. So, B is the homogeneous coordinate ring of a non-degenerate projective variety $\tilde{X} \subset \mathbb{P}_k^{r+1}$ of dimension d. By Theorem 5.3 (c) we have $\deg \tilde{X} = r+1$.

(b): By statement (a) the ring B is normal (cf Reminder 3.2 C)). In addition, B is a birational integral extension ring of A.

(c): This follows immediately from the fact that $\dim_k B_1 = \dim_k A_1 + 1 = r+2$ (cf part C) of Remark 2.5).

(e): According to Theorem 5.3 (b) we have an exact sequence of graded S-modules

$$0 \to A \to B \to (S/P)(-1) \to 0,$$

where $P := (y_{t-1}, \cdots, y_r)S$ with appropriate independent linear forms $y_{t-1}, \cdots, y_r \in S_1$. In particular $0 :_S B/A = P$ and hence $I \subset P$. It follows that

$$\mathrm{Sing}(\nu) = \operatorname{Supp}\operatorname{Coker}(\nu : \mathcal{O}_X \to \nu_* \mathcal{O}_{\tilde{X}}) = \operatorname{Supp}((B/A)^\sim) = \mathrm{Proj}(S/P)$$

and that $x := P/I \in \mathrm{Proj}(A) = X$ is the generic point of $\mathrm{Sing}(\nu)$. Localizing the above sequence at x we get an exact sequence of $\mathcal{O}_{X,x}$-modules

$$0 \to \mathcal{O}_{X,x} \to (\nu_* \mathcal{O}_{\tilde{X}})_x \to \kappa(x) \to 0 \tag{5.4}$$

in which $\mathcal{O}_{X,x}$ has dimension $d-t+1>1$. As B is a CM-module over A (cf Theorem 5.3 (a)), $(\nu_*\mathcal{O}_{\tilde{X}})_x \simeq \tilde{B}_x$ is a CM-module over the local domain $\mathcal{O}_{X,x}$. So, the sequence (5.4) shows that $H^1_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x}) \simeq \kappa(x)$ and $H^i_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x}) = 0$ for all $i \neq 1, \dim(\mathcal{O}_{X,x})$.

(d): Let $P \subset S$ be as above. We already know that $\mathrm{Sing}(\nu) = \mathrm{Proj}(S/P) = \mathbb{P}^{t-2}_k \subset \mathbb{P}^r_k$. Moreover, a closed point $q \in X$ belongs to $\mathrm{Sing}(\nu)$ if and only if the line $\varrho^{-1}(q) \subset \mathbb{P}^{r+1}_k$ is a secant line of $\tilde{X}$. So

$$\mathrm{Sec}_p(\tilde{X}) = \{p\} \cup \varrho^{-1}(\mathrm{Sing}(\nu)) = \{p\} \cup \varrho^{-1}(\mathbb{P}^{t-1}_k) = \mathbb{P}^{t-1}_k \subset \mathbb{P}^{r+1}_k$$

and $\mathrm{Sing}(\nu) = \varrho(\mathrm{Sec}_p(\tilde{X}) \setminus \{p\})$.

(f): $\nu \restriction: \tilde{X} \setminus \nu^{-1}(\mathrm{Sing}(\nu)) \to X \setminus \mathrm{Sing}(\nu)$ is an isomorphism. As $\tilde{X}$ is a normal CM-variety (cf Reminder 3.2 C)) it follows that $\mathrm{Nor}(X), \mathbf{CM}(X) \supseteq X \setminus \mathrm{Sing}(\nu)$. As $S_2(X) \supseteq \mathrm{Nor}(X) \cup \mathbf{CM}(X)$ it remains to show that $S_2(X) \subset X \setminus \mathrm{Sing}(\nu)$. As the generic point x of $\mathrm{Sing}(\nu)$ is not an S_2-point (cf statement (e)), our claim follows. □

Now, we may extend Proposition 3.4 to arbitrary non arithmetically Cohen-Macaulay varieties.

Corollary 5.7. *Let $1 \leq t \leq \dim(X)$. Then, the following statements are equivalent:*

(i) *X is of almost minimal degree and of arithmetic depth t.*

(ii) *X is the projection $\varrho(\tilde{X})$ of a variety $\tilde{X} \subset \mathbb{P}^{r+1}_k$ of minimal degree from a point $p \in \mathbb{P}^{r+1}_k \setminus \tilde{X}$ such that $\dim \mathrm{Sec}_p(\tilde{X}) = t-1$.*

Proof. (i) $\Longrightarrow$ (ii): Clear by Theorem 5.6 (d).

(ii) $\Longrightarrow$ (i): Let $\tilde{X} \subset \mathbb{P}^{r+1}_k$, p and $\varrho : \mathbb{P}^{r+1}_k \setminus \tilde{X} \to X$ be as in statement (ii). Let $\tilde{A}$ be the homogeneous coordinate ring of $\tilde{X}$. Then $\tilde{A}$ is normal (cf Reminder 3.2 C)). Observe that $\varrho \restriction: \tilde{X} \twoheadrightarrow X$ is a finite morphism (cf Remark 3.3 A)) so that $\dim \tilde{X} = d$.

As $\dim \mathrm{Sec}_p(\tilde{X}) = t-1 < d = \dim \tilde{X} < \dim \mathrm{Join}(p, \tilde{X})$ there are lines joining p and $\tilde{X}$ which are not secant lines of $\tilde{X}$. So, in view of Remark 3.3 A) and statement (3.9) of Remark 3.3 B), the morphism $\varrho \restriction: \tilde{X} \twoheadrightarrow X$ is birational and $X \subset \mathbb{P}^r_k$ is of almost minimal degree.

Moreover, the finite birational morphism $\varrho \restriction: \tilde{X} \twoheadrightarrow X$ is induced by a finite injective birational homomorphism $\delta : A \hookrightarrow \tilde{A}$ of graded rings. Thus, by Theorem 5.6 (b) we get an isomorphism of graded rings ι, which occurs in the commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "\delta"'] \arrow[dr, "\varepsilon'"] & \\ B \arrow[rr, "\iota", "\simeq"'] & & \tilde{A} \end{tikzcd}$$

Now Theorem 5.6 d) shows that $\mathrm{depth}\, A = \dim \mathrm{Sec}_p(\tilde{X}) + 1 = t$. □

As a further application of Theorem 5.6 we now have a glance at arithmetically Cohen-Macaulay varieties of almost minimal degree and show that their non-normal locus is either empty or a linear space. More precisely

Proposition 5.8. *Assume that $X \subset \mathbb{P}^r_k$ is of almost minimal degree, S_2 and not normal. Let $\dim X =: d$. Then X is arithmetically Cohen-Macaulay and the non-normal locus $X \setminus \mathrm{Nor}(X)$ is a linear space $\mathbb{P}^{d-1}_k \subset \mathbb{P}^r_k$.*

Proof. (*Induction on* d) Let $d = 1$. Then X is a curve of degree $r + 1$ and thus may have at most one singular point (cf [4, (4.7) (B)]). So, let $d > 1$. Then $\mathrm{Nor}(X) \subset X = S_2(X)$ shows that X is arithmetically Cohen-Macaulay (cf Theorem 5.6 f)). Moreover, as X is S_2 and not normal, the Serre criterion for normal points shows that the non-normal locus $X \setminus \mathrm{Nor}(X)$ of X is of pure codimension 1.

Let $Z \subset X$ be the reduced and purely $(d-1)$-dimensional closed subscheme supported by $X \setminus \mathrm{Nor}(X)$. It suffices to show that $\deg Z = 1$. Now, let $\ell \in S_1$ be a generic linear form and consider the hyperplane $\mathbb{P}_k^{r-1} := \mathrm{Proj}(S/\ell S)$. Then, the hyperplane section $X' := \mathbb{P}_k^{r-1} \cap X = \mathrm{Proj}(A/\ell A)$ is an arithmetically Cohen-Macaulay-variety of almost minimal degree in $\mathbb{P}_k^{r-1}$ with $\dim X' = d-1$, and $Z' := \mathbb{P}_k^{r-1} \cap Z \subset X'$ is a reduced purely 1-codimensional subscheme with $\deg Z' = \deg Z$. Now, let z' be one of the generic points of Z'. Then $z' \in Z$ shows that $\mathcal{O}_{X,z'}$ is not normal and hence not regular. As $\mathcal{O}_{X',z'}$ is a hypersurface ring of $\mathcal{O}_{X,z}$, the ring $\mathcal{O}_{X',x'}$ is not regular either. As $\dim \mathcal{O}_{X',z'} = 1$ it follows that $\mathcal{O}_{X',z'}$ is not normal and hence $z' \in X' \setminus \mathrm{Nor}(X')$. By induction we have $X' \setminus \mathrm{Nor}(X') = \mathbb{P}_k^{d-2}$ for some linear subspace $\mathbb{P}_k^{d-2} \subset \mathbb{P}_k^r$. It follows that $\{\bar{z'}\} = \mathbb{P}_k^{d-2}$. This shows that the closed reduced subschemes Z' and $\mathbb{P}_k^{d-2}$ of $\mathbb{P}_k^r$ coincide, hence $\deg Z = \deg Z' = 1$. □

6. Del Pezzo Varieties and Fujita's Classification

In this section we shall treat projective varieties of almost minimal degree which are arithmetically Cohen-Macaulay. We call these varieties maximal Del Pezzo varieties and make sure that this is in coincidence with Fujita's notion of Del Pezzo variety [16]. We also briefly discuss the link with Fujita's classification of varieties of Δ-genus 1.

Remark 6.1. A) Let $d := \dim(X) > 0$ and let $\omega_X = K(A)^\sim$ denote the dualizing sheaf of X. Keep in mind that a finitely generated graded A-module of $\mathrm{depth} > 1$ is determined (up to a graded isomorphism) by the sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ induced by M. So, as $K(A)$ satisfies the second Serre property S_2 (cf [28, 3.1.1]), we have for each $r \in \mathbb{Z}$:

$$\omega_X \simeq \mathcal{O}_X(r) \text{ if and only if } K(A) \simeq D(A)(r). \tag{6.1}$$

$$\text{If depth } A > 1, \text{ then } \omega_X \simeq \mathcal{O}_X(r) \text{ if and only if } K(A) \simeq A(r). \tag{6.2}$$

B) $X \subset \mathbb{P}_k^r$ is said to be *linearly complete* if the inclusion morphism $X \hookrightarrow \mathbb{P}_k^r$ is induced by the complete linear system $|\mathcal{O}_X(1)|$. It is equivalent to say that the natural monomorphism $\eta : A_1 \to H^0(X, \mathcal{O}_X(1)) = D(A)_1$ is an isomorphism hence – equivalently – that $H^1(A)_1 = 0$.

Theorem 6.2. *The following statements are equivalent:*

(i) *X is arithmetically Gorenstein and of almost minimal degree.*
(ii) *X is arithmetically Cohen-Macaulay and of almost minimal degree.*
(iii) *X is S_2, linearly complete and of almost minimal degree.*
(iv) *$\omega_X \simeq \mathcal{O}_X(1-d)$ and X is of almost minimal degree.*
(v) *$K(A) \simeq A(1-d)$ and X is arithmetically Cohen-Macaulay.*
(vi) *$\omega_X \simeq \mathcal{O}_X(1-d)$ and X is arithmetically Cohen-Macaulay.*
(vii) *$H^{d+1}(A)_{1-d} \simeq k$ and $H^1(A)_1 = H^i(A)_n = 0$ for $2 \leq i \leq d$ and $1 - d \leq n \leq 1$.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) and (v) $\Longrightarrow$ (vi) are obvious. The implication (ii) $\Longrightarrow$ (i) follows by Remark 4.5, the implication (vi) $\Longrightarrow$ (v) by (6.2). It remains to prove the implications (v) $\Longrightarrow$ (vii) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii). The implication (v) $\Longrightarrow$ (vii) is easy.

(vii) $\Longrightarrow$ (ii): $H^{d+1}(A)_{1-d} \simeq k$ implies that $H^{d+1}(A)_n = 0$ for all $n > 1-d$. As $P_A(n) = \dim_k A_k - \sum_{i=1}^{d+1}(-1)^i \dim_k H^i(A)_n$, statement (vii) implies that

$$P_A(n) = \begin{cases} (-1)^d, & \text{if } n = 1-d, \\ 0, & \text{if } 1-d < n < 0, \\ 1, & \text{if } n = 0, \\ r+1, & \text{if } n = 1. \end{cases}$$

So, we may write $P_A(n) = (r-d+2)\binom{n+d-1}{d} + \binom{n+d-2}{d-2}$. In particular, X is of almost minimal degree. But then, by Corollary 4.4 b) we see that X must be arithmetically Cohen-Macaulay.

(ii) $\Longrightarrow$ (v): Assume that X is arithmetically Cohen-Macaulay and of almost minimal degree. Then, the shape of the Hilbert series given in Corollary 4.4 (a) allows to conclude that $K(A) \simeq A(1-d)$.

(v) $\Longrightarrow$ (iv): We know that (v) $\Longrightarrow$ (vii) and (vii) $\Longrightarrow$ (ii). So, (v) implies that X is of almost minimal degree and hence induces (iv).

(iv) $\Longrightarrow$ (iii): By (6.2), statement (iv) implies $K(A) \simeq D(A)(1-d)$ and hence by Theorem 4.2 (b) that X is arithmetically Cohen-Macaulay.

(iii) $\Longrightarrow$ (ii): Assume that X is of almost minimal degree, S_2 and linearly complete. We have to show that X is arithmetically Cohen-Macaulay. Assume to the contrary that $t = \operatorname{depth} A \leq d$. As $H^1(A)_1 = 0$, Proposition 3.1 (c) yields $t > 1$. But now, by Theorem 5.6 (d) and (e), X contains a G-point and thus cannot be S_2 – a contradiction. $\square$

Definition 6.3. A) $X \subset \mathbb{P}^r_k$ is called a *maximal Del Pezzo variety* if it satisfies the equivalent conditions (i) – (vii) of Theorem 6.2.
B) $X \subset \mathbb{P}^r_k$ is called a *Del Pezzo variety*, if there is an integer $r' \geq r$, a maximal Del Pezzo variety $X' \subset \mathbb{P}^{r'}_k$ and a linear projection $\pi : \mathbb{P}^{r'}_k \setminus \mathbb{P}^{r'-r-1}_k \to \mathbb{P}^r_k$ with $X' \cap \mathbb{P}^{r'-r-1} = \emptyset$ and such that π gives rise to an isomorphism $\pi\restriction : X' \xrightarrow{\simeq} X$. So, $X \subset \mathbb{P}^r_k$ is Del Pezzo if and only if it is obtained by a non-singular projection of a maximal Del Pezzo variety.

Remark 6.4. A) As a linearly complete variety $X \subset \mathbb{P}^r_k$ cannot be obtained by a proper non-singular projection of a non-degenerate variety $X' \subset \mathbb{P}^{r'}_k$ we can say that $X \subset \mathbb{P}^r_k$ is maximally Del Pezzo if and only if it is Del Pezzo and linearly complete.
B) Keep the previous notation and let $r' = h^0(X, \mathcal{O}_X(1)) - 1$, whence

$$r' = \dim_k D(A)_1 - 1 = r + \dim_k H^1(A)_k.$$

Moreover let $\varphi : X \to \mathbb{P}^{r'}_k$ be the closed immersion defined by the complete linear system $|\mathcal{O}_X(1)|$ and set $X' := \varphi(X)$. Then $X' \subset \mathbb{P}^{r'}_k$ is linearly complete with homogeneous coordinate ring $A' = k[D(A)_1] \subset D(A)$, whereas the isomorphism $\varphi : X \xrightarrow{\simeq} X'$ is induced by the inclusion $A \hookrightarrow A'$ and inverse to an isomorphism $\pi\restriction : X' \xrightarrow{\simeq} X$ which is the restriction of a linear projection $\pi : \mathbb{P}^{r'}_k \setminus \mathbb{P}^{r'-r-1}_k \to \mathbb{P}^r_k$ with $X' \cap \mathbb{P}^{r'-r-1}_k = \emptyset$. Now, clearly $X' \subset \mathbb{P}^{r'}_k$ is linearly complete and moreover

$$D(A') = D(A), H^i(A') \simeq H^i(A), i \neq 1, K(A') \simeq K(A) \text{ and } P_{A'}(x) = P_A(x).$$

Note that $X' \subset \mathbb{P}_k^{r'}$ is called the *linear completion of* $X \subset \mathbb{P}_k^r$.
C) Observe that the linear completion of $X \subset \mathbb{P}_k^r$ is the maximal non-degenerate projective variety $X' \subset \mathbb{P}_k^{r'}$ which can be projected non-singularly onto X. More precisely: If $\tilde{X} \subset \mathbb{P}_k^{\bar{r}}$ is a non-degenerate projective variety and $\bar{\pi}\restriction : \tilde{X} \xrightarrow{\simeq} X$ is an isomorphism induced by a linear projection $\bar{\pi} : \mathbb{P}_k^{\bar{r}} \longrightarrow \mathbb{P}_k^r$, then $\bar{r} \leq r'$ and the isomorphism $\bar{\pi}\restriction^{-1} \circ \pi\restriction : X' \xrightarrow{\simeq} \tilde{X}$ comes from a linear projection $\varrho : \mathbb{P}_k^{r'} \longrightarrow \mathbb{P}_k^{\bar{r}}$. In particular, if $\tilde{X} \subset \mathbb{P}_k^{\bar{r}}$ is linearly complete we have $r' = \bar{r}$ and ϱ becomes an isomorphism so that we may identify X' with $\tilde{X}$. Consequently, by what we said in part A) it follows that $X \subset \mathbb{P}_k^r$ is Del Pezzo if and only if its linear completion $X' \subset \mathbb{P}_k^{r'}$ is (maximally) Del Pezzo.

We now shall tie the link to Fujita's classification of polarized Del Pezzo varieties.

Remark 6.5. (see [16]). A) A *polarized variety over* k is a pair $(V, \mathcal{L})$ consisting of a reduced irreducible projective variety V over k and an ample invertible sheaf of $\mathcal{O}_V$-modules $\mathcal{L}$.

B) Let $(V, \mathcal{L})$ be a polarized k-variety. For a coherent sheaf of $\mathcal{O}_V$-modules $\mathcal{F}$ and $i \in \mathbb{N}_0$ let $h^i(V, \mathcal{F})$ denote the k-dimension of the i-th Serre cohomology group $H^i(V, \mathcal{F})$ of V with coefficients in $\mathcal{F}$. Then, the function

$$n \mapsto \chi_{(V,\mathcal{L})}(n) := \sum_{i=0}^{\dim V} (-1)^i h^i(V, \mathcal{L}^{\otimes n})$$

is a polynomial of degree $\dim V$, the so called *Hilbert polynomial of the polarized variety* $(V, \mathcal{L})$.

C) Let $(V, \mathcal{L})$ be a polarized variety of dimension d. Then, there are uniquely determined integers $\chi_i(V, \mathcal{L}), i = 0, \ldots, d$ such that

$$\chi_{(V,\mathcal{L})}(n) = \sum_{i=0}^{d} \chi_i(V, \mathcal{L}) \binom{n+i-1}{i}.$$

Clearly $\chi_d(V, \mathcal{L}) > 0$. The *degree*, the Δ-*genus* and the *sectional genus* of the polarized variety $(V, \mathcal{L})$ are defined respectively by

$$\begin{aligned} \deg(V, \mathcal{L}) &:= \chi_d(V, \mathcal{L}); \\ \Delta(V, \mathcal{L}) &:= d + \deg(V, \mathcal{L}) - h^0(V, \mathcal{L}); \\ g_s(V, \mathcal{L}) &:= 1 - \chi_{d-1}(V, \mathcal{L}). \end{aligned}$$

D) According to Fujita (cf [16]) the polarized variety $(V, \mathcal{L})$ is called a *Del Pezzo variety*, if it satisfies the following conditions

(6.3) $\quad \Delta(V, \mathcal{L}) = 1,$

(6.4) $\quad g_s(V, \mathcal{L}) = 1,$

(6.5) $\quad V$ has only Gorenstein singularities and $\omega_V \simeq \mathcal{L}^{\otimes(1-\dim V)}$,

(6.6) $\quad$ For all $i \neq 0, \dim V$ and all $n \in \mathbb{Z}$ it holds $H^i(V, \mathcal{L}^{\otimes n}) = 0$.

Remark 6.6. A) We consider the polarized variety $(X, \mathcal{O}_X(1))$. For all $n \in \mathbb{Z}$ we have $H^0(X, \mathcal{O}_X(1)^{\otimes n}) = H^0(X, \mathcal{O}_X(n)) = D(A)_n$. Thus:

$$\chi_{(X,\mathcal{O}_X(1))}(n) = P_A(n), \tag{6.7}$$

where P_A is the Hilbert polynomial of A. Therefore

$$\deg(X, \mathcal{O}_X(1)) = \deg X, \tag{6.8}$$

$$\Delta(X, \mathcal{O}_X(1)) = \deg X - \operatorname{codim} X - 1 - \dim_k H^1(A)_1. \tag{6.9}$$

As a consequence of the last equality we obtain

$$\Delta(X, \mathcal{O}_X(1)) \leq \deg X - \operatorname{codim} X + 1, \tag{6.10}$$

with equality if and only if $X \subset \mathbb{P}^r_k$ is linearly complete.

B) Let $X' \subset \mathbb{P}^{r'}_k$ be the linear completion of $X \subset \mathbb{P}^r_k$. Then $(X, \mathcal{O}_X(1))$ and $(X', \mathcal{O}_{X'}(1))$ are isomorphic polarized varieties. In particular $(X', \mathcal{O}_{X'}(1))$ is Del Pezzo in the sense of Fujita if and only $(X, \mathcal{O}_X(1))$ is.

Lemma 6.7. *Let $X \subset \mathbb{P}^r_k$ be of almost minimal degree. Then*

(a)

$$\Delta(X, \mathcal{O}_X(1)) = \begin{cases} 0, & \text{if depth } A = 1, \\ 1, & \text{if depth } A > 1. \end{cases}$$

(b)

$$g_s(X, \mathcal{O}_X(1)) = \begin{cases} 0, & \text{if } X \text{ is not arithmetically Cohen-Mcaulay}, \\ 1, & \text{if } X \text{ is arithmetically Cohen-Macaulay}. \end{cases}$$

Proof. (a): This follows immediately from Proposition 3.1 c), Theorem 4.2 b) and by (6.9).

(b): This is a consequence of Corollary 4.4 b) and (6.7). □

Theorem 6.8. *Let $X \subset \mathbb{P}^r_k$ be of dimension $d > 0$. Then, the following statements are equivalent:*

(i) *X is Del Pezzo in the sense of Definition 6.3 B).*
(ii) *$(X, \mathcal{O}_X(1))$ is Del Pezzo in the sense of Fujita.*
(iii) *$\Delta(X, \mathcal{O}_X(1)) = g_s(X, \mathcal{O}_X(1)) = 1$.*
(iv) *$\Delta(X, \mathcal{O}_X(1)) = 1$ and $H^i(X, \mathcal{O}_X(n)) = 0$ for all $i \neq 0, d$ and all $n \in \mathbb{Z}$.*
(v) *$g_s(X, \mathcal{O}_X(1)) = 1$ and $H^i(X, \mathcal{O}_X(n)) = 0$ for all $i \neq 0, d$ and all $n \in \mathbb{Z}$.*
(vi) *$\Delta(X, \mathcal{O}_X(1)) = 1$ and $\omega_X \simeq \mathcal{O}_X(1-d)$.*
(vii) *$H^i(X, \mathcal{O}_X(n)) = 0$ for all $i \neq 0, d$ and all $n \in \mathbb{Z}$, and $\omega_X \simeq \mathcal{O}_X(1-d)$.*
(viii) *$\Delta(X, \mathcal{O}_X(1)) = 1$ and X is S_2.*

Proof. First let us fix a few notation. Let $r' = h^0(X, \mathcal{O}_X(1)) = \dim_k D(A)_1$ and let $X' \subset \mathbb{P}^{r'}_k$ be the linear completion of $X \subset \mathbb{P}^r_k$.

(i) $\Longrightarrow$ (ii): Let X be Del Pezzo in the sense of Definition 6.3 B). According to Remark 6.4 C) we get that $X' \subset \mathbb{P}^{r'}_k$ is maximally Del Pezzo. According to Theorem 6.2, the pair $(X', \mathcal{O}_{X'}(1))$ thus satisfies the requirements (6.3) - (6.6) of Remark 6.5 D) and hence is Del Pezzo in the sense of Fujita. By Remark 6.6 B) the same follows for $(X, \mathcal{O}_X(1))$.

Clearly statement (ii) implies each of the statements (iii) - (viii). So, it remains to show that each of the statements (iii) - (viii) implies statement (i). According to Remark 6.4 C) we may replace X by X' in statement (i). As $(X', \mathcal{O}_{X'}(1))$ and $(X, \mathcal{O}_X(1))$ are isomorphic polarized varieties we may replace X by X' in each of the statements (iii) - (viii). So, we may assume that $X \subset \mathbb{P}^r_k$ is linearly complete.

(iii) $\Longrightarrow$ (i): According to statement (6.10) of Remark 6.6 A) the equality $\Delta(X, \mathcal{O}_X(1)) = 1$ implies that $X \subset \mathbb{P}^r_k$ is of almost minimal degree. But now by Lemma 6.7 b) the equality $g_s(X, \mathcal{O}_X(1)) = 1$ implies that X is arithmetically Cohen-Macaulay.

(iv) $\Longrightarrow$ (i): By $\Delta(X, \mathcal{O}_X(1)) = 1$ we see again that $X \subset \mathbb{P}^r_k$ is of almost minimal degree. As $H^1(A)_1 = 0$ we have depth $A > 1$ (Proposition 3.1 C)). As $H^{i+1}(A)_n \simeq H^i(X, \mathcal{O}_X(n)) = 0$ for all $i \neq 0, d$, it follows that depth $A = d + 1$, so that X is arithmetically Cohen-Macaulay.

(vi) $\Longrightarrow$ (i): As we have proved the implication (iii) $\Longrightarrow$ (i) it suffices to prove that statement (v) implies the equality $\Delta(X, \mathcal{O}_X(1)) = 1$. We proceed by induction on d. Let $d = 1$. As $g_s(X, \mathcal{O}_X(1)) = 1$ implies $\chi_0(X, \mathcal{O}_X(1)) = 0$ we get $\chi_{(X,\mathcal{O}_X(1))}(n) = (\deg X)n$. As $H^1(X, \mathcal{O}_X(1)) = 0$ it follows $r + 1 = h^0(X, \mathcal{O}_X(1)) = \chi_{(X,\mathcal{O}_X(1))}(1) = \deg X$ and hence by (6.10) we get $\Delta(X, \mathcal{O}_X(1)) = 1$.

So, let $d > 1$, let $\ell \in S_1$ be a generic linear form and consider the irreducible projective variety $Y := \mathrm{Proj}(A/\ell A) \subset \mathbb{P}^r_k := \mathrm{Proj}(S/\ell S)$ of dimension $d - 1$ and with homogeneous coordinate ring $A' = (A/\ell A)/H^0(A/\ell A)$. As

$$\chi_{(Y,\mathcal{O}_Y(1))}(n) = P_{A'}(n) = \Delta P_A(x) = \Delta\chi_{(X,\mathcal{O}_X(1))}(n)$$

it follows $\Delta(Y, \mathcal{O}_Y(1)) = \Delta(X, \mathcal{O}_X(1))$ and $g_s(Y, \mathcal{O}_Y(1)) = g_s(X, \mathcal{O}_X(1)) = 1$. So, by induction it suffices to show that

$$H^i(Y, \mathcal{O}_Y(n)) = H^{i+1}(A')_n = 0 \text{ for all } i \neq 0, d-1 \text{ and all } n \in \mathbb{Z},$$

and that $Y \subset \mathbb{P}^{r-1}_k$ is linearly complete, hence that $H^1(A')_1 = 0$. As

$$H^{i+1}(A)_n = H^i(X, \mathcal{O}_X(n)) = 0$$

for all $i \neq 0, d$ and all $n \in \mathbb{Z}$ and as $H^1(A)_1 = 0$ this follows immediately if we apply cohomology to the sequence

$$0 \to A(1) \xrightarrow{\ell} A \to A/\ell A \to 0$$

and observe that $H^j(A') \simeq H^j(A/\ell A)$ for all $j > 0$.

(vi) $\Longrightarrow$ (i): This is immediate by (6.10) and Theorem 6.2.

(vii) $\Longrightarrow$ (i): As X is linearly complete, $H^1(A)_1 = 0$. Moreover by our hypothesis $H^i(A) = 0$ for all $i \neq 1, d+1$. Finally, by (6.2) we have $K(A) \simeq D(A)(1-d)$, hence $H^{d+1}(A)_{1-d} \simeq K(A)_{d-1} \simeq D(A)_0 \simeq k$. So, statement (vii) of Theorem 6.2 is true.

(viii) $\Longrightarrow$ (i): In view of (6.10), statement (viii) implies statement (iii) of Theorem 6.2. $\square$

Our next aim is to extend Theorem 5.3 to maximal Del Pezzo varieties.

Theorem 6.9. *Let $X \subset \mathbb{P}^r_k$ be a maximal Del Pezzo variety of dimension d which is non-normal. Let $B = k \oplus B_1 \oplus B_2 \oplus \cdots$ be the graded normalization of A. Then:*

(a) *There are linearly independent linear forms $y_d, y_{d+1}, \cdots, y_r \in S_1$ such that $B/A \cong (S/(y_d, y_{d+1}, \cdots, y_r))(-1)$.*

(b) *B is the homogeneous coordinate ring of a variety of minimal degree $\tilde{X} \subset \mathbb{P}^{r+1}_k$. In particular, B is a Cohen-Macaulay ring.*

Proof. We make induction on d. The case $d = 1$ is clear by Proposition 5.2 C). Therefore, let $d > 1$. Statement (b) follows easily from statement (a). So, we only shall prove this latter. According to Proposition 5.8 there are linearly independent linear forms $y_d, y_{d+1}, \cdots, y_r \in S_1$ such that $I \subset (y_d, y_{d+1}, \cdots, y_r)$ and such that $\mathfrak{s} := (y_d, y_{d+1}, \cdots, y_r)/I \subset S/I = A$ defines the non-normal locus $X \setminus \mathrm{Nor}(X)$ of X. Observe that $\mathfrak{s} \subset A$ is a prime of height 1 and that $A/\mathfrak{s} \simeq S/(y_d, y_{d+1}, \cdots, y_r)S$ is a polynomial ring in d inderminates over k. Next, we consider the canonical exact sequence

$$0 \to A \to B \to C \to 0 \tag{6.11}$$

in which $C := B/A$ is a finitely generated graded A-module such that $C_{\leq 0} = 0$ and $\mathrm{Rad}\,\mathrm{Ann}_B\, C = \mathfrak{s}$. Our aim is to show that $C \simeq (A/\mathfrak{s})(-1)$. Let $\ell \in S_1$ be a generic linear form.

Then, according to Bertini and as $\operatorname{depth} A > 1$, the ideal $\ell A \subset A$ is prime. Moreover $(A \setminus \ell A) \cap \mathfrak{s} \neq \emptyset$, so that $(A \setminus \ell A)^{-1} B = A_{\ell A}$. It follows that ℓB has a unique minimal prime $\mathfrak{p}$ and that $\mathfrak{p} B_{\mathfrak{p}} = \ell B_{\mathfrak{p}}$. As B is S_2 we get $\ell B = \mathfrak{p}$, so that ℓB is a prime ideal of B. Therefore $B/\ell B$ is a finite birational integral extension domain of $A' := A/\ell A$ and hence a subring of the graded normalization B' of A'.

As $\operatorname{depth} A > 2$ and $\operatorname{depth} B \geq 2$, the short exact sequence (6.11) yields $\operatorname{depth} C \geq 2$. In particular ℓ is C-regular. Hence we get the following commutative diagram with exact rows and columns in which $U = \operatorname{Coker} \iota$ is a graded A-module

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A/\ell A \arrow[r] \arrow[d, equal] & B/\ell B \arrow[r] \arrow[d, "\lambda"] & C/\ell C \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
 & & U \arrow[r, "\simeq"] \arrow[d] & U \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{6.12}$$

Now $X' := \mathrm{Proj}(A') \subset \mathrm{Proj}(S/\ell S) = \mathbb{P}^{r-1}_k$ is again a maximal Del Pezzo variety. Moreover X' is non-normal, since otherwise $B/\ell B = A/\ell A$, hence $B = A$. Let $\mathfrak{s}' \subset A'$ be the prime of height 1 which defines the non-normal locus of X' and keep in mind that $A'/\mathfrak{s}'$ is a polynomial ring in $d-1$ inderminates over k. By induction $C' \simeq (A'/\mathfrak{s}')(-1)$.

Our next aim is to show that $\dim U \leq 0$. As $C \neq 0$, we have $C/\ell C \neq 0$. As C' is a free $A'/\mathfrak{s}'$-module of rank one, it follows $\dim U < \dim(A'/\mathfrak{s}') = d-1$. As $\dim(B/\ell B) = \dim(B') = d$ it follows that λ is an isomorphism in codimension one. As B' is normal and hence satisfies the property R_1, it follows that $B/\ell B$ satisfies R_1, too.

Let $s \in \mathfrak{s} \setminus \{0\}$. As B is normal, it satisfies the Serre property S_2 so that B/sB satisfies S_1. Therefore the set of generic points of the (closed) non-S_2-locus of the B-module B/sB is given by

$$\mathcal{Q} := \{\mathfrak{q} \in V(sB) : \operatorname{depth}(B_{\mathfrak{q}}/sB_{\mathfrak{q}}) = 1 < \operatorname{height} \mathfrak{q} - 1\}.$$

In particular $\mathcal{Q}$ is finite, and hence ℓ avoids all members of $\mathcal{P} := (\mathrm{Ass}_B(B/sB) \cup \mathcal{Q}) \cap \mathrm{Proj}\, B$.

Now, let $\mathfrak{r} \in \mathrm{Proj}\, B \cap V(\ell B)$ such that $\operatorname{height} \mathfrak{r} > 2$. If $s \notin \mathfrak{r}$, the equality $A_s = B_s$ yields that $B_{\mathfrak{r}}$ is a Cohen-Macaulay ring, so that $\operatorname{depth}(B_{\mathfrak{r}}/\ell B_{\mathfrak{r}}) > 1$. If $s \in \mathfrak{r}$ the fact that ℓ avoids all members of $\mathcal{P}$ implies that s, ℓ is a $B_{\mathfrak{r}}$-sequence and $\operatorname{depth} B_{\mathfrak{r}}/sB_{\mathfrak{r}}) > 1$. It follows again that $\operatorname{depth}(B_{\mathfrak{r}}/\ell B_{\mathfrak{r}}) > 1$. This proves, that the scheme $\operatorname{Proj}(B/\ell B)$ is S_2. As $B/\ell B$ satisfies R_1 it follows that $\operatorname{Proj}(B/\ell B)$ is a normal scheme, hence that $\operatorname{Proj}(B/\ell B) = \operatorname{Proj} B'$. As a consequence, we get indeed that $\dim U \leq 0$, that is U is a graded A-module of finite length.

Now, let $\mathfrak{t} \subset A$ be the preimage of $\mathfrak{s}'$ under the canonical map $A \to A'$. Then $\mathfrak{t}$ and $\mathfrak{s} + \ell A$ are primes of height 2 in A and so

$$\mathfrak{s} + \ell A = \operatorname{Rad}((\operatorname{Ann}_A C) + \ell A) = \operatorname{Rad}(\operatorname{Ann}_A C/\ell C) \supseteq \operatorname{Ann}_A C' = \operatorname{Ann}_A(A'/\mathfrak{s}'(-1)) = \mathfrak{t}$$

implies that $\mathfrak{s} + \ell A = \operatorname{Ann}_A(C/\ell C) = \mathfrak{t}$. As a consequence we get $\mathfrak{s}C \subseteq \ell C$ and hence, by the genericity of ℓ, that $\mathfrak{s}C = 0$. It follows $\mathfrak{s}B \subseteq \mathfrak{s}$ and $\mathfrak{s}$ becomes an ideal of B. Now, let $a \in A$ and $c \in C \setminus \{0\}$ such that $ac = 0$. By the genericity of ℓ we may assume that $c \notin \ell C$ so that $\iota(c + \ell C) \neq 0$ and $a\iota(c + \ell C) = 0$. It follows $a \in \mathfrak{t} = \mathfrak{s} + \ell A$ and hence, by genericity, that $a \in \mathfrak{s}$. This shows that C is a torsion-free $A/\mathfrak{s}$-module and hence that $B/\mathfrak{s}$ is a torsion-free $A/\mathfrak{s}$-module.

As $\operatorname{rank}_{A/\mathfrak{s}} C = e_0(C) = e_0(C/\ell C) = e_0(C') = \operatorname{rank}_{A'/\mathfrak{s}'} C' = 1$ we get an exact sequence of graded $A/\mathfrak{s}$-modules

$$0 \to C \to (A/\mathfrak{s})(-m) \to W \to 0 \tag{6.13}$$

with $m \in \mathbb{Z}$ and $\dim W < \dim A/\mathfrak{s} = d$. We choose m maximally. Then, there is no homogeneous element $f \in A/\mathfrak{s}$ of positive degree with $C(m) \subseteq f(A/\mathfrak{s})$, so that $\dim W = \dim(W(m)) = \dim((A/\mathfrak{s})/C(m)) < d - 1$.

As $\operatorname{depth} C \geq 2$ we have $\operatorname{depth} W \geq 1$. As ℓ is generic we thus get an exact sequence

$$0 \to C/\ell C \to (A'/\mathfrak{s}')(-m) \to W/\ell W \to 0$$

with $\dim W/\ell W = \dim W - 1 < d - 2 = \dim A'/\mathfrak{s}' - 1$. Comparing Hilbert coefficients we get $e_1(C/\ell C) = e_1((A'/\mathfrak{s}')(-m)) = m$.

The diagram (6.12) contains the short exact sequence $0 \to C/\ell C \to (A'/\mathfrak{s}')(-1) \to U \to 0$ with $\dim U \leq 0$. Assume first, that $d \geq 3$ so that $\dim U < \dim A'/\mathfrak{s}' - 1$. Then, we may again compare Hilbert coefficients and get $e_1(C/\ell C) = e_1((A'/\mathfrak{s}')(-1)) = 1$, hence $m = 1$. It follows $\dim W/\ell W = \dim U \leq 0$, thus $\dim W \leq 1$. Suppose that $\dim W = 1$. Then $H^1(W)_n \neq 0$ for all $n \ll 0$ and the sequence (6.11) yields $H^2(C)_n \neq 0$ for all $n \ll 0$. As $H^3(A) = 0$, the sequence (6.11) induces that $H^2(B)_n \neq 0$ for all $n \ll 0$; but this contradicts the fact that B is S_2. So, we have $\dim W \leq 0$. Now, by the sequence (6.13) we get $W = H^0(W) \simeq H^1(C)$, and $\operatorname{depth} C > 1$ implies $W = 0$. Therefore $C \simeq A/\mathfrak{s}(-1)$.

It remains to treat the case $d = 2$. Now $A'/\mathfrak{s}' \simeq k[x]$ so that the $A'/\mathfrak{s}'$-submodule $C/\ell C$ of $(A'/\mathfrak{s}')(-1)$ is generated by a single homogeneous element of degree $m \geq 1$. By Nakayama it follows $C \simeq (A/\mathfrak{s})(-m)$. It remains to show that $m = 1$.

By the case $d = 1$ we know that B' is the homogeneous coordinate ring of a rational normal curve, so that $H^2(B')_0 = 0$. The middle column of the diagram (6.12) now implies $H^2(B/\ell B)_0 = 0$. Applying cohomology to the exact sequence $0 \to B(-1) \xrightarrow{\ell} B \to B/\ell B \to 0$ we thus get an isomorphism $H^3(B)_{-1} \simeq H^3(B)_0$ so that $H^3(B)_{-1} = 0$. If we apply cohomology to the sequence (6.11) we thus get an exact sequence

$$0 \to H^2(B)_{-1} \to H^2(A/\mathfrak{s})_{-1-m} \to H^3(A)_{-1} \to 0.$$

By Theorem 6.2 (cf statement (vii)) we have $H^3(A)_{-1} \simeq k$. As $A/\mathfrak{s}$ is a polynomial ring in two indeterminates over k we have $H^2(A/\mathfrak{s})_{-1-m} \simeq k^m$. It follows $H^2(B)_{-1} \simeq k^{m-1}$.

Moreover $\tilde{X} := \operatorname{Proj} B$ is a projective normal surface and the natural morphism $\nu : \tilde{X} \to X$ is a normalization of X. In particular $\mathcal{L} := B(1)^\sim = \nu^*\mathcal{O}_X(1)$ is an ample invertible sheaf of $\mathcal{O}_{\tilde{X}}$-modules and $\mathcal{L}^{\otimes n} = \nu^*\mathcal{O}_X(n) = B(n)^\sim$ for all $n \in \mathbb{Z}$. In addition we have $H^2(B)_{-1} \simeq H^1(\tilde{X}, \mathcal{L}^{\otimes -1})$ and $B_1 \simeq H^0(\tilde{X}, \mathcal{L})$. Moreover $Y := \operatorname{Proj} B/\ell B \simeq \operatorname{Proj} B'$ is the effective divisor on $\tilde{X}$ defined by the global section $\ell \in H^0(\tilde{X}, \mathcal{L})\setminus\{0\}$. As $H^1(Y, \mathcal{O}_Y) = H^2(B/\ell B) = 0$, the sectional genus $g_s(\tilde{X}, \mathcal{L})$ vanishes (cf [1, (5.3) B)]). As $\dim_k H^0(\tilde{X}, \mathcal{L}) = \dim_k B_1 \geq r + 1 > 1$ it follows $H^1(\tilde{X}, \mathcal{L}^{\otimes -1}) = 0$ (cf [1, Proposition (5.4)]). Therefore $k^{m-1} \simeq H^2(B)_{-1} = 0$, hence $m = 1$. □

Now, we may extend Theorem 5.6 as follows

Corollary 6.10. *Let $X \subset \mathbb{P}^r_k$ be of almost minimal degree. Assume that either $t := \operatorname{depth} A \leq \dim X =: d$ or that X is maximally Del Pezzo (that is $t = d + 1$) and non-normal. Then, there is a d-dimensional variety $\tilde{X} \subset \mathbb{P}^{r+1}_k$ of minimal degree, a point $p \in \mathbb{P}^{r+1}_k \setminus \tilde{X}$ and a projection $\varrho : \mathbb{P}^{r+1}_k \setminus \{p\} \to \mathbb{P}^r_k$ from p such that:*

(a) *$\varrho(\tilde{X}) = X$ and $\nu := \varrho \restriction : \tilde{X} \to X$ is the normalization of X.*

(b) *The secant cone $\operatorname{Sec}_p(\tilde{X}) \subset \mathbb{P}^{r+1}_k$ is a projective subspace $\mathbb{P}^{t-1}_k \subset \mathbb{P}^{r+1}_k$ and*
$$X \setminus \operatorname{Nor} X = \operatorname{Sing}(\nu) = \varrho(\operatorname{Sec}_p(\tilde{X} \setminus \{p\}) \subset X$$
is a projective subspace $\mathbb{P}^{t-2}_k \subset \mathbb{P}^r_k$.

(c) *The singular fibre $\nu^{-1}(\operatorname{Sing}(\nu)) = \operatorname{Sec}_p(\tilde{X}) \cap \tilde{X} \subset \tilde{X}$ is a quadric in $\mathbb{P}^{t-1}_k = \operatorname{Sec}_p(\tilde{X})$.*

Proof. Let B be the graded normalization of A. Then, according to Theorem 5.6 resp. Theorem 6.9 we see that B is the homogeneous coordinate ring of a variety $\tilde{X} \subset \mathbb{P}^{r+1}_k$ of minimal degree. Moreover, by Theorem 5.3 resp. Theorem 6.9 there are linearly independent linear forms $y_{t-1}, y_t, \cdots, y_r \in S_1$ such that $I \subseteq (y_{t-1}, y_t, \cdots, y_r)$ and $B/A \simeq (S/(y_{t-1}, y_t, \cdots, y_r))(-1)$. Now, all claims except statement (c) follow as in Theorem 5.6.

To prove statement (c), we consider the prime $\mathfrak{s} := (y_{t-1}, y_t, \ldots, y_r)/I \subset A$. Then $A/\mathfrak{s}$ is a polynomial ring in $t-1$ indeterminates over k and we have $B/A \simeq (A/\mathfrak{s})(-1)$. In particular $\mathfrak{s} \subset B$ is an ideal and we have an exact sequence $0 \to A/\mathfrak{s} \to B/\mathfrak{s} \to (A/\mathfrak{s})(-1) \to 0$. Therefore $B/\mathfrak{s} \simeq (A/\mathfrak{s})[z]/(f)$ for some polynomial $f = z^2 + uz + v$ with $u \in (A/\mathfrak{s})_1$ and $v \in (A/\mathfrak{s})_2$. As
$$\nu^{-1}(\operatorname{Sing}(\nu)) = \operatorname{Proj}(B/\mathfrak{s}) \subset \operatorname{Sec}_p(\tilde{X}) = \mathbb{P}^{t-1}_k = \operatorname{Proj}((A/\mathfrak{s})[z])$$
the claims of (c) follows. □

7. Varieties of almost minimal degree that are projections

We now wish to give a more detailed insight in the nature of those varieties of almost minimal degree which are projections of cones over rational normal scrolls. Let us first recall a few facts on such scrolls.

Remark 7.1. (cf [21, pp 94, 97, 108-110] and [11]) A) Let $n \in \mathbb{N}$ and let $l, a_1, \ldots, a_{l-1} \in \mathbb{N}$. Let $a_0 = -1, a_l = n$ and assume that $a_i - a_{i-1} > 1$ for $i = 1, \ldots, l$. Then, up to projective equivalence, the numbers $a_1, a_2, \ldots, a_{l-1}$ define a unique *rational normal l-fold scroll*

$S_{a_1\cdots a_{l-1}} \subset \mathbb{P}^n_k$. Keep in mind that $S_{a_1\cdots a_{l-1}}$ is smooth, rational, arithmetically Cohen-Macaulay and of dimension l.
B) Keep the notation of part A). After an appropriate linear coordinate transformation we may assume that the vanishing ideal of $S_{a_1\cdots a_{l-1}}$ in the polynomial ring $k[x_0,\ldots,x_r]$ is the ideal generated by the 2×2-minors of the $2\times(n-l+1)$-matrix

$$M_{a_1\cdots a_{l-1}} := \left(\begin{array}{ccc|ccc|c|ccc} x_0 & \cdots & x_{a_1-1} & x_{a_1+1} & \cdots & x_{a_2-1} & \cdots & x_{a_{l-1}+1} & \cdots & x_{n-1} \\ x_1 & \cdots & x_{a_1} & x_{a_1+2} & \cdots & x_{a_2} & \cdots & x_{a_{l-1}+2} & \cdots & x_n \end{array}\right).$$

C) Let $V \in GL_2(k)$ and $W \in GL_{n-l}(k)$. Then, the 2×2-minors of the *conjugate matrix* $VM_{a_1\cdots a_{l-1}}W^{-1}$ generate the same ideal as the 2×2-minors of the matrix of $M_{a_1\cdots a_{l-1}}$. So, if we subject $M_{a_1\cdots a_{l-1}}$ to regular k-linear row and column transformations, the 2×2-minors of the resulting $2\times(n-l+1)$-matrix still generate the vanishing ideal of the same scroll $S_{a_1\cdots a_{l-1}}$.
D) A $2\times(n-l+1)$-matrix N whose entries are linear forms in $k[x_0,\ldots,x_r]$ is said to be 1-*generic*, if no conjugate of N has a zero entry. Observe that the property of being 1-generic is preserved under conjugation. Moreover, if N' is obtained by deleting some columns from the 1-generic matrix N, then N' is again 1-generic.

Finally, let N be a 1-generic $2\times(n-l+1)$-matrix whose entries are linear forms in $k[x_0,\ldots,x_r]$. Let $y_0,\ldots,y_m$ be a basis of the k-vector space $L \subset k[x_0,\ldots,x_r]_1$ generated by the entries of N. Then, $m > n-l+1$, thus $h := m-n+l > 1$.

Moreover, there are integers $b_1,\ldots,b_{h-1} \in \mathbb{N}$ such that with $b_0 = -1$ and $b_h = m$ we have $b_i - b_{i-1} > 1$ for $i = 1,\ldots,h$, and such that N is conjugate to the $2\times(m-l+1)$-matrix

$$N' := \left(\begin{array}{ccc|ccc|c|ccc} y_0 & \cdots & y_{b_1-1} & y_{b_1+1} & \cdots & y_{b_2-1} & \cdots & y_{b_{h-1}+1} & \cdots & y_{m-1} \\ y_1 & \cdots & y_{b_1} & y_{b_1+2} & \cdots & y_{b_2} & \cdots & y_{b_{h-1}+2} & \cdots & y_m \end{array}\right).$$

So, by parts B) and C) the 2×2-minors of N generate the vanishing ideal of a rational normal h-fold scroll in $\mathbb{P}^m_k = \mathrm{Proj}(k[y_0,\ldots,y_m])$.

Remark 7.2. A) Let $s \in \mathbb{N}$ and let $\tilde{X} \subset \mathbb{P}^s_k = \mathrm{Proj}(R), R = k[x_0,\ldots,x_s]$, be a cone over a rational normal scroll $S_{a_1\cdots a_{l-1}} \subset \mathbb{P}^n_k$ with $n \in \{1,2,\ldots,s\}$. According to the previous remark we may assume that the vanishing ideal of $\tilde{X}$ in R is generated by the 2×2-minors of the $2\times(n-l+1)$-matrix $M_{a_1\cdots a_{l-1}}$. In this case (and with the convention that $\mathbb{P}^{-1}_k = \emptyset$ and $\dim\emptyset = -1$), the vertex $\mathrm{Sing}(\tilde{X})$ of $\tilde{X}$ is given by $\mathbb{P}^{s-n-1}_k = \mathrm{Proj}(R/(x_0,\ldots,x_n)R)$ and so $\dim\mathrm{Sing}(\tilde{X}) = s-n-1$ and $\dim\tilde{X} = l+s-n$.
B) According to 7.1 C) the 2×2-minors of any matrix obtained from $M_{a_1\cdots a_{l-1}}$ by k-linear row and column operations generate the vanishing ideal of $\tilde{X}$ in R.
C) Let N be a 1-generic $2\times(n-l+1)$-matrix whose entries are linear forms in $k[x_0,\ldots,x_n]$. Let $y_0,\ldots,y_m$ be a basis of the k-space spanned by the entries of N and let $h := m-n+l$. Then, by parts A) and B) and by Remark 7.1 C), the 2×2-minors of N generate the vanishing ideal of a cone $Y \subset \mathbb{P}^s_k$ over a rational normal h-fold scroll $Z \subset \mathbb{P}^m_k$. In particular $\dim Y = h+s-m = l+s-n$ and $\dim\mathrm{Sing}(Y) = s-m-1$.

We now prove the result which shall be crucial in the rest of this chapter.

Theorem 7.3. *Let $\tilde{X} \subset \mathbb{P}_k^{r+1}$ be a (cone over a) rational normal scroll and let $\varrho : \mathbb{P}_k^{r+1} \setminus \{p\} \to \mathbb{P}_k^r$ be a linear projection from a point $p \in \mathbb{P}_k^{r+1} \setminus \tilde{X}$. Then, there is a (cone over a) rational normal scroll $Y \subset \mathbb{P}_k^r$ such that $Y \supset \varrho(\tilde{X}), \dim Y = \dim \tilde{X} + 1$ and $\dim \operatorname{Sing}(\tilde{X}) \leq \dim \operatorname{Sing}(Y) \leq \dim \operatorname{Sing}(\tilde{X}) + 3$.*

Proof. According to Remark 7.2 A) we may assume that the vanishing ideal of $\tilde{X}$ in $S' = k[x_0, \ldots, x_{r+1}]$ is generated by the 2×2-minors of the $2 \times (n - l + 1)$-matrix

$$M := \left(\begin{array}{ccc|ccc|c|ccc} x_0 & \cdots & x_{a_1-1} & x_{a_1+1} & \cdots & x_{a_2-1} & \cdots & x_{a_{l-1}+1} & \cdots & x_{n-1} \\ x_1 & \cdots & x_{a_1} & x_{a_1+2} & \cdots & x_{a_2} & \cdots & x_{a_{l-1}+2} & \cdots & x_n \end{array} \right)$$

with appropriate integers $n, l, a_1, \ldots, a_{l-1} \in \mathbb{N}$ such that, with $a_0 = -1, a_l = n$, we have $a_i - a_{i-1} > 1$ for $i = 1, \ldots, l, n \leq r + 1, \dim \tilde{X} = l + r + 1 - n$ and $\dim \operatorname{Sing}(\tilde{X}) = r - n$.

Let $p := (c_0 : c_1 : \cdots : c_{r+1})$. As $p \notin \tilde{X}$ there are two different indices $i, j \in \{0, 1, \ldots, n - 1\} \setminus \{a_1, a_2, \ldots, a_{l-1}\}$ such that

$$\delta := \det \begin{pmatrix} c_i & c_j \\ c_{i+1} & c_{j+1} \end{pmatrix} \neq 0.$$

Without loss of generality we may assume that $c_{i+1} \neq 0$. Define

$$y_\alpha := \begin{cases} x_\alpha, & \text{if} \quad \alpha = i + 1, \\ x_\alpha - \frac{c_\alpha}{c_{i+1}} x_{i+1}, & \text{if} \quad \alpha \in \{0, \ldots, r + 1\} \setminus \{i + 1\}. \end{cases}$$

Then $S' = k[y_0, y_1, \ldots, y_{r+1}]$ and with respect to the coordinates $y_0, \ldots, y_{r+1}$ the point p may be written as $(0 : \cdots : 0 : 1 : 0 : \cdots : 0)$ with the entry "1" in the $(i + 1)$-th position. Therefore we may assume that the projection ϱ is induced by the inclusion map

$$S'' := k[y_0, \ldots, y_i, y_{i+2}, \ldots, y_{r+1}] \hookrightarrow S'.$$

We now express the indeterminates which occur in the matrix M in terms of the variables y_α :

$$x_\alpha = \begin{cases} y_\alpha, & \text{for} \quad \alpha = i + 1, \\ y_\alpha + \frac{c_\alpha}{c_{i+1}} y_{i+1},, & \text{if} \quad \alpha \in \{0, \ldots, r + 1\} \setminus \{i + 1\}. \end{cases}$$

Let us first assume that $i < j$. Then, the 2×2-submatrix U of M which contains x_i and x_j in its first row takes the form

$$U = \begin{pmatrix} y_i + \frac{c_i}{c_{i+1}} y_{i+1} & y_j + \frac{c_j}{c_{i+1}} y_{i+1} \\ y_{i+1} & y_{j+1} + \frac{c_{j+1}}{c_{i+1}} y_{i+1} \end{pmatrix}.$$

Now performing sucessively k-linear row and column operations we finally get the following transformed matrix

$$U' = \begin{pmatrix} y_i & y_j - \frac{c_i}{c_{i+1}} y_{j+1} - \frac{c_{j+1}}{c_{i+1}} y_i - \frac{\delta}{c_{i+1}^2} y_{i+1} \\ y_{i+1} & y_{j+1} \end{pmatrix}.$$

If $i + 1 = j$, then by performing k-linear row and column operations, U can be brought to the form

$$U' = \begin{pmatrix} y_i & -\frac{c_i}{c_{i+1}} y_{i+2} - \frac{c_{i+2}}{c_{i+1}} y_i - \frac{\delta}{c_{i+1}^2} y_{i+1} \\ y_{i+1} & y_{i+2} \end{pmatrix}.$$

Let M' be the $2\times(n-l+1)$-matrix which is obtained if the above row and column operations are performed with the whole matrix M. Observe that the submatrix U of M is transformed into the submatrix U' of M', which sits in the same columns as U. Now, as $-\frac{\delta}{c_{i+1}^2} \neq 0$ we may add appropriate k-multiples of the columns of U' to the columns of M' to remove the indeterminate y_{i+1} from all entries of M' which do not belong to the two columns of U'. So, we get a $2\times(n-l+1)$-matrix $\tilde{M}$, conjugate to M. In particular, $\tilde{M}$ is 1-generic (cf Remark 7.1 D)) and the entries of $\tilde{M}$ span the same k-space as the entries of M, namely $\sum_{t=0}^{n} kx_t = \sum_{t=0}^{n} ky_t$. Now, let N be the matrix of size $2 \times (n-l-1) = 2 \times (n-1-(l+1)+1)$ obtained by deleting the two columns of U' from $\tilde{M}$. Then N is 1-generic (cf Remark 7.1 D)) and y_{i+1} does not appear in N. So, the entries of N span a subspace

$$L \subseteq \Sigma_{t=0,t\neq i+1}^{n} ky_t \subset k[y_0,\ldots,y_i,y_{i+1},\ldots,y_n] \subset S'',$$

whose dimension m is such that $n-m \in \{1,2,3,4\}$.

By Remark 7.2 C) the ideal $I_2(N) \subset S''$ generated by the 2×2-minors of N is the vanishing ideal of a cone $Y \subset \mathbb{P}_k^r = \mathrm{Proj}(S'')$ over a rational normal scroll such that $\dim Y = (l+1) + r - (n-1) = \dim \tilde{X} + 1$ and $\dim \mathrm{Sing}(Y) = r - m - 1 = \dim \mathrm{Sing}(\tilde{X}) + (n-m-1)$. As $n-m-1 \in \{0,1,2,3\}$ we get $\dim \mathrm{Sing}(\tilde{X}) \leq \dim \mathrm{Sing}(Y) \leq \dim \mathrm{Sing}(\tilde{X}) + 3$. According to Remark 7.2 B) the ideal $I_2(\tilde{M}) \subset S'$ generated by the 2×2-minors of $\tilde{M}$ is the vanishing ideal of $\tilde{X}$. Therefore $I_2(\tilde{M}) \cap S''$ is the vanishing ideal of $\varrho(\tilde{X})$. As each 2×2-minor of N is a 2×2-minor of $\tilde{M}$, it follows $I_2(N) \subseteq I_2(\tilde{M}) \cap S''$ and hence $Y \supseteq \varrho(\tilde{X})$.

This settles the case $i < j$. If $j < i$ we first commute the columns of U and then conclude as above. $\square$

We now apply the previous result to varieties of almost minimal degree. We still keep the convention that $\mathbb{P}_k^{-1} = \emptyset$ and $\dim \emptyset = -1$, and begin with a preliminary remark.

Remark 7.4. (cf [21]) A) Let $l, n, a_1, \ldots, a_{l-1} \in \mathbb{N}$ be as in Remark 7.1 and consider the rational l-fold scroll $S_{a_1\cdots a_{l-1}}$. We set $a_0 = -1, a_l = n, d_i = a_i - a_{i-1} - 1$, for $i = 1, \ldots, l$ and define the linear subspaces

$$\mathbb{E}_i = \mathrm{Proj}(S/\Sigma_{j\notin\{a_{i-1}+1,\ldots,a_i\}} Sx_j) = \mathbb{P}_k^{d_i} \subset \mathbb{P}_k^n, i = 1,\ldots,l.$$

For each $i \in \{1,\ldots,l\}$ we consider the Veronese embedding

$$\nu_i : \mathbb{P}_k^1 \to \mathbb{E}_i,\ (s:t) \mapsto (s^{d_i} : s^{d_i-1}t : \ldots : st^{d_i-1} : t^{d_i}),$$

so that $\nu_i(\mathbb{P}_k^1) \subset \mathbb{E}_i = \mathbb{P}_k^{d_i}$ is a rational normal curve. Now, for each $q \in \mathbb{P}_k^1$ let

$$\mathbb{E}(q) = \langle \nu_1(q), \ldots, \nu_l(q) \rangle = \mathbb{P}_k^{l-1} \subset \mathbb{P}_k^n$$

be the projective space spanned by the l points $\nu_1(q),\ldots,\nu_l(q)$. Then $q \neq q'$ implies $\mathbb{E}(q) \cap \mathbb{E}(q') = \emptyset$ for all $q, q' \in \mathbb{P}_k^1$. Moreover, $S_{a_1\cdots a_{l-1}} = \cup_{q\in\mathbb{P}_k^1} \mathbb{E}(q)$.

B) Keep the above notation. For each $1 \leq i \leq l$ let $\mathbb{V}_i = k^{d_i+1} \subset k^{n+1}$ be the affine cone over $\mathbb{E}_i$, fix $q = (s:t) \in \mathbb{P}_k^1$ and set $v_i = (s^{d_i}, s^{d_i-1}t, \ldots, t^{d_i})$. Moreover, for each $1 \leq i \leq l$ let $w_i = (w_{i0}, \ldots, w_{id_i}) \in \mathbb{V}_i$ such that $(w_{i0} : \ldots : w_{id_i}) \in \mathbb{E}_i \setminus \nu_i(q)$ is a point on the tangent of the rational normal curve $\nu_i(\mathbb{P}_k^1) \subset \mathbb{E}_i$ in the point $\nu_i(q)$. In particular, v_i and w_i are lineraly independent. Now let $\pi : k^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{P}_k^n = \mathbb{P}(k^{n+1})$ be the canonical projection and let

$$r = \pi(\Sigma_{i=1}^{l} r_i v_i) \in \mathbb{E}(q) = \pi(\oplus_{i=1}^{l} kv_i \setminus \{0\}) = \mathbb{P}(\oplus_{i=1}^{l} kv_i),$$

where $(r_1, \ldots, r_l) \in k^l \setminus \{0\}$. Then the tangent space to the scroll $S_{a_1 \cdots a_{l-1}}$ in the point r is given by

$$T_r(S_{a_1 \cdots a_{l-1}}) = \mathbb{P}(k(\Sigma_{i=1}^l r_i w_i) + \oplus_{i=1}^l k v_i) = \langle \mathbb{E}(q) \cup \pi(\Sigma_{i=1}^l r_i w_i)\rangle.$$

From this we easily deduce that $T_r(S_{a_1 \cdots a_{l-1}}) \cap T_{r'}(S_{a_1 \cdots a_{l-1}}) = \mathbb{E}(q)$ for all $r, r' \in \mathbb{E}(q)$ with $r \neq r'$.

C) Now, let $s \in \mathbb{N}$ such that $n < s$ and let $\tilde{X} \subset \mathbb{P}_k^s = \operatorname{Proj}(R), R = k[x_0, \ldots, x_n]$, be a cone over the rational normal l-fold scroll

$$S_{a_1 \cdots a_{l-1}} \subset \mathbb{P}_k^n = \operatorname{Proj}(R/(x_{n+1}, \ldots, x_s)R = k[x_0, \ldots, x_n]).$$

Then, the vertex of $\tilde{X}$ is given by $\operatorname{Sing}(\tilde{X}) = \operatorname{Proj}(R/(x_0, \ldots, x_n)) = \mathbb{P}_k^{s-n-1} \subset \mathbb{P}_k^s$ (cf Remark 7.2 A)). Now, for each $q \in \mathbb{P}_k^1$ let

$$\mathbb{F}(q) = \langle \mathbb{E}(q) \cup \operatorname{Sing}(\tilde{X})\rangle = \mathbb{P}_k^{l+s-n-1} = \mathbb{P}_k^{\dim \tilde{X}} \subset \mathbb{P}_k^s$$

be the linear subspace spanned by $\mathbb{E}(q) = \mathbb{P}_k^{l-1} \subset \mathbb{P}_k^s$ and the vertex $\operatorname{Sing}(\tilde{X})$ of $\tilde{X}$. Then by part A), $q \neq q'$ implies that $\mathbb{F}(q) \cap \mathbb{F}(q') = \operatorname{Sing}(\tilde{X})$ for all $q, q' \in \mathbb{P}_k^1$ and moreover $\tilde{X} = \cup_{q \in \mathbb{P}_k^1} \mathbb{F}(q)$.

It also follows easily from part B), that for any $q \in \mathbb{P}_k^1$ and any $r \in \mathbb{E}(q) \setminus \operatorname{Sing}(\tilde{X})$ the tangent space of $\tilde{X}$ at r is given by $T_r(\tilde{X}) = \langle T_{\tilde{r}}(S_{a_1 \cdots a_{l-1}}) \cup \operatorname{Sing}(\tilde{X})\rangle = \mathbb{P}_k^d$, where $\tilde{r} = (r_0 : \ldots : r_n)$ is the canonical projection of r from $\operatorname{Sing}(\tilde{X})$. As a consequence of the last statement in part B) we thus get for all $q \in \mathbb{P}_k^1$ and all $r, r' \in \mathbb{F}(q) \setminus \operatorname{Sing}(\tilde{X})$: If $r \neq r'$, then $T_r(\tilde{X}) \cap T_{r'}(\tilde{X}) = \mathbb{F}(q)$.

Theorem 7.5. *Let $X \subset \mathbb{P}_k^r$ be a variety of almost minimal degree which is the projection of a (cone over a) rational normal scroll $\tilde{X} \subset \mathbb{P}_k^{r+1}$ with $\dim \operatorname{Sing}(\tilde{X}) =: h$ from a point $p \in \mathbb{P}_k^{r+1} \setminus \tilde{X}$. Then*

(a) *X is contained in a (cone over a) rational normal scroll $Y \subset \mathbb{P}_k^r$ such that $\operatorname{codim}_Y(X) = 1$ and $h \leq \dim \operatorname{Sing}(Y) \leq h + 3$.*
(b) *X is of arithmetic depth $t \leq h + 5$.*

Proof. (a): This is clear by Theorem 7.3.
(b): Assume first that X is not arithmetically Cohen-Macaulay. Then, the non CM-locus Z of X is a linear subspace $\mathbb{P}_k^{t-2}$ of $\mathbb{P}_k^r$ (cf Theorem 5.6 (d), (f)). As $\operatorname{codim}_Y(X) = 1, \mathcal{O}_{X,x}$ is a Cohen-Macaulay ring for each point $x \in X \setminus \operatorname{Sing}(Y)$. It follows that $\mathbb{P}_k^{t-2} = Z \subseteq X \cap \operatorname{Sing}(Y)$ and hence $t - 2 \leq \dim \operatorname{Sing}(Y) \leq h + 3$.

Now, let X be arithmetically Cohen-Macaulay. In this case we conclude by a geometric argument which in fact also implies in the previous case. Let $d = \dim X$. After an appropriate change of coordinates, we may assume that we are in the situation of Remark 7.4 B) and C). So, we may write $\tilde{X} = \cup_{q \in \mathbb{P}_k^1} \mathbb{F}(q)$, where $\mathbb{F}(q) = \mathbb{P}_k^{d-1} \subset \mathbb{P}_k^r$ is a linear subspace for all $q \in \mathbb{P}_k^1$. Let $U = \{(q, q') \in \mathbb{P}_k^1 \times \mathbb{P}_k^1 | q \neq q'\}$. Then, according to Remark 7.4 C) we have $\mathbb{F}(q) \cap \mathbb{F}(q') = \operatorname{Sing}(\tilde{X}) = \mathbb{P}_k^h$, whenever $(q, q') \in U$. Now, for each pair $(q, q') \in U$ consider the linear subspace

$$\mathbb{H}(q, q') = \langle \mathbb{F}(q) \cup \{p\}\rangle \cap \langle \mathbb{F}(q') \cup \{p\}\rangle \subset \mathbb{P}_k^{r+1}.$$

Observe that $\operatorname{Sing}(\tilde{X}) \subseteq \mathbb{H}(q, q')$ and $\dim \mathbb{H}(q, q') \leq h + 2$ for all $(q, q') \in U$. Moreover $\dim \mathbb{H}(q, q') = h + 2$ if and only if $p \in \langle \mathbb{F}(q) \cup \mathbb{F}(q')\rangle$. Consequently we have $\dim \mathbb{H}(q, q') = h + 2$ if and only if there is a line running through p and intersecting $\mathbb{F}(q)$ and $\mathbb{F}(q')$. Clearly such a line is contained in $\mathbb{H}(q,q')$ and its intersection points with $\mathbb{F}(q)$ and $\mathbb{F}(q')$ are different as $p \notin \mathbb{F}(q) \cup \mathbb{F}(q')$.

Let $V \subseteq U$ be the closed subset of all pairs (q,q') for which $\dim \mathbb{H}(q,q') = h+2$. It follows that the union of all proper secant lines of $\tilde{X}$ which run through p is a subset of $W = \cup_{(q,q')\in V}\mathbb{H}(q,q')$. Moreover, it follows from the last statement of Remark 7.4 C), that for each point $q \in \mathbb{P}^1_k$ there is at most one point $r(q) \in \mathbb{F}(q) \setminus \mathrm{Sing}(\tilde{X})$ such that there is a tangent line $l(q) = \mathbb{P}^1_k$ of $\tilde{X}$ at $r(q)$ running through p. Let $T \subset \mathbb{P}^1_k$ be the closed subset of all $q \in \mathbb{P}^1_k$ for which this happens. Then, all tangents to non-singular points of $\tilde{X}$ through p are contained in $Y = \cup_{q\in T} l(q)$. Finally observe that the remaining tangents are the lines running trough p and $\mathrm{Sing}(\tilde{X})$. It follows $\mathbb{P}^{t-1}_k = \mathrm{Sec}_p(\tilde{X}) \subseteq W \cup Y \cup \langle \mathrm{Sing}(\tilde{X}) \cup \{p\}\rangle$ and hence $t-1 \leq \max\{\dim W, \dim Y, h+1\}$. As $\{\mathbb{H}(q,q')|(q,q') \in V\}$ is a family of linear $(h+2)$-subspaces of $\mathbb{P}^{r+1}_k$, it follows $\dim W \leq h+2+\dim V \leq h+4$. As $\{l(q)|q \in T\}$ is a family of lines we have $\dim Y \leq 1 + \dim T \leq 2$. So we get $t-1 \leq h+4$, hence $t \leq h+5$. □

Corollary 7.6. *Let $X \subset \mathbb{P}^r_k$ be a variety of almost minimal degree which is a projection of a rational normal scroll $\tilde{X} \subset \mathbb{P}^{r+1}_k$ from a point $p \in \mathbb{P}^{r+1}_k \setminus \tilde{X}$. Then X is of arithmetic depth $t \leq 4$.*

Proof. Clear from Theorem 7.5. □

As a final comment of this section let us say something about the exceptional case of projections of the Veronese surface.

Remark 7.7. (The exceptional case) Let us recall that the Veronese surface $F \subset \mathbb{P}^5_k$ is defined by the 2×2-minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}.$$

Let $p \in \mathbb{P}^5_K \setminus F$ denote a closed point. Suppose that $\mathrm{rank}\, M|_p = 3$, i.e. the case of a generic point and remember that $\det M = 0$ defines the secant variety of F. Then the projection of F from p defines a surface $X \subset \mathbb{P}^4_K$ of almost minimal degree and $\mathrm{depth}\, A = 1$.

Recall that $\dim_k(I)_2 = 0$ (cf. Corollary 4.4 C)). Therefore the surface X is cut out by cubics, i.e. it is not contained in a variety of minimal degree.

8. Betti numbers

Our next aim is to study the Betti numbers of A if X is of almost minimal degree, non-arithmetically Cohen-Macaulay and a projection of a (cone over a) rational normal scroll.

Lemma 8.1. *Assume that $X \subset \mathbb{P}^r_k$ is of almost minimal degree and of arithmetic depth $\leq d = \dim X$. Let $B = \mathrm{Hom}_A(K(A), K(A))$. Then*

$$\mathrm{Tor}^S_i(k,B) \simeq \begin{cases} k(0) \oplus k(-1), & \textit{if} \quad i = 0, \\ k^{b_i}(-i-1), & \textit{if} \quad 0 < i \leq r-d, \\ 0, & \textit{if} \quad r-d < i, \end{cases}$$

where $b_i = (r+1-d)\binom{r-d}{i} - \binom{r-d}{i+1}$ for $1 \leq i \leq r-d$.

Proof. By Theorem 5.3 (a) the A-module B is Cohen-Macaulay and hence of depth $d+1$ over S. Therefore $\operatorname{Tor}_i^S(k,B)=0$ for all $i>r-d$. According to Theorem 5.3 (b) there is a short exact sequence of graded S-modules

$$0 \to A \to B \to C \to 0, \quad C \simeq (S/(y_{t-2},\ldots,y_r)S)(-1), \tag{8.1}$$

where $y_0,\ldots,y_r$ form a generic set of linear forms of S. In particular, C is of dimension $t-1<d$ and generated by a single element of degree 1. This already shows that $\operatorname{Tor}_0^S(k,B)\simeq k(0)\oplus k(-1)$.

Applying cohomology to the above short exact sequence we get an isomorphism $H^{d+1}(A)\simeq H^{d+1}(B)$ which shows that end $H^{d+1}(B)=-d$ (cf Theorem 4.2 (b)). As $\operatorname{depth} B=d+1$ it follows $\operatorname{reg} B=1$. Moreover the above exact sequence yields

$$\dim_k B_1=\dim_k A_1+1=r+2=\dim_k(S(0)\oplus S(-1))_1.$$

So, the graded S-module B must have a minimal free resolution of the form

$$0\to S^{b_{r-d}}(-r+d-1)\to\ldots\to S^{b_i}(-i-1)\to\ldots\to S^{b_1}(-2)\to S\oplus S(-1)\to B\to 0$$

with $b_1,\ldots,b_{r-d}\in\mathbb{N}$.

As B is a Cohen-Macaulay module of dimension $d+1$, regularity 1 and of multiplicity $\deg X=r-d+2$ (cf Theorem 5.3 (c)) its Hilbert series is given by

$$F(\lambda,B)=\frac{1+(r+1-d)\lambda}{(1-\lambda)^{d+1}}.$$

On the use of Betti numbers b_i we also may write

$$F(\lambda,B)=\frac{1}{(1-\lambda)^{d+1}}(1+\lambda+\sum_{i=0}^{r-d}(-1)^i b_i\lambda^{i+1}).$$

Comparing coefficients we obtain

$$b_i=(r+1-d)\binom{r-d}{i}-\binom{r-d}{i+1},\ i=1,\ldots,r-d,$$

as required. $\square$

Next we recall a well known result about the Betti numbers of a variety of minimal degree.

Lemma 8.2. *Let $Y\subset\mathbb{P}_k^r$ be a variety of minimal degree with $\dim Y=d+1$. Let U be the homogeneous coordinate ring of Y. Then*

$$\operatorname{Tor}_i^S(k,U)\simeq\begin{cases} k, & \text{if}\quad i=0,\\ k^{c_i}(-i-1), & \text{if}\quad 0<i<r-d,\\ 0, & \text{if}\quad r-d\le i,\end{cases}$$

where $c_i=i\binom{r-d}{i+1}$ for $1\le i<r-d$.

Proof. This is well known (cf for instance [10]). In fact the Eagon-Northcott complex provides a minimal free resolution of U over S. $\square$

Theorem 8.3. *Let $X \subset \mathbb{P}^r_k$ be a variety of almost minimal degree which is the projection of a (cone over a) rational normal scroll $\tilde{X} \subset \mathbb{P}^{r+1}_k$ from a point $p \in \mathbb{P}^{r+1}_k \setminus \tilde{X}$. Assume that $t := \operatorname{depth} A \leq d := \dim X$. Then*

$$\operatorname{Tor}^S_i(k, A) \simeq \begin{cases} k, & \text{if} \quad i = 0, \\ k^{u_i}(-i-1) \oplus k^{v_i}(-i-2), & \text{if} \quad 0 < i \leq r-t+1, \\ 0, & \text{if} \quad r-t+1 < i, \end{cases}$$

where

(a) $u_1 = t + \binom{r+1-d}{2} - d - 2,$
$i\binom{r-d}{i+1} \leq u_i \leq (r+1-d)\binom{r-d}{i} - \binom{r-d}{i+1}, \quad \textit{if } 1 < i < r-2d+t-1,$
$u_i = i\binom{r-d}{i+1}, \quad \textit{if } r-2d+t-1 \leq i < r-d,$
$u_i = 0, \quad \textit{if } r-d \leq i < r-t+1,$

(b) $\max\{0, \binom{r-t+2}{i+1} - (i+2)\binom{r-d}{i+1}\} \leq v_i \leq \binom{r-t+2}{i+1}, \quad \textit{if } 1 \leq i < r-2d+t-2,$
$v_i = \binom{r-t+2}{i+1} - (i+2)\binom{r-d}{i+1}, \quad \textit{if } r-2d+t-2 \leq i < r-d,$
$v_i = \binom{r-t+2}{i+1}, \quad \textit{if } r-d \leq i \leq r-t+1.$

Moreover, $v_i - u_{i+1} = \binom{r-t+2}{i+1} - (r-d+1)\binom{r-d}{i+1} + \binom{r-d}{i+2}$ *for all* $1 \leq i < r-d$.

Proof. As $\operatorname{depth} A = t$ and $\operatorname{reg} A = 2$ (cf Theorem 4.2) the modules $\operatorname{Tor}^S_i(k, A)$ behave as stated in the main equality. So let the numbers u_i, v_i be defined according to this main equality with the convention that $u_i = v_i = 0$ for $i > r-t+1$. Moreover let b_i and c_i be as in Lemma 8.1 resp. 8.2 with the convention that $b_i = 0$ for $i > r-d$ and $c_i = 0$ for $i \geq r-d$.

According to Corollary 7.6 there is a (cone over a) rational normal scroll $Y \subset \mathbb{P}^r_k$ of dimension $d+1$ such that $X \subset Y$. Let $J \subset S$ be the vanishing ideal of Y and let $U := S/J$ be the homogeneous coordinate ring of Y. The short exact sequence $0 \to I/J \to U \to A \to 0$ together with Lemma 8.2 implies short exact sequences

$$0 \to k^{c_i}(-i-1) \to k^{u_i}(-i-1) \oplus k^{v_i}(-i-2) \to \operatorname{Tor}^S_{i-1}(k, I/J) \to 0 \tag{8.2}$$

for all $i \geq 1$.

Keep in mind that $\operatorname{beg}(I/J) = 2, u_1 = \dim_k I_2 = t + \binom{r+1-d}{2} - d - 2$ (cf Corollary 4.4 (c)) and $\dim_k J_2 = \binom{r-d}{2}$ (cf Lemma 8.2) so that

$$\dim_k (I/J)_2 = r - 2d + t - 2.$$

Whence, by Green's Linear Syzygy Theorem (cf [11, Theorem 7.1]) we have

$$\operatorname{Tor}^S_j(k, I/J)_{j+2} = 0 \ \text{ for all } j \geq r-2d+t-2.$$

So, the sequence 8.2 yields that $u_i = c_i$ for all $i \geq r-2d+t-1$. This proves statement (a) in the range $i \geq r-2d+t-1$. The sequence 8.2 also yields that $c_i \leq u_i$ for all $i \leq r-2d+t-1$.

Next we consider the short exact sequence of graded S-modules 8.1. In particular we have $\operatorname{Tor}^S_i(k, C) \simeq k^{\binom{r-t+2}{i}}(-i-1)$ for all $i \in \mathbb{N}_0$. So, by the sequence 8.1 and in view of Lemma 8.2 we get exact sequences

$$\begin{aligned} k^{b_{i+1}}(-i-2) &\to k^{\binom{r-t+2}{i+1}}(-i-2) \to k^{u_i}(-i-1) \oplus k^{v_i}(-i-2) \\ &\to k^{b_i}(-i-1) \to k^{\binom{r-t+2}{i}}(-i-1) \to k^{u_{i-1}}(-i) \oplus k^{v_{i-1}}(-i-1) \to k^{b_{i-1}}(-i) \end{aligned} \tag{8.3}$$

for all $i \geq 2$. Now, we read off that $u_i \leq b_i$ for all $i \geq 1$ and statement (a) is proved completely.

The sequence 8.3 also yields that

$$v_i = u_{i+1} - b_{i+1} + \binom{r-t+2}{i+1} \text{ for all } i \geq 1. \tag{8.4}$$

Observe that $c_{i+1} - b_{i+1} = -(i+2)\binom{r-d}{i+1}$ for $1 \leq i < r-d$. If $1 \leq i < r-2d+t-1$, statement (a) gives $c_{i+1} \leq u_{i+1} \leq b_{i+1}$ so that

$$c_{i+1} - b_{i+1} + \binom{r-t+2}{i+1} \leq v_i \leq \binom{r-t+2}{i+1}.$$

This proves the first estimate of statement (b).

If $r-2d+t-2 \leq i < r-d$, statement (a) yields $u_{i+1} = c_{i+1}$, hence $v_i = c_{i+1} - b_{i+1} + \binom{r-t+2}{i+1}$. This proves the second claim of statement (b). Finally, if $r-d \leq i < r-t+1$ statement (a) and Lemma 8.2 yield that $u_{i+1} = c_{i+1} = 0$. Now the last claim of the statement (b) follows by 8.4. $\square$

9. Examples

In this final section we present a few examples which illustrate the previous results. All calculations of the "graded Betti numbers" u_i and v_i (cf Theorem 8.3) have been performed by means of the computer algebra system SINGULAR [20]. As for rational scrolls and their secant varieties we refer to [9] and [21].

First, we present three examples of 3-folds X of almost minimal degree in $\mathbb{P}^{11}_k$, one of them being defined by 32 quadrics, the second by 32 quadrics and 1 cubic, the third by 32 quadrics and 3 cubics. These examples show that, contrary to the number of defining quadrics (cf Corollary 4.4 (c)), the number of defining cubics may vary if the embedding dimension r, the dimension d and the arithmetic $\operatorname{depth} t$ of X are fixed. Notice that each smooth variety $X \subset \mathbb{P}^{11}_k$ of almost minimal degree which is not arithmetically Cohen-Macaulay is obtained by projecting a rational scroll $\tilde{X} \subset \mathbb{P}^{12}_k$ from a point $p \in \mathbb{P}^{12}_k \setminus \operatorname{Sec}(\tilde{X})$ (cf Theorem 5.6).

We first fix some notation. Let $l, n, d_1, \ldots, d_l \in \mathbb{N}$ such that $d_1 \leq d_2 leq \ldots d_l$ and $\sum_{i=1}^{l} d_i = n - l + 1$. Let $a_i = i - 1 + \sum_{j=1}^{i} d_j, i = 1, \ldots, l-1$. Then, we write $S(d_1, \ldots, d_l)$ for the rational normal scroll $S_{a_1 \cdots, a_{l-1}}$ (cf Remark 7.1).

Example 9.1. A) Let $\tilde{X} \subset \mathbb{P}^{12}_k$ the 3-scroll $S(2,2,6)$, thus the smooth variety of degree 10 defined by the 2×2 minors of the matrix

$$\left(\begin{array}{cc|cc|cccccc} x_0 & x_1 & x_3 & x_4 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} \\ x_1 & x_2 & x_4 & x_5 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \end{array}\right).$$

Its homogeneous coordinate ring is

$$B = k[(s,t)^2u^5, (s,t)^2v^5, (s,t)^6w] \subset k[s,t,u,v,w].$$

Projecting $\tilde{X}$ from the point

$$p_1 = (0:0:0:0:0:0:0:0:0:1:0:0:0) \in \mathbb{P}^{12}_k \setminus \tilde{X}$$

we get a non-degenerate variety $X \subset \mathbb{P}^{11}_k$ of dimension 3 and of degree ≤ 10, (cf Remark 3.3 A)). Let S denote a polynomial ring in 12 indeterminates, let $I \subset S$ be the homogeneous vanishing ideal and let $A = S/I$ be the homogeneous coordinate ring of X. Also, consider the only not necessarily vanishing graded Betti numbers

$$u_i := \dim_k \operatorname{Tor}_i^S(k, A)_{i+1},\ v_i := \dim_k \operatorname{Tor}_i^S(k, A)_{i+2}$$

of X. These numbers present themselves as shown below:

i	1	2	3	4	5	6	7	8	9	10	11
u_i	32	130	234	234	140	48	7	0	0	0	0
v_i	0	20	155	456	728	728	486	220	66	12	1

In particular $t := \operatorname{depth} A = 1$ so that X cannot be of minimal degree and hence $\deg X = 10 = 11 - 3 + 2$. Therefore, X is of almost minimal degree and of arithmetic $\operatorname{depth} 1$. In particular the projection map $\nu : \tilde{X} \to X$ is an isomorphism (cf Theorem 5.7) and so X becomes smooth. Observe, that I is generated by 32 quadrics.
B) Let $\tilde{X} \subset \mathbb{P}_k^{12}$ be as in part A) but project $\tilde{X}$ from the point

$$p_2 = (0:0:0:0:0:0:0:0:0:0:1:0:0) \in \mathbb{P}_k^{12} \setminus \tilde{X}.$$

Again let $X \subset \mathbb{P}_k^{11}$ be the image of $\tilde{X}$ under this projection and define S, I, A as in part A). Now the Betti numbers u_i, v_i present themselves as follows:

i	1	2	3	4	5	6	7	8	9	10	11
u_i	32	131	234	234	140	48	7	0	0	0	0
v_i	1	20	155	456	728	728	486	220	66	12	1

So, as above, we see that X is a smooth variety of almost minimal degree having dimension 3 and arithmetic $\operatorname{depth} 1$. Observe, that now I is minimally generated by 32 quadrics and 1 cubic. So, if the same scroll $\tilde{X} = S(2,2,6) \subset \mathbb{P}_k^{12}$ is projected from two different points $p_1, p_2 \in \mathbb{P}_k^{12} \setminus \operatorname{Sec}(\tilde{X})$, the homological nature of the projection $X \subset \mathbb{P}_k^{11}$ may differ.
C) Now, consider the scroll $\tilde{X} := S(2,4,4) \subset \mathbb{P}_k^{12}$, so that $\tilde{X}$ is the smooth variety of dimension 3 and degree 10 defined by the 2×2-minors of the matrix

$$\left(\begin{array}{cc|cccc|cccc} x_0 & x_1 & x_3 & x_4 & x_5 & x_6 & x_8 & x_9 & x_{10} & x_{11} \\ x_1 & x_2 & x_4 & x_5 & x_6 & x_7 & x_9 & x_{10} & x_{11} & x_{12} \end{array}\right).$$

Its homogeneous coordinate ring is

$$B = k[(s,t)^2u^3, (s,t)^4v, (s,t)^4w] \subset k[s,t,u,v,w].$$

Define $X \subset \mathbb{P}_k^{11}$ as the projection of $\tilde{X}$ from the point $p_2 \in \mathbb{P}_k^{12} \setminus \tilde{X}$ (cf part B)). In this case, the Betti numbers u_i and v_i of X take the values listed in the following table:

i	1	2	3	4	5	6	7	8	9	10	11
u_i	32	133	248	234	140	48	7	0	0	0	0
v_i	3	34	155	456	728	728	486	220	66	12	1

So, again $X \subset \mathbb{P}_k^{11}$ is a smooth variety of almost minimal degree having dimension 3 and arithmetic $\operatorname{depth} 1$. But this time, besides 32 quadrics three cubics are needed to generate the homogeneous vanishing ideal I of X.

The previous example where all of arithmetic $\mathrm{depth}\,1$ and of dimension 3. By projecting rational 3-scrolls in $\mathbb{P}_k^{12}$ from appropriate points we also may obtain 3-dimensional varieties $X \subset \mathbb{P}_k^{11}$ of almost minimal degree and of arithmetic depth not equal to 1. We present two examples to illustrate this.

Example 9.2. A) Next consider the 3-scroll $\tilde{X} := S(3,3,4) \subset \mathbb{P}_k^{12}$ defined by the 2×2-minors of the matrix

$$\left(\begin{array}{ccc|ccc|cccc} x_0 & x_1 & x_2 & x_4 & x_5 & x_6 & x_8 & x_9 & x_{10} & x_{11} \\ x_1 & x_2 & x_3 & x_5 & x_6 & x_7 & x_9 & x_{10} & x_{11} & x_{12} \end{array}\right).$$

$\tilde{X}$ has the homogeneous coordinate ring

$$B = k[(s,t)^3u^2, (s,t)^3v^2, (s,t)^4w] \subset k[s,t,u,v,w].$$

We project $\tilde{X}$ from the point

$$p_3 = (0:0:0:0:0:0:1:0:0:0:0:0:0) \in \mathbb{P}_k^{12} \setminus \tilde{X}.$$

Like above we get a non-degenerate variety $X \subset \mathbb{P}_k^{11}$ of degree $10 = \operatorname{codim} X + 2$ and Betti numbers:

i	1	2	3	4	5	6	7	8	9	10
u_i	33	142	278	284	155	48	7	0	0	0
v_i	1	9	40	141	266	266	156	55	11	1

So, X is of arithmetic $\mathrm{depth}\,2$.

The tangent line of the curve

$$\sigma : k \to \tilde{X};\ s \mapsto \sigma(s) := (0:0:0:0:s^3:s^2:s:1:0:0:0:0:0)$$

in the point $\sigma(0)$ contains p_3. So, the secant cone $\mathrm{Sec}_{p_3}(\tilde{X})$ – which must be a line according to Theorem 5.6 (d) – is just the line ℓ which joins p_3 and $\sigma(0)$. The projection of ℓ from p_3 to $\mathbb{P}_k^{11}$ is the point

$$q := (0:0:0:0:0:0:1:0:0:0:0:0) \in X.$$

So, in the notation of Theorem 5.6, we have $\mathrm{Sing}(\nu) = \varrho(\ell \setminus \{p_2\}) = \{q\}$. Now, let $a := \frac{u^2}{v^2}, b := \frac{s}{t}, c := \frac{tw}{v^2}$. An easy calculation shows that there is an isomorphism

$$\epsilon : X_{t^3v^2} \xrightarrow{\simeq} Y := \mathrm{Spec}(k[a, ab, b^2, b^3, b^2c, bc, c])$$

with $\epsilon(q) = \underline{0}$, where $X_{t^3v^2} \subset X$ is the affine open neighborhood of q defined by $t^3v^2 \neq 0$. It is easy to verify, that $\mathcal{O}_{Y,\underline{0}}$ is a G-ring and hence that $q \in X$ is a G-point, as predicted by Theorem 5.6.

B) Next, we project the 3-scroll $\tilde{X} := S(2,4,4) \subset \mathbb{P}_k^{12}$ of Example 9.1 C) from the point

$$p_4 = (0:1:0:0:0:0:0:0:0:0:0:0:0) \in \mathbb{P}_k^{12} \setminus \tilde{X}.$$

We get a 3-dimensional variety $X \subset \mathbb{P}_k^{11}$ of degree 10 whose non-vanishing Betti numbers are:

i	1	2	3	4	5	6	7	8	9
u_i	34	151	314	364	230	69	7	0	0
v_i	0	0	0	6	35	56	36	10	1

Now, X is of arithmetic $\mathrm{depth}\, 3 = \dim(X)$. For each pair $(s,t) \in k^2 \setminus \{(0,0)\}$ consider the point

$$\pi(s,t) := (s^2 : st : t^2 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0) \in \tilde{X}.$$

Whenever $st \neq 0$, the two points $\pi(s,t)$ and $\pi(-\sqrt{-1}s, \sqrt{-1}t)$ are different and the line joining them contains p_4 and hence belongs to the secant cone $\mathrm{Sec}_{p_4}(\tilde{X})$. Moreover the tangent line of the curve

$$\tau : k \to \tilde{X};\ s \mapsto \pi(s,1)$$

in the point $\tau(0) = \pi(0,1)$ runs through p_4 and thus belongs to $\mathrm{Sec}_{p_4}(\tilde{X})$. Altogether this shows (cf Theorem 5.6 (d)) that $\mathrm{Sec}_{p_4}(\tilde{X})$ coincides with the 2-plane

$$\{(a : b : c : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0) \,|\, (a : b : c) \in \mathbb{P}^2_k\} \subset \mathbb{P}^{12}_k.$$

Projecting this plane from p_4 we obtain the line

$$h := \{(a : c : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0) \,|\, (a : c) \in \mathbb{P}^1_k\} \subset X.$$

So, in the notation of Theorem 5.6 we have $h = \mathrm{Sing}(\nu)$. Let $a := \frac{t}{s}, b := \frac{s^2v}{u^3}, c = \frac{s^2w}{u^3}$ and let $X_{s^2u^3} \subset X$ be the affine open set defined by $s^2u^2 \neq 0$. It is easy to verify, that there is an isomorphism $\varphi : X_{s^2u^3} \xrightarrow{\cong} Y := \mathrm{Spec}(k[a^2, b, ab, c, ac])$ such that $P := (b, ab, c, ac) \in Y$ is the generic point of $\varphi(h \cap X_{s^2u^3})$. An easy calculation shows that $\mathcal{O}_{Y,P}$ is a G-ring and hence, that the generic point of h in X is again a G-point.

We now present a class of non-normal Del Pezzo varieties. Note that these varieties are in fact arithmetically Gorenstein.

Example 9.3. A) Let $r \geq 4$ and let $\tilde{X} \subset \mathbb{P}^{r+1}_k$ be the rational surface scroll $S(2, r-1)$, hence the variety which is defined by the 2×2-minors of the matrix

$$\left(\begin{array}{cc|cccc} x_0 & x_1 & x_3 & x_4 & \cdots & x_r \\ x_1 & x_2 & x_4 & x_5 & \cdots & x_{r+1} \end{array}\right).$$

$\tilde{X}$ has the homogeneous coordinate ring

$$B := k[(s,t)^2 u^{r-2}, (s,t)^{r-2} v^2] \subset k[s,t,u,v].$$

Now, let $\varrho : \mathbb{P}^{r+1}_k \setminus \{p\} \twoheadrightarrow \mathbb{P}^r_k, (x_0 : x_1 : x_2 : \cdots : x_{r+1}) \mapsto (x_0 : x_2 : x_3 : \cdots : x_{r+1})$ be the projection from the point $p = (0 : 1 : 0 : \cdots : 0) \in \mathbb{P}^{r+1}_k \setminus \tilde{X}$ and let $X := \varrho(\tilde{X}) \subset \mathbb{P}^r_k$. Then X is a surface and has the homogeneous coordinate ring

$$A := k[s^2u^{r-2}, t^2u^{r-2}, (s,t)^{r-2}v^2] \subset B.$$

As B is a birational extension of A, the morphism $\nu = \varrho \restriction : \tilde{X} \to X$ is birational, so that $\deg X = \deg \tilde{X} = r$ and $X \subset \mathbb{P}^r_k$ is a surface of almost minimal degree. Moreover, as $\tilde{X}$ is smooth, $\nu = \varrho \restriction : \tilde{X} \to X$ is a normalization of X and $\mathrm{Sing}(\nu) = \varrho(\mathrm{Sec}_p(\tilde{X}) \setminus \{p\})$ is the non-normal locus of X.

Similar as in example 9.2 B) we can check that the secant cone of $\tilde{X}$at p satisfies

$$\mathrm{Sec}_p(\tilde{X}) = \{(a : b : c : 0 : \cdots : 0) \,|\, (a : b : c) \in \mathbb{P}^2_k\}$$

and hence is a 2-plane. So, by Theorem 5.6, X cannot be of arithmetic $\mathrm{depth} \leq 2 = \dim X$ and hence is arithmetically Cohen-Macaulay.

Moreover

$$h := X \setminus \mathrm{Nor}(X) = \varrho(\mathrm{Sec}_p(\tilde{X}) \setminus \{p\}) = \{(a : c : 0 : \cdots : 0) \in \mathbb{P}^r_k \,\big|\, (a : c) \in \mathbb{P}^1_k\}.$$

So, the non-normal locus h of X is a line.

Now consider the affine open set $X_{s^2u^{r-2}} \subset X$ defined by $s^2u^{r-2} \neq 0$ and let $a := \frac{s^{r-5}v^2t}{u^{r-2}}$ and $b := \frac{s^{r-4}v^2}{u^{r-2}}$. Then, an easy calculation shows that there is an isomorphism

$$\varphi : X_{s^2u^{r-2}} \xrightarrow{\simeq} Y := \mathrm{Spec}(k[a, b, \frac{a^2}{b^2}]) = \mathrm{Spec}(k[a, b, c]/(cb^2 - a^2))$$

which maps $h \cap X_{s^2u^{r-2}}$ to the singular line $a = b = 0$ of the surface Y. The pinch point $\underline{0}$ of Y can be written as $\varphi(\varrho(\ell \setminus \{p\}))$, where ℓ is the tangent line to $\tilde{X}$ at the point $(1 : 0 : \cdots : 0)$ which contains p.

The same arguments apply to the affine open set $X_{t^2u^{r-2}} \subset X$. This allows to conclude that the open neighborhood $X_{s^2u^{r-2}} \cup X_{t^2u^{r-2}}$ of the singular line h of X is isomorphic to the blow-up $\mathrm{Proj}(k[a, b][a^2T, b^2T])$ of the affine plane $\mathbb{A}^2_k = \mathrm{Spec}(k[a, b])$ with respect to the polynomials a^2 and b^2.

B) Let $r \geq 5$ and let $\tilde{X} \subset \mathbb{P}^{r+1}_k$ be the rational normal 3-scroll $S(1, 1, r-3)$, hence the variety which is defined by the 2×2-minors of the matrix

$$\left(\begin{array}{c|c|cccc} x_0 & x_2 & x_4 & x_5 & \cdots & x_r \\ x_1 & x_3 & x_5 & x_6 & \cdots & x_{r+1} \end{array}\right).$$

$\tilde{X}$ has the homogeneous coordinate ring

$$B := k[(s, t)u^{r-4}, (s, t)v^{r-4}, (s, t)^{r-4}w] \subset k[s, t, u, v, w].$$

Now, let $\varrho : \mathbb{P}^{r+1}_k \setminus \{p\} \twoheadrightarrow \mathbb{P}^r_k, (x_0 : x_1 : x_2 : \cdots : x_{r+1}) \mapsto (x_0 : x_1 - x_2 : x_3 : \cdots : x_{r+1})$ be the projection from the point $p = (0 : 1 : 1 : 0 : \cdots : 0) \in \mathbb{P}^{r+1}_k \setminus \tilde{X}$ and let $X := \varrho(\tilde{X}) \subset \mathbb{P}^r_k$. Then X is of dimension 3 and has the homogeneous coordinate ring

$$A := k[su^{r-4}, tu^{r-4} - sv^{r-4}, tv^{r-4}, (s, t)^{r-4}w] \subset B.$$

As B is a birational extension of A, the morphism $\nu = \varrho \restriction: \tilde{X} \to X$ is birational, so that $\deg X = \deg \tilde{X} = r$ and $X \subset \mathbb{P}^r_k$ has dimension 3 and is of almost minimal degree. Moreover, as $\tilde{X}$ is smooth, $\nu = \varrho \restriction: \tilde{X} \to X$ is a normalization of X and $\mathrm{Sing}(\nu) = \varrho(\mathrm{Sec}_p(\tilde{X}) \setminus \{p\})$ is the non-normal locus of X.

Similar as in example A) above we can check that the secant cone of $\tilde{X}$at p satisfies

$$\mathrm{Sec}_p(\tilde{X}) = \{(a : b : c : d : 0 : \cdots : 0) \,\big|\, (a : b : c : d) \in \mathbb{P}^3_k\}$$

and hence is a 3-plane. So, by Theorem 5.6 the variety X cannot be of arithmetic $\mathrm{depth} \leq 3 = \dim X$ and hence is arithmetically Cohen-Macaulay, that is a non-normal Del Pezzo variety of dimension 3.

Moreover

$$\varrho(\mathrm{Sec}_p(\tilde{X}) \setminus \{p\}) = \{(a : b : d : 0 : \cdots : 0) \in \mathbb{P}^r_k \,\big|\, (a : b : d) \in \mathbb{P}^2_k\}.$$

So, the non-normal locus of X is a plane, in accordance with Proposition 5.8 and Corollary 6.10.

Finally observe that X in 9.3 A) is a divisor on the variety of minimal degree $\varrho(Z) \subset \mathbb{P}^r_k$, where $Z \subset \mathbb{P}^{r+1}_k$ is the variety defined by the 2×2 minors of the matrix

$$\begin{pmatrix} x_3 & x_4 & \cdots & x_r \\ x_4 & x_5 & \cdots & x_{r+1} \end{pmatrix}.$$

In the previous example we have met arithmetically Cohen-Macaulay varieties of almost minimal degree which occur as a subvariety of codimension one on a variety of minimal degree. We now present an example of a normal Del Pezzo variety which does not have this property.

Example 9.4. Let $X \subset \mathbb{P}^9_k$ be the smooth 6-dimensional arithmetically Gorenstein variety of degree 5 defined by the 4×4 Pfaffian quadrics F_1, F_2, F_3, F_4, F_5 of the skew symmetric matrix (cf [8])

$$M = \begin{pmatrix} 0 & x_0 & x_1 & x_2 & x_3 \\ -x_0 & 0 & x_4 & x_5 & x_6 \\ -x_1 & -x_4 & 0 & x_7 & x_8 \\ -x_2 & -x_5 & -x_7 & 0 & x_9 \\ -x_3 & -x_6 & -x_8 & -x_9 & 0 \end{pmatrix}.$$

According to [8] the columns of M provide a minimal system of generators for the first syzygy module of the homogeneous vanishing ideal $I \subset S = k[x_0, x_2, \cdots, x_9]$ of X. Assume now that there is a variety $W \subset \mathbb{P}^9_k$ of minimal degree with $\dim W = 7$ and $X \subset W$. So, W is arithmetically Cohen-Macaulay and of codimension 2 and by the Theorem of Hilbert-Burch the homogeneous vanishing ideal $J \subset S$ of W is generated by the three 2×2-minors $G_1, G_2, G_3 \in S_2$ of a 2×3-matrix with linearly independent entries in S_1 (cf [10]). So, after an eventual renumbering of the generators F_i, we may assume that $G_1, G_2, G_3, F_4, F_5 \in I_2$ is a minimal system of generators of I. As J admits two independent syzygies

$$\lambda_{i1}G_1 + \lambda_{i2}G_2 + \lambda_{i3}G_3 = 0, \ \lambda_{ij} \in S_1, i = 1, 2, j = 1, 2, 3,$$

a minimal system of generators for the first syzygy module of I would be given by the matrix of the form

$$N = \begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \in S_1^{5\times 5}.$$

On the other hand there should be a k-linear transformation which converts N into M – a contradiction.

Remark 9.5. A) The variety $X \subset \mathbb{P}^9_k$ of Example 9.4 is normal and Dell Pezzo and hence not a projection of a variety $\tilde{X} \subset \mathbb{P}^{10}_k$ of minimal degree. The non-existence of the above variety $W \subset \mathbb{P}^9_k$ of minimal degree thus is in accordance with Theorem 7.3. By Remark 7.7 the projection $X \subset \mathbb{P}^4_k$ of the Veronese surface $F \subset \mathbb{P}^5_k$ is not contained in a variety $Y \subset \mathbb{P}^4_k$ of minimal degree either, according to the fact, that F is not a scroll. So Remark 7.7 and Example 9.4 illustrate that the hypotheses of Theorem 7.3 cannot be weakened.
B) The examples of this section (with the execption of the last one) are all of relatively big codimension. It turns out, that the structure of varieties of almost minimal degree and small

codimension is fairly fixed and cannot vary very much. We study these varieties more extensively in [6].

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