ASYMPTOTIC BEHAVIOUR OF COHOMOLOGY: TAMENESS, SUPPORTS AND ASSOCIATED PRIMES

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Abstract. Let $R = \oplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring, let $X = \mathrm{Proj}(R)$ and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. We present a few results about the behaviour of the finitely generated R_0-modules $H^i(X, \mathcal{F}(n))$ if i is fixed and n tends to $-\infty$. We notably consider

a) The minimal number of generators of $H^i(X, \mathcal{F}(n))$ at the "top level", thus for the highest value of i for which these modules are not vanishing for all n, provided that the base ring R_0 is local.
b) The relations between the support of the R_0-module $H^i(X, \mathcal{F}(n))$ and the fibre dimensions of $\mathcal{F}$ under the natural morphism $X \to X_0 = \mathrm{Spec}(R_0)$.
c) The Tameness Problem, hence the vanishing and non-vanishing of the modules $H^i(X, \mathcal{F}(n))$ if i is fixed and n tends to $-\infty$.
d) The behaviour of the support $\mathrm{Supp}\big(H^i(X, \mathcal{F}(n)\big)$ of $H^i(X, \mathcal{F}(n))$ if i is fixed and n tends to $-\infty$.
e) The behaviour of the set $\mathrm{Ass}\big(H^i(X, \mathcal{F}(n))\big)$ of associated primes of $H^i(X, \mathcal{F}(n))$ if i is fixed and n tends to $-\infty$.

We make use of the Serre-Grothendieck Correspondence for Cohomology and present our results in terms of graded components of local cohomology of graded modules.

1. Introduction

Let $R = \oplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring, so that R_0 is a Noetherian ring, R is a $\mathbb{N}_0$-graded R_0-algebra and $R = R_0[\ell_0, \cdots, \ell_r]$ with finitely many elements $\ell_0, \cdots, \ell_r \in R_1$. Let $X := \mathrm{Proj}(R)$ be the projective scheme induced by R.

For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ let $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ denote the n-th twist of $\mathcal{F}$. Moreover, for $i \in \mathbb{N}_0$ let $H^i(X, \mathcal{F}(n))$ denote the i-th Serre cohomology group of X with coefficients in $\mathcal{F}(n)$. Keep in mind that $H^i(X, \mathcal{F}(n))$ carries a natural structure of R_0-module. If $\mathcal{F}$ is in addition coherent, we can say (cf [21])

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1.1 a) *For all* $i \in \mathbb{N}_0$ *and all* $n \in \mathbb{Z}$ *the* R_0*-module* $H^i(X, \mathcal{F}(n))$ *is finitely generated.*

b) *For all* $i \in \mathbb{N}$ *and all* $n \gg 0$ *we have* $H^i(X, \mathcal{F}(n)) = 0$.

In view of these two statements it is natural to fix a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, an integer $i \in \mathbb{N}$ and to ask about the *asymptotic behaviour of the* R_0*-module* $H^i(X, \mathcal{F}(n))$ *if* n *tends to* $-\infty$.

The aim of this paper is to prove and to review various results concerning this asymptotic behaviour. We prefer to do this primarily in a purely algebraic context - more precisely in the language of local cohomology. To do so, let $R_+ := \oplus_{n>0} R_n$ denote the *irrelevant graded ideal* of R, let M be a finitely generated graded R-module, let $i \in \mathbb{N}_0$ and let $H^i_{R_+}(M)$ denote the i-th local cohomology module of M with support in R_+. Keep in mind that the R-modules $H^i_{R_+}(M)$ carry a natural grading (cf [13, Chapter 12]). If $n \in \mathbb{Z}$ and $i \in \mathbb{N}_0$, we use $H^i_{R_+}(M)_n$ to denote the n-th graded component of $H^i_{R_+}(M)$. Now, according to [13, 15.1.15] we can say

1.2 a) *The* R_0*-module* $H^i_{R_+}(M)_n$ *is finitely generated for all* $i \in \mathbb{N}_0$ *and all* $n \in \mathbb{Z}$.

b) *For all* $i \in \mathbb{N}_0$ *and all* $n \gg 0$ *we have* $H^i_{R_+}(M)_n = 0$.

The apparent similarity between the statements in 1.1 on the one hand and in 1.2 on the other hand is explained by the *Serre-Grothendieck correspondence* (cf [13, Chapter 20]). Namely, assume that $\mathcal{F} = \tilde{M}$ is the coherent sheaf of $\mathcal{O}_X$-modules induced by M. Then, for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$ there are exact sequences respectively isomorphisms of R_0-modules

1.3 a) $0 \to H^0_{R_+}(M)_n \to M_n \to H^0(X, \mathcal{F}(n)) \to H^1_{R_+}(M)_n \to 0;$

b) $H^i(X, \mathcal{F}(n)) \cong H^{i+1}_{R_+}(M)_n.$

So, in the present paper we study the asymptotic behaviour of the R_0-module $H^i_{R_+}(M)_n$ if n tends to $-\infty$.

In [10] and [11] (and also in [3]) we have extensively studied the behaviour of certain numerical invariants – mainly multiplicities – of the R_0-module $H^i_{R_+}(M)_n$ if n tends to $-\infty$.

In the present paper we study numerical invariants of the module $H^i_{R_+}(M)_n$ uniquely in section 2, where we prove a result on the growth of the minimal number of generators of $H^i_{R_+}(M)_n$ "at the top level" in the case where R_0 is local (cf Theorem 2.3 and Corollary 2.4). What we prove there is actually not of purely "asymptotic nature", as we get a general estimate for the growth of the above-mentioned number of generators if n is replaced by $n-1$.

In section 3, we apply this estimate in order to get a first sample of results on the supports of the R_0-modules $H^i_{R_+}(M)_n$, for those values of i which occur as dimensions of M along the fibres of the natural morphism $\operatorname{Spec}(R) \to \operatorname{Spec}(R_0)$ (cf Theorem 3.6). We apply this at the "top level" and thus get a refinement of a statement recently shown in [33] (cf Corollaries 3.7 - 3.10).

In section 4 we consider the most fundamental question related to the asymptotic behaviour of cohomology: the *Tameness Problem*, that is the problem whether for fixed $i \in \mathbb{N}_0$ either $H^i_{R_+}(M)_n \neq 0$ for all $n \ll 0$ or $H^i_{R_+}(M)_n = 0$ for all $n \ll 0$ (cf Problem 4.3). This problem is still open in general and we decided to present the state of the art in this subject by collecting a list of cases, in which tameness has been shown to hold (cf Theorems 4.5, 4.8). We also recall the relation between the above tameness question and the nature of the so called *cohomological pattern* of a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ (cf Reminder 4.9).

To this predominantly expository fourth section, we add a fairly technical consideration. Namely, in section 5 we study tameness at "almost top levels". We prove two results which show that under certain hypotheses tameness at "high levels" holds (cf Propositions 5.1, 5.3 and Theorem 5.2). Applying this to the special case where the base ring R_0 has dimension ≤ 2 we get a generalization of a result recently shown in [30] and [33], (cf Corollary 5.4, Comments and Problems 5.5).

Section 6 is devoted to the problem of *Asymptotic Stability of Supports* (cf Problem 6.1), that is to the question whether for fixed $i \in \mathbb{N}_0$ the support $\operatorname{Supp}\big(H^i_{R_+}(M)_n\big)$ of the R_0-module $H^i_{R_+}(M)_n$ becomes ultimately constant if n tends to $-\infty$. We prove that under certain assumptions on R_0 and M we have "*Asymptotic stability of Supports in Codimension* ≤ 2" (cf Theorem 6.8). We apply this to the case where $\dim(R_0) \leq 2$ (cf Corollary 6.10) and finally show that we have asymptotic stability of supports if R_0 is a domain of dimension ≤ 2 and essentially of finite type over a field and if the graded R-module M is R_0-torsion-free (cf Corollary 6.11).

In the final section 7 we discuss the problem of *Asymptotic Stability of Associated Primes* (cf Problem 7.1), that is the question whether for fixed $i \in \mathbb{N}_0$ the set of associated primes $\operatorname{Ass}\big(H^i_{R_+}(M)_n\big)$ of the R_0-module $H^i_{R_+}(M)_n$ becomes ultimately constant if n tends to $-\infty$. This problem has some relation to another, fairly prominent question of local cohomology theory, namely: the question whether a local cohomology module has finitely many associated primes (cf Remark 7.2). There are strikingly simple examples which show that both of these questions need not have an affirmative answer in general. We list some of these examples which concern the problem of asymptotic stability of associated primes (cf Examples 7.3 - 7.5). In Theorem 7.7 we also present a list of cases in which asymptotic stability of associated primes holds. Finally we prove that under certain conditions one has "*Asymptotic Stability of Associated Primes in Codimension* ≤ 2" (cf Proposition 7.9) and apply this to the case where $\dim(R_0) \leq 2$ to show that under certain conditions one has asymptotic stability of associated primes (cf Corollary 7.10).

The present paper has a strong expository component, as it includes a survey of the state of the art in the subject it treats. We feel the need for such a survey, as a synopsis on the considerable number of partial results and striking examples which have been published in the last years at different places, may help the reader to get an overall impression of the whole subject. At the end of each section we add a few comments and present open problems with the hope that this will stimulate further research. Also, with the aim of being expository we offer a proof for the description of cohomological dimensions in terms of fibre dimensions which uses only local cohomology (cf Theorem 2.3 a), Proposition 3.4 a)). As basic references we recommend [16], [19], [20], [21].

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2. Numbers of Generators at the Top Level

All known pathologies concerning the asymptotic behaviour of local cohomology have been found in top local cohomology modules. On the other hand top local cohomology modules also have some particularly nice asymptotic properties. So, in [33] it has been shown that the minimal number of generators of the n-th graded component of the top local cohomology module is antipolynomial in n, if the base ring R_0 is local. In the present section we take up this theme and prove an additional result on the mentioned minimal number of generators. Throughout we keep the notation of the introduction.

2.1. **Notation.** A) If M is a finitely generated graded R-module, we use $\operatorname{cd}(M)$ do denote the *cohomological dimension of* M with respect to R_+, thus

$$\operatorname{cd}(M) := \sup\{i \in \mathbb{N}_0 \,|\, H^i_{R_+}(M) \neq 0\},$$

with the convention that sup is taken in $\mathbb{Z} \cup \{\pm\infty\}$ and $\sup \emptyset = -\infty$.

B) If $(R_0, \mathfrak{m}_0)$ is local and T is a finitely generated R_0-module, we use $\mu(T)$ to denote the *minimal number of generators* of T, thus

$$\mu(T) = \dim_{R_0/\mathfrak{m}_0}(T/\mathfrak{m}_0 T). \qquad \bullet$$

If M is a finitely generated graded R-module, $H^{\operatorname{cd}(M)}_{R_+}(M)$ is called the *top local cohomology module of* M with respect to R_+. We now consider the top local cohomology module of M if the base ring R_0 is local with maximal ideal $\mathfrak{m}_0$. We fix an $\mathfrak{m}_0$-primary ideal $\mathfrak{q}_0$ and consider the function $\mathbb{Z} \to \mathbb{N}_0$ given by

$$n \mapsto \operatorname{length}_{R_0}\left(R_0/\mathfrak{q}_0 \otimes_{R_0} H^{\operatorname{cd}(M)}_{R_+}(M)_n\right).$$

In the main result of this section we restate a few facts on this function which have been established in [11]. We also prove that this function increases by at least $\operatorname{cd}(M) - 1$ if

n is replaced by $n-1$ and if its value is $\neq 0$ at n. To prove this, we need the following auxiliary result.

2.2**. Proposition.** *Let $(R_0, \mathfrak{m}_0)$ be local such that $k := R_0/\mathfrak{m}_0$ is an algebraically closed field. Let P and Q be two R_0-modules such that $Q \neq 0$. Let $r \in \mathbb{N}_0$ and let $\ell_0, \cdots, \ell_r : P \to Q$ be homomorphisms of R_0-modules. Assume that there is a set $\mathcal{W} \subseteq R_0^{r+1}$ which is mapped onto $k^{r+1} \backslash \{\underline{0}\}$ under the canonical map $R_0^{r+1} \to k^{r+1}$ and such that the map $\sum_{i=0}^{r} \beta_i \ell_i : P \to Q$ is surjective for all $(\beta_0, \cdots, \beta_r) \in \mathcal{W}$. Then*

$$\mathrm{length}_{R_0}(P) \geq \mathrm{length}_{R_0}(Q) + r \cdot s,$$

where

$$s := \min\{v \in \mathbb{N} \mid \mathfrak{m}_0^v Q = 0\}.$$

Proof: If $\mathrm{length}_{R_0}(P) = \infty$, our claim is obvious. So, let $\mathrm{length}_{R_0}(P) < \infty$. Then, there is some $t \in \mathbb{N}$ such that $\mathfrak{m}_0^t P = 0$. As there is an epimorphism $P \twoheadrightarrow Q$, we have $t \geq s$. Let $\overline{R}_0 := R_0/\mathfrak{m}_0^t$ and $\overline{\mathfrak{m}}_0 := \mathfrak{m}_0/\mathfrak{m}_0^t$. Then $(\overline{R}_0, \overline{\mathfrak{m}}_0)$ is a local Artinian ring and $\overline{R}_0/\overline{\mathfrak{m}}_0 := \overline{k} \cong k$ is algebraically closed. Moreover P and Q carry a natural structure of $\overline{R}_0$-module. Finally, let $U \subseteq \overline{R}_0$ be the image of $\mathcal{W}$ under the canonical map $R_0^{r+1} \to \overline{R}_0^{r+1}$. Then U is mapped onto $\overline{k}^{r+1} \backslash \{0\}$ under the canonical map $\overline{R}_0^{r+1} \to \overline{k}^{r+1}$ and the map $\sum_{i=0}^{r} \alpha_i \ell_i : P \to Q$ is surjective for all $(\alpha_0, \cdots, \alpha_r) \in U$. So, by [8, Proposition 3.9] we obtain

$$\mathrm{length}_{\overline{R}_0}(P) \geq \mathrm{length}_{\overline{R}_0}(Q) + r \cdot \overline{s},$$

where $\overline{s} := \min\{w \in \mathbb{N} | \overline{\mathfrak{m}}_0^w Q = 0\}$. This proves our claim. ■

Now, we are ready to prove the main result of this section.

2.3**. Theorem.** *Assume that $(R_0, \mathfrak{m}_0)$ is local. Let $M \neq 0$ be a finitely generated graded R-module, let $\mathfrak{q}_0 \subseteq R_0$ be an $\mathfrak{m}_0$-primary ideal and let $d := \dim(M/\mathfrak{m}_0 M)$. Then:*

a) $\mathrm{cd}(M) = d \geq 0$.

b) *The graded R-module $H^d_{R_+}(M)/\mathfrak{q}_0 H^d_{R_+}(M)$ is Artinian.*

c) *There is a polynomial $Q \in \mathbb{Q}[x]$ of degree $\delta < d$ such that*

$$\mathrm{length}_{R_0}(R_0/\mathfrak{q}_0 \otimes_{R_0} H^d_{R_+}(M)_n) = Q(n) \text{ for all } n \ll 0;$$

moreover δ is independent of $\mathfrak{q}_0$.

d) *If $d > 0$ and $H^d_{R_+}(M)_n \neq 0$ for some $n \in \mathbb{Z}$, then*

$$\mathrm{length}_{R_0}\left(R_0/\mathfrak{q}_0 \otimes_{R_0} H^d_{R_+}(M)_{n-1}\right) \geq \mathrm{length}_{R_0}\left(R_0/\mathfrak{q}_0 \otimes_{R_0} H^d_{R_+}(M)_n\right) + (d-1)s(n),$$

where $s(n) := \min\{v \in \mathbb{N} \mid \mathfrak{m}_0^v H^d_{R_+}(M)_n \subseteq \mathfrak{q}_0 H^d_{R_+}(M)_n\}$.

Proof: "a)": This follows by [4, Lemma 3.4] and the well-known relations between ideal transforms and local cohomology (cf [13, Theorem 2.2.4]). For the reader's convenience we also offer a direct proof of this frequently used statement. We proceed by induction on d.

If $d = 0$ we have $M_n/\mathfrak{m}_0 M_n = (M/\mathfrak{m}_0 M)_n = 0$ for all $n \gg 0$. So, by Nakayama $M_n = 0$ for all $n \gg 0$. Hence M is R_+-torsion so that $H^0_{R_+}(M) \cong M \neq 0$ and $H^i_{R_+}(M) = 0$ for all $i > 0$. This proves the case $d = 0$.

So, let $d > 0$. Let x be an indeterminate and let $R'_0 := R_0[x]_{\mathfrak{m}_0 R_0[x]}$. Then R'_0 is a local Noetherian flat extension ring of R_0 with maximal ideal $\mathfrak{m}'_0 = \mathfrak{m}_0 R'_0$ and residue field $R'_0/\mathfrak{m}'_0 = (R_0/\mathfrak{m}_0)(x)$. Consider the homogeneous Noetherian ring $R' := R'_0 \otimes_{R_0} R$ and the finitely generated graded R'-module $M' = R'_0 \otimes_{R_0} M$. Then, the natural isomorphism of R'-modules $M'/\mathfrak{m}'_0 M' \cong R'_0/\mathfrak{m}'_0 \otimes_{R_0} M/\mathfrak{m}_0 M$ yields $\dim(M'/\mathfrak{m}'_0 M') = \dim(M/\mathfrak{m}_0 M) = d$.

Moreover, by the graded flat base change property of local cohomology we have natural isomorphisms of R'-modules $H^i_{R'_+}(M') \cong R'_0 \otimes_{R_0} H^i_{R_+}(M)$ for all $i \in \mathbb{N}_0$ and these show that $\mathrm{cd}(M') = \mathrm{cd}(M)$. This allows us to replace R and M respectively by R' and M' and hence to assume that $k := R_0/\mathfrak{m}_0$ is infinite.

Now, let $\overline{M} := M/\Gamma_{R_+}(M)$. Then, there is a short exact sequence of graded R-modules

$$0 \to \left(\Gamma_{R_+}(M) + \mathfrak{m}_0 M\right)/\mathfrak{m}_0 M \to M/\mathfrak{m}_0 M \to \overline{M}/\mathfrak{m}_0 \overline{M} \to 0.$$

The R-module $\left(\Gamma_{R_+}(M) + \mathfrak{m}_0 M\right)/\mathfrak{m}_0 M$ is concentrated in finitely many degrees and annihilated by $\mathfrak{m}_0$, and hence is of dimension ≤ 0. Therefore $\dim(\overline{M}/\mathfrak{m}_0\overline{M}) = \dim(M/\mathfrak{m}_0 M) = d$.

As $d > 0$, M is not concentrated in finitely many degrees so that $\overline{M} \neq 0$ and hence $\mathrm{cd}(\overline{M}) \geq 0$. As $H^0_{R_+}(\overline{M}) = 0$ and in view of the natural isomorphisms $H^i_{R_+}(M) \cong H^i_{R_+}(\overline{M})$ for all $i > 0$ we thus get $\mathrm{cd}(\overline{M}) = \mathrm{cd}(M)$.

So, we may replace M by $\overline{M}$ and hence assume that $\Gamma_{R_+}(M) = 0$, so that $R_1 \not\subseteq \mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Ass}_R(M)$. As $d > 0$ we also have $R_1 \not\subseteq \mathfrak{s}$ for each $\mathfrak{s} \in \mathrm{Min}_R(M/\mathfrak{m}_0 M)$. As k is infinite, we thus find some $x \in R_1$ which avoids all members of $\mathrm{Ass}_R(M)$ and $\mathrm{Min}_R(M/\mathfrak{m}_0 M)$. In particular we have $x \in \mathrm{NZD}_R(M)$ and $\dim\left(M/\mathfrak{m}_0 M)/x(M/\mathfrak{m}_0 M)\right) = d - 1$.

So, we end up with a short exact sequence of graded R-modules

$$0 \to M(-1) \xrightarrow{x} M \to M/xM \to 0$$

in which (by induction) $\mathrm{cd}(M/xM) = d - 1$. Now, let $i > d$. Then $H^{i-1}_{R_+}(M/xM) = 0$ and the above sequence yields a monomorphism $x : H^i_{R_+}(M)(-1) \rightarrowtail H^i_{R_+}(M)$. As $x \in R_+$ it follows $H^i_{R_+}(M) = 0$. Therefore $\mathrm{cd}(M) \leq d$.

It remains to show that $H^d_{R_+}(M) \neq 0$. To do so, we apply cohomology to the graded short exact sequence $0 \to \mathfrak{m}_0 M \to M \to M/\mathfrak{m}_0 M \to 0$ and get an exact sequence of graded R-modules

$$H^d_{R_+}(M) \to H^d_{R_+}(M/\mathfrak{m}_0 M) \to H^{d+1}_{R_+}(\mathfrak{m}_0 M).$$

As $\dim(\mathfrak{m}_0 M/\mathfrak{m}_0(\mathfrak{m}_0 M)) = \dim(\mathfrak{m}_0 M/\mathfrak{m}_0^2 M) \leq \dim(M/\mathfrak{m}_0^2 M) = \dim(M/\mathfrak{m}_0 M) = d$ we may apply what we have shown above to the R-module $\mathfrak{m}_0 M$ and get $H^{d+1}_{R_+}(\mathfrak{m}_0 M) = 0$. It thus remains to show that $H^d_{R_+}(M/\mathfrak{m}_0 M) \neq 0$.

Let $\mathfrak{m} := \mathfrak{m}_0 + R_+$ be the homogeneous maximal ideal of R. By the base ring independence property of local cohomology we have an isomorphism of R-modules $H^d_{R_+}(M/\mathfrak{m}_0 M)$ $\cong H^d_{\mathfrak{m}}(M/\mathfrak{m}_0 M)$. It hence suffices to show that $H^d_{\mathfrak{m}}(M/\mathfrak{m}_0 M) \neq 0$. As $M/\mathfrak{m}_0 M$ is a graded R-module, $\dim((M/\mathfrak{m}_0 M)_{\mathfrak{m}}) = \dim(M/\mathfrak{m}_0 M) = d$ so that $H^d_{\mathfrak{m}R_{\mathfrak{m}}}((M/\mathfrak{m}_0 M)_{\mathfrak{m}}) \neq 0$ (cf [13, Theorem 6.1.4]). It follows $H^d_{\mathfrak{m}}(M/\mathfrak{m}_0 M) \neq 0$.

"b)": See [11, Theorem 2.10], or [33, Theorem 2.1] for the special case where $\mathfrak{q}_0 = \mathfrak{m}_0$.

"c)": For the existence of Q see [11, Theorem 2.10] or [33, Corollary 2.4]. According to [11, Theorem 2.10], the degree of Q is independent of $\mathfrak{q}_0$ and $< d$.

"d)": There is a Noetherian flat local extension ring $(R'_0, \mathfrak{m}'_0)$ such that $\mathfrak{m}'_0 = \mathfrak{m}_0 R'_0$ and such that $R'_0/\mathfrak{m}'_0$ is an algebraically closed field (cf [8, Proposition 2.2]). Consider the $\mathfrak{m}'_0$-primary ideal $\mathfrak{q}'_0 = \mathfrak{q}_0 R'_0 \subseteq R'_0$, the homogeneous Noetherian ring $R' := R'_0 \otimes_{R_0} R$ and the finitely generated graded R'-module $M' := R'_0 \otimes_{R_0} M$. Then, for each $i \in \mathbb{N}_0$ and each $n \in \mathbb{Z}$ the graded flat base change property of local cohomology yields an isomorphism of R'_0-modules

$$H^i_{R_+}(M')_n \cong R'_0 \otimes_{R_0} H^i_{R_+}(M)_n.$$

This first shows that $\mathrm{cd}(M') = d$. Choosing $i = d$ and keeping in mind that $\mathfrak{m}'_0 = \mathfrak{m}_0 R'_0$ and that R'_0 is R_0-flat, we thus get

$$\mathrm{length}_{R'_0}\left(R'_0/\mathfrak{q}'_0 \otimes_{R'_0} H^d_{R'_+}(M')_n\right) = \mathrm{length}_{R_0}\left(R_0/\mathfrak{q}_0 \otimes_{R_0} H^d_{R_+}(M)_n\right)$$

for all $n \in \mathbb{Z}$. This allows us to replace $\mathfrak{q}_0, R$ and M respectively by $\mathfrak{q}'_0, R'$ and M'. So, we may assume that $k := R_0/\mathfrak{m}_0$ is algebraically closed.

Now, let

$$\mathcal{P} := \{\mathfrak{p} \in \mathrm{Ass}_R(M) \,\big|\, \dim(R/(\mathfrak{m}_0 R + \mathfrak{p})) < d\}$$

and let

$$\mathfrak{a} := \bigcap_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}.$$

Then $\mathfrak{a} \subseteq R$ is a graded ideal (which equals R if $\mathcal{P} = \emptyset$) and the $\mathfrak{a}$-torsion submodule $\Gamma_{\mathfrak{a}}(M)$ of M is graded and satisfies

$$\mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M)) = \mathcal{P} \text{ and } \mathrm{Ass}(M/\Gamma_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(M)\backslash\mathcal{P}.$$

This first shows that $\dim(\Gamma_{\mathfrak{a}}(M)/\mathfrak{m}_0\Gamma_{\mathfrak{a}}(M)) < d$ and hence, by statement a), that $\mathrm{cd}(\Gamma_{\mathfrak{a}}(M)) < d$. By the exact sequence $0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M/\Gamma_{\mathfrak{a}}(M) \to 0$ we thus get isomorphisms of R_0-modules

$$H^d_{R_+}(M/\Gamma_{\mathfrak{a}}(M))_n \cong H^d_{R_+}(M)_n$$

for all $n \in \mathbb{Z}$. These show in particular, that $\mathrm{cd}(M/\Gamma_{\mathfrak{a}}(M)) \geq d$. On on use of statement a) we also get

$$\mathrm{cd}(M/\Gamma_{\mathfrak{a}}(M)) = \dim((M/\Gamma_{\mathfrak{a}}(M))/\mathfrak{m}_0(M/\Gamma_{\mathfrak{a}}(M)) = \dim\left(M/(\mathfrak{m}_0 M + \Gamma_{\mathfrak{a}}(M))\right) \leq d$$

and hence $\mathrm{cd}(M/\Gamma_{\mathfrak{a}}(M)) = d$. But now, the above isomorphisms allow us to replace M by $M/\Gamma_{\mathfrak{a}}(M)$ and hence to assume that

$$\dim(R/(\mathfrak{m}_0 R + \mathfrak{p})) \geq d \text{ for all } \mathfrak{p} \in \mathrm{Ass}_R(M).$$

Now, consider the non-empty finite set of graded primes

$$Q := \{\mathfrak{p} \in \mathrm{Min}_R(M/\mathfrak{m}_0 M) \mid \dim(R/\mathfrak{p}) = d\}.$$

Then for all $\mathfrak{p} \in \mathrm{Ass}_R(M) \cup Q$ we have $\dim(R/(\mathfrak{m}_0 R + \mathfrak{p})) \geq d$. But this means that the first graded component $(R/(\mathfrak{m}_0 R + \mathfrak{p}))_1 \cong R_1/(\mathfrak{m}_0 R_1 + \mathfrak{p}_1)$ of $R/(\mathfrak{m}_0 R + \mathfrak{p})$ is a k-vector space of dimension $\geq d$, hence that the k-vector subspace $(\mathfrak{m}_0 R_1 + \mathfrak{p}_1)/\mathfrak{m}_0 R_1 \subseteq R_1/\mathfrak{m}_0 R_1$ is of codimension $\geq d$ for all such $\mathfrak{p}$. As $\mathrm{Ass}_R(M) \cup Q$ is finite and as k is infinite, we thus find a k-vector space $V \subseteq R_1/\mathfrak{m}_0 R_1$ of dimension d such that

$$(*) \qquad V \cap ((\mathfrak{m}_0 R_1 + \mathfrak{p}_1)/\mathfrak{m}_0 R_1) = 0 \text{ for all } \mathfrak{p} \in \mathrm{Ass}_R(M) \cup Q.$$

Now, let $h_1, \cdots, h_d \in R_1$ be such that the classes $h_i + \mathfrak{m}_0 R_1 \in R_1/\mathfrak{m}_0 R_1$ $(i = 1, \cdots, d)$ form a k-basis of V. Then, for each

$$(\beta_1, \cdots, \beta_d) \in R_0^d \backslash \mathfrak{m}_0^d =: \mathcal{W}$$

we have $\left(\sum_{i=1}^d \beta_i h_i\right) + \mathfrak{m}_0 R_1 \in V\backslash\{0\}$. In view of $(*)$ this means that $\sum_{i=1}^d \beta_i h_i$ avoids all $\mathfrak{p} \in \mathrm{Ass}_R(M) \cup Q$ for all $(\beta_1, \cdots, \beta_d) \in \mathcal{W}$. Hence, for all such $(\beta_1, \cdots, \beta_d)$ we have

$$\sum_{i=1}^d \beta_i h_i \in \mathrm{NZD}_R(M) \text{ and } \dim\left((M/\sum_{i=1}^d \beta_i h_i M) \Big/ \mathfrak{m}_0(M/\sum_{i=1}^d \beta_i h_i M)\right) < d.$$

So, for all $(\beta_1, \cdots, \beta_d) \in \mathcal{W}$ there is an exact sequence

$$0 \longrightarrow M(-1) \xrightarrow{\sum_{i=1}^d \beta_i h_i} M \longrightarrow M/\sum_{i=0}^d \beta_i h_i M \longrightarrow 0$$

and we have $\mathrm{cd}(M/\sum_{i=1}^d \beta_i h_i M) < d$, (cf statement a)). Hence, for all $(\beta_1, \cdots, \beta_d) \in \mathcal{W}$ we get an epimorphism

$$H^d_{R_+}(M)_{n-1} \xrightarrow{\sum_{i=1}^d \beta_i h_i} H^d_{R_+}(M)_n \longrightarrow 0.$$

Now, let

$$\ell_i = 1 \otimes_{R_0} h_i : R_0/\mathfrak{q}_0 \otimes_{R_0} H^d_{R_+}(M)_{n-1} \to R_0/\mathfrak{q}_0 \otimes_{R_0} H^d_{R_+}(M)_n$$

denote the homomorphism of $R_0/\mathfrak{q}_0$-modules induced by the multiplication map $h_i : H^d_{R_+}(M)_{n-1} \to H^d_{R_+}(M)_n$. Then for each $(\beta_1, \cdots, \beta_d) \in \mathcal{W}$ we get an epimorphism of R_0-modules

$$\sum_{i=1}^{d} \beta_i \ell_i : R_0/\mathfrak{q}_0 \otimes_{R_0} H^d_{R_+}(M)_{n-1} \twoheadrightarrow R_0/\mathfrak{q}_0 \otimes_{R_0} H^d_{R_+}(M)_n.$$

By Nakayama, the right hand side module does not vanish. Now Proposition 2.2 gives our claim. ■

As an application, we now may draw the following conclusion on the minimal number of generators of the graded components of the top local cohomology module.

2.4. **Corollary.** *Assume that $(R_0, \mathfrak{m}_0)$ is local and let M be a finitely generated graded R-module such that $d := \mathrm{cd}(M) > 0$. Then:*

a) *(Cf* [33, Corollary 2.4]*) There is a polynomial $\overline{Q} \in \mathbb{Q}[x]$ of degree $< d$ such that $\mu\left(H^d_{R_+}(M)_n\right) = \overline{Q}(n)$ for all $n \ll 0$.*

b) *If $\mu\left(H^d_{R_+}(M)_n\right) \neq 0$ for some $n \in \mathbb{Z}$, then*

$$\mu\left(H^d_{R_+}(M)_{n-1}\right) \geq \mu\left(H^d_{R_+}(M)_n\right) + d - 1.$$

Proof: Apply Theorem 2.3 c), d) with $\mathfrak{q}_0 = \mathfrak{m}_0$. ■

2.5. **Comments and Problems.** A) Keep the above hypotheses and notations. It follows immediately by Corollary 2.4:

a) *If $d = 1$, then $\overline{Q}$ is a non-zero constant.*

b) *If $d = 2$, then $\overline{Q} = ex + c$ with $e \in \mathbb{N}$ and $c \in \mathbb{Z}$.*

B) Assume that $\dim(R_0) = 0$. Then $d = \dim(M)$ and there is a polynomial $P \in \mathbb{Q}[x]$ of degree $d - 1$ such that $\mathrm{length}_{R_0}\left(H^d_{R_+}(M)_n\right) = P(n)$ for all $n \ll 0$, (cf [13, Exercise 17.1.10]). As

$$\mu\left(H^d_{R_+}(M)_n\right) \leq \mathrm{length}_{R_0}\left(H^d_{R_+}(M)_n\right) \leq \mathrm{length}(R_0)\mu\left(H^d_{R_+}(M)_n\right)$$

for all $n \in \mathbb{Z}$ it follows $\overline{Q}(n) \leq P(n) \leq \mathrm{length}(R_0)\overline{Q}(n)$ for all $n \ll 0$. In particular we have $\deg(\overline{Q}) = d - 1$.

C) In the cases considered in parts A) and B) we have $\deg(\overline{Q}) = d - 1$. This raises the question, whether in general we have $\deg(\overline{Q}) = \mathrm{cd}(M) - 1$.

D) We do not know of an example such that for an arbitrary $i \in \mathbb{N}_0$ the function given by $n \mapsto \mu\left(H^i_{R_+}(M)_n\right)$ is not antipolynomial, that is, not presented by a polynomial for $n \ll 0$. If $\dim(R_0) \leq 1$ this function is indeed antipolynomial (cf [3, Theorem (3.5) a)]). This is an immediate consequence of the fact that the graded R-module

$R_0/\mathfrak{q}_0 \otimes_{R_0} H^i_{R_+}(M)$ is Artinian whenever $\dim(R_0) \leq 1$ and $\mathfrak{q}_0 \subseteq R_0$ is an $\mathfrak{m}_0$-primary ideal (cf [3, Corollary 2.6 a)]).

On the other hand this latter statement fails in general, namely: (cf [3, Example 4.1]) if k is a field, x, y, t are indeterminates, $R_0 := k[x,y]_{(x,y)}, \mathfrak{m}_0 := (x,y)R_0$ and if $R := R_0[\mathfrak{m}_0 t]$ is the Rees ring of $\mathfrak{m}_0$, then the R-module $R_0/\mathfrak{m}_0 \otimes_{R_0} H^1_{R_+}(R)$ is not Artinian. So, the main ingredient of all known proofs of the antipolynomiality of the function $n \mapsto \mu\left(H^i_{R_+}(M)_n\right)$ is lacking in general: the fact that the R-module $R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M)_n$ is Artinian. •

3. Conclusions on the Supports

In [27] and [33] the supports of top local cohomology modules have been studied extensively. In this section we take up this theme. We keep the previous notation.

If $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)$, we write $\kappa(\mathfrak{p}_0)$ for the function field $R_{0\mathfrak{p}_0}/\mathfrak{p}_0 R_{0\mathfrak{p}_0}$ of $\mathrm{Spec}(R_0)$ at $\mathfrak{p}_0$.

3.1. **Proposition.** *Let $M \neq 0$ be a finitely generated graded R-module and let $\mathfrak{p}_0 \in$ Spec(R_0) such that $i := \dim_R(\kappa(\mathfrak{p}_0) \otimes_{R_0} M) > 0$. Then, there is an integer n_0 such that $\mathfrak{p}_0 \in$ Supp$\left(H^i_{R_+}(M)_n\right)$ if and only if $n \leq n_0$.*

Proof: By the graded flat base change theorem for local cohomology there are isomorphisms of $(R_0)_{\mathfrak{p}_0}$-modules

$$H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n \cong \left(H^i_{R_+}(M)_n\right)_{\mathfrak{p}_0}$$

which allow to replace R and M respectively by $R_{\mathfrak{p}_0}$ and $M_{\mathfrak{p}_0}$, and hence to assume that R_0 is local with maximal ideal $\mathfrak{p}_0$. As $H^i_{R_+}(M)_n = 0$ for all $n \gg 0$ and as $i = \dim(M/\mathfrak{p}_0 M) = \mathrm{cd}(M)$ (cf Theorem 2.3 a))) we get our claim by Corollary 2.4 b). ■

3.2. **Definition.** A) Let M be a finitely generated graded R-module and let $i \in \mathbb{Z}$. We set

$$U^i(M) := \{\mathfrak{p}_0 \in \mathrm{Spec}(R_0) \mid \dim_R(\kappa(\mathfrak{p}_0) \otimes_{R_0} M) \leq i\}.$$

B) Let M and i be as in part A). We define the i-th *fibre skeleton of* M as the set

$$\begin{aligned} F^i(M) :=& \{\mathfrak{p}_0 \in \mathrm{Spec}(R_0) \mid \dim_R(\kappa(\mathfrak{p}_0) \otimes_{R_0} M) = i\} \\ =& U^i(M) \setminus U^{i-1}(M). \end{aligned}$$

•

3.3. **Proposition.** *Let $M \neq 0$ be a finitely generated graded R_0-module and let $i \in \mathbb{Z}$. Then:*

a) *(Semicontinuity of fibre dimensions) The set $U^i(M)$ is open in* $\operatorname{Spec}(R_0)$.

b) *The sets $U^i(M)$ and $F^i(M)$ depend only on* $\operatorname{Supp}_R(M)$.

Proof: "a)": See [33, Proposition 3.4] for example.

"b)": Let $\mathfrak{a} := 0 \underset{R}{:} M$. It suffices to show that $U^i(M)$ and $F^i(M)$ are determined by $\sqrt{\mathfrak{a}}$. Let $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)$. Then

$$\dim_R\left(\kappa(R_0/\mathfrak{p}_0) \otimes_{R_0} M\right) = \dim_R(M_{\mathfrak{p}_0}/\mathfrak{p}_0 M_{\mathfrak{p}_0}) =$$
$$= \dim\left(R_{\mathfrak{p}_0} \Big/ \sqrt{0 \underset{R_{\mathfrak{p}_0}}{:} M_{\mathfrak{p}_0}/\mathfrak{p}_0 M_{\mathfrak{p}_0}}\right) = \dim\left(R_{\mathfrak{p}_0} \Big/ \left(\sqrt{0 \underset{R}{:} M/\mathfrak{p}_0 M}\right)_{\mathfrak{p}_0}\right).$$

As $\sqrt{0 : M/\mathfrak{p}_0 M} = \sqrt{(0 \underset{R}{:} M) + \mathfrak{p}_0 R} = \sqrt{\sqrt{\mathfrak{a} + \mathfrak{p}_0 R}}$, we get our claim. ∎

3.4. **Proposition.** *Let $M \neq 0$ be a finitely generated graded R-module. Then:*

a) $\operatorname{cd}(M) = \sup\{i \in \mathbb{Z} \mid F^i(M) \neq \emptyset\} = \sup\{\dim(M/\mathfrak{m}_0 M) | \mathfrak{m}_0 \in \operatorname{Max}(R_0)\}$.

b) *(Cf* [33, 1.2]*)* $\bigcup_{n \in \mathbb{Z}} \operatorname{Supp}\left(H^{\operatorname{cd}(M)}_{R_+}(M)_n\right) = F^{\operatorname{cd}(M)}(M)$.

Proof: "a)": In view of Proposition 3.3 a) it suffices to prove the first equality. We have to show that

$$\operatorname{cd}(M) = \sup\{\dim_R\left(\kappa(\mathfrak{p}_0) \otimes M\right) \mid \mathfrak{p}_0 \in \operatorname{Spec}(R_0)\}.$$

But this is immediate by Theorem 2.3 a) and the graded flat base change property of local cohomology.

"b)": This is again clear by Theorem 2.3 a) and the graded flat base change property of local cohomology. ∎

3.5. **Corollary.** *Let $M \neq 0$ be a finitely generated graded R-module and let N be a graded subquotient of M. Then* $\operatorname{cd}(N) \leq \operatorname{cd}(M)$.

Proof: According to Proposition 3.4 a) and Proposition 3.3 b) the invariants $\operatorname{cd}(N)$ and $\operatorname{cd}(M)$ are determined by the sets $\operatorname{Supp}(N)$ resp. $\operatorname{Supp}(M)$. As $\operatorname{Supp}(N) \subseteq \operatorname{Supp}(M)$ our claim follows easily. ∎

Now we shall prove the main result of the present section.

3.6. **Theorem.** *Let $M \neq 0$ be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Then:*

a) *For each $n \in \mathbb{Z}$ we have*

$$\operatorname{Supp}\left(H^i_{R_+}(M)_n\right) \cap F^i(M) \subseteq \operatorname{Supp}\left(H^i_{R_+}(M)_{n-1}\right) \cap F^i(M).$$

b) *There is an integer n_0 such that*

$$F^i(M) \subseteq \operatorname{Supp}\left(H^i_{R_+}(M)_n\right) \text{ for all } n \leq n_0.$$

Proof: "a)": This follows immediately from Proposition 3.1.

"b)": For each $n \in \mathbb{Z}$, the set $W_n := \operatorname{Supp}\left(H^i_{R_+}(M)_n\right) \cap F^i(M)$ is a relatively closed subset of $F^i(M)$. According to Proposition 3.3 a) the set $U^i(M)$ is open in $\operatorname{Spec}(R_0)$, whereas $\operatorname{Spec}(R_0) \backslash U^{i-1}(M)$ is of the form $\operatorname{Var}(\mathfrak{a}_0)$ with some ideal $\mathfrak{a}_0 \subseteq R_0$. If $\mathfrak{q}_1, \cdots, \mathfrak{q}_r$ are the different minimal primes of $\mathfrak{a}_0$ which belong to $U^i(M)$ and if $\overline{\bullet}$ denotes the formation of topological closures in $U^i(M)$, we thus have

$$F^i(M) = U^i(M) \backslash U^{i-1}(M) = \overline{\{\mathfrak{q}_1\}} \cup \cdots \cup \overline{\{\mathfrak{q}_r\}}.$$

By statement a) we have $W_n \subseteq W_{n-1}$ for all $n \in \mathbb{Z}$ and by Proposition 3.1 we have $\cup_{n \in \mathbb{Z}} W_n = F^i(M)$. So, there is an integer n_0 such that $\mathfrak{q}_1, \cdots, \mathfrak{q}_r \in W_{n_0}$. By the closedness of W_{n_0} in $F^i(M)$ it follows $W_{n_0} = F^i(M)$. ■

If we apply the previous result to the top local cohomology module we get the following refinement of [33, Theorem 1]:

3.7. **Corollary.** *Let M be a finitely generated graded R-module such that $c = \operatorname{cd}(M) > 0$. Then:*

a) *For each $n \in \mathbb{Z}$ we have*

$$\operatorname{Supp}\left(H^c_{R_+}(M)_n\right) \subseteq \operatorname{Supp}\left(H^c_{R_+}(M)_{n-1}\right).$$

b) *There is an integer $n_0 \in \mathbb{Z}$ such that*

$$\operatorname{Supp}\left(H^c_{R_+}(M)_n\right) = F^c(M) \text{ for all } n \leq n_0.$$

Proof: This is immediate by Proposition 3.4 b) and Theorem 3.6. ■

Now, we get back easily the finiteness statement shown in [33, Theorem 1]:

3.8. **Lemma.** *Let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Then*

$$\operatorname{Ass}_R\left(H^i_{R_+}(M)\right) = \{\mathfrak{p}_0 + R_+ \mid \mathfrak{p}_0 \in \bigcup_{n \in \mathbb{Z}} \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\}.$$

Proof: See [4, Remark 5.5 (A)]. ■

3.9. **Corollary.** *(Cf* [33, Theorem 1(1)]*) Let M be a finitely generated graded R-module. Then $H^{\operatorname{cd}(M)}_{R_+}(M)$ has only finitely many minimal associated primes.*

Proof: This is immediate from Corollary 3.7 and Lemma 3.8. ■

Let us notice, that on the top level we have the following *rigidity result* for the supports:

3.10**. Corollary.** *Let $M \neq 0$ be a finitely generated graded R-module. Then:*

a) *For all sufficiently small values of n, the set* $\operatorname{Supp}\left(H^{\operatorname{cd}(M)}_{R_+}(M)_n\right)$ *depends only on* $\operatorname{Supp}_R(M)$.

b) *The (finite) set of minimal primes of* $H^{\operatorname{cd}(M)}_{R_+}(M)$ *depends only on* $\operatorname{Supp}_R(M)$.

Proof: This is immediate by Proposition 3.3 b), Corollary 3.7 and Lemma 3.8. ■

The results of this section apply for those $i \in \mathbb{N}_0$ for which $F^i(M) \neq \emptyset$. As for the set of these values of i we have

3.11**. Proposition.** *Let $M \neq 0$ be a finitely generated graded R-module. Set*

$$\Phi(M) := \{i \in \mathbb{N}_0 \mid F^i(M) \neq \emptyset\} \text{ and}$$
$$\Psi(M) := \{\operatorname{height}\big(\mathfrak{p}/(0 \underset{R}{:} M)\big) \mid \mathfrak{p} \in \min\big((0 \underset{R}{:} M) + R_+\big)\}.$$

Then $\Psi(M) = \{\operatorname{height}\big((\mathfrak{p}_0 + R_+)/(0 \underset{R}{:} M)\big) \mid \mathfrak{p}_0 \in \min(0 \underset{R_0}{:} M)\}$ *and*

a) $\Psi(M) \subseteq \Phi(M)$.

b) $\min \Phi(M) = \min \Psi(M)$.

c) $\max \Phi(M) = \operatorname{cd}(M)$.

Proof: "a)": Let $\mathfrak{p}_0 \in \min(0 \underset{R_0}{:} M)$. Then $\mathfrak{p}_0 + R_+ \in \min\left((0 \underset{R}{:} M) + R_+\right)$. Moreover, each $\mathfrak{q} \in \operatorname{Var}(0 \underset{R}{:} M)$ with $\mathfrak{q} \subseteq \mathfrak{p}_0 + R_+$ satisfies $\mathfrak{q} \cap R_0 = \mathfrak{p}_0$. Therefore $\operatorname{height}\left((\mathfrak{p}_0 + R_+)/(0 \underset{R}{:} M)\right) = \dim_R(\kappa(\mathfrak{p}_0) \otimes_{R_0} M) \in \Phi(M)$.

"b)": In view of statement a) it suffices to show that $n := \min \Phi(M) \in \Psi(M)$. So, let $\mathfrak{q}_0 \in F^n(M)$ and let $\mathfrak{p}_0 \in \min(0 \underset{R_0}{:} M)$ with $\mathfrak{p}_0 \subseteq \mathfrak{q}_0$. As $\mathfrak{q}_0 \in U^n(M)$, Proposition 3.3 a) yields $\mathfrak{p}_0 \in U^n(M)$ and the last equality in the proof of part a) implies $\operatorname{height}\left((\mathfrak{p}_0 + R_+)/(0 \underset{R}{:} M)\right) = \dim_R(\kappa(\mathfrak{p}_0) \otimes_{R_0} M) \leq n$. By the minimality of n we must have equality on the right hand side, and this proves our claim.

"c)": This is clear from Proposition 3.4 a). ■

3.12. **Comments and Problems.** A) Let $i \in \mathbb{N}_0$. According to Theorem 3.6 b) we have

$$\overline{F^i(M)} \subseteq \operatorname{Supp}\left(H^i_{R_+}(M)_n\right) \text{ for all } n \ll 0.$$

As may be seen in the case $\dim(R_0) = 0$ it may occur that $F^i(M) = \emptyset$ whereas $H^i_{R_+}(M)_n \neq 0$ for all $n \ll 0$. So, Theorem 3.6 is far away from furnishing precise information on the sets $\operatorname{Supp}\left(H^i_{R_+}(M)_n\right)$ in general.

B) It seems rather natural to expect that there must be a closer relation between the sets $F^i(R)$ and $\operatorname{Supp}\left(H^i_{R_+}(R)_n\right)$ than stated in Theorem 3.6 b) in the case where $R = R_0(\mathfrak{a}_0) = R_0[\mathfrak{a}_0 t]$ is the Rees algebra of an ideal $\mathfrak{a}_0 \subseteq R_0$. •

4. Tameness

In this section we discuss the most fundamental concept related to the asymptotic behaviour of cohomology: the concept of tameness. We keep the previous notation and hypotheses.

4.1. **Definition.** Let $T = \oplus_{n\in\mathbb{Z}} T_n$ be a graded R-module. We say that T is *tame* or *asymptotically gap free* (cf [4, 4.1]) if

$$\text{either } T_n \neq 0 \text{ for all } n \ll 0 \text{ or else } T_n = 0 \text{ for all } n \ll 0.$$

•

4.2. **Remark.** It is easy to see that a graded Artinian R-module $T = \oplus_{n\in\mathbb{Z}}$ is tame. (Hint: As R is homogeneous, $T_{n_0} = 0$ implies that $T_{\leq n_0} = \oplus_{n \leq n_0} T_n$ is an R-submodule of T.) •

Now, we can formulate the most fundamental question concerning the asymptotic behaviour of cohomology.

4.3. **Problem.** *(Tameness Problem) Let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module. Is the graded R-module $H^i_{R_+}(M)$ tame?* •

Although the tameness of $H^i_{R_+}(M)$ has been shown in many special cases (see Theorem 4.5 and Theorem 4.8), the Tameness Problem is still open in general.

4.4. **Remark.** A) Let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. In various cases, the tameness of $H^i_{R_+}(M)$ is a consequence of a stronger property, namely the *asymptotic stability of supports* or even the *asymptotic stability of associated primes*.

Let us recall these notions in a precise way. Given a sequence $(\mathcal{S}_n)_{n\in\mathbb{Z}}$ of sets $\mathcal{S}_n \subseteq \operatorname{Spec}(R_0)$ we say that $\mathcal{S}_n$ is *asymptotically stable for* $n \to -\infty$ if there is some $n_0 \in \mathbb{Z}$ such that $\mathcal{S}_n = \mathcal{S}_{n_0}$ for all $n \le n_0$.

In this terminology we clearly have the following implications:

$\operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ *is asymptotically stable for* $n \to -\infty \implies$
$\operatorname{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right)$ *is asymptotically stable for* $n \to -\infty \implies$
$H^i_{R_+}(M)$ *is tame.*

B) Let M and i be as in part A). Then, according to Theorem 3.6 b) we can say:

If $F^i(M) \neq \emptyset$, *then* $H^i_{R_+}(M)$ *is tame.* •

In the following result we review the state of the art in the Tameness Problem for the cases where the base ring R_0 has dimension ≤ 2.

4.5. **Theorem.** *Let* $M \neq 0$ *be a finitely generated graded* R*-module. Then, the graded* R*-module* $H^i_{R_+}(M)$ *is tame for all* $i \in \mathbb{N}_0$ *in each of the following cases:*

Case	$\dim(R_0)$	Conditions on R_0	Conditions on M
a)	0		
b)	1	semilocal	
c)	1	finite integral extension of a domain	
d)	1	essentially of finite type over a field	
e)	1		Cohen-Macaulay
f)	2	semilocal	
g)	2	domain, essentially of finite type over a field	torsion-free over R_0
h)	2		Cohen-Macaulay

Proof and Comments: "a)": If $\dim(R_0) = 0$, the R-modules $H^i_{R_+}(M)$ are Artinian (cf [13, Theorem 17.1.9]) and so we may conclude by Remark 4.2.

"b)": See [4, Lemma 4.2].

"c), d)": Let $i \in \mathbb{N}_0$. If R_0 is of dimension ≤ 1 and either a finite integral extension of a domain or essentially of finite type over a field, the set $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ is asymptotically stable for $n \to -\infty$ (cf [2, Corollary (3.10) b)]). Now, we may conclude by Remark 4.4 A).

"e), h)": See [30] or [33, Theorem 5.6]. For further information also consult [28] and [29].

"f)": See [2, Theorem 4.7 a)].

"g)": Let $i \in \mathbb{N}_0$. As we shall see later (cf Corollary 6.11) the set $\mathrm{Supp}\left(H^i_{R_+}(M)_n\right)$ is asymptotically stable for $n \to -\infty$ under the imposed conditions. Now, we may conclude by Remark 4.4 A). ■

In addition tameness holds for $H^i_{R_+}(M)$ without restriction on the base ring R_0 but for certain specific levels i. In order to formulate the corresponding result, let us recall two cohomological invariants of a finitely generated graded R-module M.

4.6**. Definition.** A) Let M be a finitely generated graded R-module. The *finiteness dimension of* M (*with respect to* R_+) is defined as (cf [13, 9.1.3])

$$f(M) := \inf\{i \in \mathbb{N} \mid H^i_{R_+}(M) \text{ is not finitely generated }\}.$$

B) Let M be as above. Then, the *finite length dimension of* M is defined as (cf [11, (3.1)])

$$g(M) := \inf\{i \in \mathbb{N} \mid \mathrm{length}_{R_0}\left(H^i_{R_+}(M)_n\right) = \infty \text{ for infinitely many } n\}.$$

•

4.7**. Remark.** Let M be a finitely generated graded R-module. Then, the R_0-modules $H^i_{R_+}(M)_n$ are all finitely generated and vanish for all $n \gg 0$. Therefore

$$f(M) := \inf\{i \in \mathbb{N} \mid H^i_{R_+}(M)_n \neq 0 \text{ for infinitely many } n\}$$

and hence

$$f(M) \leq g(M).$$

•

4.8**. Theorem.** *Let $M \neq 0$ be a finitely generated and graded R-module. Let $i \in \mathbb{N}_0$. Then, the R-module $H^i_{R_+}(M)$ is tame in the following cases:*

a) $i \leq 1$
b) $i \leq f(M)$.
c) R_0 *is semilocal and* $i \leq g(M)$.
d) $i \in \Psi(M)$, *where* $\Psi(M)$ *is defined as in Proposition 3.11.*
e) $i = \mathrm{cd}(M)$.

Proof and Comment: "a)": The case $i = 0$ is obvious as $H^0_{R_+}(M)$ is finitely generated. The case $i = 1$ is a special case of b).

"b)": If $i < f(M)$ we conclude by Remark 4.7. If $i = f(M)$, the set $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ is asymptotically stable for $n \to -\infty$ (cf [4, Proposition 5.6]). So we conclude by Remark 4.4 A).

"c)": By the flat base change property it suffices to treat the local case. But if $(R_0, \mathfrak{m}_0)$ is local and $i < g(M)$, there is a polynomial $P \in \mathbb{Q}[x]$ such that $\mathrm{length}_{R_0}\left(H^i_{R_+}(M)_n\right) = P(n)$ for all $n \ll 0$ (cf [11, Theorem 3.6 a)]). So $H^i_{R+}(M)$ is tame in this case.

If $i = g(M)$, the set $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ is again asymptotically stable for $n \to -\infty$ (cf [11, Theorem (4.10) a)]) and hence $H^i_{R_+}(M)$ is tame.

"d), e)": Are clear by Theorem 3.6, Remark 4.4 A), B) and Corollary 3.7 b). ∎

The Tameness Problem arose in the geometric context which we shall recall now briefly.

4.9**. Reminder.** (cf [4]) A) Let $X = \mathrm{Proj}(R)$ and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. The set

$$\mathcal{P}(\mathcal{F}) := \{(i,n) \in \mathbb{N}_0 \times \mathbb{Z} \,\big|\, H^i\left(X, \mathcal{F}(n)\right) \neq 0\}$$

is called the *cohomological pattern of* $\mathcal{F}$. The *cohomological dimension of* $\mathcal{F}$ is defined by

$$\mathrm{cd}(\mathcal{F}) := \sup\{i \in \mathbb{N}_0 \,\big|\, \exists n : (i,n) \in \mathcal{P}(\mathcal{F})\}.$$

If $\mathcal{F} \neq 0$ is induced by the finitely generated graded R-module M, the *Serre-Gothendieck correspondence* yields (cf (1.3))

$$\mathcal{P}(\mathcal{F}) = \begin{cases} \{(i,n) \in \mathbb{N} \times \mathbb{Z} \,\big|\, H^{i+1}_{R_+}(M)_n \neq 0\} \cup \\ \{(0,n) \,\big|\, n \in \mathbb{Z} : H^1_{R_+}(M)_n \neq 0 \text{ or } \Gamma_{R_+}(M)_n \neq M_n\}. \end{cases}$$

B) A set $P \subseteq \mathbb{N}_0 \times \mathbb{Z}$ is called a *combinatorial pattern of width* w if

(1) $\exists m, n \in \mathbb{Z} : (0,m), (w,n) \in P$;
(2) $(i,n) \in P \Longrightarrow i \leq w$;
(3) $(i,n) \in P \Longrightarrow \exists j \leq i : (j, n+i-j+1) \in P$;
(4) $(i,n) \in P \Longrightarrow \exists k \geq i : (k, n+i-k-1) \in P$;

(5) $\quad i > 0 \Longrightarrow \forall n \gg 0 : (i,n) \notin P.$

The combinatorial pattern P is called *tame*, if for each $i > 0$ we have

(6) $\quad (\forall n \ll 0 : (i,n) \in P)$ or $(\forall n \ll 0 : (i,n) \notin P)$.

C) According to [4, Proposition 3.5] we have:

If $\mathcal{F} \neq 0$, then $\mathcal{P}(\mathcal{F})$ is a combinatorial pattern of width $\mathrm{cd}(\mathcal{F})$.

Moreover (cf [4, Corollary 4.7]):

If $X = \mathbb{P}^r_{R_0} = \mathrm{Proj}(R_0[x_0, \cdots, x_r])$, then each tame combinatorial pattern of width $\leq r$ is the cohomological pattern of a coherent sheaf of $\mathcal{O}_X$-modules.

D) We do not know of any example of a non-tame cohomological pattern of a coherent sheaf of $\mathcal{O}_X$-modules. This raises the question, whether the cohomological pattern of a coherent sheaf of $\mathcal{O}_X$-modules is always tame. According to the observations made in part A) this question is equivalent to the Tameness Problem 4.3. •

4.10**. Comments and Problems.** A) To our best knowledge, the Tameness Problem is still open in the particular case, where $R = M$ and $R = \mathcal{R}(\mathfrak{a}_0) := R_0[\mathfrak{a}_0 t]$ is the Rees-ring of an ideal $\mathfrak{a}_0 \subseteq R_0$.

Similarly, one could ask about the Tameness-Problem in the case where $R = M$ and $R = \mathrm{Gr}(\mathfrak{a}_0) := \mathcal{R}(\mathfrak{a}_0)/\mathfrak{a}_0\mathcal{R}(\mathfrak{a}_0)$ is the associated graded ring of an ideal $\mathfrak{a}_0 \subseteq R_0$.

B) Another open case concerns the level $i = 2$:

Is $H^2_{R_+}(M)$ tame for each finitely generated graded R-module M?

If $x \in R_1$ is a non-zero divisor on M we have an exact sequence of graded R-modules

$$D_{R_+}(M) \to D_{R_+}(M/xM) \to H^2_{R_+}(M)(-1) \xrightarrow{x} H^2_{R_+}(M)$$

in which D_{R_+} denotes the functor of taking R_+-transforms (cf [13, Theorem 2.2.4 (i), Exercise 12.4.5 (iii)]). So, in this case it would be sufficient to show that the cokernel of the natural map $D_{R_+}(M) \to D_{R_+}(M/xM)$ is tame.

One could be tempted to show this latter statement by proving that graded quotients of the R_+-transform of a finitely generated graded R-module are tame. Unfortunately this is not true in general, as illustrated by the following example:

Let $R_0 := k[x,y]_{(x,y)}, \mathfrak{m}_0 := (x,y)R_0$ and $R := R_0[\mathfrak{m}_0 t] = \mathcal{R}(\mathfrak{m}_0)$ where k is a field and x, y, t are indeterminates. Then $D_{R_+}(R)_n = R_0 t^n$ for all $n \leq 0$ (cf [3, Example 4.1]) so that $R_1 \cdot D_{R_+}(R)_n = \mathfrak{m}_0 D_{R_+}(R)_{n+1}$ for all $n < 0$. Consequently the graded R-module

$$D_{R_+}(R)/\sum_{n\in\mathbb{Z}} R\left(D_{R_+}(R)_{2n}\right) =: Q$$

satisfies $Q_{2n} = 0$ and $Q_{2n+1} \cong R_0/\mathfrak{m}_0$ for all $n < 0$ and hence is not tame. •

5. Tameness at almost Top Levels

In Theorem 4.8 statements a) - d) tameness at "low" levels is given, whereas statement e) concerns the top level. Nothing is said there on tameness at levels "near the top". In the present section we shall prove a few results which concern tameness at "almost top levels".

Our first result concerns the case where the base ring R_0 is local. It is only of interest in those cases, where the depth of the R_0-module M_n is 0 for all $n \gg 0$. It is meaningful as the cohomological dimension does not increase if one passes to quotients (cf Corollary 3.5).

5.1. **Proposition.** *Let $(R_0, \mathfrak{m}_0)$ be local, let $\mathfrak{q}_0 \subseteq R_0$ be an $\mathfrak{m}_0$-primary ideal, let M be a finitely generated graded R-module and set $\overline{c} := \mathrm{cd}\,(M/\Gamma_{\mathfrak{m}_0}(M))$. Then:*

- *a) For each $i > \overline{c}$, the R-module $H^i_{R_+}(M)$ is Artinian.*
- *b) If $\overline{c} \geq 0$, the R-module $H^{\overline{c}}_{R_+}(M)/\mathfrak{q}_0 H^{\overline{c}}_{R_+}(M)$ is Artinian.*
- *c) For each $i \geq \overline{c}$, the R-module $H^i_{R_+}(M)$ is tame.*

Proof: "a)": Let $\overline{M} := M/\Gamma_{\mathfrak{m}_0}(M)$ and consider the exact sequences of graded R-modules

$$H^i_{R_+}\left(\Gamma_{\mathfrak{m}_0}(M)\right) \to H^i_{R_+}(M) \to H^i_{R_+}(\overline{M}) \to H^{i+1}_{R_+}\left(\Gamma_{\mathfrak{m}_0}(M)\right).$$

As the R_0-module $\Gamma_{\mathfrak{m}_0}(M)$ is annihilated by some power of $\mathfrak{m}_0$, the modules at both ends of the above sequences are Artinian. So, for each $i \in \mathbb{N}_0$ we get an exact sequence of graded R-modules

$$0 \to B \to H^i_{R_+}(M) \to H^i_{R_+}(\overline{M}) \xrightarrow{\pi} A \to 0$$

in which A and B are Artinian. Choosing $i > \overline{c}$ we get our claim.

"b)": Consider the previous sequence with $i = \overline{c}$ and set $C := \mathrm{Ker}(\pi)$. Then, the exact sequence $0 \to C \to H^{\overline{c}}_{R_+}(\overline{M}) \to A \to 0$ yields an exact sequence of graded R-modules

$$\mathrm{Tor}^R_1(R/\mathfrak{q}_0 R, A) \to C/\mathfrak{q}_0 C \to H^{\overline{c}}_{R_+}(\overline{M})/\mathfrak{q}_0 H^{\overline{c}}_{R_+}(\overline{M}).$$

The first module in this sequence is Artinian as A is. The last module in the sequence is Artinian by Theorem 2.3 b). So $C/\mathfrak{q}_0 C$ is Artinian.

Finally, the exact sequence $0 \to B \to H^{\overline{c}}_{R_+}(M) \to C \to 0$ induces an exact sequence of graded R-modules

$$B/\mathfrak{q}_0 B \to H^{\overline{c}}_{R_+}(M)/\mathfrak{q}_0 H^{\overline{c}}_{R_+}(M) \to C/\mathfrak{q}_0 C$$

in which the modules at both ends are Artinian. So, the middle module is Artinian.

"c)": If $i > \overline{c}$, the desired tameness follows from statement a) and Remark 4.2. So, we may assume that $\overline{c} \geq 0$ and restrict ourselves to showing that $H^{\overline{c}}_{R_+}(M)$ is tame.

Assume that $H^{\overline{c}}_{R_+}(M)_n \neq 0$ for infinitely many negative integers n. Then, by Nakayama

$$(*) \qquad \left(H^{\overline{c}}_{R_+}(M)/\mathfrak{q}_0 H^{\overline{c}}_{R_+}(M)\right)_n = H^{\overline{c}}_{R_+}(M)_n/\mathfrak{q}_0 H^{\overline{c}}_{R_+}(M)_n \neq 0$$

for infinitely many integers $n < 0$. According to statement b) and Remark 4.2, the R-module $H^{\overline{c}}_{R_+}(M)/\mathfrak{q}_0 H^{\overline{c}}_{R_+}(M)$ is tame. Therefore $(*)$ holds for all $n \ll 0$ and thus $H^{\overline{c}}_{R_+}(M)_n \neq 0$ for all $n \ll 0$. ■

Next, we want to draw a conclusion from the previous result which also applies if R_0 is not local. If $\mathfrak{A}_0 \neq \emptyset$ is a set of ideals of R_0 and M is a graded R_0-module, we write $\Gamma_{\mathfrak{A}_0}(M)$ for the $\mathfrak{A}_0$-torsion submodule of M so that $\Gamma_{\mathfrak{A}_0}(M)$ is the set of all $m \in M$ which are annihilated by a product of ideals of $\mathfrak{A}_0$. Keep in mind that $\Gamma_{\mathfrak{A}_0}(M)$ is a graded submodule of M.

5.2. **Theorem.** *Let $M \neq 0$ be a finitely generated graded R-module such that*

$$F^{\mathrm{cd}(M)}(M) \subseteq \mathrm{Max}(R_0).$$

Set $\overline{c} := \mathrm{cd}\left(M/\Gamma_{F^{\mathrm{cd}(M)}(M)}(M)\right)$. Then:

- *a) For each $i > \overline{c}$, the R-module $H^i_{R_+}(M)$ is Artinian.*
- *b) If $\overline{c} \geq 0$ and $\mathfrak{q}_0 \subseteq R_0$ is an ideal with $\mathrm{Var}(\mathfrak{q}_0) \subseteq F^{\mathrm{cd}(M)}(M)$, the R-module $H^{\overline{c}}_{R_+}(M)/\mathfrak{q}_0 H^{\overline{c}}_{R_+}(M)$ is Artinian.*
- *c) For each $i \geq \overline{c}$, the R-module $H^i_{R_+}(M)$ is tame.*

Proof: "a)": According to Corollary 3.7 b) the set $\mathcal{Z} := F^{\mathrm{cd}(M)}(M)$ is a non-empty finite set of maximal ideals of R_0. Let $\overline{M} := M/\Gamma_{\mathcal{Z}}(M)$. Then, for each $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)\setminus\mathcal{Z}$ we have an isomorphism of graded $R_{\mathfrak{p}_0}$-modules $\overline{M}_{\mathfrak{p}_0} \cong M_{\mathfrak{p}_0}$. So, for each $j \in \mathbb{N}_0$ the graded flat base change property of local cohomology yields isomorphisms of graded $R_{\mathfrak{p}_0}$-modules

$$(*) \qquad H^j_{R_+}(M)_{\mathfrak{p}_0} \cong H^j_{(R_{\mathfrak{p}_0})_+}(\overline{M}_{\mathfrak{p}_0}) \cong H^j_{R_+}(\overline{M})_{\mathfrak{p}_0}.$$

These show that $H^i_{R_+}(M)_{\mathfrak{p}_0} = 0$ for each $i > \overline{c}$ and each $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)\setminus\mathcal{Z}$, so that $\mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)\right) \subseteq \mathcal{Z}$ for all $i > \overline{c}$. Therefore it is enough to show that for $i > \overline{c}$ the $R_{\mathfrak{m}_0}$-module $H^i_{R_+}(M)_{\mathfrak{m}_0} \cong H^i_{(R_{\mathfrak{m}_0})_+}(M_{\mathfrak{m}_0})$ is Artinian for each $\mathfrak{m}_0 \in \mathcal{Z}$. But for each such $\mathfrak{m}_0$ there is an isomorphism of graded $R_{\mathfrak{m}_0}$-modules $M_{\mathfrak{m}_0}/\Gamma_{\mathfrak{m}_0 R_{0\mathfrak{m}_0}}(M_{\mathfrak{m}_0}) \cong \overline{M}_{\mathfrak{m}_0}$, so that

$\operatorname{cd}\big(M_{\mathfrak{m}_0}/\Gamma_{\mathfrak{m}_0 R_{0\mathfrak{m}_0}}(M_{\mathfrak{m}_0})\big) \leq \overline{c}$. If we apply Proposition 5.1 a) to the graded $R_{\mathfrak{m}_0}$-module $M_{\mathfrak{m}_0}$ we get our claim.

"b)": If $\operatorname{Var}(\mathfrak{q}_0) = \emptyset$, our claim is trivial. So let $\operatorname{Var}(\mathfrak{q}_0) \neq \emptyset$. As

$$\operatorname{Supp}_{R_0}\big(H^{\overline{c}}_{R_+}(M)/\mathfrak{q}_0 H^{\overline{c}}_{R_+}(M)\big) \subseteq \operatorname{Var}(\mathfrak{q}_0)$$

it suffices to show that the graded R_0-module

$$\big(H^{\overline{c}}_{R_+}(M)/\mathfrak{q}_0 H^{\overline{c}}_{R_+}(M)\big)_{\mathfrak{m}_0} \cong H^{\overline{c}}_{(R_{\mathfrak{m}_0})_+}(M_{\mathfrak{m}_0})/\mathfrak{q}_0 R_{0\mathfrak{m}_0} H^{\overline{c}}_{(R_{\mathfrak{m}_0})_+}(M_{\mathfrak{m}_0})$$

is Artinian for each $\mathfrak{m}_0 \in \operatorname{Var}(\mathfrak{q}_0)$. From the proof of statement a) and as $\operatorname{Var}(\mathfrak{q}_0) \subseteq \mathcal{Z}$ we know that $\operatorname{cd}\Big(M_{\mathfrak{m}_0}/\Gamma_{\mathfrak{m}_0 R_{0\mathfrak{m}_0}}(M_0)\Big) \leq \overline{c}$ for all such $\mathfrak{m}_0$.

Now, we conclude by Proposition 5.1 applied to the graded $R_{\mathfrak{m}_0}$-module $M_{\mathfrak{m}_0}$ and the $\mathfrak{m}_0 R_{0\mathfrak{m}_0}$-primary ideal $\mathfrak{q}_0 R_{0\mathfrak{m}_0}$.

"c)": Assume first, that $\operatorname{Supp}_{R_0}\big(H^{\overline{c}}_{R_+}(M)\big) \nsubseteq \mathcal{Z}$. Then, there is a $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)\backslash\mathcal{Z}$ such that $H^{\overline{c}}_{R_+}(M)_{\mathfrak{p}_0} \neq 0$. By the previous isomorphisms $(*)$ we get $H^{\overline{c}}_{R_+}(\overline{M})_{\mathfrak{p}_0} \neq 0$, hence $\big(H^{\overline{c}}_{R_+}(\overline{M})_{\mathfrak{p}_0}\big)_n \neq 0$ for all $n \ll 0$ (cf Corollary 3.7 a)). Another use of $(*)$ gives $\big(H^{\overline{c}}_{R_+}(M)_n\big)_{\mathfrak{p}_0} \cong \big(H^{\overline{c}}_{R_+}(M)_{\mathfrak{p}_0}\big)_n \neq 0$ for all $n \ll 0$ and thus $H^{\overline{c}}_{R_+}(M)_n \neq 0$ for all $n \ll 0$. So, the R-module $H^{\overline{c}}_{R_+}(M)$ is tame. We therefore may assume that $\operatorname{Supp}_{R_0}\big(H^{\overline{c}}_{R_+}(M)\big) \subseteq \mathcal{Z}$. If we apply Proposition 5.1 c) to the graded $R_{\mathfrak{m}_0}$-module $M_{\mathfrak{m}_0}$ we see that the $R_{\mathfrak{m}_0}$-module $H^{\overline{c}}_{R_+}(M)_{\mathfrak{m}_0} \cong H^{\overline{c}}_{(R_{\mathfrak{m}_0})_+}(M_{\mathfrak{m}_0})$ is tame for all $\mathfrak{m}_0 \in \mathcal{Z}$. As $\mathcal{Z}$ is finite, we get our claim. ■

As an application of the previous result we now may prove that under some restrictions on the support of M, the module $H^i_{R_+}(M)$ is tame on the "highest two levels i at which non-vanishing of $H^i_{R_+}(M)$ may occur at all".

5.3. **Proposition.** *Let* $\dim(R_0) =: d < \infty$, *let* $M \neq 0$ *be a finitely generated graded* R-*module and let* $m := \max \Psi(M)$, *where* $\Psi(M)$ *is defined according to Proposition 3.11. Assume that* $\operatorname{Supp}_R(M)$ *is catenarian and locally equidimensional. Then:*

a) $\operatorname{cd}(M) \leq d + m$.

b) *If* $\operatorname{cd}(M) = d + m$, *then* $F^{\operatorname{cd}(M)}(M) \subseteq \operatorname{Max}(R_0)$.

c) $H^i_{R_+}(M)$ *is tame for* $i \geq d + m - 1$.

Proof: By the graded base-ring independence property of local cohomology, by Proposition 3.3 b) and by the fact that $\dim(R_0)$ does not increase if R is replaced by one of its graded homomorphic images, we may replace R by $R/(0 \underset{R}{:} M)$ and hence assume that $0 \underset{R}{:} M = 0$. In particular R is catenarian and locally equidimensional.

Now, let $\mathfrak{m}_0 \in \operatorname{Spec}(R_0)$, let $\mathfrak{p}_0 \subseteq \mathfrak{m}_0$ be a minimal prime of R_0 and let $\mathfrak{q} \subseteq R$ be a graded prime of R such that $\mathfrak{q} \cap R_0 = \mathfrak{m}_0$. There is a minimal prime $\mathfrak{p}$ of R such that $\mathfrak{p} \cap R_0 = \mathfrak{p}_0$.

As R is catenarian and locally equidimensional we have

$$\operatorname{height}\left((\mathfrak{m}_0 + R_+)/\mathfrak{q}\right) = \operatorname{height}(\mathfrak{m}_0 + R_+) - \operatorname{height}(\mathfrak{q})$$

and

$$\begin{aligned}\operatorname{height}(\mathfrak{m}_0 + R_+) &= \operatorname{height}\left((\mathfrak{m}_0 + R_+)/\mathfrak{p}\right)\\ &= \operatorname{height}\big((\mathfrak{m}_0 + R_+)/(\mathfrak{p}_0 + R_+)\big) - \operatorname{height}\left((\mathfrak{p}_0 + R_+)/\mathfrak{p}\right)\\ &= \operatorname{height}(\mathfrak{m}_0/\mathfrak{p}_0) + \operatorname{height}(\mathfrak{p}_0 + R_+)\end{aligned}$$

and hence

$$(*) \qquad \operatorname{height}\left((\mathfrak{m}_0 + R_+)/\mathfrak{q}\right) = \operatorname{height}(\mathfrak{m}_0/\mathfrak{p}_0) + \operatorname{height}(\mathfrak{p}_0 + R_+) - \operatorname{height}(\mathfrak{q}).$$

After these preparations we prove our claims:

"a), b)": In view of Proposition 3.3 b) we have

$$\begin{aligned}\dim_R\left(\kappa(\mathfrak{m}_0) \otimes_{R_0} M\right) &= \dim_R\left(\kappa(\mathfrak{m}_0) \otimes_{R_0} R\right)\\ &= \operatorname{height}\left((\mathfrak{m}_0 + R_+)/\mathfrak{m}_0 R\right) = \operatorname{height}\left((\mathfrak{m}_0 + R_+)/\mathfrak{q}\right),\end{aligned}$$

where $\mathfrak{q} \subseteq R$ is an appropriate minimal prime of $\mathfrak{m}_0 R$ such that $\mathfrak{q} \cap R_0 = \mathfrak{m}_0$. As $\operatorname{height}(\mathfrak{m}_0/\mathfrak{p}_0) \leq d$ and $\operatorname{height}(\mathfrak{p}_0 + R_+) \in \Psi(M)$ statements a) and b) follow easily from the equality $(*)$.

"c)": If $\operatorname{cd}(M) < d + m$ we conclude by statement a) and by Theorem 4.8 e). So, let $\operatorname{cd}(M) = d + m$. By statement b) we have $F^{\operatorname{cd}(M)}(M) \subseteq \operatorname{Max}(R_0)$. Let $\overline{M} := M/\Gamma_{F^{\operatorname{cd}}(M)}(M)$. According to Theorem 5.2 c) it suffices to show that $\overline{c} := \operatorname{cd}(\overline{M}) < d+m$. Choose $\mathfrak{m}_0 \in \operatorname{Spec}(R_0)$ such that $\overline{c} = \dim_R\left(\kappa(\mathfrak{m}_0) \otimes_{R_0} \overline{M}\right)$. Then, there is a graded prime $\mathfrak{q} \in \operatorname{Supp}_R(\overline{M})$ such that $\mathfrak{q} \cap R_0 = \mathfrak{m}_0$ and $\overline{c} = \operatorname{height}\left((\mathfrak{m}_0 + R_+)/\mathfrak{q}\right)$. By $(*)$ we get $\overline{c} = \operatorname{height}(\mathfrak{m}_0/\mathfrak{p}_0) + \operatorname{height}(\mathfrak{p}_0 + R_+) - \operatorname{height}(\mathfrak{q})$ for each minimal prime $\mathfrak{p}_0 \subseteq \mathfrak{m}_0$ of R_0. Assuming $\overline{c} = d+m$ we would get $\mathfrak{m}_0 \in F^{\operatorname{cd}(M)}(M)$ and $\operatorname{height}(\mathfrak{q}) = 0$. As $\mathfrak{q} \in \operatorname{Supp}_R(\overline{M})$ and $\mathfrak{q} \cap R_0 = \mathfrak{m}_0$ the last equality would imply $\Gamma_{\mathfrak{m}_0}(\overline{M}) \neq 0$, a contradiction. ■

As an application for low-dimensional base rings R_0 we get

5.4. **Corollary.** *Let $\dim(R_0) \leq 2$ and let $M \neq 0$ be a finitely generated graded R-module such that Supp(M) is catenarian and locally equidimensional. Then:*

$$H^i_{R_+}(M) \text{ is tame for all } i \geq \max \Psi(M).$$

Proof: This is clear from Theorem 4.8 d) and Proposition 5.3 c). ■

5.5. **Comments and Problems.** A) Let R and M be as in Corollary 5.4. Assume in addition that

$$\text{height}\left(\mathfrak{p}/(0 \underset{R}{:} M)\right) = \text{grade}_M(\mathfrak{p}) \text{ for all } \mathfrak{p} \in \min\left((0 \underset{R}{:} M) + R_+\right).$$

Then, Corollary 5.4 combined with an argument similar to that used in [28] or [30] shows that all the R-modules $H^i_{R_+}(M)$ are tame. If M is a Cohen-Macaulay module, the hypotheses of Corollary 5.4 as well as the above requirement are satisfied. Therefore Corollary 5.4 may be viewed as an extension of [33, Theorem 5.6] or the main result of [30], which both say that $H^i_{R_+}(M)$ is tame if $\dim(R_0) \leq 2$ and M is a Cohen-Macaulay module.

B) We do not know, whether the conclusion of Corollary 5.4 holds, if R_0 is local and of dimension 3. An affirmative answer to this would imply that $H^i_{R_+}(M)$ is tame for all $i \geq 0$ if R_0 is semilocal and of dimension ≤ 3 and the finitely generated graded R-module M is Cohen-Macaulay. This would improve [33, Theorem 5.4], which states the finiteness of the set of minimal associated primes of the R-module $H^i_{R_+}(M)$. $\bullet$

6. Asymptotic Stability of Supports

We keep the hypotheses and notation of the previous sections and consider the following question.

6.1. **Problem.** *(Asymptotic Stability of Supports): Let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module. Is the set* $\text{Supp}\left(H^i_{R_+}(M)_n\right)$ *asymptotically stable for $n \to -\infty$, e.g. is there some integer n_0 such that* $\text{Supp}\left(H^i_{R_+}(M)_n\right) = \text{Supp}\left(H^i_{R_+}(M)_{n_0}\right)$ *for all $n \leq n_0$?* $\bullet$

This problem still is open in general: as observed already in Remark 4.4 A) asymptotic stability of supports implies tameness. It is an open problem, whether tameness implies asymptotic stability of supports.

6.2. **Remark.** A) Let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Observe that $\cup_{n\in\mathbb{Z}}\text{Supp}\left(H^i_{R_+}(M)_n\right) = \text{Supp}_{R_0}\left(H^i_{R_+}(M)\right)$ and that $\text{Supp}\left(H^i_{R_+}(M)_n\right) = \emptyset$ for all $n \gg 0$. So, if $\text{Supp}\left(H^i_{R_+}(M)_n\right)$ is asymptotically stable for $n \to -\infty$, the R_0-support of $H^i_{R_+}(M)$ is closed.

In view of Lemma 3.8 the R_0-support of $H^i_{R_+}(M)$ is closed if and only if $\text{Ass}_R\left(H^i_{R_+}(M)\right)$ has only finitely many minimal members.

B) Keep the above notations and hypotheses. In view of the observations made in part A) it might be of interest to attack Problem 6.1 by studying first the question:

(i) *(Closedness of R_0-Supports) Is* $\mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)\right)$ *a closed subset of* $\mathrm{Spec}(R_0)$*?*

or – equivalently –

(ii) *(Finiteness of Minimal Associated Primes) Has* $\mathrm{Ass}_R\left(H^i_{R_+}(M)\right)$ *only finitely many minimal members?*

Question (ii) is a special case of one asked by Katzman-Sharp in [27, Remark 1.9]. •

Now, we shall prove that "Asymptotic Stability of Supports" holds "in codimension ≤ 2" provided R_0 is a domain and essentially of finite type over a field and the finitely generated graded R-module M satisfies a certain "purity condition" (which holds for example, if M is torsion-free over R_0).

6.3. **Notation.** A) If $\mathcal{P} \subseteq \mathrm{Spec}(R_0)$ and $k \in \mathbb{N}_0$ we write

$$\mathcal{P}^{\leq k} := \{\mathfrak{p}_0 \in \mathcal{P} \,|\, \mathrm{height}(\mathfrak{p}_0) \leq k\}.$$

B) Let M be a finitely generated graded R-module, let $i \in \mathbb{N}_0$ and let $n \in \mathbb{Z}$. We use $\overline{\bullet}$ to denote topological closure in $\mathrm{Spec}(R_0)$ and set

$$V^i_n(M) := \left(\,\overline{\mathrm{Ass}(H^i_{R_+}(M)_n)^{\leq 1}}\,\right)^{\leq 2}$$

so that $V^i_n(M)$ consists of all $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)$ which satisfy $\mathrm{height}(\mathfrak{p}_0) \leq 2$ and contain some $\mathfrak{q}_0 \in \mathrm{Ass}\left(H^i_{R_+}(M)_n\right)$ with $\mathrm{height}(\mathfrak{q}_0) \leq 1$.

C) Let M, i and n be as in part B). We define

$$W^i_n(M) := \mathrm{Supp}\left(H^i_{R_+}(M)_n\right)^{\leq 2} \setminus V^i_n(M).$$

•

6.4. **Remark.** A) Keep the previous hypotheses and notations. Then, clearly we have

$$\dim(R_0) \leq 1 \Longrightarrow W^i_n(M) = \emptyset.$$

B) It is immediate from the above definition, that

$$\mathrm{Supp}\left(H^i_{R_+}(M)\right)^{\leq 2} = V^i_n(M) \,\dot{\cup}\, W^i_n(M).$$

C) If R_0 is a domain, there is some $s \in R_0\setminus\{0\}$ such that the $(R_0)_s$-modules M_s and $H^j_{R_+}(M)_s$ are torsion-free (or zero) for all $j \in \mathbb{N}_0$ ([2, Theorem 2.5]). Therefore we have

$$\mathrm{Ass}\left(H^j_{R_+}(M)_n\right) \subseteq \{0\} \cup \mathrm{Var}(sR_0) \text{ for all } j \in \mathbb{N}_0 \text{ and all } n \in \mathbb{Z}.$$

As an immediate consequence we get

$$W_n^i(M) \subseteq \mathrm{Var}(sR_0) \text{ for all } i \in \mathbb{N}_0 \text{ and all } n \in \mathbb{Z}.$$

D) Clearly, by its definition, the set $W_n^i(M)$ consists of primes of height 2 which are minimal associated primes of $H^i_{R_+}(M)_n$. This gives the following alternative descriptions of $W_n^i(M)$

$$W_n^i(M) = \{\mathfrak{p}_0 \in \mathrm{Spec}(R_0) \,|\, \mathrm{height}(\mathfrak{p}_0) = 2 \text{ and } 0 \neq \mathrm{length}_{(R_0)_{\mathfrak{p}_0}}((H^i_{R_+}(M)_n)_{\mathfrak{p}_0}) < \infty\}$$

$$W_n^i(M) = \mathrm{Ass}\left(H^i_{R_+}(M)\right)^{\leq 2} \setminus V_n^i(M).$$

•

6.5. **Lemma.** *Let $i \in \mathbb{N}_0$ assume that R_0 is essentially of finite type over a field and let M be a finitely generated graded R-module. Then, there is an integer n_0 such that*

$$V_n^i(M) = V_{n_0}^i(M) \text{ for all } n \leq n_0.$$

Proof: According to [2, Theorem 3.7] the set $\mathcal{T}^i(M)_n := \mathrm{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 1}$ is asymptotically stable for $n \to -\infty$ (cf Remark 4.4 A) for this notion). Taking closures and then restricting to codimensions ≤ 2 we get our claim. ■

6.6. **Lemma.** *Let $i \in \mathbb{N}_0$, assume that R_0 is essentially of finite type over a field and let M be a finitely generated graded R-module. Then, the following statements are equivalent:*

(i) *The set $\bigcup_{n\in\mathbb{Z}} W_n^i(M)$ is finite.*

(ii) *There is an integer n_1 such that $W_n^i(M) = W_{n_1}^i(M)$ for all $n \leq n_1$.*

Proof: "(ii) $\Longrightarrow$ (i)": According to Remark 6.4 D) the sets $W_n^i(M)$ are finite and vanish for all $n \gg 0$. This gives our claim.

"(i) $\Longrightarrow$ (ii)": Assume that $W := \cup_{n\in\mathbb{Z}} W_n^i(M)$ is finite and choose $\mathfrak{p}_0 \in W$. According to Remark 6.4 D) we have $\mathrm{height}(\mathfrak{p}_0) = 2$. So $(R_0)_{\mathfrak{p}_0}$ is of dimension 2 and essentially of finite type over a field. Applying [2, Corollary (4.8)] to the graded $R_{\mathfrak{p}_0}$-module $M_{\mathfrak{p}_0}$ we see that the set $\mathrm{Ass}_{R_0}\left(H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n\right)$ is asymptotically stable for $n \to -\infty$.

By the graded flat base change property of local cohomology we thus find an integer $k(\mathfrak{p}_0)$ such that

$$(*) \qquad \mathrm{Ass}\left((H^i_{R_+}(M)_n)_{\mathfrak{p}_0}\right) = \mathrm{Ass}\left((H^i_{R_+}(M)_{k(\mathfrak{p}_0)})_{\mathfrak{p}_0}\right) \text{ for all } n \leq k(\mathfrak{p}_0).$$

Now, let

$$n_1 := \min\{k(\mathfrak{p}_0) \,|\, \mathfrak{p}_0 \in W\} \text{ and } W' = \cup_{n\leq n_1} W_n^i(M).$$

It suffices to show that $W' \subseteq W^i_n(M)$ for all $n \leq n_1$. So, let $\mathfrak{p}_0 \in W'$. According to Remark 6.4 D) there is some $m \leq n_1$ such that $\mathfrak{p}_0$ has height 2 and is a minimal member of $\mathrm{Ass}\left(H^i_{R_+}(M)_m\right)$. As $m, n \leq n_1 \leq k(\mathfrak{p}_0)$ it follows by $(*)$ that $\mathfrak{p}_0$ is a minimal member of $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right)$ for all $n \leq n_1$. As $\mathrm{height}(\mathfrak{p}_0) = 2$ we conclude by Remark 6.4 D) that $\mathfrak{p}_0 \in W^i_n(M)$ for all $n \leq n_1$. ■

6.7. **Lemma.** *Let $i \in \mathbb{N}_0$ and assume that R_0 is a domain and essentially of finite type over a field. Let M be a finitely generated graded R-module which is torsion-free over R_0. Then, the set $\cup_{n\in\mathbb{Z}} W^i_n(M)$ is finite.*

Proof: Choose $s \in R_0 \backslash \{0\}$ according to Remark 6.4 C) so that the $(R_0)_s$-modules $H^j_{R_+}(M)_s$ are torsion-free or vanishing for all $j \in \mathbb{N}_0$. If s is invertible in R_0 we have $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right) \subseteq \{0\}$ and hence $W^i_n(M) = \emptyset$ for all $n \in \mathbb{Z}$. So, we may assume that $R_0 \neq sR_0$. As M is torsion-free over R_0, there is an exact sequence of graded R-modules $0 \to M \xrightarrow{s} M \to M/sM \to 0$ which yields a monomorphism of graded R-modules

$$(*) \qquad 0 \to H^i_{R_+}(M)/sH^i_{R_+}(M) \to H^i_{R_+}(M/sM).$$

If we apply [2, Theorem 3.7] to the graded R/sR-module M/sM, we get that the set

$$\bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{R_0/sR_0}\left(H^i_{(R/sR)_+}(M/sM)_n\right)^{\leq 1} \text{ is finite.}$$

In view of the graded base ring independence of local cohomology and as R_0 is a catenarian domain, it follows that the set

$$\widetilde{W} := \bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M/sM)_n\right)^{\leq 2}$$

is finite.

Now, let $\mathfrak{p}_0 \in \cup_{n\in\mathbb{Z}} W^i_n(M)$. Then $\mathrm{height}(\mathfrak{p}_0) = 2, s \in \mathfrak{p}_0$ and there is some $n \in \mathbb{Z}$ such that $\left(H^i_{R_+}(M)_n\right)_{\mathfrak{p}_0}$ is a non-zero $(R_0)_{\mathfrak{p}_0}$-module of finite length (cf Remark 6.4 C), D)).

Hence – by Nakayama $\left(H^i_{R_+}(M)_n/sH^i_{R_+}(M)_n\right)_{\mathfrak{p}_0} \cong \left(H^i_{R_+}(M)_n\right)_{\mathfrak{p}_0}/s\left(H^i_{R_+}(M)_n\right)_{\mathfrak{p}_0}$ is an $(R_0)_{\mathfrak{p}_0}$-module of finite length $\neq 0$. It follows that

$$\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n/sH^i_{R_+}(M)_n\right).$$

Therefore the fact that $\mathrm{height}(\mathfrak{p}_0) = 2$ together with $(*)$ yields $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M/sM)_n\right)^{\leq 2} \subseteq \widetilde{W}$. This proves our claim. ■

Now, we prove the announced asymptotic stability result for supports in codimension ≤ 2. Unfortunately we could find no way to remove the hypotheses $(*)$, although we believe that it is not necessary.

6.8. **Theorem.** *Let $i \in \mathbb{N}_0$ and assume that R_0 is a domain and essentially of finite type over a field. Let M be a finitely generated graded R-module such that*

$$(*) \qquad \text{height}(\mathfrak{p} \cap R_0) \neq 1 \textit{ for all } \mathfrak{p} \in \text{Ass}(M).$$

Then, the set $\text{Supp}\left(H^i_{R_+}(M)_n\right)^{\leq 2}$ *is asymptotically stable for $n \to -\infty$.*

Proof: According to Lemma 6.6, Lemma 6.5 and Remark 6.4 B) it suffices to show that the set $W^i(M) := \cup_{n\in\mathbb{Z}} W^i_n(M)$ is finite. Let $\mathcal{P} := \{\mathfrak{p} \in \text{Ass}(M) | \text{height}(\mathfrak{p} \cap R_0) \geq 2\}$, let $\mathfrak{a}_0 := \cap_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p} \cap R_0$ and consider the finitely generated graded R-module $\overline{M} := M/\Gamma_{\mathfrak{a}_0}(M)$. Observe that

$$\begin{aligned} \text{Ass}\big(\Gamma_{\mathfrak{a}_0}(M)\big) &= \text{Ass}(M) \cap \text{Var}\big(\mathfrak{a}_0 R\big) = \mathcal{P} \text{ and} \\ \text{Ass}(\overline{M}) &= \text{Ass}(M) \backslash \text{Var}(\mathfrak{a}_0 R) = \text{Ass}(M) \backslash \mathcal{P}. \end{aligned}$$

Assume first, that $\overline{M} = 0$. Then $M = \Gamma_{\mathfrak{a}_0}(M)$ and thus M, and hence $H^i_{R_+}(M)$ is annihilated by some power of $\mathfrak{a}_0$. Therefore $W^i_n(M) \subseteq \text{Var}(\mathfrak{a}_0)^{\leq 2}$ and we may conclude as this latter set is finite (or empty) by the definition of $\mathfrak{a}_0$.

So, let $\overline{M} \neq 0$. Then $\text{Ass}(\overline{M}) = \text{Ass}(M) \backslash \mathcal{P} \neq \emptyset$. By hypothesis $(*)$ it follows that $\text{Ass}(\overline{M}) = \{0\}$ and hence $\overline{M}$ is torsion-free over R_0. So, by Lemma 6.7 the set $W^i(\overline{M}) := \cup_{n\in\mathbb{Z}} W^i(\overline{M})$ is finite.

Now, let $\mathfrak{p}_0 \in W^i(M)$. Then, $\mathfrak{p}_0 \in W^i_n(M)$ for some $n \in \mathbb{Z}$ so that $\left(H^i_{R_+}(M)_n\right)_{\mathfrak{p}_0}$ is an $(R_0)_{\mathfrak{p}_0}$-module of finite length $\neq 0$.

Now, let $j \in \mathbb{N}_0$. As $\text{Supp}\left(H^j_{R_+}(\Gamma_{\mathfrak{a}_0}(M))\right) \subseteq \text{Var}(\mathfrak{a}_0)$ and $\text{height}(\mathfrak{p}_0) = 2 \leq \text{height}(\mathfrak{a}_0)$, the $(R_0)_{\mathfrak{p}_0}$ -module $\left(H^j_{R_+}(\Gamma_{\mathfrak{a}_0}(M))_n\right)_{\mathfrak{p}_0}$ is of finite length. Applying this to the exact sequence

$$\left(H^i_{R_+}(\Gamma_{\mathfrak{a}_0}(M))_n\right)_{\mathfrak{p}_0} \to \left(H^i_{R_+}(M)_n\right)_{\mathfrak{p}_0} \to \left(H^i_{R_+}(\overline{M})_n\right)_{\mathfrak{p}_0} \to \left(H^{i+1}_{R_+}(\Gamma_{\mathfrak{a}_0}(M))_n\right)_{\mathfrak{p}_0}$$

we see that $\left(H^i_{R_+}(\Gamma_{\mathfrak{a}_0}(M))_n\right)_{\mathfrak{p}_0}$ or $\left(H^i_{R_+}(\overline{M})_n\right)_{\mathfrak{p}_0}$ is a non-zero $(R_0)_{\mathfrak{p}_0}$-module of finite length.

In the first case we have $\mathfrak{p}_0 \in \text{Var}(\mathfrak{a}_0)^{\leq 2}$, in the second case we have $\mathfrak{p}_0 \in W^i_n(\overline{M})$ (cf Remark 6.4 D)). Altogether, this shows that $W^i(M) \subseteq W^i(\overline{M}) \cup \text{Var}(\mathfrak{a}_0)^{\leq 2}$; but this latter set is finite. ∎

6.9. **Corollary.** *Let $i \in \mathbb{N}_0$ and assume that R_0 is a domain of dimension 2 and essentially of finite type over a field. Let M be a finitely generated graded R-module such that there is no $\mathfrak{p} \in \text{Ass}(M)$ with* $\text{height}(\mathfrak{p} \cap R_0) = \dim(R_0/\mathfrak{p} \cap R_0) = 1$. *Then the set* $\text{Supp}\left(H^i_{R_+}(M)_n\right)$ *is asymptotically stable for $n \to -\infty$.*

Proof: Let $Q_0 := \{\mathfrak{p} \cap R_0 | \mathfrak{p} \in \mathrm{Ass}(M) \text{ and } \mathrm{height}(\mathfrak{p} \cap R_0) = 1\}$. By our hypotheses we have $Q_0 \subseteq \mathrm{Max}(R_0)$. If $Q_0 = \emptyset$, the hypotheses $(*)$ of Theorem 6.8 holds and we are done.

Otherwise let $S_0 := R_0 \setminus \bigcup_{\mathfrak{m}_0 \in Q_0} \mathfrak{m}_0$. Let $t_1, \cdots, t_r \in R_0 \setminus \{0\}$ be a set of generators for the ideal $\bigcap_{\mathfrak{m}_0 \in Q_0} \mathfrak{m}_0$. Then we have the canonical identification

$$\mathrm{Spec}(R_0) = \mathrm{Spec}(S_0^{-1} R_0) \cup \bigcup_{j=1}^{r} \mathrm{Spec}\left((R_0)_{t_i}\right),$$

which yields

$$\mathrm{Supp}(T) = \mathrm{Supp}(S_0^{-1} T) \cup \bigcup_{j=1}^{r} \mathrm{Supp}(T_{t_i})$$

for each R_0-module T. It thus suffices to show that the sets

$$\mathrm{Supp}\left(S_0^{-1} H^i_{R_+}(M)_n\right) \text{ and } \mathrm{Supp}\left((H^i_{R_+}(M)_n)_{t_j}\right) \ (j = 1, \cdots, r)$$

are asymptotically stable for $n \to -\infty$.

As $S_0^{-1} R_0$ is semilocal of dimension 1, the set $\mathrm{Ass}\left(H^i_{(S_0^{-1}R)_+}(S_0^{-1}M)_n\right)$ is asymptotically stable for $n \to -\infty$ (cf [2, Corollary (3.10)]) and hence so is $\mathrm{Supp}\big(H^i_{(S_0^{-1}R)_+}(S_0^{-1}M)_n\big)$. By our choice of t_j the graded R_{t_j}-module M_{t_j} satisfies the hypotheses $(*)$ of Theorem 6.8 so that the sets $\mathrm{Supp}\left(H^i_{(R_{t_j})_+}(M_{t_j})_n\right)$ are asymptotically stable for $n \to -\infty$. Now the graded flat base change property of local cohomology allows us to conclude. ■

6.10. **Corollary.** *Keep the hypotheses of Corollary 6.9. Then* $\mathrm{Ass}_R\left(H^i_{R_+}(M)\right)$ *has only finitely many minimal members.*

Proof: This is immediate by Corollary 6.9 and Remark 6.2 A). ■

6.11. **Corollary.** *Let $i \in \mathbb{N}$ and assume that R_0 is a domain of dimension ≤ 2 and essentially of finite type over a field. Let M be a finitely generated graded R-module which is torsion-free over R_0. Then*

a) *The set* $\mathrm{Supp}\left(H^i_{R_+}(M)_n\right)$ *is asymptotically stable for $n \to -\infty$.*

b) $\mathrm{Ass}_R\left(H^i_{R_+}(M)\right)$ *has only finitely many minimal members.*

■

6.12**. Comments and Problems.** A) As mentioned already above, we believe that by some refinement of the arguments in the proof of Theorem 6.8 the hypothesis $(*)$ may be avoided.

B) The hypotheses made in Corollary 6.11 seem to be to restrictive. So, we like to give encouragement to try to prove the desired "Asymptotic Stability of Supports" under the single condition that $\dim(R_0) \leq 2$.

C) There is some further evidence for the above suggestion: namely, as an easy consequence of [33, Theorem 5.4] the set $\operatorname{Supp}\big(H^i_{R_+}(M)_n\big)$ is asymptotically stable for $n \to -\infty$ as soon as $\dim(R_0) \leq 2$ and the finitely generated graded R-module M is Cohen-Macaulay. •

7. Asymptotic Stability of Associated Primes

We keep the previous hypotheses and notation and devote the present section to the following question

7.1**. Problem.** *(Asymptotic Stability of Associated Primes): Let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module. Is the set $\operatorname{Ass}\big(H^i_{R_+}(M)_n\big)$ asymptotically stable for $n \to -\infty$, e.g. is there some integer n_0 such that $\operatorname{Ass}\big(H^i_{R_+}(M)_n\big) = \operatorname{Ass}\big(H^i_{R_+}(M)_{n_0}\big)$ for all $n \leq 0$.* •

7.2**. Remark.** A) Let M be a finitely generated and graded R-module and let $i \in \mathbb{N}_0$. Then, according to Lemma 3.8 and the fact that $H^i_{R_+}(M)_n = 0$ for all $n \gg 0$ we can say:

> *If $\operatorname{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)$ is asymptotically stable for $n \to -\infty$, then the set $\operatorname{Ass}_R\big(H^i_{R_+}(M)\big)$ is finite.*

B) The above implication shows that the Problem of Asymptotic Stability of Associated Primes can be viewed as a refinement of the question, whether the R-module $H^i_{R_+}(M)$ has only finitely many associated primes. This latter question is a special case of the more general problem whether for an arbitrary finitely generated module M over an arbitrary Noetherian ring R and an arbitrary ideal $\mathfrak{a} \subseteq R$ the set $\operatorname{Ass}_R\big(H^i_{\mathfrak{a}}(M)\big)$ is finite. This *Finiteness Problem for Associated Primes (of Local Cohomology Modules)* has been posed originally in [23] and has been answered affirmatively in many special cases (cf [6], [12], [22], [24], [25], [31], [32]) and inspired interesting further investigations (cf [17], [18]). Nevertheless, as shown by Singh (cf [34], [35]), the formulated Finiteness Problem for Associated Primes cannot be answered affirmatively in general. •

We now briefly review some of the known counter-examples to illustrate the kind of "pathologies" that may occur.

7.3. **Example.** (cf [3], [5]) Let x, y, z, u, v, w be indeterminates and fix a prime number p. Let $R_0 := \mathbb{Z}[x, y, z]_{(p,x,y,z)}, \mathfrak{m}_0 := (p, x, y, z)R_0$ and $\mathfrak{p}_0 := (x, y, z)R_0$. Consider $S = R_0[u, v, w]$ as a homogeneous R_0-algebra and let $R := S/(xu + yv + zw)S$. Then, for each $n \leq -3$ we have

$$\mathrm{Ass}_{R_0}\left(H^3_{R_+}(R)_n\right) = \begin{cases} \{\mathfrak{p}_0, \mathfrak{m}_0\}, & \textit{if } p \,\big|\, \Pi_{i=1}^{-n-3} \begin{pmatrix} -n-2 \\ i \end{pmatrix}; \\ \{\mathfrak{p}_0\} & \textit{otherwise} . \end{cases}$$

B) As $p|\begin{pmatrix} pm \\ 1 \end{pmatrix}$ for all $m \in \mathbb{Z}$ and $p \not| \begin{pmatrix} p^k - 1 \\ i \end{pmatrix}$ for all $k \in \mathbb{N}$ and all $i \in \{1, \cdots, p^k - 2\}$, the two sets

$$\mathbb{A} :=\{n \leq -3 \,\big|\, \mathrm{Ass}_{R_0}\left(H^3_{R_+}(R)_n\right) = \{\mathfrak{p}_0, \mathfrak{m}_0\}\} \text{ and}$$
$$\mathbb{B} :=\{n \leq -3 \,\big|\, \mathrm{Ass}_{R_0}\left(H^3_{R_+}(R)_n\right) = \{\mathfrak{p}_0\}\}$$

are both infinite. This shows in particular that the set $\mathrm{Ass}\left(H^3_{R_+}(R)_n\right)$ is not asymptotically stable for $n \to -\infty$.

Moreover, by Lemma 3.8 we have

$$\mathrm{Ass}_R\left(H^3_{R_+}(R)\right) = \{\mathfrak{p}_0 + R_+, \mathfrak{m}_0 + R_+\}$$

and this shows that the converse of the implication given in Remark 7.2 A) does not hold in general. •

7.4. **Example.** A) (cf [3], [26]) Let x, y, s, t, u, v be indeterminates, let k be a field and set $R_0 := k[x, y, s, t]_{(x,y,s,t)}$. Consider $S := R_0[u, v]$ as a homogeneous R_0-algebra and set $R := S/(sx^2v^2 - (t + s)xyuv + ty^2u^2)S$. Then, the set

$$\bigcup_{n \leq 0} \mathrm{Ass}_{R_0}\left(H^2_{R_+}(R)_n\right) \textit{ is infinite} .$$

B) Clearly, the set $\mathrm{Ass}\left(H^2_{R_+}(R)_n\right)$ is not asymptotically stable for $n \to -\infty$ and according to Lemma 3.8 the set $\mathrm{Ass}_R\left(H^2_{R_+}(R)\right)$ is infinite. •

7.5. **Example.** A) (cf [36, Remark 4.2]) Let x, y, z, u, v be indeterminates and let k be a field. Let $R_0 := k[x, y, z]$, consider the homogeneous R_0-algebra $S = R_0[u, v]$ and set $R := S/(y^2u^2 + xyzuv + z^2v^2)S$. Then, the set $\mathrm{Ass}_R(H^2_{R_+}(R))$ is infinite. So by Lemma 3.8

$$\bigcup_{n\leq 0} \mathrm{Ass}_{R_0}\left(H^2_{R_+}(R)_n\right) \textit{ is infinite.}$$

B) Here again, the set $\mathrm{Ass}\left(H^2_{R_+}(M)_n\right)$ is not asymptotically stable for $n \to -\infty$ and the set $\mathrm{Ass}_R\left(H^2_{R_+}(M)\right)$ is infinite. •

7.6. **Remark.** A) Observe that in Examples 7.3 and 7.4 the base ring R_0 is local regular of dimension 4. In Example 7.5 the ring R_0 is regular non-local and of dimension 3. In all three examples R is a complete intersection ring and hence Cohen-Macaulay. In Examples 7.3 and 7.5 the ring R is a domain. In Example 7.4 R is reduced with two minimal primes. So, the somehow unexpected pathologies occur in surprisingly simple cases.

B) Examples 7.3 and 7.4 show in particular that over a regular local base ring of dimension 4 asymptotic stability of associated primes may fail in two different ways. In particular in Example 7.3 the behaviour of the set $\mathrm{Ass}\left(H^3_{R_+}(R)_n\right)$ is not governed by a periodic or "polynomial" pattern, but rather by a self-similar pattern. This is in accordance with the fact that in fairly simple situations standard numerical invariants of the components $H^i_{R_+}(R)_n$ need not behave (anti-)polynomially (cf [11], [27]). •

Although asymptotic stability of associated primes does not hold in general, there are many cases in which it does hold. In the following theorem we review the most important of theses cases. We use the notation $f(M)$ and $g(M)$ introduced in Definition 4.6.

7.7. **Theorem.** *Let $M \neq 0$ be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Then, the set Ass$\left(H^i_{R_+}(M)_n\right)$ is asymptotically stable for $n \to -\infty$ in the following cases:*

Case	$\dim(R_0)$	Conditions on R_0	Conditions on i
a)	0		
b)	1	semilocal	
c)	1	finite integral extension of a domain	
d)	1	essentially of finite type over a field	
e)	2	semilocal and finite integral extension of a domain	
f)	2	semilocal and essentially of finite type over a field	
g)			$i \leq f(M)$
h)		semilocal	$i \leq g(M)$

Proof and Comments: "a)": By the flat base change property of local cohomology one has only to consider the case where the Artinian base ring R_0 is local. Now Theorem 4.5 a) allows us to conclude.

"b)": See [3, Theorem (3.5) e)].

"c), d)": See [2, Corollary (3.10) b)].

"e), f)": See [2, Corollary (4.8)].

"g)": See [4, Proposition 5.6] for the case $i = f(M)$ and observe that the claim is obvious if $i < f(M)$ (cf Remark 4.7).

"h)": As R_0 is semilocal the flat base change property of local cohomology allows us to reduce to the case where the base ring R_0 is local.

Now, if $i < g(M)$, by [11, Theorem (3.6) a)] there is a polynomial $P \in \mathbb{Q}[x]$ such that

$$\text{length}_{R_0}\left(H^i_{R_+}(M)_n\right) = P(n) \text{ for all } n \ll 0,$$

and this gives the requested asymptotic stability of the set $\text{Ass}\left(H^i_{R_+}(M)_n\right)$ for $n \to -\infty$. If $i = g(M)$ we conclude by [11, Theorem (4.10) a)]. ■

If the set $\operatorname{Ass}\left(H^i_{R_+}(M)_n\right)$ takes a stable value $\mathcal{P}_0$ for $n \to -\infty$, it is natural to ask whether $\mathcal{P}_0$ may be expressed "by local cohomological data of M on $\operatorname{Proj}(M)$". In the cases g) and h) this is indeed possible under mild conditions on R_0. We make this more precise in the following reminder.

7.8**. Remark.** A) Let M be a finitely generated graded R-module and let $\mathfrak{p} \in \operatorname{Proj}(R)$. We define the ($R_+$-) *adjusted depth of M at $\mathfrak{p}$* by

$$\operatorname{adj}\operatorname{depth}_{\mathfrak{p}}(M) := \operatorname{depth}_{R_{\mathfrak{p}}}(M) + \operatorname{height}\left((R_+ + \mathfrak{p})/\mathfrak{p}\right)$$

(cf [13, Definition (9.2.2)]). If $X := \operatorname{Proj}(R)$ and $\mathcal{F}$ is the coherent sheaf of $\mathcal{O}_X$-modules induced by M and if $\pi : X \to X_0 := \operatorname{Spec}(R_0)$ is the natural morphism, then

$$\operatorname{adj}\operatorname{depth}_{\mathfrak{p}}(M) = \operatorname{depth}_{\mathcal{O}_{X,\mathfrak{p}}}(\mathcal{F}_{\mathfrak{p}}) + \dim\left(\overline{\{\mathfrak{p}\}} \cap \pi^{-1}(\pi(\mathfrak{p}))\right).$$

B) For $i \in \mathbb{N}_0$, we now set

$$Q^i(M) := \{\mathfrak{p} \cap R_0 \,\big|\, \mathfrak{p} \in \operatorname{Proj}(R) \text{ and } \operatorname{adj}\operatorname{depth}_{\mathfrak{p}}(M) = i\}.$$

Assume in addition that R_0 is a homomorphic image of a regular ring and that $f := f(M) < \infty$. Then, according to [5, Theorem (1.8)] we have

$$\operatorname{Ass}_{R_0}\left(H^f_{R_+}(M)_n\right) = Q^f(M) \text{ for all } n \ll 0.$$

C) Assume, that R_0 is as in part B) and in addition local and that $g := g(M) < \infty$. Then, according to [11, Theorem (4.10) c)] and by the flat base change property of local cohomology we have

$$\operatorname{Ass}_{R_0}\left(H^g_{R_+}(M)_n\right) \setminus \operatorname{Max}(R_0) = Q^g(M) \setminus \operatorname{Max}(R_0) \text{ for all } n \ll 0.$$

•

In [2, Theorem 3.7] it is shown that "Asymptotic Stability of Associated Primes" holds "in codimension ≤ 1" if the base ring R_0 is essentially of finite type over a field. We now shall prove that under certain additional assumptions "Asymptotic Stability of Associated Primes holds in codimension ≤ 2".

7.9**. Proposition.** *Let $i \in \mathbb{N}_0$. Assume that R_0 is a domain which is essentially of finite type over a field. Let M be a finitely generated graded R-module which is torsion-free over R_0. Assume that*

$$(*) \qquad \operatorname{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 1} \subseteq \{0\} \text{ for all } n \ll 0.$$

Then, the set $\operatorname{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 2}$ *is asymptotically stable for $n \to -\infty$.*

Proof: According to [2, Theorem (3.7)] the set $\mathrm{Ass}\left(H^i_{R_+}(M)\right)^{\leq 1}$ is asymptotically stable for $n \to -\infty$, so that either $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 1} = \emptyset$ for all $n \ll 0$ or $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 1} = \{0\}$ for all $n \ll 0$.

Assume first that $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 1} = \emptyset$ for all $n \ll 0$. Then, $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 2} = \mathrm{Supp}\left(H^i_{R_+}(M)_n\right)^{\leq 2}$ for all $n \ll 0$ and Proposition 6.8 gives our claim.

So, we may assume that there is an integer n_0 such that $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 1} = \{0\}$ for all $n \leq n_0$. According to Remark 6.4 C) there is some $s \in R_0 \setminus \{0\}$ such that $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right) \setminus \{0\} \subseteq \mathrm{Var}(sR_0)$ for all $i \in \mathbb{Z}$.

Now, let $n \leq n_0$ and let $\mathfrak{p}_0 \in \mathrm{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 2} \setminus \{0\}$. Then $s \in \mathfrak{p}_0$ and

$$\mathrm{Ass}_{R_{0_{\mathfrak{p}_0}}}\left((H^i_{R_+}(M)_n)_{\mathfrak{p}_0}\right) = \{0, \mathfrak{p}_0 R_{0_{\mathfrak{p}_0}}\}$$

so that $\Gamma_{\mathfrak{p}_0 R_{0_{\mathfrak{p}_0}}}\left((H^i_{R_+}(M)_n)_{\mathfrak{p}_0}\right) \neq 0$ and $s \in \mathfrak{p}_0 R_{0_{\mathfrak{p}_0}}$ is a non-zero divisor with respect to the $R_{0_{\mathfrak{p}_0}}$-module $\left(H^i_{R_+}(M)_n\right)_{\mathfrak{p}_0} / \Gamma_{\mathfrak{p}_0 R_{0_{\mathfrak{p}_0}}}\left((H^i_{R_+}(M)_n)_{\mathfrak{p}_0}\right)$. Therefore (cf [2, Lemma (4.4)]),

$$\Gamma_{\mathfrak{p}_0 R_{\mathfrak{p}_0}}\left((H^i_{R_+}(M)_n)_{\mathfrak{p}_0}\right) \nsubseteq s\left(H^i_{R_+}(M)_n\right)_{\mathfrak{p}_0}$$

hence $\Gamma_{\mathfrak{p}_0}\left(H^i_{R_+}(M)_n\right) \nsubseteq sH^i_{R_+}(M)_n$, so that $\mathfrak{p}_0 \in \mathrm{Ass}\left(H^i_{R_+}(M)_n / sH^i_{R_+}(M)_n\right)$. In view of the natural monomorphism $H^i_{R_+}(M)_n / sH^i_{R_+}(M)_n \rightarrowtail H^i_{R_+}(M/sM)_n$ we get $\mathfrak{p}_0 \in \mathrm{Ass}\left(H^i_{R_+}(M/sM)_n\right)$. As $\mathrm{height}(\mathfrak{p}_0/sR_0) = 1$ and $sH^i_{R_+}(M/sM)_n = 0$ we conclude that $\mathfrak{p}_0/sR_0 \in \mathrm{Ass}_{R_0/sR_0}\left(H^i_{R_+}(M/sM)_n\right)^{\leq 1}$.

According to [2, Theorem (3.7)] the set $\mathrm{Ass}_{R_0/sR_0}\left(H^i_{R_+}(M/sM)_n\right)^{\leq 1}$ is asymptotically stable for $n \to -\infty$, so that $\bigcup_{n \leq n_0} \mathrm{Ass}_{R_0/sR_0}\left(H^i_{R_+}(M/sM)_n\right)^{\leq 1}$ is a finite set. As a consequence, the union $\bigcup_{n \leq n_0} \mathrm{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 2} =: \mathcal{S}$ is finite.

Moreover, for each $\mathfrak{p}_0 \in \mathcal{S}$ the local ring $R_{0_{\mathfrak{p}_0}}$ is of dimension ≤ 2 and essentially of finite type over a field. Therefore, the set $\mathrm{Ass}_{R_{0_{\mathfrak{p}_0}}}\left(H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n\right)$ is asymptotically stable for $n \to -\infty$ (cf [2, Corollary (4.8)] for all $\mathfrak{p}_0 \in \mathcal{S}$. So, by the local flat base change property, the set $\mathrm{Ass}_{R_{0_{\mathfrak{p}_0}}}\left((H^i_{R_+}(M)_n)_{\mathfrak{p}_0}\right)$ is asymptotically stable for $n \to -\infty$ whenever $\mathfrak{p}_0 \in \mathcal{S}$. As $\mathcal{S}$ is finite, it follows that $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 2}$ is asymptotically stable for $n \to -\infty$. ■

7.10. **Corollary.** *Let $i \in \mathbb{N}_0$. Assume that R_0 is a domain of dimension ≤ 2 and essentially of finite type over a field. Let M be a finitely generated graded R-module which is torsion-free over R_0. Assume that*

$$(*) \qquad \mathrm{Ass}\left(H^i_{R_+}(M)_n\right) \subseteq \{0\} \cup \mathrm{Max}(R_0) \text{ for all } n \ll 0.$$

Then, the set $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right)$ *is asymptotically stable for $n \to -\infty$.*

Proof: As the set $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right)^{\leq 1}$ is asymptotically stable (cf [2, Theorem (3.7)], the set

$$\mathcal{P}_0 := \mathrm{Max}(R_0)^{\leq 1} \cap \bigcup_{n\in\mathbb{Z}} \mathrm{Ass}\left(H^i_{R_+}(M)_n\right)$$

is finite and it suffices to show that the set $\mathcal{S}_n := \mathrm{Ass}\left(H^i_{R_+}(M)_n\right) \backslash \mathcal{P}_0$ is asymptotically stable for $n \to -\infty$. If $\dim(R_0) \leq 1$, this is obvious. So, let $\dim(R_0) = 2$. Then $\bigcap_{\mathfrak{m}_0 \in \mathcal{P}_0} \mathfrak{m}_0 \subseteq R_0$ is a non-zero ideal and thus generated by finitely many elements $t_1, \cdots, t_r \in R_0 \backslash \{0\}$.

By our hypothesis $(*)$ we have $\mathcal{S}_n^{\leq 1} \subseteq \{0\}$ for all $n \ll 0$ and hence by the graded flat base change property of local cohomology there is a $n_0 \in \mathbb{Z}$ such that

$$\mathrm{Ass}_{(R_0)_{t_j}}\left(H^i_{(R_{t_j})_+}(M_{t_j})_n\right)^{\leq 1} \subseteq \{0\} \text{ for all } j \in \{1, \cdots, r\} \text{ and all } n \leq n_0.$$

By Proposition 7.9 the sets $\mathrm{Ass}_{(R_0)_{t_j}}\left(H^i_{(R_{t_j})_+}(M_{t_j})_n\right)$ become asymptotically stable for $n \to -\infty$. Another use of the graded flat base change property now yields that the set $\mathcal{S}_n$ is asymptotically stable for $n \to -\infty$. ■

7.11. **Comments and Problems.** A) We would be rather surprised to learn that the Examples 7.3 and 7.5 do not present the simplest situation in which asymptotic stability of associated primes does not hold. So, we conjecture that the following two questions have affirmative answers:

> *Let R_0 be regular local and of dimension ≤ 3 and assume that the finitely generated graded R-module M is R_0-torsion-free. Let $i \in \mathbb{N}_0$. Is the set* $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right)$ *asymptotically stable for $n \to -\infty$?*
>
> *Let R_0 be a domain of dimension ≤ 2, which is essentially of finite type over a field and assume that the finitely generated graded R-module M is R_0-torsion-free. Let $i \in \mathbb{N}_0$. Is the set* $\mathrm{Ass}\left(H^i_{R_+}(M)_n\right)$ *asymptotically stable for $n \to -\infty$?*

B) An affirmative answer to the second of these questions would just mean that in Corollary 7.10 the hypotheses $(*)$ may be dropped. We believe that the hypothesis $(*)$ may be dropped also in Proposition 7.9.

C) Finally we would not be too surprised to learn that the two questions of part A) have an affirmative answer if the condition of R_0-torsion-freeness is dropped. •

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