A DIAGONAL BOUND FOR COHOMOLOGICAL POSTULATION NUMBERS OF PROJECTIVE SCHEMES

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Abstract. Let X be a projective scheme over a field K and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. We show that the cohomological postulation numbers $\nu^i_{\mathcal{F}}$ of $\mathcal{F}$, e.g. the ultimate places at which the cohomological Hilbert functions $n \mapsto \dim_K \left(H^i\left(X, \mathcal{F}(n)\right)\right) =: h^i_{\mathcal{F}}(n)$ start to be polynomial for $n \ll 0$, are (polynomially) bounded in terms of the cohomology diagonal $\left(h^i_{\mathcal{F}}(-i)\right)_{i=0}^{\dim(\mathcal{F})}$ of $\mathcal{F}$. As a consequence we obtain that there are only finitely many different cohomological Hilbert functions $h^i_{\mathcal{F}}$ if $\mathcal{F}$ runs through all coherent sheaves of $\mathcal{O}_X$-modules with fixed cohomology diagonal. In order to prove these results we extend the regularity bound of Bayer-Mumford [1] from graded ideals to graded modules. Moreover we prove that the Castelnuovo-Mumford regularity of the dual $\mathcal{F}^\vee$ of a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$-modules $\mathcal{F}$ is (polynomially) bounded in terms of the cohomology diagonal of $\mathcal{F}$.

1. Introduction

Let X be a projective scheme over a field K with twisting sheaf $\mathcal{O}_X(1)$ and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. For $i \in \mathbb{N}_0$, the i-th *cohomological Hilbert function of (X with respect to) $\mathcal{F}$* is defined as the function

$$h^i_{\mathcal{F}} : \mathbb{Z} \to \mathbb{N}_0,\ n \mapsto h^i_{\mathcal{F}}(n) := \dim_K \left(H^i(X, \mathcal{F}(n))\right), \tag{1.1}$$

where $H^i(X, \mathcal{F}(n))$ denotes the i-th cohomology group of X with coefficients in the n-th twist $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes n}$ of $\mathcal{F}$.

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It is well known that the function $n \mapsto h^i_{\mathcal{F}}(n)$ is polynomial for all $n \ll 0$. The corresponding polynomial

$$p^i_{\mathcal{F}} \in \mathbb{Q}[\mathbf{x}] \text{ with } p^i_{\mathcal{F}}(n) = h^i_{\mathcal{F}}(n), \quad \forall n \ll 0 \tag{1.2}$$

is called the i-th *cohomological Hilbert polynomial of (X with respect to)* $\mathcal{F}$ and is of degree $\leq i$ (cf [5, 20.4.14]).
Now, for each $i \in \mathbb{N}_0$, we may define the i-th *cohomological postulation number (of* X *with respect to)* $\mathcal{F}$ by

$$\nu^i_{\mathcal{F}} := \inf\{n \in \mathbb{Z} \mid h^i_{\mathcal{F}}(n) \neq p^i_{\mathcal{F}}(n)\} - 1 \tag{1.3}$$

with the usual convention that $\inf \emptyset = \infty$. The basic aim of the present paper is to establish the following bounding result (cf Theorem 4.6)

> *For each $r \in \mathbb{N}_0$ there is a polynomial $M_r \in \mathbb{Q}[\mathbf{u}_0, \ldots, \mathbf{u}_r]$ such that for each $i \in \mathbb{N}_0$, each field K, each projective scheme X over K and each coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ with* $\dim(\mathcal{F}) \leq r$ *we have*
>
> $$\nu^i_{\mathcal{F}} \geq M_r(h^0_{\mathcal{F}}(0), \ldots, h^r_{\mathcal{F}}(-r)).$$

This extends a result of Matteotti [7], who has calculated bounding functions for the numbers $\nu^i_{\mathcal{F}}$ in terms of the "cohomology diagonal $(h^j_{\mathcal{F}}(-j))_{j \leq i}$ at and below level i" and in terms of the corresponding cohomological Hilbert polynomials $(p^j_{\mathcal{F}})_{j \leq i}$. So, what we shall prove is that one may bound the numbers $\nu^i_{\mathcal{F}}$ without knowing the polynomials $p^i_{\mathcal{F}}$, only in terms of the "full" cohomology diagonal.
As a consequence of the mentioned bounding result we shall prove the following finiteness result (cf Theorem 5.4)

> *Let $r \in \mathbb{N}_0$ and let $h_0, \ldots, h_r \in \mathbb{N}_0$. Let $(X, \mathcal{F})$ run through all pairs in which X is a projective scheme over some field and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules such that*
>
> $$\dim \mathcal{F} \leq r \text{ and } h^j_{\mathcal{F}}(-j) \leq h_j \text{ for } j = 0, \ldots, r.$$
>
> *Then only finitely many different cohomological Hilbert functions $h^i_{\mathcal{F}}$ may occur.*

Keep the previous notation and hypothesis and let $k \in \mathbb{N}_0$. Then, the *Calstelnuovo-Mumford regularity of $\mathcal{F}$ above level k* is defined by (cf [4, 1.11])

$$reg_k(\mathcal{F}) := \inf\{t \in \mathbb{Z} \mid h^i_{\mathcal{F}}(n - i) = 0, \quad \forall n \geq t, \ \forall i > k\}, \tag{1.4}$$

so that

$$reg_0(\mathcal{F}) =: reg(\mathcal{F}) \tag{1.5}$$

is the usual *Castelnuovo-Mumford regularity* of $\mathcal{F}$.

One of the main steps towards the bounding result mentioned above is to show that the regularity of the dual $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^r_K}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^r_K})$ of a coherent sheaf $\mathcal{F}$ over a projective space $\mathbb{P}^r_K$ is bounded in terms of the cohomology diagonal of $\mathcal{F}$. More precisely (cf Theorem 3.8)

> *For each $r \in \mathbb{N}_0$ there is a polynomial $L_r \in \mathbb{Q}[\mathbf{u}_0, \ldots, \mathbf{u}_r]$ such that for each field K and each coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$-modules $\mathcal{F}$ we have*
>
> $$reg(\mathcal{F}^\vee) \leq L_r\left(h^0_{\mathcal{F}}(0), \ldots, h^r_{\mathcal{F}}(-r)\right).$$

One important ingredient in order to prove the preceeding result is the following bounding result which relates the Castelnuovo-Mumford regularity $reg(M)$ and the generating degree $d(M)$ (s. 2.2 B) resp. 2.1 C) for the definitions) of a graded submodule M of a graded free module over a polynomial ring (cf Theorem 2.6).

> *For each $r \in \mathbb{N}_0$ there is a polynomial $F_r \in \mathbb{Q}[\mathbf{s}, \mathbf{t}]$ such that for each $s \in \mathbb{N}$, for each field K, for each polynomial ring $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \ldots, \mathbf{x}_r]$ and for each graded submodule $M \subseteq K[\underline{\mathbf{x}}]^{\oplus s}$ we have*
>
> $$reg(M) \leq F_r(s, d(M)).$$

In fact, polynomial regularity bounds of the above type may be deduced by classical results on syzygies, (cf [3, Sec. 4]). We include a proof of the above bounding result mainly because of our choice of the bounding polynomial F_r: namely, if $s = 1$ our bound coincides with the regularity bound of Bayer-Mumford (s. [1, 2.3]). The classical syzygetic method furnishes much weaker bounds.

The results mentioned above, partly are modified versions of the main results of the thesis of the second author [6] and have been announced without proofs in [3].

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2. A Regularity Bound of Bayer-Mumford Type

Let $K[\underline{\mathbf{x}}] := K[\mathbf{x}_0, \ldots, \mathbf{x}_r]$ be a polynomial ring over a field K. The principal aim of this section is to extend the regularity bound of Bayer-Mumford [1] for graded ideals $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ to graded submodules $M \subseteq K[\underline{\mathbf{x}}]^{\oplus s}$ for all $s \in \mathbb{N}$. We begin with some preliminaries on graded rings and modules.

2.1. **Definition and Remark.** A) Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring (so that $R = R_0[R_1]$) and let $R_+ := \bigoplus_{n>0} R_n$ denote the irrelevant ideal of R. If T is a graded R-module and $n \in \mathbb{Z}$, we denote by T_n the n-th homogeneous part of T, so that $T = \bigoplus_{n\in\mathbb{Z}} T_n$. Using this notation we define the *beginning* and the *end* of T resprectively by

$$beg(T) := \inf\{n \in \mathbb{Z} \mid T_n \neq 0\},$$
$$end(T) := \sup\{n \in \mathbb{Z} \mid T_n \neq 0\},$$

with the usual convention that inf and sup are formed in $\mathbb{Z} \cup \{\pm\infty\}$ with $\inf \emptyset = \infty,\ \sup \emptyset = -\infty$.

B) Let R and T be as above. For $m \in \mathbb{Z}$ we define the m-th *left-* resp. *right-truncation* of T as the R_0-submodules

$$T_{\geq m} := \bigoplus_{n\geq m} T_n\ , \quad T_{\leq m} := \bigoplus_{n\leq m} T_n.$$

Clearly, $T_{\geq m}$ is a graded submodule of T.

C) Let R and T be as above. The *generating degree* of T is defined by

$$d(T) := \inf \big\{ m \in \mathbb{Z} \,\big|\, T = (T_{\leq m})R \big\}.$$

D) Let R and T be as above. Let R'_0 be a noetherian R_0-algebra. Then $R' := R'_0 \underset{R_0}{\otimes} R = \bigoplus_{n\geq 0} R'_0 \underset{R_0}{\otimes} R_n$ carries a natural grading which turns it into a homogeneous noetherian ring with irrelevant ideal $R'_+ = R_+R'$. Moreover $T' = R' \underset{R}{\otimes} T = R'_0 \underset{R_0}{\otimes} T = \bigoplus_{n\in\mathbb{Z}} R'_0 \underset{R_0}{\otimes} T_n$ becomes a graded R'-module.
If R'_0 is faithfully flat over R_0, then R' is faithfully flat over R and moreover

$$beg(T') = beg(T),\ end(T') = end(T),\ d(T') = d(T). \qquad \bullet$$

2.2. **Reminder and Remark.** A) Let $R = \bigoplus_{n\geq 0} R_n$ be as in 2.1 and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded R-module. Let $i \in \mathbb{N}_0$. Then, the i-th *local cohomology module* $H^i_{R_+}(M)$ of M with respect to the irrelevant ideal $R_+ \subseteq R$ carries a natural grading (cf [5, Chap. 12]).

B) Let R and M be as in part A) and assume in addition, that M is finitely generated. Let $i \in \mathbb{N}_0$ and let $n \in \mathbb{Z}$. Then, the n-th homogeneous part $H^i_{R_+}(M)_n$ of $H^i_{R_+}(M)$ is a finitely generated R_0-module and vanishes if n is sufficiently large. This allows to define the

(*Castelnuovo-Mumford*) *regularity of* M *at and above level* k by

$$reg^k(M) := \sup\{end\left(H^i_{R_+}(M)\right) + i \,\big|\, i \geq k\} \in \mathbb{Z} \cup \{-\infty\}$$

for each $k \in \mathbb{Z}_0$ (cf [5, (15.2.9)]). Then

$$reg(M) := reg^0(M)$$

is the usual (*Castelnuovo-Mumford*) *regularity* of M.

C) Keep the above notations and hypotheses. Assume in addition that R'_0 is a faithfully flat R_0-algebra. Then $R'_0 \underset{R_0}{\otimes} M$ is a finitely generated graded module over $R'_0 \underset{R_0}{\otimes} R$ and the graded flat base change theorem gives rise to natural isomorphisms of R'_0-modules

$$H^i_{(R'_0 \underset{R_0}{\otimes} R)_+}(R'_0 \underset{R_0}{\otimes} M)_n \cong R'_0 \underset{R_0}{\otimes} H^i_{R_+}(M)_n$$

for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ (cf [5, (15.2.2)]).
In particular we have

$$reg^k(R'_0 \underset{R_0}{\otimes} M) = reg^k(M)$$

for all $k \in \mathbb{Z}_0$.

D) Assume now, that $R = K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \ldots, \mathbf{x}_r]$ is a polynomial ring over the field K. Let M be a finitely generated and graded R-module and let

$$0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to M \to 0$$

be a minimal graded free resolution of M. Then we have the following well known "*syzygetic characterization*" of regularity (cf [5, (15.3.7)])

$$reg(M) = \max\{d(F_j) - j \,\big|\, 0 \leq j \leq p\}. \qquad \bullet$$

2.3. **Reminder and Remark.** A) Let $R = \underset{n\geq 0}{\oplus} R_n$ be a homogeneous noetherian ring and let M be a finitely generated and graded R-module. A homogeneous element $f \in R_n$ of R is said to be (R_+-)*filter-regular with respect to* M, if it is a non-zero divisor with respect to the module $M/H^0_{R_+}(M)$.

B) It is easy to see that $f \in R_n$ is filter-regular with respect to M if and only if the annihilator $(0 \underset{M}{:} f)$ of f in M is contained in $H^0_{R_+}(M)$. So, $f \in R_n$ is filter-regular with respect to M iff $e := end(0 \underset{M}{:} f) < \infty$. Moreover, if this is the case, we have $e \leq end\left(H^0_{R_+}(M)\right)$. If in addition $n > 0$, we have $e = end\left(H^0_{R_+}(M)\right)$.

C) If R_0 is an infinite field and if $R_1 \neq 0$, then there is an element $f \in R_1 \setminus \{0\}$ which is filter-regular with respect to M (cf [5, 15.1.4]). •

2.4. **Lemma.** *Let $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \ldots, \mathbf{x}_r]$ be a polynomial ring over the field K, let U be a finitely generated and graded $K[\underline{\mathbf{x}}]$-module, let $m \in \mathbb{Z}$ and let $M, N \subseteq U$ be two graded submodules such that $d(M), d(N) \leq m$ and $reg(M+N) < m$. Then $d(M \cap N) \leq m$.*

Proof: There are graded epimorphisms $\pi : F \to M, \varrho : G \to N$ in which F and G are graded free $K[\underline{\mathbf{x}}]$-modules of finite rank with $d(F), d(G) \leq m$. In particular we have $reg(F \oplus G) \leq m$. So, the graded short exact sequence

$$0 \to Ker(\pi + \varrho) \to F \oplus G \xrightarrow{\pi+\varrho} M + N \to 0$$

yields $reg\,(Ker(\pi+\varrho)) \leq m$ (cf [5, (15.2.15) (i)]), thus $d\,(Ker(\pi+\varrho)) \leq m$. The commutative diagram

$$\begin{array}{ccc} M \oplus N & \xrightarrow{\sigma := id_M + id_N} & M + N \\ \uparrow{\scriptstyle \pi \oplus \varrho} & & \uparrow{\scriptstyle \pi + \varrho} \\ F \oplus G & = & F \oplus G \end{array}$$

shows that $(\pi \oplus \varrho)\,(Ker(\pi+\varrho)) = Ker(\sigma)$, hence $d\,(Ker(\sigma)) \leq m$. In view of the graded isomorphism $M \cap N \cong Ker(\sigma)$ we get our claim. ■

2.5. **Definition and Remark.** A) We define a sequence of polynomials $(F_r)_{r \in \mathbb{N}_0} \subseteq \mathbb{Q}[\mathbf{s}, \mathbf{t}]$ as follows:

$$F_0(\mathbf{s}, \mathbf{t}) := \mathbf{t};$$

$$F_r(\mathbf{s}, \mathbf{t}) := F_{r-1}(\mathbf{s}, \mathbf{t}) + \mathbf{s} \binom{F_{r-1}(\mathbf{s}, \mathbf{t}) + r}{r}, \quad \forall r \in \mathbb{N}.$$

We call F_r the r-th *Bayer-Mumford* polynomial.

B) We define a sequence of integers $(e_r)_{r \in \mathbb{N}_0}$ by

$$e_0 := 0; \; e_r := r\, e_{r-1} + 1, \quad \forall r \in \mathbb{N}.$$

It follows easily by induction, that

$$t \leq F_r(s,t) \leq F_r(s',t') \text{ if } 0 \leq s \leq s', \; 0 \leq t \leq t';$$

$$F_r(s,t) < s^{e_r}(2t)^{r!} \text{ if } s, t \in \mathbb{N}.$$

•

Now, we are ready to formulate and to prove the main result of this section.

2.6. **Theorem.** *Let* $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \cdots, \mathbf{x}_r]$ *be a polynomial ring over the field* K. *Let* $s, d \in \mathbb{N}$ *and let* $M \subseteq K[\underline{\mathbf{x}}]^{\oplus s}$ *be a non-zero graded submodule with* $d(M) \leq d$. *Then*

$$reg(M) \leq F_r(s, d).$$

Proof: In view of 2.1 D) and 2.2 C) we may replace K by any of its extension fields and thus assume that K is infinite.

We proceed by induction on r. If $r = 0, M$ is a graded free $K[\mathbf{x}_0]$-module of finite rank, so that $reg(M) \leq d = F_0(s, d)$, (cf 2.2 D)).

So, let $r > 0$. We write $R := K[\underline{\mathbf{x}}]$ and $T := R^{\oplus s}/M$. As $reg(R^{\oplus s}) = 0$ and in view of the graded short exact sequence $0 \to M \to R^{\oplus s} \to T \to 0$ it suffices to show that $reg(T) \leq F_r(s, d) - 1$ (cf [5, (15.2.15) (i)]).

By 2.3 C) there is an element $f \in R_1 \setminus \{0\}$ which is filter-regular with respect to T. After a linear change of coordinates we may assume that $f = \mathbf{x}_r$.

Now, let $\overline{R} := R/\mathbf{x}_r R = K[\mathbf{x}_0, \cdots, \mathbf{x}_{r-1}]$. Then, the graded $\overline{R}$-module $\overline{M} := (M + \mathbf{x}_r R^{\oplus s})/\mathbf{x}_r R^{\oplus s} \subseteq \overline{R}^{\oplus s}$ satisfies $d(\overline{M}) \leq d$. So, by induction we have $reg(\overline{M}) \leq F_{r-1}(s, d)$. By the base ring independence of local cohomology, this inequality remains valid if we consider $\overline{M}$ as an R-module. As $reg(\mathbf{x}_r R^{\oplus s}) = 1$, the graded short exact sequence $0 \to \mathbf{x}_r R^{\oplus s} \to (M + \mathbf{x}_r R^{\oplus s}) \to \overline{M} \to 0$ therefore gives (cf [5, (15.2.15) (iii)]

$$(1) \qquad reg(M + \mathbf{x}_r R^{\oplus s}) \leq F_{r-1}(s, d).$$

As $reg(R^{\oplus s}) = 0$, the graded short exact sequence $0 \to (M + \mathbf{x}_r R^{\oplus s}) \to R^{\oplus s} \to T/\mathbf{x}_r T \to 0$ gives $reg(T/\mathbf{x}_r T) \leq reg(M + \mathbf{x}_r R^{\oplus s}) - 1$ and hence

$$(2) \qquad reg(T/\mathbf{x}_r T) \leq F_{r-1}(s, d) - 1.$$

By [5, (18.3.11)] we also have $reg^1(T) \leq reg(T/\mathbf{x}_r T)$, so that $reg^1(T) \leq F_{r-1}(s, d) - 1$, hence $reg^1(T) \leq F_r(s, d) - 1$. It therefore remains to show that $end\left(H^0_{R_+}(T)\right) \leq F_r(s, d) - 1$.
Applying cohomology to the graded short exact sequence $0 \to T/(0 \underset{T}{:} \mathbf{x}_r) \xrightarrow{\mathbf{x}_r} T(1) \to (T/\mathbf{x}_r T)(1) \to 0$ we get exact sequences

$$0 \to H^0_{R_+}\left(T/(0 \underset{T}{:} \mathbf{x}_r)\right)_n \to H^0_{R_+}(T)_{n+1} \to H^0_{R_+}(T/\mathbf{x}_r T)_{n+1}.$$

In view of the inequality (2) we thus get isomorphims

$$(3) \qquad H^0_{R_+}\left(T/(0 \underset{T}{:} \mathbf{x}_r)\right)_n \cong H^0_{R_+}(T)_{n+1}, \quad \forall n \geq F_{r-1}(s, d) - 1.$$

If we apply cohomology to the graded short exact sequence $0 \to (0 \underset{T}{:} \mathbf{x}_r) \to T \to T/(0 \underset{T}{:} \mathbf{x}_r) \to 0$ and keep in mind that $(0 \underset{T}{:} \mathbf{x}_r) \subseteq H^0_{R_+}(T)$

(cf 2.3 B)) we get exact sequences $0 \to (0 \underset{T}{:} \mathbf{x}_r)_n \to H^0_{R_+}(T)_n \to H^0_{R_+}\Big(T/(0 \underset{T}{:} \mathbf{x}_r)\Big)_n \to 0$ for all $n \in \mathbb{Z}$. So, in view of the isomorphism (3) we obtain short exact sequences

$$(4) \quad 0 \to (0 \underset{T}{:} \mathbf{x}_r)_n \to H^0_{R_+}(T)_n \xrightarrow{\pi_n} H^0_{R_+}(T)_{n+1} \to 0, \ \forall n \geq F_{r-1}(s,d).$$

If we apply Lemma 2.4 to the submodules $M, \mathbf{x}_r R^{\oplus s} \subseteq R^{\oplus s}$ and keep in mind that $d(M), d(\mathbf{x}_r R^{\oplus s}) \leq d \leq F_{r-1}(s,d)$ and $reg(M + \mathbf{x}_r R^{\oplus s}) \leq F_{r-1}(s,d)$, (see (1)), we get $d(M \cap \mathbf{x}_r R^{\oplus s}) \leq F_{r-1}(s,d) + 1$. As $M \cap \mathbf{x}_r R^{\oplus s} = \mathbf{x}_r(M \underset{R^{\oplus s}}{:} \mathbf{x}_r)$, we obtain $d(M \underset{R^{\oplus s}}{:} \mathbf{x}_r) \leq F_{r-1}(s,d)$. But this means that $d(0 \underset{T}{:} \mathbf{x}_r) \leq F_{r-1}(s,d)$. So, if the epimorphism π_n in (4) is injective for some $n \geq F_{r-1}(s,d)$, the map π_m is an isomorphism for all $m \geq n$, hence $H^0_{R_+}(T)_m = 0$ for all $m \geq n$. So, in the range $n \geq F_{r-1}(s,d)$, the function $n \mapsto dim_K \left(H^0_{R_+}(T)_n\right)$ is strictly decreasing until it reaches the value 0. But this implies $end\left(H^0_{R_+}(T)\right) \leq F_{r-1}(s,d) + \dim_K \left(H^0_{R_+}(T)_{F_{r-1}(s,d)}\right) - 1$. As $\dim_K(T_{F_{r-1}(s,d)}) \leq \dim_K\left((R^{\oplus s})_{F_{r-1}(s,d)}\right) = s\binom{F_{r-1}(s,d)+r}{r}$ and $H^0_{R_+}(T)_{F_{r-1}(s,d)} \subseteq T_{F_{r-1}(s,d)}$ it follows

$$end\left(H^0_{R_+}(T)\right) \leq F_{r-1}(s,d) + s\binom{F_{r-1}(s,d)+r}{r} - 1 = F_r(s,d) - 1,$$

and this concludes our proof. ∎

2.7. **Corollary.** *Let $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \cdots, \mathbf{x}_r]$ be a polynomial ring over the field K, let $s \in \mathbb{N}$ and let $M \subseteq K[\underline{\mathbf{x}}]^{\oplus s}$ be a graded submodule. Then,*

$$reg(M) \leq s^{e_r} \left(2d(M)\right)^{r!}.$$

Proof: As $reg(0) = -\infty$, we may assume that $M \neq 0$. If $d(M) = 0$, by Nakayama, there is a graded isomorphism $M \cong K[\underline{\mathbf{x}}]^{\oplus u}$ with some $u \in \{1, \cdots, s\}$, so that $reg(M) = 0$. Therefore we may assume that $d(M) > 0$. Then, we conclude by Theorem 2.6 and the estimate at the end of 2.5 B). ∎

2.8. **Remark.** A) For $s = 1$, Corollary 2.7 gives the regularity bound of Bayer-Mumford [1].
C) In [6, (2.1)] it is shown that under the hypothesis of Corollary 2.7 one has $reg(M) \leq (2d(M))^{s^r r!}$. There, a different approach is used: First, the regularity criterion of Bayer-Stillman [2] is extended to graded submodules of free modules (cf [6, (1.10)]). Then, this extended criterion is used to prove the mentioned bound. Actually a slight modification of the proof of [6, (2.1)] gives the bound of Corollary 2.7. •

3. Regularity of Dual Sheaves

Let $r \in \mathbb{N}$, let K be a field and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$-modules. The aim of this section is to show that the regularity $reg(\mathcal{F}^\vee)$ of the dual sheaf $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^r_K}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^r_K})$ is bounded in terms of the full cohomology diagonal $(h^i_{\mathcal{F}}(-i))_{i=0}^r$ of $\mathcal{F}$ by a universal polynomial. We first give a few preliminaries.

3.1. **Reminder and Remark.** A) Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring and let $X = Proj(R)$ be the projective scheme defined by R. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules, and let M be a finitely generated graded R-module such that $\mathcal{F} = \widetilde{M}$, e.g. $\mathcal{F}$ is the sheaf of $\mathcal{O}_X$-modules induced by M. Then, the *Serre-Grothendieck correspondence* (cf [5, (20.4.4)] yields an exact sequence of graded R-modules

$$(1) \qquad 0 \to H^0_{R_+}(M) \to M \to \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{F}(n)) \to H^1_{R_+}(M) \to 0$$

and isomorphisms of graded R-modules

$$(2) \qquad H^{i+1}_{R_+}(M) \cong \bigoplus_{n\in\mathbb{Z}} H^i(X, \mathcal{F}(n)), \quad (\forall i \in \mathbb{N}).$$

For the regularity of $\mathcal{F}$ (cf (1.5)) we thus get

$$reg(\mathcal{F}) = reg^2(M).$$

B) Keep the previous notations and hypotheses and consider the graded R-module $T(\mathcal{F}) := \bigoplus_{n\geq 0} H^0(X, \mathcal{F}(n))$. Then, the exact sequence (1) of A) shows that $T(\mathcal{F})_n = M_n$ for all $n \gg 0$, so that $T(\mathcal{F})$ is finitely generated and $T(\mathcal{F})^\sim \cong \mathcal{F}$. Applying the sequence (1) with $T(\mathcal{F})$ instead of M we now see that

$$H^0_{R_+}(T(\mathcal{F})) = 0, \; end\left(H^1_{R_+}(T(\mathcal{F}))\right) < 0$$

and hence, if $\mathcal{F} \neq 0$:

$$\max\left\{0, reg(\mathcal{F})\right\} = reg(T(\mathcal{F})).$$

C) Keep the above notations and hypotheses and let $f \in R_1$. Then, f is filter-regular with respect to M if and only if it is a non-zero divisor with respect to $T(\mathcal{F})$ or - equivalently - if the homomorphism of sheaves $f : \mathcal{F} \to \mathcal{F}(1)$ is injective. In this situation, we also say that f is *regular with respect to* $\mathcal{F}$. If this is the case - with $Y := Proj(R/fR)$ - we have exact sequences

$$(3)\quad \begin{array}{c} H^i\left(X,\mathcal{F}(n)\right) \to H^i\left(X,\mathcal{F}(n+1)\right) \to H^i\left(Y,\mathcal{F}\restriction_Y(n+1)\right) \to \\ H^{i+1}\left(X,\mathcal{F}(n)\right) \to H^{i+1}\left(X,\mathcal{F}(n+1)\right) \to H^{i+1}\left(Y,\mathcal{F}\restriction_Y(n+1)\right) \end{array}$$

in which $\mathcal{F}\restriction_Y$ denotes the restriction of $\mathcal{F}$ to Y.

D) Keep the previous hypothesis and notations and assume in addition, that $R_0 = K$ is a field. Let K' be an extension field of K. Then $X' = Proj(K' \underset{K}{\otimes} R)$ is a projective scheme over $K', \mathcal{F}' := (K' \underset{K}{\otimes} M)^{\sim}$ is a coherent sheaf of $\mathcal{O}_{X'}$-modules and

$$H^i\left(X',\mathcal{F}'(n)\right) \cong K' \underset{K}{\otimes} H^i\left(X,\mathcal{F}(n)\right)$$

for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{N}$ (cf 2.2 C), 3.1 A)). So in the notation of (1.1) we have

$$h^i_{\mathcal{F}'}(n) = h^i_{\mathcal{F}}(n) \text{ for all } i \in \mathbb{N}_0 \text{ and all } n \in \mathbb{Z}.$$

E) Keep the above notation and hypotheses. Let $\bullet^* =^* Hom_R(\bullet, R)$ denote the functor of taking duals in the category of graded R-modules. Then $(M^*)^{\sim} \cong (M^{\sim})^{\vee}$ and hence $(T(\mathcal{F})^*)^{\sim} \cong \mathcal{F}^{\vee}$. •

3.2. **Definition and Remark.** A) For $r \in \mathbb{N}_0$ we introduce the polynomials $G_r \in \mathbb{Q}[\mathbf{u}_0, \cdots, \mathbf{u}_r, \mathbf{v}]$ defined by

$$\begin{aligned} G_0(\mathbf{u}_0; \mathbf{v}) :=& \mathbf{u}_0 \\ G_r(\mathbf{u}_0, \cdots, \mathbf{u}_r; v) :=& \mathbf{u}_0 + \sum_{w=1}^{v} G_{r-1}(\mathbf{u}_0 + \mathbf{u}_1, \cdots, \mathbf{u}_{r-1} + \mathbf{u}_r; w) \end{aligned}$$

for all $v \in \mathbb{N}_0$ and all $r > 0$.

B) Moreover, for each $r \in \mathbb{N}$ we consider the polynomial $H_r \in \mathbb{Q}[\mathbf{u}_1, \cdots, \mathbf{u}_r]$ defined by

$$H_r(\mathbf{u}_1, \cdots, \mathbf{u}_r) := \left(2 \sum_{j=1}^{r} \binom{r-1}{j-1} \mathbf{u}_j\right)^{2^{r-1}}.$$

Finally, let $H_0 := 0$.

C) Let $u_0, \cdots, u_r, v, u'_0, \cdots, u'_r, v' \in \mathbb{N}_0$ such that $u'_j \leq u'_j$ for all $j \in \{0, \cdots, r\}$ and $v \leq v'$. Then

$$\begin{aligned} 0 \leq G_r(u_0, \cdots, u_r, v) \leq G_r(u'_0, \cdots, u'_r, v'), \\ 0 \leq H_r(u_1, \cdots, u_r) \leq H_r(u'_1, \cdots, u'_r). \end{aligned}$$

•

3.3. **Lemma.** *Let $r \in \mathbb{N}_0$, let K be a field and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$-modules. Then:*

a) $h^0_{\mathcal{F}}(n) \leq G_r(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r); n)$ *for all* $n \geq 0$*;*

b) $reg(\mathcal{F}) \leq reg\,(T(\mathcal{F})) \leq H_r\,(h^1_{\mathcal{F}}(-1), \cdots, h^r_{\mathcal{F}}(-r))$.

Proof: For $r = 0$, both statements are obvious. So let $r > 0$.
"a)": In view of 3.1 D) we may assume that K is infinite. We write $\mathbb{P}^r_K = Proj(K[\underline{\mathbf{x}}])$, with a polynomial ring $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \cdots, \mathbf{x}_r]$ and choose a finitely generated graded $K[\underline{\mathbf{x}}]$-module M with $\tilde{M} = \mathcal{F}$. Then, there is an element $f \in K[\underline{\mathbf{x}}]_1 \backslash \{0\}$ which is filter-regular with respect to M, hence regular with respect to $\mathcal{F}$ (cf 2.3 C), 3.1 C)). We may assume that $f = \mathbf{x}_r$ and write $\mathbb{P}^{r-1}_K = Proj\,(K[\underline{\mathbf{x}}]/(\mathbf{x}_r))$ and $\mathcal{G} := \mathcal{F} \restriction_{\mathbb{P}^{r-1}_K}$. In view of the sequences (3) of 3.1 C) we thus get the inequalities

$$h^0_{\mathcal{F}}(n) \leq h^0_{\mathcal{F}}(0) + \sum_{m=1}^{n} h^0_{\mathcal{G}}(m), \qquad \text{for all } n \in \mathbb{N}_0;$$
$$h^i_{\mathcal{G}}(-i) \leq h^i_{\mathcal{F}}(-i) + h^{i+1}_{\mathcal{F}}\left(-(i+1)\right), \quad \text{for all } i \in \mathbb{N}_0.$$

By induction and in view of the monotony statement of 3.2 C), we now get

$$\begin{aligned} h^0_{\mathcal{F}}(n) &\leq h^0_{\mathcal{F}}(0) \\ &+ \sum_{m=1}^{n} G_{r-1}(h^0_{\mathcal{F}}(0) + h^1_{\mathcal{F}}(-1), \cdots, h^{r-1}_{\mathcal{F}}\left(-(r-1)\right) + h^r_{\mathcal{F}}(-r); m) \\ &= G_r\left(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r); n\right). \end{aligned}$$

"b)": See [4, Rem. 6] and observe 3.2 C). ∎

3.4. **Definition and Remark.** A) Let $R = \bigoplus_{n \geq 0} R_n$ be a noetherian, homogeneous ring such that $R_0 = K$ is a field. If T is a finitely generated and graded R-module, we use $\mu(T)$ to denote the minimal number of homogeneous elements needed to generate T.

B) Keep the notations and hypothesis of part A) and assume in addition that there is an element $f \in R_1$ which is a non-zero divisor with respect to T. Then

$$\mu(T) \leq dim_K(T_n) \text{ for all } n \geq d(T). \qquad \bullet$$

3.5. **Notation.** For $r \in \mathbb{N}_0$, let us introduce the polynomial

$$U_r := \binom{H_r(\mathbf{u}_1, \cdots, \mathbf{u}_r) + r + 1}{r} G_r\left(\mathbf{u}_0, \cdots, \mathbf{u}_r; H_r(\mathbf{u}_1, \cdots, \mathbf{u}_r)\right) \in \mathbb{Q}[\mathbf{u}_0, \cdots, \mathbf{u}_r],$$

where G_r and H_r are defined according to 3.2 A), B). •

3.6. **Lemma.** *Let $r \in \mathbb{N}_0$, let K be a field, let $\mathcal{F} \neq 0$ be a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$-modules and let $F_1 \to F_0 \xrightarrow{\pi} T(\mathcal{F}) \to 0$ be a minimal graded free presentation of the module $T(\mathcal{F})$ over the polynomial ring $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \cdots, \mathbf{x}_r]$. Then*

a) $rank(F_0) \leq G_r\Big(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r); H_r\left(h^1_{\mathcal{F}}(-1), \ldots, h^r_{\mathcal{F}}(-r)\right)\Big);$

b) $$rank(F_1) \leq U_r\left(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r)\right).$$

Proof: Let K' be an arbitrary extension field of K and consider the coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_{K'}}$- modules $\mathcal{F}' := \left(K' \underset{K}{\otimes} T(\mathcal{F})\right)^{\sim}$. Then the graded isomorphism $T(\mathcal{F}') \cong K' \underset{K}{\otimes} T(\mathcal{F})$ and the equalities $h^i_{\mathcal{F}'}(-i) = h^i_{\mathcal{F}}(-i)$ (cf 3.1 D)) allow to replace K and $\mathcal{F}$ by K' and $\mathcal{F}'$. So, we may assume that K is infinite. Thus, there is an element $f \in K[\underline{\mathbf{x}}]_1 \setminus \{0\}$ which is filter-regular with respect to $T(\mathcal{F})$. As $H^0_{K[\underline{\mathbf{x}}]_+}(T(\mathcal{F})) = 0$ (cf 3.1 B)), f is a non-zero divisor with respect to $T(\mathcal{F})$. As $0 \leq d\,(T(\mathcal{F})) \leq reg\,(T(\mathcal{F})) \leq H_r\,(h^1_{\mathcal{F}}(-1), \cdots, h^r_{\mathcal{F}}(-r))$ (cf 2.2 D), Lemma 3.3 b)) we conclude by 3.4 B) that $rank(F_0) = \mu\,(T(\mathcal{F})) \leq dim_K\Big(T(\mathcal{F})_{H_r(h^1_{\mathcal{F}}(-1),\cdots,h^r_{\mathcal{F}}(-r))}\Big)$. So, by Lemma 3.3 a) and the definition of $T(\mathcal{F})$ we obtain:

$$(4) \quad rank(F_0) \leq G_r\left(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r); H_r\left(h^1_{\mathcal{F}}(-1), \cdots, h^r_{\mathcal{F}}(-r)\right)\right).$$

This proves in particular statement a). In view of 2.2 D) and Lemma 3.3 b) we have $d\,(Ker(\pi)) = d(F_1) \leq reg\,(T(\mathcal{F})) + 1 \leq H + 1$, where

$$(5) \quad H := H_r\left(h^1_{\mathcal{F}}(-1), \cdots, h^r_{\mathcal{F}}(-r)\right).$$

As $Ker(\pi) \subseteq F_0$ is torsion free, 3.4 B) now gives $rank(F_1) = \mu\,(Ker(\pi)) \leq dim\,(Ker(\pi)_{H+1}) \leq dim_K\,((F_0)_{H+1})$. As $beg(F_0) = beg\,(T(\mathcal{F})) = 0$, we have $dim_K\,((F_0)_{H+1}) \leq rank(F_0)\binom{H+r+1}{r}$, thus $rank(F_1) \leq rank(F_0)\binom{H+r+1}{r}$. In view of (4) and (5) this proves statement b). ■

3.7. **Definition and Remark.** A) Let $r \in \mathbb{N}_0$. Using the notation of 2.5 A), 3.2 B) and 3.5 we define the polynomial

$$L_r := F_r\left(U_r(\mathbf{u}_0, \cdots, \mathbf{u}_r), H_r(\mathbf{u}_1, \cdots, \mathbf{u}_r) + 1\right) \in \mathbb{Q}[\mathbf{u}_0, \cdots, \mathbf{u}_r].$$

B) By the monotony statements of 3.2 C) and 2.5 B) it follows

$$0 \leq L_r(u_0, \cdots, u_r) \leq L_r(u'_0, \cdots, u'_r)$$

for all $u_0, \cdots, u_r, u'_0, \cdots, u'_r \in \mathbb{N}_0$ with $u_j \leq u'_j$ for $j = 0, \cdots, r$. •

Now, we are ready to formulate and to prove the main result of the present section.

3.8. **Theorem.** *Let $r \in \mathbb{N}_0$, let K be a field and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$-modules. Then, for the dual $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^r_K}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^r_K})$ of $\mathcal{F}$ we have*

$$reg(\mathcal{F}^\vee) \leq L_r\left(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r)\right).$$

Proof: We may assume that $\mathcal{F} \neq 0$. Let $\underline{h} := (h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r))$ and consider a minimal graded free presentation $F_1 \to F_0 \to T(\mathcal{F}) \to 0$ of the graded module $T(\mathcal{F})$ (cf 3.1 B)) over the polynomial ring $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \cdots, \mathbf{x}_r]$.
Applying the functor $\bullet^* =^* Hom_{K[\underline{\mathbf{x}}]}(\bullet, K[\underline{\mathbf{x}}])$ of graded duals to the above presentation, we get a graded exact sequence

$$0 \to T(\mathcal{F})^* \to F_0^* \to Q \to 0, \tag{6}$$

in which Q is a graded submodule of F_1^*. As $-beg(F_1^*) = d(F_1) \leq reg\,(T(\mathcal{F})) + 1 \leq H_r(\underline{h}) + 1$, (cf 2.2 D), Lemma 3.3 b)), we have a graded embedding $F_1^* \hookrightarrow K[\underline{\mathbf{x}}]^{\oplus rank(F_1)}\,(H_r(\underline{h}) + 1)$. By Lemma 3.6 we know that $rank(F_1) \leq U_r(\underline{h})$. So $Q\,(-H_r(\underline{h}) - 1)$ becomes a graded submodule of $K[\underline{\mathbf{x}}]^{\oplus U_r(\underline{h})}$. As $d(Q) \leq d(F_0^*) = -beg(F_0) = 0$, we have $d\,(Q(-H_r(\underline{h}) - 1)) \leq H_r(\underline{h}) + 1$. So, by Theorem 2.6 we obtain $reg(Q(-H_r(\underline{h}) - 1)) \leq F_r\left(U_r(\underline{h}), H_r(\underline{h}) + 1\right)$ and hence

$$reg(Q) \leq F_r(U_r(\underline{h}), H_r(\underline{h}) + 1) - H_r(\underline{h}) - 1 \leq L_r(\underline{h}) - 1.$$

As $reg(F_0^*) = d(F_0^*) = -beg(F_0) = 0$, the exact sequence (6) yields the estimate $reg\,(T(\mathcal{F}^*)) \leq \max\{0, reg(Q) + 1\} \leq L_r(\underline{h})$. As $(T(\mathcal{F})^*)^\sim = \mathcal{F}^\vee$, (cf 3.1 E)), this proves our claim in view of 3.1 B). ■

3.9. **Corollary.** *Keep the notations and hypotheses of* Theorem 3.8 *and let $h_0, \cdots, h_r \in \mathbb{N}_0$ be such that $h^i_{\mathcal{F}}(-i) \leq h_i$ for $i = 0, \cdots, r$. Then*

$$reg(\mathcal{F}^\vee) \leq reg\,(T(\mathcal{F}^\vee)) \leq L_r(h_0, \cdots, h_r).$$

Proof: As $reg\,(T(\mathcal{F}^\vee)) = \max\{0, reg(\mathcal{F}^\vee)\}$, (cf 3.1 B)) our statement follows from Theorem 3.8 and the inequalities of 3.7 B). ■

4. Bounding Cohomological Postulation Numbers

Let $r \in \mathbb{N}_0$ and let X be a projective scheme over the field K and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules such that $dim(\mathcal{F})\big(:= dim(Supp(\mathcal{F}))\big) \leq r$. We shall prove that the cohomological postulation numbers $\nu^i_{\mathcal{F}}$ (see (1.3)) are bounded in terms of the cohomology diagonal $(h^i_{\mathcal{F}}(-i))^r_{i=0}$ by a universal polynomial.
We begin with the following auxilary result, in which the notation introduced in 3.1 B) and 3.2 is used.

4.1. **Lemma.** *Let $r \in \mathbb{N}$, let K be a field, let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$-modules. Let F be a graded free module of finite rank over the polynomial ring $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \ldots, \mathbf{x}_r]$ and let $\pi : F \twoheadrightarrow T(\mathcal{F})$ be a minimal graded epimorphism. Let $\mathcal{G} := Ker(\pi)^\sim$ be the sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$-modules induced by $Ker(\pi)$. Then:*

a) $T(\mathcal{G}(1)) = T(\mathcal{G})(1) \cong Ker(\pi)(1)$;

b) $h^0_{\mathcal{G}(1)}(0) \leq r h^0_{\mathcal{F}}(0) + G_r(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r); 1)$;

c) $h^0_{\mathcal{G}(1)}(n) = 0$ *for all* $n < 0$;

d) if $r > 1$, *then* $h^1_{\mathcal{G}(1)}(n) = 0$ *for all* $n < 0$ *and moreover* $h^i_{\mathcal{G}(1)}(n) = h^{i-1}_{\mathcal{F}}(n+1)$ *for* $1 < i < r$ *and all* $n \in \mathbb{Z}$;

e) $h^r_{\mathcal{G}(1)}(-r) \leq h^{r-1}_{\mathcal{F}}(-(r-1)) + \binom{H+r}{r} G_r(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r); H)$,

where $H := H_r(h^1_{\mathcal{F}}(-1), \cdots, h^r_{\mathcal{F}}(-r))$.

Proof: Consider the exact sequence of sheaves of $\mathcal{O}_{\mathbb{P}^r_K}$-modules

$$0 \to \mathcal{G} \to \tilde{F} \xrightarrow{\tilde{\pi}} \mathcal{F} \to 0. \tag{1}$$

"a)": Applying cohomology to (1) and keeping in mind that $r > 0$ and $beg(F) \geq 0$, we get a commutative diagram of graded $K[\underline{\mathbf{x}}]$-modules with first exact row

$$\begin{array}{ccccc} 0 \longrightarrow T(\mathcal{G}) \longrightarrow & T(\tilde{F}) & \longrightarrow & T(\mathcal{F}) \\ & \cong \uparrow & & \| \\ & F & \xrightarrow{\pi} & T(\mathcal{F}) \end{array}$$

So, there is a graded isomorphism $T(\mathcal{G}) \cong Ker(\pi)$. As π is minimal, we have $Ker(\pi)_0 = 0$ and hence $T(\mathcal{G})_0 = 0$. It follows $T(\mathcal{G}(1)) = T(\mathcal{G})(1)_{\geq 0} = T(\mathcal{G})(1) \cong Ker(\pi)(1)$.

"b)": As usual we may assume that K is infinite. In view of statement a) we have $h^0_{\mathcal{G}(1)}(0) = dim_K(Ker(\pi)_1) \leq dim(F_1) = dim_K(K[\underline{\mathbf{x}}]_1 F_0) + dim_K(F_1/K[\underline{\mathbf{x}}]_1 F_0)$. As π is minimal, $dim_K(F_0) = h^0_{\mathcal{F}}(0)$, hence

$dim_K(K[\underline{\mathbf{x}}]_1 \, F_0) \leq (r+1)dim_K(F_0) = (r+1)h^0_{\mathcal{F}}(0)$. Also, by the minimality of π we have $dim_K(F_1/K[\underline{\mathbf{x}}]_1F_0) = dim_K(T(\mathcal{F})_1/K[\underline{\mathbf{x}}]_1T(\mathcal{F})_0) = h^0_{\mathcal{F}}(1) - dim_K(K[\underline{\mathbf{x}}]_1 \, H^0(X, \mathcal{F}))$. As K is infinite, there is an element $f \in K[\underline{\mathbf{x}}]_1$ which is filter-regular with respect to $T(\mathcal{F})$ (cf 2.3 C)) and the resulting monomorphism $f : H^0(X, \mathcal{F}) \rightarrowtail H^0(X, \mathcal{F}(1))$ (cf 3.1 C)) implies that $dim_K(K[\underline{\mathbf{x}}]_1H^0(X, \mathcal{F})) \geq h^0_{\mathcal{F}}(0)$.
Altogether we obtain $h^0_{\mathcal{G}(1)}(0) \leq rh^0_{\mathcal{F}}(0)+h^0_{\mathcal{F}}(1)$ and the stated inequality follows by Lemma 3.3 a).

"c)": Clear from statement a) which gives $h^0_{\mathcal{G}}(0) = 0$.

"d)": Observe that $h^0_{\mathcal{G}}(n) = 0$ for all $n \leq 0$ (by c)), that $h^0_{\tilde{F}}(0) = h^0_{\mathcal{F}}(0)$ by the minimality of π and that $h^j_{\tilde{F}}(m) = 0$ for all $i \in \{1, \cdots, r-1\}$ and all $m \in \mathbb{Z}$. Then apply cohomology to (1).

"e)": If we apply cohomology to (1) we obtain

$$h^r_{\mathcal{G}(1)}(-r) \leq h^{r-1}_{\mathcal{F}}(-(r-1)) + h^r_{\tilde{F}}(-r+1).$$

There is a graded exact sequence

$$0 \to K[\underline{\mathbf{x}}]^{\oplus rank(F)}(-d(F)) \to F \to N \to 0$$

with $\dim(N) \leq r$. Passing to induced sheaves and then to cohomology, we get $h^r_{\tilde{F}}(-r+1) \leq h^r_{\mathcal{O}_{\mathbb{P}^r_K}}(-d(F)-r+1)rank(F) = \binom{d(F)+r-2}{r}rank(F)$. As $rank(F) \leq G_r(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r), H)$ (cf Lemma 3.6 a)) and $d(F) = d(T(\mathcal{F})) \leq reg(T(\mathcal{F})) \leq H$ (cf 2.2 D) and Lemma 3.3 b)) we get our claim. ■

In our next Lemma, we use the notation of (1.3) and of 3.7.

4.2. **Lemma.** *Let $r \in \mathbb{N}_0$ let K be a field and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$-modules. Then $\nu^r_{\mathcal{F}} \geq -r-1-L_r(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r))$.*

Proof: Let $n \in \mathbb{Z}$. By duality we have $Hom_K(H^r(\mathbb{P}^r_K, \mathcal{F}(n)), K) \cong H^0(\mathbb{P}^r, \mathcal{F}^\vee(-n-r-1))$, hence $h^r_{\mathcal{F}}(n) = h^0_{\mathcal{F}^\vee}(-n-r-1)$. So, if $-n-r-1 \geq reg(\mathcal{F}^\vee)$, we have $h^r_{\mathcal{F}}(n) = \chi_{\mathcal{F}^\vee}(-n-r-1)$. As $n \mapsto \chi_{\mathcal{F}^\vee}(-n-r-1)$ is a polynomial function, it follows that $\nu^r_{\mathcal{F}} \geq -r-1-reg(\mathcal{F}^\vee)$ and our claim results from Theorem 3.8 . ■

4.3. **Lemma.** *Let K be a field, let $r \in \mathbb{N}_0$, let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$-modules and let $e := \sum_{x \in \mathbb{P}^r_K, x\, closed} length_{\mathcal{O}_{\mathbb{P}^r_K,x}}(H^0_{\mathfrak{m}_{\mathbb{P}^r_K,x}}(\mathcal{F}_x))$. Then,*

$$e \leq h^0_{\mathcal{F}}(n-1) \leq \max\{e, h^0_{\mathcal{F}}(n) - 1\},$$

for each $n \in \mathbb{Z}$.

Proof: Let $\mathcal{H} \subseteq \mathcal{F}$ be the (unique) maximal coherent subsheaf with finite support. Then $\mathcal{H}$ is of length e and $\mathcal{G} := \mathcal{F}/\mathcal{H}$ has no closed associated points. Therefore $h^0_{\mathcal{H}}(m) = e$ and $h^1_{\mathcal{H}}(m) = 0$, so that $h^0_{\mathcal{F}}(m) = e + h^0_{\mathcal{G}}(m)$ for all $m \in \mathbb{Z}$. As $h^0_{\mathcal{G}}(n-1) \leq \max\{0, h^0_{\mathcal{G}}(n) - 1\}$ (cf [4, 5.3]), we get our claim. ■

Now, in order to prove and to formulate the main result of this section, let us define one more class of bounding polynomials.

4.4. **Definition and Remark.** A) Let $r \in \mathbb{N}_0$. Then, for $p = 0, \cdots, r$ we define polynomials $M_{r,p} \in \mathbb{Q}[\mathbf{u}_0, \cdots, \mathbf{u}_r]$ by

$$M_{r,0} := -\,\mathbf{u}_0, \text{ and for } 1 \leq p \leq r$$
$$M_{r,p} := -\,r - L_r(\mathbf{u}_0, \cdots, \mathbf{u}_r) + M_{r,p-1}(V, 0, \mathbf{u}_1, \cdots, \mathbf{u}_{r-2}, W),$$

where

$$V := r\mathbf{u}_0 + G_r(\mathbf{u}_0, \cdots, \mathbf{u}_r; 1)$$
$$W := \mathbf{u}_{r-1} + \binom{H_r(\mathbf{u}_1, \cdots, \mathbf{u}_r) + r}{r} G_r(\mathbf{u}_0, \cdots, \mathbf{u}_r;\; H_r(\mathbf{u}_1, \cdots, \mathbf{u}_r)),$$

(and where G_r, H_r, L_r are defined as in 3.2 A), B) and 3.7 A) respectively). Finally, we set

$$M_r := M_{r,r}.$$

B) In view of the monotony statements of 3.2 C) and 3.7 B) it follows that

$$-u_0 \geq M_{r,p}(u_0, \cdots, u_r) \geq M_{r,p}(u'_0, \cdots, u'_r)$$

for all $u_0, \cdots, u_r, u'_0, \cdots, u'_r \in \mathbb{N}_0$ with $u_j \leq u'_j$ for $j = 0, \cdots, r$ and for all $p \in \{0, \cdots, r\}$. •

In the following lemma, $pd_R(M)$ is used to denote the projective dimension of an R-module M.

4.5. **Lemma.** *Let $r \in \mathbb{N}$, let $p \in \{0, \cdots, r\}$, let K be a field and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$-modules with $pd_{K[\underline{\mathbf{x}}]}(T(\mathcal{F})) \leq p$, where $K[\mathbf{x}_0, \cdots, \mathbf{x}_r] = K[\underline{\mathbf{x}}]$ is a polynomial ring. Then*

$$\nu^i_{\mathcal{F}} \geq M_{r,p}(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r)) \textit{ for } i = 0, \cdots, r.$$

Proof: (Induction on p) Let $p = 0$. Then $T(\mathcal{F})$ is a graded free module with $beg(T(\mathcal{F})) \geq 0$. As $\mathcal{F} = T(\mathcal{F})^{\sim}$ we have $h^i_{\mathcal{F}}(n) = 0$ for all $i \neq 0, r$ and all $n \in \mathbb{Z}$. But this implies that $\nu^i_{\mathcal{F}} \geq \nu^0_{\mathcal{F}}$ for all $i \in \mathbb{N}_0$. In view of Lemma 4.3 we have $\nu^0_{\mathcal{F}} \geq -h^0_{\mathcal{F}}(0) = M_{r,0}(h^0_{\mathcal{F}}(0), \cdots, h^r_{\mathcal{F}}(-r))$.

So, let $p > 0$, let $\pi : F \to T(\mathcal{F})$ be a minimal graded epimorphism from a graded free $K[\underline{\mathbf{x}}]$-module F and let $\mathcal{G} := Ker(\pi)^{\sim}$. By statement a) of Lemma 4.1 we have $pd_{K[\underline{\mathbf{x}}]}(T(\mathcal{G}(1))) < p$. So, by induction $\nu^i_{\mathcal{G}(1)} \geq M_{r,p-1}(h^0_{\mathcal{G}(1)}(0), \cdots, h^r_{\mathcal{G}(1)}(-r))$ for all $i \in \mathbb{N}_0$. By statements b), c), d) and e) of Lemma 4.1 and in the notation of 4.4 we thus get $\nu_{\mathcal{F}}^{i-1} = \nu_{\mathcal{G}(1)}^{i} \geq M$ for $i = 2, \cdots, r-1$, where

$$M := M_{r,p-1}(V(h_{\mathcal{F}}^0(0), \cdots, h_{\mathcal{F}}^r(-r)), 0, h_{\mathcal{F}}^1(-1), \cdots, h_{\mathcal{F}}^{r-2}(-(r-2)),\\ W(h_{\mathcal{F}}^0(0), \cdots, h_{\mathcal{F}}^r(-r))).$$

By Lemma 4.3, the definition of V and the lefthand side inequality of 4.4 B) we have $\nu_{\mathcal{F}}^0 \geq -h_{\mathcal{F}}^0(0) \geq -V(h_{\mathcal{F}}^0(0), \cdots, h_{\mathcal{F}}^r(-r)) \geq M$. So $\nu_{\mathcal{F}}^i \geq M_{r,p}(h_{\mathcal{F}}^0(0), \cdots, h_{\mathcal{F}}^r(-r))$ for $i = 0, \cdots, r-2$. By Lemma 4.2 we also have $\nu_{\mathcal{F}}^r \geq -r-1-L_r(h_{\mathcal{F}}^0(0), \cdots, h_{\mathcal{F}}^r(-r)) \geq -r - L_r(h_{\mathcal{F}}^0(0), \cdots, h_{\mathcal{F}}^r(-r)) + M = M_{r,p}(h_{\mathcal{F}}^0(0), \cdots, h_{\mathcal{F}}^r(-r))$. As the function $n \mapsto \sum_{i=0}^r (-1)^i h_{\mathcal{F}}^i(n) = \chi_{\mathcal{F}}(n)$ is polynomial, we get $\nu_{\mathcal{F}}^{r-1} \geq M_{r,p}(h_{\mathcal{F}}^0(0), \cdots, h_{\mathcal{F}}^r(-r))$. This concludes our proof. ■

4.6. **Theorem.** *Let $r \in \mathbb{N}_0$, let X be a projective scheme over the field K and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules such that* $\dim(\mathcal{F}) \leq r$. *Then $\nu_{\mathcal{F}}^i \geq M_r(h_{\mathcal{F}}^0(0), \cdots, h_{\mathcal{F}}^r(-r))$ for all $i \in \mathbb{N}_0$.*

Proof: As usual, we may assume that K is infinite. Let $\mathcal{J} \subseteq \mathcal{O}_X$ be the annihilator sheaf of $\mathcal{F}$ and let $Y \subseteq X$ be the closed subscheme defined by $\mathcal{J}$. Then $h_{\mathcal{F}}^i(n) = h_{\mathcal{F}\restriction Y}^i(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. This allows to replace X by Y and hence to assume that $\dim(X) = \dim(\mathcal{F}) \leq r$. As K is infinite we thus find a finite morphism $\varphi : X \to \mathbb{P}_K^r$ induced by global sections of $\mathcal{O}_X(1)$. It follows that $h_{\mathcal{F}}^i(n) = h_{\varphi_*\mathcal{F}}^i(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{N}$. This allows to replace $\mathcal{F}$ by $\varphi_*\mathcal{F}$ and hence to assume that $X = \mathbb{P}_K^n = Proj(K[\underline{\mathbf{x}}])$, where $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \cdots, \mathbf{x}_r]$ is a polynomial ring. As $H_{K[\underline{\mathbf{x}}]_+}^0(T(\mathcal{F})) = 0$ (cf 3.1 B)), we have $pd_{K[\underline{\mathbf{x}}]}(T(\mathcal{F})) \leq r$. So, Lemma 4.5 gives that $\nu_{\mathcal{F}}^i \geq M_{r,r}(h_{\mathcal{F}}^0(0), \cdots, h_{\mathcal{F}}^r(-r))$ for all $i \in \mathbb{N}_0$. As $M_r = M_{r,r}$, this proves our claim. ■

4.7. **Corollary.** *Keep the notations and hypotheses of Theorem 4.6 and let $h_0, \cdots, h_r \in \mathbb{N}_0$ be such that $h_{\mathcal{F}}^i(-i) \leq h_i$ for $i = 0, \cdots, r$. Then $\nu_{\mathcal{F}}^i \geq M_r(h_0, \cdots, h_r)$ for all $i \in \mathbb{N}_0$.*

Proof: Clear from Theorem 4.6 and the monotony property of 4.4 B). ■

5. A Finiteness Result

Again, let $r \in \mathbb{N}_0$, let X be a projective scheme over a field K and let $h_0, \cdots, h_r \in \mathbb{N}_0$. In this section we shall prove that there are only finitely many possible cohomological Hilbert functions $h_{\mathcal{F}}^i : \mathbb{Z} \to \mathbb{N}_0$ if $\mathcal{F}$ runs through all coherent sheaves of $\mathcal{O}_X$-modules with $\dim(\mathcal{F}) \leq r$ and satisfying $h_{\mathcal{F}}^i(-i) = h_i$ for $i = 0, \cdots, r$. So, what we prove is that the cohomology diagonal $(h^i_{\mathcal{F}}(-i))_{i=0}^{\dim \mathcal{F}}$ of a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules bounds the number of possible cohomological Hilbert functions $h^i_{\mathcal{F}}$ of $\mathcal{F}$. First, we prove the following result, in which the polynomials $H_k \in \mathbb{Q}[\mathbf{u}_1, \cdots, \mathbf{u}_r]$ and $G_k \in \mathbb{Q}[\mathbf{u}_0, \cdots, \mathbf{u}_k]$ are defined according to 3.2.

5.1. **Lemma.** *Let $r \in \mathbb{N}_0$, let K be a field, let X be a projective scheme over K. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules with* $\dim \mathcal{F} \leq r$. *Then:*

a) $h^i_{\mathcal{F}}(n) \leq \frac{1}{2} H_{r-i+1}(h^i_{\mathcal{F}}(-i), \cdots h^r_{\mathcal{F}}(-r))$ *for* $1 \leq i \leq r$ *and for all* $n \geq -i$.

b) $h^i_{\mathcal{F}}(n) \leq G_i(h^i_{\mathcal{F}}(-i), \cdots, h^0_{\mathcal{F}}(0); -n-i)$ *for* $0 \leq i \leq r$ *and for all* $n \leq -i$.

Proof: "a)" See [4, Rem. 6].

"b)" As usual, we may assume that K is infinite. Let $X = Proj(R)$, where $R = \underset{n \geq 0}{\oplus} R_n$ is a homogeneous noetherian ring with $R_0 = K$. Then, there is an element $f \in R_1$ which is filter-regular with respect to $T(\mathcal{F})$ and hence regular with respect to $\mathcal{F}$. The induced monomorphisms $H^0(X, \mathcal{F}(n)) \overset{f}{\rightarrowtail} H^0(H, \mathcal{F}(n+1))$ prove the case $i = 0$. So, let $i > 0$ and set $Y = Proj(R/fR)$ and $\mathcal{G} := \mathcal{F} \restriction_Y$. Then, $\dim(\mathcal{G}) < r$ and by induction we have $h^{i-1}_{\mathcal{G}}(m) \leq G_{i-1}\big(h^{i-1}_{\mathcal{G}}(-(i-1)), \cdots, h^0_{\mathcal{G}}(0); -m-i+1\big), (\forall m \leq -i+1)$. Moreover the sequences (3) of 3.1 C) show that $h^i_{\mathcal{F}}(n) \leq h^i_{\mathcal{F}}(-i) + \sum_{m=n+1}^{-i} h^{i-1}_{\mathcal{G}}(m)$ for all $n \leq -i$ and $h^j_{\mathcal{G}}(-j) \leq h^j_{\mathcal{F}}(-j) + h^{j+1}_{\mathcal{F}}(-(j+1))$ for all $j \leq i-1$. In view of the monotony property of G_{i-1} and the definition of G_i (cf 3.2), we get our claim. ■

5.2. **Notation.** Let $r \in \mathbb{N}_0$ and let $\underline{h} = (h_0, \cdots, h_r) \in \mathbb{N}_0^{r+1}$. By $\overline{\mathcal{C}}^{(r)}_{\leq \underline{h}}$ we denote the class of all pairs $(X, \mathcal{F})$ in which X is a projective scheme over some field K and in which $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules such that

$$\dim(\mathcal{F}) \leq r \text{ and } h^i_{\mathcal{F}}(-i) \leq h_i \text{ for } i = 0, \cdots, r. \qquad \bullet$$

Now, we have the following finiteness result for cohomological Hilbert polynomials and characteristic polynomials. (We use the symbol # to denote cardinality.)

5.3. **Proposition.** *Let $r \in \mathbb{N}_0$ and let $\underline{h} = (h_0, \cdots, h_r) \in \mathbb{N}_0^{r+1}$. Then*

a) $\forall i \in \{0, \cdots, r\} \ : \# \left\{ p^i_{\mathcal{F}} \,\middle|\, (X, \mathcal{F}) \in \overline{\mathcal{C}}^{(r)}_{\leq \underline{h}} \right\} < \infty.$

b) $\# \left\{ \chi_{\mathcal{F}} \,\middle|\, (X, \mathcal{F}) \in \overline{\mathcal{C}}^{(r)}_{\leq \underline{h}} \right\} < \infty.$

Proof: "a)": Fix $i \in \{0, \cdots, r\}$. Let $\mathcal{C} := \overline{\mathcal{C}}^{(r)}_{\leq \underline{h}}$. We define in the notation of 4.4 and 3.2

$$N_{r,i}(\underline{h}) := \prod_{n=M_r(\underline{h})-i}^{M_r(\underline{h})} \left(G_i(h_i, \cdots, h_0; -n-i) + 1\right).$$

By 4.7 we have $\nu^i_{\mathcal{F}} \geq M_r(\underline{h})$ whenever $(X, \mathcal{F}) \in \mathcal{C}$. So, by 5.1 b) and the monotony properties of G_i we obtain $0 \leq p^i_{\mathcal{F}}(n) \leq G_i(h_i, \cdots, h_0; -n - i)$ for all $(X, \mathcal{F}) \in \mathcal{C}$ and all $n \leq M_r(\underline{h})$. As $p^i_{\mathcal{F}}$ is a polynomial of degree $\leq i$, it is determined by the values $p^i_{\mathcal{F}}(n)$ with $M_r(\underline{h}) - i \leq n \leq M_r(\underline{h})$. So, at most $N_{r,i}(\underline{h})$ different cohomological Hilbert polynomials $p^i_{\mathcal{F}}$ occur, if $(X, \mathcal{F})$ runs through $\mathcal{C}$.
"b)": Follows from a) as $\chi_{\mathcal{F}} = \sum_{i=0}^r (-1)^i p^i_{\mathcal{F}}$ whenever $\mathcal{F}$ is a coherent sheaf of dimension $\leq r$ over a projective scheme over a field. ■

5.4. **Theorem.** *Let $0 \leq i \leq r$ and let $\underline{h} = (h_0, \cdots, h_r) \in \mathbb{N}_0^{r+1}$. Then*

$$\#\left\{h^i_{\mathcal{F}} \,\middle|\, (X, \mathcal{F}) \in \overline{\mathcal{C}}^{(r)}_{\leq \underline{h}}\right\} < \infty.$$

Proof: Let $\mathcal{C} := \overline{\mathcal{C}}^{(r)}_{\leq \underline{h}}$. Assume first that $i > 0$. Let

$$\underline{h}_{\geq k} := (h_k, \cdots, h_r), (k = 1, \cdots, r),$$
$$S_{r,i}(\underline{h}) := \Pi_{n=M_r(\underline{h})+1}^{-i} \left(G_i(h_i, \cdots, h_0; -n + i) + 1\right),$$
$$T_{r,i}(\underline{h}) := \left(\frac{1}{2} H_{r-i+1}(\underline{h}_{\geq i}) + 1\right)^{H_r(\underline{h}_{\geq 1})-1}.$$

By Lemma 5.1 and the monotony properties of G_i and of H_{r-i+1} we see that at most $S_{r,i}(\underline{h}) T_{r,i}(\underline{h})$ different functions

$$h^i_{\mathcal{F}} \restriction : [M_r(\underline{h}) + 1, H_r(\underline{h}_{\geq 1}) - 1] \to \mathbb{N}_0$$

may occur if $(X, \mathcal{F})$ runs through $\mathcal{C}$.
For each pair $(X, \mathcal{F}) \in \mathcal{C}$ we have $\nu^i_{\mathcal{F}} \geq M_r(\underline{h})$ and $reg(\mathcal{F}) \leq H_r(\underline{h}_{\geq 1})$, (s. 4.7 and 3.3 b)), so that $h^i_{\mathcal{F}}(n) = p^i_{\mathcal{F}}(n)$ for all $n \leq M_r(\underline{h})$ and $h^i_{\mathcal{F}}(n) = 0$ for all $n \geq H_r(\underline{h}_{\geq 1})$. By Proposition 5.3 it follows that $\{h^i_{\mathcal{F}} \,|\, (X, \mathcal{F}) \in \mathcal{C}\}$ is a finite set.
As $\sum_{i=0}^r (-1) h^i_{\mathcal{F}}(n) = \chi_{\mathcal{F}}(n)$ for all $n \in \mathbb{Z}, h^0_{\mathcal{F}}(n) = \chi_{\mathcal{F}}(n)$ for all $n \geq H_r(\underline{h}_{\geq 1})$ and $h^0_{\mathcal{F}}(n) = p^0_{\mathcal{F}}(n)$ for all $n \leq M_r(\underline{h})$, the finiteness of the set $\{h^0_{\mathcal{F}} \,|\, (X, \mathcal{F}) \in \mathcal{C}\}$ follows by Proposition 5.3 and the finiteness of the sets $\{h^i_{\mathcal{F}} \,|\, (X, \mathcal{F}) \in \mathcal{C}\}$ for $i > 0$. ■

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