MULTIPLICITIES OF GRADED COMPONENTS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let M be a finitely generated graded module over a Noetherian homogeneous ring R with local base ring $(R_0, \mathfrak{m}_0)$. Then, the n-th graded component $H^i_{R_+}(M)_n$ of the i-th local cohomology module of M with respect to the irrelevant ideal R_+ of R is a finitely generated R_0-module which vanishes for all $n \gg 0$. In various situations we show that, for an $\mathfrak{m}_0$-primary ideal $\mathfrak{q}_0 \subseteq R_0$, the multiplicity $e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n)$ of $H^i_{R_+}(M)_n$ is antipolynomial in n of degree less than i. In particular we consider the following three cases:

a) $i < g(M)$, where $g(M)$ is the so called cohomological finite length dimension of M;
b) $i = g(M)$;
c) $\dim(R_0) = 2$.

In the cases a) and b) we express the degree and the leading coefficient of the representing polynomial in terms of local cohomological data of M (e.g. the sheaf induced by M) on $\mathrm{Proj}(R)$.

We also show that the lengths of the graded components of various graded submodules of $H^i_{R_+}(M)$ are antipolynomial of degree less than i and prove invariance results on these degrees.

1. Introduction

Throughout this paper we use the following notation.

1.1. **Notation and Convention.** A) Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$. So R_0 is a Noetherian ring and there are finitely many elements $l_1, \ldots, l_r \in R_1$ such that $R = R_0[l_1, \ldots, l_r]$. Let $R_+ := \bigoplus_{n\in\mathbb{N}} R_n$ denote the irrelevant ideal of R and let $\mathfrak{m} := \mathfrak{m}_0 \oplus R_+$ denote the graded maximal ideal of R. Moreover let $\mathfrak{q}_0 \subseteq R_0$ be an $\mathfrak{m}_0$-primary ideal. Finally let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module.

B) Let $(\widehat{R_0}, \widehat{\mathfrak{m}_0})$ denote the $\mathfrak{m}_0$-adic completion of the local ring $(R_0, \mathfrak{m}_0)$. By $\widehat{R}$ we shall denote the Noetherian homogeneous ring $\widehat{R_0} \otimes_{R_0} R$ and by $\widehat{M}$ we denote the finitely generated graded $\widehat{R}$-module $\widehat{R_0} \otimes_{R_0} M$.

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C) For $i \in \mathbb{N}_0$ and for any ideal $\mathfrak{a} \subseteq R$ let $H^i_{\mathfrak{a}}(M)$ denote the i-th local cohomology module of M with respect to $\mathfrak{a}$. Keep in mind that if the ideal $\mathfrak{a}$ is graded, $H^i_{\mathfrak{a}}(M)$ carries a natural R-grading. In this case, for $n \in \mathbb{Z}$, let $H^i_{\mathfrak{a}}(M)_n$ denote the n-th graded component of $H^i_{\mathfrak{a}}(M)$.

D) If $\mathbb{M} \subseteq \mathbb{Z}$, $\inf(\mathbb{M})$ and $\sup(\mathbb{M})$ are formed in $\mathbb{Z} \cup \{\pm\infty\}$. We convene that the degree of the 0-polynomial and the Krull dimensions of the empty set and of the zero module are equal to $-\infty$, thus $\deg(0) = \dim(\emptyset) = \dim(0) = -\infty$.

E) We say that a numerical function $h : \mathbb{Z} \to \mathbb{Z}$ is *antipolynomial (of degree d)*, if there is a polynomial $p \in \mathbb{Q}[\mathbf{x}]$ (of degree d) such that $h(n) = p(n)$ for all $n \ll 0$. In this situation, p is called the *representing polynomial of h.*

F) If $\emptyset \neq \mathcal{P} \subseteq \mathrm{Spec}(R_0)$, we write $\min \mathcal{P}$ for the set of all minimal members of $\mathcal{P}$ with respect to inclusion.

In this paper we are interested in the graded components $H^i_{R_+}(M)_n$ of the i-th local cohomology module $H^i_{R_+}(M)$ of M with respect to R_+. In geometric terms, we aim to study the Serre cohomology modules $H^{i-1}(X, \mathcal{F}(n))$ of the projective scheme $X = \mathrm{Proj}(R)$ with coefficients in the n-th twist $\mathcal{F}(n)$ of the coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F} = \widetilde{M}$ induced by M (cf. [B]).
It is well known that the R_0-module $H^i_{R_+}(M)_n$ is finitely generated for all $n \in \mathbb{Z}$ and vanishes for all sufficiently large values of n (cf. [B-S, Proposition 15.1.5]). On the other hand not much is known in general on the asymptotic behaviour of the R_0-module $H^i_{R_+}(M)_n$ if n tends to $-\infty$, except that this behaviour may be unexpectedly complicated (cf. [B-K-S], [K-S], [Si-Sw]). In this paper, we aim to study the mentioned asymptotic behaviour from the point of view of numerical invariants.

If $\dim(R_0) = 0$, this asymptotic behaviour is well understood. In this case the graded R-module $H^i_{R_+}(M)$ is Artinian, $h^i_M(n) := \mathrm{length}_{R_0}(H^i_{R_+}(M)_n)$ is finite for all $n \in \mathbb{Z}$ and hence there is a polynomial $p^i_M \in \mathbb{Q}[\mathbf{x}]$ such that $h^i_M(n) = p^i_M(n)$ for all $n \ll 0$ (cf. [B-S, Theorem 17.1.9]). Moreover, the degree and the leading coefficient of the polynomial p^i_M can be expressed in terms of local cohomological data of M on $X = \mathrm{Proj}(R)$ (hence of the coherent sheaf $\mathcal{F} = \widetilde{M}$ induced by M on X): Namely, the degree of p^i_M equals the dimension of the so called i-th *cohomological pseudo-support*

$$^{+}\mathrm{Psupp}^i(M) := \{\mathfrak{p} \in \mathrm{Proj}(R) | H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}$$

of M. Moreover, the leading coefficient of p^i_M can be expressed by an "associativity formula" in terms of the lengths of the $R_{\mathfrak{p}}$-modules $H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ where $\mathfrak{p}$ runs through all $\mathfrak{p} \in {}^{+}\mathrm{Psupp}^i(M)$ for which $\dim(R/\mathfrak{p})$ is maximal (cf. [B-M-M, Theorem 1.1]).
Our aim is to find extensions of these results which apply if $\dim(R_0) > 0$. We let the case $\dim(R_0) = 0$ set the standard for our "maximal expectation". We shall show

that in certain cases some numerical invariants of the R_0-modules $H^i_{R_+}(M)_n$ are of antipolynomial growth and express the degrees and the leading coefficients of the representing polynomials in terms of local cohomological data of M on $\mathrm{Proj}(R)$.
If $\dim(R_0) > 0$, the R_0-modules $H^i_{R_+}(M)_n$ need not be of finite length and so other numerical invariants must be used. There are three ways of doing so which could lead to natural extensions of the results which hold if $\dim(R_0) = 0$. Namely instead of the function given by h^i_M, one could consider

- *the function given by* $n \mapsto \mathrm{length}_{R_0}(\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n))$*, where* $\Gamma_{\mathfrak{m}_0}$ *denotes* $\mathfrak{m}_0$*-torsion;*
- *the function given by* $n \mapsto \mathrm{length}_{R_0}(0 :_{H^i_{R_+}(M)_n} \mathfrak{q}_0)$*, where* $(0 :_{H^i_{R_+}(M)_n} \mathfrak{q}_0) \subseteq H^i_{R_+}(M)_n$ *is the submodule of all elements annihilated by* $\mathfrak{q}_0$*;*
- *the function given by* $n \mapsto e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n)$*, where* $e_{\mathfrak{q}_0}$ *denotes the Hilbert-Samuel multiplicity.*

Instead of the length one also could think on the minimal number of generators as a numerical invariant and consider

- *the function given by* $n \mapsto \mathrm{length}_{R_0}(H^i_{R_+}(M)_n/\mathfrak{q}_0 H^i_{R_+}(M)_n)$.

If $\dim(R_0) = 1$, the four numerical functions mentioned above are antipolynomial of degree less than i (cf. [B-F-T, Theorem 3.5]). Moreover in this case, the first Hilbert-Samuel coefficient of the R_0-module $H^i_{R_+}(M)_n$ is antipolynomial in n of degree less than i, whereas the corresponding postulation numbers have a common upper bound (cf. [B-R, Theorems 3.1 and 3.3]). In the present paper we shall see that the degrees of the polynomials which represent the functions

$$n \mapsto \mathrm{length}_{R_0}(0 :_{H^i_{R_+}(M)_n} \mathfrak{q}_0) \text{ and } n \mapsto \mathrm{length}_{R_0}(H^i_{R_+}(M)_n/\mathfrak{q}_0 H^i_{R_+}(M)_n)$$

are independent on the $\mathfrak{m}_0$-primary ideal $\mathfrak{q}_0$ (cf. Theorem 2.8). This follows easily from the fact that the graded R-modules $(0 :_{H^i_{R_+}(M)} \mathfrak{m}_0)$ and $H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M)$ are K-Artinian, e.g. Artinian with graded components of finite length in all degrees $n \ll 0$ (cf. Remark 2.1).
The same independence result holds for the function

$$n \mapsto \mathrm{length}_{R_0}(H^c_{R_+}(M)_n/\mathfrak{q}_0 H^c_{R_+}(M)_n)$$

if R_0 is of arbitrary dimension and c is the cohomological dimension of M with respect to R_+, thus $c = \dim(M/\mathfrak{m}_0 M)$. (The fact that the graded R-module $H^c_{R_+}(M)_n/\mathfrak{m}_0 H^c_{R_+}(M)_n$ is K-Artinian for arbitrary local base ring R_0 has been shown independently by Rotthaus and Şega (cf. [Ro-Se, Theorem 2.1]).)
To return to the case $\dim(R_0) = 1$ let us notice that we do not give a general description of the degrees and the leading coefficients of the representing polynomials of the above four numerical functions.

In the case $\dim(R_0) = 2$, the situation changes drastically. Here, the graded R-modules $(0 :_{H^i_{R_+}(M)} \mathfrak{m}_0)$ and $H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M)$ need not be Artinian in general (cf. [B-F-T, Examples 4.1, 4.2]). Moreover the above numerical functions need not be antipolynomial in this case, as shown by examples of Katzman and Sharp. Let us quote one of their examples in slightly modified form.

1.2. **Example.** (cf. [K-S, Theorem 3.2]) Let K be any field of characteristic zero. Let $R_0 := K[\mathbf{x}, \mathbf{y}]_{(\mathbf{x},\mathbf{y})}$ and let $S := R_0[\mathbf{u}, \mathbf{v}]$ where $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}$ are indeterminates over K. Define a grading on S by setting $\deg(\mathbf{x}) = \deg(\mathbf{y}) = 0$ and $\deg(\mathbf{u}) = \deg(\mathbf{v}) = 1$. Let $f := 2\mathbf{x}^2\mathbf{v}^2 + 2\mathbf{x}\mathbf{y}\mathbf{u}\mathbf{v} + \mathbf{y}^2\mathbf{u}^2$ and set $R := S/fS$. Notice that f is homogeneous of degree 2 and R is a Noetherian homogeneous ring whose base ring R_0 is regular local and of dimension 2. According to Katzman and Sharp, for each $n < 0$

$$\operatorname{length}_{R_0}(H^2_{R_+}(R)_n) = \begin{cases} n^2, & \text{if } n \equiv 0 \mod 4; \\ n^2 - 1, & \text{if } n \not\equiv 0 \mod 4. \end{cases}$$

This example somehow covers all the bad cases that are to be expected if $\dim(R_0) = 2$. We namely prove (cf. Theorem 5.7):

If $\dim(R_0) = 2$ *and* $\dim_{R_0}(H^i_{R_+}(M)_n) \geq 1$ *for all* $n \ll 0$*, then the function given by* $n \mapsto e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n)$ *is antipolynomial of degree less than* i*.*

Under fairly mild additional hypotheses on R_0, this result finds a natural extension to the case $\dim(R_0) > 2$ (cf. Theorem 5.9). Also, if $\dim(R_0) = 2$ and $x \in \mathfrak{m}_0$ is $\mathfrak{m}_0$-filter regular with respect to $H^i_{R_+}(M)_n$ for all $n \ll 0$, the function given by

$$n \mapsto \operatorname{length}_{R_0}(0 :_{H^i_{R_+}(M)_n} x)$$

is antipolynomial of degree less than i (cf. Proposition 5.4). It should be noted that here the graded R-module $(0 :_{H^i_{R_+}(M)} x)$ need not be Artinian (cf. Remark 5.5 B)). Moreover we shall see that the graded R-modules $H^1_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ are Artinian if $\dim(R_0) = 2$ but not necessarily if $\dim(R_0) = 3$ (cf. Proposition 5.10 and Example 5.11).

If R_0 is of arbitrary dimension one still has a chance to study the asymptotic behaviour of some of our numerical functions for special choices of i. We already gave a comment of this type concerning the case $i = \dim(M/\mathfrak{m}_0 M)$. In Section 3 we follow this idea and first study the function

$$n \mapsto \operatorname{length}_{R_0}(H^i_{R_+}(M)_n)$$

for values of i which are strictly smaller than the *cohomological finite length dimension of* M, which we define as

$$g(M) := \inf\{j \in \mathbb{N}_0 | \operatorname{length}_{R_0}(H^j_{R_+}(M)_n) = \infty \text{ for infinitely many } n \in \mathbb{Z}\}.$$

If $i < g(M)$, we shall see that there are isomorphisms $H^i_{R_+}(M)_n \cong H^i_{\mathfrak{m}}(M)_n$ for all $n \ll 0$ (cf. Proposition 3.4). This allows to apply local duality and hence to generalize what has been said on the case $\dim(R_0) = 0$ (cf. Theorem 3.6 and Remark 3.7 A)):

> *If $i < g(M)$, the function given by $n \mapsto \mathrm{length}_{R_0}(H^i_{R_+}(M)_n)$ is antipolynomial and the degree and the leading coefficient of the representing polynomial may be expressed by local cohomological data of M on* $\mathrm{Proj}(R)$.

As an application we can show that for arbitrary i, but under some assumption on the supports of the R_0-modules $H^i_{R_+}(M)_n$, the function given by

$$n \mapsto e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n)$$

is antipolynomial with a representing polynomial whose degree and leading coefficient are expressed in terms of local cohomological data of M on $\mathrm{Proj}(R)$ (cf. Corollary 3.10).

In Section 4 we study the case where i equals the finite length dimension $g = g(M)$ of M. As a first result we prove that the graded R-module $\Gamma_{\mathfrak{m}_0 R}(H^g_{R_+}(M))$ is K-Artinian (cf. Proposition 4.2) and draw the conclusion that the numerical functions given by

$$n \mapsto \mathrm{length}_{R_0}(\Gamma_{\mathfrak{m}_0}(H^g_{R_+}(M)_n)) \text{ and } n \mapsto \mathrm{length}_{R_0}(0 :_{H^g_{R_+}(M)_n} \mathfrak{q}_0)$$

are both antipolynomial of degree less than g (cf. Corollaries 4.3 and 4.4). Now, by a mere use of the Associativity Formula one might conclude that the function given by

$$n \mapsto e_{\mathfrak{q}_0}(H^g_{R_+}(M)_n)$$

is antipolynomial of degree less than g.

But we do better and apply the results of Section 3 in order to get a description of the degree and the leading coefficient of the representing polynomial in terms of local cohomological data of M (cf. Theorem 4.11):

> *The function given by $n \mapsto e_{\mathfrak{q}_0}(H^g_{R_+}(M)_n)$ is antipolynomial and the degree and the leading coefficient of the representing polynomial may be expressed by local cohomological data of M on* $\mathrm{Proj}(R)$.

To reach this goal, we first prove a result on the asymptotic behaviour of the set $\mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n)$ and of the set $\mathrm{Supp}_0(H^g_{R_+}(M)_n)$ of all $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n)$ for which $\dim(R_0/\mathfrak{p}_0)$ is maximal (cf. Theorem 4.10).

As for the unexplained terminology we refer to [E].

2. Hilbert-Kirby Polynomials

In this section we give a few preliminaries on Hilbert-Kirby polynomials and some first applications to local cohomology modules.

2.1. **Remark.** A) Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be an Artinian R-module. We keep in mind the following fact:

a) *If* $(B^{(m)} = \bigoplus_{n\in\mathbb{Z}} B_n^{(m)})_{m\in\mathbb{N}_0}$ *is a descending sequence of graded submodules* $B^{(m)}$ *of* A *such that for each* $n \in \mathbb{Z}$ *there is some* $m_n \in \mathbb{N}_0$ *with* $B_n^{(m_n)} = 0$*, then* $B^{(m)} = 0$ *for all* $m \gg 0$*.*

B) Let A be as above. Then all the R_0-modules A_n are Artinian and the following statements are equivalent:

(i) *The* R_0*-module* A_n *is finitely generated for all* $n \ll 0$*.*
(ii) $\mathrm{length}_{R_0}(A_n) < \infty$ *for all* $n \ll 0$*.*
(iii) *There is some* $m \in \mathbb{N}$ *with* $\mathfrak{m}_0^m A_n = 0$ *for all* $n \ll 0$*.*

(To prove (ii) $\Rightarrow$ (iii), apply statement A)a) with $B^{(m)} = \mathfrak{m}_0^m A$).
If A satisfies the equivalent requirements (i), (ii) and (iii), there is a (uniquely determined) polynomial $\widetilde{P}_A \in \mathbb{Q}[\mathbf{x}]$ such that

$$\mathrm{length}_{R_0}(A_n) = \widetilde{P}_A(n) \text{ for all } n \ll 0,$$

the *Hilbert-Kirby polynomial of* A (cf. [Ki]).

C) A graded R-module A which is Artinian and satisfies the equivalent conditions (i), (ii) and (iii) of part B) is said to be a *graded Artinian module with Hilbert-Kirby polynomial* or just a *graded* K*-Artinian* R*-module.* Clearly:

The property of being a graded K*-Artinian* R*-module is inherited by graded submodules and by graded homomorphic images and is preserved under shifting.*

Moreover one has the following base change property:

If $(R'_0, \mathfrak{m}'_0)$ *is a Noetherian flat local* R_0*-algebra with* $\mathfrak{m}'_0 = \mathfrak{m}_0 R'_0$ *and if* A *is a graded* K*-Artinian* R*-module, then* $A' := R'_0 \otimes_{R_0} A$ *is a graded* K*-Artinian module over* $R' := R'_0 \otimes_{R_0} R$*. Moreover* $\widetilde{P}_{A'} = \widetilde{P}_A$*.*

2.2. **Remark.** A) Let $N = \bigoplus_{n\in\mathbb{Z}} N_n$ be a Noetherian graded R-module. Then, it is easy to see that the following statements are equivalent:

(i) $\mathrm{length}_{R_0}(N_n) < \infty$ *for all* $n \gg 0$*.*
(ii) *There is some* $m \in \mathbb{N}$ *with* $\mathfrak{m}_0^m N_n = 0$ *for all* $n \gg 0$*.*

If N satisfies these two equivalent requirements, there is a (uniquely determined) polynomial $P_N \in \mathbb{Q}[\mathbf{x}]$ such that

$$\mathrm{length}_{R_0}(N_n) = P_N(n) \text{ for all } n \gg 0,$$

the *Hilbert polynomial of* N.

B) A graded R-module N which is Noetherian and satisfies the equivalent conditions (i) and (ii) of part A) is said to be a *graded Noetherian module with Hilbert polynomial* or just a *graded H-Noetherian R-module.* Observe the following facts:

> *The property of being a graded H-Noetherian R-module is inherited by graded submodules and by graded homomorphic images and is preserved under shifting.*

In addition we have the following base change property:

> *If $(R_0', \mathfrak{m}_0')$ is a Noetherian flat local R_0-algebra with $\mathfrak{m}_0' = \mathfrak{m}_0 {R_0}'$ and if N is a graded H-Noetherian R-module, then $N' := R_0' \otimes_{R_0} N$ is a graded H-Noetherian module over $R' := R_0' \otimes_{R_0} R$. Moreover $P_{N'} = P_N$.*

C) Let N be a graded H-Noetherian R-module. Then the projective support of N,

$$\mathrm{ProjSupp}(N) := \mathrm{Proj}(R) \cap \mathrm{Supp}(N),$$

is contained in $\mathrm{Var}(\mathfrak{m}_0 R)$. Moreover the degree of the Hilbert polynomial of N is given by the dimension of the projective support of N, thus

a) $\deg(P_N) = \dim(\mathrm{ProjSupp}(N))$.

If $\mathrm{length}_{R_0}(N_n) < \infty$ for all $n \in \mathbb{Z}$, we have $\mathrm{Supp}(N) \subseteq \mathrm{Var}(\mathfrak{m}_0 R)$, so the graded support of N satisfies ${}^*\mathrm{Supp}(N) \subseteq \mathrm{ProjSupp}(N) \cup \{\mathfrak{m}\}$ and hence $\dim(\mathrm{ProjSupp}(N)) = \dim(N) - 1$. Thus we get:

b) *If* $\mathrm{length}_{R_0}(N_n) < \infty$ *for all $n \in \mathbb{Z}$, then* $\deg(P_N) = \dim(N) - 1$.

D) Let $P_N \neq 0$. By $e(N)$ we denote the multiplicity of N. So, the leading coefficient of the Hilbert polynomial of N has the form

a) $LC(P_N) = \frac{e(N)}{\deg(P_N)!}$ *with* $e(N) \in \mathbb{N}$.

Finally, let $\mathrm{ProjSupp}_0(N)$ denote the (finite) set of all $\mathfrak{p} \in \mathrm{ProjSupp}(N)$ whose closure has maximal dimension, thus

$$\mathrm{ProjSupp}_0(N) := \{\mathfrak{p} \in \mathrm{ProjSupp}(N) | \dim(R/\mathfrak{p}) = \deg(P_N) + 1\}.$$

Then, the so-called *Associativity Formula for multiplicities* says (cf. [Br-He, Corollary 4.7.9])

$$e(N) = \sum_{\mathfrak{p} \in \mathrm{ProjSupp}_0(N)} \mathrm{length}_{R_\mathfrak{p}}(N_\mathfrak{p}) e(R/\mathfrak{p}).$$

For later use let us recall the relation between graded K-Artinian and H-Noetherian modules given by graded Matlis duality.

2.3. **Remark.** A) (cf. [B-S, Exercise 13.4.5], [Br-He, Theorem 3.6.17]) Let ${}^*E := {}^*E_R(R/\mathfrak{m})$ be the *injective envelope of the graded R-module $R/\mathfrak{m}$, and let $E_0 := E_{R_0}(R_0/\mathfrak{m}_0)$ be the injective envelope of the R_0-module $R_0/\mathfrak{m}_0$. Moreover, for a graded R-module $T = \bigoplus_{n\in\mathbb{Z}} T_n$ and an R_0-module U, let ${}^*D(T) := {}^*\mathrm{Hom}_R(T, {}^*E)$ and $D_0(U) := \mathrm{Hom}_{R_0}(U, E_0)$ denote the *Matlis dual of T and the Matlis dual of U respectively. Keep in mind that there are isomorphisms of R_0-modules

a) ${}^*D(T)_n \cong D_0(T_{-n})$ *for all* $n \in \mathbb{Z}$.

As the exact contravariant functor D_0 preserves lengths of R_0-modules, we thus get

b) $\mathrm{length}_{R_0}({}^*D(T)_n) = \mathrm{length}_{R_0}(T_{-n})$ *for all* $n \in \mathbb{Z}$,

and this includes the case where the occuring lengths are infinite.

B) Assume now in addition that $(R_0, \mathfrak{m}_0)$ is complete. Then by graded Matlis duality and by the statements A)a), b) we have:

a) *The graded R-module T is H-Noetherian if and only if its *Matlis dual* ${}^*D(T)$ *is K-Artinian. Moreover, in this case* $\widetilde{P}_{{}^*D(T)}(\mathbf{x}) = P_T(-\mathbf{x})$.

b) *The graded R-module T is K-Artinian if and only if its *Matlis dual* ${}^*D(T)$ *is H-Noetherian. Moreover, in this case* $\widetilde{P}_T(\mathbf{x}) = P_{{}^*D(T)}(-\mathbf{x})$.

C) Let A be a graded K-Artinian R-module. Then A carries a natural structure as a graded $\widehat{R}$-module. Clearly, as an $\widehat{R}$-module A is again K-Artinian with $\mathrm{length}_{\widehat{R_0}}(A_n) = \mathrm{length}_{R_0}(A_n)$ for all $n \in \mathbb{Z}$. In particular, as an $\widehat{R}$-module A has the same Hilbert-Kirby polynomial as it has over R.
We now take the *Matlis dual of the $\widehat{R}$-module A and denote it by ${}^*\widehat{D}(A)$. Then according to statements B)b) and A)b) we can say

The graded $\widehat{R}$-module ${}^*\widehat{D}(A)$ *is H-Noetherian with*

$$\mathrm{length}_{R_0}(A_n) = \mathrm{length}_{\widehat{R_0}}({}^*\widehat{D}(A)_{-n})$$

for all $n \in \mathbb{Z}$ *and* $\widetilde{P}_A(\mathbf{x}) = P_{{}^*\widehat{D}(A)}(-\mathbf{x})$.

Moreover in view of Remark 2.2 C)b) we can say:

a) *If* $\mathrm{length}_{R_0}(A_n) < \infty$ *for all* $n \in \mathbb{Z}$, *then* $\deg(\widetilde{P}_A) = \dim_{\widehat{R}}({}^*\widehat{D}(A)) - 1$.

2.4. **Proposition.** *Let T be a graded R-module such that the R-module $(0 :_T \mathfrak{q}_0)$ is Artinian. Then*

$$\mathrm{length}_{R_0}(0 :_{T_n} \mathfrak{q}_0) < \infty \text{ for all } n \in \mathbb{Z}$$

and

$$\deg(\widetilde{P}_{(0:_T\mathfrak{q}_0)}) = \dim_{\widehat{R}}({}^*\widehat{D}(0 :_T \mathfrak{m}_0)) - 1.$$

Proof. As $(0 :_{T_n} \mathfrak{q}_0)$ is an Artinian $R_0/\mathfrak{q}_0$-module, we have $\mathrm{length}_{R_0}(0 :_{T_n} \mathfrak{q}_0) < \infty$ for all $n \in \mathbb{Z}$. By Remark 2.3 C)a) it follows

$$\deg(\widetilde{P}_{(0:_T\mathfrak{q}_0)}) = \dim_{\widehat{R}}({}^*\widehat{D}(0 :_T \mathfrak{q}_0)) - 1.$$

It thus suffices to show that the $\widehat{R}$-modules ${}^*\widehat{D}(0 :_T \mathfrak{m}_0)$ and ${}^*\widehat{D}(0 :_T \mathfrak{q}_0)$ have the same dimension. As $(0 :_T \mathfrak{m}_0) = (0 :_T \mathfrak{m}_0\widehat{R_0})$ and $(0 :_T \mathfrak{q}_0) = (0 :_T \mathfrak{q}_0\widehat{R_0})$ we may replace R by $\widehat{R}$, hence assume that R_0 is complete and identify the functors ${}^*\widehat{D}$ and ${}^*D$.

As $\mathfrak{q}_0$ is $\mathfrak{m}_0$-primary, there is some $n \in \mathbb{N}$ with $\mathfrak{m}_0^n \subseteq \mathfrak{q}_0$. As ${}^*D$ is contravariant and exact, the monomorphisms of graded R-modules

$$(0 :_T \mathfrak{m}_0) \rightarrowtail (0 :_T \mathfrak{q}_0) \rightarrowtail (0 :_T \mathfrak{m}_0^n)$$

yield epimorphisms of graded R-modules

$${}^*D(0 :_T \mathfrak{m}_0^n) \twoheadrightarrow {}^*D(0 :_T \mathfrak{q}_0) \twoheadrightarrow {}^*D(0 :_T \mathfrak{m}_0).$$

It thus suffices to show that

$$\mathrm{Supp}({}^*D(0 :_T \mathfrak{m}_0^n)) \subseteq \mathrm{Supp}({}^*D(0 :_T \mathfrak{m}_0)).$$

If $n = 1$, this is obvious. If $n > 1$, let $\mu := \dim_{R_0/\mathfrak{m}_0}(\mathfrak{m}_0^{n-1}/\mathfrak{m}_0^n)$. If we apply the functor $\mathrm{Hom}_{R_0}(\bullet, T)$ to the short exact sequence of R_0-modules

$$(\alpha) \qquad 0 \to (R_0/\mathfrak{m}_0)^{\oplus\mu} \to R_0/\mathfrak{m}_0^n \to R_0/\mathfrak{m}_0^{n-1} \to 0$$

we get an exact sequence of graded R-modules

$$0 \to \mathrm{Hom}_{R_0}(R_0/\mathfrak{m}_0^{n-1}, T) \to \mathrm{Hom}_{R_0}(R_0/\mathfrak{m}_0^n, T) \to \mathrm{Hom}_{R_0}(R_0/\mathfrak{m}_0, T)^{\oplus\mu}$$

and hence

$$0 \to (0 :_T \mathfrak{m}_0^{n-1}) \to (0 :_T \mathfrak{m}_0^n) \to U \to 0,$$

where U is a graded submodule of $(0 :_T \mathfrak{m}_0)^{\oplus\mu}$. If we apply the functor ${}^*D$ we thus obtain the following diagram of graded R-modules with exact row and column

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \uparrow & & & & \\
0 & \longrightarrow & {}^*D(U) & \longrightarrow & {}^*D(0 :_T \mathfrak{m}_0^n) & \longrightarrow & {}^*D(0 :_T \mathfrak{m}_0^{n-1}) \\
 & & \uparrow & & & & \\
 & & {}^*D(0 :_T \mathfrak{m}_0)^{\oplus\mu} & & & &
\end{array}$$

which induces

$$\mathrm{Supp}({}^*D(0 :_T \mathfrak{m}_0^n)) \subseteq \mathrm{Supp}({}^*D(0 :_T \mathfrak{m}_0)) \cup \mathrm{Supp}({}^*D(0 :_T \mathfrak{m}_0^{n-1}))$$

and hence allows to conclude by induction. □

2.5. **Corollary.** *Let A be a graded K-Artinian R-module and let $\mathfrak{a} \subseteq \mathfrak{m}_0$ be an ideal. Then* $\deg(\widetilde{P}_{(0:_A\mathfrak{a})}) = \deg(\widetilde{P}_A)$.

Proof. According to Remark 2.1 B) there is some $m \in \mathbb{N}$ with $\mathfrak{m}_0^m A_n = 0$ for all $n \ll 0$. Consequently $\widetilde{P}_A = \widetilde{P}_{(0:_A \mathfrak{m}_0^m)}$ and $\widetilde{P}_{(0:_A \mathfrak{a})} = \widetilde{P}_{(0:_A \mathfrak{a}+\mathfrak{m}_0^m)}$. If we apply Proposition 2.4 to the $\mathfrak{m}_0$-primary ideals $\mathfrak{m}_0^m$ and $\mathfrak{a} + \mathfrak{m}_0^m$ we get

$$\deg(\widetilde{P}_{(0:_A \mathfrak{m}_0^m)}) = \deg(\widetilde{P}_{(0:_A \mathfrak{a}+\mathfrak{m}_0^m)}) = \dim_{\widehat{R}}({}^*\widehat{D}(0 :_A \mathfrak{m}_0)) - 1.$$

This proves our claim. □

2.6. **Proposition.** *Let T be a graded R-module such that the R-module $T/\mathfrak{q}_0 T$ is Artinian. Then*

$$\mathrm{length}_{R_0}(T_n/\mathfrak{q}_0 T_n) < \infty \textit{ for all } n \in \mathbb{Z}$$

and

$$\deg(\widetilde{P}_{T/\mathfrak{q}_0 T}) = \dim_{\widehat{R}}({}^*\widehat{D}(T/\mathfrak{m}_0 T)) - 1.$$

Proof. As $T_n/\mathfrak{q}_0 T_n$ is an Artinian $R_0/\mathfrak{q}_0$-module, we have $\mathrm{length}_{R_0}(T_n/\mathfrak{q}_0 T_n) < \infty$ for all $n \in \mathbb{Z}$. By Remark 2.3 C)a) it follows

$$\deg(\widetilde{P}_{T/\mathfrak{q}_0 T}) = \dim_{\widehat{R}}({}^*\widehat{D}(T/\mathfrak{q}_0 T)) - 1.$$

It thus suffices to show that the $\widehat{R}$-modules ${}^*\widehat{D}(T/\mathfrak{m}_0 T)$ and ${}^*\widehat{D}(T/\mathfrak{q}_0 T)$ have the same dimension. As $T/\mathfrak{m}_0 T = T/\mathfrak{m}_0 \widehat{R_0} T$ and $T/\mathfrak{q}_0 T = T/\mathfrak{q}_0 \widehat{R_0} T$, we may replace R by $\widehat{R}$, hence assume that R_0 is complete and identify the functors ${}^*\widehat{D}$ and ${}^*D$.
As $\mathfrak{q}_0$ is $\mathfrak{m}_0$-primary, there is some $n \in \mathbb{N}$ with $\mathfrak{m}_0^n \subseteq \mathfrak{q}_0$. As ${}^*D$ is contravariant and exact, the epimorphisms of graded R-modules

$$T/\mathfrak{m}_0^n T \twoheadrightarrow T/\mathfrak{q}_0 T \twoheadrightarrow T/\mathfrak{m}_0 T$$

yield monomorphisms of graded R-modules

$${}^*D(T/\mathfrak{m}_0 T) \rightarrowtail {}^*D(T/\mathfrak{q}_0 T) \rightarrowtail {}^*D(T/\mathfrak{m}_0^n T).$$

It thus suffices to show that $\mathrm{Supp}({}^*D(T/\mathfrak{m}_0^n T)) \subseteq \mathrm{Supp}({}^*D(T/\mathfrak{m}_0 T))$. This is done similarly as in the proof of Proposition 2.4 on use of the exact sequence (α) of that proof. □

2.7. **Corollary.** *Let A be a graded K-Artinian R-module and let $\mathfrak{a} \subseteq \mathfrak{m}_0$ be an ideal. Then* $\deg(\widetilde{P}_{A/\mathfrak{a}A}) = \deg(\widetilde{P}_A)$.

Proof. According to Remark 2.1 B) there is some $m \in \mathbb{N}$ with $\mathfrak{m}_0^m A_n = 0$ for all $n \ll 0$. Therefore $\widetilde{P}_A = \widetilde{P}_{A/\mathfrak{m}_0^m A}$ and $\widetilde{P}_{A/\mathfrak{a}A} = \widetilde{P}_{A/(\mathfrak{a}+\mathfrak{m}_0^m)A}$. Applying Proposition 2.6 to the $\mathfrak{m}_0$-primary ideals $\mathfrak{m}_0^m$ and $\mathfrak{a} + \mathfrak{m}_0^m$ we get

$$\deg(\widetilde{P}_{A/\mathfrak{m}_0^m A}) = \deg(\widetilde{P}_{A/(\mathfrak{a}+\mathfrak{m}_0^m)A}) = \dim_{\widehat{R}}({}^*\widehat{D}(A/\mathfrak{m}_0 A)) - 1.$$

This proves our claim. □

Our next theorem is a supplement to the main result of [B-F-T].

2.8. **Theorem.** *Let* $i \in \mathbb{N}_0$ *and assume that* $\dim(R_0) \leq 1$. *Let*

$$d := \dim_{\widehat{R}}({}^*\widehat{D}(H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M))) - 1$$

and

$$\bar{d} := \dim_{\widehat{R}}({}^*\widehat{D}(0 :_{H^i_{R_+}(M)} \mathfrak{m}_0)) - 1.$$

Then there are polynomials $P, \bar{P}, \bar{Q} \in \mathbb{Q}[\mathbf{x}]$ *such that*

a) $\deg(P) = d$ *and* $\mathrm{length}_{R_0}(H^i_{R_+}(M)_n/\mathfrak{q}_0 H^i_{R_+}(M)_n) = P(n)$ *for all* $n \ll 0$.

b) $\deg(\bar{P}) = \bar{d}$ *and* $\mathrm{length}_{R_0}(0 :_{H^i_{R_+}(M)_n} \mathfrak{q}_0) = \bar{P}(n)$ *for all* $n \ll 0$.

c) $\deg(\bar{Q}) = \bar{d}$ *and* $\mathrm{length}_{R_0}(\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n)) = \bar{Q}(n)$ *for all* $n \ll 0$.

Proof. According to [B-F-T, Corollary 2.6] the graded R-modules $H^i_{R_+}(M)/\mathfrak{q}_0 H^i_{R_+}(M)$ and $(0 :_{H^i_{R_+}(M)} \mathfrak{q}_0)$ are Artinian. So, if we apply Proposition 2.6 respectively Proposition 2.4 we get statements a) and b).
By [B-F-T, Theorem 2.5] the graded R-module $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ is Artinian. As all the R_0-modules $H^i_{R_+}(M)_n$ are finitely generated, it follows that $\mathrm{length}_{R_0}(\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n))$ is finite for all $n \in \mathbb{Z}$. So, there is some $t \in \mathbb{N}$ with $\mathfrak{m}_0^t \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)) = 0$ (cf. Remark 2.1 B)). It follows $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)) = (0 :_{H^i_{R_+}(M)} \mathfrak{m}_0^t)$. If we apply Proposition 2.4 with $\mathfrak{q}_0 = \mathfrak{m}_0^t$, we get statement c). □

Next, we give a supplement to Theorem 2.1 and Corollary 2.4 of [Ro-Se].

2.9. **Remark.** We denote by $c(M)$ the *cohomological dimension of* M *with respect to* R_+, thus

$$c := c(M) := \sup\{i \in \mathbb{N}_0 | H^i_{R_+}(M) \neq 0\}.$$

According to [B-H, Lemma 3.4] one has

$$c(M) = \dim_R(M/\mathfrak{m}_0 M).$$

2.10. **Theorem.** *Let* $c := c(M) \in \mathbb{N}_0$ *and let*

$$d := \dim_{\widehat{R}}({}^*\widehat{D}(H^c_{R_+}(M)/\mathfrak{m}_0 H^c_{R_+}(M))) - 1.$$

Then, there is a polynomial $Q \in \mathbb{Q}[\mathbf{x}]$ *of degree* d *such that*

$$\mathrm{length}_{R_0}(H^c_{R_+}(M)_n/\mathfrak{q}_0 H^c_{R_+}(M)_n) = Q(n) \text{ for all } n \ll 0.$$

Proof. Let $T := H^c_{R_+}(M)$. According to Proposition 2.6 if suffices to show that $T/\mathfrak{q}_0 T$ is Artinian. The graded R-module $T/\mathfrak{m}_0 T$ is Artinian (cf. [Ro-Se, Theorem 2.1]).
As $\mathfrak{q}_0$ is $\mathfrak{m}_0$-primary, there is some $n \in \mathbb{N}$ such that $\mathfrak{m}_0^n \subseteq \mathfrak{q}_0$. It therefore suffices to show that $T/\mathfrak{m}_0^n T$ is an Artinian R-module. If $n = 1$, this is clear by what is said in the first paragraph of this proof. If $n > 1$, we set $\mu := \dim_{R_0/\mathfrak{m}_0}(\mathfrak{m}_0^{n-1}/\mathfrak{m}_0^n)$ and use the exact sequence of R-modules

$$(T/\mathfrak{m}_0 T)^{\oplus \mu} \to T/\mathfrak{m}_0^n T \to T/\mathfrak{m}_0^{n-1} T \to 0$$

to conclude by induction. □

3. The case $i < g(M)$

In this section we introduce the cohomological finite length dimension $g = g(M)$ of a finitely generated graded R-module M and we study the asymptotic behaviour of the R_0-modules $H^i_{R_+}(M)_n$ for $n \to -\infty$ in the range $i < g$.

3.1**. Definition.** We define the *cohomological finite length dimension* $g(M)$ *of* M as the least integer i such that the R_0-module $H^i_{R_+}(M)_n$ is of infinite length for infinitely many integers n:

$$g := g(M) := \inf\{i \in \mathbb{N}_0 | \sharp\{n \in \mathbb{Z} | \mathrm{length}_{R_0}(H^i_{R_+}(M)_n) = \infty\} = \infty\}.$$

3.2**. Remark.** Clearly $g > 0$ and:

a) $\mathrm{length}_{R_0}(H^i_{R_+}(M)_n) < \infty$ *for all* $i < g$ *and all* $n \ll 0$.
b) If $g < \infty$, $\mathrm{length}_{R_0}(H^g_{R_+}(M)_n) = \infty$ *for infinitely many* $n < 0$.
c) *If* M *is annihilated by some power of* $\mathfrak{m}_0$ *(thus in particular if* $\dim(R_0) = 0$*) we have* $g = \infty$.
d) *Let* $(R_0', \mathfrak{m}_0')$ *be a Noetherian flat local* R_0*-algebra with* $\mathfrak{m}_0' = \mathfrak{m}_0 R_0$*. Then* $g(R_0' \otimes_{R_0} M) = g(M)$.

(Statement d) follows immediately by the flat base change property of local cohomology and by the fact that $\mathrm{length}_{R_0'}(R_0' \otimes_{R_0} T) = \mathrm{length}_{R_0}(T)$ for each R_0-module T.)

Our next aim is to show that in the range $i < g(M)$ the R_0-modules $H^i_{R_+}(M)_n$ and $H^i_{\mathfrak{m}}(M)_n$ coincide for all $n \ll 0$.

3.3**. Lemma.** *Let* $i \in \mathbb{N}_0$*, let* $n_0 \in \mathbb{Z}$*, let* $\mathfrak{a} \subseteq R_0$ *be an ideal and let* $b \in \mathfrak{m}_0$*. Assume that the* R_0*-module* $H^j_{\mathfrak{a}\oplus R_+}(M)_n$ *is of finite length for all* $n \leq n_0$ *and all* $j \leq i$*. Then, there are isomorphisms of* R_0*-modules*

$$H^j_{(\mathfrak{a},b)\oplus R_+}(M)_n \cong H^j_{\mathfrak{a}\oplus R_+}(M)_n \textit{ for all } n \leq n_0 \textit{ and all } j \leq i.$$

Proof. Let $j \leq i$. Then, for each $n \in \mathbb{Z}$ there is an exact sequence of R_0-modules

$$(H^{j-1}_{\mathfrak{a}\oplus R_+}(M)_n)_b \to H^j_{(\mathfrak{a},b)\oplus R_+}(M)_n \to H^j_{\mathfrak{a}\oplus R_+}(M)_n \to (H^j_{\mathfrak{a}\oplus R_+}(M)_n)_b$$

(cf. [B-S, Exercise 13.1.12]). For $n \leq n_0$ the R_0-modules $H^{j-1}_{\mathfrak{a}\oplus R_+}(M)_n$ and $H^j_{\mathfrak{a}\oplus R_+}(M)_n$ are b-torsion. Therefore, we have $(H^{j-1}_{\mathfrak{a}\oplus R_+}(M)_n)_b = (H^j_{\mathfrak{a}\oplus R_+}(M)_n)_b = 0$. □

3.4. **Proposition.** *Let $i \in \mathbb{N}_0$, let $n_0 \in \mathbb{Z}$ and assume that the R_0-module $H^j_{R_+}(M)_n$ is of finite length for all $n \leq n_0$ and all $j \leq i$. Then:*

a) *For each $n \leq n_0$ and each $j \leq i$ there is an isomorphism of R_0-modules*

$$H^j_{R_+}(M)_n \cong H^j_{\mathfrak{m}}(M)_n.$$

b) *For each $j \leq i$, the graded R-module $H^j_{\mathfrak{m}}(M)$ is K-Artinian.*

Proof. a) Let $b_1, \ldots, b_t \in R_0$ be such that $\mathfrak{m}_0 = \sum_{l=1}^t R_0 b_l$. For each $k \in \{0, \ldots, t\}$ let $\mathfrak{a}_k := \sum_{l=1}^k R_0 b_l$. Observe that $\mathfrak{a}_0 \oplus R_+ = R_+$ and $\mathfrak{a}_t \oplus R_+ = \mathfrak{m}$. Now, a repeated application of Lemma 3.3 gives our claim.
b) Clear by statement a), as the R-modules $H^j_{\mathfrak{m}}(M)$ are Artinian. □

As an application we get the announced comparison result, the basic tool needed to prove the main result of this section.

3.5. **Corollary.** *Let $j < g(M)$. Then:*

a) *$H^j_{R_+}(M)_n \cong H^j_{\mathfrak{m}}(M)_n$ for all $n \ll 0$.*
b) *$H^j_{\mathfrak{m}}(M)$ is K-Artinian.*

3.6. **Theorem.** *Let $i < g$. Then:*

a) *There is a polynomial $\widetilde{P} \in \mathbb{Q}[\mathbf{x}]$ such that*

$$\mathrm{length}_{R_0}(H^i_{R_+}(M)_n) = \widetilde{P}(n) \textit{ for all } n \ll 0.$$

b) *The set*

$$\mathcal{S} := \mathcal{S}^i(M) := \{\mathfrak{p} \in \mathrm{Proj}(R) | \mathfrak{p} \cap R_0 = \mathfrak{m}_0 \textit{ and } H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}$$

is a closed subset of $\mathrm{Proj}(R)$ *and* $\deg(\widetilde{P}) = \dim(\mathcal{S}) < i$.

c) *If $\widetilde{P} \neq 0$, then*

$$\mathcal{S}_0 := \mathcal{S}^i_0(M) := \{\mathfrak{p} \in \mathcal{S} | \dim(R/\mathfrak{p}) = \dim(\mathcal{S}) + 1\}$$

is a finite non-empty set and the leading coefficient of $\widetilde{P}$ is given by

$$LC(\widetilde{P}) = \frac{(-1)^{\dim(\mathcal{S})}}{\dim(\mathcal{S})!} \sum_{\mathfrak{p} \in \mathcal{S}_0} \mathrm{length}_{R_{\mathfrak{p}}}(H^{i-\dim(\mathcal{S})-1}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}))e(R/\mathfrak{p}).$$

Proof. a) In view of Corollary 3.5 it suffices to set $\widetilde{P} := \widetilde{P}_{H^i_{\mathfrak{m}}(M)}$.
b), c) By the flat base change property of local cohomology we have $\mathrm{length}_{\widehat{R_0}}(H^j_{\widehat{R}_+}(\widehat{M})_n) = \mathrm{length}_{R_0}(H^j_{R_+}(M)_n)$ for all $j \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. This shows that neither g nor $\widetilde{P}(\mathbf{x})$ are affected if we replace R and M by $\widehat{R}$ and $\widehat{M}$ respectively. Moreover, if we set

$$\mathcal{F} := \{\mathfrak{p} \in \mathrm{Proj}(R) | \mathfrak{p} \cap R_0 = \mathfrak{m}_0\} \text{ and } \widehat{\mathcal{F}} := \{\mathfrak{P} \in \mathrm{Proj}(\widehat{R}) | \mathfrak{P} \cap \widehat{R_0} = \widehat{\mathfrak{m}_0}\},$$

by the natural isomorphism of graded rings $R/\mathfrak{m}_0R \cong \widehat{R}/\widehat{\mathfrak{m}_0}\widehat{R}$, the map

$$\Phi : \mathcal{F} \to \widehat{\mathcal{F}}, \mathfrak{p} \mapsto \widehat{\mathfrak{p}}$$

is a homeomorphism (whose inverse is given by $\mathfrak{P} \mapsto \mathfrak{P} \cap R$) such that

$$(\alpha) \qquad \widehat{R}/\widehat{\mathfrak{p}} \cong R/\mathfrak{p} \text{ for all } \mathfrak{p} \in \mathcal{F}.$$

In particular we have $\dim(\widehat{R}/\widehat{\mathfrak{p}}) = \dim(R/\mathfrak{p})$ for all $\mathfrak{p} \in \mathcal{F}$ and the flat base change property of local cohomology gives rise to isomorphisms of $\widehat{R}_{\widehat{\mathfrak{p}}}$-modules

$$(\beta) \qquad \widehat{R}_{\widehat{\mathfrak{p}}} \otimes_{R_\mathfrak{p}} H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \cong H^{i-\dim(\widehat{R}/\widehat{\mathfrak{p}})}_{\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}}(\widehat{M}_{\widehat{\mathfrak{p}}}).$$

The statements $(\alpha), (\beta)$ show that Φ induces bijections

$$\mathcal{S} \to \widehat{\mathcal{S}} := \{\mathfrak{P} \in \widehat{\mathcal{F}} | H^{i-\dim(\widehat{R}/\mathfrak{P})}_{\mathfrak{P}\widehat{R}_\mathfrak{P}}(\widehat{M}_\mathfrak{P}) \neq 0\},$$

$$\mathcal{S}_0 \to \widehat{\mathcal{S}}_0 := \{\mathfrak{P} \in \widehat{\mathcal{S}} | \dim(\widehat{R}/\mathfrak{P}) = \dim(\mathcal{S}) + 1\}.$$

By statement (α) we have $\widehat{R}_{\widehat{\mathfrak{p}}}/\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}} \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and hence by statement (β)

$$\text{length}_{\widehat{R}_{\widehat{\mathfrak{p}}}}(H^{i-\dim(\widehat{R}/\widehat{\mathfrak{p}})}_{\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}}(\widehat{M}_{\widehat{\mathfrak{p}}})) = \text{length}_{R_\mathfrak{p}}(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) \text{ for all } \mathfrak{p} \in \mathcal{S}_0.$$

So, to prove the remaining claims we may replace R and M by $\widehat{R}$ and $\widehat{M}$ respectively and hence assume that $(R_0, \mathfrak{m}_0)$ is complete.
Now, by Cohen's Structure Theorem for complete local rings there is a polynomial ring $R' = R'_0[\mathbf{x}_1, \ldots, \mathbf{x}_s]$ over a complete regular local ring R'_0 and a surjective homomorphism $h : R' \to R$ of graded rings. We write $d' := \dim(R')$ and keep in mind that $R'(-s)$ is the *canonical module of R'. Now, let $\Omega := \text{Ext}^{d'-i}_{R'}(M, R'(-s))$. Then, by the Graded Local Duality Theorem (cf. [Br-He, Theorem 3.6.19]) and the graded base ring independence of local cohomology we get an isomorphism of graded R-modules $H^i_\mathfrak{m}(M) \cong {}^*D(\Omega)$. As $H^i_\mathfrak{m}(M)$ is K-Artinian, Ω is H-Noetherian and

$$(\gamma) \qquad \widetilde{P}(\mathbf{x}) = \widetilde{P}_{H^i_\mathfrak{m}(M)}(\mathbf{x}) = P_\Omega(-\mathbf{x})$$

(cf. Remark 2.3 B)a)). Now, let $\mathfrak{p} \in \text{Proj}(R)$ and let $\mathfrak{p}' := h^{-1}(\mathfrak{p}) \in \text{Proj}(R')$. Then $\dim(R'/\mathfrak{p}') = \dim(R/\mathfrak{p})$ shows that $\dim(R'_{\mathfrak{p}'}) = d' - \dim(R/\mathfrak{p})$. Moreover

$$\Omega_\mathfrak{p} = \Omega_{\mathfrak{p}'} \cong \text{Ext}^{d'-i}_{R'_{\mathfrak{p}'}}(M_{\mathfrak{p}'}, R'(-s)_{\mathfrak{p}'}) = \text{Ext}^{\dim(R'_{\mathfrak{p}'})+\dim(R/\mathfrak{p})-i}_{R'_{\mathfrak{p}'}}(M_{\mathfrak{p}'}, R'_{\mathfrak{p}'}).$$

Hence by local duality $H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \cong D(\Omega_\mathfrak{p})$, where $D := \text{Hom}_{R_\mathfrak{p}}(\bullet, E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}))$ denotes the functor of taking the Matlis duals over $R_\mathfrak{p}$. As the functor D is length preserving, we get

$$(\delta) \qquad \text{length}_{R_\mathfrak{p}}(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) = \text{length}_{R_\mathfrak{p}}(\Omega_\mathfrak{p}) \text{ for all } \mathfrak{p} \in \text{Proj}(R).$$

As Ω is H-Noetherian, we have $\Omega_{\mathfrak{p}} = 0$ for each $\mathfrak{p} \in \mathrm{Proj}(R)$ with $\mathfrak{p} \cap R_0 \neq \mathfrak{m}_0$. Together with statement (δ) this proves that

$$(\varepsilon) \qquad \mathcal{S} = \mathrm{ProjSupp}(\Omega).$$

First of all this shows that $\mathcal{S}$ is a closed subset of $\mathrm{Proj}(R)$. Moreover, by statement (γ) and by Remark 2.2 C)a) we have $\deg(\widetilde{P}) = \dim(\mathcal{S})$.
By the statements $(\gamma), (\delta), (\varepsilon)$, by the fact that $\dim(R/\mathfrak{p}) = \dim(\mathcal{S}) + 1$ for all $\mathfrak{p} \in \mathcal{S}_0$ and by Remark 2.2 C)b), D)a) we get our claim. □

3.7. **Remark.** A) Assume that $\dim(R_0) = 0$. Then $g = g(M) = \infty$ (cf. Remark 3.2 c)). Let $i \in \mathbb{N}_0$. Then the set $\mathcal{S}$ of Theorem 3.6 b) is precisely the i-th cohomological pseudo-support ${}^{+}\mathrm{Psupp}^i(M)$ of M (cf. [B-M-M, Definition and Remark 1.2 A)]) and so Theorem 3.6 generalizes Theorem 1.1 of [B-M-M].

B) As shown by Example 1.2 it may happen that for a fixed $i \in \mathbb{N}$ the R_0-modules $H^i_{R_+}(M)_n$ are all of finite length but that these lengths are not of antipolynomial growth. In such a case we obviously must have $i > g(M)$.

Next, we apply Theorem 3.6 to prove a result on the asymptotic growth of the multiplicity $e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n)$ of the R_0-module $H^i_{R_+}(M)_n$ with respect to $\mathfrak{q}_0$ for $n \to -\infty$. We first recall a few preliminaries.

3.8. **Remark.** A) For a finitely generated R_0-module T there is a (uniquely determined) polynomial $\bar{P}_{\mathfrak{q}_0,T} \in \mathbb{Q}[\mathbf{x}]$ such that

$$\mathrm{length}_{R_0}(T/\mathfrak{q}_0^{n+1}T) = \bar{P}_{\mathfrak{q}_0,T}(n) \text{ for all } n \gg 0,$$

the *Hilbert-Samuel polynomial of* T *with respect to* $\mathfrak{q}_0$.

B) In the notation of part A) we have

$$\deg(\bar{P}_{\mathfrak{q}_0,T}) = \dim(T).$$

By $e_{\mathfrak{q}_0}(T)$ we shall denote the *Hilbert-Samuel multiplicity of* T *with respect to* $\mathfrak{q}_0$. So, if $T \neq 0$, the leading coefficient of $\bar{P}_{\mathfrak{q}_0,T}$ is given by

$$LC(\bar{P}_{\mathfrak{q}_0,T}) = \frac{e_{\mathfrak{q}_0}(T)}{\dim(T)!} \text{ with } e_{\mathfrak{q}_0}(T) \in \mathbb{N}.$$

C) Finally, let $\mathrm{Supp}_0(T)$ denote the (finite) set of all $\mathfrak{p}_0 \in \mathrm{Supp}(T)$ such that $R_0/\mathfrak{p}_0$ is of maximal dimension:

$$\mathrm{Supp}_0(T) := \{\mathfrak{p}_0 \in \mathrm{Supp}(T) | \dim(R_0/\mathfrak{p}_0) = \dim(T)\}.$$

Then, the so called *Associativity Formula for Hilbert-Samuel multiplicities* says (cf. [Br-He, Corollary 4.7.8])

a) $e_{\mathfrak{q}_0}(T) = \sum_{\mathfrak{p}_0 \in \mathrm{Supp}_0(T)} \mathrm{length}_{(R_0)_{\mathfrak{p}_0}}(T_{\mathfrak{p}_0}) e_{\mathfrak{q}_0}(R_0/\mathfrak{p}_0)$.

3.9. **Lemma.** *Let $i \in \mathbb{N}_0$ and assume that there is a non-empty set $\mathcal{W} \subseteq \mathrm{Spec}(R_0)$ with $\mathrm{Supp}_0(H^i_{R_+}(M)_n) = \mathcal{W}$ for all $n \ll 0$. Assume that for each $\mathfrak{p}_0 \in \mathcal{W}$ there is some polynomial $\widetilde{P}^{[\mathfrak{p}_0]} \in \mathbb{Q}[\mathbf{x}]$ such that $\mathrm{length}_{(R_0)_{\mathfrak{p}_0}}(H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n) = \widetilde{P}^{[\mathfrak{p}_0]}(n)$ for all $n \ll 0$ and set $\widetilde{Q} := \sum_{\mathfrak{p}_0 \in \mathcal{W}} \widetilde{P}^{[\mathfrak{p}_0]} e_{\mathfrak{q}_0}(R_0/\mathfrak{p}_0)$. Then*

a) *$e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n) = \widetilde{Q}(n)$ for all $n \ll 0$.*
b) *$\deg(\widetilde{Q}) = \max\{\deg(\widetilde{P}^{[\mathfrak{p}_0]}) | \mathfrak{p}_0 \in \mathcal{W}\}$.*
c) *$LC(\widetilde{Q}) = \sum_{\mathfrak{p}_0 \in \mathcal{W}_0} LC(\widetilde{P}^{[\mathfrak{p}_0]}) e_{\mathfrak{q}_0}(R_0/\mathfrak{p}_0)$, where $\mathcal{W}_0 := \{\mathfrak{p}_0 \in \mathcal{W} | \dim(R_0/\mathfrak{p}_0) = \deg(\widetilde{Q})\}$.*

Proof. By the graded flat base change property of local cohomology there are isomorphisms of R_0-modules $H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n \cong (H^i_{R_+}(M)_n)_{\mathfrak{p}_0}$ for all $n \in \mathbb{Z}$. Now, we conclude by the Associativity Formula of Remark 3.8 C)a). □

3.10. **Corollary.** *Let $i \in \mathbb{N}_0$ and assume that there is a non-empty set $\mathcal{W} \subseteq \mathrm{Spec}(R_0)$ with $\mathrm{Supp}_0(H^i_{R_+}(M)_n) = \mathcal{W}$ for all $n \ll 0$. Assume that $i < g(M_{\mathfrak{p}_0})$ for all $\mathfrak{p}_0 \in \mathcal{W}$. Then:*

a) *There is a polynomial $\widetilde{Q} \in \mathbb{Q}[\mathbf{x}]$ such that*
$$e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n) = \widetilde{Q}(n) \text{ for all } n \ll 0.$$
b) *In the notation of Theorem 3.6 we have*
$$d := \deg(\widetilde{Q}) = \max\{\dim(\mathcal{S}^i(M_{\mathfrak{p}_0})) | \mathfrak{p}_0 \in \mathcal{W}\}.$$
c) *Let $\mathcal{W}_0 := \{\mathfrak{p}_0 \in \mathcal{W} | \dim(\mathcal{S}^i(M_{\mathfrak{p}_0})) = d\}$. Then*
$$\mathcal{U} := \mathcal{U}^i(M) := \{\mathfrak{p} \in \mathrm{Proj}(R) | \mathfrak{p} \cap R_0 \in \mathcal{W}_0 \text{ and } \mathfrak{p} \in \mathcal{S}^i_0(M_{\mathfrak{p} \cap R_0})\}$$
is a finite non-empty set and the leading coefficient of $\widetilde{Q}$ is given by
$$LC(\widetilde{Q}) = \frac{(-1)^d}{d!} \sum_{\mathfrak{p} \in \mathcal{U}} \mathrm{length}_{R_{\mathfrak{p}}}(H^{i-d-1}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})) e((R/\mathfrak{p})_{\mathfrak{p} \cap R_0}) e_{\mathfrak{q}_0}(R_0/(\mathfrak{p} \cap R_0)).$$

Proof. Statement a) is immediate by Lemma 3.9 a). By Theorem 3.6 b) and in the notation of Lemma 3.9 we have $\deg(\widetilde{P}^{[\mathfrak{p}_0]}) = \dim(\mathcal{S}^i(M_{\mathfrak{p}_0}))$. In view of Lemma 3.9 b) this proves statement b).
Now, let $\mathfrak{p}_0 \in \mathcal{W}_0$ and let
$$\mathcal{U}^{[\mathfrak{p}_0]} := \{\mathfrak{p} \in \mathrm{Proj}(R) | \mathfrak{p} \cap R_0 = \mathfrak{p}_0 \text{ and } \mathfrak{p} \in \mathcal{S}^i_0(M_{\mathfrak{p}_0})\}.$$
Then, by Theorem 3.6 c) we have
$$LC(\widetilde{P}^{[\mathfrak{p}_0]}) = \frac{(-1)^d}{d!} \sum_{\mathfrak{p} \in \mathcal{U}^{[\mathfrak{p}_0]}} \mathrm{length}_{R_{\mathfrak{p}}}(H^{i-d-1}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})) e((R/\mathfrak{p})_{\mathfrak{p}_0}).$$
As $\mathcal{U} = \bigcup_{\mathfrak{p}_0 \in \mathcal{W}_0} \mathcal{U}^{[\mathfrak{p}_0]}$ statement c) follows by Lemma 3.9 c). □

3.11. **Remark.** A) It is not known yet, whether the set $\mathrm{Supp}_0(H^i_{R_+}(M)_n)$ is always asymptotically stable for $n \to -\infty$. So we have to assume this.

B) The hypothesis that $i < g(M_{\mathfrak{p}_0})$ for all $\mathfrak{p}_0 \in \mathcal{W}$ cannot be omitted in general, as shown again by Example 1.2.

C) Let $i < g(M)$. Then the requested asymptotic stability holds with $\mathcal{W} \subseteq \{\mathfrak{m}_0\}$ by Theorem 3.6. If $\mathcal{W} = \{\mathfrak{m}_0\}$ Corollary 3.10 is a mere restatement of Theorem 3.6.

3.12. **Corollary.** *Let $i \in \mathbb{N}_0$ and let $\dim(H^i_{R_+}(M)_n) = \dim(R_0)$ for infinitely many $n \in \mathbb{Z}$. Then there is a non-empty set $\mathcal{W} \subseteq \mathrm{Supp}_0(R_0)$ for which Corollary 3.7 holds.*

Proof. By our hypothesis $\mathcal{V} := \bigcup_{n\in\mathbb{Z}} \mathrm{Supp}_0(H^i_{R_+}(M)_n) \cap \mathrm{Supp}_0(R_0)$ is a non-empty finite set and $\mathrm{Supp}_0(H^i_{R_+}(M)_n) \cap \mathrm{Supp}_0(R_0) \neq \emptyset$ for infinitely many integers n. So, there is a $\mathfrak{p}_0 \in \mathcal{V}$ such that

$$(\alpha) \qquad \mathfrak{p}_0 \in \mathrm{Supp}_0(H^i_{R_+}(M)_n) \text{ for infinitely many } n \in \mathbb{Z}.$$

For each $\mathfrak{p}_0 \in \mathcal{V}$ we have $\dim((R_0)_{\mathfrak{p}_0}) = 0$ and the graded $R_{\mathfrak{p}_0}$-module $H^i_{R_+}(M)_{\mathfrak{p}_0} \cong H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})$ is Artinian. Hence $(H^i_{R_+}(M)_n)_{\mathfrak{p}_0} \cong (H^i_{R_+}(M)_{\mathfrak{p}_0})_n$ is non-vanishing for infinitely many $n < 0$ if and only if it is for all $n \ll 0$. So, for each $\mathfrak{p}_0 \in \mathcal{V}$ the above condition (α) is equivalent to the condition

$$\mathfrak{p}_0 \in \mathrm{Supp}_0(H^i_{R_+}(M)_n) \text{ for all } n \ll 0.$$

This shows that $\dim(H^i_{R_+}(M)_n) = \dim(R_0)$ for all $n \ll 0$ and then that there is a non-empty set $\mathcal{W} \subseteq \mathcal{V}$ such that $\mathrm{Supp}_0(H^i_{R_+}(M)_n) = \mathcal{W}$ for all $n \ll 0$. As $g(M_{\mathfrak{p}_0}) = \infty$ for each $\mathfrak{p}_0 \in \mathcal{W}$ (cf. Remark 3.2 c)), we may apply Corollary 3.7. □

4. The Case $i = g(M)$

4.1. **Remark.** A) Let us recall the definition of the R_+*-finiteness dimension of* M (cf. [B-S, Definition 9.1.3]):

$$f := f(M) := \inf\{j \in \mathbb{N}_0 | H^j_{R_+}(M) \text{ is not finitely generated}\}.$$

B) As the R_0-modules $H^j_{R_+}(M)_n$ are finitely generated and vanish for all $n \gg 0$ we also may write

a) $f = \inf\{j \in \mathbb{N}_0 | H^j_{R_+}(M)_n \neq 0 \text{ for infinitely many } n < 0\}$.

So, we may say

b) *$f \leq g$ with strict inequality if* $\mathrm{Ass}_{R_0}(H^f_{R_+}(M)_n) = \{\mathfrak{m}_0\}$ *for all* $n \ll 0$.

C) Assume that $x \in R_1$ is a non-zerodivisor with respect to M. Then applying cohomology to the exact sequence $0 \to M \xrightarrow{x\cdot} M(1) \to M/xM \to 0$ we see that

a) $f(M/xM) \geq f(M) - 1$; $g(M/xM) \geq g(M) - 1$.

4.2**. Proposition.** *Let $i \in \mathbb{N}_0$ with $i \leq g$. Then the graded R-module $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ is K-Artinian.*

Proof. We choose $i \leq g$. As the R_0-modules $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))_n = \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n) \subseteq H^i_{R_+}(M)_n$ are finitely generated, it suffices to show that $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ is Artinian. This we do by induction on i.

As $H^0_{R_+}(M)_n = 0$ for all $n \ll 0$, the case $i = 0$ is obvious. So, let $i > 0$. Let $\mathbf{x}$ be an indeterminate and let $R'_0 := R_0[\mathbf{x}]_{\mathfrak{m}_0 R_0[\mathbf{x}]}$, $\mathfrak{m}'_0 := \mathfrak{m}_0 R'_0$, $R' := R'_0 \otimes_{R_0} R$ and $M' := R'_0 \otimes_{R_0} M$. Then by the flat base change property of local cohomology $R'_0 \otimes_{R_0} \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)) \cong \Gamma_{\mathfrak{m}'_0}(H^i_{R'_+}(M'))$. So, by the Remarks 2.1 C) and 3.2 d) it suffices to show that the R'-module $\Gamma_{\mathfrak{m}'_0}(H^i_{R'_+}(M'))$ is K-Artinian. Therefore, we may replace R and M by R' and M' respectively and hence assume that $R_0/\mathfrak{m}_0$ is infinite. As $i > 0$ we may replace M by $M/\Gamma_{R_+}(M)$ and thus assume that $\Gamma_{R_+}(M) = 0$. So there is an element $x \in R_1$ which is a non-zerodivisor with respect to M.

Now, consider the graded R-module $U^i := H^{i-1}_{R_+}(M)/xH^{i-1}_{R_+}(M)$ and the exact sequence of graded R-modules

$$0 \to U^i \to H^{i-1}_{R_+}(M/xM) \to (0 :_{H^i_{R_+}(M)} x)(-1) \to 0$$

which induces an exact sequence of graded R-modules

$$(\alpha) \qquad \Gamma_{\mathfrak{m}_0 R}(H^{i-1}_{R_+}(M/xM)) \to \Gamma_{\mathfrak{m}_0 R}((0 :_{H^i_{R_+}(M)} x)(-1)) \to H^1_{\mathfrak{m}_0 R}(U^i).$$

As $i \leq g$, the R_0-module $H^{i-1}_{R_+}(M)_n$ is of finite length for all but finitely many degrees n and hence the same holds for U^i_n. Therefore the graded R-module

$$H^1_{\mathfrak{m}_0 R}(U^i) \cong \bigoplus_{n \in \mathbb{Z}} H^1_{\mathfrak{m}_0 R}(U^i)_n \cong \bigoplus_{n \in \mathbb{Z}} H^1_{\mathfrak{m}_0}(U^i_n)$$

is concentrated in finitely many degrees and has graded components which are Artinian R-modules. So, $H^1_{\mathfrak{m}_0 R}(U^i)$ is an Artinian R-module. As

$$i - 1 \leq g - 1 = g(M) - 1 \leq g(M/xM),$$

(cf. Remark 4.1 C)a)) by induction, $\Gamma_{\mathfrak{m}_0 R}(H^{i-1}_{R_+}(M/xM))$ is an Artinian R-module. Therefore, by the sequence (α) the R-module

$$\Gamma_{\mathfrak{m}_0 R}(0 :_{H^i_{R_+}(M)} x) = (0 :_{\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))} x)$$

is Artinian. So by Melkersson's Lemma the R-module $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ is Artinian, too. □

4.3. **Corollary.** *Let $g < \infty$. Then there is a polynomial $\widetilde{P} \in \mathbb{Q}[\mathbf{x}]$ of degree less than g such that*

$$\mathrm{length}_{R_0}(\Gamma_{\mathfrak{m}_0}(H^g_{R_+}(M)_n)) = \widetilde{P}(n) \textit{ for all } n \ll 0.$$

Proof. By Proposition 4.2 the graded R-module

$$\Gamma_{\mathfrak{m}_0 R}(H^g_{R_+}(M)) = \bigoplus_{n\in\mathbb{Z}} \Gamma_{\mathfrak{m}_0}(H^g_{R_+}(M)_n)$$

is K-Artinian and so has a Hilbert-Kirby polynomial $\widetilde{P}$. It remains to show that $\deg(\widetilde{P}) < g$. This we do by induction on g. As usually we may assume that $R_0/\mathfrak{m}_0$ is infinite and – as $g > 0$ – that $\Gamma_{R_+}(M) = 0$. So, there is an element $x \in R_1$ which is a non-zerodivisor with respect to M. As $\mathrm{length}_{R_0}(H^{g-1}_{R_+}(M)_n) < \infty$ for all $n \ll 0$ we thus get for all $n \ll 0$ exact sequences of R_0-modules

$$0 \to H^{g-1}_{R_+}(M/xM)_{n+1}/U_{n+1} \to H^g_{R_+}(M)_n \to H^g_{R_+}(M)_{n+1}$$

with an R_0-submodule $U_{n+1} \subseteq H^{g-1}_{R_+}(M/xM)_{n+1}$ of finite length. In particular $\Gamma_{\mathfrak{m}_0}(H^{g-1}_{R_+}(M/xM)_{n+1}/U_{n+1}) \cong \Gamma_{\mathfrak{m}_0}(H^{g-1}_{R_+}(M/xM)_{n+1})/U_{n+1}$ for all $n \ll 0$. So, we get exact sequences

$$0 \to \Gamma_{\mathfrak{m}_0}(H^{g-1}_{R_+}(M/xM)_{n+1})/U_{n+1} \to \Gamma_{\mathfrak{m}_0}(H^g_{R_+}(M)_n) \to \Gamma_{\mathfrak{m}_0}(H^g_{R_+}(M)_{n+1})$$

for all $n \ll 0$. It follows that

$$0 \leq \widetilde{P}(n) \leq \widetilde{P}(n+1) + \mathrm{length}_{R_0}(\Gamma_{\mathfrak{m}_0}(H^{g-1}_{R_+}(M/xM))_{n+1}) \text{ for all } n \ll 0.$$

If $g = 1$, we have $H^{g-1}_{R_+}(M/xM)_{n+1} = H^0_{R_+}(M/xM)_{n+1} = 0$ for all $n \ll 0$ and hence $\deg(\widetilde{P}) < 1 = g$. Now, let $g > 1$. Observe that $g(M/xM) \geq g-1$ (cf. Remark 4.1 C)a)). If $g(M/xM) = g-1$ we can use induction to find a polynomial $\widetilde{Q} \in \mathbb{Q}[\mathbf{x}]$ of degree less than $g-1$ such that

$$\mathrm{length}_{R_0}(\Gamma_{\mathfrak{m}_0}(H^{g-1}_{R_+}(M/xM)_{n+1})) = \widetilde{Q}(n+1) \text{ for all } n \ll 0.$$

If $g(M/xM) > g-1$, we find such a polynomial $\widetilde{Q}$ by Theorem 3.6. Thus, for all $n \ll 0$ we have $0 \leq \widetilde{P}(n) \leq \widetilde{P}(n+1) + \widetilde{Q}(n+1)$, and this proves that $\deg(\widetilde{P}) < g$. □

4.4. **Corollary.** *Let $g < \infty$ and let $\widetilde{P}$ be as in Corollary 4.3. Then, there is an polynomial $\bar{P} \in \mathbb{Q}[\mathbf{x}]$ such that*

$$\deg(\bar{P}) = \deg(\widetilde{P})$$

and

$$\mathrm{length}_{R_0}(0 :_{H^g_{R_+}(M)_n} \mathfrak{q}_0) = \bar{P}(n) \textit{ for all } n \ll 0.$$

Proof. Apply Corollary 2.5 with $A = \Gamma_{\mathfrak{m}_0 R}(H^g_{R_+}(M))$. □

4.5. **Remark.** A) Let $\mathfrak{p} \in \mathrm{Proj}(R)$. The ($R_+$-)*adjusted depth of* M *at* $\mathfrak{p}$ is defined as

$$\mathrm{adjdepth}_{\mathfrak{p}}(M) := \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \mathrm{height}((R_+ + \mathfrak{p})/\mathfrak{p})$$

(cf. [B-S, Definition 9.2.2], where a different notation is used).

B) Observe that with $\mathfrak{p}_0 := \mathfrak{p} \cap R_0$ we have

a) $\mathrm{height}((R_+ + \mathfrak{p})/\mathfrak{p}) = \dim((R/\mathfrak{p})_{\mathfrak{p}_0})$.

As a consequence of this,

b) $\mathrm{adjdepth}_{\mathfrak{p}R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}) = \mathrm{adjdepth}_{\mathfrak{p}}(M)$.

C) Let us also recall that according to [B-S, Theorems 9.3.5 and 13.1.17] we have

a) $f(M) \leq \inf\{\mathrm{adjdepth}_{\mathfrak{p}}(M) | \mathfrak{p} \in \mathrm{Proj}(R)\}$

with equality if R_0 is a homomorphic image of a regular local ring.

4.6. **Lemma.** *Assume that* $\mathrm{Ass}_{R_0}(H^f_{R_+}(M)_n) = \{\mathfrak{m}_0\}$ *for all* $n \ll 0$. *Then, in the notation of Theorem 3.6 we have*

$$\{\mathfrak{p} \in \mathrm{Proj}(R) | \mathrm{adjdepth}_{\mathfrak{p}}(M) = f\} = \mathcal{S}^f(M) \neq \emptyset.$$

Proof. As $f < g = g(M)$ (cf. Remark 4.1 B)b)) and $H^f_{R_+}(M)_n \neq 0$ for infinitely many $n < 0$ (cf. Remark 4.1 B)a)) Theorem 3.6 shows that $\mathcal{S}^f(M) \neq \emptyset$. It thus remains to prove the stated equality. Let $\mathfrak{p} \in \mathrm{Proj}(R)$ with $\mathrm{adjdepth}_{\mathfrak{p}}(M) = f$. We wish to show that $\mathfrak{p} \in \mathcal{S}^f(M)$. As a first step we prove that $\mathfrak{p}_0 := \mathfrak{p} \cap R_0 = \mathfrak{m}_0$. Assume to the contrary that $\mathfrak{p}_0 \subsetneqq \mathfrak{m}_0$. Then, by Remark 4.5 B)b), C)a) we have

$$f(M_{\mathfrak{p}_0}) \leq \mathrm{adjdepth}_{\mathfrak{p}R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}) = f.$$

As $\mathrm{Supp}(H^f_{R_+}(M)_n) \subseteq \{\mathfrak{m}_0\}$ for all $n \ll 0$ we get $(H^f_{R_+}(M)_n)_{\mathfrak{p}_0} = 0$ for all $n \ll 0$. On use of the graded flat base change property of local cohomology we thus get $f(M_{\mathfrak{p}_0}) > f$, a contradiction. So we must have $\mathfrak{p}_0 = \mathfrak{m}_0$. Now, on use of Remark 4.5 B)a) we have $\mathrm{height}((R_+ + \mathfrak{p})/\mathfrak{p}) = \dim(R/\mathfrak{p})$, hence $\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = f - \dim(R/\mathfrak{p})$. It follows $H^{f-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$, thus $\mathfrak{p} \in \mathcal{S}^f(M)$.

Conversely, let $\mathfrak{p} \in \mathcal{S}^f(M)$. Then $\mathfrak{p} \cap R_0 = \mathfrak{m}_0$ and $H^{f-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$. So, in view of Remark 4.5 B)a)

$$\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq f - \dim(R/\mathfrak{p}) = f - \mathrm{height}((R_+ + \mathfrak{p})/\mathfrak{p}).$$

It follows $\mathrm{adjdepth}_{\mathfrak{p}}(M) \leq f$ and hence $\mathrm{adjdepth}_{\mathfrak{p}}(M) = f$ (cf. Remark 4.5 C)a)). □

4.7. **Proposition.** *Let* $f < \infty$. *Then for all* $n \ll 0$ *we have*

$$\min \mathrm{Ass}_{R_0}(H^f_{R_+}(M)_n) = \min\{\mathfrak{p} \cap R_0 | \mathfrak{p} \in \mathrm{Proj}(R) \text{ and } \mathrm{adjdepth}_{\mathfrak{p}}(M) = f\}.$$

Proof. By [B-H, Proposition 5.6] we know that the set $\mathrm{Ass}_{R_0}(H^f_{R_+}(M)_n)$ is asymptotically stable for $n \to -\infty$. So, there is a non-empty set $\mathcal{W} \subseteq \mathrm{Spec}(R_0)$ such that $\mathcal{W} = \min \mathrm{Ass}_{R_0}(H^f_{R_+}(M)_n)$ for all $n \ll 0$.
Let $\mathfrak{p}_0 \in \mathcal{W}$. Then $(H^f_{R_+}(M)_n)_{\mathfrak{p}_0} \neq 0$ for all $n \ll 0$. So, on use of the flat base change property of local cohomology we get $f(M_{\mathfrak{p}_0}) = f$ and $\mathrm{Ass}_{(R_0)_{\mathfrak{p}_0}}(H^f_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n) = \{\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}\}$ for all $n \ll 0$. Now, if we apply Lemma 4.6 to the $R_{\mathfrak{p}_0}$-module $M_{\mathfrak{p}_0}$, we find some $\mathfrak{q} \in \mathrm{Proj}(R_{\mathfrak{p}_0})$ with $\mathfrak{q} \cap (R_0)_{\mathfrak{p}_0} = \mathfrak{p}_0(R_0)_{\mathfrak{p}_0}$. Let $\mathfrak{p} = \mathfrak{q} \cap R$. Then $\mathfrak{p}_0 = \mathfrak{p} \cap R_0$ and $\mathrm{adjdepth}_{\mathfrak{p}}(M) = \mathrm{adjdepth}_{\mathfrak{q}}(M_{\mathfrak{p}_0}) = f$ (cf. Remark 4.5 B)b)).
Now, conversely, let $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)$ minimal with the property that there is some $\mathfrak{p} \in \mathrm{Proj}(R)$ with $\mathfrak{p} \cap R_0 = \mathfrak{p}$ and $\mathrm{adjdepth}_{\mathfrak{p}}(M) = f$. Then $\mathrm{adjdepth}_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}) = f$ (cf. Remark 4.5 B)b)) and hence $f(M_{\mathfrak{p}_0}) \leq f$. Again, by the flat base change property of local cohomology $f(M_{\mathfrak{p}_0}) \geq f$ and hence $f(M_{\mathfrak{p}_0}) = f$. Another use of the flat base change property implies that $\mathfrak{p}_0 \in \mathrm{Supp}(H^f_{R_+}(M)_n)$ for infinitely many $n < 0$ (cf. Remark 4.1 B)a)). Therefore we find some $\mathfrak{p}_0' \in \mathcal{W}$ with $\mathfrak{p}_0' \subseteq \mathfrak{p}_0$. By the first part of our proof there is some $\mathfrak{p}' \in \mathrm{Proj}(R)$ with $\mathfrak{p}' \cap R_0 = \mathfrak{p}_0'$ and $\mathrm{adjdepth}_{\mathfrak{p}'}(M) = f$. By the minimality of $\mathfrak{p}_0$ we get $\mathfrak{p}_0' = \mathfrak{p}_0$ and hence $\mathfrak{p}_0 \in \mathcal{W}$. □

4.8. **Remark.** In [B-K-S, Theorem 1.8] it is shown that, under the hypothesis that R_0 is a homomorphic image of a regular ring and $f < \infty$,

$$\mathrm{Ass}_{R_0}(H^f_{R_+}(M)_n) = \{\mathfrak{p} \cap R_0 | \mathfrak{p} \in \mathrm{Proj}(R) \text{ and } \mathrm{adjdepth}_{\mathfrak{p}}(M) = f\}$$

for all $n \ll 0$. Proposition 4.7 shows that if one considers only the subsets of minimal members no extra hypothesis on R_0 is needed to get equality.

4.9. **Lemma.** *Let $g < \infty$ and $S_0 \subseteq R_0$ be a multiplicatively closed set such that $S_0 \cap \mathfrak{m}_0 \neq \emptyset$ and such that the set $\mathcal{P}_n := \{\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n) | S_0 \cap \mathfrak{p}_0 = \emptyset\}$ is non-empty for infinitely many $n \in \mathbb{Z}$. Then:*

a) $f(S_0^{-1}M) = g$.
b) $\mathrm{Ass}_{S_0^{-1}R_0}(H^g_{(S_0^{-1}R)_+}(S_0^{-1}M)_n) = \{S_0^{-1}\mathfrak{p}_0 | \mathfrak{p}_0 \in \mathcal{P}_n\}$ *for all* $n \in \mathbb{Z}$.

Proof. By the graded flat base change property of local cohomology we have isomorphisms of $S_0^{-1}R_0$-modules

$$H^i_{(S_0^{-1}R)_+}(S_0^{-1}M)_n \cong S_0^{-1}H^i_{R_+}(M)_n$$

for all $i \in \mathbb{N}_0$ and for all $n \in \mathbb{Z}$. Choosing $i = g$, we get claim b).
If $i < g$, $\mathrm{Supp}(H^i_{R_+}(M)_n) \subseteq \{\mathfrak{m}_0\}$ for all $n \ll 0$. As $S_0 \cap \mathfrak{m}_0 \neq \emptyset$ it follows that $S_0^{-1}H^i_{R_+}(M)_n = 0$ for all $n \ll 0$ and hence the above isomorphisms yield claim a). □

Our next result is an extension of the corresponding results in [B-H] and [B-K-S] obtained by replacing $f(M)$ by $g(M)$.

4.10. **Theorem.** *Let $g < \infty$ and $\mathcal{T} := \mathcal{T}(M) := \{\mathfrak{p} \in \mathrm{Proj}(R) | \mathrm{adjdepth}_{\mathfrak{p}}(M) = g\}$ and let $\mathcal{W} = \mathcal{W}(M)$ be the set of all $\mathfrak{p}_0 \in \mathcal{Q} := \mathcal{Q}(M) := \{\mathfrak{p} \cap R_0 | \mathfrak{p} \in \mathcal{T}\}$ for which $\dim(R_0/\mathfrak{p}_0)$ is maximal. Then:*

a) *There is some $n_0 \in \mathbb{Z}$ such that $\mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n) = \mathrm{Ass}_{R_0}(H^g_{R_+}(M)_{n_0})$ for all $n \leq n_0$.*

b) *If n_0 is as in statement a), then $\mathrm{Supp}_0(H^g_{R_+}(M)_n) = \mathcal{W}$ for all $n \leq n_0$.*

c) *If R_0 is a homomorphic image of a regular local ring, then*

$$\mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n) \setminus \{\mathfrak{m}_0\} = \mathcal{Q} \setminus \{\mathfrak{m}_0\} \text{ for all } n \ll 0.$$

If $g = f$, the same equalities hold without removing $\mathfrak{m}_0$ from $\mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n)$ respectively $\mathcal{Q}$.

Proof. a) If $f = g$, we conclude by [B-H, Proposition 5.6]. So, let $f < g$. By the graded flat base change property there are isomorphisms of $\widehat{R_0}$-modules $H^g_{\widehat{R}_+}(\widehat{M})_n \cong \widehat{R_0} \otimes_{R_0} H^g_{R_+}(M)_n$ and these show by flatness that

$$\mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n) = \{\widehat{\mathfrak{p}_0} \cap R_0 | \widehat{\mathfrak{p}_0} \in \mathrm{Ass}_{\widehat{R_0}}(H^g_{\widehat{R}_+}(\widehat{M})_n)\}$$

for all $n \in \mathbb{Z}$. This allows to replace R and M respectively by $\widehat{R}$ and $\widehat{M}$ and hence to assume that $(R_0, \mathfrak{m}_0)$ is complete.
Now, by the definition of g, the set

$$(\alpha) \qquad \mathcal{P} := \bigcup_{n \in \mathbb{Z}} \mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n) \setminus \{\mathfrak{m}_0\}$$

is non-empty. As $\mathcal{P}$ is countable and R_0 is complete, by the Countable Prime Avoidance Principle (cf. [S-V]), we find an element

$$(\beta) \qquad s \in \mathfrak{m}_0 \setminus \bigcup \mathcal{P}.$$

As $\mathrm{length}_{R_0}(H^g_{R_+}(M)_n) = \infty$ for infinitely many $n \in \mathbb{Z}$, we have $\mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n) \setminus \{\mathfrak{m}_0\} \neq \emptyset$ for all such n. If we apply Lemma 4.9 with $S = \{s^k | k \in \mathbb{N}\}$ we get $f(M_s) = g$. So, by [B-H, Proposition 5.6] there is some $m_0 \in \mathbb{Z}$ such that

$$\mathrm{Ass}_{(R_0)_s}(H^g_{(R_s)_+}(M_s)_n) = \mathrm{Ass}_{(R_0)_s}(H^g_{(R_s)_+}(M_s)_{m_0}) \text{ for all } n \leq m_0.$$

In the notation of Lemma 4.9 we have $\mathcal{P}_n = \mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n) \setminus \{\mathfrak{m}_0\}$. So, by Lemma 4.9 b) the set $\mathcal{P}_n$ is asymptotically stable for $n \to -\infty$. Moreover, by Proposition 4.2 the set $\mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n) \cap \{\mathfrak{m}_0\}$ is asymptotically stable for $n \to -\infty$. This proves claim a).
b) According to statement a) there is a non-empty set $\mathcal{V} \subseteq \mathrm{Spec}(R_0)$ such that

$$\mathrm{Supp}_0(H^g_{R_+}(M)_n) = \mathcal{V} \text{ for all } n \ll 0.$$

It suffices to show that $\mathcal{V} = \mathcal{W}$. If $f = g$, this follows by Proposition 4.7. So, let $f < g$. Let $v := \dim(R_0/\mathfrak{p}_0)$ and $w := \dim(R_0/\mathfrak{p}'_0)$, where $\mathfrak{p}_0 \in \mathcal{V}$ and $\mathfrak{p}'_0 \in \mathcal{W}$.

First let $\mathfrak{p}_0 \in \mathcal{V}$. As $\mathrm{length}_{R_0}(H^g_{R_+}(M)_n) = \infty$ for infinitely many $n < 0$, we have $\mathfrak{p}_0 \subsetneqq \mathfrak{m}_0$. If we apply Lemma 4.9 with $S = R_0 \setminus \mathfrak{p}_0$ we get $f(M_{\mathfrak{p}_0}) = g$ and $\mathrm{Ass}_{(R_0)_{\mathfrak{p}_0}}(H^g_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n) = \{\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}\}$ for all $n \ll 0$. So, by Proposition 4.7 there is some $\mathfrak{q} \in \mathrm{Proj}(R_{\mathfrak{p}_0})$ such that $\mathfrak{q} \cap (R_0)_{\mathfrak{p}_0} = \mathfrak{p}_0(R_0)_{\mathfrak{p}_0}$ and $\mathrm{adjdepth}_{\mathfrak{q}}(M_{\mathfrak{p}_0}) = g$. Setting $\mathfrak{p} := \mathfrak{q} \cap R$ it follows $\mathfrak{p} \cap R_0 = \mathfrak{p}_0$, $\mathfrak{q} = \mathfrak{p}R_{\mathfrak{p}_0}$ and $\mathrm{adjdepth}_{\mathfrak{p}}(M) = g$ (cf. Remark 4.5 B)b)). So, we have shown

$$(\gamma) \qquad \mathcal{V} \subseteq \mathcal{Q} \text{ and } v > 0.$$

Next, let $\mathfrak{p}_0 \in \mathcal{W}$. By (α) we have $\dim(R_0/\mathfrak{p}_0) = w \geq v > 0$ so that $\mathfrak{p}_0 \subsetneqq \mathfrak{m}_0$ and Lemma 4.9 a) yields again that $f(M_{\mathfrak{p}_0}) = g$. As $\mathfrak{p}_0 \in \min \mathcal{Q}$ it follows by Remark 4.5 B)b) that $\mathfrak{p}_0(R_0)_{\mathfrak{p}_0} \in \min \mathrm{Ass}_{(R_0)_{\mathfrak{p}_0}}(H^g_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n)$ for all $n \ll 0$ and hence (by Lemma 4.9 b)) that $\mathfrak{p}_0 \in \min \mathrm{Ass}_{R_0}(H^g_{R_+}(M)_n)$ for all $n \ll 0$. As $\dim(R_0/\mathfrak{p}_0) \geq v$ it follows $\mathfrak{p}_0 \in \mathrm{Supp}_0(H^g_{R_+}(M)_n)$ for all $n \ll 0$. This shows that $\mathcal{W} \subseteq \mathcal{V}$ and $v = w$. In view of statement (γ) we get $\mathcal{V} = \mathcal{W}$.
c) If $g = f$ we may conclude by [B-K-S, Theorem 1.8]. So, let $f < g$. Now, the set $\mathcal{P}$ defined in (α) is finite by statement a) and in addition non-empty. So, we find an element s as in (β). Again we get $f(M_s) = g$. If we apply [B-K-S, Theorem 1.8] to the graded R_s-module M_s we get our claim by Lemma 4.9 b) and Remark 4.5 B)b). □

4.11. **Theorem.** *Let $g < \infty$, and let $\mathcal{T} = \mathcal{T}(M)$ and $\mathcal{W} = \mathcal{W}(M)$ be as in Theorem 4.10.*

a) *There is a polynomial $\widetilde{Q} \in \mathbb{Q}[\mathbf{x}]$ such that $e_{\mathfrak{q}_0}(H^g_{R_+}(M)_n) = \widetilde{Q}(n)$ for all $n \ll 0$.*
b) $d := \deg(\widetilde{Q}) = \max\{\mathrm{height}((R_+ + \mathfrak{p})/\mathfrak{p}) | \mathfrak{p} \in \mathcal{T}$ *and* $\mathfrak{p} \cap R_0 \in \mathcal{W}\} - 1 > 0$.
c) *Let*

$$\mathcal{T}_0 := \mathcal{T}_0(M) := \{\mathfrak{p} \in \mathcal{T} | \mathfrak{p} \cap R_0 \in \mathcal{W} \textit{ and } \mathrm{height}((R_+ + \mathfrak{p})/\mathfrak{p}) = d + 1\}.$$

Then, the leading coefficient of $\widetilde{Q}$ is given by

$$LC(\widetilde{Q}) = \frac{(-1)^d}{d!} \sum_{\mathfrak{p} \in \mathcal{T}_0} \mathrm{length}_{R_\mathfrak{p}}(H^{g-d-1}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) e((R/\mathfrak{p})_{\mathfrak{p} \cap R_0}) e_{\mathfrak{q}_0}(R_0/(\mathfrak{p} \cap R_0)).$$

Proof. By Theorem 4.10 it is clear that $\mathrm{Supp}_0(H^g_{R_+}(M)_n) = \mathcal{W}$ for all $n \ll 0$. Let $\mathfrak{p}_0 \in \mathcal{W}$. Then, $\mathfrak{p}_0 \subsetneqq \mathfrak{m}_0$ and by Lemma 4.9 and Remark 4.1 B)b)

$$(\alpha) \qquad f(M_{\mathfrak{p}_0}) = g < g(M_{\mathfrak{p}_0}).$$

Altogether we thus may apply Corollary 3.10. This obviously gives statement a).
Moreover, by (α) we may apply Lemma 4.6 and Remark 4.5 B)b) in order to see that

$$(\beta) \qquad \{\mathfrak{p}R_{\mathfrak{p}_0} | \mathfrak{p} \in \mathcal{T} \text{ and } \mathfrak{p} \cap R_0 = \mathfrak{p}_0\} = \mathcal{S}^g(M_{\mathfrak{p}_0}) \text{ for all } \mathfrak{p}_0 \in \mathcal{W}.$$

Now by Remark 4.5 B)a) and Corollary 3.10 we get

$$\max\{\mathrm{height}((R_+ + \mathfrak{p})/\mathfrak{p}) | \mathfrak{p} \in \mathcal{T} \text{ and } \mathfrak{p} \cap R_0 \in \mathcal{W}\} - 1$$

$$= \max\{\dim((R/\mathfrak{p})_{\mathfrak{p}\cap R_0}) - 1 | \mathfrak{p} \in \mathcal{T} \text{ and } \mathfrak{p} \cap R_0 \in \mathcal{W}\}$$
$$= \max\{\dim(\mathcal{S}^g(M_{\mathfrak{p}_0})) | \mathfrak{p}_0 \in \mathcal{W}\} = \deg(\widetilde{Q}).$$

This proves statement b).
Finally from (β) and in view of Remark 4.5 B)a) we obtain $\mathcal{T}_0(M) = \mathcal{U}(M)$, where $\mathcal{U}(M)$ is defined as in Corollary 3.10. Now statement c) of Corollary 3.10 allows to complete our proof. □

5. The case $\dim(R_0) = 2$

In this section we primarily study the asymptotic growth of the multiplicities $e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n)$ for $n \to -\infty$ with respect to $\mathfrak{q}_0$ in the case where $\dim(R_0) = 2$. As shown by Example 1.2 we cannot expect that the function $n \mapsto e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n)$ behaves antipolynomially in general. But as we shall show below, antipolynomiality holds if $\dim_{R_0}(H^i_{R_+}(M)_n) \geq 1$ for all $n \ll 0$.
Also, under fairly mild conditions on the ring R_0, there is a natural extension of the mentioned antipolynomiality result to the case where R_0 is of dimension greater than 2. In addition we show that in the case $\dim(R_0) = 2$ the function $n \mapsto \text{length}_{R_0}(0 :_{H^i_{R_+}(M)_n} x)$ is antipolynomial if $x \in \mathfrak{m}_0$ is choosen appropriately and that the R-modules $H^1_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ are Artinian.

5.1. **Remark.** A) Let T be an R_0-module and let $x \in \mathfrak{m}_0$. We recall that x is said to be $\mathfrak{m}_0$-*filter regular with respect to* T if it satisfies the following equivalent conditions:

(i) *x is a non-zero divisor with respect to $T/\Gamma_{\mathfrak{m}_0}(T)$.*
(ii) $(0 :_T x) \subseteq \Gamma_{\mathfrak{m}_0}(T)$.
(iii) $x \notin \bigcup \text{Ass}_{R_0}(T) \setminus \{\mathfrak{m}_0\}$.

B) Let T be finitely generated. Then in view of the above characterisation (iii), each $\mathfrak{m}_0$-filter regular element $x \in \mathfrak{m}_0$ is a parameter with respect to T.

5.2. **Lemma.** *Let $\dim(R_0) = 2$, let $i \in \mathbb{N}_0$ and let $x \in \mathfrak{m}_0$ be a parameter for R_0. Then, the graded R-module $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M/xM))$ is K-Artinian with Hilbert-Kirby polynomial of degree less than i.*

Proof. By the graded base ring independence of local cohomology, there is an isomorphism of graded R-modules $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M/xM)) \cong \Gamma_{\mathfrak{m}_0 R/xR}(H^i_{(R/xR)_+}(M/xM))$. But according to [B-F-T, Theorems 2.5 b) and 3.5 d)] the right hand side module is K-Artinian with Hilbert-Kirby polynomial of degree less than i. □

5.3. **Lemma.** *Let $i \in \mathbb{N}_0$. Then the graded R-module $H^i_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M))$ is K-Artinian with Hilbert-Kirby polynomial of degree less than i.*

Proof. As $\Gamma_{\mathfrak{m}_0 R}(M)$ is annihilated by some power of $\mathfrak{m}_0$, we have $g(\Gamma_{\mathfrak{m}_0 R}(M)) = \infty$ (cf. Remark 3.2 c)) and moreover, by the graded base ring independence of local cohomology, an isomorphism of graded R-modules $H^i_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M)) \cong H^i_{\mathfrak{m}}(\Gamma_{\mathfrak{m}_0 R}(M))$. Now we conclude by Corollary 3.5 b). □

Next, we prove a result which seems of a certain interest for its own.

5.4. **Proposition.** *Let $\dim(R_0) = 2$, let $i \in \mathbb{N}_0$ and let $x \in \mathfrak{m}_0$ be a parameter for R_0 and $\mathfrak{m}_0$-filter regular with respect to M and with respect to $H^i_{R_+}(M)_n$ for all $n \ll 0$. Then there exists a polynomial $\widetilde{S} \in \mathbb{Q}[\mathbf{x}]$ such that $\deg(\widetilde{S}) < i$ and*

$$\operatorname{length}_{R_0}(0 :_{H^i_{R_+}(M)_n} x) = \widetilde{S}(n) \text{ for all } n \ll 0.$$

Proof. Consider the exact sequence $0 \to \Gamma_{\mathfrak{m}_0 R}(M) \to M \to \overline{M} \to 0$ where $\overline{M} := M/\Gamma_{\mathfrak{m}_0 R}(M)$ and the induced exact sequence

$$H^i_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M)) \to H^i_{R_+}(M) \xrightarrow{\mu} H^i_{R_+}(\overline{M}) \to H^{i+1}_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M)).$$

As $H^i_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M))$ and $H^{i+1}_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M))$ are K-Artinian (cf. Lemma 5.3) so are $\operatorname{Ker}(\mu)$ and $\operatorname{Coker}(\mu)$. Consider the exact sequences

$$0 \to \operatorname{Ker}(\mu) \to H^i_{R_+}(M) \to \operatorname{Im}(\mu) \to 0,$$

$$0 \to \operatorname{Im}(\mu) \to H^i_{R_+}(\overline{M}) \to \operatorname{Coker}(\mu) \to 0.$$

Now we apply the functor $\operatorname{Hom}_R(R/x_0R, \bullet)$ to get the following exact sequences of graded R-modules

$$0 \to (0 :_{\operatorname{Ker}(\mu)} x) \to (0 :_{H^i_{R_+}(M)} x) \to (0 :_{\operatorname{Im}(\mu)} x) \to \operatorname{Ext}^1_R(R/x_0R, \operatorname{Ker}(\mu)),$$

$$0 \to (0 :_{\operatorname{Im}(\mu)} x) \to (0 :_{H^i_{R_+}(\overline{M})} x) \to (0 :_{\operatorname{Coker}(\mu)} x).$$

As $(0 :_{\operatorname{Ker}(\mu)} x)$ and $(0 :_{\operatorname{Coker}(\mu)} x)$ are graded submodules of $\operatorname{Ker}(\mu)$ and $\operatorname{Coker}(\mu)$, they both are K-Artinian. Moreover, $\operatorname{Ext}^1_R(R/x_0R, \operatorname{Ker}(\mu))$ is a graded subquotient of a finite direct sum of copies of $\operatorname{Ker}(\mu)$ and thus is K-Artinian. So, the modules $(0 :_{\operatorname{Ker}(\mu)} x)$, $(0 :_{\operatorname{Coker}(\mu)} x)$ and $\operatorname{Ext}^1_R(R/x_0R, \operatorname{Ker}(\mu))$ admit Hilbert-Kirby polynomials. It thus suffices to show that the function given by $n \mapsto \operatorname{length}_{R_0}(0 :_{H^i_{R_+}(\overline{M}_n)} x)$ is antipolynomial of degree less than i. Therefore we may replace M by $\overline{M}$ and hence assume that $\Gamma_{\mathfrak{m}_0 R}(M) = 0$. Now, x is regular with respect to M. By this argument and the fact that x is $\mathfrak{m}_0$-filter regular with respect to $H^i_{R_+}(M)_n$ for all $n \ll 0$, we get for all such n the exact sequences

$$(\alpha) \qquad 0 \to (0 :_{H^i_{R_+}(M)_n} x) \to \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n) \to \Gamma_{\mathfrak{m}_0}(xH^i_{R_+}(M)_n) \to 0,$$

$$(\beta)\qquad 0 \to \Gamma_{\mathfrak{m}_0 R}(xH^i_{R_+}(M)) \xrightarrow{\nu} \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)) \to \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)/xH^i_{R_+}(M)),$$

$$0 \to \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)/xH^i_{R_+}(M)) \to \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M/xM)).$$

By Lemma 5.2, the R-module $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M/xM))$ is K-Artinian with Hilbert-Kirby polynomial of degree less than i. Now in view of (β), $\mathrm{Coker}(\nu)$ is K-Artinian with $\deg(\widetilde{P}_{\mathrm{Coker}(\nu)}) < i$. Let $\widetilde{S} := \widetilde{P}_{\mathrm{Coker}(\nu)}$. Then $\deg(\widetilde{S}) < i$ and on use of $(\alpha), (\beta)$ we get

$$\mathrm{length}_{R_0}(0 :_{H^i_{R_+}(M)_n} x) = \mathrm{length}_{R_0}(\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n)) - \mathrm{length}_{R_0}(\Gamma_{\mathfrak{m}_0}(xH^i_{R_+}(M))_n)$$

$$= \mathrm{length}_{R_0}(\mathrm{Coker}(\nu)_n) = \widetilde{S}(n)$$

for all $n \ll 0$ and this completes our proof. □

5.5**. Remark.** A) If $(R_0, \mathfrak{m}_0)$ is complete, there are elements $x \in \mathfrak{m}_0$ which satisfy all the requirements of Proposition 5.4. Namely, let

$$\mathcal{P} := \{\mathfrak{p} \cap R_0 | \mathfrak{p} \in \mathrm{Ass}_R(M) \cup \mathrm{Ass}(R_0) \cup \bigcup_{n\in\mathbb{Z}, i\in\mathbb{N}_0} \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)\} \setminus \{\mathfrak{m}_0\}.$$

Then, by the Countable Prime Avoidance Principle (cf. [S-V]) we have $\mathfrak{m}_0 \setminus \bigcup \mathcal{P} \neq \emptyset$, and each element in this set satisfies the requirements of Proposition 5.4 (cf. Remark 5.1).

B) Although the function $n \mapsto \mathrm{length}_{R_0}(0 :_{H^i_{R_+}(M)} x)$ is antipolynomial in the situation of Proposition 5.4, the graded R-module $(0 :_{H^i_{R_+}(M)} x)$ need not be Artinian. Indeed if this module is Artinian, then by Melkersson's Lemma $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ is Artinian. But there are many examples for which this is not the cases. To see a concrete case let $R = M$ be as in Example 1.2 and choose $i = 2$.

5.6**. Lemma.** *Let* $\dim(R_0) = 2$ *and let* $x \in \mathfrak{m}_0$ *be a parameter which is a non-zero divisor with respect to* M*. Then for each* $i \in \mathbb{N}_0$*, the graded module*

$$\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))/(xH^i_{R_+}(M) \cap \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)))$$

is K*-Artinian with Hilbert-Kirby polynomial of degree less than* i*.*

Proof. The short exact sequence $0 \to M \xrightarrow{x\cdot} M \to M/xM \to 0$ yields a monomorphism of graded R-modules $H^i_{R_+}(M)/xH^i_{R_+}(M) \rightarrowtail H^i_{R_+}(M/xM)$. As

$$\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))/(xH^i_{R_+}(M) \cap \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)))$$

$$\cong (\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)) + xH^i_{R_+}(M))/xH^i_{R_+}(M) \subseteq \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)/xH^i_{R_+}(M)),$$

we may conclude by the left-exactness of the functor $\Gamma_{\mathfrak{m}_0 R}$ and by the fact that $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)/xH^i_{R_+}(M))$ is K-Artinian with Hilbert-Kirby polynomial of degree less than i (cf. Lemma 5.2). □

5.7. **Theorem.** *Let* $\dim(R_0) = 2$ *and let* $\dim_{R_0}(H^i_{R_+}(M)_n) \geq 1$ *for all* $n \ll 0$. *Then there exists a polynomial* $\widetilde{Q} \in \mathbb{Q}[\mathbf{x}]$ *such that* $\deg(\widetilde{Q}) < i$ *and* $e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n) = \widetilde{Q}(n)$ *for all* $n \ll 0$.

Proof. If $\dim_{R_0}(H^i_{R_+}(M)_n) = 2$ for infinitely many n, we conclude by Corollary 3.10. So, we may assume that $\dim_{R_0}(H^i_{R_+}(M)_n) = 1$ for all $n \ll 0$.
By the graded flat base change property of local cohomology there are isomorphisms of $\widehat{R_0}$-modules

$$\widehat{R_0} \otimes_{R_0} H^i_{R_+}(M)_n \cong H^i_{\widehat{R}_+}(\widehat{M})_n$$

for all $n \in \mathbb{Z}$. Moreover, $\mathfrak{q}_0\widehat{R_0}$ is $\widehat{\mathfrak{m}_0}$-primary and for each finitely generated R_0-module T we have $\dim_{R_0}(\widehat{R_0} \otimes_{R_0} T) = \dim_{R_0}(T)$ and $e_{\mathfrak{q}_0\widehat{R_0}}(\widehat{R_0} \otimes_{R_0} T) = e_{\mathfrak{q}_0}(T)$. Altogether, this allows to replace R and M by $\widehat{R}$ and $\widehat{M}$ respectively and hence to assume that $(R_0, \mathfrak{m}_0)$ is complete. Consider the exact sequence $0 \to \Gamma_{\mathfrak{m}_0}(M) \to M \to \overline{M} \to 0$ with $\overline{M} := M/\Gamma_{\mathfrak{m}_0R}(M)$ and the induced exact sequence

$$H^i_{R_+}(\Gamma_{\mathfrak{m}_0R}(M)) \to H^i_{R_+}(M) \xrightarrow{\beta} H^i_{R_+}(\overline{M}) \to H^{i+1}_{R_+}(\Gamma_{\mathfrak{m}_0R}(M)).$$

Since $H^i_{R_+}(\Gamma_{\mathfrak{m}_0}(M))$ and $H^{i+1}_{R_+}(\Gamma_{\mathfrak{m}_0}(M))$ are K-Artinian, the R-modules $\mathrm{Ker}(\beta)$ and $\mathrm{Coker}(\beta)$ are K-Artinian too. Hence $\dim_{R_0}(\mathrm{Ker}(\beta)_n) = \dim_{R_0}(\mathrm{Coker}(\beta)_n) \leq 0$, and so $e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n) = e_{\mathfrak{q}_0}(H^i_{R_+}(\overline{M})_n)$ for all $n \ll 0$. So we may replace M by $\overline{M}$ and hence assume that $\Gamma_{\mathfrak{m}_0R}(M) = 0$. Now, according to Remark 5.5 A) there is an element $x \in \mathfrak{m}_0$ which is a parameter for R_0 and $\mathfrak{m}_0$-filter regular with respect to M and all R_0-modules $H^i_{R_+}(M)_n$.
As x is a non-zero divisor with respect to $T_n := H^i_{R_+}(M)_n/\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n)$ and this R_0-module has dimension 1 for all $n \ll 0$, we thus get

$$e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n) = e_{\mathfrak{q}_0}(T_n) = e_{\mathfrak{q}_0}(T_n/xT_n)$$

$$= \mathrm{length}_{R_0}(T_n/xT_n) = \mathrm{length}_{R_0}(H^i_{R_+}(M)_n/(xH^i_{R_+}(M)_n + \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n)))$$

for all $n \ll 0$.
So, in view of Lemma 5.6 and the short exact sequence

$$0 \to \Gamma_{\mathfrak{m}_0R}(H^i_{R_+}(M))/(xH^i_{R_+}(M) \cap \Gamma_{\mathfrak{m}_0R}(H^i_{R_+}(M))) \to H^i_{R_+}(M)/xH^i_{R_+}(M)$$

$$\to H^i_{R_+}(M)/(xH^i_{R_+}(M) + \Gamma_{\mathfrak{m}_0R}(H^i_{R_+}(M))) \to 0$$

it suffices to find a polynomial $\widetilde{P} \in \mathbb{Q}[\mathbf{x}]$ of degree less than i such that

$$\mathrm{length}_{R_0}(H^i_{R_+}(M)_n/xH^i_{R_+}(M)_n) = \widetilde{P}(n)$$

for all $n \ll 0$.
As x is a non-zero divisor with respect to M, we have a monomorphism of graded R-modules

$$H^i_{R_+}(M)/xH^i_{R_+}(M) \rightarrowtail H^i_{R_+}(M/xM).$$

If we apply the functor $\Gamma_{\mathfrak{m}_0 R}$ and keep in mind that

$$H^i_{R_+}(M)_n/xH^i_{R_+}(M)_n = (H^i_{R_+}(M)/xH^i_{R_+}(M))_n$$

is $\mathfrak{m}_0$-torsion for all $n \ll 0$, we may use Lemma 5.2 and set $\widetilde{P} := \widetilde{P}_{\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)/xH^i_{R_+}(M))}$. □

5.8**. Remark.** A) Let R be as in Example 1.2. Then the function $n \mapsto e_{\mathfrak{q}_0}(H^2_{R_+}(R)_n) = \mathrm{length}_{R_0}(H^2_{R_+}(R)_n)$ is not antipolynomial. This shows that the conclusion of Theorem 5.7 need not hold if $\dim(H^i_{R_+}(M)_n) = 0$ for all $n \ll 0$.

B) Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be as in Example 1.2, let $R'_0 := R_0[\mathbf{z}]_{(\mathfrak{m}_0,\mathbf{z})}$, where $\mathbf{z}$ is an indeterminante. Let $R' := R'_0 \otimes_{R_0} R$ and let $\mathfrak{m}'_0$ be the maximal ideal of R'_0. Then, by the graded flat base change property of local cohomology there are isomorphisms of R'_0-modules $H^2_{R'_+}(R')_n \cong R'_0 \otimes_{R_0} H^2_{R_+}(R)_n$ for all $n \in \mathbb{Z}$. On use of the Associativity Formula (cf. Remark 3.8 C)a)) it easily follows that

$$e_{\mathfrak{m}'_0}(H^2_{R'_+}(R')_n) = \mathrm{length}_{R_0}(R'_0 \otimes_{R_0} H^2_{R_+}(R)_n) \text{ for all } n \in \mathbb{Z}$$

so that the function $n \mapsto e_{\mathfrak{m}'_0}(H^2_{R'_+}(R')_n)$ is not antipolynomial. On the other hand we have $\dim_{R'_0}(H^2_{R'_+}(R')_n) = 1$ for all $n \ll 0$.
This shows that the conclusion of Theorem 5.7 need not hold if $\dim(R_0) > 2$.

C) Observe that in Theorem 5.7 we did not impose that the set $\mathrm{Supp}_0(H^i_{R_+}(M)_n)$ is asymptotically stable for $n \to -\infty$ but only that $\dim_{R_0}(H^i_{R_+}(M)_n)$ ultimately takes a positive value if n tends to $-\infty$.

Notice that in Theorem 5.7 the only restriction on the base ring R_0 was that $\dim(R_0) = 2$. But under reasonably mild conditions on the structure of R_0, one has the following extension of Theorem 5.7.

5.9**. Theorem.** *Let $i \in \mathbb{N}_0$ and assume that R_0 is a finite integral extension of a domain or essentially of finite type over a field. Assume that $\dim_{R_0}(H^i_{R_+}(M)_n) \geq \dim(R_0) - 1$ for infinitely many integers n. Then the set $\mathrm{Supp}_0(H^i_{R_+}(M)_n)$ is asymptotically stable for $n \to -\infty$ and there is a polynomial $\widetilde{S} \in \mathbb{Q}[\mathbf{x}]$ such that $\deg(\widetilde{S}) < i$ and $e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n) = \widetilde{S}(n)$ for all $n \ll 0$.*

Proof. If $\dim_{R_0}(H^i_{R_+}(M)_n) = \dim(R_0)$ for infinitely many $n \in \mathbb{Z}$, we may conclude by Corollary 3.12. So, we may assume that $\dim_{R_0}(H^i_{R_+}(M)_n) = \dim(R_0) - 1$ for infinitely many $n \in \mathbb{Z}$. Now, by [B-F-L, Theorem 3.7 and Proposition 3.9] the set

$$\{\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n) | \dim(R_0/\mathfrak{p}_0) \geq \dim(R_0) - 1\}$$

is asymptotically stable for $n \to -\infty$. This shows that $\dim_{R_0}(H^i_{R_+}(M)_n) = \dim(R_0) - 1$ and that there is a non-empty set $\mathcal{W} \subseteq \mathrm{Spec}(R_0)$ with $\mathrm{Supp}_0(H^i_{R_+}(M)_n) = \mathcal{W}$ for

all $n \ll 0$.
Now, let $\mathfrak{p}_0 \in \mathcal{W}$. Then $\dim(R_0/\mathfrak{p}_0) = \dim(R_0) - 1$ and hence $\dim((R_0)_{\mathfrak{p}_0}) \leq 1$. As $\mathfrak{p}_0 \in \mathrm{Supp}_0(H^i_{R_+}(M)_n)$ for all $n \ll 0$ and in view of the graded flat base change property of local cohomology we see that $H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n \cong (H^i_{R_+}(M)_n)_{\mathfrak{p}_0}$ is an $(R_0)_{\mathfrak{p}_0}$-module of finite length and hence $H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n = \Gamma_{\mathfrak{p}_0}(H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n)$ for all $n \ll 0$. But according to [B-F-T, Theorem 3.5 d)] there is a polynomial $\widetilde{P}^{[\mathfrak{p}_0]} \in \mathbb{Q}[\mathbf{x}]$ of degree less than i such that $\mathrm{length}_{(R_0)_{\mathfrak{p}_0}}(H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n) = \widetilde{P}^{[\mathfrak{p}_0]}(n)$ for all $n \ll 0$. Now we get our claim by Theorem 3.6. □

Our last result is an extension of [B-F-T, Theorem 2.5 b)] to the case $\dim(R_0) = 2$.

5.10. **Proposition.** *Let $i \in \mathbb{N}_0$ and let $\dim(R_0) \leq 2$. Then the graded R-module $H^1_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ is Artinian.*

Proof. If $\dim(R_0) \leq 1$, this statement is the same as [B-F-T, Theorem 2.5 b)]. Let $\dim(R_0) = 2$. There is a system of parameters (x, y) of R_0 and hence $H^1_{\mathfrak{m}_0 R}(H^i_{R_+}(M)) = H^1_{(x,y)R}(H^i_{R_+}(M))$. Consider the exact sequence of R-modules

$$0 \to H^1_{yR}(H^{i-1}_{R_+}(M)) \to H^i_{(y,R_+)}(M) \to \Gamma_{yR}(H^i_{R_+}(M)) \to 0,$$

which yields an epimorphism $H^1_{xR}(H^i_{(y,R_+)}(M)) \twoheadrightarrow H^1_{xR}(\Gamma_{yR}(H^i_{R_+}(M)))$. Furthermore, there is a monomorphism

$$H^1_{xR}(H^i_{(y,R_+)}(M)) \rightarrowtail H^{i+1}_{(x,y,R_+)}(M) = H^{i+1}_{\mathfrak{m}}(M),$$

and the latter module is Artinian. So $H^1_{xR}(\Gamma_{yR}(H^i_{R_+}(M)))$ is Artinian.
Application of the functor Γ_{xR} to the monomorphism $H^1_{yR}(H^i_{R_+}(M)) \rightarrowtail H^{i+1}_{(y,R_+)}(M)$ yields a monomorphism $\Gamma_{xR}(H^1_{yR}(H^i_{R_+}(M))) \rightarrowtail \Gamma_{xR}(H^{i+1}_{(y,R_+)}(M))$. Furthermore, there is an epimorphism

$$H^{i+1}_{\mathfrak{m}}(M) = H^{i+1}_{(x,y,R_+)}(M) \twoheadrightarrow \Gamma_{xR}(H^{i+1}_{(y,R_+)}(M)),$$

and the first module is Artinian. So $\Gamma_{xR}(H^1_{yR}(H^i_{R_+}(M)))$ is Artinian. Now, by the exact sequence

$$0 \to H^1_{xR}(\Gamma_{yR}(H^i_{R_+}(M))) \to H^1_{(x,y)R}(H^i_{R_+}(M)) \to \Gamma_{xR}(H^1_{yR}(H^i_{R_+}(M))) \to 0$$

we get our claim. □

The following example shows that the above result need not hold if $\dim(R_0) > 2$.

5.11. **Example.** Let K be a field, let $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{t}$ be indeterminates and let $R_0 := K[\mathbf{x}, \mathbf{y}, \mathbf{z}]_{(\mathbf{x},\mathbf{y},\mathbf{z})}$ and $\mathfrak{m}_0 := (\mathbf{x}, \mathbf{y}, \mathbf{z})R_0$. Furnish the polynomial ring $S := R_0[\mathbf{u}, \mathbf{v}]$ with its standard grading and consider the Noetherian homogeneous R_0-algebra $R := S/(\mathbf{xv} - \mathbf{yu})S$, which is canonically isomorphic to the Rees ring $R_0[(\mathbf{x}, \mathbf{y})\mathbf{t}]$ of R_0 with respect to the ideal $(\mathbf{x}, \mathbf{y}) \subseteq R_0$. Let $\overline{R_0} := K[\mathbf{x}, \mathbf{y}]_{(\mathbf{x},\mathbf{y})} \cong R_0/\mathbf{z}R_0$, $\overline{\mathfrak{m}_0} :=$

$\mathfrak{m}_0\overline{R_0} = (\mathbf{x}, \mathbf{y})\overline{R_0}$ and $\overline{R} := R/\mathbf{z}R = \overline{S}/(\mathbf{xv} - \mathbf{yu})\overline{S}$, where $\overline{S} := \overline{R_0}[\mathbf{u}, \mathbf{v}]$. If we apply cohomology to the exact sequence $0 \to R \xrightarrow{\mathbf{z}\cdot} R \to \overline{R} \to 0$ and observe that R_+ is generated by two elements, we get an exact sequence of graded R-modules

$$(\alpha) \qquad H^2_{R_+}(R) \xrightarrow{\mathbf{z}\cdot} H^2_{R_+}(R) \to H^2_{\overline{R}_+}(\overline{R}) \to 0.$$

As $R \cong R_0 \otimes_{\overline{R_0}} \overline{R}$, the graded flat base change property of local cohomology gives rise to isomorphisms of R_0-modules $H^2_{R_+}(R) \cong R_0 \otimes_{\overline{R_0}} H^2_{\overline{R}_+}(\overline{R})$. As R_0 is isomorphic to a localization of the polynomial ring $\overline{R_0}[\mathbf{z}]$, it follows that the map $\mathbf{z}\cdot$ in (α) is injective. so applying local cohomology with support in $\mathfrak{m}_0 R$ we get a monomorphism of R-modules $\Gamma_{\mathfrak{m}_0 R}(H^2_{\overline{R}_+}(\overline{R})) \rightarrowtail H^1_{\mathfrak{m}_0 R}(H^2_{R_+}(R))$. But according to [B-F-T, Example 4.2], the R-module $\Gamma_{\mathfrak{m}_0 R}(H^2_{\overline{R}_+}(\overline{R})) = \Gamma_{\overline{\mathfrak{m}_0}}(H^2_{\overline{R}_+}(\overline{R}))$ is not Artinian. So $H^1_{\mathfrak{m}_0 R}(H^2_{R_+}(R))$ is not Artinian.

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