ASYMPTOTIC DEPTH OF TWISTED HIGHER DIRECT IMAGE SHEAVES

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Abstract

Let $\pi : X \to X_0$ be a projective morphism of schemes, such that X_0 is noetherian and essentially of finite type over a field K. Let $i \in \mathbb{N}_0$, let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $\mathcal{L}$ be an ample invertible sheaf over X. Let $Z_0 \subseteq X_0$ be a closed set. We show that the depth of the higher direct image sheaf $\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F})$ along Z_0 ultimately becomes constant if n tends to $-\infty$, provided X_0 has dimension ≤ 2. There are various examples which show that the mentioned asymptotic stability may fail if $\dim(X_0) \geq 3$. To prove our stability result, we show that for a finitely generated graded module M over a homogeneous noetherian ring $R = \bigoplus_{n\geq 0} R_n$ for which R_0 is essentially of finite type over a field and an ideal $\mathfrak{a}_0 \subseteq R_0$ the $\mathfrak{a}_0$-depth of the n-th graded component $H^i_{R_+}(M)_n$ of the i-th local cohomology module of M with respect to $R_+ := \bigoplus_{k>0} R_k$ ultimately becomes constant in codimension ≤ 2 if n tends to $-\infty$.

1 Introduction

Let $\pi : X \to X_0$ be a projective morphism of schemes, such that X_0 is noetherian and essentially of finite type over a field. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules. Let $i \in \mathbb{N}_0$. In [2, Theorem 5.5] we did show:

(1.1) *If $n \to -\infty$, the set*

$$\begin{aligned} & \mathrm{Ass}_{X_0}(\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))^{\leq 2} \\ := \; & \{x_0 \in \mathrm{Ass}_{X_0}(\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F})) \mid \dim(\mathcal{O}_{X_0,x_0}) \leq 2\} \end{aligned}$$

ultimately becomes constant.

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Our aim is to prove a corresponding but stronger stability result for the depths in codimension ≤ 2 of the sheaves $\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F})$ along a closed subset $Z_0 \subseteq X_0$.

To make this precise we introduce the following notion: If $Z_0 \subseteq X_0$ is a closed set, $\mathcal{G}$ is a coherent sheaf of $\mathcal{O}_{X_0}$-modules and $t \in \mathbb{N}_0$, we define the *depth of* $\mathcal{G}$ *along* Z_0 and the *depth in codimension* $\leq t$ *of* $\mathcal{G}$ *along* Z_0 respectively by

(1.2) $\operatorname{depth}(Z_0, \mathcal{G}) := \inf\{\operatorname{depth}(\mathcal{G}_{x_0}) \mid x_0 \in Z_0\};$

(1.3) $\operatorname{depth}(Z_0, \mathcal{G})^{\leq t} := \inf\{\operatorname{depth}(\mathcal{G}_{x_0}) \mid x_0 \in Z_0, \dim(\mathcal{O}_{X_0,x_0}) \leq t\}.$

We convene that $\inf(\emptyset) := \infty$, so that

$$\operatorname{depth}(Z_0, \mathcal{G})^{\leq t} = \infty \text{ if } \operatorname{codim}(Z_0, X_0) > t.$$

Our main result now says (cf. Theorem 3.5)

(1.4) *If* $n \to -\infty$, *the number*

$$\operatorname{depth}(Z_0, \mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))^{\leq 2}$$

ultimately becomes constant.

As an immediate consequence we obtain (cf. Corollary 3.6)

(1.5) *Let* $\dim(X_0) \leq 2$. *Then, if* $n \to -\infty$, *the number*

$$\operatorname{depth}(Z_0, \mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))$$

ultimately becomes constant.

The examples constructed by Chardin-Cutkosky-Herzog-Srinivasan [5] illustrate that the conclusion of (1.5) need not hold if $\dim(X_0) \geq 3$.

The basic tool to prove our main result is a corresponding stability result for the depths of graded components of certain local cohomology modules. We shall establish this result in the next section.

2 Depth and Local Cohomology

By $\mathbb{N}_0$ we denote the set of non-negative integers, by $\mathbb{N}$ the set of positive integers.

Notations and Conventions 2.1. (A) Throughout this paper let $R = R_0 \oplus R_1 \oplus \cdots$ be a *homogeneous noetherian ring.* So, R ist $\mathbb{N}_0$-graded, R_0 ist noetherian and there are finitely many elements $a_1, \ldots, a_k \in R_1$ such that $R = R_0[a_1, \ldots, a_k]$. By R_+ we denote the *irrelevant ideal* of R, thus $R_+ = R_1 \oplus R_2 \oplus \cdots$.

(B) If $i \in \mathbb{N}_0$ and M is a graded R-module, we write $H^i_{R_+}(M)$ for the i-th *local cohomology module of M with respect to R_+*, and we always furnish this module with its natural grading.
For $n \in \mathbb{Z}$ we denote by $H^i_{R_+}(M)_n$ the *n-th graded component of* $H^i_{R_+}(M)$. Keep in mind that $H^i_{R_+}(M)_n$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$ and vanishes for all $n \gg 0$, provided that the graded R-module M ist finitely generated.

(C) Now, fix an ideal $\mathfrak{a}_0 \subseteq R_0$. We write $\mathrm{Var}(\mathfrak{a}_0)$ for the *variety* $\{\mathfrak{p}_0 \in \mathrm{Spec}(R_0) \mid \mathfrak{a}_0 \subseteq \mathfrak{p}_0\}$ of $\mathfrak{a}_0$. Keep in mind that for a finitely generated R_0-module T we always have

$$\mathrm{depth}(\mathfrak{a}_0, T) = \inf\{\mathrm{depth}(T_{\mathfrak{p}_0}) \mid \mathfrak{p}_0 \in \mathrm{Var}(\mathfrak{a}_0)\}.$$

Now, for any finitely generated R_0-module T and any $t \in \mathbb{N}_0$ we define the *depth in codimension $\leq t$ of T with respect to* $\mathfrak{a}_0$ by:

$$\mathrm{depth}(\mathfrak{a}_0, T)^{\leq t} = \inf\{\mathrm{depth}(T_{\mathfrak{p}_0}) \mid \mathfrak{p}_0 \in \mathrm{Var}(\mathfrak{a}_0), \mathrm{height}(\mathfrak{p}_0) \leq t\}.$$

Again we use the convention that $\inf(\emptyset) = \infty$, so that

$$\mathrm{depth}(\mathfrak{a}_0, T)^{\leq t} \in \{0, 1, \ldots, t, \infty\}$$

with

$$\mathrm{depth}(\mathfrak{a}_0, T)^{\leq t} = \infty \Longleftrightarrow \forall \mathfrak{p}_0 \in \mathrm{Var}(\mathfrak{a}_0) \cap \mathrm{Supp}(T) : \mathrm{height}(\mathfrak{p}_0) > t.$$

(D) We say that a graded R-module $U = \bigoplus_{n \in \mathbb{Z}} U_n$ is *tame* if

$$U_n = 0 \text{ for all } n \ll 0 \text{ or } U_n \neq 0 \text{ for all } n \ll 0.$$

(E) Let $(\mathcal{S}_n)_{n \in \mathbb{Z}}$ be a family of numbers or sets. We say that $\mathcal{S}_n$ is *asymptotically stable for $n \to -\infty$* if there is some $n_0 \in \mathbb{Z}$ such that $\mathcal{S}_n = \mathcal{S}_{n_0}$ for all $n \leq n_0$.

Lemma 2.2. *Assume that* $(R_0, \mathfrak{m}_0)$ *is local and of dimension* ≤ 2. *Let* $i \in \mathbb{N}_0$ *and let* M *be a finitely generated graded* R*-module such that* $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ *is asymptotically stable for* $n \to -\infty$.
Then $\mathrm{depth}_{R_0}(H^i_{R_+}(M)_n)$ *is asymptotically stable for* $n \to -\infty$.

Proof. By the asymptotic stability of $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ for $n \to -\infty$ we have

$$\mathrm{depth}_{R_0}(H^i_{R_+}(M)_n) = 0 \textit{ for all } n \ll 0, \text{ or else}$$

$$\mathrm{depth}_{R_0}(H^i_{R_+}(M)_n) > 0 \textit{ for all } n \ll 0.$$

In the first case we are done. So, assume that we are in the second case. According to [3, Proposition 5.10] the graded R-module $H^1_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ is Artinian and hence tame. Therefore either

$$H^1_{\mathfrak{m}_0}(H^i_{R_+}(M)_n) = 0 \text{ for all } n \ll 0, \text{ or else}$$

$$H^1_{\mathfrak{m}_0}(H^i_{R_+}(M)_n) \neq 0 \text{ for all } n \ll 0.$$

As $\dim(R_0) \leq 2$ we thus respectively have either

$$\mathrm{depth}_{R_0}(H^i_{R_+}(M)_n) = 2 \text{ for all } n \ll 0, \text{ or}$$

$$\mathrm{depth}_{R_0}(H^i_{R_+}(M)_n) = 1 \text{ for all } n \ll 0.$$

□

Proposition 2.3. *Let* $\mathfrak{a}_0 \subseteq R_0$ *be an ideal, let* $i \in \mathbb{N}_0$ *and let* M *be a finitely generated graded* R*-module such that* $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2}$ *is asymptotically stable for* $n \to -\infty$.
Then $\mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n)^{\leq 2}$ *is asymptotically stable for* $n \to -\infty$.

Proof. Let $t := \liminf_{n \to -\infty} \mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n)^{\leq 2}$. Observe that $t \in \{0, 1, 2, \infty\}$ (cf. 2.1 (C)). We have to show that $t(n) := \mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n)^{\leq 2} = t$ for all $n \ll 0$.
If $t = \infty$, this is clear. So, let $t \in \{0, 1, 2\}$. We set

$$V := \{\mathfrak{p}_0 \in \mathrm{Var}(\mathfrak{a}_0) \mid \mathrm{height}(\mathfrak{p}_0) \leq 2\}.$$

By our hypothesis the set $V \cap \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ takes a constant value U for all $n \ll 0$. If $U \neq \emptyset$, we have $\mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n)^{\leq 2} = 0$ for all $n \ll 0$. If $U = \emptyset$ we have $\mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n)^{\leq 2} > 0$ for all $n \ll 0$. This gives our claim if $t = 0$.
So, assume that $t \in \{1, 2\}$. Then $\mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n)^{\leq 2} \geq 1$ for all $n \ll 0$. By our hypothesis, the set $V \cap \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n) = V \cap \overline{\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)}$

takes a constant value W for all $n \ll 0$. If $\mathrm{height}(\mathfrak{p}_0) \leq 1$ for some $\mathfrak{p}_0 \in W$, we have $\mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n) \leq \mathrm{height}(\mathfrak{p}_0) \leq 1$ for all $n \ll 0$, so that $\mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n) = 1$ for all $n \ll 0$. Thus, our claim follows in this case.

Therefore, we may assume that $\mathrm{height}(\mathfrak{p}_0) = 2$ for all $\mathfrak{p}_0 \in W$. As W is closed in V and V is closed under generalization, the set W must be finite. As $t < \infty$ we must have $\mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n)^{\leq 2} < \infty$ for infinitely many $n < 0$. Therefore $W \neq \emptyset$ (cf. 2.1 (C)).

Now by Lemma 2.2 and the Flat Base-Change Property of local cohomology (cf. [4]) we get that $\mathrm{depth}_{(R_0)_{\mathfrak{p}_0}}((H^i_{R_+}(M)_n)_{\mathfrak{p}_0})$ is asymptotically stable for $n \to -\infty$ for all $\mathfrak{p}_0 \in W$. As W is finite it follows that $\mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n) = \min\{\mathrm{depth}_{(R_0)_{\mathfrak{p}_0}}((H^i_{R_+}(M)_n)_{\mathfrak{p}_0}) \mid \mathfrak{p}_0 \in W\}$ is asymptotically stable for $n \to -\infty$. □

Corollary 2.4. *Assume that R_0 is essentially of finite type over a field. Let $i \in \mathbb{N}_0$, let $\mathfrak{a}_0 \subseteq R_0$ be an ideal and let M be a finitely generated graded R-module.*
Then $\mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n)^{\leq 2}$ *is asymptotically stable for $n \to -\infty$.*

Proof. According to [2, Proposition 3.5] the set $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2}$ is asymptotically stable for $n \to -\infty$. Now, we may conclude by Proposition 2.3. □

Corollary 2.5. *Assume that* $\dim(R_0) \leq 2$ *and R_0 is essentially of finite type over a field. Let $i \in \mathbb{N}_0$, let $\mathfrak{a}_0 \subseteq R_0$ be an ideal and let M be a finitely generated graded R-module.*
Then $\mathrm{depth}(\mathfrak{a}_0, H^i_{R_+}(M)_n)$ *is asymptotically stable for $n \to -\infty$.* □

This result actually is shown in [1].

Remarks 2.6. (A) According to [5], there is a normal homogeneous domain $R = R_0 \oplus R_1 \oplus \cdots$, such that $(R_0, \mathfrak{m}_0)$ is local of dimension 3, essentially of finite type over $\mathbb{C}$ and such that $H^2_{R_+}(R)$ is not tame. In particular $\mathrm{depth}(\mathfrak{m}_0, H^2_{R_+}(M)_n)$ is not asymptotically stable for $n \to -\infty$. So in codimensions ≥ 3 the depth of $H^i_{R_+}(M)_n$ need not be asymptotically stable any more.

(B) Let $R = R_0 \oplus R_1 \oplus \cdots$ be as in 2.1 (A), let $\mathfrak{a}_0 \subseteq R_0$ be an ideal and let M be a finitely generated graded R-module. Let

$$c := \sup\{i \in \mathbb{N}_0 \mid H^i_{R_+}(M) \neq 0\}$$

be the *cohomological dimension of M with respect to R_+* and let

$$f := \inf\{i \in \mathbb{N} \mid H^i_{R_+}(M) \text{ not finitely generated }\}$$

be the *cohomological finiteness dimension of M with respect to R_+*.
If $f < \infty$, then clearly $f \leq c$. Moreover it is well known that $\mathrm{Ass}_{R_0}(H^c_{R_+}(M)_n)$ need not be asymptotically stable for $n \to -\infty$ (cf. [8] for example) and that $\mathrm{Ass}_{R_0}(H^f_{R_+}(M)_n)$ is asymptotically stable for $n \to -\infty$. In [6] it is shown:
If $f = c$, then $\mathrm{depth}(\mathfrak{a}_0, H^c_{R_+}(M)_n)$ is asymptotically stable for $n \to -\infty$.

3 Depth and Higher Direct Images

For the unexplained terminology of this section we refer to [7].

Notations and Conventions 3.1. (A) For the rest of this note let X_0 denote a noetherian scheme, let $\pi : X \to X_0$ denote a projective scheme over X_0 with very ample sheaf $\mathcal{O}_X(1)$ and let $Z_0 \subseteq X_0$ be a closed set.

(B) If $\mathcal{G}$ is a coherent sheaf of $\mathcal{O}_{X_0}$-modules and $t \in \mathbb{N}_0$ we always use the notation introduced in (1.2) and (1.3).

(C) If X_0 is affine and $I(Z_0) \subseteq \mathcal{O}(X_0)$ is the vanishing ideal of Z_0, we write

$$\mathrm{depth}(Z_0, T) := \mathrm{depth}(I(Z_0), T) = \mathrm{depth}(Z_0, \widetilde{T}) \text{ and}$$

$$\mathrm{depth}(Z_0, T)^{\leq t} := \mathrm{depth}(I(Z_0), T)^{\leq t} = \mathrm{depth}(Z_0, \widetilde{T})^{\leq t}$$

for each finitely generated $\mathcal{O}(X_0)$-module T.

Proposition 3.2. *Assume that X_0 is affine and essentially of finite type over a field. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$.*
Then $\mathrm{depth}(Z_0, H^i(X, \mathcal{F}(n)))^{\leq 2}$ *is asymptotically stable for $n \to -\infty$.*

Proof. Let $R_0 := \mathcal{O}(X_0)$, $\mathfrak{a}_0 := I(Z_0) \subseteq R_0$. Then, there is a homogeneous noetherian R_0-algebra $R = R_0 \oplus R_1 \oplus \cdots$ with $X = \mathrm{Proj}(R)$ and $\mathcal{O}_X(1) = R(1)^{\sim}$. Moreover there is a finitely generated graded R-module M such that $\mathcal{F} = \widetilde{M}$. Now, for each $n \in \mathbb{Z}$ the Serre-Grothendieck Correspondence gives rise to a short exact sequence of R_0-modules

$$0 \to H^0_{R_+}(M)_n \to M_n \to H^0(X, \mathcal{F}(n)) \to H^1_{R_+}(M)_n \to 0$$

and to isomorphisms of R_0-modules

$$H^j(X, \mathcal{F}(n)) \cong H^{j+1}_{R_+}(M)_n \text{ for all } j > 0.$$

Therefore, our claim follows by Corollary 2.4. $\square$

Proposition 3.3. *Assume that X_0 is affine and essentially of finite type over a field. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules, let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$.*
Then, $\operatorname{depth}(Z_0, H^i(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))^{\leq 2}$ *is asymptotically stable for $n \to -\infty$.*

Proof. This follows from Proposition 3.2 by essentially the same arguments as used in the proof of [2, Theorem 5.3]. □

Corollary 3.4. *Let X_0, $\mathcal{L}$, $\mathcal{F}$ and i be as in Proposition 3.3. Assume in addition that* $\dim(X_0) \leq 2$*.*
Then, $\operatorname{depth}(Z_0, H^i(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))$ *is asymptotically stable for $n \to -\infty$.* □

Theorem 3.5. *Let X_0 be essentially of finite type over a field. Let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $i \in \mathbb{N}_0$.*
Then $\operatorname{depth}(Z_0, \mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))^{\leq 2}$ *is asymptotically stable for $n \to -\infty$.*

Proof. We may assume that X_0 is affine. Now, we can conclude by Proposition 3.3 as $\mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}) \cong H^i(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F})^{\sim}$. □

Corollary 3.6. *Let X_0, $\mathcal{L}$, $\mathcal{F}$ and i be as in Theorem 3.5. Assume in addition that* $\dim(X_0) \leq 2$*.*
Then $\operatorname{depth}(Z_0, \mathcal{R}^i\pi_*(\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}))$ *is asymptotically stable for $n \to -\infty$.* □

Remark 3.7. The observations made in 2.6 (A) show that the conclusions of the Corollaries 3.4 and 3.6 need not hold if $\dim(X_0) \geq 3$.

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