ON PROJECTIVE CURVES OF MAXIMAL REGULARITY

MARKUS BRODMANN AND PETER SCHENZEL

Abstract. Let $\mathcal{C} \subseteq \mathbb{P}^r_K$ be a non-degenerate projective curve of degree $d > r+1$ of maximal regularity so that $\mathcal{C}$ has an extremal secant line $\mathbb{L}$. We show that $\mathcal{C} \cup \mathbb{L}$ is arithmetically Cohen Macaulay if $d < 2r-1$ and we study the Betti numbers and the Hartshorne-Rao module of the curve $\mathcal{C}$.

1. Introduction

Let $\mathcal{C} \subseteq \mathbb{P}^r_K$ denote a non-degenerate projective curve, where K is an algebraically closed field. Two basic numerical invariants related to $\mathcal{C}$ are the degree $\deg \mathcal{C}$ and the Castelnuovo-Mumford regularity $\operatorname{reg} \mathcal{C}$. In their fundamental paper (cf. [5]) Gruson, Lazarsfeld and Peskine have shown that

$$\operatorname{reg} \mathcal{C} \leq \deg \mathcal{C} - r + 2.$$

The degree of the curve $\mathcal{C}$ reflects its geometric behaviour. The Castelnuovo-Mumford regularity of the curve $\mathcal{C} \subseteq \mathbb{P}^r_K$ defined in terms of the vanishing of local cohomology, can be expressed by the degree of the generators of the higher syzygy modules of the defining ideal $I_{\mathcal{C}}$ and thus reflects the cohomological and homological behaviour of $\mathcal{C}$.

In [1] we have studied non-degenerate curves of degree $r+2$ in $\mathbb{P}^r_K$. For $r \geq 4$ we were lead to distinguish four different main cases I - IV, according to the structure of the Hartshorne-Rao module of the considered curve. In geometric terms, case IV is precisely the case in which an extremal secant occurs or – in (co-)homological terms – the case of maximal regularity. In this paper, we investigate this latter geometric or homological situation in arbitrary degrees.

So, we consider a non-degenerate projective irreducible curve $\mathcal{C} \subseteq \mathbb{P}^r_K$ of degree $d > r+1$ (with $r \geq 3$) whose Castelnuovo-Mumford regularity takes the maximally possible value $d-r+2$. In this case, $\mathcal{C}$ is smooth and rational and has a $(d-r+2)$-secant line $\mathbb{L}$ (cf. [5]).

We study the Betti numbers and the Hartshorne-Rao module of the curve $\mathcal{C}$. To do so, we investigate the relation between the two curves $\mathcal{C}$ and $\mathcal{C} \cup \mathbb{L}$ from the cohomological and the homological point of view (cf. 2.7 resp. 4.1).

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Quite often $\mathcal{C} \cup \mathbb{L}$ is an arithmetically Cohen-Macaulay (CM) curve. Then much can be said on $\mathcal{C}$, mainly as the structure of the Hartshorne-Rao module is known in this case (cf. 3.3 (ix), 4.5). Obviously, to make use of this fact, one needs to know under which circumstances $\mathcal{C} \cup \mathbb{L}$ is arithmetically CM. We give various necessary and sufficient conditions for this (see 3.3, 3.6). Also, we show that $\mathcal{C} \cup \mathbb{L}$ is arithmetically CM if $d < 2r-1$ (cf. 3.5).

In the case where $\mathcal{C} \cup \mathbb{L}$ is arithmetically CM we give an approximation of the Betti numbers of $\mathcal{C}$ (cf. 4.6), which extends the corresponding result for $d = r+2$ (cf. [1]) to arbitrary degrees d.

Also we briefly discuss the "exceptional case" in which $d = r+1$ (cf. 5.1) and we present several examples calculated by means of the computer algebra system *Singular* (cf. [4]). By these example we notably illustrate:

- The occurrence and non-occurrence of a trisecant line in the exceptional case $d = r+1$ (cf. 5.1).
- The fact that in the general case, $\mathcal{C}$ need not lie on a surface of minimal degree (cf. 5.3).
- The fact that in the case $d = 2r-1$ the curve $\mathcal{C} \cup \mathbb{L}$ need not but can be arithmetically CM (cf. 5.4 resp. 5.5), while this is true in general for $d < 2r-1$.
- The variability of the (socle of) the Harthshorne-Rao module $\mathcal{C}$, if $\mathcal{C} \cup \mathbb{L}$ is not arithmetically CM (cf. 5.6).

In Section 2 we prove a few results about the secant lines. The main result 2.3 gives an estimate on the dimension of the space of global sections of $\mathcal{O}_{\mathcal{C}\cup\mathbb{L}}(1)$. Section 3 is devoted to the study of the structure of the coordinate ring of $\mathcal{C} \cup \mathbb{L}$. In Section 4 we continue with the homological aspect describing the Betti numbers of the coordinate rings of $\mathcal{C}$ and $\mathcal{C} \cup \mathbb{L}$ and their interaction.

2. Extremal Secants

We fix a few notations, which we use throughout this paper.

Notation and Remark 2.1. A) Let $R = \oplus_{n\geq 0} R_n$ be a non-negatively graded Noetherian ring. By R_+ we shall denote the irrelevant homogeneous ideal $\oplus_{n>0} R_n$ of R. If M is a graded R-module, and if $n \in \mathbb{Z}$, we use M_n to denote the n-th graded component of M. Also, for $n \in \mathbb{Z}$, we use $M_{\leq n}$ to denote the R_0-submodule $\oplus_{m\leq n} M_m$ of M and $M_{\geq n}$ to denote the graded R-submodule $\oplus_{m\geq n} M_m$ of M. All polynomial rings are furnished with their standard grading.

B) Let r be an integer ≥ 4, let K be an algebraically closed field and let $S = K[x_0, \ldots, x_r]$ be a polynomial ring. Let $\mathcal{C} \subseteq \mathbb{P}^r_K = \operatorname{Proj}(S)$ be a non-degenerate curve, hence a non-degenerate closed connected integral subscheme of dimension 1. Moreover, let $\mathcal{J} = \mathcal{J}_{\mathcal{C}} \subseteq \mathcal{O}_{\mathbb{P}^r_K}$ the sheaf of vanishing ideals of $\mathcal{C}$, let $I = I_{\mathcal{C}} = \oplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{J}(n)) \subseteq S$ denote the vanishing ideal of $\mathcal{C}$ and let $A = A_{\mathcal{C}} := S/I$ denote the homogeneous coordinate ring of $\mathcal{C}$.

C) Keep the above notation. We use d to denote the degree of $\mathcal{C}$ and $\overline{d}$ to denote the generating degree of I, thus $\overline{d} = \min\{n \in \mathbb{N} | I = (I_{\leq n})S\}$. Keeping in mind these definitions we have the inequalities

$$\overline{d} \leq \operatorname{reg} \mathcal{C} = \operatorname{reg} I \leq d - r + 2,$$

in which reg is used to denote Castelnuovo-Mumford regularities (cf. [5]).
D) Let $\mathbb{L} \subseteq \mathbb{P}^r_K$ be a line, let $L \subseteq S$ be the vanishing ideal of $\mathbb{L}$ and let

$$J = J_{\mathcal{C}\cup\mathbb{L}} = L \cap I \subseteq S$$

be the vanishing ideal of the union $\mathcal{C} \cup \mathbb{L} \subseteq \mathbb{P}^r_K$. Also let μ denote the degree (thus the length) of the scheme $\mathcal{C} \cap \mathbb{L} = \operatorname{Proj} S/(I+L) \subseteq \mathbb{P}^r_K$, so that $\mathbb{L}$ is a μ-secant of $\mathcal{C}$. As S/L is isomorphic to a polynomial ring in two indeterminates, the vanishing ideal $(I+L)^{\mathrm{sat}} = \cup_{n\in\mathbb{N}}(I+L) :_S (S_+)^n \subseteq S$ of the intersection $\mathcal{C} \cap \mathbb{L} \subseteq \mathbb{P}^r_K$ can be written in the following form

$$(I+L)^{\mathrm{sat}} = L + fS \text{ for some } f \in S_\mu.$$

Lemma 2.2. *In the notations of 2.1, we have*

$$\mu \leq \overline{d} \leq \operatorname{reg} \mathcal{C} \leq d - r + 2.$$

Moreover, if $\mu = \overline{d}$, we may choose $f \in I_\mu = I \cap S_\mu$. Finally, if $f \in I_\mu$, then

$$(I+L)^{\mathrm{sat}} = L + fS = I + L \textit{ and } I = J + fS.$$

Proof. As L is a prime ideal with $I, S_+ \nsubseteq L$, we have $I_{\overline{d}} \nsubseteq L$. As $I_{\overline{d}} \subseteq L + fS$ it follows $\mu \leq \overline{d}$. Moreover, if $\mu = \overline{d}$, then

$$L_\mu \subsetneqq I_\mu + L_\mu \subseteq (L+fS)_\mu = L_\mu + fK,$$

thus $I_\mu + L_\mu = L_\mu + fK$, and this allows to choose $f \in I_\mu$. Finally, whenever $f \in I_\mu$, we have $L + fS \subseteq I + L \subseteq (I+L)^{\mathrm{sat}} = L + fS$. This proves the stated equalities. □

Theorem 2.3. *If $\mu \geq 2$, then $h^0(\mathcal{C} \cup \mathbb{L}, \mathcal{O}_{\mathcal{C}\cup\mathbb{L}}(1)) \leq d - \mu + 3$.*

Proof. Let $\delta := h^0(\mathcal{C} \cup \mathbb{L}, \mathcal{O}_{\mathcal{C}\cup\mathbb{L}}(1))$ and consider the graded K-algebra

$$D := \oplus_{n\geq 0} H^0(\mathcal{C} \cup \mathbb{L}, \mathcal{O}_{\mathcal{C}\cup\mathbb{L}}(n)),$$

so that $\delta = \dim_K D_1 \geq r+1$. As $S/J \subseteq K[D_1] \subseteq D$ and $(S/J)_n = D_n$ for all $n \gg 0$, the inclusion $S/J \hookrightarrow K[D_1]$ yields an isomorphism of schemes

$$\varepsilon : \operatorname{Proj} K[D_1] \xrightarrow{\sim} \operatorname{Proj}(S/J) = \mathcal{C} \cup \mathbb{L}.$$

Now, set $\mathbb{P}^{\delta-1}_K = \operatorname{Proj} T$, where T is the polynomial ring $S[x_{r+1}, \ldots, x_{\delta-1}]$ and let π be a surjective homomorphism of graded K-algebras, which appears in the commutative diagram

$$\begin{array}{ccc} S & \hookrightarrow & T \\ \downarrow & & \downarrow \pi \\ S/J & \hookrightarrow & K[D_1]. \end{array}$$

Consider $\operatorname{Proj} K[D_1]$ as a closed non-degenerate subscheme of $\mathbb{P}_K^{\delta-1}$ by means of π and let $Z = \operatorname{Proj}(T/S_+T)$. Then, $Z \cap \operatorname{Proj} K[D_1] = \emptyset$ and we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Proj} K[D_1] & \xrightarrow[\varepsilon]{\sim} & \mathcal{C} \cup \mathbb{L} \\ \downarrow & & \downarrow \\ \mathbb{P}_K^{\delta-1} \setminus Z & \overset{p}{\twoheadrightarrow} & \mathbb{P}^r \end{array}$$

in which p is the projection centered at Z. So, for any closed subscheme $Y \subseteq \mathcal{C} \cup \mathbb{L}$, we know that $\varepsilon^{-1}(Y)$ is a closed subscheme of $\mathbb{P}_K^{\delta-1}$, isomorphic to Y and of the same degree as Y. Therefore $\mathcal{C}' := \varepsilon^{-1}(\mathcal{C}) \subseteq \mathbb{P}_K^{\delta-1}$ is a reduced irreducible curve of degree d, $\mathbb{L}' := \varepsilon^{-1}(\mathbb{L}) \subseteq \mathbb{P}_K^{\delta-1}$ is a line and $\mathcal{C}' \cup \mathbb{L}' = \operatorname{Proj} K[D_1]$. As

$$\mathcal{C}' \cap \mathbb{L}' = \varepsilon^{-1}(\mathcal{C} \cap \mathbb{L}) \simeq \mathcal{C} \cap \mathbb{L},$$

we see that $\mathbb{L}'$ is a μ-secant of $\mathcal{C}'$.

Moreover, $\mathcal{C}'$ is non-degeneratedly embedded into $\mathbb{P}_K^{\delta-1}$. Otherwise, we could find a hyperplane $H \subseteq \mathbb{P}_K^{\delta-1}$ with $\mathcal{C}' \subseteq H$. As $\mathcal{C}' \cup \mathbb{L}' \subseteq \mathbb{P}_K^{\delta-1}$ is non-degenerate, this would imply $\mathbb{L}' \nsubseteq H$ and hence $\mathcal{C}' \cap \mathbb{L}' \subseteq H \cap \mathbb{L}' \simeq \operatorname{Spec}(K)$, a contradiction to the assumption $\mu \geq 2$.

But now, by 2.2 we have $\mu \leq d - (\delta - 1) + 2$, thus $\delta \leq d - \mu + 3$. $\square$

Remark and Definition 2.4. A) We keep the notation of 2.1. In accordance with 2.1 C) we say that $\mathcal{C}$ is of *maximal regularity* if $\operatorname{reg} \mathcal{C} = d - r + 2$.

B) We say that $\mathbb{L}$ is an *extremal secant* of $\mathcal{C}$ if $\mu = \operatorname{reg} \mathcal{C}$. (This is justified in view of 2.2.)

C) On use of the table of [5, p. 504] we can say:

If $d > r + 1$ and if $\mathcal{C}$ is of maximal regularity, then $\mathcal{C}$ is smooth, rational and has an extremal secant line.

In particular we can say the following (cf. 2.2):

If $d > r + 1$, the curve $\mathcal{C}$ is of maximal regularity if and only if it has a $(d - r + 2)$-secant line. In this case $\mathcal{C}$ is smooth and rational.

Convention and Remark 2.5. A) We are interested in the case where $\mathcal{C}$ is of maximal regularity. For the moment, we do not focus on the two particular cases $d = r$ and $d = r + 1$. So, in view of 2.4 C) it is natural to convene from now on, that $d > r + 1$ and $\mu = d - r + 2$.

B) In view of 2.2 we now can write

$$I = J + fS \quad \text{with } f \in I_{d-r+2} \setminus L,$$

where L denotes the ideal defining the secant line. As $L \subseteq S$ is a prime ideal, we have $J :_S f = (I \cap L) :_S f = L :_S f = L$ and hence get graded isomorphisms

$$I/J \simeq fS/f(J :_S f) \simeq (S/L)(-d + r - 2)$$

which yield the short exact sequence of graded S-modules

$$0 \to S/L(-d + r - 2) \to S/J \to A \to 0.$$

C) In view of 2.4 C), the curve $\mathcal{C} \subseteq \mathbb{P}^r_K$ is smooth and rational. Therefore, the graded K-algebra

$$\Gamma(\mathcal{C}) := \oplus_{n\geq 0} H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n))$$

may be viewed as the homogeneous coordinate ring of a rational normal curve $\mathcal{C}' \subseteq \mathbb{P}^d_K$ of degree d, (cf. [5]). So, if $K[s,t]$ is a polynomial ring, we have an isomorphism of graded K-algebras, $\Gamma(\mathcal{C}) \simeq K[s,t]^{(d)}$, where $R^{(d)}$ is used to denote the d-th Veronesean subring $\oplus_{n\geq 0} R_{nd}$ of the graded ring $R = \oplus_{n\geq 0} R_n$.

Notation and Remark 2.6. A) Let R be a non-negatively graded Noetherian K-algebra and let M be a graded R-module. For $i \in \mathbb{N}_0$, let $H^i(M) = H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to the ideal R_+. Then $H^i(M)$ is a graded R-module. If M is finitely generated, each graded component of $H^i(M)$ is of finite dimension over K. In this case, we set

$$h^i(M)_n = h^i_{R_+}(M)_n := \dim_K H^i(M)_n.$$

Finally, we use $D(M)$ or $D_{R_+}(M)$ to denote the R_+-transform $\varinjlim \operatorname{Hom}_R((R_+)^n, M)$ of M, which is again a graded R-module.
B) Let $X = \operatorname{Proj}(R)$ and let $\mathcal{F} = \tilde{M}$ be the sheaf of $\mathcal{O}_X$-modules induced by M. Then, the *Serre-Grothendieck correspondence* yields natural isomorphisms of graded R-modules

$$D(M) \simeq \oplus_{n\in\mathbb{Z}} H^0(X, \mathcal{F}(n)) \text{ and } H^{i+1}(M) \simeq \oplus_{n\in\mathbb{Z}} H^i(X, \mathcal{F}(n)) \text{ for all } i > 0.$$

C) Let $Y \subseteq \mathbb{P}^r_K$ be a closed subscheme, $\mathcal{J}_Y \subseteq \mathcal{O}_{\mathbb{P}^r_K}$ its sheaf of vanishing ideals, $N \subseteq S$ its homogeneous vanishing ideal and $C = S/N$ its homogeneous coordinate ring. Then, by part B), the *Hartshorne-Rao module*

$$\oplus_{n\in\mathbb{Z}} H^1(\mathbb{P}^r_K, \mathcal{J}_Y(n))$$

of Y is naturally isomorphic to the graded S-module $H^1(C)$.

Proposition 2.7. *In the notation of 2.6 and under the convention 2.5 we have the following results*

a) $h^1(A)_n = \begin{cases} 0, & \textit{for } n \notin \{1,\ldots,d-r\}, \\ d-r, & \textit{for } n = 1, \\ 1, & \textit{for } n = d-r. \end{cases}$

b) $h^2(A)_n = \max\{0, -dn-1\}$ *for all* $n \in \mathbb{Z}$.

c) $h^1(S/J)_n = \begin{cases} 0, & \textit{for } n \notin \{2,\ldots,d-r-1\}, \\ h^1(A)_n - d + r + n - 1, & \textit{for } 2 \leq n \leq d-r-1. \end{cases}$

d) $h^2(S/J)_n = \begin{cases} 0, & \textit{for all } n > 0, \\ d-r+1, & \textit{for } n = 0, \\ -n(d+1) + d - r, & \textit{for all } n < 0. \end{cases}$

Proof. a): As A is a domain of dimension > 1 and K is algebraically closed, $h^1(A)_n = 0$ for all $n \leq 0$. As $\operatorname{reg} A = \operatorname{reg} \mathcal{C} - 1 = d - r + 1$, we have $h^1(A)_n = 0$ for all $n \geq d - r + 1$. In view of 2.6 C) and the table of [5, p. 504], we have

$$h^1(A)_{d-r} = h^1(\mathbb{P}^r_K, \mathcal{J}(d-r)) = 1.$$

By 2.5 C) and the Serre-Grothendieck correspondence 2.6 B) we get isomorphisms of graded K-algebras

$$D(A) \simeq \Gamma(\mathcal{C}) \simeq K[s,t]^{(d)}.$$

So, the graded exact sequence $0 \to A \to D(A) \to H^1(A) \to 0$ yields $h^1(A)_1 = \dim_K(K[s,t]^{(d)})_1 - \dim_K A_1 = d - r$.
b): Keep in mind the natural isomorphisms of graded A-modules

$$H^2(A) \simeq H^2(D(A)) \simeq H^2_{A_+}(D(A)) \simeq H^2_{A_+D(A)}(D(A)) = H^2_{\operatorname{Rad} A_+D(A)}(D(A))$$

which follow from the fact that $D(A)/A$ is of finite length and from the base ring independence of local cohomology. Because of the isomorphism $D(A) \simeq K[s,t]^{(d)}$ and as $\operatorname{Rad} A_+D(A) = D(A)_+$ we obtain graded isomorphisms

$$H^2(A) \simeq H^2_{D(A)_+}(D(A)) \simeq H^2_{(K[s,t]^{(d)})_+}(K[s,t]^{(d)}) \simeq H^2_{K[s,t]_+}(K[s,t])^{(d)}.$$

Here we have to remind the fact that local cohomology commutes with taking Veroneseans. So it follows $h^2(A)_n = h^2_{K[s,t]_+}(K[s,t])_{dn} = \max\{0, -dn-1\}$.
c), d): If we apply local cohomology to the short exact sequence of 2.5 B) and keep in mind that $H^i(S/L) = 0$ for all $i \neq 2$ and $h^2(S/L)_n = \max\{0, -n-1\}$ for all $n \in \mathbb{Z}$, we obtain exact sequences

$$0 \to H^1(S/J)_n \to H^1(A)_n \to K^{\max\{0,-n+d-r+1\}} \to H^2(S/J)_n \to H^2(A)_n \to 0.$$

It follows from a) that $h^1(S/J)_n = 0$ for all $n \notin \{1, \ldots, d-r\}$. By 2.3 and on use of the Serre-Grothendieck correspondence we have

$$\dim_K D(S/J)_1 = h^0(\mathcal{C} \cup \mathbb{L}, \mathcal{O}_{\mathcal{C}\cup\mathbb{L}}(1)) \leq r + 1.$$

Hence, the graded short exact sequence $0 \to S/J \to D(S/J) \to H^1(S/J) \to 0$ induces $h^1(S/J)_1 \leq r + 1 - \dim_K(S/J)_1$. As $J_1 \subseteq I_1 = 0$, we get $h^1(S/J)_1 = 0$.
If we apply the above sequence with $n = 1$ and keep in mind statements a) and b), it follows $h^2(S/J)_1 = 0$ and hence $h^2(S/J)_n = 0$ for all $n > 0$. Now another use of statements a), b) and the above exact sequences proves statements d) and c). □

3. On the Structure of S/J

We keep the notation introduced in 2.1 and the convention made in 2.5. Our aim is to study the homogeneous coordinate ring S/J of the union $\mathcal{C} \cup \mathbb{L} \subseteq \mathbb{P}^r_K$ and to relate it to the curve $\mathcal{C}$ and its Hartshorne-Rao module $H^1(A)$. We start with a few general observations.

Remark 3.1. A) For a finitely generated graded module M over a non-negatively graded Noetherian ring $R = \oplus_{n\geq 0} R_n$ and for $i \in \mathbb{N}_0$ let

$$a_i(M) := \sup\{n \in \mathbb{Z} | H^i_{R_+}(M)_n \neq 0\},$$

with the convention that $\sup\emptyset = -\infty$. In this notation it follows from 2.7 c), d), that

$$\operatorname{reg} S/J = \max\{2, a_1(A)+1\} \leq d-r.$$

B) As a consequence of this last observation we have $\operatorname{reg} J \leq d-r+1$. By 2.5 B) we have $I = J + fS$ for some $f \in I_{d-r+2}$. As J is generated in degrees $\leq \operatorname{reg} J$, it follows

$$J = (I_{\leq \operatorname{reg} J})S = (I_{\leq d-r+1})S.$$

C) As a consequence of the previous equalities we have

$$L = (J :_S f) = (J :_S (J_{d-r+2} + fK)) = ((I_{\leq d-r+1}) :_S I_{d-r+2}),$$

so that L is determined by I and hence $\mathbb{L}$ by $\mathcal{C}$. This shows that $\mathbb{L}$ is the unique extremal secant line of $\mathcal{C}$.

We add a few more observations concerning the Hartshorne-Rao modules $H^1(A)$ and $H^1(S/J)$ of the curves $\mathcal{C} \subseteq \mathbb{P}^r_K$ resp. $\mathcal{C} \cup \mathbb{L} \subseteq \mathbb{P}^r_K$.

Remark 3.2. A) As $H^i(D(A)) = 0$ for all $i \neq 2$ and as $H^2(D(A)) \simeq H^2(A)$, it follows from 2.7 b) that the A-module $D(A)$ satisfies $\operatorname{reg} D(A) = 1$. So, as a graded A-module, $D(A)$ is generated in degrees ≤ 1. In view of the natural epimorphism

$$D(A) \to H^1(A) \to 0$$

and as $H^1(A)_0 = 0$ (cf. 2.7 a)) it follows, that $H^1(A)$ is generated in degree 1, and thus $H^1(A) = (H^1(A)_1)S$.

B) As $H^i(D(S/J)) = 0$ for all $i \neq 2$ and as $H^2(D(S/J)) \simeq H^2(S/J)$, it follows from 2.7 d) that $\operatorname{reg} D(S/J) = 2$, so that the S/J-module $D(S/J)$ is generated in degrees ≤ 2. In view of the natural epimorphism

$$D(S/J) \to H^1(S/J) \to 0$$

and as $H^1(S/J)_{\leq 1} = 0$ (cf. 2.7 c)), it follows that $H^1(S/J)$ is generated in degree 2, thus $H^1(S/J) = (H^1(S/J)_2))S$.

C) For a graded S-module T, let $\operatorname{soc} T$ denote the *socle* $0 :_S S_+ = \operatorname{Hom}_S(K, T)$ of T. If we apply cohomology to the sequence of 2.5 B) and keep in mind the left-exactness of the functor soc, we get an exact sequence of graded S-modules

$$0 \to \operatorname{soc} H^1(S/J) \to \operatorname{soc} H^1(A) \to \operatorname{soc} H^2(S/L(-d+r-2)).$$

As $\operatorname{soc} H^2(S/L(-d+r-2)) = K(-d+r)$ and as $H^1(S/J)_{d-r} = 0$, (cf. 2.7 c)), we thus get an isomorphism of graded S-modules

$$\operatorname{soc} H^1(A) \simeq \operatorname{soc}(H^1(S/J)) \oplus K(-d+r),$$

which relates the socles of the Hartshorne-Rao modules of the two curves $\mathcal{C}$ and $\mathcal{C} \cup \mathbb{L} \subseteq \mathbb{P}^r_K$.

Resuming our previous results, we get the following Cohen-Macaulay (CM) criterion for the ring S/J.

Theorem 3.3. *The following statements are equivalent:*

(i) S/J *is CM.*
(ii) $H^1(S/J) = 0$.
(iii) $h^1(S/J)_2 = 0$.
(iv) $\operatorname{reg} S/J = 2$.
(v) $h^1(A)_n = d - r + 1 - n$, *for* $n = 1, \ldots, d - r$.
(vi) $h^1(A)_2 \leq d - r - 1$.
(vii) $\operatorname{soc} H^1(A) = K(r - d)$.
(viii) *There is an isomorphism of graded* S*-modules*

$$H^1(A) \simeq H^2(S/L)(-d + r - 2)_{\geq 1}.$$

(ix) *There are independent linear forms* $y_0, \ldots, y_r \in S_1$ *and an isomorphism of graded* S*-modules*

$$H^1(A) \simeq \operatorname{Hom}_K(S/((y_0, y_1)^{d-r} + (y_2, \ldots, y_r)), K)(r - d).$$

Proof. (i) $\Longleftrightarrow$ (ii): This is clear, as $\dim S/J = 2$ and $H^0(S/J) = 0$.
(ii) $\Longleftrightarrow$ (iii): Clear, as $H^1(S/J)$ is generated in degree 2 (cf. 3.2 B)).
(ii) $\Longleftrightarrow$ (iv): Clear by the estimate of 3.1 A) and the fact that $H^1(S/J)_{\leq 1} = 0$.
(ii) $\Longleftrightarrow$ (v): Clear by 2.7 a) and c).
(iii) $\Longleftrightarrow$ (vi): Clear by 2.7 c).
(ii) $\Longleftrightarrow$ (vii): As $H^1(S/J)_{\geq d-r} = 0$ (cf. 2.7 c)) and as $H^1(S/J) = 0$ if and only if $\operatorname{soc} H^1(S/J) = 0$, we conclude by the isomorphism of 3.2 C).
(ii) $\Longrightarrow$ (viii): If we apply the cohomology functor to the exact sequence of 2.5 B), keep in mind that $H^1(S/J) = 0$ and $H^1(A)_{\leq 0} = H^2(S/J)_{\geq 1} = 0$ (cf. 2.7 a), d)) we get an isomorphism of graded S-modules $H^1(A) \simeq H^2(S/L)(-d + r - 2)_{\geq 1}$.
(viii) $\Longrightarrow$ (ix): Choose linear forms $y_0, y_1, y_2, \ldots, y_r \in S_1$ such that $L = (y_2, \ldots, y_r)$ and $S_+ = (y_0, y_1) + L$. Then, there is an isomorphism of graded S-modules

$$S/((y_0, y_1)^{d-r} + (y_2, \ldots, y_r)) \simeq (S/L)/(S/L)_{\geq d-r}.$$

Moreover, there is an isomorphism of graded S-modules

$$\operatorname{Hom}_K((S/L)/(S/L)_{\geq d-r}, K) \simeq \operatorname{Hom}_K(S/L, K)_{\geq r-d+1}.$$

By graded local duality, there is an isomorphism of graded S-modules

$$\operatorname{Hom}_K(S/L, K) \simeq H^2(S/L)(-2).$$

So, there are isomorphisms of graded S-modules

$$H^2(S/L)(-d + r - 2)_{\geq 1} \simeq \operatorname{Hom}_K(S/L, K)(-d + r)_{\geq 1} \simeq$$
$$\operatorname{Hom}_K(S/L, K)_{\geq r-d+1}(r - d) \simeq \operatorname{Hom}_K(S/((y_0, y_1)^{d-r} + (y_2, \ldots, y_r)), K)(r - d).$$

(ix) $\Longrightarrow$ (vi): This follows by a direct calculation. □

Our next aim is to prove a sufficient criterion for S/J to be CM. We begin with a preliminary remark.

Remark 3.4. A) Let $s \in \mathbb{N}$ and let $X \subseteq \mathbb{P}^s_K$ a scheme of d points in semi-uniform position (cf. [7]), let $N \subseteq T := K[x_0, \dots, x_s]$ be the homogeneous vanishing ideal of X so that T/N is the homogeneous coordinate ring of X. Then, according to [3] or [7], we have the estimate

$$\operatorname{reg} T/N = \operatorname{reg} N - 1 \leq \left[\frac{d-1}{s}\right].$$

B) Let X as in A) and assume in addition that $d \leq 2s$. Then, the ideal N satisfies the condition N_p of Green-Lazarsfeld with $p = 2s+1-d$ (cf. [3, Theorem 1]). Thus, in particular N is generated by quadrics.

Now, we are ready to prove the announced Cohen-Macaulay criterion.

Proposition 3.5. *If $d < 2r - 1$, then S/J is a Cohen-Macaulay ring.*

Proof. Let $\ell \in S_1$ be a generic linear form so that $\operatorname{Proj}(A/\ell A) = \operatorname{Proj}(S/(I, \ell))$ is a scheme of d points in semi-uniform position. After a linear coordinate transformation we may assume that $\ell = x_r$ and set $S/x_r S = K[x_0, \dots, x_{r-1}] = T$. Then,

$$\operatorname{Proj}(A/x_r A) = \operatorname{Proj}(T/IT) \subseteq \operatorname{Proj} T = \mathbb{P}^{r-1}_K$$

is a scheme of d points in semi-uniform position with vanishing ideal

$$N = (IT)^{\text{sat}} = \cup_{n \geq 0}(IT :_T (T_+)^n).$$

By 3.4 B) we know that N is generated by quadrics. In view of the natural isomorphism $H^0(A/x_r A) \simeq N/IT$ it follows that $H^0(A/x_r A)$ is generated in degree two. If we apply cohomology to the exact sequence $0 \to A(-1) \xrightarrow{x_r} A \to A/x_r A \to 0$, we get exact sequences

$$0 \to H^0(A/x_r A)_n \to H^1(A)_{n-1} \to H^1(A)_n \to H^1(A/x_r A)_n$$

for all $n \in \mathbb{Z}$. As

$$H^1(A/x_r A)_n \simeq H^1(T/IT)_n \simeq H^1(T/N)_n = 0$$

for all $n \geq \operatorname{reg} T/N$, 3.4 A) yields that $H^1(A/x_r A)_n = 0$ for all $n \geq \left[\frac{d-1}{r-1}\right] = 2$. If we apply the above sequence with $n = d - r + 1$ and observe 2.7 a), it follows $H^0(A/x_r A)_{d-r+1} \neq 0$. As $H^0(A/x_r A)$ is generated in degree 2 we get $H^0(A/x_r A)_2 \neq 0$. Applying the above sequence with $n = 2$ and observing 2.7 a) we get $h^1(A)_2 \leq h^1(A)_1 - 1 = d - r - 1$. So, by 3.3 we get that S/J is CM. □

We know by 3.2 B) that $H^1(S/J)$ is generated in degree 2. We close this section with a result on the number of generators of the module $H^1(S/J)$.

Proposition 3.6. *$H^1(S/J)$ is minimally generated by*

$$\dim_K I_2 - \binom{r+1}{2} + d + 1$$

homogeneous elements of degree 2. In particular we have

$$\dim_K I_2 \geq \binom{r+1}{2} - d - 1$$

with equality if and only if S/J is CM.

Proof. By 2.7 c) we have $h^1(S/J)_2 = h^1(A)_2 - d + r + 1$. In view of the graded exact sequence $0 \to A \to D(A) \to H^1(A) \to 0$ we have

$$h^1(A)_2 = \dim_K D(A)_2 - \dim_K A_2.$$

Moreover the graded isomorphism $D(A) \simeq K[s,t]^{(d)}$ (cf. 2.5 C), 2.6 B)) yields $\dim_K D(A)_2 = 2d+1$. As $\dim_K A_2 = \dim_K S_2 - \dim_K I_2$ and $\dim_K S_2 = \binom{r+1}{2}$ we obtain $h^1(S/J)_2 = \dim_K I_2 - \binom{r+1}{2} + d + 1$. On use of this equality and of 3.3 all our claims follow. □

4. Betti Numbers

We keep our previous notation and hypothesis. We shall relate now the Betti numbers of the S-module A to the Betti numbers of the S-modules S/J and $H^1(A)$. Our interest shall be focused to the case in which S/J is CM. Nevertheless we begin with a few more general considerations.
First of all, we have the following relation between the Betti modules of A and of S/J.

Proposition 4.1. *Let $t := \operatorname{reg} S/J$. Then $t \leq d - r$ and for all $i \in \{1, \ldots, r\}$ we have*

$$\operatorname{Tor}_i^S(K, A) \simeq \operatorname{Tor}_i^S(K, S/J) \oplus K^{\binom{r-1}{i-1}}(-i-d+r+1).$$

Proof. As $\operatorname{depth} S/J > 0$ we have

$$\operatorname{Tor}_i^S(K, S/J)_{i+j} = 0 \text{ if } (i,j) \notin \{1, \ldots, r\} \times \{1, \ldots, t\}.$$

Moreover we have $\operatorname{Tor}_i^S(K, S/L) \simeq K^{\binom{r-1}{i}}(-i)$ for all $i \in \mathbb{N}_0$. By the sequence of 2.5 B), for all $i, j \in \mathbb{N}$, we get an exact sequence

$$K^{\binom{r-1}{i}}(-i-d+r-2)_{i+j} \to \operatorname{Tor}_i^S(K, S/J)_{i+j} \to$$
$$\operatorname{Tor}_i^S(K, A)_{i+j} \to K^{\binom{r-1}{i-1}}(-i-d+r-1)_{i+j} \to \operatorname{Tor}_{i-1}^S(K, S/J)_{i+j}.$$

It follows that $\operatorname{Tor}_i^S(K, S/J)_{i+j} \simeq \operatorname{Tor}_i^S(K, A)_{i+j}$ for all $j \notin \{d-r+1, d-r+2\}$. As $t = \operatorname{reg} S/J \leq d - r$ (cf. 3.1 a)), we have $\operatorname{Tor}_i^S(K, S/J)_{i+j} = \operatorname{Tor}_{i-1}^S(K, S/J)_{i+j} = 0$ for all $j \geq d - r + 1$. Now, our claim follows easily. □

We convene that $\binom{a}{b} = 0$ for all $a \in \mathbb{N}_0$ and all $b \in \mathbb{Z} \setminus \{0, \dots, a\}$. Concerning the Betti modules of the S-module $D(A)$ we have the following auxiliary result, which shall be used later to determine the Betti numbers of $H^1(A)$. It generalizes and simplifies the corresponding result in [1, (5.3) (b)].

Lemma 4.2. *For $i \in \{1, \dots, r-1\}$ let $c_i := (d-1)\binom{r-1}{i} - \binom{r-1}{i-1}$. Then*

$$\operatorname{Tor}_i^S(K, D(A)) = \begin{cases} K(0) \oplus K^{d-r}(-1), & \text{for } i = 0 \\ K^{c_i}(-i-1), & \text{for } i \in \{1, \dots, r-1\}. \end{cases}$$

Proof. In view of the natural exact sequence

$$0 \to A \to D(A) \to H^1(A) \to 0$$

and as $H^1(A)$ is minimally generated by $d-r$ homogeneous elements of degree 1 (cf. 2.7 a), 3.1), we have $\operatorname{Tor}_0^S(K, D(A)) \simeq K(0) \oplus K^{d-r}(-1)$. As $H^1(A)$ is generated in degree 1, we also have $\operatorname{Tor}_1^S(K, H^1(A))_1 = 0$. As I is generated in degrees ≥ 2, the exact sequence $0 \to I \to S \to A \to 0$ gives $\operatorname{Tor}_1^S(K, A)_1 = 0$. It follows that $\operatorname{Tor}_1^S(K, D(A))_1 = 0$. As $\operatorname{reg} D(A) = 1$ (cf. 3.2 A)) and as $\operatorname{depth} D(A) = 2$, it follows that $\operatorname{Tor}_i^S(K, D(A)) \simeq K^{c_i}(-i-1)$ for all $i \in \mathbb{N}$, with $c_i \in \mathbb{N}$ for $i \leq r-1$ and $c_i = 0$ for $i \geq r$.
As $\dim_K D(A)_n = \min\{0, 1+nd\}$ for all $n \in \mathbb{Z}$, the Hilbert series of the S-module $D(A)$ is $F(t, D(A)) = \sum_{n \geq 0}(1+nd)t^n = (1-t)^{-2}Q(t)$,with $Q(t) = 1 + (d-1)t$. As

$$1 + (d-r)t - c_1t^2 + c_2t^3 - \ldots + (-1)^{r-1}c_{r-1}t^r =$$
$$\sum_{i=0}^{r-1}(-1)^i \dim_K \operatorname{Tor}_i^S(K, D(A))_j \, t^j = (1-t)^{r-1}Q(t)$$

we obtain $c_i = (d-1)\binom{r-1}{i} - \binom{r-1}{i+1}$ for $i = 1, \dots, r-1$. □

Now, we consider the Betti numbers of the Hartshorne-Rao module $H^1(A)$.

Remark 4.3. In view of the exact sequence $0 \to A \to D(A) \to H^1(A) \to 0$, for all $i \in \mathbb{N}$ and all $j \in \mathbb{Z}$, we get exact sequences of K-vector spaces

$$\operatorname{Tor}_{i+1}^S(K, H^1(A))_{i+j} \to \operatorname{Tor}_i^S(K, A)_{i+j} \to \operatorname{Tor}_i^S(K, D(A))_{i+j}$$
$$\to \operatorname{Tor}_i^S(K, H^1(A))_{i+j} \to \operatorname{Tor}_{i-1}^S(K, A)_{i+j} \to \operatorname{Tor}_{i-1}^S(K, D(A))_{i+j}$$

Proposition 4.4. *In our previous notation we have with $H := H^1(A)$:*

a) $\operatorname{Tor}_0^S(K, H) \simeq K^{d-r}(-1)$.
b) $\operatorname{Tor}_1^S(K, H) \simeq K^{c_1 - \dim_K I_2}(-2)$.
c) $\operatorname{Tor}_i^S(K, H) \simeq K^{a_i}(-i-1) \oplus \operatorname{Tor}_{i-1}^S(K, A)_{\geq i+2}$ *with* $a_i \leq \binom{r+1}{i}(d-r)$ *for* $i \in \{2, \dots, r\}$.
d) $\operatorname{Tor}_{r+1}^S(K, H) \simeq \operatorname{Tor}_r^S(K, A)_{\geq r+3}$.

Proof. As H is minimally generated by $d-r$ elements of degree 1 (cf. 2.7 a), 3.2 A)), we get statement a) and the fact that $\mathrm{Tor}_i^S(K,H)_{\leq i} = 0$ for all $i \in \mathbb{N}_0$. If we apply the six term exact sequence of 4.3 with $i = 1$ and keep in mind that $\mathrm{Tor}_0^S(K,A) = K, \mathrm{Tor}_1^S(K,A)_2 \simeq I_2$ and $\mathrm{Tor}_1^S(K,D(A)) = K^{c_1}(-2)$ (cf. 4.2) we thus get statement b).
Also, in view of the exact sequence of 4.3 and observing 4.2, we obtain graded isomorphisms $\mathrm{Tor}_i^S(K,H) \simeq K^{a_i}(-i-1) \oplus \mathrm{Tor}_{i-1}^S(K,A)_{\geq i+2}$ for all $i \geq 2$. It remains to show that $a_i \leq \binom{r+1}{i}(d-r)$ for $i = \{2,\ldots,r\}$ and that $a_{r+1} = 0$.
To derive the stated inequality, write $\mathrm{Tor}_i^S(K,H)$ as the i-th cohomology module $\mathrm{Ker}(\partial_i)/\mathrm{Im}(\partial_{i+1})$ of the Koszul complex

$$\ldots \to H^{\binom{r+1}{i+1}}(-i-1) \xrightarrow{\partial_{i+1}} H^{\binom{r+1}{i}}(-i) \xrightarrow{\partial_i} H^{\binom{r+1}{i-1}}(-i+1) \to \ldots$$

of H with respect to $x_0,\ldots,x_r$ and observe that $H^{\binom{r+1}{i+1}}(-i-1)_{i+1} = H_0^{\binom{r+1}{i+1}} = 0$ and $H^{\binom{r+1}{i}}(-i)_{i+1} = H_1^{\binom{r+1}{i}} \simeq (K^{d-r})^{\binom{r+1}{i}}$ (cf. 2.7 a)).
Finally, the natural graded isomorphism $\mathrm{soc}\, H \simeq \mathrm{Tor}_{r+1}^S(K,H)(r+1)$, the socle isomorphism of 3.2 C) and the vanishing of $h^1(S/J)_1$ (cf. 2.7 c)) show that

$$K^{a_{r+1}} \simeq \mathrm{Tor}_{r+1}^S(K,H)_{r+2} \simeq (\mathrm{soc}\, H)_1 = 0.$$

□

In case S/J is CM, the Betti modules of the Hartshorne-Rao module $H^1(A)$ can be determined precisely:

Proposition 4.5. *For $i \in \{0,\ldots,r+1\}$ let $a_i := (d-r+1)\binom{r-1}{i-1} + (d-r)\binom{r-1}{i}$ and $b_i := \binom{r-1}{i-2}$. Assume that S/J is CM. Then, for all $i \in \{0,\ldots,r+1\}$ we have*

$$\mathrm{Tor}_i^S(K,H^1(A)) \simeq K^{a_i}(-i-1) \oplus K^{b_i}(-i-d+r).$$

Proof. For $a \in \mathbb{N}, m \in \{1,\ldots,r\}$ and $i \in \{0,\ldots,m\}$ we set

$$M_m := S/((x_0,x_1)^a + (x_2,\ldots,x_m))$$

and $u_i := (a+1)\binom{m-1}{i-1} + a\binom{m-1}{i-2}$ and $v_i := \binom{m-1}{i}$. By induction on m we wish to show:

$$\mathrm{Tor}_i^S(K,M_m) \simeq K^{u_i}(-i+1-a) \oplus K^{v_i}(-i).$$

By the Hilbert-Burch Theorem $M_1 = S/(x_0,x_1)^a$ has a minimal free resolution of the shape

$$0 \to S^a(-a-1) \to S^{a+1}(-a-1) \to S \to M_1 \to 0.$$

This proves the above claim if $m = 1$. So, let $m > 1$. Then, the graded short exact sequence

$$0 \to M_{m-1}(-1) \xrightarrow{x_m} M_{m-1} \to M_m \to 0$$

induces isomorphisms

$$\mathrm{Tor}_i^S(K,M_m) \simeq \mathrm{Tor}_i^S(K,M_{m-1}) \oplus \mathrm{Tor}_{i-1}^S(K,M_{m-1})(-1)$$

for all $i \in \{0, \ldots, m\}$. On use of the Pascal formulae for binomial coefficients we may perform the induction step needed to prove the above claim.
Now, choose $m = r, a = d - r$ and set $M := M_r$. Then, by the previous claim we obtain

$$\mathrm{Tor}_i^S(K, M) \simeq K^{u_i}(-i + 1 - d + r) \oplus K^{v_i}(-i),$$

with $u_i = (d-r+1)\binom{r-1}{i-1} + (d-r)\binom{r-1}{i-2}$ and $v_i = \binom{r-1}{i}$ for each $i \in \{0, \ldots, r\}$. After a linear change of coordinates, we may assume by Theorem 3.3 that

$$H := H^1(A) = \mathrm{Hom}_K(M, K)(r - d),$$

hence by graded local duality, that

$$H \simeq \mathrm{Ext}_S^{r+1}(M, S(-r-1))(r-d) \simeq \mathrm{Ext}_S^{r+1}(M, S)(-d-1).$$

Let $0 \to F_{r+1} \to F_r \to \ldots \to F_1 \to F_0 \to M \to 0$ be a graded minimal free resolution of M. As M is of finite length over S, we have $\mathrm{Ext}_S^i(M, S) = 0$ for all $i \neq r+1$ and thus get a minimal free resolution

$$0 \to \mathrm{Hom}_S(F_0, S) \to \ldots \to \mathrm{Hom}_S(F_{r+1}, S) \to \mathrm{Ext}_S^{r+1}(M, S) \to 0$$

of $H(d+1)$. It follows

$$\mathrm{Tor}_i^S(K, H) \simeq \mathrm{Tor}_i^S(K, \mathrm{Ext}_S^{r+1}(M, S))(-d-1) \simeq$$
$$K \otimes_S \mathrm{Hom}_S(F_{r+1-i}, S)(-d-1) \simeq K^{u_{r+1-i}}(-i-1) \oplus K^{v_{r+1-i}}(-i-d-r).$$

As $u_{r+1-i} = a_i, v_{r+1-i} = b_i$ we get our claim. □

Finally, if S/J is CM, the Betti numbers of A can be approximated as follows:

Theorem 4.6. *Assume that S/J is CM and that $d > r + 1$. Then, for each $i \in \{1, 2, \ldots, r\}$ we have*

$$\mathrm{Tor}_i^S(K, A) \simeq K^{u_i}(i-1) \oplus K^{v_i}(-i-2) \oplus K^{\binom{r-1}{i-1}}(-i-d+r-1),$$

where u_i and v_i are given resp. bounded according to the following table

i	1	$2 \leq i \leq r-2$	$r-1$	r
u_i	$\binom{r+1}{2} - d - 1$	$\leq c_i$	$\leq d-1$	0
v_i	$\leq (d-1)\binom{r}{2} + (r-1)$	$\leq a_{i+1}$	$d-r+1$	0

in which c_i and a_{i+1} are defined according to 4.3 resp. 4.5. Moreover $u_i - v_{i-1} = c_i - a_i$ for all $i \in \{2, \ldots, r-1\}$.

Proof. The stated general shape of the Betti module $\mathrm{Tor}_i^S(K, A)$ follows from Proposition 4.1, as I is generated in degree ≥ 2 and as $\mathrm{reg}\, S/J = 2$ (cf. Theorem 3.3). The requested value of u_1 is a consequence of Proposition 3.6. The vanishing of u_r and v_r is a consequence of Proposition 4.1.

By Proposition 4.5 and Lemma 4.2 we have

$$\mathrm{Tor}^S_{i+1}(K, H^1(A))_{i+1} \simeq \mathrm{Tor}^S_{i-1}(K, D(A))_{i+1} = 0$$

for all $i \in \mathbb{N}$, $\mathrm{Tor}^S_i(K, H^1(A))_{i+1} \simeq K^{a_i}$ for all $i \in \{0, \ldots, r\}$ and $\mathrm{Tor}^S_i(K, D(A))_{i+1} \simeq K^{c_i}$ for all $i \in \{1, \ldots, r-1\}$. So, the sequences of 4.3 imply that $u_i \leq c_i, v_i \leq a_{i+1}$ for all $i \in \{1, \ldots, r-1\}, u_i - v_{i-1} = c_i - a_i$ for all $i \in \{2, \ldots, r-1\}$ and $v_{r-1} = a_r$. □

5. Examples

We keep the hypotheses and notations of the previous sections. We also introduce the notation

$$\beta_{ij} := \dim_K \mathrm{Tor}^S_i(K, A)_{i+j}$$

for the Betti numbers of $\mathcal{C}$.

Remark 5.1. A) We first consider the "exceptional case" in which $d = r + 1$, a case which has been excluded previously by the convention made in 2.5 A). In this case we know that $\mathcal{C} \subseteq \mathbb{P}^r_K$ is either an elliptic normal curve, or the projection of a rational normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+1}_K$ from a generic point or else a singular rational curve, obtained by projecting a rational normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+1}_K$ from a point which lies precisely on one secant line of $\tilde{\mathcal{C}}$, (cf. [1, (4.7) B)]). In the first and the third case, $\mathcal{C}$ is of arithmetic genus 1, so that $H^2(A)_0 \neq 0$. In the second case we have $H^1(A)_1 \neq 0$. So, in all three cases we have $\mathrm{reg}\,\mathcal{C} = \mathrm{reg}\,A + 1 \geq 3$ and hence $\mathrm{reg}\,\mathcal{C} = 3$, (cf. 2.2). So $\mathcal{C}$ is of maximal regularity in any case.
B) If $\mathcal{C}$ has a 3-secant line $\mathbb{L}$, then by 2.2 we know that I is generated by quadrics and one cubic. According to [5, p. 504] or to [6] this only may occur in the case where $\mathcal{C}$ is smooth and rational. From the proof of Theorem (3.1) in [5] (cf. p. 503) it follows that our curve $\mathcal{C}$ always lies on a rational surface scroll $\mathbb{S}_{r-1-2a,a} \subseteq \mathbb{P}^r_K$, $(0 \leq a \leq \frac{r-1}{2})$ (we use the notation of [1, (6.1)]). Moreover, by [5, Remark (2), p. 504], $\mathcal{C}$ has a trisecant line if and only if $a = 1$.
C) Let us assume now, that $\mathcal{C}$ has a trisecant line $\mathbb{L}$. Then, for some $f \in S_3 \setminus L$ we have $I = J + fS, I + L = L + fS$ (cf. 2.2) and the resulting exact sequence

$$0 \to S/J \to S/L \oplus A \to S/(L + fS) \to 0$$

together with the fact that $H^1(A)_n = 0$ for all $n \neq 1$ shows that $H^1(S/J)_n = 0$ for all $n \neq 1$. If we apply 2.3 with $\mu = 3$ we also obtain

$$r + 1 + h^1(S/J)_1 = \dim(S/J)_1 + h^1(S/J)_1 = h^0(\mathcal{C} \cup \mathbb{L}, \mathcal{O}_{\mathcal{C}\cup\mathbb{L}}(1)) \leq r + 1,$$

hence $h^1(S/J)_1 = 0$. Therefore $H^1(S/J) = 0$ and S/J becomes CM, too. So, the statement of 3.5 remains valid.

Examples 5.2. We consider the two non-degenerate rational curves $\mathcal{C}_k \subseteq \mathbb{P}^{10}_K$ of degree $11, (k = 1, 2)$ given parametrically by

$$\mathcal{C}_1 :(s^{11} : s^{10}t : s^9t^2 : s^7t^4 : s^6t^5 : s^5t^6 : s^4t^7 : s^3t^8 : s^2t^9 : st^{10} : t^{11}),$$
$$\mathcal{C}_2 :(s^{11} : s^{10}t : s^8t^3 : s^7t^4 : s^6t^5 : s^5t^6 : s^4t^7 : s^3t^8 : s^2t^9 : st^{10} : t^{11}).$$

It is easily seen, that $\mathcal{C}_k$ lies on the rational surface scroll $\mathbb{S}_{9-2k,k}$ for $k = 1, 2$. Both curves are obviously smooth and obtained by projecting a rational normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+1}_K$ from a point (which avoids all secant lines). In particular, both curves are of regularity 3. Moreover for the Betti numbers β_{ij} of $A = A_{\mathcal{C}_k}$ we have

k	i	1	2	3	4	5	6	7	8	9	10
	β_{i1}	43	221	550	812	742	398	91	8	0	0
1	β_{i2}	0	1	8	28	56	70	84	45	11	1
	β_{i1}	43	222	558	840	798	468	147	8	0	0
2	β_{i2}	1	9	36	84	126	126	84	45	11	1

The non-vanishing of the Betti number $\beta_{8,1}$ is in perfect coincidence with the observation that both curves $\mathcal{C}_k$ lie on a rational surface scroll, e.g. a surface of minimal degree (cf. [2, 3.C.1]).
In the case $k = 1$ we have $\beta_{12} = 0$, so that I is generated by quadrics. In view of 5.1 B) $\mathcal{C}_1$ has no trisecant line. In the case $k = 2$ we have $\beta_{12} = 1$, so that I is generated by quadrics and one cubic and we may expect that $\mathcal{C}$ has a trisecant line $\mathbb{L}$. This holds indeed by [5, p. 504]. In particular S/J must be CM by 5.1 C). In fact, for the Betti numbers

$$\gamma_{ij} := \dim_K \operatorname{Tor}^S_i(K, S/J)_{i+j}$$

of S/J we get the following values

i	1	2	3	4	5	6	7	8	9	10
γ_{i1}	43	222	558	840	798	468	147	8	0	0
γ_{i2}	0	0	0	0	0	0	0	9	2	0

Observe also, that in both cases the number of generating quadrics is $\binom{12}{2} - 12 = 43$, in accordance with 3.6.

It is easy to see that for each $r \geq 3$ and each $d > r + 1$ there are non-degenerate curves $\mathcal{C} \subseteq \mathbb{P}^r_K$ of maximal regularity and of degree d lying on a rational surface scroll $\mathbb{S} \subseteq \mathbb{P}^r_K$. But in general, curves of maximal regularity need not lie on a scroll.

Example 5.3. Let $\mathcal{C} \subseteq \mathbb{P}^8_K$ be the non-degenerate rational curve given parametrically by

$$\mathcal{C} : (s^{11} : s^{10}t : s^9t^2 : s^8t^3 : s^7t^4 : s^6t^5 : s^5t^6 : (st^{10} + s^2t^9) : t^{11}).$$

Calculating the Betti numbers β_{ij} we get

i	1	2	3	4	5	6	7	8
β_{i1}	24	84	126	84	20	0	0	0
β_{i2}	0	0	0	20	36	21	4	0
β_{i3}	0	0	0	0	0	0	0	0
β_{i4}	1	7	21	35	35	21	7	1

In particular we get $\operatorname{reg} A = 4$, thus $\operatorname{reg} \mathcal{C} = 5 = 11 - 8 - 2$, so that $\mathcal{C}$ is of maximal regularity. As $\beta_{61} = \dim_K \operatorname{Tor}^6_K(K, A)_7 = 0$, Green's $K_{p,1}$-Theorem shows that $\mathcal{C}$ does not lie on a surface scroll (cf. [2]).

By 3.5 – and in accordance with 3.6 – S/J is CM. Moreover, by 4.1 and 3.3 (iv), the first two lines of the previous table describe the Betti numbers of S/J.

By 3.5 we know that S/J is CM if $d < 2r - 1$. The next example illustrates that this result is sharp: There are non-degenerate curves $\mathcal{C} \subseteq \mathbb{P}^r_K$ of maximal regularity and of degree $d = 2r - 1$ such that S/J is not a CM ring.

Example 5.4. Let $\mathcal{C} \subseteq \mathbb{P}^6_K$ be given by

$$\mathcal{C} : (s^{11} : s^{10}t : s^9t^2 : s^8t^3 : s^7t^4 : st^{10} : t^{11}),$$

so that $\mathcal{C}$ is non-degenerate and of degree $11 = 2 \cdot 6 - 1$. Here, the Betti numbers β_{ij} take the following values

i	1	2	3	4	5	6
β_{i1}	10	20	15	4	0	0
β_{i2}	3	10	10	0	0	0
β_{i3}	1	5	10	15	7	1
β_{i4}	0	0	0	0	0	0
β_{i5}	0	0	0	0	0	0
β_{i6}	1	5	10	10	5	1

We see that $\operatorname{reg} A = 6$, so that $\operatorname{reg} \mathcal{C} = 7 = 11 - 6 + 2 = d - r + 2$. Hence $\mathcal{C}$ is of maximal regularity. But now S/J is not a CM-ring. One way to see this, is to apply 4.1 and to observe 3.3 (iv). One also could observe that the number β_{11} of generating quadrics of I is $10 \neq \binom{6+1}{2} - 11 - 1 = 9$ and apply 3.6. Moreover, the first three lines of the above table provide the Betti numbers of S/J (see 4.1). Also, by 3.6, $H^1(S/J)$ is minimally generated by one element of degree 2 and the socle formula of 3.2 C) shows that $H^1(S/J)_n = 0$ for all $n > 2$. So, S/J is a Buchsbaum ring with $H^1(S/J) = K(-2)$.

One might ask, whether the converse of 3.5 is true. We shall give an example showing that this is not the case in general: There are non-degenerate curves $\mathcal{C} \subseteq \mathbb{P}^r$ of maximal regularity of degree $d \geq 2r - 1$ and such that S/J is a CM ring.

Example 5.5. Let $\mathcal{C} \subseteq \mathbb{P}^6_K$ be the curve of degree 11 defined parametrically by

$$\mathcal{C} : (s^{11} : s^{10}t : s^9t^2 : s^8t^3 : s^7t^4 : (s^2t^9 + st^{10}) : t^{11}).$$

Here, the Betti numbers of $\mathcal{C}$ are as listed below:

i	1	2	3	4	5	6
β_{i1}	9	16	9	0	0	0
β_{i2}	6	24	36	25	6	0
β_{i3}	0	0	0	0	0	0
β_{i4}	0	0	0	0	0	0
β_{i5}	0	0	0	0	0	0
β_{i6}	1	5	10	10	5	1

So, first of all we have $\operatorname{reg} \mathcal{C} = \operatorname{reg} A + 1 = 7 = 11 - 6 + 2 = d - r + 2$ so that $\mathcal{C}$ is again of maximal regularity. The number of generating quadrics is $9 = \binom{6+1}{2} - 11 - 1 = \binom{r+1}{2} - d - 1$ so that S/J is a CM-ring by 3.6. On the other hand we have $d = 11 = 2 \cdot 6 - 1 = 2r - 1$. Again, by 4.1 the first two lines of the above diagram furnishes the Betti numbers of $\mathcal{C} \cup \mathbb{L}$.

By 3.3 we know that S/J is CM if and only if $\operatorname{soc} H^1(A) \simeq K(r-d)$, whereas in general we have $\operatorname{soc} H^1(A) \simeq \operatorname{soc}(H^1(S/J)) \oplus K(r-d)$ (see 3.2 C)). We present two examples which illustrate how much $\operatorname{soc} H^1(A)$ may vary in general.

Examples 5.6. We consider two curves $\mathcal{C}_k \subseteq \mathbb{P}^6_K$ of degree $13, (k = 1, 2)$ given parametrically by

$$\mathcal{C}_1 : (s^{13} : s^{12}t : s^{11}t^2 : s^{10}t^3 : s^9t^4 : st^{12} : t^{13}),$$
$$\mathcal{C}_2 : (s^{13} : s^{12}t : s^{11}t^2 : s^{10}t^3 : s^9t^4 : (st^{12} - s^2t^{11}) : t^{13}).$$

The Betti numbers are listed below

k	i	1	2	3	4	5	6
	β_{i1}	10	20	15	4	0	0
	β_{i2}	1	0	0	0	0	0
	β_{i3}	0	10	20	15	4	0
1	β_{i4}	0	0	0	0	0	0
	β_{i5}	1	5	10	10	5	1
	β_{i6}	0	0	0	0	0	0
	β_{i7}	0	0	0	0	0	0
	β_{i8}	1	5	10	10	5	1
	β_{i1}	9	16	9	0	0	0
	β_{i2}	2	4	0	1	0	0
	β_{i3}	2	14	36	34	14	2
2	β_{i4}	0	0	0	0	0	0
	β_{i5}	0	0	0	0	0	0
	β_{i6}	0	0	0	0	0	0
	β_{i7}	0	0	0	0	0	0
	β_{i8}	1	5	10	10	5	1

In both cases we have $\operatorname{reg} \mathcal{C}_k = 9 = 13 - 6 + 2 = d - r + 2$, so that $\mathcal{C}$ is of maximal regularity.
Moreover, by 4.1, we read off that $\operatorname{reg} S/J = 6$ resp. 4 in the case $k = 1$ resp. $k = 2$. So, by 3.3 clearly S/J is not CM in either case. Keeping in mind that (cf. 4.4 d) for the second isomorphism)

$$\operatorname{soc} H^1(A) \simeq \operatorname{Tor}^S_{r+1}(K, H^1(A)) \simeq \operatorname{Tor}^S_r(K, A)_{r \geq 3}(r+1) \simeq \oplus_{j \geq 3} K^{\beta_{rj}}(-j+1)$$

we thus get

$$\operatorname{soc} H^1(A) \simeq \begin{cases} K(-4) \oplus K(-7), & \text{if } k = 1, \\ K^2(-2) \oplus K(-7), & \text{if } k = 2. \end{cases}$$

Moreover, it follows that S/J is a Buchsbaum ring with $H^1(S/J) \simeq K^2(-2)$ in the case $k = 2$ while it is not a Buchsbaum ring in the case $k = 1$.

Further examples in higher degrees show that $\operatorname{soc} H^1(A)$ may indeed vary rather strongly.

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Universität Zürich, Institut für Mathematik, Winterthurer Str. 190, CH — 8057 Zürich, Switzerland

E-mail address: brodmann@math.unizh.ch

Martin-Luther-Universität Halle-Wittenberg, Fachbereich Mathematik und Informatik, D — 06 099 Halle (Saale), Germany

E-mail address: schenzel@mathematik.uni-halle.de