Sparse Convolution Quadrature for Time Domain Boundary Integral Formulations of the Wave Equation by Cutoff and Panel-Clustering

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1 Introduction

This paper is concerned with the numerical solution of the wave equation in an unbounded domain. Problems governed by the wave equation arise in many physical applications such as electromagnetic wave propagation or the computation of transient acoustic waves. When such problems are formulated in unbounded domains, the approach of *retarded potentials* allows a transformation of partial differential equations into space-time integral equations on the bounded surface of the scatterer.

Although this approach goes back to the early 1960s (cf. [11]) the development of fast numerical methods for integral equations in the field of hyperbolic problems is still in its infancies compared to the vast of fast methods for elliptic boundary integral equations (cf. [24] and references therein). Existing numerical discretisation methods include collocation methods with some stabilisation techniques (cf. [2], [3], [6], [7], [8], [22], [23]) and Laplace-Fourier methods coupled with Galerkin boundary elements in space (cf. [1], [5], [9], [12]). Numerical experiments can be found, e.g., in [13]. In [10], a fast version of the *marching-on-in-time* (MOT) method is presented which is based on a suitable plane wave expansion of the arising potential which reduces the storage and computational costs.

In this paper, we consider the convolution quadrature method for the time discretisation (cf. [18], [19], [20], [21]), and develop a panel-clustering method to obtain a data-sparse approximation of the underlying boundary integral equations. In [14], we have developed and analysed a simple cut-off strategy which reduces the number of entries in the system matrix which have to be computed while the rest is set to zero. The use of panel-clustering will further reduce the storage and computational complexity.

In [25], [26], and [27] Lubich's convolution quadrature method is applied to problems such as viscoelastic and poroelastic continua.

2 Formulation of the Problem

We consider a scattering problem in an exterior domain. For this, let $\Omega \subset \mathbb{R}^3$ be an unbounded Lipschitz domain with boundary Γ. Let $\bar{u}$ be the solution to the wave equation

$$\begin{aligned} \partial_t^2 \bar{u} &= \Delta \bar{u} + f \text{ , in } \Omega \times (0,T) \text{ ,} \\ \bar{u}(\cdot,0) &= u_0 \text{ in } \Omega \text{ ,} \\ \bar{u}_t(\cdot,0) &= u_1 \text{ in } \Omega \text{ ,} \\ \bar{u} &= 0 \text{ on } \Gamma \times (0,T) \text{ ,} \end{aligned}$$

for some time interval $(0,T)$ and given data f, u_0 and u_1.

To formulate the differential equation as a boundary integral equation, we introduce an incident solution v and a diffracted solution u in the whole $\mathbb{R}^3$, with $\bar{u}|_\Omega = (u+v)|_\Omega$, where v solves the open space problem

$$\begin{aligned} \partial_t^2 v &= \Delta v + f_p \text{ in } \mathbb{R}^3 \times (0,T) \text{ ,} \\ v(\cdot,0) &= u_{0p} \text{ in } \mathbb{R}^3 \text{ ,} \\ v_t(\cdot,0) &= u_{1p} \text{ in } \mathbb{R}^3 \text{ ,} \end{aligned}$$

where f_p, u_{ip} are prolongations of f and u_i to the whole $\mathbb{R}^3$, respectively. Given the solution to the above problem, v, u solves the homogeneous wave equation

$$\partial_t^2 u = \Delta u \text{ in } \Omega \times (0,T) \text{ ,} \tag{1a}$$

$$u(\cdot,0) = \partial_t u(\cdot,0) = 0 \text{ in } \Omega \text{ ,} \tag{1b}$$

$$u = g \text{ on } \Gamma \times (0,T) \text{ ,} \tag{1c}$$

where $g = -v|_{\Gamma \times (0,T)}$.

When considering a discretisation of the above partial differential equation on the unbounded domain Ω, one has to introduce an artificial boundary with additional boundary conditions. This is avoided by transforming the partial differential equation into a boundary integral equation. For this, we employ an ansatz as a *single layer potential*

$$u(x,t) = \int_0^t \int_\Gamma k(\|x-y\|, t-\tau)\phi(y,\tau) d\Gamma_y d\tau \text{ ,} \quad (x,t) \in \Omega \times (0,T) \text{ ,} \tag{2}$$

where $k(d,t)$ is the fundamental solution of the wave equation,

$$k(d,t) = \frac{\delta(t-d)}{4\pi d} \text{ ,} \tag{3}$$

$\delta(t)$ being the Dirac delta distribution. Inserting (2) into (1a), we see that the differential equation is satisfied. Also, the initial conditions are satisfied. An equation for the unknown density ϕ is obtained by taking the limit to the

boundary. Since the single layer potential is continuous across the boundary, we obtain the following boundary integral equation for ϕ,

$$\int_0^t \int_\Gamma k(\|x-y\|, t-\tau)\phi(y,\tau)d\Gamma_y d\tau = g(x,t) \qquad \forall (x,t) \in \Gamma \times (0,T)\,. \quad (4)$$

Note that only the two-dimensional surface Γ is involved in this equation and not the three-dimensional domain Ω. This is one major advantage for the numerical solution process compared to finite element or finite volume methods.

3 Convolution Quadrature Method

Discretising (4) directly in space and time, e.g., with a Galerkin method in space and a collocation method in time, involves the treatment of the Dirac delta distribution. The resulting integration domains for a boundary element method are given by the intersection of the light cone (of finite width) with the triangles or quadrilaterals of the surface mesh which can be of quite general shape and, hence, numerical quadrature becomes rather complicated. In addition, care needs to be taken to obtain an unconditionally stable scheme.

The convolution quadrature approach for the time discretisation leads to an unconditionally stable scheme (see [20]). The resulting integration domains are just the boundary elements themselves. Furthermore, the approach allows a data-sparse approximation of the system matrix by panel-clustering.

To explain the convolution quadrature method, we consider a convolution of the form

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau\,, \quad t \geq 0\,. \quad (5)$$

Choosing a stepsize Δt, (5) can be approximated by a discrete convolution $(f \star_{\Delta t} g)(t_n)$ which will be based on the inverse Laplace transform

$$f(t) = \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} \hat{f}(s)\mathrm{e}^{st}ds$$

for some $\sigma > 0$. The inverse Laplace transform is defined if $\hat{f}$ is analytic and for $\mathrm{Re}\, s > \sigma$, $|\hat{f}(s)| \leq c|s|^{-\mu}$ for some $c < \infty$ and $\mu > 0$. Inserting this representation of $f(t)$ into (5), we obtain

$$(f \star g)(t) = \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} \hat{f}(s)y_g(s,t)ds \quad \text{with} \quad y_g\,(s,t) := \int_0^t \mathrm{e}^{s(t-\tau)}g(\tau)d\tau\,.$$

Observe that the function $y_g(s,\cdot)$ satisfies the differential equation $\partial_t y(s,\cdot) = sy(s,\cdot) + g$, which can be approximated by a p-th order linear multistep method,

$$\sum_{j=0}^{k} \alpha_j y_{n+j-k}(s) = \Delta t \sum_{j=0}^{k} \beta_j \left(s y_{n+j-k}(s) + g((n+j-k)\Delta t) \right) , \qquad (6)$$

with starting values $y_{-k}(s) = \ldots = y_{-1}(s) = 0$. We assume that sufficiently many time derivatives of g vanish at $t = 0$. Formally, a p -th order approximation of (5) is then given by

$$(f \star_{\Delta t} g)(t_n) = \frac{1}{2\pi i} \int_{\sigma + i\mathbb{R}} \hat{f}(s) y_n(s) ds \, . \qquad (7)$$

To see that (7) can be written as a discrete convolution, we multiply (6) by ζ^n for $|\zeta| < 1$ and $\frac{\operatorname{Re} \gamma(\zeta)}{\Delta t} > \sigma$ and sum over n to obtain

$$\sum_{n=0}^{\infty} y_n \zeta^n = \left(\frac{\gamma(\zeta)}{\Delta t} - s \right)^{-1} \sum_{n=0}^{\infty} g(n\Delta t) \zeta^n \, ,$$

with $\gamma(\zeta) := \frac{\sum_{j=0}^{k} \alpha_j \zeta^{k-j}}{\sum_{j=0}^{k} \beta_j \zeta^{k-j}}$. Doing the same for (7), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} (f \star_{\Delta t} g)(t_n) \zeta^n &= \frac{1}{2\pi i} \int_{\sigma + i\mathbb{R}} \frac{\hat{f}(s)}{\frac{\gamma(\zeta)}{\Delta t} - s} ds \sum_{n=0}^{\infty} g(n\Delta t) \zeta^n \\ &= \sum_{n=0}^{\infty} \hat{f} \left(\frac{\gamma(\zeta)}{\Delta t} \right) g(n\Delta t) \zeta^n \, , \end{aligned}$$

where we have employed Cauchy's integral formula in the last step. If we define $\omega_n^{\Delta t}$ by

$$\hat{f} \left(\frac{\gamma(\zeta)}{\Delta t} \right) = \sum_{n=0}^{\infty} \omega_n^{\Delta t} \zeta^n \, ,$$

we have

$$\sum_{n=0}^{\infty} (f \star_{\Delta t} g)(t_n) \zeta^n = \sum_{n=0}^{\infty} \omega_n^{\Delta t} \zeta^n \sum_{m=0}^{\infty} g(m\Delta t) \zeta^m = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \omega_{n-j}^{\Delta t} g(j\Delta t) \right) \zeta^n \, .$$

Thus

$$(f \star_{\Delta t} g)(t_n) = \sum_{j=0}^{n} \omega_{n-j}^{\Delta t} g(j\Delta t) \, ,$$

which has the form of a discrete convolution.

4 Time Discretisation: Convolution Quadrature Method

In our case, the convolution coefficients are spatial boundary integral operators. The continuous convolution in (4) is approximated by the discrete convolution,

$$\sum_{j=0}^{n} \int_{\Gamma} \omega_{n-j}^{\Delta t}(\|x-y\|)\phi_{\Delta t}^{j}(y)d\Gamma_y = g(x,t_n)\,, \qquad n=1,\ldots,N, \quad x\in\Gamma\,, \quad (8)$$

where the convolution coefficients $\omega_n^{\Delta t}(d)$ are functions of $d=\|x-y\|$ determined by

$$\hat{k}\left(d, \frac{\gamma(\zeta)}{\Delta t}\right) = \sum_{n=0}^{\infty} \omega_n^{\Delta t}(d)\zeta^n. \quad (9)$$

As a multistep method, we use the second order accurate, A-stable BDF2 method with

$$\gamma(\zeta) = \frac{1}{2}(\zeta^2 - 4\zeta + 3)\,.$$

The coefficients of the power series (9) can be obtained by the Taylor expansion of $\hat{k}(d, \frac{\gamma(\zeta)}{\Delta t}) = \frac{\mathrm{e}^{-\frac{\gamma(\zeta)}{\Delta t}d}}{4\pi d}$ about $\zeta = 0$,

$$\omega_n^{\Delta t}(d) = \frac{1}{n!}\left.\frac{\partial^n \hat{k}(d, \frac{\gamma(\zeta)}{\Delta t})}{\partial \zeta^n}\right|_{\zeta=0} = \frac{1}{n!}\,\frac{1}{4\pi d}\left.\frac{\partial^n \mathrm{e}^{-\frac{\gamma(\zeta)}{\Delta t}d}}{\partial \zeta^n}\right|_{\zeta=0}.$$

It can also be shown that

$$\omega_n^{\Delta t}(d) = \frac{1}{n!}\frac{1}{4\pi d}\left(\frac{d}{2\Delta t}\right)^{n/2} \mathrm{e}^{-\frac{3d}{2\Delta t}} H_n\left(\sqrt{\frac{2d}{\Delta t}}\right), \quad (10)$$

where H_n are the Hermite polynomials.

5 Space Discretisation. Galerkin Boundary Element Methods

For the space discretisation, we employ a standard Galerkin boundary element method with piecewise constant or piecewise linear basis functions. Let $\mathcal{G}$ be a regular (in the sense of Ciarlet [4]) boundary element mesh on Γ consisting of shape regular, possibly curved triangles τ_i. Let $\mathbb{P}_0$ and $\mathbb{P}_1$ denote the space of constant and linear functions, respectively. We denote by

$$S_{-1,0} := \left\{ u \in L^{\infty}(\Gamma) \quad : \quad \forall \tau_i \in \mathcal{G} : u|_{\tau_i} \in \mathbb{P}_0 \right\}$$

the space of piecewise constant, discontinuous functions, and by

$$S_{0,1} := \left\{ u \in C^0(\Gamma) \quad : \quad \forall \tau_i \in \mathcal{G} : (u \circ \chi_i)|_{\tau_i} \in \mathbb{P}_1 \right\}$$

the space of continuous, piecewise linear functions, where χ_i denotes a regular mapping of the curved triangle τ_i to a planar reference triangle.

As a basis for $S_{-1,0}$ we choose

$$b_i(x) = \delta_{ij}, \text{ if } x \in \tau_j$$

and the basis for $S_{0,1}$ consists of the standard hat functions on the planar reference triangle, lifted to the surface Γ by the mapping χ_i. We generally refer to the boundary element space by S and its basis by $(b_i)_{i=1}^M$. The mesh width h is given by the maximum triangle diameter in $\mathcal{G}$.

For the Galerkin boundary element method, we replace $\phi^n_{\Delta t}$ in (8) by some $\phi^n_{\Delta t,h} \in S$ and impose the integral equation in a weak form. The fully discrete problem consists of finding $\phi^n_{\Delta t,h} \in S$, $n = 1, 2, \ldots, N$, of the form

$$\phi^n_{\Delta t,h}(y) = \sum_{i=1}^{M} \phi_{n,i} b_i(y) \,,$$

such that

$$\sum_{j=0}^{n} \sum_{i=1}^{M} \phi_{j,i} \int_\Gamma \int_\Gamma \omega^{\Delta t}_{n-j}(\|x-y\|) b_i(y) b_k(x) d\Gamma_y d\Gamma_x = \int_\Gamma g(x,t_n) b_k(x) d\Gamma_x \tag{11}$$

for all $1 \le k \le M$ and $n = 1, \ldots, N$. This can be written as a linear system

$$\sum_{j=0}^{n} \mathbf{A}_{n-j} \boldsymbol{\phi}_j = \mathbf{g}_n \,, \qquad n = 1, \ldots, N \,, \tag{12}$$

with the vectors $\boldsymbol{\phi}_j = (\phi_{j,i})_{i=1}^M$ and the matrices

$$(\mathbf{A}_n)_{k,i} := \int_\Gamma \int_\Gamma \omega^{\Delta t}_n(\|x-y\|) b_i(y) b_k(x) d\Gamma_y d\Gamma_x \,,$$

and

$$(\mathbf{g}_n)_k = \int_\Gamma g(x,t_n) b_k(x) d\Gamma_x \,.$$

6 Sparse Approximation of the Matrices $\mathbf{A}_n$ by Cutoff

6.1 Cutoff Strategy and Perturbation Analysis

The matrices $\mathbf{A}_n$ are full matrices. Thus, storage requirements and computational complexity for the solution of the fully discrete problem using fast

iterative methods are proportional to M^2. However, a substantial part of the matrix consists of small entries and can be replaced by 0. To see this, we recall the definition of the convolution coefficients

$$\omega_n^{\Delta t}(d) = \frac{1}{n!}\frac{1}{4\pi d}\left(\frac{d}{2\Delta t}\right)^{n/2} \mathrm{e}^{-\frac{3d}{2\Delta t}} H_n\left(\sqrt{\frac{2d}{\Delta t}}\right) . \tag{13}$$

For $n = 0$, we have

$$\omega_0^{\Delta t}(d) = \frac{\mathrm{e}^{-\frac{3}{2}\frac{d}{\Delta t}}}{4\pi d} ,$$

with a singularity at $d = 0$ and, for $n = 1$,

$$\omega_1^{\Delta t}(d) = \frac{1}{\Delta t}\frac{\mathrm{e}^{-\frac{3}{2}\frac{d}{\Delta t}}}{2\pi} .$$

In Fig. 1, we plot $\omega_n^{\Delta t}(d)$ for $\Delta t = 1$ and different n. For general Δt, we have the relation

$$\omega_n^{\Delta t}(d) = \Delta t^{-1}\omega_n^1\left(\frac{d}{\Delta t}\right) .$$

The convolution functions have their maximum near $d = t_n$. Away from this maximum, the coefficients decay fast. Using bounds for the Hermite polynomials, it can be shown (cf. [14]) that outside the interval

$$I_{n,\varepsilon}^{\Delta t} := \left[t_n - 3\sqrt{\Delta t}\sqrt{t_n}|\log\varepsilon|, t_n + 3\sqrt{\Delta t}\sqrt{t_n}|\log\varepsilon|\right] \tag{14}$$

we have

$$|\omega_n^{\Delta t}(d)| \le \frac{\varepsilon}{4\pi d} \quad \forall d \notin I_{n,\varepsilon}^{\Delta t} . \tag{15}$$

Given an error tolerance ε, we only consider those entries of $\mathbf{A}_n$, for which the possible values of $\|x - y\|$ lie inside $I_{n,\varepsilon}^{\Delta t}$. The remaining entries are set to zero. Let $\mathcal{P}_\varepsilon \subset \{1, \ldots, M\} \times \{1, \ldots, M\}$ be defined by

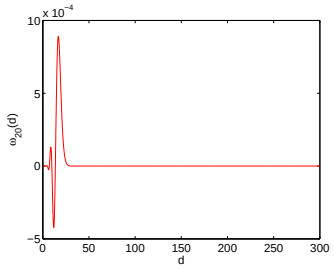
$$\mathcal{P}_\varepsilon := \left\{(i,j) : \exists\, (x,y) \in \operatorname{supp} b_i \cap \operatorname{supp} b_j, \text{ s.t. } \|x - y\| \in I_{n,\varepsilon}^{\Delta t}\right\} . \tag{16}$$

This induces a sparse approximation $\tilde{\mathbf{A}}_n$ by

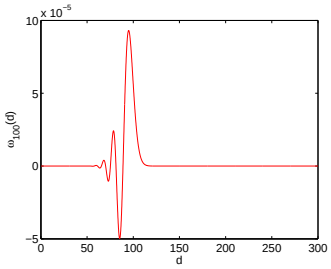
$$(\tilde{\mathbf{A}}_n)_{i,j} := \begin{cases} (\mathbf{A}_n)_{i,j} & \text{if } (i,j) \in \mathcal{P}_\varepsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Instead of solving (12), we solve for an approximate solution $\tilde{\boldsymbol{\phi}}_j = \left(\tilde{\phi}_{j,i}\right)_{i=1}^M$,

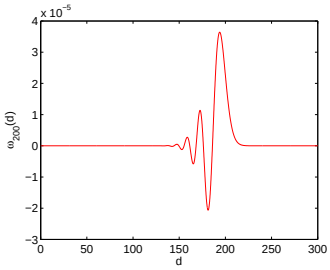
$$\sum_{j=0}^{n} \tilde{\mathbf{A}}_{n-j}\tilde{\boldsymbol{\phi}}_j = \mathbf{g}_n , \qquad n = 1, \ldots, N , \tag{18}$$



(a) $n = 20$



(b) $n = 100$



(c) $n = 200$

Fig. 1. The convolution weights $\omega_n^{\Delta t}(d)$ for $\Delta t = 1$ and different values of n.

and we have the approximate solution

$$\tilde{\phi}^n_{\Delta t,h}(y) := \sum_{i=1}^{M} \tilde{\phi}_{n,i} b_i(y)\,. \tag{19}$$

In [14], the following theorem is proven.

Theorem 1. *Let the exact solution $\phi(\cdot,t)$ of (4) be in $H^{m+1}(\Gamma)$ for any $t \in [0,T]$. There exists a constant $C > 0$ such that, for all cutoff parameters ε in (17) with $0 < \varepsilon < Ch\Delta t^3$, the solution $\tilde{\phi}_{\Delta t,h}$ in (19) exists and satisfies the error estimate*

$$\left\| \tilde{\phi}^n_{\Delta t,h} - \phi(\cdot,t_n) \right\|_{H^{-1/2}(\Gamma)} \le C_g(T) \left(\varepsilon h^{-1} \Delta t^{-5} + \Delta t^2 + h^{m+3/2} \right),$$

where C_g depends on the boundary data g.

Corollary 1. *Let the assumptions in Theorem 1 be satisfied. Let*

$$\Delta t^2 \sim h^{m+3/2}\,, \tag{20}$$

and choose

$$\varepsilon \sim h^{7m/2+25/4}.$$

Then the solution $\tilde{\phi}^n_{\Delta t,h}$ *exists and converges with optimal rate*

$$\left\| \tilde{\phi}^n_{\Delta t,h} - \phi\left(\cdot, t_n\right) \right\|_{H^{-1/2}(\Gamma)} \leq C_g(T) h^{m+3/2} \sim C_g(T) \Delta t^2.$$

6.2 Efficient Algorithmic Realisation

Before we present a way to further reduce the storage requirements, we take a look at the solution procedure. The problem to be solved is

$$\tilde{\boldsymbol{\phi}}_n = \tilde{\mathbf{A}}_0^{-1} \left(\mathbf{g}_n - \sum_{i=0}^{n-1} \tilde{\mathbf{A}}_{n-i} \tilde{\boldsymbol{\phi}}_i \right), \quad n = 0, 1, \ldots, N. \tag{21}$$

A straightforward way to solve (21) is to compute $\left(\mathbf{g}_n - \sum_{i=0}^{n-1} \tilde{\mathbf{A}}_{n-i} \tilde{\boldsymbol{\phi}}_i \right)$ and then to solve the system for each n. The required work is however proportional to N^2. When using the following algorithm (cf. [16]) the computational costs are proportional to $N \log^2 N$. The procedure depends on a (small) control parameter r.

Algorithm 1 (Recursive solver for block triangular system)

Comment: Main program
begin
solve_triangular$(0, N)$;
end;
Comment: The recursive subroutine **solve_triangular** is defined as follows.

procedure solve_triangular$(a, b : \text{integer})$;
begin
  if $b - a \leq r - 1$ **then**
    for $n := a$ **to** b **do**

$$\tilde{\boldsymbol{\phi}}_n := \tilde{\mathbf{A}}_0^{-1} \left(\mathbf{g}_n - \sum_{i=a}^{n-1} \tilde{\mathbf{A}}_{n-i} \tilde{\boldsymbol{\phi}}_i \right) \tag{22}$$

    end
  else begin
    $m := \lceil \frac{b+a}{2} \rceil$;
    solve_triangular$(a, m-1)$;
    for $n := m$ **to** b **do**

$$\mathbf{g}_n := \mathbf{g}_n - \sum_{i=a}^{m-1} \tilde{\mathbf{A}}_{n-i} \tilde{\boldsymbol{\phi}}_i \tag{23}$$

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        end;
        solve_triangular(m, b);
    end;
end;
```

When using fast iterative methods, the computational costs for (22) are proportional to r^2 matrix vector multiplications. It will turn out (cf. Theorem 2 and Table 1) that the matrix vector multiplications have the complexity $\mathcal{O}\left(M^{1+\frac{13}{16}-\frac{m}{8}}\right)$. The special form of (23) allows the use of the discrete fast Fourier transform (see, e.g., [17]) and the updates of $\mathbf{g}$ can be done in $\mathcal{O}\left(M^{1+\frac{13}{16}-\frac{m}{8}}\,(b-a)\log(b-a)\right)$ operations. The **procedure solve_triangular** calls itself twice with half the dimension. The total computational cost sums up $\mathcal{O}\left(M^{1+\frac{13}{16}-\frac{m}{8}}N\log^2 N\right)$ (cf. [17]).

6.3 Storage Requirements

The approximation of the matrices $\mathbf{A}_n$ by sparse approximations $\tilde{\mathbf{A}}_n$ results in reduced storage requirements, and consequently reduced computational complexity for the solution of the wave equation. To determine the storage requirements for the sparse matrices, assume that the dimension M of the boundary element space satisfies

$$c_1 h^{-2} \le M \le C_1 h^{-2}. \tag{24}$$

We further assume that there is a moderate constant C such that for any $1 \le i \le M$, the subset

$$\mathcal{P}_i := \{j \in \{1,\dots,M\} : (i,j) \in \mathcal{P}_\varepsilon\}\,,$$

with $\mathcal{P}_\varepsilon$ as in (16), satisfies

$$\sharp\mathcal{P}_i \le C \max\left\{1, \frac{\sqrt{\Delta t}\, t_n^{3/2}\log M}{h^2}\right\}. \tag{25}$$

This assumption can be derived from the assumption that $ch^2 \le \operatorname{supp} b_j \le Ch^2$ and that the area of

$$R_{i,n} := \left\{y \in \Gamma \quad : \quad \exists x \in \operatorname{supp} b_i : \|x-y\| \in I_{n,\varepsilon}^{\Delta t}\right\}$$

satisfies $|R_{i,n}| \le C\sqrt{\Delta t}\, t_n^{3/2}|\log(\varepsilon)|$ ($R_{i,n}$ is part of a ring with radius t_n and the same width as the interval $I_{n,\varepsilon}^{\Delta t}$.). Due to Corollary 1, $|\log\varepsilon| \sim \log M$.

With these assumptions, the number of nonzero matrix entries in $\tilde{\mathbf{A}}$ can be estimated by

$$\sum_{i=1}^{M} \sharp\mathcal{P}_i \le CM \max\left\{1, \sqrt{\Delta t}\, t_n^{3/2} h^{-2}\log M\right\}.$$

Relation (20) allows to substitute $\sqrt{\Delta t}$ and the combination with (24) yields

Table 1. Storage requirements for $\tilde{\mathbf{A}}_n$

	$m=0$	$m=1$
$t_n = \mathcal{O}(\Delta t \log M)$	$CM^{1+\frac{1}{4}} \log^2 M$	CM
$t_n = \mathcal{O}(1)$	$Ct_n^{3/2} M^{1+\frac{13}{16}} \log M$	$Ct_n^{3/2} M^{1+\frac{11}{16}} \log M$

Theorem 2. *The number of nonzero entries in the sparse approximation* $\tilde{\mathbf{A}}_n$ *is bounded from above by*

$$CM \max\left\{1, t_n^{3/2} M^{\frac{13}{16}-\frac{1}{8}m} \log M\right\}.$$

We distinguish between four cases: The case of piecewise constant and piecewise linear boundary elements ($m = 0$, and $m = 1$, respectively) and small and large n ($t_n = \mathcal{O}(\Delta t \log M)$ and $t_n = \mathcal{O}(1)$, respectively). The storage requirements for the different cases are summarised in Table 1. For small n, the storage requirements are significantly decreased. In Section 7, we present a method for further reducing the storage requirements even when $t_n > \mathcal{O}(\Delta t \log M)$.

7 Panel-Clustering

The panel-clustering method was developed in [15] for the data-sparse approximation of boundary integral operators which are related to elliptic boundary value problems. Since then, the field of sparse approximations of non-local operators has grown rapidly and nowadays advanced versions of the panel-clustering method are available and a large variety of alternative methods such as wavelet discretisations, multipole expansions, $\mathcal{H}$-matrices etc. exist. However, these fast methods (with the exception of $\mathcal{H}$-matrices) are developed mostly for problems of elliptic type while the data-sparse approximation of retarded potentials is to our knowledge still in its infancies. In this section, we develop the panel-clustering method for retarded potentials.

7.1 The Algorithm

The panel-clustering can be applied as soon as $t_n > \mathcal{O}(\Delta t\,|\log \varepsilon|)$. (Note that for the first time steps the simple cutoff strategy reduces the computational complexity much more significantly than for the later time steps, see Table 1.)

For $t_n > \mathcal{O}(\Delta t\,|\log \varepsilon|)$, the matrices $\mathbf{A}_n$ in (12) are partitioned into subblocks $\mathbf{B} := \mathbf{A}_n|_{s\times t}$ for some index set $s \times t \subset \{1,\dots,M\} \times \{1,\dots,M\}$. The subblocks are either replaced by zero, if the block entries are sufficiently small, or they are replaced by low rank matrices. To explain this approach in detail we first introduce the basic notation.

Let $\mathcal{I} := \{1, 2, \dots, M\}$ denote the degrees of freedoms for the space discretisation.

Definition 1 (Cluster). *A* cluster t *is a subset of* $\mathcal{I}$. *If* t *is a cluster, the corresponding subdomain of* Γ *is* $\Gamma_t := \bigcup_{i\in t} \operatorname{supp}(b_i)$. *The* cluster box $Q_t \subset \mathbb{R}^3$ *is the minimal axisparallel cuboid which contains* Γ_t *and the* cluster size L_t *is the maximal side length of* Q_t.

The clusters are collected in a hierarchical cluster tree $T_{\mathcal{I}}$.

Definition 2 (Cluster Tree). *A tree* $T_{\mathcal{I}}$ *is a* cluster tree *if the following conditions are satisfied.*

1. *The nodes in* $T_{\mathcal{I}}$ *are clusters.*
2. *The root of* $T_{\mathcal{I}}$ *is* $\mathcal{I}$.
3. *The leaves of* $T_{\mathcal{I}}$ *are the degrees of freedom, i.e.,* $\mathcal{L}(T_{\mathcal{I}}) = \mathcal{I}$ *and the tree hierarchy is given by a father/son relation: For each interior node* $t \in T_{\mathcal{I}} \setminus \mathcal{L}(T_{\mathcal{I}})$, *the set of* sons $\operatorname{sons}(t)$ *of* t *is the minimal subset in* $T_{\mathcal{I}} \setminus \{t\}$ *such that*

$$t = \bigcup_{s\in\operatorname{sons}(t)} s$$

holds. Vice versa, the father *of any* $s \in \operatorname{sons}(t)$ *is* t.

The standard construction of the cluster tree $T_{\mathcal{I}}$ is based on a recursive bisection of an axisparallel cuboid $\tilde{\mathcal{B}}$ which contains Γ. The bisection of $\tilde{\mathcal{B}}$ yields an auxiliary binary tree $T_{\tilde{\mathcal{B}}}$. Then, the clusters in $T_{\mathcal{I}}$ are given by collecting, for any box $\tilde{B} \in T_{\tilde{\mathcal{B}}}$, the indices $i \in \mathcal{I}$ which satisfies $\xi_i \in \tilde{B}$, where ξ_i denotes the nodal point for the i-th degree of freedom. Clusters in $T_{\mathcal{I}}$ which coincide with their father are removed from $T_{\mathcal{I}}$ and empty clusters are removed as well.

The kernel function k is approximated on $\Gamma_t \times \Gamma_s$, where (t, s) is a pair of clusters which satisfy the following condition. Recall the definition of the interval $I_{n,\varepsilon}^{\Delta t}$ as in (14).

Definition 3. *Let* $\varepsilon > 0$ *and* $n > C\,|\log\varepsilon|$. *Let* $0 < \eta < 1$ *be some control parameter. A pair of clusters* $(t, s) \in T_{\mathcal{I}} \times T_{\mathcal{I}}$ *is* admissible *at time step* t_n *if*

$$\forall\,(x, y) \in Q_t \times Q_s : \|x - y\| \notin I_{n,\varepsilon}^{\Delta t} \tag{26a}$$

or

$$\textit{(26a) is violated and } \max\{L_t, L_s\} \le \eta \Delta t n^b. \tag{26b}$$

The power b *in (26b) is a fixed number which is related to the accuracy of resulting discretisation.*

A theoretical bound on b is $b \ge 1/4$ under the condition $n \ge C\,|\log\varepsilon|$. Numerical experiments indicate that the choice $b \approx 0.3$ also preserves the optimal convergence rates. This is shown in a forthcoming paper.

The following algorithm subdivides $\mathcal{I} \times \mathcal{I}$ into a matrix part P^{sparse}, corresponding to pairs of indices where the matrix has to be assembled in the

conventional way, a zero part P^0 where the corresponding matrix entries are set to zero and a panel-clustering part P^{pc}, where the system matrix is approximated by panel-clustering. Note that the father/son relation of the cluster tree induces a father/son structure for pairs of clusters $\mathbf{b} = (c, s)$ by

$$\mathrm{sons}\,(\mathbf{b}) := \begin{cases} \mathrm{sons}\,(c) \times \mathrm{sons}\,(s) & \text{if } \mathrm{sons}\,(c) \neq \emptyset \text{ and } \mathrm{sons}\,(s) \neq \emptyset \\ c \times \mathrm{sons}\,(s) & \text{if } \mathrm{sons}\,(c) = \emptyset \text{ and } \mathrm{sons}\,(s) \neq \emptyset \\ \mathrm{sons}\,(c) \times s & \text{if } \mathrm{sons}\,(c) \neq \emptyset \text{ and } \mathrm{sons}\,(s) = \emptyset \\ \emptyset & \text{if } \mathrm{sons}\,(c) \neq \emptyset \text{ and } \mathrm{sons}\,(s) \neq \emptyset \end{cases}$$

Algorithm 2 *Let $n > C\,|\log\varepsilon|$. The minimal admissible block partitioning of $\mathcal{I} \times \mathcal{I}$ at time step t_n is obtained as the result of the procedure* ***divide***$\left((\mathcal{I},\mathcal{I}), P^{\mathrm{sparse}}, P^{\mathrm{pc}}, P^0\right)$ *defined by (cf. [15])*

procedure divide$\left(\mathbf{b}, P^{\mathrm{sparse}}, P^{\mathrm{pc}}, P^0\right)$;
begin
  if ($\mathbf{b}$ is non-admissible and $\mathrm{sons}\,(\mathbf{b}) = \emptyset$) **then** $P^{\mathrm{sparse}} := P^{\mathrm{sparse}} \cup \{\mathbf{b}\}$
  else if ($\mathbf{b}$ satisfies (26a) **then** $P^0 := P^0 \cup \{\mathbf{b}\}$
  else if ($\mathbf{b}$ satisfies (26b) **then** $P^{\mathrm{pc}} := P^{\mathrm{pc}} \cup \{\mathbf{b}\}$
  else for all $\widetilde{\mathbf{b}} \in \mathrm{sons}\,(\mathbf{b})$ **do divide**$\left(\widetilde{\mathbf{b}}, P^{\mathrm{sparse}}, P^{\mathrm{pc}}, P^0\right)$;
end;

Remark 1. The set P^{sparse} is empty in most cases since the cluster sizes of the leaves satisfy

$$L_{\{i\}} = O\,(h)$$

while relation (20) implies for the bound in (26b)

$$\eta \Delta t n^b = O\left(\eta h^{m/2+3/4} n^b\right),$$

where $m = 0$ for constant and $m = 1$ for linear elements. Hence after a few time steps, $\eta \Delta t n^b \geq Ch$ and any pair $\mathbf{b}$ with $\mathrm{sons}\,(\mathbf{b}) = \emptyset$, i.e., $i, j \in \mathcal{I}$, satisfies (26a) or (26b).

Next, we explain the data sparse approximation on the blocks $\mathbf{b} = (c, s) \in P^{\mathrm{pc}}$. Since $\omega_n^{\Delta t}\,(\|\mathbf{x} - \mathbf{y}\|)$ is defined in $Q_c \times Q_s$ we may define its approximation by Čebyšev interpolation:

$$\omega_n^{\Delta t}(\|x - y\|) \approx \breve{\omega}_n^{\Delta t}(\|x - y\|) = \sum_{\mu,\nu \in \left(\mathbb{N}_{\leq q}\right)^3} \mathcal{L}_c^{(\mu)}(x)\,\mathcal{L}_s^{(\nu)}(y)\,\omega_n^{\Delta t}(\|x^\mu - y^\nu\|), \tag{27}$$

where $\mathcal{L}_c^{(\mu)}$ (resp. $\mathcal{L}_s^{(\nu)}$) are the tensorised versions of the $q-$th order Lagrange polynomials (properly scaled and translated to Q_c resp. Q_s) corresponding to the tensorised Čebyšev nodes x^μ for Q_c resp. y^ν for. Q_s).

The matrix $\mathbf{A}_n$ is the representation of the bilinear form $a_n : S \times S \to \mathbb{R}$,

$$a_n(\phi,\psi) := \int_\Gamma \int_\Gamma \omega_n^{\Delta t}(\|x-y\|)\phi(y)\psi(x)d\Gamma_y d\Gamma_x$$

with respect to the nodal basis $(b_i)_{i=1}^M$. We introduce the convention that, for any function $\varphi \in S$, the coefficient vector in the basis representation is denoted by $\boldsymbol{\varphi} = (\varphi_i)_{i=1}^M$, i.e., $\varphi = \sum_{i=1}^M \varphi_i b_i$.

The sparse approximation of a_n by our combined cutoff and panel-clustering strategy is given by

$$\begin{aligned} a_n(\phi,\psi) \approx & \sum_{(i,j)\in P^{\mathrm{sparse}}} \psi_i \phi_j \left(\mathbf{A}_n^{\mathrm{sparse}}\right)_{i,j} \\ & + \sum_{\mathbf{b}=(\sigma,s)\in P^{\mathrm{pc}}} \sum_{\mu,\nu\in\left(\mathbb{N}_{\le q}\right)^3} \left(\mathbf{S}_{\mathbf{b}}^{(n)}\right)_{\mu,\nu} J_\sigma^{(\mu)}(\psi)\, J_s^{(\nu)}(\phi), \end{aligned}$$

with the sparse matrix part of $\mathbf{A}_n$

$$\left(\mathbf{A}_n^{\mathrm{sparse}}\right)_{i,j} := \begin{cases} \int_{\Gamma_{\{i\}}} \int_{\Gamma_{\{j\}}} \omega_n^{\Delta t}(\|x-y\|) b_j(y)\, b_i(x)\, d\Gamma_y d\Gamma_x & \text{if } (i,j) \in P^{\mathrm{sparse}}, \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

the *interaction matrix* $\mathbf{S}_{\mathbf{b}}^{(n)}$

$$\left(\mathbf{S}_{\mathbf{b}}^{(n)}\right)_{\mu,\nu} := \omega_n^{\Delta t}(\|x^\mu - y^\nu\|) \quad 0 \le \mu_i, \nu_i \le q,\ 1 \le i \le 3$$

and the *influence coefficients*

$$J_\sigma^{(\mu)}(\psi) := \sum_{i\in\sigma} \psi_i \int_{\Gamma_\sigma} \mathcal{L}_\sigma^{(\mu)}(x) b_i(x)\, d\Gamma_x, \quad 0 \le \mu_i, \nu_i \le q,\ 1 \le i \le 3.$$

The algorithmic realisation of the sparse matrix multiplication based on this approximation of the bilinear form and the recursive computation of the influence coefficients $J_\sigma^{(\mu)}(\psi)$ are structured as follows.

Phase 1: Computation and storage of the Galerkin operator

(a) Generate and store the cluster tree and the partitioning of $\mathcal{I} \times \mathcal{I}$ into P^{sparse}, P^{pc}, and P^0.
Introduce recursive tree levels $0 \le \ell \le \ell_{\max}$ by $T_{\mathcal{I}}(0) = \{\mathcal{I}\}$ and

$$T_{\mathcal{I}}(\ell+1) := \{\sigma \in T_{\mathcal{I}} : \exists s \in T_{\mathcal{I}}(\ell) \text{ with "}\sigma \text{ is son of } s\text{"}\}.$$

Let $\ell_{\min}$ denote the minimal index such that (i) there exists $\sigma \in T_{\mathcal{I}}(\ell_{\min})$ with $L_\sigma \le \eta\Delta t n^b$ and (ii) for all $0 \le \ell < \ell_{\min}$ and $\sigma \in T_{\mathcal{I}}(\ell)$ there holds $L_\sigma > \eta\Delta t n^b$.

(b) Compute and store the nonzero entries of the matrix $\mathbf{A}_n^{\text{sparse}}$.
(c) Compute and store the basis influence coefficients

$$J_{\{i\}}^{(\mu)}(b_i) := \int_{\text{supp}(b_i)} \mathcal{L}_{\{i\}}^{(\mu)}(x)\, b_i(x)\, d\Gamma_x, \quad 1 \le i \le M,\ \mu \in (\mathbb{N}_{\le q})^3. \quad (29)$$

(d) Compute and store the interaction matrices $\mathbf{S}_{\mathbf{b}}^{(n)}$ for all $\mathbf{b} \in P^{\text{pc}}$.

Phase 2: Evaluation of a matrix-vector multiplication $\boldsymbol{\varphi} = \mathbf{A}_n \boldsymbol{\psi}$

(a) For all $\sigma \in T_{\mathcal{I}}(\ell_{\max})$, for all $\mu \in (\mathbb{N}_{\le q})^3$ compute

$$J_\sigma^{(\mu)}(\psi) = \psi_i J_{\{i\}}^{(\mu)}(b_i)\ .$$

For $\ell = \ell_{\max}-1, \ell_{\max}-2, \dots, \ell_{\min}$, for all $\sigma \in T_{\mathcal{I}}(\ell)$ and all $\mu \in (\mathbb{N}_{\le q})^3$ compute

$$J_\sigma^{(\mu)}(\psi) = \sum_{s \in \text{sons}(\sigma)} \sum_{\nu \in (\mathbb{N}_{\le q})^3} \gamma_{\mu,\nu,s} J_s^{(\nu)}(\psi) \quad \text{with} \quad \gamma_{\mu,\nu,s} := \mathcal{L}_\sigma^{(\mu)}(x_s^{(\nu)})\,.$$

(b) Let

$$T_{\mathcal{I}}^{\text{pc}} := \{c \in T_{\mathcal{I}} \mid \exists s \in T_{\mathcal{I}} : (c,s) \in P^{\text{pc}}\}$$

and, for $c \in T_{\mathcal{I}}^{\text{pc}}$, let

$$P_{\text{right}}^{\text{pc}}(c) := \{s \in T_{\mathcal{I}} \mid (c,s) \in P^{\text{pc}}\}.$$

For all $c \in T_{\mathcal{I}}^{\text{pc}}$ and all $\mu \in (\mathbb{N}_{\le q})^3$ compute

$$R_c^{(\mu)}(\psi) := \sum_{s \in P_{\text{right}}^{\text{pc}}(c)} \sum_{\nu \in (\mathbb{N}_{\le q})^3} \left(\mathbf{S}_{\mathbf{b}}^{(n)}\right)_{\mu,\nu} J_s^{(\nu)}(\psi)\,.$$

(c) For $\ell = \ell_{\min}, \ell_{\min}+1, \dots, \ell_{\max}-1$, $\sigma \in T_{\mathcal{I}}(\ell)$, $s \in \text{sons}(\sigma)$, and all $\nu \in (\mathbb{N}_{\le q})^3$ compute

$$R_s^{(\nu)}(\psi) := R_s^{(\nu)}(\psi) + \sum_{\mu \in (\mathbb{N}_{\le q})^3} \gamma_{\mu,\nu,s} R_\sigma^{(\mu)}(\psi)\ .$$

For all $\{i\} \in T_{\mathcal{I}}(\ell_{\max})$ do

$$\varphi_i := \sum_{\nu \in (\mathbb{N}_{\le q})^3} R_{\{i\}}^{(\nu)}(\psi)\, J_{\{i\}}^{(\nu)}(b_i)\ .$$

(d) Evaluate (by taking into account the sparsity of $\mathbf{A}_n$)

$$\boldsymbol{\varphi} := \boldsymbol{\varphi} + \mathbf{A}_n^{\text{sparse}} \boldsymbol{\psi}.$$

7.2 Error Analysis

We proceed with the error analysis of the resulting perturbed Galerkin discretisation which leads to an a-priori choice of the interpolation order q such that the convergence rate of the unperturbed discretisation is preserved.

Standard estimates for tensorised Čebyšev-interpolation yield

$$\sup_{z\in Q_c-Q_s}\left|\omega_n^{\Delta t}(\|z\|)-\breve{\omega}_n^{\Delta t}(\|z\|)\right| \leq C\frac{L^{q+1}\left(1+\log^5 q\right)}{2^{2q+1}\,(q+1)!}\max_{i\in\{1,2,3\}}\sup_{z\in Q_c-Q_s}\left|\partial_{z_i}^{q+1}\omega\left(\|z\|\right)\right|, \tag{30}$$

where $C>0$ is some constant independent of all parameters, L denotes the maximal side length of the boxes Q_c and Q_s and Q_c-Q_s is the difference domain $\{x-y:(x,y)\in Q_c\times Q_s\}$.

Theorem 3. *For $\mathbf{b}=(c,s)\in P^{\mathrm{pc}}$, let $(x,y)\in \Gamma_c\times\Gamma_s$ and $n\geq C\left|\log\varepsilon\right|$. Assume that the partial derivatives of $\omega_n^{\Delta t}(\|x-y\|)$ satisfy*

$$\max_{1\leq i\leq 3}\left|\partial_{z_i}^q\omega_n^{\Delta t}\left(\|z\|\right)\right|\,|\leq c^q q!\|z\|^{-1}\left(\frac{1}{\Delta t n^b}\right)^q \quad \forall z\in Q_c-Q_s \tag{31a}$$

with b as in Definition 3. Then

$$|\breve{\omega}_n^{\Delta t}(\|x-y\|)-\omega_n^{\Delta t}(\|x-y\|)|\leq\frac{C_1}{\operatorname{dist}\left(Q_c,Q_s\right)}\left(C_2\frac{L}{\Delta t n^b}\right)^{q+1} \tag{31b}$$

with L as in (30).

Note that in a forthcoming paper, the validity of assumption (31a) will be derived.

Theorem 4. *Let $\varepsilon>0$ and $n>C|\log^2\varepsilon|$ for some C. Let the assumptions of Theorem 3 be satisfied and the interpolation order chosen according to $q\geq|\log\varepsilon|\,/\log 2$.*

(a) Let $\mathbf{b}=(c,s)\in P^{\mathrm{pc}}$ be admissible for some $0<\eta\leq\eta_0$ and sufficiently small $\eta_0=O\left(1\right)$. Then

$$|\omega_n^{\Delta t}(\|x-y\|)-\breve{\omega}_n^{\Delta t}(\|x-y\|)|\leq C\frac{\varepsilon}{\|x-y\|}\quad\forall\,(x,y)\in\Gamma_c\times\Gamma_s \tag{32a}$$

for some C independent of n and Δt.

(b) Let $\mathbf{b}=(c,s)\in P^0$. Then

$$\left|\omega_n^{\Delta t}(\|x-y\|)\right|\leq\frac{\varepsilon}{\|x-y\|}\quad\forall\,(x,y)\in\Gamma_c\times\Gamma_s\,. \tag{32b}$$

Proof. Assume that $(c,s) \in P^{\mathrm{pc}}$. Then, due to Condition (26b), we obtain from Theorem 3 the estimate

$$|\breve{\omega}_n^{\Delta t}(\|x-y\|) - \omega_n^{\Delta t}(\|x-y\|)| \leq \frac{C_1}{\operatorname{dist}(Q_c,Q_s)} (C_2\eta)^{q+1}.$$

The distance can be estimated by means of Condition (26b). For all $(x,y) \in Q_c \times Q_s$, there holds

$$\|x-y\| \leq \operatorname{dist}(Q_c,Q_s) + \sqrt{3}(L_c+L_s) \leq \operatorname{dist}(Q_c,Q_s) + 2\sqrt{3}\eta\Delta t n^b. \quad (33)$$

Because $(c,s) \in P^{\mathrm{pc}}$, Condition (26a) is violated and there exists $(x,y) \in Q_c \times Q_s$ such that $\|x-y\| \in I_{n,\varepsilon}^{\Delta t}$. Thus, by taking into account $n^b \leq n$, we obtain

$$\begin{aligned}\operatorname{dist}(Q_c,Q_s) &\geq \|x-y\| - \sqrt{3}(L_c+L_s) \geq t_n - 3\sqrt{\Delta t}\sqrt{t_n}|\log\varepsilon| - 2\sqrt{3}\eta\Delta t n^b \\ &= t_n\left(1 - 3\frac{|\log\varepsilon|}{\sqrt{n}} - 2\sqrt{3}\eta\right) \geq \frac{t_n}{10}\end{aligned}$$

for $n > 15|\log^2\varepsilon|$ and $0 \leq \eta \leq \eta_0$ with $\eta_0 = \left(40\sqrt{3}\right)^{-1}$. Hence,

$$\operatorname{dist}(Q_c,Q_s) \geq \frac{t_n}{10} \geq 2\left(2\sqrt{3}\eta\Delta t n^b\right) \quad (34)$$

for all $0 \leq \eta \leq \eta_0$.

The combination of (33) and (34) yields

$$\frac{1}{\operatorname{dist}(Q_c,Q_s)} \leq \frac{3}{2\|x-y\|}$$

and

$$|\breve{\omega}_n^{\Delta t}(\|x-y\|) - \omega_n^{\Delta t}(\|x-y\|)| \leq \frac{3C_1}{2\|x-y\|} (C_2\eta)^{q+1}.$$

Finally, the condition $\eta_0 \leq (2C_2)^{-1}$ implies that the interpolation order

$$q \geq \frac{|\log\varepsilon|}{\log 2}$$

leads to an approximation which satisfies

$$|\breve{\omega}_n^{\Delta t}(\|x-y\|) - \omega_n^{\Delta t}(\|x-y\|)| \leq \frac{C_1\varepsilon}{2\|x-y\|}.$$

For $(c,s) \in P^0$, the assertion follows from (15).

In [14] an analysis of the perturbation error has been derived. Since it is only based on abstract approximations which satisfy an error estimate of type (32a) and (32b), we directly obtain a similar convergence theorem also for the panel clustering method. In the following, we denote by $\tilde{\phi}_{\Delta t,k}^n \in S$ the solution at time t_n of the Galerkin discretization with cutoff strategy and panel-clustering.

Theorem 5. *Let the assumption of Theorem 4 be satisfied. We assume that the exact solution $\phi(\cdot,t)$ is in $H^{m+1}(\Gamma)$ for any $t\in[0,T]$. Then there exists $C>0$, such that for all cutoff parameters ε in (14) such that $0<\varepsilon<Ch\Delta t^3$ and interpolation orders $q\geq|\log\varepsilon|/\log 2$, the solution $\tilde{\phi}_{\Delta t,h}$ with cutoff and panel-clustering satisfies the error estimate*

$$\left\|\tilde{\phi}^n_{\Delta t,h}-\phi(\cdot,t_n)\right\|_{H^{-1/2}(\Gamma)}\leq C_g(T)\left(\varepsilon h^{-1}\Delta t^{-5}+\Delta t^2+h^{m+3/2}\right).$$

Corollary 2. *Let the assumptions of Theorem 5 be satisfied. Let $\Delta t\sim h^{m+3/2}$ and choose $\varepsilon\sim h^{7m/2+25/4}$. Then, the solution $\tilde{\phi}_{\Delta t,h}$ exists and converges with optimal rate*

$$\left\|\tilde{\phi}^n_{\Delta t,h}-\phi(\cdot,t_n)\right\|_{H^{-1/2}(\Gamma)}\leq C_g(T)\,h^{m+3/2}\sim C_g(T)\,\Delta t^2.$$

7.3 Complexity Estimates

In this subsection, we investigate the complexity of our data-sparse approximation of the wave discretisation. Since we will introduce numerical quadrature methods for approximating the integrals (28) and (29) (for possibly curved panels) in a forthcoming paper, we here restrict ourselves to the storage complexity of our data-sparse approximation scheme and discuss the computational complexity in a forthcoming paper. In this section, we always employ the theoretical value $1/4$ for the exponent b in (26b).

Sparse approximation of the system matrix $\tilde{\mathbf{A}}_n$.

To simplify the complexity analysis we assume that only the simple cutoff strategy and not the panel-clustering method is applied for the first time steps:

$$1\leq n\leq C\max\left\{\log M, M^{m-\frac{1}{2}}\right\}, \tag{35}$$

where the constant C depends only on the control parameter η. Note that the second argument in $\max\{\cdot,\cdot\}$ ensures that $P^{\mathrm{sparse}}=\emptyset$ and the matrix $\mathbf{A}^{\mathrm{sparse}}$ vanishes (cf. Remark 1). By using Theorem 2 and (20), the number of nonzero entries of $\tilde{\mathbf{A}}_n$ in this case is of order

$$M\max\left\{M^{m-\frac{1}{2}}\log M, M^{\frac{1}{4}-\frac{1}{2}m}\log^{5/2}M\right\}=\begin{cases}M^{1+\frac{1}{4}}\log^{5/2}M & m=0,\\ M^{1+\frac{1}{2}}\log M & m=1,\end{cases}$$

where the leading constant in $O(\cdot)$-estimate depends only on η. Note that $\Delta t=O\left(N^{-1}\right)$. Hence, relation (20) implies $N\sim M^{\frac{m}{4}+\frac{3}{8}}$ allows to estimate the number of n's in (35) by

$$\max\left\{\log M, M^{m-\frac{1}{2}}\right\}\leq N\max\left\{M^{-\frac{m}{4}-\frac{3}{8}}\log M, M^{\frac{3}{4}m-\frac{7}{8}}\right\}.$$

Hence, the total cost for storing these matrices $\tilde{\mathbf{A}}_n$ is given by

$$\left(NM^{\frac{7}{8}+\frac{m}{2}}\log^{\kappa_m} M\right) \quad \text{with} \quad \kappa_m := \begin{cases} 7/2 \text{ if } m=0, \\ 1 \quad \text{if } m=1. \end{cases}$$

Basis influence coefficients.

The number of basis influence coefficients (cf. (29)) is bounded by

$$O\left(M\log^3 M\right).$$

Since this step has to be computed and stored only once for all all time steps the cost for this step (and the generation of the cluster tree as well) is negligible compared to the minimal cost $O(NM)$ of the whole algorithm.

Influence Matrices.

First, we compute the cardinality of P^{pc}. Note that the maximal diameter of a cluster $t \in T_{\mathcal{I}}$ satisfying condition (26b) is bounded by

$$L_t \leq \eta \Delta t n^b.$$

An assumption on the cluster tree and the geometric shape of the surface is that

$$\left|\left\{(x,y) \in \Gamma\times\Gamma \mid \|x-y\| \in I_{n,\varepsilon}^{\Delta t}\right\}\right| = O\left(\sqrt{\Delta t}\, t_n^{3/2}\left|\log\varepsilon\right|\right),$$

where $|\omega|$ denotes the area measure of some $\omega \subset \Gamma\times\Gamma$. Hence, for sufficiently small Δt the number of pairs of clusters satisfying (26b) is bounded by

$$O\left(\frac{\sqrt{\Delta t}\, t_n^{3/2}\left|\log\varepsilon\right|}{\left(\eta\Delta t n^b\right)^4}\right). \tag{36}$$

The storage requirements per matrix $\mathbf{S}_{\mathbf{b}}^{(n)}$ are given by $q^6 \sim \log^6\varepsilon$ and this leads to a storage complexity of

$$O\left(\frac{n^{3/2-4b}\left|\log\varepsilon\right|^7}{\Delta t^2}\right). \tag{37}$$

Using the relations as in Corollary 2

$$\Delta t^2 \sim h^{m+3/2}, \qquad \varepsilon \sim h^{7m/2+25/4}, \qquad M = O\left(h^{-2}\right)$$

we see that (37) is equivalent to (we here use $4b=1$)

$$O\left(n^{1/2}\left|\log M\right|^7 M^{m/2+3/4}\right).$$

Table 2. Storage requirements for the panel clustering approximation and sparse approximation

	full matrix representation	cutoff strategy	panel clustering+cutoff strategy
$m = 0$	$O\left(NM^2\right)$	$O\left(NM^{1+\frac{13}{16}} \log M\right)$	$O\left(NM^{1-\frac{1}{16}} \left\|\log M\right\|^7\right)$
$m = 1$	$O\left(NM^2\right)$	$O\left(NM^{1+\frac{11}{16}} \log M\right)$	$O\left(NM^{1+\frac{9}{16}} \left\|\log M\right\|^7\right)$

To compute the total storage cost we sum over all $n \in \{0, 1, \ldots, N\}$ to obtain

$$\sum_{n=0}^{N} n^{\frac{1}{2}} \left|\log \varepsilon\right|^7 M^{\frac{m}{2}+\frac{3}{4}} \leq CN^{\frac{3}{2}} \left|\log M\right|^7 M^{\frac{m}{2}+\frac{3}{4}} \leq CNM^{\frac{5m}{8}+\frac{15}{16}} \left|\log M\right|^7$$

$$\leq C \begin{cases} NM^{\frac{15}{16}} \left|\log M\right|^7 & m = 0, \\ NM^{1+\frac{9}{16}} \left|\log M\right|^7 & m = 1. \end{cases}$$

Note that the storage cost for the temporary quantities in Phase 2 of the panel-clustering algorithm is proportionally to $M \log^3 M$ and, hence, negligible compared to the other components of the algorithm.

The total storage requirements are summarised in Table 2. The table shows that the panel-clustering method combined with the cutoff strategy reduces the storage amount very significantly. For piecewise constant boundary elements we even get a storage complexity with behaves better than linearly, i.e., $O\left(NM\right)$.

8 Conclusions

In this paper, we have followed the convolution quadrature approach by Lubich and combined it with Galerkin BEM for solving the retarded potential boundary integral formulation of the wave equation. The main goal was to develop fast and sparse algorithms for this purpose, i.e., a simple a-priori cutoff strategy where the number of matrix elements which have to be computed is substantially reduced and a significant portion of the matrix is replaced by zero. The panel-clustering method is applied to the remaining blocks which further reduces the computational costs.

In a forthcoming paper, we will introduce an efficient quadrature method and analyse the effect of these additional perturbations.

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