Statistical properties of dynamical systems (MAT 734)

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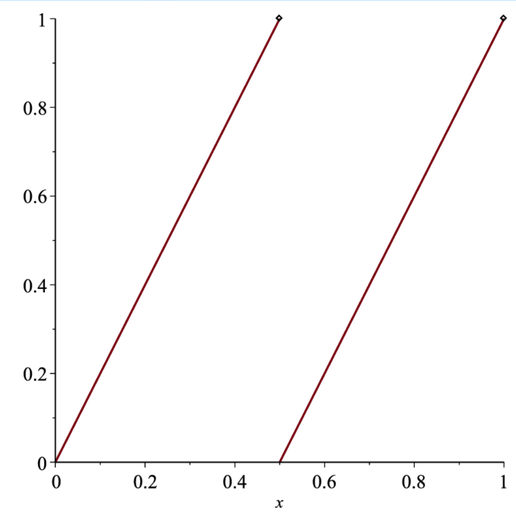
Which are the dynamical systems we are interested in?

- A measure space $(X, \mathscr{B}, \mu)$
- A measurable map $T\colon X \to X$
- **Assumption 1:** T is measure preserving, i.e., $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathscr{B}$.
- **Assumption 2:** T shows a chaotic behavior (positive topological entropy)

Two examples

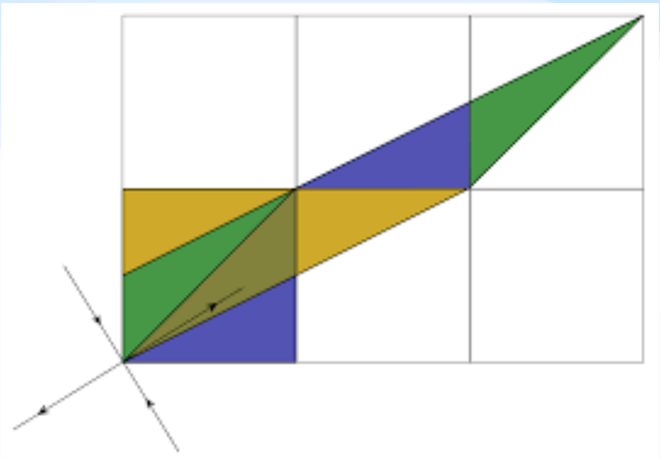
Doubling map

$$T\colon S^1 \to S^1$$
$$x \mapsto 2x \mod 1$$



Arnold CAT map

$$T\colon \mathbb{T}^2 \to \mathbb{T}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1$$



Some statistical properties we want to study

Statistical (ergodic) properties **Probabilistic properties**

Ergodicity:

For all $A \in \mathscr{B}$, $\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{+\infty} \chi_A \circ T^k(x) = \mu(A)$. ⟺ Strong law of large numbers

Mixing:

For all $A, B \in \mathscr{B}$,

$\lim_{n\to+\infty} \int \chi_A \circ T^n \cdot \chi_B d\mu = \mu(A)\mu(B)$. ⟺ Decay of correlation function

Other questions we are going to answer:

- How fast does the ergodic average $\frac{1}{n}\sum_{k=0}^{+\infty}\chi_A \circ T^k(x)$ converge to $\mu(A)$?

- What is the speed of mixing? I.e., how fast does $\int \chi_A \circ T^n \cdot \chi_B d\mu$ converge to $\mu(A) \cdot \mu(B)$?

- Are there other probabilistic properties that show the chaotic behavior of these dynamical systems? For instance, is there a Central Limit Theorem? A Local Limit Theorem?

The main tool of our analysis will be the Transfer Operator and its spectral properties. In particular, we will use it to investigate expanding map and uniformly hyperbolic automorphisms. The previous examples are the easiest cases.

References:

Transfer operators in Hyperbolic Dynamics An introduction. Carlangelo Liverani Mark Demers and Niloofar Kiamari. 33 Colloquio Brasilero de Matematica. Brazilian Mathematics Colloquiums sereis, Editora do IMPA. pp.252 (2021). ISBN 978-65-89124-26-9. ***(Available online)***

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