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Joint work with Diego Napp, Alessandro Neri and Veronica Requena

Zurich June 7th, 2023



Contents

- Notation
- MDP Convolutional Codes
- Weighted Reed-Solomon Convolutional Codes
- Field Size Consideration
- Conclusion

- \bullet \mathbb{F}_q finite field of q elements, q prime power
- *n*, *k* positive integers
- ullet $\mathbb{F}_q[z]$ polynomial ring over \mathbb{F}_q

Definition

An $(n,k)_q$ convolutional code is a rank $k \mathbb{F}_q[z]$ -submodule $\mathcal{C} \subseteq \mathbb{F}_q[z]^n$.

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$$\mathcal{C} := \{u(z)G(z) \mid u(z) \in \mathbb{F}_q[z]^k\} \subseteq \mathbb{F}_q[z]^n.$$

- the *i*th row degree δ_i is the largest degree among the entries in the *i*th row of G(z).
- the **degree** δ is the highest degree of the $k \times k$ minors of G(z).

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An $(n, k, \delta)_q$ convolutional code is a rank $k \mathbb{F}_q[z]$ -submodule $C \subseteq \mathbb{F}_q[z]^n$.

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- the **degree** δ is the highest degree of the $k \times k$ minors of G(z).
- G(z) is **reduced** if the sum of its row degrees attains the minimum possible value among the generator matrices (δ is equal to sum of row degrees).
- G(z) is **basic** if its Smith-form is given by $(I_k \quad 0)$.

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jth Column Distances

Definition

$$d_{j}^{c}(\mathcal{C}) := \min \left\{ \operatorname{wt}(v_{[0,j]}(z)) = \operatorname{wt}(v_{0} + v_{1}z + \cdots + v_{j}z^{j}) \mid v(z) \in \mathcal{C}, \ v_{0} \neq 0 \right\}$$

Definition

Let

$$G(z) := \sum_{i=0}^{m} G_i z^i.$$

For every $j \in \mathbb{N}_0$ we define the *j*-th truncated sliding generator

$$G_j^c := egin{bmatrix} G_0 & G_1 & \cdots & G_j \ & G_0 & \cdots & G_{j-1} \ & & \ddots & dots \ & & & G_0 \ \end{pmatrix}$$

MDP Convolutional Codes

- $ullet d_j^c \leq \mathrm{d}_{\mathsf{free}} \ \mathsf{for} \ \mathsf{all} \ j \in \mathbb{N}_0$
- $d_0^c \le d_1^c \le d_2^c \le \dots$
- ullet For every $j\in\mathbb{N}_0$, we have $d_i^c\leq (n-k)(j+1)+1$.
- If $d_j^c = (n-k)(j+1)+1$ for some $j \in \mathbb{N}_0$, then $d_i^c = (n-k)(i+1)+1$ for all $i \leq j$.

Definition

Let $L:=\lfloor \frac{\delta}{k} \rfloor + \lfloor \frac{\delta}{n-k} \rfloor$. An $(n,k,\delta)_q$ convolutional code with column distances $d_j^c, j \in \mathbb{N}_0$ is said to have **maximum distance profile** (MDP) if

$$d_j^c = (n-k)(j+1) + 1$$
, for $j = 1, ..., L$.

Characterization of MDP Convolutional Codes

Theorem (Gluesing-Luerssen, Rosenthal, Smarandache, '06)

Let $G(z) = \sum_{i=0}^{m} G_i z^i$ be a basic generator matrix of an (n, k, δ) convolutional code C. The following statements are equivalent:

- $d_L^c(C) = (n-k)(L+1) + 1.$
- every $(L+1)k \times (L+1)k$ full-size minor of G_L^c formed by columns with indices $1 \le t_1 < \cdots < t_{(L+1)k}$, where $t_{sk+1} > sn$ for $s=1,\ldots,L$, is nonzero .



H. Gluesing-Luerssen, J. Rosenthal, and R. Smarandache. "Strongly MDS convolutional codes.", IEEE Transactions on Information Theory, 2006

G(z) has the **MDP property** if (2) holds.

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G(z) has the **MDP property** if (2) holds.

- It is not necessary to have G(z) basic in order to show that the code generated by G(z) is MDP.
- If $\delta = km$ and G(z) has the MDP property, then the convolutional code generated by G(z) is noncatastrophic and the code is MDP.



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Generalized Reed-Solomon Codes

Definition

Let $n \leq q$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q$ pairwise distinct elements, $b_1, \ldots, b_n \in \mathbb{F}_q^*$. The code

$$C := \{(b_1 f(\alpha_1), \ldots, b_n f(\alpha_n)) \mid f \in \mathbb{F}_q[x]_{< k}\}$$

is called *Generalized Reed-Solomon (GRS) code* and it is denoted by $\mathcal{GRS}_k(\alpha, b)$, where $\alpha := (\alpha_1, \dots, \alpha_n)$ and $b = (b_1, \dots, b_n)$.

$$G:=egin{pmatrix} b_1 & b_2 & \cdots & b_n \ b_1lpha_1 & b_2lpha_2 & \cdots & b_nlpha_n \ b_1lpha_1^2 & b_2lpha_2^2 & \cdots & b_nlpha_n^2 \ dots & dots & dots \ b_1lpha_1^{k-1} & b_2lpha_2^{k-1} & \cdots & b_nlpha_n^{k-1} \end{pmatrix}=V_k(lpha)\mathrm{diag}(b)$$

- Let $\alpha := (\alpha_1, \dots, \alpha_n) \in (\mathbb{F}_q^*)^n$, α_i 's pairwise distinct
- ullet Let γ be root of an irreducible polynomial of **degree** s in $\mathbb{F}_q[z]$

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- Let γ be root of an irreducible polynomial of **degree** s in $\mathbb{F}_q[z]$
- For any $0 \le r \le m$, let G_r be the following matrix

$$\begin{pmatrix} \gamma^{\frac{r(r+1)}{2}k-r}\alpha_{1}^{(r+1)k-1} & \gamma^{\frac{r(r+1)}{2}k-r}\alpha_{2}^{(r+1)k-1} & \cdots & \gamma^{\frac{r(r+1)}{2}k-r}\alpha_{n}^{(r+1)k-1} \\ \vdots & & \vdots & & \vdots \\ \gamma^{\frac{r(r-1)}{2}k+r}\alpha_{1}^{rk+1} & \gamma^{\frac{r(r-1)}{2}k+r}\alpha_{2}^{rk+1} & \cdots & \gamma^{\frac{r(r-1)}{2}k+r}\alpha_{n}^{rk+1} \\ \gamma^{\frac{r(r-1)}{2}k}\alpha_{1}^{rk} & \gamma^{\frac{r(r-1)}{2}k}\alpha_{2}^{rk} & \cdots & \gamma^{\frac{r(r-1)}{2}k}\alpha_{n}^{rk} \end{pmatrix}$$

Remark

 $\bullet \ \mathbb{F}_q(\gamma) \cong \mathbb{F}_{q^s}$

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- Gi's are all generator matrices of a GRS code

Definition

A weighted RS convolutional code $C_{k,n}^m(\gamma,\alpha)$ is the code whose generator matrix is $G(z) = \sum_{r=0}^m G_r z^r$.



G.N. Alfarano, D. Napp, V. Requena and A. Neri "Weighted Reed-Solomon Convolutional Codes.", Linear and Multilinear Algebra, 2022.

Example: $C_{3,5}^2$

$$k = 3$$
, $m = 2$, $\delta = 6$, $L = 5$.

$$\begin{split} G_0 &= \begin{pmatrix} \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad G_1 = \begin{pmatrix} \gamma^2 \alpha_1^5 & \gamma^2 \alpha_2^5 & \gamma^2 \alpha_3^5 & \gamma^2 \alpha_4^5 & \gamma^2 \alpha_5^5 \\ \gamma \alpha_1^4 & \gamma \alpha_2^4 & \gamma \alpha_3^4 & \gamma \alpha_4^4 & \gamma \alpha_5^4 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_3^3 & \alpha_3^3 \end{pmatrix} \\ G_2 &= \begin{pmatrix} \gamma^7 \alpha_1^8 & \gamma^7 \alpha_2^8 & \gamma^7 \alpha_3^8 & \gamma^7 \alpha_4^8 & \gamma^7 \alpha_5^8 \\ \gamma^5 \alpha_1^7 & \gamma^5 \alpha_2^7 & \gamma^5 \alpha_3^7 & \gamma^5 \alpha_4^7 & \gamma^5 \alpha_5^7 \\ \gamma^3 \alpha_1^6 & \gamma^3 \alpha_2^6 & \gamma^3 \alpha_3^6 & \gamma^3 \alpha_4^6 & \gamma^3 \alpha_5^6 \end{pmatrix} \end{split}$$

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Parameters of $C_{k,n}^m(\gamma,\alpha)$

Proposition

The code $C_{k,n}^m(\gamma,\alpha)$ is a $(n,k,km)_{q^s}$ convolutional code.

- $G_0 = M_0$ full rank hence k is the dimension of the code.
- ullet The degree δ is equal to the sum of the row-degrees km.

Corollary

The generator matrix of $C_{k,n}^m(\gamma,\alpha)$ is reduced.

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 $\mathcal{C}^m_{k,n}(\gamma,\alpha)$ is an MDP convolutional code in $\mathbb{F}_{q^s}[\mathbf{z}]^n$.

Proof.

Difficult!

- Let $Y = (y_i)_i$ be a vector of algebraically independent variables
- Let $G(x, Y, B, \Lambda)$ be an upper triangular block matrix: every block is a generalized Vandermonde matrix $V(\Lambda, Y)$, where each row is multiplied by a suitable power of an another algebraically independent variable x
- ullet B is a vector whose components are the involved powers of x
- A is the vector of the exponents involved in the generalized Vandermonde matrices.

$$G(x, Y, B, \Lambda) := \begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,m} \\ & A_{1,1} & \cdots & A_{1,m} \\ & & \ddots & \vdots \\ & & & A_{m,m} \end{pmatrix}$$

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$$\textit{A}_{i,j} = \textit{A}^{\left(\beta^{(i,j)}, \lambda^{(i,j)}\right)} := \operatorname{diag}\left(x^{\beta^{(i,j)}}\right) \textit{V}\left(\lambda^{(i,j)}, y^{(j)}\right) \in \mathbb{F}[x,Y]^{k_i \times \ell_j} \text{, where}$$

$$\operatorname{diag}\left(x^{\beta^{(i,j)}}\right) = \begin{pmatrix} x^{\beta_1^{(i,j)}} & 0 & \cdots & 0 \\ 0 & x^{\beta_2^{(i,j)}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x^{\beta_{k_i}^{(i,j)}} \end{pmatrix}.$$

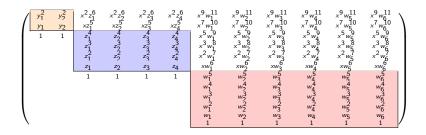
- Observe that the determinant of the matrix $G(x, Y, B, \Lambda)$ is a polynomial p(x, Y).
- We show that the monomial of minimal degree in x of p is a polynomial in Y non identically zero.
- det $G(x, Y, B, \Lambda) = p(x, Y) = p_0(Y)x^t + p_1(x, Y)x^{t+1}$.

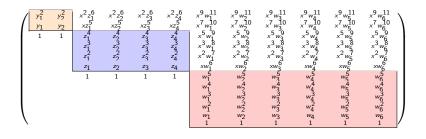
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- det $G(x, Y, B, \Lambda) = p(x, Y) = p_0(Y)x^t + p_1(x, Y)x^{t+1}$.

In our construction:

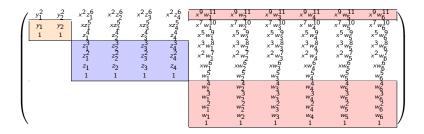
- All the minors we have to check have the shape of $G(\gamma, \alpha, B, \Lambda)$
- det $G(\gamma, \alpha, B, \Lambda) = p(\gamma, \alpha) = p_0(\alpha)\gamma^t + p_1(\gamma, \alpha)\gamma^{t+1}$:
- $p_0(\alpha)$ is given by the product of some determinants of Vandermonde matrices
- γ is such that it is not a zero of det $G(\gamma, \alpha, B, \Lambda)$

$$G(x,Y,B,\Lambda) = \begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} \\ A_{1,1} & A_{1,2} \\ A_{2,2} \end{pmatrix} = \begin{pmatrix} y_1^2 & y_2^2 & x^2 z_1^6 & x^2 z_2^6 & x^2 z_2^6 & x^2 z_2^6 & x^9 w_1^{11} & x^9 w_1^{11$$





$$y_1y_2z_1z_2z_3z_4(y_1-y_2)\prod_{1\leq i< j\leq 4}(z_i-z_j)\prod_{1\leq i< j\leq 6}(w_i-w_j).$$



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Replace γ with an indeterminate x and consider $G_L^c(x)$

$$\mathcal{P}(k, n, m, \alpha) := \{q(x) \in \mathbb{F}_q[x] \mid q(x) \text{ is a full size minor of } G_L^c \text{ formed as stated (2) } \}.$$

Remark

Then $C_{k,n}^m(\gamma,\alpha)$ is MDP if and only if $q(\gamma) \neq 0$ for every $q(x) \in \mathcal{P}(k,n,m,\alpha)$.

Let $G:=\sum_{r=0}^m G_r z^r$, where G_r is

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Remark

Then $\mathcal{C}^m_{k,n}(\gamma,\alpha)$ is MDP if and only if $q(\gamma) \neq 0$ for every $q(x) \in \mathcal{P}(k,n,m,\alpha)$.

- Let $\nu(q(x)) := \max\{\ell \in \mathbb{N} \mid x^{\ell} \text{ divides } q(x)\}$
- $D(k, n, m, \alpha) := \max\{\deg q(x) \nu(q(x)) \mid 0 \neq q(x) \in \mathcal{P}(k, n, m, \alpha)\}.$

Theorem

For any integer $s > D(k, n, m, \alpha)$, $C_{k,n}^m(\gamma, \alpha)$ is an MDP convolutional code in $\mathbb{F}_{q^s}[z]^n$.

Remark

We need to estimate *D* to determine the field size of our code.

Theorem

For every k, n, m integers with 0 < k < n, there exists an $(n, k, km)_{q^s}$ MDP convolutional code where $q^s = \mathcal{O}(n^{\frac{(km)^3}{3t}})$, and $t = \min\{k, n - k\}$.

Comparison with Known Constructions

[n, k, δ] L, m, μ, r	ANP	GRS	MAK*	AN*	HST [†]	L†	$\mathcal{C}_{k,n}^m$	GRS [‡]
[2, 1, 1] 2, 1, 1, 3	28	43	2 ⁵	3	3	55	3	-
[2, 1, 2] 4, 2, 2, 5	232	434692	27	7	11	1261	27	23
[3, 2, 2] 3, 1, 2, 8	2 ⁵¹²	$5^87^82^{12} + 1$	2 ¹¹	31	233	1981	256	26
[3, 1, 2] 3, 2, 1, 8	2 ⁵¹²	$5^87^82^{12} + 1$	2 ¹¹	31	233	3961	16	26
[3, 2, 1] 1, 1, 1, 4	232	$2^43^4 + 1$	-	5	5	3	4	22
[4, 2, 2] 2, 1, 1, 7	2128	~ 10 ¹²	-	17	77	5545	125	25
[4, 1, 3] 4, 3, 1, 15	2217	\sim 7 · 10 ⁶¹	-	-	1338936	232561	3125	213
[5, 2, 2] 1, 1, 1, 7	229	~ 10 ¹²	-	17	77	35	49	32
[6, 2, 2] 1, 1, 1, 9	2211	$\sim 7 \cdot 10^{20}$	-	59	751	71	49	128
[6, 2, 2] 1, 2, 1, 9	2211	$\sim 7 \cdot 10^{20}$	-	59	751	71	49	128
[7, 2, 2] 1, 1, 1, 11	2213	~ 10 ³²	-	-	8525	126	64	512
[7, 3, 3] 1, 1, 1, 10	2212	~ 10 ²⁶	-	127	2495	532	512	256

- *: result found by computer search
- †: not constructive
- ‡: based on a conjecture
- -: there are no constructions for such parameters



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- Conclusion

Summarizing

- We defined a new family of convolutional codes
- We showed that they are MDP exploiting a more general point of view
- We compare the obtained field size with the existing constructions

Remarks and Future Research

 Recently, for memory 1 combining Vandermonde and Moore matrices a construction of MDP convolutional codes has been obtained.



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IEEE Transactions on Information Theory, 2023.

 Recently it has been shown that in order to construct an MDP convolutional code we need a field size

$$q \geq \Omega_L(n^{L-1}).$$



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- Check if this family of codes is closed under duality
- Use the algebraic structure to develop a decoding algorithm

Thank You!