$R_3 = 1/8$

On the Plotkin Construction for Convolutional Codes

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The Plotkin Construction

Given $\mathcal{C}_0(n, k_0, d_0)$ and $\mathcal{C}_1(n, k_1, d_1)$ both $\subset \mathbb{F}_2^n$

$$\mathcal{C}(2n, k_0 + k_1, \min\{2d_0, d_1\}) = \{c = (u_0|u_0 + u_1), u_i \in \mathcal{C}_i\}$$

The length $2n$ and the dimension $k = k_0 + k_1$ are obvious.
Possible decoder of $\mathcal{C}$:
BSC $r = c + e = (u_0 + e_0|u_0 + u_1 + e_1)$.
Addition: $u_0 + e_0 + u_0 + u_1 + e_1 = u_1 + e_0 + e_1$.
Since $\mathrm{wt}(e) \geq \mathrm{wt}(e_0 + e_1)$: u_1 correct if $\tau = \mathrm{wt}(e) \leq \frac{d_1 - 1}{2}$.
Known: $u_0 + e_0$ add u_1 : $u_0 + e_1$.
$d_0 - 1$ errors in both halfs: either $u_0 + e_0$ or $u_0 + e_1$ contains $\leq \frac{d_0 - 1}{2}$ errors.

M. Plotkin, Binary codes with specific minimum distances , IEEE Trans. on Inf. Theory, vol. 6, pp. 445-450, 1960.

Recursive Plotkin Construction

Given two Plotkin construction of length $2n$ (codes of length n) $|\mathbf{u}_0|\mathbf{u}_0 + \mathbf{u}_1|$ and $|\mathbf{u}_2|\mathbf{u}_2 + \mathbf{u}_3|$ The Plotkin construction for these two creates a code of length $4n$

$$|\mathbf{u}_0|\mathbf{u}_0 + \mathbf{u}_1|\mathbf{u}_0 + \mathbf{u}_2|\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3|$$

the double Plotkin construction. A tripple Plotkin construction using a second double Plotkin constructed code

$$|\mathbf{u}_4|\mathbf{u}_4 + \mathbf{u}_5|\mathbf{u}_4 + \mathbf{u}_6|\mathbf{u}_4 + \mathbf{u}_5 + \mathbf{u}_6 + \mathbf{u}_7|$$

gives a code of length $8n$

$$|\mathbf{u}_0|\mathbf{u}_0 + \mathbf{u}_1|\mathbf{u}_0 + \mathbf{u}_2|\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3|$$

$$\mathbf{u}_0 + \mathbf{u}_4|\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_4 + \mathbf{u}_5|\mathbf{u}_0 + \mathbf{u}_2 + \mathbf{u}_4 + \mathbf{u}_6|\sum_{i=0}^{7}\mathbf{u}_i|.$$

Any tripple Plotkin construction can be viewed as a double Plotkin or a Plotkin construction.

Example double Plotkin

Consider the four binary codes $\mathcal{C}_{\mathbf{u}_0}(8,7,2)$, $\mathcal{C}_{\mathbf{u}_1}(8,4,4)$, $\mathcal{C}_{\mathbf{u}_2}(8,4,4)$, and $\mathcal{C}_{\mathbf{u}_3}(8,1,8)$ and the double Plotkin construction

$$\mathbf{u}_0|\mathbf{u}_0+\mathbf{u}_1|\mathbf{u}_0+\mathbf{u}_2|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3|$$

is a $(32,16,8)$ code.
Adding all four blocks we get the code $\mathcal{C}_{\mathbf{u}_3}(8,1,8)$
Adding the first to the third block and the second to the fourth we get the code

$$|\mathbf{u}_2|\mathbf{u}_2+\mathbf{u}_3|$$

which has the parameters $(16,5,8)$.
The first and the second block are the code

$$|\mathbf{u}_0|\mathbf{u}_0+\mathbf{u}_1|$$

which has the parameters $(16,11,4)$.

Reed–Muller Codes

The RM code $\mathcal{R}(r,m)$ with order r has length $n=2^m$, dimension $k = 1 + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{r}$ and minimum distance $d = 2^{m-r}$, $\mathcal{R}(r,m) = (2^m, k, 2^{m-r})$.

We can use RM codes in the Plotkin construction. If we use $\mathcal{C}_{\mathbf{u}_0} = \mathcal{R}(r+1,m)$ and $\mathcal{C}_{\mathbf{u}_1} = \mathcal{R}(r,m)$ we get $\mathcal{C} = \mathcal{R}(r+1,m+1)$, thus

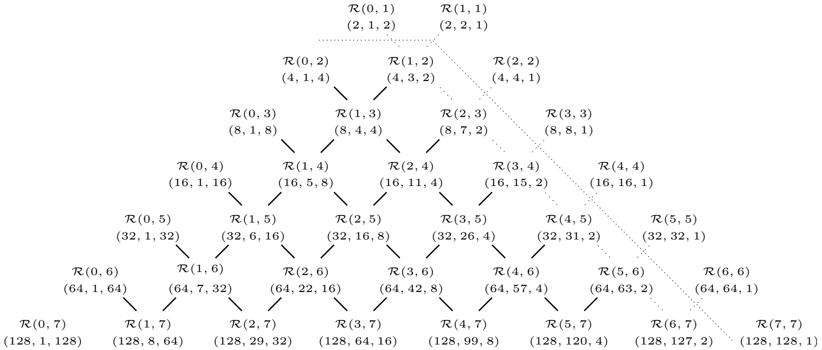
$$\mathcal{R}(r{+}1,m{+}1) = \{|\mathbf{u}_0|\mathbf{u}_0+\mathbf{u}_1| : \ \mathbf{u}_0 \in \mathcal{R}(r+1,m), \ \mathbf{u}_1 \in \mathcal{R}(r,m)\}.$$

Since $d_0 = 2^{m-r-1}$ and $d_1 = 2^{m-r}$ we get $d = 2^{m-r}$. For the length we get $2 \cdot 2^m = 2^{m+1}$. For the dimension it holds that

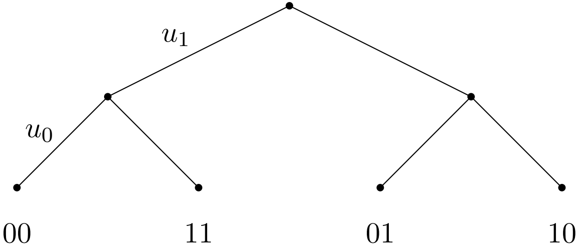
$$\binom{m+1}{r+1} = \binom{m}{r+1} + \binom{m}{r}$$

and thus $k = k_0 + k_1$ and the parameters are those of the RM code $\mathcal{R}(r+1,m+1)$.

Recursive Construction of RM Codes



Generalized Code Concatenation, Partitioning



$$u_0 | u_0 + u_1$$

The Plotkin Construction for BPSK: $(1, -1)$

Define: $\mathbf{x}_0\mathbf{x}_1 = (x_{0,0}x_{1,0}, x_{0,1}x_{1,1}, \ldots, x_{0,n-1}x_{1,n-1})$

Operation $x_i = (-1)^{c_i}$

$$\{c = (u_0|u_0 + u_1)\} \Longleftrightarrow \{x = (x_0|x_0x_1)\}$$

AWGN $y = x + n = (y_0|y_1) = (x_0 + n_0|x_0x_1 + n_1)$.
Def: $y_i \bowtie y_j \bowtie \ldots \bowtie y_\ell = \mathrm{sign}(y_iy_j \ldots y_\ell)\min\{|y_i|, |y_j|, \ldots, |y_\ell|\}$
Join operation is cumutative and associative
Then $y_0 \bowtie y_1 \leftrightarrow \hat{x}_0y_1 \rightarrow \hat{x}_1$
Assume $\hat{x}_1$ is correct then

Gain 3 dB

$$y_0 + y_1x_1 = x_0 + n_0 + x_0 + n_1$$

Proof: $x_0 + n_0$ and $x_0 + n_1 \in \mathcal{N}(1, \sigma^2)$ or $\in \mathcal{N}(-1, \sigma^2)$
Sum is $\in \mathcal{N}(\pm 2, 2\sigma^2)$, Signal power: 4, Noise variance: 2 $\longrightarrow$ 3 dB

Double Plotkin with Convolutional Codes

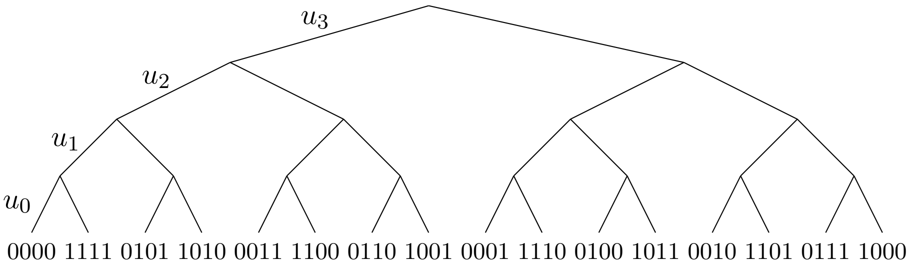
Covolutional codes $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$
Encoding $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$
Transmission $\mathbf{x}_0$, $\mathbf{x}_0\mathbf{x}_1$, $\mathbf{x}_0\mathbf{x}_2$, and $\mathbf{x}_0\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3$
Receive $\mathbf{y}_0$, $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$

Decoding:
First step: decode $\mathbf{y}_0 \bowtie \mathbf{y}_1 \bowtie \mathbf{y}_2 \bowtie \mathbf{y}_3$ with $\mathcal{C}_3 \to \mathbf{x}_3$
Second step: decode $(\mathbf{y}_0 \bowtie \mathbf{y}_2) + (\mathbf{y}_1 \bowtie \mathbf{x}_3\mathbf{y}_3)$ with $\mathcal{C}_2 \to \mathbf{x}_2$
Third step: decode $(\mathbf{y}_0 + \mathbf{x}_2\mathbf{y}_2) \bowtie (\mathbf{y}_1 + \mathbf{x}_2\mathbf{x}_3\mathbf{y}_3)$ with $\mathcal{C}_1 \to \mathbf{x}_1$
Fourth step: decode $\mathbf{y}_0 + \mathbf{x}_1\mathbf{y}_1 + \mathbf{x}_2\mathbf{y}_2 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{y}_3$ with $\mathcal{C}_0 \to \mathbf{x}_0$

Note, last step has gain of 6 dB!
Example Coderates: $R_3 = 1/8$, $R_2 = 4/8$, $R_1 = 4/8$, $R_0 = 7/8$

Generalized Code Concatenation, Partitioning

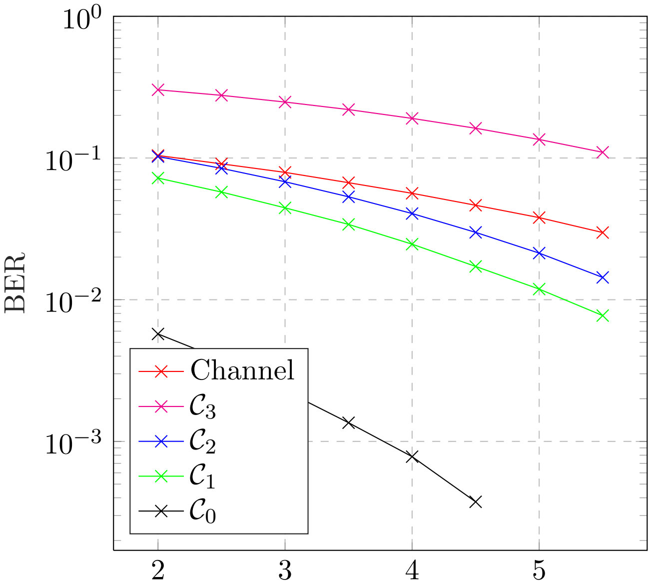
Double Plotkin construction



$$u_0 | u_0 + u_1 | u_0 + u_2 | u_0 + u_1 + u_2 + u_3$$

Inner codes: $(4,4,1) \to (4,3,2) \to (4,3,2) \to (4,1,4)$

AWGN BER for Double Plotkin Construction



Literature for 3 dB gain

IEEE IT Paper, Gottfried Schnabl and Martin Bossert, 1995

$y_0 + y_1 x_1$ used for decoding PC code in GMC decoder for RM codes
Schritt 2b, p. 418 in Bossert, Kanalcodierung, BG Teubner, 2. Auflage 1998
Step 2b, p. 376 in Bossert, Channel Coding for Telcommunications, Wiley, 1999

Dissertation (in German), Norbert Stolte, 2002

Rekursive Codes mit der Plotkin-Konstruction und ihre Decodierung
Equivalent channel SNR, $SNR_u = 2SNR_v$ sections 3.2.3 and 3.2.4, recursive Plotkin, OCBM (=polar) Fig. 3.11, p. 29

IEEE IT Paper, Erdal Arikan, 2009

Capacity based analysis, novel result: asymptotic capacity achieving.