Successive enlargements of filtrations and Application to Insider Modeling.

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Introduction

- Information modelling and Insider's information are classical questions in finance:
    - Grorud and Pontier (1998, 2001),
    - Amendinger, Imkeller and Schweizer (1998),
    - after a lot of papers...
- In the classical context, insider has extra-infomations at the begining.
- Tools are based on theory of initial enlargement of filtration.
- A natural model will be that insider can obtain or adjust her private information over time
- Our approach is different from previous works with 'dynamical' enlargmen (Corcuera et al., Khia Larsson Protter), less general than 'local approach' but tractable formulas.
-

Aim of our work

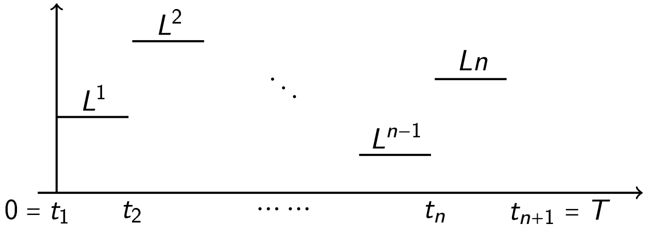
Aim : Propose a model for the dynamic flow of insider's information

- Classicaly, initial enlargment is used;
- Here, we consider succesive enlargments
- Our objectives are:
  - develop a theoretical framework for successive enlargments
  - application to credit risk: insider's information in a default risk model.

Model setup

$(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ a filtered probability space.

- common information: $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a reference filtration satisfying the usual conditions, $\mathcal{F}_0$ is trivial.
- horizon time: $T > 0$
- time intervals: $\{t_i, t = 1, \cdots, n\}$ with $t_1 = 0$ and $t_{n+1} = T$
- private information flow: $\{L^i, i = 1, \cdots, n\}$ where L^i is $\mathcal{A}$-measurable and valued in a measurable space $(E, \mathcal{E})$



Insider's information flow

Insider's information contains both common and private information described by the filtration $\mathbb{G}^I = (\mathcal{G}_t^I)_{t\geq 0}$ where

$$\mathcal{G}_t^I := \mathcal{F}_t \vee \sigma(L^1) \vee \cdots \vee \sigma(L^i), \quad t \in [t_i, t_{i+1}), \ i = 1, \cdots n.$$

- By definition,

$$\mathcal{G}_t^I = \mathcal{G}_t^i, \quad t \in [t_i, t_{i+1})$$

  where

$$\mathcal{G}_t^i := \mathcal{F}_t \vee \sigma(L^1) \vee \cdots \vee \sigma(L^i), \quad t \in [0, T],$$

  is a family of successive initial enlargement of filtrations s.t.

$$\mathcal{G}_t^i = \mathcal{G}_t^{i-1} \vee \sigma(L^i)$$

- Introduce the private information process

$$L_t = \sum_{i=1}^{n} L^i \mathbf{1}_{[t_i, t_{i+1})}(t), \quad t \in [0, T]$$

  Then $\mathcal{G}_t^I = \mathcal{F}_t \vee \sigma(L_s, s \leq t)$.

Reminder on initial enlargement of filtration

Assumption H' (Jacod 1985) :

We assume that $\mathbb{P}(L \in \cdot | \mathcal{F}_t)(\omega) \sim \mathbb{P}(L \in \cdot)$ for all $t \geq 0$ and for $\mathbb{P}$-almost all $\omega \in \Omega$.

- (Grorud-Pontier 1998) : Assumption (H') is equivalent to : There exists a unique probability measure $\mathbb{P}^L$,

$$\begin{cases} \mathbb{P}^L \text{ equivalent to } \mathbb{P} \\ \mathbb{P}^L \text{ identical to } \mathbb{P} \text{ on } \mathcal{F}_\infty \\ \text{the law of } L \text{ is unchanged under } \mathbb{P} \text{ and } \mathbb{P}^L \\ \text{for any } t \geq 0,\ \mathcal{F}_t \text{ and } \sigma(L) \text{ are independent under } \mathbb{P}^L \end{cases}$$

Reminder (cont.)

- Let the conditional density process be defined by

$$\mathbb{P}(L \in dx|\mathcal{F}_t) = p_t(x)\mathbb{P}(L \in dx).$$

- $(p_t(L), t \geq 0)$ is a $(\mathbb{P}^L, \mathbb{F} \vee \sigma(L))$-martingale and is the Radon-Nikodym derivative of the probability $\mathbb{P}$ w.r.t. $\mathbb{P}^L$, i.e.

$$\left.\frac{d\mathbb{P}^L}{d\mathbb{P}}\right|_{\mathcal{F}_t \vee \sigma(L)} = \frac{1}{p_t(L)}.$$

Successive density hypothesis

Assumption 1:

For any $i \in \{1, \cdots, n\}$, the $\mathcal{G}_T^{i-1}$-conditional law of L^i is equivalent to its $\mathcal{G}_{t_i}^{i-1}$-conditional law under the probability $\mathbb{P}$, namely there exists a positive $\mathcal{G}_T^{i-1} \otimes \mathcal{E}$-measurable function $\alpha_T^{i|i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$ such that

$$\mathbb{P}(L^i \in dx \,|\, \mathcal{G}_T^{i-1}) = \alpha_T^{i|i-1}(\boldsymbol{L}^{(i-1)}, x)\mathbb{P}(L^i \in dx \,|\, \mathcal{G}_{t_i}^{i-1}).$$

Remarks about this assumption

- Here the conditional law $\mathbb{P}(L^i \in dx \,|\, \mathcal{G}^{i-1}_{t_i})$ itself is a random measure instead of a deterministic probability law as in Jacod's hypothesis. So the existence of martingale version of density process can't be obtain by same arguments. $\Longrightarrow$ We work with a terminal time T.
- Under Assumption 1, the filtration $\mathbb{G}^i$ is right-continuous on $[t_i, T]$, and also is $\mathbb{G}^I$ on $[0, T]$.
- Assumption 1 is invariant under a change of probability measure.

One step enlargement

Let $\mathbb{H} = (\mathcal{H}_u)_{u \in [t,T]}$ be a filtration of $\mathcal{A}$, $0 \le t < T$, X an $\mathcal{A}$-measurable r.v., $\mathcal{J}_u = \mathcal{H}_u \vee \sigma(X)$, $u \in [t, T]$ Assume that there exists a positive $\mathcal{H}_T \otimes E$-measurable function $q_T(\cdot)$ such that

$$\mathbb{P}(X \in dx \,|\, \mathcal{H}_T) = q_T(x)\, \mathbb{P}(X \in dx \,|\, \mathcal{H}_t). \tag{1}$$

One step enlargement (Follow.)

Simple but illustrative example which satisfies the hypothesis (1) but not Jacod's hypothesis.

Let Y_1 and Y_2 two independent r.v. which both follow the standard normal distribution. Let $X = \max(Y_1, Y_2)$ and $\mathcal{H}_u = \sigma(Y_1)$ for all $u \in [t, T]$.

- $\mathbb{P}(X \in dx | \mathcal{H}_T) = 1\, \mathbb{P}(X \in dx | \mathcal{H}_t) \Rightarrow$ hypothesis (1) is true.
- $\mathbb{P}(X \in dx) = 2\Phi(x)\phi(x)dx$ where Φ and ϕ the probability distribution function and the probability density function of the standard normal distribution.
- $\mathbb{P}(X \in dx | \mathcal{H}_t) = \Phi(Y_1)\delta_{Y_1}(du) + \mathbf{1}_{[Y_1, +\infty)}\phi(u)du$, which is not absolutely continuous w.r.t. the Lebesgue measure. This is a typical situation which we can not handle within the classical framework of Jacod's density hypothesis.

Change of probability

Under hypothesis (1), there exists an unique equivalent probability measure $\mathbb{Q}$ to $\mathbb{P}$ such that

1) $\mathbb{Q}$ coincides with $\mathbb{P}$ on $\mathbb{H}$,
2) X and $\mathbb{H}$ are conditionally independent under $\mathbb{Q}$ given $\mathcal{H}_t$,
3) X has the same conditional law given $\mathcal{H}_t$ under $\mathbb{P}$ and $\mathbb{Q}$,
4) $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_T} = q_T(X)^{-1}$

Suppose Assumption 1 is true. Let $\mathbb{P}^0 := \mathbb{P}$, and for any $i \in \{1, \cdots, n\}$, let $\mathbb{P}^i$ be the probability measure on $(\Omega, \mathcal{A})$ such that

$$\frac{d\mathbb{P}^i}{d\mathbb{P}^{i-1}} = \frac{1}{\alpha_T^{i|i-1}(\boldsymbol{L}^{(i)})}.$$

For any $\boldsymbol{x}^{(i)} \in E^i$, let $\psi_t^i(\boldsymbol{x}^{(i)}) := \prod_{k=1}^i \frac{1}{\alpha_t^{k|k-1}(\boldsymbol{x}^{(k)})}, \quad t \in [t_i, T]$.

Change of Probability-Forward construction

All $(\mathbb{P}^i)_{i=1}^n$ are well defined and equivalent to $\mathbb{P}$. For $i \in \{1, \ldots, n\}$,

1) $\mathbb{P}^i$ and $\mathbb{P}^{i-1}$ coincide on $\mathcal{G}_T^{i-1}$, all $(\mathbb{P}^i)_{i=1}^n$ coincide with $\mathbb{P}$ on $\mathcal{F}_T$,
2) $\boldsymbol{L}^{(i)}$ and $\mathcal{F}_T$ are conditionally independent given $\mathcal{F}_{t_i}$ under $\mathbb{P}^i$,
3) for any $t \in [t_i, T]$, the R-N density of $\mathbb{P}^i$ w.r.t. $\mathbb{P}^{i-1}$ is given by $\alpha_t^{i|i-1}(\boldsymbol{L}^{(i)})^{-1}$ on $\mathcal{G}_t^i$ and hence the Radon-Nikodym density of $\mathbb{P}^i$ w.r.t. $\mathbb{P}$ is given by $\psi_t^i(\boldsymbol{L}^{(i)})$ on $\mathcal{G}_t^i$.

- Warning:

$$\mathbb{F} \longrightarrow \mathbb{G}^1 \longrightarrow \mathbb{G}^2 \longrightarrow \cdots \longrightarrow \mathbb{G}^n$$

$$\mathbb{P} \xrightarrow{L^1} \mathbb{P}^1 \xrightarrow{L^2} \mathbb{P}^2 \longrightarrow \cdots \xrightarrow{L^n} \mathbb{P}^n$$

The probability laws of $L^{i+1}, \cdots, L^n$ are changed during the procedure $\mathbb{P}^{i-1} \xrightarrow{L^i} \mathbb{P}^i$

- It's important to work with a family of probability measures under which the law of L^i remains unchanged
- Main idea: backward change of probability measures

$$\mathbb{F} \xrightarrow{L^1} \mathbb{G}^1 \xrightarrow{L^2} \cdots \xrightarrow{L^{n-3}} \mathbb{G}^{n-3} \xrightarrow{L^{n-2}} \mathbb{G}^{n-2} \xrightarrow{L^{n-1}} \mathbb{G}^{n-1} \xrightarrow{L^n} \mathbb{G}^n$$

$$\mathbb{P} \rightsquigarrow^{L^n} \mathbb{Q}^n$$

$$\mathbb{Q}^n \rightsquigarrow^{L^{n-1}} \mathbb{Q}^{n-1}$$

$$\mathbb{Q}^{n-1} \rightsquigarrow^{L^{n-2}} \mathbb{Q}^{n-2}$$

$$\cdots\cdots$$

$$\mathbb{Q}^2 \rightsquigarrow^{L^1} \mathbb{Q}^1$$

A backward construction

Let $\mathbb{Q}^{n+1} = \mathbb{P}$, and for $i \in \{1, \ldots, n\}$, let $\mathbb{Q}^i$ be a probability measure on $(\Omega, \mathcal{A})$ such that

$$\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}} := \frac{1}{\alpha_T^{i|i-1}(\boldsymbol{L}^{(i)})} \tag{2}$$

All$(\mathbb{Q}^i)_{i=1}^n$ are well defined and verify the following properties for any $i \in \{1, \cdots, n\}$

1) $\mathbb{Q}^i$ coincides with $\mathbb{P}$ on $\mathcal{G}_T^{i-1}$,
2) for any $k \in \{i, \cdots, n\}$, L^k and $\mathcal{G}_T^{k-1}$ are conditionally independent given $\mathcal{G}_{t_k}^{k-1}$ under $\mathbb{Q}^i$,
3) for any $k \in \{1, \cdots, n\}$, L^k has the same conditional law given $\mathcal{G}_{t_k}^{k-1}$ under all $(\mathbb{Q}^i)_{i=1}^n$ and $\mathbb{P}$.

- Although the marginal law of each L^i is unchanged, the joint law is modified.
- The Radon-Nikodym derivative is given by

$$\left.\frac{d\mathbb{Q}^k}{d\mathbb{P}}\right|_{\mathcal{G}_t^n} = \varphi^i_T(\boldsymbol{L}) = \prod_{k=i}^{n} \frac{1}{\alpha_T^{k|k-1}(\boldsymbol{L}^{(k)})}$$

- Probabilities $(\mathbb{Q}^i)_{i=1}^n$ are crucial in the evaluation of financial claims

Conditionnal expectation with successive information

Recall that $\psi_t^i(\boldsymbol{L}^{(i)})$ is the RN density of $\mathbb{P}^i$ w.r.t. $\mathbb{P}$ on $\mathcal{G}_t^i$

Theorem

Let $Y_T(\boldsymbol{L})$ be a bounded or non-negative $\mathcal{G}_T^I$-measurable random variable. For any $t \in [0, T]$, we have

$$\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \,|\, \mathcal{G}_t^I] = \sum_{i=1}^{n} 1_{[t_i, t_{i+1})}(t) \left. \frac{\mathbb{E}^{\mathbb{P}}[Y_{t_{i+1}}(\boldsymbol{x}^{(i)})\psi^i_{t_{i+1}}(\boldsymbol{x}^{(i)})^{-1} \,|\, \mathcal{F}_t]}{\psi^i_{t_i}(\boldsymbol{x}^{(i)})^{-1}} \right|_{\boldsymbol{x}^{(i)} = \boldsymbol{L}^{(i)}}$$

where $Y_{t_{i+1}}(\cdot)$ is $\mathcal{F}_{t_{i+1}} \otimes \mathcal{E}^{\otimes i}$-measurable such that $Y_{t_{i+1}}(\boldsymbol{L}^{(i)}) = \mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \,|\, \mathcal{G}^i_{t_{i+1}}]$.

Several stronger density hypothesis

Density hypothesis with different initial σ-algebras

Assumption 2

For any $i \in \{1, \cdots, n\}$, the $\mathcal{G}_T^{i-1}$-conditional law of L^i is equivalent to its $\mathcal{G}_0^{i-1}$-conditional law under the probability $\mathbb{P}$, namely there exists a positive $\mathcal{G}_T^{i-1} \otimes \mathcal{E}$-measurable function $\beta_T^{i|i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$ such that

$$\mathbb{P}(L^i \in dx \,|\, \mathcal{G}_T^{i-1}) = \beta_T^{i|i-1}(\boldsymbol{L}^{(i-1)}, x)\, \mathbb{P}(L^i \in dx \,|\, \mathcal{G}_0^{i-1}).$$

Jacod Hypothesis in successive framework

Assumption 3

For any $i \in \{1, \cdots, n\}$, the $\mathcal{G}_T^{i-1}$-conditional law of L^i is equivalent to its conditional law under the probability $\mathbb{P}$, namely there exists a positive $\mathcal{G}_T^{i-1} \otimes \mathcal{E}$-measurable function $p_T^{i|i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$ such that

$$\mathbb{P}(L^i \in dx | \mathcal{G}_T^{i-1}) = p_T^{i|i-1}(\boldsymbol{L}^{(i-1)}, x)\mathbb{P}(L^i \in dx).$$

Relation

$$\text{Assumption 3} \Rightarrow \text{Assumption 2} \Rightarrow \text{Assumption 1}.$$

Examples show that the reciprocal statements are false.

- Trival examples L^i which is a deterministic function of $\boldsymbol{L}^{(i-1)}$ satisfies Assumption 2 but not Assumption 3; L_i, which is a $\mathcal{G}^{i-1}_{t_i}$-measurable random variable but not $\mathcal{G}^{i-1}_0$-measurable satisfies Assumption 1 and not Assumption 2.
- More interesting examples exist.

- Under assumptions 2 (resp. 3), we can simplify the conditional expectations computations.
- Under assumptions 2 (resp. 3), we can define a family of probability measures $(\bar{\mathbb{Q}}^i)_{i=1}^n$ in the same spirit of $(\mathbb{Q}^i)_{i=1}^n$

Global enlargement approach

Considering $(L_1, \dots, L_n)$ as a vector and a global approach.

Assumption

Let $\boldsymbol{L} = (L^1, \dots, L^n)$. Assume

$$\mathbb{P}(\boldsymbol{L} \in \cdot | \mathcal{F}_t)(\omega) \sim \mathbb{P}(\boldsymbol{L} \in \cdot), \quad t \in [0, T], \ \mathbb{P} - a.s.$$

and denote the conditional $\mathbb{F}$-density of $\boldsymbol{L}$ (w.r.t. its law) by

$$\mathbb{P}(\boldsymbol{L} \in d\boldsymbol{x} | \mathcal{F}_t) = p_t(\boldsymbol{x}) \mathbb{P}(\boldsymbol{L} \in d\boldsymbol{x}), \quad \boldsymbol{x} = (x_1, \cdots, x_n) \in E^n$$

- The filtration $\mathbb{G}^{\boldsymbol{L}} = \mathbb{F} \vee \sigma(\boldsymbol{L})$ coincides with $\mathbb{G}^n$.
- Under $\mathbb{P}^{\boldsymbol{L}}$ defined by $\frac{d\mathbb{P}^{\boldsymbol{L}}}{d\mathbb{P}}\Big|_{\mathcal{G}_t^{\boldsymbol{L}}} = \frac{1}{p_t(\boldsymbol{L})}$,

  $\boldsymbol{L}$ is independent of $\mathbb{F}$, but $(L^1, \dots, L^n)$ are not mutually independent.

Global vs. successive enlargement of filtrations

- Assumption 4 $\Leftrightarrow$ Assumption 2.
- We introduce another probability measure $\mathbb{Q}^{\boldsymbol{L}}$ to "de-correlate" the random variables $L^1, \cdots, L^n$ Let

$$\frac{d\mathbb{Q}^{\boldsymbol{L}}}{d\mathbb{P}}\Big|_{\mathcal{G}_T^{\boldsymbol{L}}} = \frac{\xi(\boldsymbol{L})}{p_T(\boldsymbol{L})} \text{ where } \xi(\boldsymbol{x}) = \frac{\prod_{i=1}^n \mathbb{P}(L^i \in dx_i)}{\mathbb{P}(\boldsymbol{L} \in d\boldsymbol{x})}$$

- Assumption 4 $+$ $\mathbb{P}(\boldsymbol{L} \in d\boldsymbol{x}) \sim \prod_{i=1}^n \mathbb{P}(L^i \in dx_i)$ $\Leftrightarrow$ Assumption 3.
- $\mathbb{Q}^L$ coincide $\bar{\mathbb{Q}}^1$ constructed under Assumption 3 .

Application to default model

$(\Omega, \mathcal{A}, \mathbb{P})$ probability space with a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions.

- We consider a default time defined as

$$\tau = \inf\{t \geq 0 : X_t \leq L_t\}$$

where X_t is an $\mathbb{F}$-adapted process represent the value of the firm and the default barrier is $L_t = \sum_{i=1}^{n} L^i \mathbf{1}_{[t_i, t_{i+1})}(t)$ with L^i r.v. $\mathcal{A}$-measurable and $\mathbb{R}$-valued.

Application to default model (cont)

- It's a not a classical model as
  - Structural approach : L is a constant or a deterministic function $L(t)$, then τ is an $\mathbb{F}$-stopping time.
  - In reduced-form credit risk models (e.g. Cox model): $n = 1$ and L^1 is a uni-exponential r.v. independent of $\mathbb{F}$.
- Extention of the model in Hillairet-Jiao (2011), where a manager knows the default barrier from initial time $t = 0$ and the random variable L^1 can be correlated with $\mathbb{F}$.

Conditional survival probability for insider

- In this default model, we have

$$\{\tau > t\} = \left(\cap_{k=1}^{i-1}\{X^*_{[t_k,t_{k+1}[} < L_k\}\right)\cap\{X^*_{[t_i t[} < L_i\}, \text{ for } t \in [t_i, t_{i+1})$$

where $X^*_{[t,s[} := \inf_{t\le u<s} X_u$

- Our objective is to calculate the conditional survival/default probability given the insider's information flow

$$\mathbb{P}(\tau > s|\mathcal{G}^I_t), \text{ for } s > t.$$

- The information flow of a standard investor in the credit risk analysis is $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(1_{\{\tau\le u\}}, u \le t)$, $t \in [0, T]$.
- Compare with the conditional probability $\mathbb{P}(\tau > s|\mathcal{G}_t)$

Conditional survival probability-Full information

Let $t \leq s$, we denote by i and j the indexes such that $t_i \leq t < t_{i+1}$ and $t_j \leq s < t_{j+1}$. If $i < j$, then

$$\mathbb{P}(\tau > s|\mathcal{G}_t^I) = \mathbf{1}_{\tau > t} \frac{E^{\mathbb{Q}^L}\left(\frac{p_s}{\varphi}(\boldsymbol{L})\mathbf{1}_{X^*_{[t,t_{i+1}[}>L^i} \dots \mathbf{1}_{X^*_{[t_j,s[}>L^j}|\mathcal{G}_t^i\right)}{E^{\mathbb{Q}^L}\left(\frac{p_t}{\varphi}(\boldsymbol{L})|\mathcal{G}_t^i\right)}$$

else if $i = j$

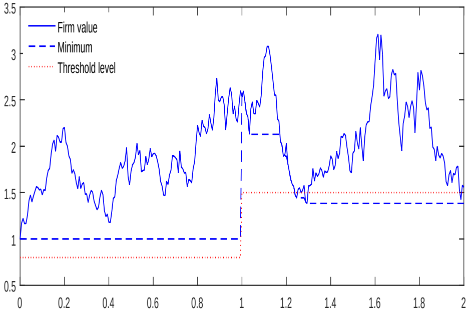
$$\mathbb{P}(\tau > s|\mathcal{G}_t^I) = \mathbf{1}_{\tau > t} \frac{E^{\mathbb{Q}^L}\left(\frac{p_s}{\varphi}(\boldsymbol{L})\mathbf{1}_{X^*_{[t,s[}>L^i}|\mathcal{G}_t^i\right)}{E^{\mathbb{Q}^L}\left(\frac{p_t}{\varphi}(\boldsymbol{L})|\mathcal{G}_t^i\right)}.$$

Numerical illustrations

- X is a geometric Brownian motion
- L^1, L^2 are exponential r.v. independant of $\mathbb{F}$.
- Joint law of (L^1, L^2) is given by a Gumbel-Barnett Copula with parameter $\theta \in [0, 1]$.

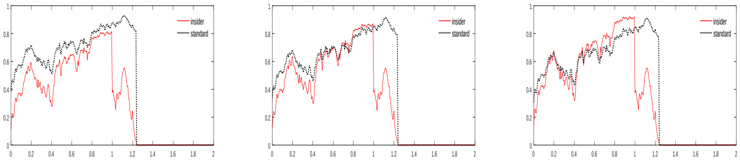
First Example

Figure: First Case : Default in $[1,2]$, $L^0 = 0.8$, $L^1 = 1.5$



- Threshold level L_2 larger than expected value.
- High risk of default after t_2.

Figure: Survival Probability $t \to P(\tau > T|\mathcal{G}_t^M)$ and $t \to P(\tau > T|\mathcal{G}_t)$for $\theta = 0, 0.5$ and 1



Remarks:

- Insider modifies immediately her estimation.
- Insider's estimation is better.
- Higher estimation when strong correlation than when independence.

Second example

Figure: Second Case : No Default , $L^0 = 0.8, L^1 = 0.6$

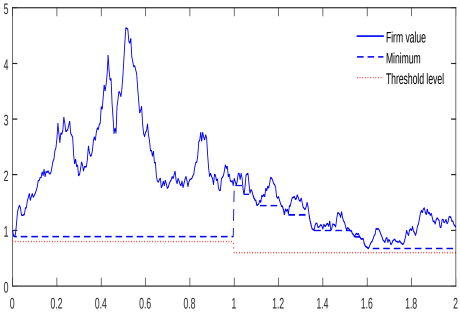
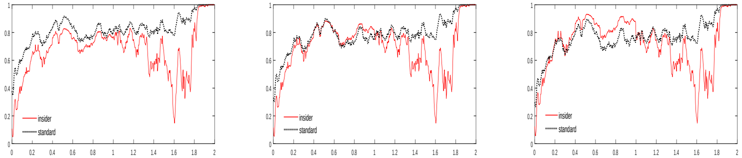


Figure: Trajectory of the firm

Figure: Survival Probability $t \to P(\tau > T|\mathcal{G}_t^M)$ and $t \to P(\tau > T|\mathcal{G}_t)$ for $\theta = 0, 0.5$ and 1



Remark:

- Threshold level L_2 close than expected value $\Rightarrow$ no important readjustment.
- Insider's estimation drops when firm values approaches level L_2 and increases close to maturity.
- Comparison between θ similar.

R eferences

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Thanks for your attention !