An enlargement of filtration formula with application to progressive enlargement with multiple random times

Libo Li
University of New South Wales

Enlargement of Filtration and Financial Applications, Part I

A joint work with M. Jeanblanc and S. Song

1. Introduction and Motivation

Motivating Question

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the usual filtered probability space with a filtration $\mathbb{F}$ satisfying the usual conditions.

Well Know Fact

The minimum and maximum of two $\mathbb{F}$-stopping time is once again a $\mathbb{F}$-stopping time.

Question:

Which properties of random times (in view of enlargement of filtration) are preserved under minimum $\wedge$ and maximum $\vee$.

Enlargement of Filtration

Definition

Let $\mathbb{G}$ be any filtration such that $\mathcal{F}_t \subset \mathcal{G}_t$ for every $t \geq 0$. Then we write $\mathbb{F} \subset \mathbb{G}$ and we say that $\mathbb{G}$ is an *enlargement* of $\mathbb{F}$.

Hypothesis (H):

- Any $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale.

Hypothesis (H'):

- Any $\mathbb{F}$-semimartingale is a $\mathbb{G}$-semimartingale.

If the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{G}$, we write

$$\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{G}.$$

Definition (Progressive enlargement)

Given a random time τ, we denote by $\mathbb{F}^\tau$ the smallest enlargement of $\mathbb{F}$ such that τ is a stopping time.

Definition (Initial enlargement)

Given a random time τ, we denote by $\mathbb{F}^{\sigma(\tau)}$ the smallest enlargement of $\mathbb{F}$ such that τ is measurable wrt to the filtration at time zero.

Motivating Question

Suppose that T_1 and T_2 are random times such that

$$\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^{T_1} \quad \text{and} \quad \mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^{T_2}$$

then

- Do we have $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^{T_1 \wedge T_2}$ (or $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^{T_1 \vee T_2}$)?
- How to compute the drift of $\mathbb{F}$-martingales in $\mathbb{F}^{T_1 \wedge T_2}$?

Classification of the existing techniques
Category I

The technique in the current literature, to proof hypothesis (H') and compute of the semimartingale decomposition in the large filtration can essentially be divided into three categories;

1. Measurable

Exploit the relationship between the $\mathbb{F}$-predictable sets and $\mathbb{G}$-predictable sets. This technique is used in:

- the work of Barlow [B] in treating the progressive enlargement with honest times.

Classification of the existing techniques
Category I

The technique in the current literature, to proof hypothesis (H') and compute of the semimartingale decomposition in the large filtration can essentially be divided into three categories;

1. Measurable

Exploit the relationship between the $\mathbb{F}$-predictable sets and $\mathbb{G}$-predictable sets. This technique is used in:

- the work of Barlow [B] in treating the progressive enlargement with honest times.
- The work of Jeulin and Yor [JY] and progressive enlargement up to a random time.

Category II

Distributional

By introducing distributional hypothesis under which the $\mathbb{F}$-conditional expectations of $\mathbb{G}$-adapted process can be computed.

This includes;

- The work of Yor [Y] on discrete enlargement.

Category II

Distributional

By introducing distributional hypothesis under which the $\mathbb{F}$-conditional expectations of $\mathbb{G}$-adapted process can be computed.

This includes;

- The work of Yor [Y] on discrete enlargement.
- The absolute continuity hypothesis of Jacod in [J] and its extensions .

Category II

Distributional

By introducing distributional hypothesis under which the $\mathbb{F}$-conditional expectations of $\mathbb{G}$-adapted process can be computed.

This includes;

- The work of Yor [Y] on discrete enlargement.
- The absolute continuity hypothesis of Jacod in [J] and its extensions .
- The kernel method of Yor [Y2] and the subsequent computations using Malliavin calculus .

Category II

Distributional

By introducing distributional hypothesis under which the $\mathbb{F}$-conditional expectations of $\mathbb{G}$-adapted process can be computed.

This includes;

- The work of Yor [Y] on discrete enlargement.
- The absolute continuity hypothesis of Jacod in [J] and its extensions .
- The kernel method of Yor [Y2] and the subsequent computations using Malliavin calculus .
- Independence or conditional independence i.e. Hypothesis (H).

Category III

Extension/Linkage

By linking the known method of solutions in categories (i) and (ii), and another filtration of interest.

This includes;

- The projection method used in Callegrao et al. [CJZ].

Category III

Extension/Linkage

By linking the known method of solutions in categories (i) and (ii), and another filtration of interest.

This includes;

- The projection method used in Callegrao et al. [CJZ].
- The local solution method of Song [S].

Category III

Extension/Linkage

By linking the known method of solutions in categories (i) and (ii), and another filtration of interest.

This includes;

- The projection method used in Callegrao et al. [CJZ].
- The local solution method of Song [S].
- Kchia and Protter [KP] on progressive enlargement with processes.

Initial and Progressive

In the well studied case of initial and progressive enlargement of $\mathbb{F}$ with a random time τ.

- We have

$$\{\tau > t\} \cap \mathcal{F}_t^\tau = \{\tau > t\} \cap \mathcal{F}_t$$
$$\{\tau \leq t\} \cap \mathcal{F}_t^\tau = \{\tau \leq t\} \cap \mathcal{F}_t^{\sigma(\tau)}.$$

Initial and Progressive

In the well studied case of initial and progressive enlargement of $\mathbb{F}$ with a random time τ.

- We have

$$\{\tau > t\} \cap \mathcal{F}_t^\tau = \{\tau > t\} \cap \mathcal{F}_t$$
$$\{\tau \leq t\} \cap \mathcal{F}_t^\tau = \{\tau \leq t\} \cap \mathcal{F}_t^{\sigma(\tau)}.$$

- The three levels of filtrations

$$\mathbb{F} \subset \mathbb{F}^\tau \subset \mathbb{F}^{\sigma(\tau)}.$$

Initial and Progressive

In the well studied case of initial and progressive enlargement of $\mathbb{F}$ with a random time τ.

- We have

$$\{\tau > t\} \cap \mathcal{F}_t^\tau = \{\tau > t\} \cap \mathcal{F}_t$$
$$\{\tau \leq t\} \cap \mathcal{F}_t^\tau = \{\tau \leq t\} \cap \mathcal{F}_t^{\sigma(\tau)}.$$

- The three levels of filtrations

$$\mathbb{F} \subset \mathbb{F}^\tau \subset \mathbb{F}^{\sigma(\tau)}.$$

- The $\mathbb{F}^\tau$ semimartingale decomposition can be computed by computing 'projections' from $\mathbb{F}^{\sigma(\tau)}$.

Sketch of the solution:

Given two random times T_1 and T_2.

- Dividing the space Ω into $D_1 := \{T_1 < T_2\}$ and $D_2 := \{T_1 \geq T_2\}$

Sketch of the solution:

Given two random times T_1 and T_2.

- Dividing the space Ω into $D_1 := \{T_1 < T_2\}$ and $D_2 := \{T_1 \geq T_2\}$
- We can show

$$\mathcal{F}_t^{T_1 \wedge T_2} \cap \{T_1 < T_2\} = \mathcal{F}_t^{T_1} \cap \{T_1 < T_2\}$$
$$\mathcal{F}_t^{T_1 \wedge T_2} \cap \{T_1 \geq T_2\} = \mathcal{F}_t^{T_2} \cap \{T_1 \geq T_2\}$$

Sketch of the solution:

Given two random times T_1 and T_2.

- Dividing the space Ω into $D_1 := \{T_1 < T_2\}$ and $D_2 := \{T_1 \geq T_2\}$
- We can show

$$\mathcal{F}_t^{T_1 \wedge T_2} \cap \{T_1 < T_2\} = \mathcal{F}_t^{T_1} \cap \{T_1 < T_2\}$$
$$\mathcal{F}_t^{T_1 \wedge T_2} \cap \{T_1 \geq T_2\} = \mathcal{F}_t^{T_2} \cap \{T_1 \geq T_2\}$$

- We construct auxiliary filtration $\widehat{\mathbb{F}}$ using D_i and $\mathbb{F}^{T_i}$ such that

$$\mathbb{F} \subset \mathbb{F}^{T_1 \wedge T_2} \subset \widehat{\mathbb{F}}$$

  and show that $\mathbb{F} \overset{H'}{\hookrightarrow} \widehat{\mathbb{F}}$ (using that $\mathbb{F} \overset{H'}{\hookrightarrow} \widehat{\mathbb{F}}^{T_i}$ for all $i = 1, \dots, k$)

Sketch of the solution:

Given two random times T_1 and T_2.

- Dividing the space Ω into $D_1 := \{T_1 < T_2\}$ and $D_2 := \{T_1 \geq T_2\}$
- We can show

$$\mathcal{F}_t^{T_1 \wedge T_2} \cap \{T_1 < T_2\} = \mathcal{F}_t^{T_1} \cap \{T_1 < T_2\}$$
$$\mathcal{F}_t^{T_1 \wedge T_2} \cap \{T_1 \geq T_2\} = \mathcal{F}_t^{T_2} \cap \{T_1 \geq T_2\}$$

- We construct auxiliary filtration $\widehat{\mathbb{F}}$ using D_i and $\mathbb{F}^{T_i}$ such that

$$\mathbb{F} \subset \mathbb{F}^{T_1 \wedge T_2} \subset \widehat{\mathbb{F}}$$

  and show that $\mathbb{F} \overset{H'}{\hookrightarrow} \widehat{\mathbb{F}}$ (using that $\mathbb{F} \overset{H'}{\hookrightarrow} \widehat{\mathbb{F}}^{T_i}$ for all $i = 1, \dots, k$)
- This shows that $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^{T_1 \wedge T_2}$.

Sketch of the solution:

Given two random times T_1 and T_2.

- Dividing the space Ω into $D_1 := \{T_1 < T_2\}$ and $D_2 := \{T_1 \geq T_2\}$
- We can show

$$\mathcal{F}_t^{T_1 \wedge T_2} \cap \{T_1 < T_2\} = \mathcal{F}_t^{T_1} \cap \{T_1 < T_2\}$$
$$\mathcal{F}_t^{T_1 \wedge T_2} \cap \{T_1 \geq T_2\} = \mathcal{F}_t^{T_2} \cap \{T_1 \geq T_2\}$$

- We construct auxiliary filtration $\widehat{\mathbb{F}}$ using D_i and $\mathbb{F}^{T_i}$ such that

$$\mathbb{F} \subset \mathbb{F}^{T_1 \wedge T_2} \subset \widehat{\mathbb{F}}$$

  and show that $\mathbb{F} \overset{H'}{\hookrightarrow} \widehat{\mathbb{F}}$ (using that $\mathbb{F} \overset{H'}{\hookrightarrow} \widehat{\mathbb{F}}^{T_i}$ for all $i = 1, \ldots, k$)
- This shows that $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^{T_1 \wedge T_2}$.
- The decomposition of $\mathbb{F}$-martingales in $\mathbb{F}^{T_1 \wedge T_2}$ can be computed using the projections from $\widehat{\mathbb{F}}$.

General Framework and Results

The Framework

The filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is an enlargement of $\mathbb{F}$ satisfying;

Assumption

The filtration $\mathbb{G}$ is such that there exists an $\mathcal{F}$-measurable partition of Ω given by $\{D_1, \ldots, D_k\}$ and a family of right continuous filtration $\{\mathbb{F}^1, \ldots, \mathbb{F}^k\}$ where for every $i = 1, \ldots, k$

(i) $\mathbb{F} \subset \mathbb{F}^i$ and $\mathcal{F}^i_\infty \subset \mathcal{F}$,
(ii) $\forall t \geq 0$, $\mathcal{G}_t \cap D_i = \mathcal{F}^i_t \cap D_i$
(iii) $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^i$ and the $\mathbb{F}^i$-decomposition of a $\mathbb{F}$-martingale M is known.

Example

- $\mathbb{G} = \mathbb{F}^{T_1 \wedge T_2}$
- $D_1 = \{T_1 < T_2\}$ and $D_2 = \{T_2 \leq T_1\}$
- $\mathbb{F}^1 = \mathbb{F}^{T_1}$ and $\mathbb{F}^2 = \mathbb{F}^{T_2}$

The hypothesis (H')

To show that hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{G}$. We first construct an auxiliary filtration $\widehat{\mathbb{F}}$ such that $\mathbb{G} \subset \widehat{\mathbb{F}}$.

Definition (The direct sum filtration)

For every $t \geq 0$, we define the following family of sets

$$\widehat{\mathcal{F}}_t := \{A \in \mathcal{F} \,|\, \forall i,\ \exists A_t^i \in \mathcal{F}_t^i \text{ such that } A \cap D_i = A_t^i \cap D_i\},$$

Lemma

The family of σ-algebras $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{t \geq 0}$ is a right continuous filtration such that $\mathbb{F} \subset \mathbb{G} \subset \widehat{\mathbb{F}}$ holds.

Methodology

- The space is decomposed in to $\Omega = D_1 \cup \cdots \cup D_k$.

Methodology

- The space is decomposed in to $\Omega = D_1 \cup \cdots \cup D_k$.
- Construct $\widehat{\mathbb{F}}$ so that $\mathbb{F} \subset \mathbb{G} \subset \widehat{\mathbb{F}}$ and on each set D_i,

$$\widehat{\mathcal{F}}_t \cap D_i = \mathcal{G}_t \cap D_i = \mathcal{F}^i_t \cap D_i \quad \forall t \geq 0 \tag{1}$$

Methodology

- The space is decomposed in to $\Omega = D_1 \cup \cdots \cup D_k$.
- Construct $\widehat{\mathbb{F}}$ so that $\mathbb{F} \subset \mathbb{G} \subset \widehat{\mathbb{F}}$ and on each set D_i,

$$\widehat{\mathcal{F}}_t \cap D_i = \mathcal{G}_t \cap D_i = \mathcal{F}^i_t \cap D_i \quad \forall t \geq 0 \tag{1}$$

- Using $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^i$ for $i = 1, \ldots, k$ to show that $\mathbb{F} \overset{H'}{\hookrightarrow} \widehat{\mathbb{F}}$. (Bichteler-Dellacherie-Mokobodzki Theorem)

Methodology

- The space is decomposed in to $\Omega = D_1 \cup \cdots \cup D_k$.
- Construct $\widehat{\mathbb{F}}$ so that $\mathbb{F} \subset \mathbb{G} \subset \widehat{\mathbb{F}}$ and on each set D_i,

$$\widehat{\mathcal{F}}_t \cap D_i = \mathcal{G}_t \cap D_i = \mathcal{F}^i_t \cap D_i \quad \forall t \geq 0 \tag{1}$$

- Using $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^i$ for $i = 1, \ldots, k$ to show that $\mathbb{F} \overset{H'}{\hookrightarrow} \widehat{\mathbb{F}}$. (Bichteler-Dellacherie-Mokobodzki Theorem)
- By Stricker's theorem $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{G}$.

Methodology

- The space is decomposed in to $\Omega = D_1 \cup \cdots \cup D_k$.
- Construct $\widehat{\mathbb{F}}$ so that $\mathbb{F} \subset \mathbb{G} \subset \widehat{\mathbb{F}}$ and on each set D_i,

$$\widehat{\mathcal{F}}_t \cap D_i = \mathcal{G}_t \cap D_i = \mathcal{F}^i_t \cap D_i \quad \forall t \geq 0 \tag{1}$$

- Using $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^i$ for $i = 1, \ldots, k$ to show that $\mathbb{F} \overset{H'}{\hookrightarrow} \widehat{\mathbb{F}}$. (Bichteler-Dellacherie-Mokobodzki Theorem)
- By Stricker's theorem $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{G}$.
- Compute the $\widehat{\mathbb{F}}$-semimartingale decomposition of $\mathbb{F}$-martingales.

Methodology

- The space is decomposed in to $\Omega = D_1 \cup \cdots \cup D_k$.
- Construct $\widehat{\mathbb{F}}$ so that $\mathbb{F} \subset \mathbb{G} \subset \widehat{\mathbb{F}}$ and on each set D_i,

$$\widehat{\mathcal{F}}_t \cap D_i = \mathcal{G}_t \cap D_i = \mathcal{F}_t^i \cap D_i \quad \forall t \geq 0 \tag{1}$$

- Using $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^i$ for $i = 1, \ldots, k$ to show that $\mathbb{F} \overset{H'}{\hookrightarrow} \widehat{\mathbb{F}}$. (Bichteler-Dellacherie-Mokobodzki Theorem)
- By Stricker's theorem $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{G}$.
- Compute the $\widehat{\mathbb{F}}$-semimartingale decomposition of $\mathbb{F}$-martingales.
- Compute the $\mathbb{G}$-semimartingale decomposition from the $\widehat{\mathbb{F}}$-semimartingale decomposition.

$\widehat{\mathbb{F}}$-semimartingale Decomposition

Theorem

For every $i = 1, \ldots, k$, suppose that M^i are $\mathbb{F}^i$-local martingales and $N^i_t := \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{D_i} \,\middle|\, \mathcal{F}^i_t\right)$ then

$$\widehat{M}^i = \left(M^i - \frac{\mathbb{1}_{D_i}}{N^i_-} \bullet \langle N^i, M^i \rangle^i \right) \mathbb{1}_{D_i}$$

is an $\widehat{\mathbb{F}}$-local martingale. If for every $i = 1, \ldots, k$, the $\mathbb{F}^i$-semimartingale decomposition of a $\mathbb{F}$-martingale M is given by $M = M^i + K^i$, then

$$M = \widehat{M} + \sum_{i=1}^{k} \mathbb{1}_{D_i} \left(K^i \mathbb{1}_{D_i} + \frac{\mathbb{1}_{D_i}}{N^i_-} \bullet \langle N^i, M \rangle^i \right)$$

where $\widehat{M}$ is an $\widehat{\mathbb{F}}$-local martingale.

Summary

Remark

If we take $\mathbb{F}^i = \mathbb{F}$, we recover the results of Yor [Y] on discrete enlargement.

Remark

Theoretically one can always calculate the $\mathbb{G}$-semimartingale decomposition of $\mathbb{F}$-martingales by computing the "$\mathbb{G}$-dual predictable projections" from its $\widehat{\mathbb{F}}$ semimartingale decomposition.

Problem:

The dual predictable projection can rarely be computed explicitly.

$\mathbb{G}$-Semimartingale Decomposition

We have $\mathbb{F} \subset \mathbb{G} \subset \widehat{\mathbb{F}}$ and the $\widehat{\mathbb{F}}$-semimartingale decomposition of a $\mathbb{F}$-martingale M is known.

Aim:
Compute the $\mathbb{G}$-semimartingale decomposition of $\mathbb{F}$-martingale a M.

Method:
Compute the $\mathbb{G}$-optional projection of the $\widehat{\mathbb{F}}$-semimartingale decomposition of M.

Technical Difficulties

Given $\mathbb{G} \subset \widehat{\mathbb{F}}$:

- Shown by Föllmer and Protter [FP] that the optional $\mathbb{G}$-optional projection of a $\widehat{\mathbb{F}}$ local martingale is in general not a $\mathbb{G}$ local martingale.

Technical Difficulties

Given $\mathbb{G} \subset \widehat{\mathbb{F}}$:

- Shown by Föllmer and Protter [FP] that the optional $\mathbb{G}$-optional projection of a $\widehat{\mathbb{F}}$ local martingale is in general not a $\mathbb{G}$ local martingale.
- A $\widehat{\mathbb{F}}$-localizing sequence of stopping times is not necessary a $\mathbb{G}$-localizing sequence.

Technical Difficulties

Given $\mathbb{G} \subset \widehat{\mathbb{F}}$:

- Shown by Föllmer and Protter [FP] that the optional $\mathbb{G}$-optional projection of a $\widehat{\mathbb{F}}$ local martingale is in general not a $\mathbb{G}$ local martingale.
- A $\widehat{\mathbb{F}}$-localizing sequence of stopping times is not necessary a $\mathbb{G}$-localizing sequence.
- For ease of presentation, we do not distinguish between local martingales and martingales.

Technical Results

Proposition

Let $D \in \mathcal{F}$ be a set such that $\mathcal{G}_t \cap D = \widehat{\mathcal{F}}_t \cap D$ for all $t \geq 0$, then for any $\widehat{\mathbb{F}}$-predictable process of finite variation $\widehat{V}$, there exists map ψ such that $\psi(\widehat{V})$ is a $\mathbb{G}$-predictable process of finite variation and $\widehat{V}\mathbb{1}_D = \psi(\widehat{V})\mathbb{1}_D$.

Remark

- *One show that every positive $\widehat{\mathbb{F}}$-predictable increasing processes $\widehat{V}$ can be map to a $\mathbb{G}$-predictable increasing process $\psi(\widehat{V})$.*
- *In general one set $\psi(\widehat{V})$ to be $\psi(\widehat{V}^+) - \psi(\widehat{V}^-)$.*
- *Problem: The processes $\psi(\widehat{V}^+)$ and $\psi(\widehat{V}^-)$ can both take value infinity, but we assume here that $\psi(\widehat{V}^+) - \psi(\widehat{V}^-)$ always make sense.*

Notation

Let us introduce some notation and apply the previous results.

- For every $i = 1, \dots, k$,

$$N_t^i := \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{D_i} \,\middle|\, \mathcal{F}_t^i\right)$$
$$\widetilde{N}_t^i := \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{D_i} \,\middle|\, \mathcal{G}_t\right)$$
$$\widehat{V}^i := K^i \mathbb{1}_{D_i} + \frac{\mathbb{1}_{D_i}}{N_-^i} \bullet \langle N^i, M \rangle^i$$

Note that :

- $\widehat{V}^i$ is $\widehat{\mathbb{F}}$-adapted
- There exists a map ψ such that $\mathbb{1}_{D_i}\widehat{V}^i = \mathbb{1}_{D_i}\psi(\widehat{V}^i)$.
- $\psi(\widehat{V}^i)$ is $\mathbb{G}$-adapted.

Main Result

Theorem

For any bounded $\mathbb{F}$-martingale M the process

$$M - \sum_{i=1}^{k} \widetilde{N}^i_- \bullet \psi(\widehat{V}^i)$$

is a $\mathbb{G}$-local martingale.

Remark

This formula is interesting as the drift part resembles a weighted average of the processes $\psi(\widehat{V}^i)$ which are constructed from modified drifts of M in each filtration $\mathbb{F}^i$.

Proof.

For every $i = 1, \ldots, k$, the process

$$\begin{aligned}\widehat{M}^i &= M\mathbb{1}_{D_i} - \mathbb{1}_{D_i}\left(K^i\mathbb{1}_{D_i} + \frac{\mathbb{1}_{D_i}}{N^i_-}\bullet\langle N^i, M\rangle^i\right)\\ &= M\mathbb{1}_{D_i} - \mathbb{1}_{D_i}\widehat{V}^i\\ &= M\mathbb{1}_{D_i} - \mathbb{1}_{D_i}\psi(\widehat{V}^i)\end{aligned}$$

is a $\widehat{\mathbb{F}}$-(local) martingale. Taking the $\mathbb{G}$-conditional expectation

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left(\widehat{M}^i_t \,\middle|\, \mathcal{G}_t\right) &= M_t\widetilde{N}^i_t - \widetilde{N}^i_t\psi(\widehat{V}^i)_t\\ &= M_t\widetilde{N}^i_t - \int_{(0,t]}\widetilde{N}^i_{s-}d\psi(\widehat{V}^i)_s\\ &\quad - \int_{(0,t]}\psi(\widehat{V}^i)_{s-}d\widetilde{N}^i_s - \left[\widetilde{N}^i, \psi(\widehat{V}^i)\right]_t\end{aligned}$$

□

Proof.

Using Yœurp's Lemma we conclude that

$$M_t \widetilde{N}^i_t - \int_{(0,t]} \widetilde{N}^i_{s-} d\psi(\widehat{V}^i)_s$$

is a $\mathbb{G}$-martingale. Since $\widetilde{N}^i$ are $\mathbb{G}$-conditional probabilities and $\sum_{i=1}^k \widetilde{N}^i = 1$, this shows

$$M - \sum_{i=1}^{k} \widetilde{N}^i_- \bullet \psi(\widehat{V}^i)$$

is a $\mathbb{G}$-martingale. $\square$

Applications to Multiple Random Times

Application I

Let $\tau_1, \ldots, \tau_n$ be random times and $\tau_{(1)}, \tau_{(2)}, \ldots, \tau_{(n)}$ be their ordered counter part.

Aim: To show that $\mathbb{F} \stackrel{H'}{\hookrightarrow} \mathbb{F}^{\tau_{(j)}}$ for $j = 1, \ldots, n$.

Assumption

- *For every $j = 1, \ldots, n$, the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{F}^{\tau_j}$.*
- *The $\mathbb{F}^{\tau_j}$-semimartingale decomposition of any $\mathbb{F}$-martingale M is known.*

To apply our frame

If we let $D_i = \{\tau_i = \tau_{(j)}\}$ and $\mathbb{F}^i = \mathbb{F}^{\tau_i}$ then the filtration $\mathbb{F}^{\tau_{(j)}}$ satisfies the hypothesis on $\mathbb{G}$ introduced before.

Application I

Example

Given 3 random times τ_1, τ_2 and τ_3, then if the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{F}^{\tau_i}$ for $i = 1, 2, 3$, then the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{F}^{\tau(j)}$ for all j.

In other words,

$$\forall i = 1, 2, 3, \quad \mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^{\tau_i}$$

implies

$$\forall j = 1, 2, 3. \quad \mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^{\tau(j)}$$

Application II

Let $\tau_1, \dots, \tau_n$ be random times and $\{\tau_{(1)}, \tau_{(2)}, \dots, \tau_{(k)}\}$ be the first k smallest random times.

Assumption

For every $j = 1, \dots, n$, the hypothesis (H') is satisfied between $\mathbb{F}$ and the progressive enlargement with any subgroup of k random times.

• Under the above assumptions, the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{F}^{\tau_{(1)}, \cdots, \tau_{(k)}}$.

Explicit Computations

Multiple Random Times

Suppose $\tau_1, \ldots \tau_n$ exhibits a $\mathbb{F}$-conditional density with respect to Lebesgue measure, then the $\mathbb{G}$-drift term

$$\sum_{i=1}^{k} \widetilde{N}^i_- \bullet \psi(\widehat{V}^i)$$

can be computed explicitly.

Conclusion

We have shown that if $\mathbb{G}$ is any enlargement of $\mathbb{F}$ satisfying

Assumption

There exists an $\mathcal{F}$-measurable partition of Ω given by $\{D_1, \ldots, D_k\}$ and a family of right continuous filtration $\{\mathbb{F}^1, \ldots, \mathbb{F}^k\}$ where $\forall i = 1, \ldots, k$;

(i) $\mathbb{F} \subset \mathbb{F}^i$ and $\mathcal{F}^i_\infty \subset \mathcal{F}$,
(ii) $\forall t \geq 0$, $\mathcal{G}_t \cap D_i = \mathcal{F}^i_t \cap D_i$
(iii) $\mathbb{F} \overset{H'}{\hookrightarrow} \mathbb{F}^i$ and the $\mathbb{F}^i$-decomposition of a $\mathbb{F}$-martingale M is known.

Results:

- every $\mathbb{F}$-semimartingale is again a $\mathbb{G}$-semimartingale.
- the $\mathbb{G}$-drift can be interpreted as a *'weighted average'* of the *'modified'* drift of M in $\mathbb{F}^i$.

References I

[B] Barlow, M.:
Study of a filtration expanded to include an honest time.
Zeitschrift für Wahrscheinlichkeitstheorie and Verwandte Gebiete 44 (1978), 307–323.

[CJZ] Callegaro, G. and Jeanblanc, M. and Zargari, B.:
Carthaginian enlargement of filtrations. ESAIM: Probability and Statistics, 17, (2013), pp 550-566.

[FP] Follmer, H. and Protter, P:
Local Martingales and Filtration Shrinkage
ESAIM: Probability and Statistics, 2011.

[HWY] He, S.W., Wang, J.G. and Yan. J.A.:
Semimartingale Theory and Stochastic Calculus.
Chinese edition, Science Press, Beijing, 1995.

References II

[J] Jeulin, T.:
Semi-martingales et grossissement d'une filtration.
Lecture Notes in Mathematics 833. Springer, Berlin Heidelberg New York, 1980.

[JY] Jeulin, T. and Yor, M.:
Grossissement d'une filtration et semi-martingales: formules explicites.
In: *Séminaire de Probabilités XII, Lecture Notes in Mathematics 649*. Springer, Berlin, 1978, pp. 78–97.

[KP] Kchia, Y. and Protter, P.:
On progressive filtration expansion via a process, with applications to insider trading.
International Journal of Theoretical and Applied Finance. 18, 4 (2015), 1550027

[S] Song S.:
Grossissement d'une filtration et problèmes connexes
Thesis Université Paris IV (1987)

References III

[Y] Yor, M.:
Entropie d'une partition, et grossissement initial d'une filtration
Grossissement de filtrations: exemples et applications. Lecture Notes 1118. T. Jeulin and M. Yor, eds. Springer-Verlag, Berlin Heidelberg New York, 1980, 45–58.

[Y2] Yor, M.:
Grossissement de filtrations et absolue continuité de noyaux.
Grossissement de filtrations: exemples et applications. Lecture Notes 1118. Th. Jeulin and M. Yor, eds. Springer-Verlag, Berlin Heidelberg New York, 1980, pp. 6–14.

Thank you all for listening!