Path transformations for local times of one-dimensional diffusions

Umut Çetin

London School of Economics

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A little background

- X is a strong Markov process taking values in (l, r), where $-\infty \leq l < r < \infty$.
- X cannot be killed inside (l, r) and is continuous on $[0, \zeta[$, where
$$\zeta := \{t \geq 0 : X_t = l \text{ or } r\}$$
is its *lifetime*.
- Define
$$T_y := \inf\{t > 0 : X_t = y\}$$
for $y \in [l, r]$. We assume that X is regular, i.e.
$$P^x(T_y < \infty) > 0$$
for all $x, y \in (l, r)$, where P^x corresponds to the law of X with $X_0 = x$.

Scale function and the speed measure

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- The scale function is continuous and strictly increasing and satisfies for $l < x < y < z < r$
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$$P^y(T_x < T_z) = \frac{s(z) - s(y)}{s(z) - s(x)}.$$

  Consequently its left derivative, s', exists.
- The behaviour of X near the endpoints l and r are to a large extent determined by s and m.

Classification of boundaries

Regular It is possible to start the diffusion from a regular boundary and it can be reached in finite time.

Exit It is not possible to start the diffusion from an exit boundary but it can be reached in finite time.

Entrance A diffusion can be started at an entrance boundary but it is inaccessible from the interior of the state space.

Natural This is an inaccessible boundary like an entrance one. The difference is that one cannot start from a natural boundary.

There exist conditions on s and m that determine whether an endpoint is regular, exit, entrance, or natural. A regular boundary can be *absorbing* or *reflecting*, depending on extra assumptions that cannot be determined by s and m alone.

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- *Reflected Brownian motion on* $(0, \infty)$: 0 is regular and ∞ is natural.
- 3-*dimensional Bessel process on* $(0, \infty)$: 0 is entrance and ∞ is natural.

Further assumptions

- X is transient so that at least one of $s(l)$ and $s(r)$ is finite. Without loss of generality we will assume that $s(l) = 0$ and $s(r) = 1$ when finite.

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- X is transient so that at least one of $s(l)$ and $s(r)$ is finite. Without loss of generality we will assume that $s(l) = 0$ and $s(r) = 1$ when finite.
- The Engelbert-Schmidt conditions hold. That is, for any $x \in (l, r)$

$$\sigma(x) > 0 \text{ and } \exists \varepsilon > 0 \text{ s.t. } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |b(y)|}{\sigma^2(y)} dy < \infty. \qquad (1)$$

  Thus, for any $x \in (l, r)$, P^x is the law of the unique weak solution to

$$X_t = x + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds.$$

- The above implies we may choose

$$s(x) = \int_C^x \exp\left(-2 \int_c^z \frac{b(u)}{\sigma^2(u)} du\right) dz \quad \text{and} \quad m(dx) = \frac{2}{s'(x)\sigma^2(x)} dx, \qquad (2)$$

  for some $(c, C) \in (l, r)^2$.

Further consequences of transience

- There exists a finite *symmetric* potential density, u, with respect to m. That is, for any test function, f, vanishing at accessible boundaries

$$Uf(x) := \int_0^\infty E^x[f(X_t)]dt = \int_l^r f(y)u(x,y)m(dy).$$

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$$P^y(L^y_\infty > t) = \exp\left(-\frac{s'(y)t}{2u(y,y)}\right). \tag{3}$$
Thus, the strong Markov property implies
$$P^x(L^y_\infty \in E|\mathcal{F}_t) = \mathbf{1}_{[L^y_t \in E]}(1-\psi(X_t,y)) + \frac{s'(y)\psi(X_t,y)}{2u(y,y)}\int_E \mathbf{1}_{[a>L^y_t]}\exp\left(-\frac{s'(y)(a-L^y_t)}{2u(y,y)}\right)da,$$
where
$$\psi(x,y) := P^x(T_y < \infty).$$

Limiting values of X

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- X_∞ exists and equals l or r depending on the values of $s(l)$ and $s(r)$.
- When only one of $s(l)$ and $s(r)$ is finite, X_∞ equals either l or r, in which case X_∞ and L^y_∞ are trivially independent no matter where the diffusion has started. If both $s(l)$ and $s(r)$ are finite, the situation is more delicate. The following result that illustrates this must be well-known.

Proposition 1

Suppose that X is a regular transient diffusion on (l, r) with $s(l) = 0 = 1 - s(r)$. Then, X_∞ and L^y_∞ are independent under P^x if and only if $x = y$.

What are we after?

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- We are interested in finding the SDE representation for the dynamics of X when conditioned on the set $[L^y = a]$ for a given $a \geq 0$.
- The classical recipe consists of 1) finding an appropriate *excessive* function, h, 2) defining the transition probabilities of the conditioned process via h, and 3) constructing on the canonical space a Markov process, X^h, with these new transition probabilities using standard techniques. This procedure is called an h-transform and its origins go back to Doob and his study of boundary limits of Brownian motion.

Some sample h-transforms

- Suppose that $1 = s(r) = 1 - s(l)$ and let $h(x) := s(x)$. h is harmonic and it can be used to condition X to converge to r with probability 1. If $P^{h,x}$ denote the law of the conditioned process, the following absolute continuity relationship holds:

$$\frac{dP^{h,x}}{dP^x} = \frac{h(X_\infty)}{h(x)} = \frac{\mathbf{1}_{[X_\infty = r]}}{P^x(X_\infty = r)}.$$

- Fix $y \in (l, r)$ and consider $h(x) := u(x, y)$. Then, h is excessive ($\sim h(X)$ is a supermartingale) and is *minimal* with a *pole* at y. This h-transform conditions that X converges to y and is *killed* at its last hitting time of y.

A different approach

- If we would like to apply an h-transform to achieve our conditioning, we need to find a minimal excessive function of the pair (X, L^y) with a suitable pole so that the local time of the X^h equals $a \geq 0$ at its lifetime. The problem with this approach is that it requires the knowledge of the potential density of the Markov pair (X, L^y), which is in general not easily obtained or characterised. Moreover, as in every h-transform, it requires a killing procedure.
- We shall follow a different approach and construct the conditioned process as a weak solution to a stochastic differential equation (SDE) with a suitably chosen drift.

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- As any conditioning with $a > 0$ will make sure that X first hits y, we will assume $X_0 = y$ to ease the exposition.
- The construction of the conditioned process should be achieved in two steps: 1) make sure that X keeps hitting y before L^y reaches a and 2) as soon as L^y becomes a never let X hit y again.
- In order to achieve the first step we need to change the behaviour of X in such a way that the process is *recurrent* since, otherwise, there will be a positive probably that it will drift towards one of its endpoints before L^y becomes a.To this end we will introduce the concept of a *recurrent transformation* and consider a particular example which allows us to complete the first step of our conditioning.

The second step

- The next step is to prevent X from hitting y after $\tau^y_{a-} := \inf\{t \geq 0 : L^y_t \geq a\}$. Since $X_{\tau^y_{a-}} = y$, on $[\tau^y_{a-} < \infty]$, this means that we need to keep X above or below y after τ^y_{a-}.

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- Recall that we are not merely interested in creating a process with the property that $L^y_\infty = a$, but a conditioned version of X whose law coincides with the regular conditional probability $P^y(\cdot | L^y_\infty = a)$.

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- Recall that we are not merely interested in creating a process with the property that $L^y_\infty = a$, but a conditioned version of X whose law coincides with the regular conditional probability $P^y(\cdot | L^y_\infty = a)$.
- This necessitates, in particular, that the conditioned process should also have the same set of possible limiting values for its limiting value. If $s(l) = 0$ and $s(r) = \infty$ (resp. $s(l) = -\infty$ and $s(r) = 1$), our task is relatively simple: keep X below (resp. above) y at all times after τ^y_{a-}.

- On the other hand, if $s(l) = 0 = 1 - s(r)$, the original process could drift towards r as well as l. As we are only conditioning on L^y_∞ and not on X_∞, we will have to appropriately randomise the coefficients of the SDE for the bridge process to allow our solution have l and r as possible limit points.
- We shall see that the drift term that will correspond to the second step of the conditioning will be of *Bessel-type* and provide a connection with the excursions of X.

Step 1 and recurrent transformations

Definition 1

Let $h : (l, r) \mapsto (0, \infty)$ be an absolutely continuous function such that the limits $h(l+) := \lim_{x\to l} h(x)$ and $h(r-) := \lim_{x\to r} h(x)$ exists. Then, (h, M) is said to be a recurrent transform (of X) if the following are satisfied:

1. M is an adapted process of finite variation.
2. $h(X)M$ is a nonnegative local martingale.
3. There exists a unique weak solution to
$$X_t = y + \int_0^t \sigma(X_s)dB_s + \int_0^t \left\{ b(X_s) + \sigma^2(X_s)\frac{h'(X_s)}{h(X_s)} \right\} ds. \quad (4)$$
4. r_s is finite for all $x \in (l, r)$ with $-r_s(l+) = r_s(r-) = \infty$, where
$$r_s(x) := \int_c^x \frac{s'(y)}{h^2(y)} dy, \qquad x \in (l, r) \quad (5)$$

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- First note that $dh'(x) = h''(x)dx + n(dx)$, where n is a locally finite signed measure on (l, r) that is singular with respect to the Lebesgue measure.
- Therefore, the integral

$$\mathbf{1}_{[t<\zeta]}\left(\int_0^t |Ah(X_s)|\, ds + \int_l^r \frac{L_t^x}{2}|n(dx)|\right) < \infty, \ P^y\text{-a.s.},$$

for every $y \in (l, r)$, where $Ah = \frac{\sigma^2(x)}{2}h''(x) + b(x)h'(x)$ with an abuse of notation.

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for every $y \in (l, r)$, where $Ah = \frac{\sigma^2(x)}{2}h''(x) + b(x)h'(x)$ with an abuse of notation.
- Let us also define on $[t < \zeta]$

$$\begin{aligned} M_t &:= \exp\left(-\int_0^t \frac{Ah(X_s)}{h(X_s)}ds - \int_0^t \frac{1}{h(X_s)}d\Lambda_s(h)\right) \text{ and} \\ \Lambda_t(h) &:= \int_l^r \frac{L_t^x}{2}n(dx). \end{aligned}$$

Theorem 2

Let h and M be as above. Then, the following statements are valid.

1. *(h, M) is a recurrent transform.*
2. *Let $R^{h,y}$ be the law of the solution of (4) and $F \in \mathcal{F}_T$ for some $(\mathcal{F}_t)$-stopping time, T, such that $h(X^T)M^T$ is a uniformly integrable P^y-martingale. Then,*

$$R^{h,y}(F) = \frac{1}{h(y)} E^y \left[\mathbf{1}_F h(X_T) M_T \right]. \tag{6}$$

An example

Suppose $\delta > 2$ and consider a δ-dimensional Bessel process on $(0, \infty)$, i.e. a one-dimensional diffusion with the dynamics

$$dX_t = 2\sqrt{X_t}dB_t + \delta dt.$$

Then (h, M) is a recurrent transform if $h(x) := x^{\frac{2-\delta}{4}}$ and

$$M_t := \exp\left(\frac{(\delta-2)^2}{8}\int_0^t \frac{1}{X_s}ds\right), \qquad t \geq 0.$$

The transformation yields the following SDE for the resulting process

$$dX_t = 2\sqrt{X_t}dB_t + 2dt,$$

which is the SDE for a 2-dimensional squared Bessel process. Recall that 0 is polar for a 2-dimensional squared Bessel process.

Proposition 2

(h, M) is a recurrent transform if

$$h(x) := u(x, y), \; x \in (l, r), \; \textit{and } M_t = \exp\left(\frac{s'(y)L_t^y}{2u(y, y)}\right).$$

Consequently, there exists a unique weak solution to

$$X_t = y + \int_0^t \sigma(X_s)dB_s + \int_0^t \left\{b(X_s) + \sigma^2(X_s)\frac{u_x(X_s, y)}{u(X_s, y)}\right\} ds, \quad (7)$$

where u_x denotes the first partial left derivative of $u(x, y)$ with respect to x. Moreover, if $R^{h,y}$ denotes the law of the solution, then, for all $a > 0$, we have $R^{h,y}(L_\infty^y \geq a) = R^{h,y}(\tau_{a-}^y < \infty) = 1$ and

$$\frac{dR^{h,y}}{dP^y}\Big|_{\mathcal{F}_{\tau_{a-}^y}} = \exp\left(\frac{as'(y)}{2u(y, y)}\right)\mathbf{1}_{[\tau_{a-}^y < \zeta]}. \quad (8)$$

Bessel-type motions and Step 2

- Proposition 2 tells us what to do in our first step: We run the (h, M)-recurrent transformation given in the proposition until τ^{y}_{a-}, which is finite with probability 1.

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- In the second step we will keep X away from y after τ^y_{a-} using an h-transform.
- Such an h-transform makes the point-y an *entrance boundary* for the transformed diffusion.
- This is a problem if one wants to use the Engelbert-Schmidt theory for the existence and uniqueness of the solutions for the associated SDE starting at the entrance boundary.
- Indeed, Engelbert and Schmidt constructs the weak solution by applying a change of time and scale to a standard Brownian motion.
- As the scale function is infinite at the entrance boundary, this no longer works.

An extension of Engelbert-Schmidt theory

- To check the aforementioned impossibility try constructing the 3-dimensional Bessel process starting from 0 by applying a change of time and scale to a Brownian motion.
- Note that such a construction were possible if the Bessel process had started at $x > 0$.

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- Note that such a construction were possible if the Bessel process had started at $x > 0$.
- We nevertheless can extend the theory to entrance boundaries in the following theorem by applying a change of time and scale to a 3-dimensional Bessel process.

Theorem 3

Suppose that X is a regular transient diffusion on (l, r) such that its scale function and speed measure are defined by (2), where $\sigma : (l, r) \mapsto \mathbb{R}$ and $b : (l, r) \mapsto \mathbb{R}$ are measurable functions satisfiying (1). Assume further that X has an entrance boundary. Then there exists a unique weak solution to

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds, \qquad t < \zeta, \tag{9}$$

where $\zeta = \inf\{t \geq 0 : X_{t-} \in \{l, r\}\}$ and x is the entrance boundary.

N.B. The fact that X is transient implies there exists at most one entrance boundary.

Candidate drifts for Step 2

Suppose $s(l) = 0$. There exists a regular diffusion on (l, y) with the scale function s_l and the speed measure, m_l, defined by

$$s_l(x) := \frac{1}{s(y) - s(x)}, \qquad m_l(dx) := (s(y) - s(x))^2 m(dx).$$

y is an entrance boundary for this diffusion, which is also the unique weak solution to

$$R_t = x + \int_0^t \sigma(R_s) dB_s + \int_0^t \left\{ b(R_s) - \frac{s'(R_s)\sigma^2(R_s)}{s(y) - s(R_s)} \right\} ds, \qquad t < \zeta, \tag{10}$$

where $x \in (l, y]$ and $\zeta = \inf\{t \geq 0 : R_{t-} = l\}$. Moreover, $\lim_{t\to\infty} R_t = l$, $Q^{x,0}$-a.s., where $Q^{x,0}$ is the law of the weak solution to the SDE above.

Suppose $s(r) = 1$. There exists a regular diffusion on (y, r) with the scale function s_r and the speed measure, m_r, defined by

$$s_r(x) := \frac{1}{s(y) - s(x)}, \qquad m_r(dx) := (s(y) - s(x))^2 m(dx).$$

y is an entrance boundary for this diffusion, which is also the unique weak solution to

$$R_t = x + \int_0^t \sigma(R_s) dB_s + \int_0^t \left\{ b(R_s) + \frac{s'(R_s)\sigma^2(R_s)}{s(R_s) - s(y)} \right\} ds, \qquad t < \zeta, \tag{11}$$

where $x \in [y, r)$ and $\zeta = \inf\{t \geq 0 : R_{t-} = r\}$. Moreover, $\lim_{t\to\infty} R_t = r$, $Q^{x,1}$-a.s., where $Q^{x,1}$ is the law of the weak solution to the SDE above.

Proposition 3

Denote by X^0 (resp. X^1) the killed diffusion process on (l, y) (resp. (y, r)) with the scale function s and the speed measure m. Suppose $s(l) = 0$ (resp. $(s(r) = 1)$. Then, for any bounded and measurable f and $x \neq y$

$$Q_t^0 f(x) = \frac{P_t^0 f(s(y) - s)(x)}{s(y) - s(x)} \left(\textit{resp. } Q_t^1 f(x) = \frac{P_t^1 f(s - s(y))(x)}{s(x) - s(y)} \right),$$

where $(Q_t^0)_{t \geq 0}$ (resp. $(Q_t^1)_{t \geq 0}$) is the semigroup associated to the solutions of (10) (resp. (11)) while $(P_t^0)_{t \geq 0}$ (resp.$(P_t^1)_{t \geq 0}$) is the transition semigroup of X^0 (resp. X^1).

Note that when X is a Brownian motion and $y = 0$, the above defines the transition density of 3-dimensional Bessel process.

Time reversal

Proposition 4

Let R be the solution of (10) (resp. (11)) with $x = y$. Pick a $z \in (l, y)$ (resp. $z \in (y, r)$) and define the last passage time

$$G_z := \sup\{t : R_t = z\}.$$

Next, let Y be the diffusion on (l, y) (resp. (y, r)) obtained by conditioning X^0 (resp. X^1) converge to y with $Y_0 = z$. Then, the processes

$$\{R_{G_z - t}, 0 < t < G_z\} \text{ and } \{Y_t, 0 < t < S_y\}$$

have the same law, where

$$S_y = \inf\{t : Y_t = y\}.$$

In particular, G_z and S_y are finite and have the same distribution.

Main result

Theorem 4

There exists a filtered probability space, $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P^{L,a})$, which contains a Bernoulli random variable, θ, with $\rho(y) := P^y(X_\infty = r) = P^{L,a}(\theta = 1) = 1 - P^{L,a}(\theta = 0)$ and the adapted pair (X, B) such that $(\mathcal{G}_t)_{t\geq 0}$ is right-continuous, B is a standard Brownian motion independent of θ, and X satisfies

$$X_t = y + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \int_0^{t\wedge\tau^y_{a-}} \sigma^2(X_s)\frac{u_x(X_s,y)}{u(X_s,y)}ds$$
$$+ \int_{t\wedge\tau^y_{a-}}^t \sigma^2(X_s)\left\{\theta\mathbf{1}_{[X_s>y]}\frac{s'(X_s)}{s(X_s)-s(y)} - (1-\theta)\mathbf{1}_{[X_s<y]}\frac{s'(X_s)}{s(y)-s(X_s)}\right\}ds,$$

Moreover, weak uniqueness holds for the solutions of the above SDE.

We shall, with a slight abuse of notation, denote the law induced by the above solution on $C(\mathbb{R}_+, I)$ by $P^{L,a}$.

We can also condition L^y_∞ equal a random variable. Indeed, there exists a filtered probability space, $(\tilde{\Omega}, \tilde{\mathcal{G}}, (\tilde{\mathcal{G}}_t)_{t\geq 0}, P^{L,g})$, which contains a Bernoulli random variable, θ, with $\rho(y) = P^{L,g}(\theta = 1) = 1 - P^{L,g}(\theta = 0)$, another $\mathbb{R}_{++}$-valued random variable Γ with distribution g, and the adapted pair (X, B) such that i) $(\tilde{\mathcal{G}}_t)_{t\geq 0}$ is right-continuous; ii) B is a standard Brownian motion; iii) B, θ and Γ are mutually independent; and iv) X solves

$$X_t = y + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \int_0^{t\wedge\tau^y_{\Gamma-}} \sigma^2(X_s)\frac{u_x(X_s,y)}{u(X_s,y)}ds$$
$$+ \int_{t\wedge\tau^y_{\Gamma-}}^t \sigma^2(X_s)\left\{\theta\mathbf{1}_{[X_s>y]}\frac{s'(X_s)}{s(X_s)-s(y)} - (1-\theta)\mathbf{1}_{[X_s<y]}\frac{s'(X_s)}{s(y)-s(X_s)}\right\}ds,$$

A disintegration formula

- Similarly, the uniqueness in law holds for the solutions of the above SDE with properties i)-iv).
- Furthermore, denoting the law of its solutions by $P^{L,g}$, we have the following disintegration formula:

$$P^{L,g} = \int_0^\infty g(da) P^{L,a}. \tag{14}$$

- $P^{L,a}(L^y_\infty = a) = P^{L,g}(L^y_\infty = \Gamma) = 1.$

Corollary 5

Suppose X lives in the filteres probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ and $X_0 = y$. Consider the filtration $(\mathcal{H}_t)_{t\geq 0}$, where $\mathcal{H}_t = \mathcal{G}_t \vee \sigma(L^y_\infty)$. Then,

$$
\begin{aligned}
X_t &= y + \int_0^t \sigma(X_s)d\beta_s + \int_0^t b(X_s)ds + \int_0^{t\wedge\tau^y_{\Gamma-}} \sigma^2(X_s)\frac{u_x(X_s,y)}{u(X_s,y)}ds \\
&\quad + \int_{t\wedge\tau^y_{\Gamma-}}^t \sigma^2(X_s)\frac{s'(X_s)}{s(X_s)-s(y)}ds, \qquad t<\zeta,
\end{aligned} \tag{15}
$$

where $\Gamma := L^y_\infty$ and β is an $(\mathcal{H}_t)_{t\geq 0}$-Brownian motion stopped at ζ.
In particular, X is a weak solution of (13), where $\Gamma = L^y_\infty$, $g(da) = \frac{s'(y)}{2u(y,y)}\exp\left(-\frac{as'(y)}{2u(y,y)}\right)da$, and $\theta = \mathbf{1}_{[R_\infty = r]}$.

Final touch

Corollary 6

Let $P^{L,a}$ be the law on $C(\mathbb{R}_+, I)$ induced by solutions of (12) and F be a test function. Then

$$\begin{aligned} & E^y\left[F(X_s; s \le t) h(L^y_\infty)\right] \\ = \quad & \int_0^\infty E^{L,a}\left[F(X_s; s \le t)\right] h(a) \frac{1}{u(y,y)} \exp\left(-\frac{a}{u(y,y)}\right) da. \end{aligned} \tag{16}$$

That is, $P^{L,a}$ is a regular conditional probability of $\mathcal{B}$ given $L^y_\infty = a$, where $\mathcal{B}$ is the Borel σ-algebra on $C(\mathbb{R}_+, I)$. Consequently, if X is a solution of (13) with $g(da) = \frac{s'(y)}{2u(y,y)} \exp\left(-\frac{as'(y)}{2u(y,y)}\right) da$, then, in its own filtration, it's a regular diffusion on (l, r) with scale function s and speed measure m.

A new path decomposition result a la Williams

Theorem 7

Pick a $y \in (l, r)$ *and on a suitable probability space set up the following four independent elements:*

1. *An exponential random variable,* Γ*, with mean* $\frac{2u(y,y)}{s'(y)}$.
2. *A Bernoulli random variable,* θ*, with* $\mathbb{P}(\theta = 1) = \rho(y)$.
3. *A process* Y*, which is a* $(u(\cdot, y), M)$ *recurrent transform of* X *run upto* $\tau^y_{\Gamma-}$*, where* $M_t = \exp\left(\frac{L_t}{u(y,y)}\right)$.
4. *A pair of Bessel-type motions,* (R^0, R^1) *with laws* $(Q^{y,0}, Q^{y,1})$ *and lifetimes* (ζ^0, ζ^1).

Then, the process defined by

$$\tilde{X}_t := \begin{cases} Y_t, & t \leq \tau^y_{\Gamma-} \\ R^\theta_{t-\tau^y_{\Gamma-}}, & 0 < t - \tau^y_{\Gamma-} \leq \zeta^\theta, \end{cases}$$

has the same law as X.

- Suppose X is a Brownian motion on $(-\infty, b)$ killed at $b > 0$. Taking $y = 0$ the equation (12) reads as

$$X_t = B_t - \int_0^{t \wedge \tau_{a-}^0} \frac{1}{b - X_s} \mathbf{1}_{[X_s > 0]} ds + \int_{t \wedge \tau_{a-}^0}^t \frac{1}{X_s} ds, \qquad t < \zeta, \tag{17}$$

where ζ is the first hitting time of b, which occurs in finite time.

- The first integral represents the recurrent transform, Y, stopped at τ_{a-}^0, where

$$Y_t = B_t - \int_0^t \frac{1}{b - Y_s} \mathbf{1}_{[Y_s > 0]} ds.$$

- If we let $U := b - Y$, then

$$U_t = b + \beta_t + \int_0^t \frac{1}{U_s} \mathbf{1}_{[U_s < b]} ds, \tag{18}$$

where $\beta = -B$.

A path decomposition for the killed Brownian motion

On a suitable probability space set up the following three independent elements:

1. An exponential random variable, Γ, with mean $2b$.
2. A weak solution, U, of (18).
3. A 3-dimensional Bessel process, R, with $R_0 = 0$.

Consider

$$\tilde{X}_t := \begin{cases} b - U_t, & t \leq \tau^b_{\Gamma-} \\ R_{t-\tau^b_{\Gamma-}}, & 0 < t - \tau^y_{\Gamma-} \leq S_b, \end{cases}$$

where $(\tau^b_t)_{t \geq 0}$ is the right-continuous inverse of the local time of U at level b and $S_b := \inf\{t \geq 0 : R_t = b\}$. Then, $\tilde{X}$ has the same law as the Brownian motion starting at 0 and killed at b.

Conclusion

- We have characterised the SDE that corresponds to the conditioning of a transient diffusion on the terminal value of its local time at a fixed level.

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Conclusion

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- The results of Engelbert and Schmidt have been extended to the case of entrance boundaries as initial condition.
- The SDE representations for Bessel-type motions associated to the transient diffusions are given.
- A new path decomposition for transient diffusions is proven.