Local Martingale Defaltors Parametrisation for a Stopped Model with Applications

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Talk is based on joint work with

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The plan of the talk

- **Our Ultimate Financial Goal and Motivation**

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- **Applications to Numéraire Portfolio**

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- $$G_t := \mathbb{P}(\tau > t|\mathcal{F}_t) \quad \text{and} \quad \widetilde{G} := \mathbb{P}(\tau \geq t|\mathcal{F}_t).$$
- $$m := G + D^{o,\mathbb{F}}, \quad R := \inf\{t \geq 0 \mid G_t = 0\}. \tag{2.2}$$

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- Decomposition of $\mathbb{G}$-risk as follows

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- What means this decomposition (mathematically speaking)?
- Decomposition of $\mathbb{G}$-martingales into the sum of **Orthogonal Martingales** associated to different kind of risk (**the first milestone**)!
- How this decomposition will be used? For the "quadratic Hedging" this is enough!

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- This is our second major milestone!!

This decomposition is based on constructing three orthogonal spaces of $\mathbb{G}$-martingales: $\mathcal{M}^{(1)}(\mathbb{G})$, $\mathcal{M}^{(2)}(\mathbb{G})$ and $\mathcal{M}^{(3)}(\mathbb{G})$.

$$\widehat{M} = M^{\tau} - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \cdot [M, m] + I_{]\!]0,\tau]\!]} \cdot \left(\Delta M_{\widetilde{R}} I_{[\![\widetilde{R},+\infty[\![}\right)^{p,\mathbb{F}}, \quad (4.1)$$

$$\mathcal{M}^{(1)}(\mathbb{G}) := \left\{ \widehat{M} \text{ defined in } (4.1) \mid \widehat{M}_{\infty} \in L^1(P),\ M \in \mathcal{M}_{0,loc}(\mathbb{F}) \right\}. \quad (4.2)$$

it goes back to Aksamit/Choulli/Jeanblanc(2015).

Theorem 1:

Consider the following process

$$N^{\mathbb{G}} := D - \left(\widetilde{G}\right)^{-1} I_{[\![0,\tau]\!]} \bullet D^{o,\mathbb{F}}. \tag{4.3}$$

Then, the following assertions hold.

- $N^{\mathbb{G}}$ is a $\mathbb{G}$-martingale with integrable variation.

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Then, the following assertions hold.

- $N^{\mathbb{G}}$ is a $\mathbb{G}$-martingale with integrable variation.
- Let K be an $\mathbb{F}$-optional process, which is Lebesgue-Stieljes integrable with respect to $N^{\mathbb{G}}$. Then, $K \bullet N^{\mathbb{G}} \in \mathcal{A}(\mathbb{G})$ if and only if K belongs to $\mathcal{I}^o(N^{\mathbb{G}}, \mathbb{G})$, where

$$\mathcal{I}^o(N^{\mathbb{G}}, \mathbb{G}) := \left\{K \in \mathcal{O}(\mathbb{F}) \;\Big|\; E\left(|K| G \widetilde{G}^{-1} 1_{\{\widetilde{G}>0\}} \bullet D_\infty\right) < +\infty\right\} \tag{4.4}$$

- The elements of

$$\mathcal{M}^{(2)}(\mathbb{G}) := \left\{ K \centerdot N^{\mathbb{G}} \;\middle|\; K \in \mathcal{I}^{o}(N^{\mathbb{G}}, \mathbb{G}) \right\}, \tag{4.5}$$

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- If all $\mathbb{F}$-martingales are continuous, then $N^{\mathbb{G}} = \overline{N}^{\mathbb{G}}$, as in this case $\widetilde{G} = G_-$ and $D^{o,\mathbb{F}} = D^{p,\mathbb{F}}$ continuous.

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- It is not possible to go further in weakening the space of integrands for $N^{\mathbb{G}}$!!

on $(\Omega \times [0, +\infty), \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+))$, we consider $\mu := P \otimes D$ and on $(\Omega, \mathcal{F})$, we consider the following σ-fields

$$\mathcal{F}_{\tau-} := \sigma\left(X_\tau \;\middle|\; X \text{ is } \mathbb{F}\text{-predictable}\right),$$

$$\mathcal{F}_{\tau} := \sigma\left(X_\tau \;\middle|\; X \text{ is } \mathbb{F}\text{-optional}\right),$$

$$\mathcal{F}_{\tau+} := \sigma\left(X_\tau \;\middle|\; X \text{ is } \mathbb{F}\text{-progressively measurable}\right).$$

For $\mathcal{H} \in \{\mathcal{P}(\mathbb{F}), \mathcal{O}(\mathbb{F}), \mathcal{P}_{rog}(\mathbb{F})\}$, we define

$$L^1(\mathcal{H}, P \otimes D) := \left\{ X\ \mathcal{H}-\text{measurable} \;\middle|\; \begin{array}{l} E(|X_\tau| I_{\{\tau<+\infty\}}) \\ =: E_{P\otimes D}(|X|) < +\infty \end{array} \right\}$$

Theorem 2:

The following assertions hold.

- For any $k \in L^1\left(\mathcal{P}_{prog}(\mathbb{F}), P \otimes D\right)$, there exists a unique $h \in L^1\left(\mathcal{O}(\mathbb{F}), P \otimes D\right)$ satisfying

$$E\left(k_\tau \;\middle|\; \mathcal{F}_\tau\right) = h_\tau \quad P-a.s. \quad \text{on} \quad \{\tau < +\infty\}. \tag{4.7}$$

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- The elements of the set

$$\mathcal{M}^{(3)}(\mathbb{G}) := \left\{ k \centerdot D \;\middle|\; k \in L^1\left(\mathcal{P}_{prog}(\mathbb{F}), P \otimes D\right) \;\&\; E(k_\tau | \mathcal{F}_\tau) = 0 \right\}$$

are $\mathbb{G}$-martingales that are orthogonal to locally bounded elements of both $\mathcal{M}^{(1)}(\mathbb{G})$ and $\mathcal{M}^{(2)}(\mathbb{G})$.

A Jeulin's Martingale Space:

$$\mathcal{M}^{(4)}(\mathbb{G}) := \left\{ k \centerdot D \;\middle|\; \begin{array}{l} k \in L^1\left(\mathcal{P}_{prog}(\mathbb{F}), P \otimes D\right) \\ \text{and } E(k_\tau | \mathcal{F}_{\tau-}) = 0 \;\; P-a.s \end{array} \right\}$$

Then,

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- The elements of this space are NOT orthogonal to those of $\mathcal{M}^{(i)}(\mathbb{G})$, $i = 1, 2$.

Theorem 3:

Consider $h \in L^1(\mathcal{O}(\mathbb{F}), P \otimes D)$. Then, the $\mathbb{G}$-martingale $H_t := \mathbb{E}[h_\tau | \mathcal{G}_t]$ admits the following representation

$$H := {}^{o,\mathbb{G}}(h_\tau) = H_0 + G_-^{-1} I_{[\![0,\tau]\!]} \centerdot \widehat{M^h} - J_- G_-^{-1} I_{[\![0,\tau]\!]} \centerdot \widehat{m} + (h - J) \centerdot N^{\mathbb{G}}.$$

Here M^h and J are given by

$$M^h = {}^{o,\mathbb{F}}\left(\int_0^\infty h_u dD_u^{o,\mathbb{F}}\right) \quad \text{and} \quad J = \left(M^h - h \centerdot D^{o,\mathbb{F}}\right) G^{-1} I_{[\![0,R[\![}. \tag{4.8}$$

Consider $h \in L^1(\mathcal{P}(\mathbb{F}), P \otimes D)$. Then

$$\begin{aligned} H := {}^{o,\mathbb{G}}(h_\tau) = \mathbb{E}[h_\tau | \mathcal{G}_t] \quad &= \quad H_0 + \frac{1}{G_-} I_{[\![0,\tau]\!]} \bullet \widehat{m^h} \\ &+ \quad \frac{h - J_-}{G_-} I_{[\![0,\tau]\!]} \bullet \widehat{m} + (h - J) \bullet N^{\mathbb{G}}. \end{aligned}$$

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Let $h \in L^1(\mathcal{O}(\mathbb{F}), P \otimes D)$, and consider M^h and J, defined in (4.8). Then, the following hold

- If τ is an $\mathbb{F}$-pseudo stopping time, then it holds that

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- In case τ avoids $\mathbb{F}$-stopping times, we get

$$H = H_0 + G_-{}^{-1} I_{[\![0,\tau]\!]} \centerdot \overline{M^h} - J_-(G_-)^{-1} I_{[\![0,\tau]\!]} \centerdot \bar{m} + (h - J_-) \centerdot \overline{N}^{\mathbb{G}}.$$

Here $\overline{N}^{\mathbb{G}}$ is given by (4.6), and for any $\mathbb{F}$-local martingale, M, $\bar{M}$ is defined by

$$\bar{M} := M^\tau - \widetilde{G}^{-1} I_{[\![0,\tau]\!]} \centerdot [M, m]. \qquad (4.10)$$

- If all $\mathbb{F}$-martingales be continuous, then

$$H = H_0 + \frac{I_{[\![0,\tau]\!]}}{G_-} \cdot \overline{M^h} - \frac{J_-}{G_-} I_{[\![0,\tau]\!]} \cdot \overline{m} + (h - J_-) \cdot \overline{N}^{\mathbb{G}},$$

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where $\overline{N}^{\mathbb{G}}$ and $\overline{M}$ are given by (4.6) and (4.10) respectively.

- If G is strictly positive, then

$$H = H_0 + G_-{}^{-1} I_{[\![0,\tau]\!]} \cdot \overline{M^h}^{\mathbb{G}} - J_- G_-{}^{-1} I_{[\![0,\tau]\!]} \cdot \overline{m} + (h - J) \cdot N^{\mathbb{G}},$$

where for any $\mathbb{F}$-local martingale M,

$$\overline{M}^{\mathbb{G}} := M^\tau - \widetilde{G}^{-1} I_{[\![0,\tau]\!]} \cdot [M, m].$$

Comparing with Blanchet/Jeanblanc(2004):

Suppose that $h \in L^1(\widetilde{\Omega}, \mathcal{P}(\mathbb{F}), P \otimes D)$, and either τ avoids $\mathbb{F}$-stopping times or all $\mathbb{F}$-martingales are continuous. Then

$$\begin{aligned} H &= H_0 + {G_-}^{-1} I_{[\![0,\tau]\!]} \centerdot \overline{m^h} + h - J_- {G_-}^{-1} I_{[\![0,\tau]\!]} \centerdot \overline{m} \\ &+ (h - J_-) G_- G^{-1} I_{[\![0,\tau]\!]} \centerdot \overline{N}^{\mathbb{G}}. \end{aligned}$$

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$$m^h = {}^{o,\mathbb{F}}\left(\int_0^\infty h_u dF_u\right) \quad \text{and} \quad J = \left(m^h - h \centerdot F\right)(G^\tau)^{-1}.$$

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- $D^{o,\mathbb{F}} = D^{p,\mathbb{F}}$ implies that $m = 1 - Z$, such that

$$\begin{aligned}\overline{M} &= M^\tau - G^{-1} I_{[\![0,\tau]\!]} \cdot [M, m] = M^\tau + G^{-1} I_{[\![0,\tau]\!]} \cdot [M, Z],\\ \bar{m} &= 1 - Z^\tau - G^{-1} I_{[\![0,\tau]\!]} \cdot [Z, Z] = 1 - \bar{Z}.\end{aligned}$$

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- The previous martingale representation coincides with the one established in Theorem 1 of Blanchet/Jeanblanc(2004). However, in our case, G is allowed to vanish as opposed to the setting considered by Blanchet/Jeanblanc(2004).

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- If τ is an $\mathbb{F}$-stopping time, then

$$H = {}^{o,\mathbb{G}}(h_\tau) = {}^{o,\mathbb{F}}(h_\tau) = {}^{o,\mathbb{F}}\left(\int_0^{\infty} h_u dD_u\right) = m^h.$$

Can we extend it to

$E(h_\tau \mid \mathcal{G}_t)$ **with** $h \in L^1(\widetilde{\Omega}, \mathcal{P}_{prog}(\mathbb{F}), P \otimes D)$**?**

How far can we go in extending this optional representation?

Theorem 4:

If $(\mathcal{M}(\mathbb{G}))^{\tau}$ is the set of $\mathbb{G}$-martingales stopped at τ, then

$$(\mathcal{M}(\mathbb{G}))^{\tau} = \mathcal{M}^{(1)}(\mathbb{G}) \oplus \mathcal{M}^{(2)}(\mathbb{G}) \oplus \mathcal{M}^{(3)}(\mathbb{G}). \tag{4.11}$$

In other words, for any $\mathbb{G}$-martingale, $M^{\mathbb{G}}$, there exist $h \in L^1\left(P \otimes D, \mathcal{O}(\mathbb{F})\right)$ and $k \in L^1\left(P \otimes D, \mathcal{P}_{rog}(\mathbb{F})\right)$, such that $E(k_\tau | \mathcal{F}_\tau) = 0$ on $\{\tau < +\infty\}$, $k_\tau + h_\tau = M^{\mathbb{G}}_\tau$, and

$$\begin{aligned}\left(M^{\mathbb{G}}\right)^{\tau} &= M_0^{\mathbb{G}} + (G_-)^{-1} I_{[\![0,\tau]\!]} \bullet \widehat{M^h} - J_-^h (G_-)^{-1} I_{[\![0,\tau]\!]} \bullet \widehat{m} \\ &+ (h - J^h) \bullet N^{\mathbb{G}} + k \bullet D.\end{aligned}$$

Here M^h and J^h are defined in (4.8) and m is given by (2.2).

Jeulin's Theorem:

Suppose that τ is honest. Then,

$$\mathcal{M}(\mathbb{G}) = \text{generated by } \mathcal{M}^{(1bis)}(\mathbb{G}) \bigcup \mathcal{M}^{(2bis)}(\mathbb{G}) \bigcup \mathcal{M}^{(4)}(\mathbb{G}),$$

where

$$\begin{aligned}
\mathcal{M}^{(1bis)}(\mathbb{G}) : &= \{M^\tau - G_-^{-1} I_{[\![0,\tau]\!]} \centerdot \langle M, m\rangle^{\mathbb{F}} \mid M \in \mathcal{M}^2(\mathbb{F})\} \\
\mathcal{M}^{(2bis)}(\mathbb{G}) : &= \{H \centerdot \overline{N}^{\mathbb{G}} \mid H \text{ is } \mathbb{F}-\text{predictable and ``integrable''}\}
\end{aligned}$$

- **Assumption:** $G > 0$ or equivalently $R = +\infty \ P - a.s.$

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(Aksamit/Choulli/Deng/Jeanblanc(2014))
Let $(\sigma_n^{\mathbb{G}})$ *be a sequence of* $\mathbb{G}$*-stopping times that increases to infinity almost surely. Then, there exists a sequence of* $\mathbb{F}$*-stopping times,* $(\sigma_n^{\mathbb{G}})$*, that increases to infinity almost surely and*

$$\sigma_n^{\mathbb{G}} \wedge \tau = \sigma_n^{\mathbb{F}} \wedge \tau, \qquad \forall\ n \geq 1.$$

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- $L^1_{loc}\left(P \otimes D, \mathcal{P}_{prog}(\mathbb{F})\right) :=$
$\left\{\varphi \in \mathcal{P}_{prog}(\mathbb{F}) \ : \ |\varphi| \centerdot D \in \mathcal{A}^+_{loc}(\mathbb{G})\right\}$

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- $\mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) := \left\{\varphi \in \mathcal{O}(\mathbb{F}) \ : \ |\varphi| \frac{G}{\widetilde{G}} \centerdot D \in \mathcal{A}^+_{loc}(\mathbb{G})\right\}$

- We have

$$(\mathcal{M}_{loc}(\mathbb{G}))^{\tau} = \mathcal{M}^{(1)}_{loc}(\mathbb{G}) \oplus \mathcal{M}^{(2)}_{loc}(\mathbb{G}) \oplus \mathcal{M}^{(3)}_{loc}(\mathbb{G}).$$

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$$(\mathcal{M}_{loc}(\mathbb{G}))^{\tau} = \mathcal{M}^{(1)}_{loc}(\mathbb{G}) \oplus \mathcal{M}^{(2)}_{loc}(\mathbb{G}) \oplus \mathcal{M}^{(3)}_{loc}(\mathbb{G}).$$
- For any $M^{\mathbb{G}} \in \mathcal{M}_{0,loc}(\mathbb{G})$, there exist $M^{\mathbb{F}} \in \mathcal{M}_{0,loc}(\mathbb{F})$, $\varphi^{(o)} \in \mathcal{I}^{o}_{loc}(N^{\mathbb{G}}, \mathbb{G})$ and $\varphi^{(pr)} \in L^1_{loc}\left(P \otimes D, \mathcal{P}_{prog}(\mathbb{F})\right)$ such that
$$M^{\mathbb{G}} = \widehat{M^{\mathbb{F}}} + \varphi^{(o)} \centerdot N^{\mathbb{G}} + \varphi^{pr} \centerdot D.$$

The case where $(S, \mathbb{F}, P)$ is complete

Let $Z = \mathcal{E}(N)$ be a positive $\mathbb{G}$-local martingale. Then, Z is a $\mathbb{G}$-local martingale deflator for S^τ if and only if there exists a pair

$$\left(\varphi^{(o)}, \varphi^{(pr)}\right) \in \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}\left(P \otimes D, \mathcal{P}_{rog}(\mathbb{F})\right)$$

satisfying the following

$$1 + \varphi^{(o)}_\tau + \varphi^{(pr)}_\tau > 0 \quad \text{on } \{\tau < +\infty\} \quad P - a.s$$

$$N = -G_-^{-1} \bullet \widehat{m} + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D.$$

The General Case

Let $Z=\mathcal{E}(N)$ be a positive $\mathbb{G}$-local martingale. Then, Z is a $\mathbb{G}$-local martingale deflator for S^τ if and only if there exists a triplet

$$\left(K^{\mathbb{K}},\varphi^{(o)},\varphi^{(pr)}\right)\in\mathcal{M}_{0,loc}(\mathbb{F})\times\mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G})\times L^1_{loc}\left(P\otimes D,\mathcal{P}_{rog}(\mathbb{F})\right)$$

such that $\mathcal{E}(K^{\mathbb{F}})$ is a positive and orthogonal to S, and

$$1+\Delta K^{\mathbb{K}}>\varphi^{(o)}\Delta D^{o,\mathbb{F}}G_{-}^{-1}$$

$$\varphi^{(pr)}_\tau\widetilde{G}_\tau>-\left[\widetilde{G}_\tau+\Delta N^{\mathbb{F}}_\tau+\varphi^{(o)}_\tau G_\tau\right]\quad\text{on }\{\tau<+\infty\},$$

$$N=\widehat{K^{\mathbb{F}}}-G_{-}^{-1}\centerdot\widehat{m}+\varphi^{(o)}\centerdot N^{\mathbb{G}}+\varphi^{(pr)}\centerdot D.$$

Theorem

Let $Z = \mathcal{E}(N)$ be a positive $\mathbb{G}$-local martingale. Then, Z is a $\mathbb{G}$-local martingale deflator for S^τ if and only if there exists a triplet

$$\left(K^{\mathbb{K}}, \varphi^{(o)}, \varphi^{(pr)}\right) \in \mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}\left(P \otimes D, \mathcal{P}_{rog}(\mathbb{F})\right)$$

such that $\mathcal{E}(K^{\mathbb{F}})$ is a positive and orthogonal to S, and

$$1 + \Delta K^{\mathbb{K}} > \varphi^{(o)} \Delta D^{o,\mathbb{F}} G_-^{-1}$$

$$\varphi^{(pr)}_\tau \widetilde{G}_\tau > -\left[\widetilde{G}_\tau + \Delta N^{\mathbb{F}}_\tau + \varphi^{(o)}_\tau G_\tau\right] \quad \text{on } \{\tau < +\infty\},$$

$$N = \widehat{K^{\mathbb{F}}} - G_-^{-1} \centerdot \widehat{m} + \varphi^{(o)} \centerdot N^{\mathbb{G}} + \varphi^{(pr)} \centerdot D.$$

Theorem

Suppose that S is a continuous $\mathbb{F}$-semimartingale, $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ be a positive $\mathbb{G}$-local martingale. Then, $Z^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale deflator for S^{τ} if and only if there exists a triplet $\left(K^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right) \in \mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}^{o}_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}\left(P \otimes D, \mathcal{P}_{rog}(\mathbb{F})\right)$ satisfying:

(i) $\mathcal{E}(K^{\mathbb{K}})$ is local martingale defaltor for $(S, \mathbb{F}, P)$.

(ii) $\varphi^{(o)} \frac{\Delta D^{o,\mathbb{F}}}{G_{-}} < 1 + \Delta K^{\mathbb{F}}$,

$$\varphi^{(pr)}_{\tau} \widetilde{G}_{\tau} > -\left[\widetilde{G}_{\tau} + \Delta N^{\mathbb{F}}_{\tau} + \varphi^{(o)}_{\tau} G_{\tau}\right] \quad on\ \{\tau < +\infty\},$$

$$K^{\mathbb{G}} = \widehat{K^{\mathbb{F}}} - G_{-}^{-1} \centerdot \widehat{m} + \varphi^{(o)} \centerdot N^{\mathbb{G}} + \varphi^{(pr)} \centerdot D.$$

Theorem

Suppose that S is an $\mathbb{F}$-semimartingale, $G > 0$, and $Z^{\mathbb{G}} := \mathcal{E}\left(K^{\mathbb{G}}\right)$ is a positive $\mathbb{G}$-local martingale. Then, $Z^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale deflator for S^τ if and only if There exists a triplet $\left(K^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)}\right) \mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) \times L^1_{loc}\left(P \otimes D, \mathcal{P}_{rog}(\mathbb{F})\right)$ such that:

(a) $\mathcal{E}\left(K^{\mathbb{F}}\right)$ *is a local martingale deflator for* $(S, \mathbb{F}, P)$.

(b) $\varphi^{(o)} \frac{\Delta D^{o,\mathbb{F}}}{G_-} < 1 + \Delta K^{\mathbb{F}}$,

$$\varphi^{(pr)}_\tau \widetilde{G}_\tau > -\left[\widetilde{G}_\tau + \Delta N^{\mathbb{F}}_\tau + \varphi^{(o)}_\tau G_\tau\right] \quad on\ \{\tau < +\infty\},$$

$$K^{\mathbb{G}} = \widehat{K^{\mathbb{F}}} - G_-^{-1} \centerdot \widehat{m} + \varphi^{(o)} \centerdot N^{\mathbb{G}} + \varphi^{(pr)} \centerdot D,$$

Definition

Let $(X, \mathbb{H}, P)$ be a model.
(i) Then, the numéraire portfolio of this model is the $\mathbb{H}$-predictable process $\widetilde{\theta} \in L(X, \mathbb{H})$ such that $\mathcal{E}(\widetilde{\theta} \bullet X) > 0$ and for any $\theta \in L(X, \mathbb{H})$ such that $\mathcal{E}(\theta \bullet X) \geq 0$, we have $\mathcal{E}(\theta \bullet X)/\mathcal{E}(\widetilde{\theta} \bullet X)$ is a supermartingale.
(ii) the local martingale numéraire portfolio for $(X, \mathbb{H}, P)$, when it exists, is the portfolio $\widetilde{\theta} \in L(X, \mathbb{H})$ such that $\mathcal{E}(\theta \bullet X)/\mathcal{E}(\widetilde{\theta} \bullet X)$ is a local martingale, for any $\theta \in L(X, \mathbb{H})$ such that $\mathcal{E}(\theta \bullet X) \geq 0$.

Theorem

Suppose that S is continuous, and let $\varphi^{\mathbb{F}}$ be its local martingale numéraire. Then, the local martingale nuéraire of $(S^{\tau}, \mathbb{G})$ is given by

$$\varphi^{\mathbb{G}} = \varphi^{\mathbb{F}} - \varphi^{(m)} G_{-}^{-1}, \tag{8.1}$$

$\varphi^{(m)}$ is the integrand in the Galtchouk-Kunita-Watanabe decomposition of m with respect to the martingale part of S, M (i.e. $m = m_0 + \varphi^{(m)} \centerdot M + L^{(m)}$).

Theorem

Suppose that $(S,\mathbb{F},P)$ is a complete market, and let $\varphi^{\mathbb{F}}$ be its local martingale numéraire. Then, the model $(S^\tau,\mathbb{G})$ admits a local martingale nuéraire given by

$$\varphi^{\mathbb{G}} = \varphi^{\mathbb{F}} - \varphi^{(m)} G_-^{-1}. \tag{8.2}$$

Here, $\varphi^{(m)}$ is the integrand in the representation of m with respect to the martingale part of S.

Thank you for your attention