Dynamics of multivariate default system in random environment

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Enlargement of Filtrations and Financial Applications

Introduction

- In the reliability system, we consider a system of multi-components and study the dynamics of failure times based on the history of failure process: Knight (1975) and Arjas & Norros (1985, 1991).
- In the credit risk analysis, we are also interested in failure times – the defaults on the financial market. However the environmental information is important.
- Default modelling is based on the theory the enlargement of filtrations developped by Jacod, Jeulin & Yor...in the 70s-80s.
- Literature on multi-default modelling with two main approaches: bottom-up and top-down models for respectively non-ordered and ordered defaults.

Plan of our work

- Consider a multivariate system in a general setting of enlargement of filtrations in presence of environmental information.
- Use a random variable χ to describe all default risks and to study the dependence between the multi-default system and the environmental information.
- This general setting can be applied flexibly to diverse situations, including bottom-up and top-down models.
- The dependence structure between the default system and the market environment can be described in a dynamic manner and represented by a change of probability.

The multi-default system

Basic setting

- $(\Omega, \mathcal{A}, \mathbb{P})$: probability space for the market
- E: a Polish space
- $\chi : \Omega \to E$ random variable describing all default uncertainties

Very flexible framework

- Permits to consider both bottom-up and top-down models:
  - $E = R_+^n$, $\chi = (\tau_1, \cdots, \tau_n) : \Omega \to \mathbb{R}_+^n$ models the default times of n firms.
  - $E = (\mathbb{R}_+ \times \mathbb{R})^n$, $\chi = (\tau_i, L_i)_{i=1}^n : \Omega \to E$ models the default times with corresponding loss (or gain).
  - $E = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n \,|\, x_1 \le \cdots \le x_n\}$, $\chi = (\sigma_1, \cdots, \sigma_n) : \Omega \to E$ models the successive default times.

The prediction process

- Complete information on χ is not observable.
- $(\mathcal{N}_t)_{t\geq 0}$: filtration of $\mathcal{A}$ representing the information related to defaults observable on the market.

Definition (Norros)

Let η_t be the $\mathcal{N}_t$-conditional law of χ, $t \geq 0$. The measure-valued process $(\eta_t)_{t\geq 0}$ is called the prediction process of χ.

- $(\eta_t)_{t\geq 0}$ is an $(\mathcal{N}_t)_{t\geq 0}$-adapted process valued in the space $\mathcal{P}(E)$ of Borel probability measures on E.
- Existence of a càdlàg version which is unique up to indistinguishability.
- Martingale with respect to the weak topology on $\mathcal{P}(E)$: for any bounded Borel function h on E

$$\left(\int_E h(x)\eta_t(dx),\ t \geq 0 \right) \quad \text{is an } (\mathcal{N}_t)_{t\geq 0}\text{-martingale.}$$

Example: one default

- $E = \mathbb{R}_+$, $\chi = \tau : \Omega \to E$.
- $(\mathcal{N}_t)_{t\geq 0}$: filtration generated by $(\mathbf{1}_{\{\tau\leq t\}} = \mathbf{1}_{[0,t]} \circ \chi)_{t\geq 0}$.
- η: probability law of τ.

Prediction process

$$\eta_t(dx) = \frac{\mathbf{1}_{]t,+\infty[}(x)\eta(dx)}{\eta(\,]t,+\infty[\,)}\mathbf{1}_{\{\tau>t\}} + \delta_\tau(dx)\mathbf{1}_{\{\tau\leq t\}}.$$

Remark

Let η_t^E be the random measure

$$\frac{\mathbf{1}_{]t,+\infty[}(x)\eta(dx)}{\eta(\,]t,+\infty[\,)}\mathbf{1}_{]t,+\infty[}(\cdot) + \delta_{(\cdot)}(dx)\mathbf{1}_{[0,t]}(\cdot) \quad \text{on } E.$$

One has $\int_E h(x)\eta_t(dx) = \Big(\int_E h(x)\eta_t^E(dx)\Big) \circ \tau$ for any bounded Borel function h on E.

Example: successive defaults

- $E = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n \,|\, x_1 \leq \cdots \leq x_n\}$, $\chi = (\sigma_1, \cdots, \sigma_n) : \Omega \to E$.
- $(\mathcal{N}_t)_{t \geq 0}$: filtration generated by $\sum_{i=1}^n \mathbf{1}_{\{\sigma_i \leq t\}}$

Prediction process

$$\eta_t(dx) = \sum_{i=0}^n \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \frac{\eta_{|\boldsymbol{\sigma}_{(i)}}(\mathbf{1}_{\{t < u_{i+1}(x)\}} \cdot dx)}{\eta_{|\boldsymbol{\sigma}_{(i)}}(\mathbf{1}_{\{t < u_{i+1}(\cdot)\}})}$$

or equivalently

$$\eta_t(dx) = \frac{\eta_{|\boldsymbol{\sigma}_{(N_t)}}(\mathbf{1}_{\{t < u_{N_t+1}(x)\}} \cdot dx)}{\eta_{|\boldsymbol{\sigma}_{(N_t)}}(\mathbf{1}_{\{t < u_{N_t+1}(\cdot)\}})}.$$

- $\eta_{|\boldsymbol{\sigma}_{(i)}}$ is the conditional law of χ given $\boldsymbol{\sigma}_{(i)}$
- $\eta_{|\boldsymbol{\sigma}_{(i)}}(\mathbf{1}_{\{t < u_{i+1}(x)\}} \cdot dx)$ denotes the random measure on E sending a bounded Borel function $h : E \to \mathbb{R}$ to

$$\int_E h(x)\, \eta_{|\boldsymbol{\sigma}_{(i)}}(\mathbf{1}_{\{t < u_{i+1}(x)\}} \cdot dx) := \mathbb{E}[h(\chi) \mathbf{1}_{\{t < \sigma_{i+1}\}} \,|\, \boldsymbol{\sigma}_{(i)}].$$

Modelling of the default filtration

- The observable default information can be induced by a filtration $(\mathcal{N}_t^E)_{t\geq 0}$ of $\mathcal{B}(E)$.
- The default filtration $(\mathcal{N}_t)_{t\geq 0}$ is given by $\mathcal{N}_t = \chi^{-1}(\mathcal{N}_t^E)$.
- If $(\mathcal{N}_t^E)_{t\geq 0}$ is generated by some observation process $(N_t)_{t\geq 0}$, then $(\mathcal{N}_t)_{t\geq 0}$ is generated by $(N_t \circ \chi)_{t\geq 0}$.
- Let η_t^E be the $\mathcal{N}_t^E$-conditional law of χ, for bounded Borel function h on E, one has

$$\int_E h(x)\eta_t(dx) = \Big(\int_E h(x)\eta_t^E(dx)\Big) \circ \chi$$

Revisit on successive defaults

- $E = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n \,|\, x_1 \leq \cdots \leq x_n\}$, $\chi = (\sigma_1, \cdots, \sigma_n) : \Omega \to E$
- Observation process: $N_t = \sum_{i=1}^n \mathbf{1}_{\{x_i \leq t\}}$, $t \geq 0$.
- The filtration $(\mathcal{N}_t)_{t\geq 0}$ is generated by the process $\sum_{i=1}^n \mathbf{1}_{\{\sigma_i \leq t\}}$.

Interaction with environmental information

We distinguish two sources of risks

- $(\Omega^\circ, (\mathcal{F}_t^\circ)_{t\geq 0}, \mathbb{P}^\circ)$: the market without default
- χ valued in $(E, \mathcal{B}(E))$: the default information

We model the global market by the product space

$$(\Omega, \mathcal{A}) := (\Omega^\circ \times E, \mathcal{F}_\infty^\circ \otimes \mathcal{B}(E))$$

- $\chi : \Omega = \Omega^\circ \times E \to E$ given by projection to the 2nd coordinate.

Filtrations of $\mathcal{A}$

- Default-free filtration: $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_t = \mathcal{F}_t^\circ \otimes \{\emptyset, E\}$.
- Default filtration: $(\mathcal{N}_t)_{t\geq 0} = (\chi^{-1}(\mathcal{N}_t^E))_{t\geq 0}$, where $(\mathcal{N}_t^E)_{t\geq 0}$ is a filtration of $\mathcal{B}(E)$.
- Market filtration: $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}_t = \mathcal{F}_t^\circ \otimes \mathcal{N}_t^E$. (progressive enlargement of filtrations)
- Global filtration: $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$, $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(\chi) = \mathcal{F}_t^\circ \otimes \mathcal{B}(E)$. (initial enlargement of filtrations)

The probability measures

- Let the law of χ be a Borel probability measure η on E.
- Let $\overline{\mathbb{P}} = \mathbb{P}^\circ \otimes \eta$ be the product measure on Ω.
  - η and $\mathbb{F}$ are independent under $\overline{\mathbb{P}}$.
  - The probability law of χ under $\overline{\mathbb{P}}$ is η.

Change of probability measure

Consider the probability $\mathbb{P}$ given by a change of probability measure

$$\mathbb{P}(d\omega, dx) = \beta_t(\omega, x)\overline{\mathbb{P}}(d\omega, dx) \quad \text{on } \mathcal{F}_t \vee \sigma(\chi)$$

where β is a positive $(\mathbb{H}, \overline{\mathbb{P}})$-martingale with $\mathbb{E}_{\mathbb{P}^\circ}[\beta_t(x)] = 1$ for $x \in E$ and $t \geq 0$ (in particular, $\beta_0(x) = 1$).

- The probability law of χ under $\mathbb{P}$ is unchanged and remains η.
- The $\mathcal{N}_t$-conditional law η_t of χ (the prediction process) is the same under $\mathbb{P}$ and $\overline{\mathbb{P}}$.
- The marginal law of $\mathbb{P}$ on Ω° equals $\mathbb{P}^\circ$ if and only if $\int_E \beta_t(x)\eta(dx) = 1$ for any $t \geq 0$.

$\mathbb{G}$-conditional law under $\overline{\mathbb{P}}$

- Under $\overline{\mathbb{P}}$, the $\mathcal{G}_t$-conditional law of χ is still η_t by the independence of χ and $\mathbb{F}$.
- For any bounded $\mathcal{H}_t$-mesurable function $Y_t(\cdot)$,

$$\eta_t(Y_t(\cdot)) := \int_E Y_t(x)\eta_t(dx) = \mathbb{E}_{\overline{\mathbb{P}}}[Y_t(\chi)|\mathcal{G}_t].$$

- More generally, for any bounded $\mathcal{H}_\infty$-mesurable function $Y_\infty(\cdot)$

$$\begin{tikzcd}
Y_\infty(\chi) \arrow[r, "\mathcal{H}_t=\mathcal{F}_t\vee\sigma(\chi)"] \arrow[d, "\mathcal{F}_\infty\vee\mathcal{N}_t"'] & \mathbb{E}_{\overline{\mathbb{P}}}[Y_\infty(\chi)\,|\,\mathcal{H}_t] \arrow[d, "\mathcal{G}_t"] \\
\mathbb{E}_{\overline{\mathbb{P}}}[Y_\infty(\chi)\,|\,\mathcal{F}_\infty\vee\mathcal{N}_t] \arrow[r, "\mathcal{G}_t"'] & \mathbb{E}_{\overline{\mathbb{P}}}[Y_\infty(\chi)\,|\,\mathcal{G}_t]
\end{tikzcd}$$

leading to

$$\mathbb{E}_{\overline{\mathbb{P}}}[Y_\infty(\chi)|\mathcal{G}_t] = \eta_t\Big(\mathbb{E}_{\mathbb{P}^\circ}[Y_\infty(\cdot)\,|\,\mathcal{F}_t^\circ]\Big)$$

$\mathbb{G}$-conditional law under $\mathbb{P}$

Proposition

- The $\mathcal{G}_t$-conditional law $\eta_t^{\mathbb{G}}$ under $\mathbb{P}$ is given by

$$\eta_t^{\mathbb{G}}(dx) = \frac{\eta_t(\beta_t(x) \cdot dx)}{\eta_t(\beta_t(\cdot))}.$$

- Let $T \geq t \geq 0$ and $Y_T(\cdot)$ be a bounded $\mathcal{H}_T$-mesurable function. One has

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi)|\mathcal{G}_t] = \int_E \mathbb{E}_{\mathbb{P}^\circ}\Big[\frac{Y_T(x)\beta_T(x)}{\beta_t(x)}\Big|\mathcal{F}_t^\circ\Big]\eta_t^{\mathbb{G}}(dx),$$

or equivalently,

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi)|\mathcal{G}_t] = \frac{\eta_t(\mathbb{E}_{\mathbb{P}^\circ}[Y_T(\cdot)\beta_T(\cdot)|\mathcal{F}_t^\circ])}{\eta_t(\beta_t(\cdot))}.$$

Exemple: non-ordered defaults

- $E = \mathbb{R}_+^n$, $\chi = (\tau_1, \cdots, \tau_n)$
- For $t \geq 0$ and $J \subset \{1, \cdots, n\}$, define the set E_t^J to be

$$\{x_J \leq t, x_{J^c} > t\} := \{(x_1, \cdots, x_n) \in E \,|\, x_i \leq t \text{ for } i \in J,\ x_j > t \text{ for } j \in J^c\}.$$

- Default observations: $\mathcal{N}_t^E$ generated by $(E_s^J, s \leq t)$ on E, and $\mathcal{N}_t = \chi^{-1}(\mathcal{N}_t^E)$ generated by $(\mathbf{1}_{E_s^J} \circ \chi, s \leq t)$ on $\mathcal{A}$
- Prediction process :

$$\eta_t(dx) = \sum_{J \subset \{1,\ldots,n\}} \mathbf{1}_{\{\tau_J \leq t,\, \tau_{J^c} > t\}} \frac{\eta_J(\mathbf{1}_{\{\tau_{J^c} > t\}} \cdot dx)}{\eta_J(\mathbf{1}_{\{\tau_{J^c} > t\}})},$$

where η_J is the conditional law of χ with respect to τ_J.

- $\mathbb{G}$-conditional law of χ

$$\eta_t^{\mathbb{G}} = \sum_{J \subset \{1,\ldots,n\}} \mathbf{1}_{\{\tau_J \leq t,\, \tau_{J^c} > t\}} \frac{\eta_J(\mathbf{1}_{\{\tau_{J^c} > t\}} \beta_t(x) \cdot dx)}{\eta_J(\mathbf{1}_{\{\tau_{J^c} > t\}} \beta_t(\cdot))}.$$

Martingale characterization

The previous proposition allows to characterize $\mathbb{G}$ martingale processes. Recall that $\eta_t(\beta_t(\cdot)) = \int_E \beta_t(x)\eta_t(dx)$.

Theorem

Let $(M_t(\cdot))_{t\geq 0}$ be a $\mathbb{G}$-adapted process. It is a $(\mathbb{G},\mathbb{P})$-martingale if the process

$$\widetilde{M}_t(\cdot) := M_t(\cdot)\int_E \beta_t(x)\eta_t(dx), \quad t \geq 0$$

is a $(\mathbb{G},\overline{\mathbb{P}})$-martingale, or equivalently if

$$\forall\ T \geq t \geq 0, \quad \int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(x)|\mathcal{F}_t]\eta_t(dx) = \widetilde{M}_t(\cdot).$$

Change of probability vs Density

Change of probability $\Rightarrow$ $\mathbb{F}$-conditional density

The $\mathbb{F}$-conditional law of χ under $\mathbb{P}$ admit a density w.r.t. η.

$$\mathbb{P}(\chi \in dx | \mathcal{F}_t) = \frac{\beta_t(x)\eta(dx)}{\int_E \beta_t(x)\eta(dx)}$$

Proof. Under $\overline{\mathbb{P}}$, we have by independence of η and $\mathbb{F}$

$$\mathbb{E}_{\overline{\mathbb{P}}}[\beta_t(\chi)|\mathcal{F}_t] = \int_E \beta_t(x)\,\eta(dx).$$

For a non-negative Borel function f on E, by Bayes formula

$$\mathbb{E}_{\mathbb{P}}[f(\chi)|\mathcal{F}_t] = \frac{\mathbb{E}_{\overline{\mathbb{P}}}[f(\chi)\beta_t(\chi)|\mathcal{F}_t]}{\mathbb{E}_{\overline{\mathbb{P}}}[\beta_t(\chi)|\mathcal{F}_t]} = \frac{\int_E f(x)\beta_t(x)\,\eta(dx)}{\int_E \beta_t(x)\,\eta(dx)}.$$

Density $\Rightarrow$ change of probability

Jacod's hypothesis

We suppose that the $\mathcal{F}_t$-conditional law of χ has a density $\alpha_t(\chi)$ w.r.t. a σ-finite Borel measure ν on E, i.e.

$$\mathbb{P}(\chi \in dx | \mathcal{F}_t) = \alpha_t(x)\nu(dx).$$

In multi-default modelling, adopted by El Karoui, Jeanblanc & J. and Kchia, Larsson & Protter.

Consequences:

- $\eta(dx) = \alpha_0(x)\nu(dx)$.
- The density $(\alpha_t(\cdot), t \geq 0)_{x \in E}$ is an $(\mathbb{F}, \mathbb{P})$-martingale.

Proposition

The probability measure $\mathbb{P}$ is absolutely continuous w.r.t. $\overline{\mathbb{P}}$, with Radon-Nikodym derivative on $\mathcal{H}_t$ is

$$\beta_t(\chi) := \frac{\alpha_t(\chi)}{\alpha_0(\chi)}.$$

Example: ordered defaults

- $E = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n \,|\, x_1 \le \cdots \le x_n\}$, $\chi = (\sigma_1, \cdots, \sigma_n) : \Omega \to E$
- Assume that the probability measure η admits a density $\alpha_0(x)$ w.r.t. Lebesgue measure. For any $t \ge 0$,

$$\alpha_t(x) = \alpha_0(x)\beta_t(x), \quad x \in \mathbb{R}_+^n$$

is the $(\mathbb{F}, \mathbb{P})$-conditional density of $\chi = (\sigma_1, \cdots \sigma_n)$.

Proposition

The $\mathbb{G}$-intensity of the counting process $(\sum_{i=1}^n \mathbf{1}_{\{\sigma_i \le t\}})_{t \ge 0}$ is

$$\lambda_t = \sum_{i=0}^{n-1} \mathbf{1}_{\{\sigma_i \le t < \sigma_{i+1}\}} \frac{\int_t^\infty \cdots \int_t^\infty \alpha_t(\sigma_{(i)}, t, x_{i+2}, \cdots, x_n) dx_{i+2} \cdots dx_n}{\int_t^\infty \cdots \int_t^\infty \alpha_t(\sigma_{(i)}, x_{i+1}, \cdots, x_n) dx_{i+1} \cdots dx_n},$$

where $\sigma_{(i)} = (\sigma_1, \cdots, \sigma_i)$.

Impact of the default events

Proposition

Let $Y_T(\cdot)$ be a bounded $\mathcal{H}_T$-mesurable function. One has

$$\mathbb{E}[Y_T(\chi)\,|\,\mathcal{G}_t] = \sum_{i=0}^{n} \mathbf{1}_{\{\sigma_i \le t < \sigma_{i+1}\}} \frac{\int_t^\infty \mathbb{E}[Y_T(x)\alpha_T(x)\,|\,\mathcal{F}_t]dx_{(i+1:n)}}{\int_t^\infty \alpha_t(x)dx_{(i+1:n)}}\Big|_{x_{(i)}=\sigma_{(i)}},$$

where $x_{(i+1:n)} = (x_{i+1}, \cdots, x_n)$.

- Regime switching on each scenario of default.
- Jump of the firm value at the default time σ_i.

Back to the general setting

In practice, the market together with different types of information is modelled by a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ which is not necessarily a product space:

- $(\Omega, \mathcal{A}, \mathbb{P})$: probability space for the market
- $\chi : \Omega \to E$ default uncertainty random variable
- $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ filtration of $\mathcal{A}$ with $\mathcal{F}_0$ being the trivial σ-algebra: default free informations.
- $(\mathcal{N}_t^E)_{t\geq 0}$ filtration of $\mathcal{B}(E)$ and $(\mathcal{N}_t)_{t\geq 0} = (\chi^{-1}(\mathcal{N}_t^E))_{t\geq 0}$ filtration of $\mathcal{A}$: default informations.
- $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ the enlargement of $\mathbb{F}$ by $(\mathcal{N}_t)_{t\geq 0}$: global market information.

However, the previous results with the product space can be useful tools in the general setting.

Main tool to link the product space

Definition

Let $\Gamma_\chi : \Omega \to \Omega \times E$ be the graph map sending $\omega \in \Omega$ to $(\omega, \chi(\omega))$.

- $\forall \mathcal{F} \subset \mathcal{A}$, any $\mathcal{F} \vee \sigma(\chi)$-measurable function can be written as $Y(\chi) := Y(\cdot) \circ \Gamma_\chi$, where $Y(\cdot)$ is an $\mathcal{F} \otimes \mathcal{B}(E)$-measurable function on $\Omega \times E$.

$$\begin{tikzcd} \Omega \arrow[r, "\Gamma_\chi"] \arrow[rr, bend right, "Y(\chi)"'] & \Omega \times E \arrow[r, "Y(\cdot)"] & \mathbb{R} \end{tikzcd} .$$

Proposition

Let $(Y_t(\cdot), t \geq 0)$ be a process adapted to the filtration $\mathbb{F} \otimes \mathcal{N}^E$, then $(Y_t(\chi), t \geq 0)$ is a $\mathbb{G}$-adapted process.

Remarks on $\mathbb{G}$-adapted processes

Recall that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}_t, t \geq 0$ where $\mathcal{N}_t = \chi^{-1}(\mathcal{N}_t^E)$.

- If $\mathcal{N}_t^E = \mathcal{B}(E)$ for all t, then $\mathcal{N}_t = \chi^{-1}(\mathcal{E}) = \sigma(\chi)$. So $\mathbb{G}$ coincides with the initial enlargement of filtration. The previous lemma is similar as in Grorud & Pontier.
- If $\chi = \tau$ and $\mathcal{N}_t^E$ is generated by the functions of the form $\mathbf{1}_{[0,s]}$ with $s \leq t$, then $\mathbb{G}$ is the progressive enlargement.
- In general, $\mathbb{G}$ can be different since $(\mathcal{N}_t^E)_{t\geq 0}$ can be any filtration on $(E, \mathcal{E})$. So the measurability of $Y_t(\cdot)$ is possibly different with the classic initial and progressive enlargements.

The induced probability of Γ_χ

- Let $\mathbb{P}'$ be the probability related to $\Gamma_\chi : \omega \to (\omega, \chi(\omega))$, i.e., for non-negative $\mathcal{A} \otimes \mathcal{E}$-measurable function f on $\Omega \times E$,

$$\int_{\Omega \times E} f(\omega, x)\mathbb{P}'(d\omega, dx) = \mathbb{E}_{\mathbb{P}}[f(\chi)].$$

Density assumption

The $\mathbb{F}$-conditional law of χ has a positive density $(\alpha_t(\cdot))_{t\geq 0}$ with respect to a σ-finite Borel measure ν on E.

- Recall that $\overline{\mathbb{P}}$ denotes the product probability measure $\mathbb{P} \otimes \eta$, then

$$\frac{d\mathbb{P}'}{d\overline{\mathbb{P}}} = \frac{\alpha_t(x)}{\alpha_0(x)} =: \beta_t(x), \qquad \text{on } \mathcal{F}_t \otimes \mathcal{B}(E).$$

Evaluation formula in the general setting

Lemma

Let $Y(\cdot)$ be a bounded $\mathcal{A} \otimes \mathcal{B}(E)$-measurable function on $\Omega \times E$. One has

$$\mathbb{E}_{\mathbb{P}}[Y(\chi)|\mathcal{G}_t] = \mathbb{E}_{\mathbb{P}'}[Y(\cdot)|\mathcal{F}_t \otimes \mathcal{N}_t^E](\chi).$$

Theorem

Let $Y_T(\cdot)$ be a non-negative $\mathcal{F}_T \otimes \mathcal{E}$-measurable function on $\Omega \times E$ and $t \le T$. Then

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi)|\mathcal{G}_t] = \frac{\int_E \mathbb{E}_{\mathbb{P}}[Y_T(x)\beta_T(x)|\mathcal{F}_t]\eta_t(dx)}{\int_E \beta_t(x)\eta_t(dx)}(\chi),$$

Example: non-ordered multi-defaults

We consider $E = \mathbb{R}_+^n$ and $\chi = (\tau_1, \cdots, \tau_n) : \Omega \to E$, with $\nu = dx$.

- progressive enlargement of filtration: $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}_t$ where $\mathcal{N}_t = \chi^{-1}(\mathcal{N}_t^E)$ is generated by $(\mathbf{1}_{E_s^J} \circ \chi, s \le t)$ with

$$E_t^J = \{x \in E \,|\, x_J \le t,\ x_{J^c} > t\}$$

- prediction process:

$$\eta_t(dx) = \sum_{J \subset \{1, \cdots, n\}} \frac{\mathbf{1}_{\{x_{J^c} > t\}} \alpha_0(\cdot, x_{J^c}) \delta_{(\cdot)}(dx_J) dx_{J^c}}{\int_t^\infty \alpha_0(\cdot, x_{J^c}) dx_{J^c}} \mathbf{1}_{E_t^J} \circ \chi,$$

- evaluation formula:

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi) | \mathcal{G}_t] = \sum_{J \subset \{1, \cdots, n\}} \mathbf{1}_{\{\tau_J \le t, \tau_{J^c} > t\}} \frac{\int_t^\infty \mathbb{E}_{\mathbb{P}}[Y_T(x) \alpha_T(x) | \mathcal{F}_t] dx_{J^c}}{\int_t^\infty \alpha_t(x) dx_{J^c}} \Big|_{x_J = \tau_J}.$$

Martingale characterization in the general setting

Theorem

Let $(M_t(\cdot), t \geq 0)$ be an $(\mathcal{F}_t \otimes \mathcal{N}_t^E)_{t\geq 0}$-adapted process. Then $(M_t(\chi), t \geq 0)$ is a $(\mathbb{G}, \mathbb{P})$-martingale if

$$\widetilde{M}_t(\cdot) := M_t(\cdot) \int_E \frac{\alpha_t(x)}{\alpha_0(x)} \eta_t(dx), \quad t \geq 0$$

is an $((\mathcal{F}_t \otimes \mathcal{N}_t^E)_{t\geq 0}, \overline{\mathbb{P}})$-martingale, or equivalently

$$\forall\ T \geq t \geq 0, \quad \int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(x) | \mathcal{F}_t] \eta_t(dx) = \widetilde{M}_t(\cdot). \qquad (*)$$

Example: non-ordered defaults

- We write $M_t(x)$ as $\displaystyle\sum_{J\subset\{1,\cdots,n\}} M_t^J(x_J)\mathbf{1}_{E_t^J}(x)$.
- One has $\displaystyle\int_E \frac{\alpha_t(x)}{\alpha_0(x)}\eta_t(dx) = \sum_{J\subset\{1,\cdots,n\}} \frac{\int_t^\infty \alpha_t(\cdot, x_{J^c})dx_{J^c}}{\int_t^\infty \alpha_0(\cdot, x_{J^c})dx_{J^c}}\mathbf{1}_{E_t^J}(\cdot)$.
- Get

$$\widetilde{M}_t(\cdot) = \sum_{J\subset\{1,\cdots,n\}} M_t^J(\cdot)\frac{\int_t^\infty \alpha_t(\cdot, x_{J^c})dx_{J^c}}{\int_t^\infty \alpha_0(\cdot, x_{J^c})dx_{J^c}}\mathbf{1}_{E_t^J}(\cdot)$$

- $(*)$ becomes

$$\sum_{J\subset\{1,\cdots,n\}} \sum_{I\supset J} \frac{\int_t^T \mathbb{E}_{\mathbb{P}}[M_T^I(x_I)\int_T^\infty \alpha_T(x)dx_{I^c}|\mathcal{F}_t]dx_{I\setminus J}}{\int_t^\infty \alpha_0(x)dx_{J^c}}\mathbf{1}_{E_t^J}(x)$$

$$= \sum_{J\subset\{1,\cdots,n\}} M_t^J(x_J)\frac{\int_t^\infty \alpha_t(x)dx_{J^c}}{\int_t^\infty \alpha_0(x)dx_{J^c}}\mathbf{1}_{E_t^J}(x).$$

Corollary

With the notation of the example of non-ordered defaults.
The $\mathbb{G}$-adapted process $(M_t(\chi))_{t\geq 0}$ is a $(\mathbb{G},\mathbb{P})$-martingale if for any $J \subset \{0,\cdots,n\}$ and any $x_J \in \mathbb{R}_+^J$, the process

$$M_t^J(x_J)\int_t^\infty \alpha_t(x)dx_{J^c} - \sum_{k\in J^c}\int_{x_J^{\max}}^t M_{x_k}^{J\cup\{k\}}(x_{J\cup\{k\}})\int_{x_k}^\infty \alpha_{x_k}(x)dx_{J^c\setminus\{k\}}du_k$$

is an $(\mathbb{F},\mathbb{P})$-martingale on $[x_J^{\max},+\infty[$, where $x_J^{\max} = \max_{j\in J} x_j$.

Conclusion

- We consider a general multi-default system with environmental information.
- Two key elements are:
  - the prediction process $(\eta_t, t \geq 0)$ conditional on the observable default information;
  - the Radon-Nikodym derivative $(\beta_t(\cdot), t \geq 0)$ w.r.t. the product measure.
- We establish a link with the density approach modelling.
- The technical tools by using the product space allows to obtain general results in a unified setting.

Thanks for your attention !