A Way Martingales Can Become Strict Local Martingales

Philip Protter, Columbia University

Enlargement of Filtration and Financial Applications

Based on work with

Aditi Dandapani, now at Columbia, at ETH next fall

May 2, 2016

The Problem We Address

- We want to describe financial bubbles mathematically
- On a compact time interval $[0, T]$ we will see that this tantamount to the price process being a strict local martingale on $[0, T]$, instead of a martingale, under a given risk neutral measure
- Given a fairly general framework, can we describe how bubbles might form?
- That is, we want a reasonable framework where martingales on $[0, T]$ become strict local martingales

- We are given a filtered complete probability space: $(\Omega, \mathcal{F}, P, \mathbb{F})$ where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions, and contains at least one Brownian motion
- We let S denote our nonnegative stock price process, & assume interest rates are zero
- Let $\mathbb{Q}$ denote all risk neutral measures Q
- The **Fundamental Price** of a stock, denoted $S^* = (S_t^*)_{0 \leq t \leq T}$, is the conditional expectation

$$S_t^* = E_Q\{ \textbf{ All cash flows after time } t | \mathcal{F}_t \} \tag{1}$$

- It is impossible really to know S_t^*

Mathematics to the Rescue

- NFLVR$\Rightarrow S_t \geq S_t^*$ a.s.
- $\beta_t = S_t - S_t^* \geq 0$ is the bubble process
- **Theorem[Jarrow, P^2, Shimbo] 2010:** On a compact time interval $[0, T]$ a stock price is undergoing bubble pricing if and only if the bubble process $\beta_t > 0$ and $\beta_t = S_t - S_t^*$ is a strict local martingale under $Q \in \mathbb{Q}$
- Since S^* is a martingale, β a strict local martingale is equivalent to S itself being a strict local martingale
- This theorem builds on work of **Lowenstein & Willard**, and **Cox & Hobson**
- There is a lot of subsequent work by **E. Bayraktar, F. Biagini, H. Föllmer, C. Kardaras, A. Nikeghbali, A. Roch, M. Schweitzer,** and many others

Strict Local Martingales

- A Strict Local Martingale is a local martingale which is not a martingale
- In finance, the price process is a martingale or at least a local martingale under a risk neutral measure

Why do Bubbles Begin?

- There is a vast literature in the world of economics on the causes of financial bubbles
- A simple way they can begin is via an infusion of news to the market that investors find exciting, leading to "irrational exuberance," in the infamous words of Alan Greenspan
- Mathematically we can model this phenomenon as an expansion of the filtration
- We use the idea of initial expansion; such an "initial" expansion can also occur at a stopping time

TheDelbaen-Shirakawa Framework

- Let X be a solution of a stochastic differential equation (SDE) of the form

$$dX_t = \sigma(X_t)dB_t; \quad X_0 = 1 \tag{2}$$

where B is standard Brownian motion

- **Delbaen & Shirakawa (2002)** and **Mijatovic & Urusov (2012)** give us deterministic necessary and sufficient criteria that we can use to check whether X is a strict local martingale:
- X is a positive **Strict Local Martingale** if for any $\varepsilon > 0$:

$$\int_0^{\varepsilon} \frac{x}{\sigma(x)^2} dx = \infty \text{ and } \int_{\varepsilon}^{\infty} \frac{x}{\sigma(x)^2} dx < \infty$$

Remarks on TheDelbaen-Shirakawa Framework

- In the Delbaen-Shirakawa framework, one can be in an incomplete market with an SDE of the form

$$\begin{aligned} dX_t &= \sigma(X_t)dB_t + b(X_t, Y_t)dt; \quad X_0 = 1 \\ dY_t &= f(Y_t)dW_t + c(Y_t)dt \end{aligned} \tag{3}$$

  where Y is another source of randomness, but in the drift, not the volatility
- The beauty of the form (3) is that under any of the infinite choice of risk neutral measures Q the equation (3) reduced to the form (2), which has a unique solution in law, and hence is the same under all risk neutral measures
- This feature makes it especially easy to analyze
- One problems with the form (2) however is that it does not lend itself to bubble birth via a change to an equivalent risk neutral measure, at least not easily

Do Bubbles Really Happen? Are Strict Local Martingales Important?

- Y. Kchia and I developed a test to determine if a local martingale is in fact a martingale, or is a strict local martingale
- We use the idea of Delbaen & Shriakawa
- This a fairly good model for short periods of time, but it is not realistic for a long period of time.
- It should be replaced (at least) by adding some time dependence, to obtain a model of the form

$$dX_t = \sigma(t, X_t)dB_t + b(t, X_t, Y_t)dt; \quad X_0 = 1 \tag{4}$$

- Such a model, however, is a bit too general to lend itself to a good analysis.
- We make a compromise by assuming the stock prices locally evolve according to (3) and globally evolve according to (4).
- More precisely we make the assumption that for a compact time interval $[0, T]$ we have a finite partition $0 = t_1, \ldots, t_n = T$, and the volatility coefficient σ has the form:

$$\sigma(t, x) = \sigma_1(x)1_{[t_1,t_2]}(t) + \sum_{i=2}^{n} \sigma_i(x)1_{(t_i,t_{i+1}]}(t) \tag{5}$$

- This is in effect a type of regime change model.

Work with Shihao Yang, Harvard

- We studied 3,500 stocks traded in the New York area, from 2000 to 2013, and we applied our test
- We computed the lifetimes of bubbles
- We got a lot of false readings, and instability of the test, so we smoothed the results using a Hidden Markov Model technique (HMM)
- We got a large number of fleeting bubble readings, so we imposed a 5% filter: The stock price must rise more than 5% to signify the birth of a bubble, and it must later fall at least 5% to signify the death of a bubble, given that the test reads positive for a bubble
- The imposition of the 5% filter distorts a bit the results, and they should be interpreted with that in mind
- Using this technique, **we can compute the empirical distribution of the lifetimes of financial bubbles**

The Results

We get a histogram of the results which is well fit by a **generalized gamma distribution**

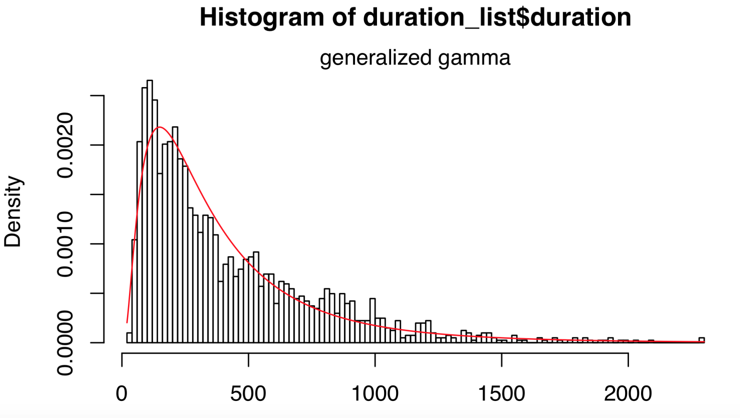


Figure: Histogram of bubble lifetimes

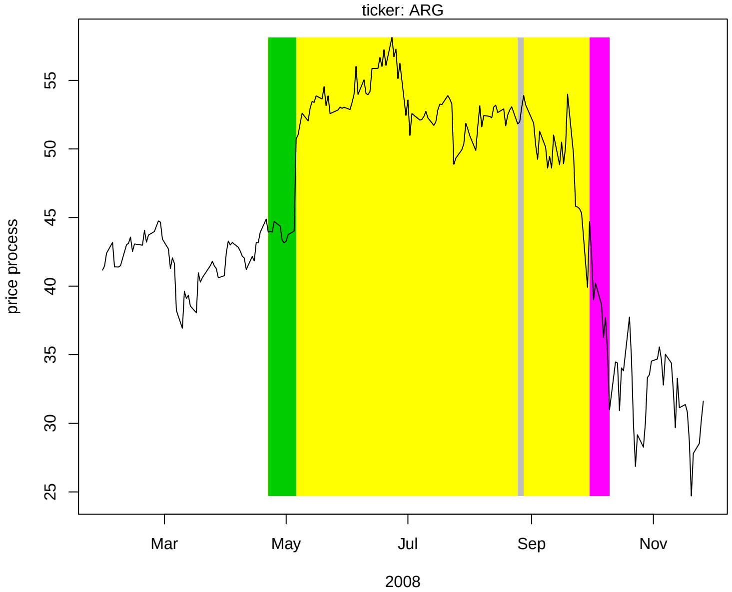
Median and Mean of Bubble Lifetime

- Our computer search found a little over 13,000 bubbles
- The mean of the lifetimes is 367 days
- The median of the lifetimes is 241 days (calendar days)
- The disparity is due to the heavy tails of the generalized gamma distribution
- Also the 5% filter makes bubble lifetimes artificially longer in duration

Examples

- We give here an example of a long-lived and an example of a short-lived short-lived bubble.
- We do this via graphs.
- The green areas are the bubble birth periods, and the purple areas are the bubble deaths.
- Yellow means that bubble is ongoing yet there is no signal detected, and gray means there is indeed a bubble signal, but it is suppressed by the 5% change requirement.
- **Short-lived bubble** Our example is for Airgas, Inc.(ticker ARG) from May to October, 2008.
- **Long-lived bubble** Our example is for The Macerich Company (ticker MAC) from late 2004 to mid 2008. Note that this bubble has two deaths, a first tentative one, and then a dramatic one.

mle path of the hidden states after HMM smoothing




mle path of the hidden states after HMM smoothing
ticker: MAC
price process
20
30
40
50
60
2005
2006
2007
2008

The PL Lions-M Musiela Framework

- P-L Lions and M. Musiela studied SDEs with stochastic volatility (Heston type SDEs) to see when the solution S was a local martingale, and when it was a strict local martingale (2007)
- L. Andersen and V. Piterbarg simultaneously published a similar result in 2007
- **Lions-Musiela framework:**

$$
\begin{aligned}
dS_t &= S_t v_t dB_t; \qquad S_0 = 1 && (6) \\
dv_t &= \sigma(v_t)dW_t + b(v_t)dt; \quad v_0 = 1 && (7)
\end{aligned}
$$

- B and W are correlated Brownian motions, with correlation coefficient ρ and our time interval is compact, $[0, T]$.

The PL Lions-M Musiela Framework, Continued

- If

$$\limsup_{x\to+\infty}\frac{\rho\ x\sigma(x)+b(x)}{x}<\infty \tag{8}$$

  holds, **then S is an integrable non negative martingale.**

- If

$$\liminf_{x\to+\infty}(\rho\ x\sigma(x)+b(x))\phi(x)^{-1}>0 \tag{9}$$

  holds, **then S is a strict local martingale.**

- $\phi(x)$ is an increasing positive smooth function that satisfies

$$\int_a^\infty \frac{1}{\phi(x)}dx<\infty$$

- The Lions-Musiela paradigm extends to processes driven by Lévy noise
- We assume that S and v follow SDEs of the form:

$$\begin{aligned} dS_t &= S_{t_-} v_t^{\alpha} dM_t \qquad (10)\\ dv_t &= \sigma(v_t) dB_t + b(v_t) dt \qquad (11)\end{aligned}$$

- M is a Lévy martingale, with Lévy measure ν, such that $[M, M]$ is locally in L^1
- A sufficient condition for S to be a martingale on $[0, T]$ is that

$$E[e^{\int_0^T (\frac{1}{2} + \int_{\mathbb{R}} x^2 \nu(dx)) v_s^{2\alpha} ds}] < \infty \qquad (12)$$

- The condition

$$\liminf_{x \to +\infty} (\rho \, x\sigma(x) + b(x)) \phi(x)^{-1} > 0$$

is sufficient for S to be a strict local martingale.
- A similar analysis applies for martingales M that are not necessarily Lévy, but are such that $d\langle M, M \rangle_t = \lambda_t dt$.

The Basic Idea

- Recall the Delbaen-Shirakawa framework:

$$dX_t = \sigma(X_t)dB_t; \quad X_0 = 1 \tag{13}$$

- This becomes a strict local martingale (and not a martingale) as soon as

$$\int_{\varepsilon}^{\infty} \frac{x}{\sigma(x)^2} dx < \infty$$

- This means that $\sigma(x)$ has to go to ∞ with x, and do so significantly faster than x
- In the Lions-Musiela framework, it is the process v that must tend to ∞ at a fast enough rate
- The equation (13) has to blow up, and in a fairly precise way

The Basic Idea, Continued

- For the Lions-Musiela paradigm we have:

$$
\begin{aligned}
dS_t &= S_t v_t dB_t; \qquad S_0 = 1 && (14) \\
dv_t &= \sigma(v_t)dW_t + b(v_t)dt; \quad v_0 = 1 && (15)
\end{aligned}
$$

- In equation (14) we have $f(x) = x$ as our coefficient of S; this will not by itself make S a strict local martingale; the process v has to play a key role
- By analogy with the Delbaen-Shirakawa paradigm we need to have v grow quickly to ∞
- How v behaves depends on the equation (15), and it becomes a question of using Feller's test for explosions in one dimensional SDEs

- One way bubbles can form is that news come to the market that causes "irrational exuberance"
- We can model this infusion of information as a filtration expansion (Grossissement de la filtration)
- We use the idea of **initial expansions:** the underlying filtration $\mathbb{F}$ is expanded by the addition of a random variable L
- Typically such an expansion takes place at time $t = 0$, but of course it can happen at any stopping time
- The new, enlarged filtration, which we will call $\mathbb{G}$ can be denoted (slightly informally) as

$$\mathcal{G}_t = \bigcap_{\epsilon>0} (\mathcal{F}_{t+\epsilon} \vee \sigma(L))$$

- We use the results of **Jean Jacod (1985)** on the initial expansion of filtrations:
- Let η be the distribution of L and $Q_t(\omega, dx)$ be the regular conditional distribution of L given $\mathcal{F}_t$.
- There exists an $\mathbb{F}$ martingale $q^x(t,\omega)\eta(dx)$ which is a version of $Q_t(\omega, dx)$.
- If S is an $\mathbb{F}$ martingale, there exists an $\mathbb{F}$ predictable process $k^x(t,\omega)$, such that $[q^x, S] = (k^x q^x_-) \cdot [S, S]$.
- Jacod's theorem tells us the following process is a $\mathbb{G}$ local martingale:

$$\widetilde{S}_t = S_t - \int_0^t k_s^L d[S, S]_s$$

- The filtration enlargement changes the semimartngale decompositions of both S and v
- We lose the local martingale property of S, but via Girsanov's theorem we can regain it by changing to an equivalent probability measure Q
- $Z_t = E[\frac{dQ}{dP}|\mathcal{G}_t]$.
- We write $Z_t = 1 + ZH \cdot B_t + ZJ \cdot W_t$ for $\mathbb{G}$ predictable processes J and H
- Under $(Q, \mathbb{G})$, S possesses the following decomposition:

$$\begin{aligned} S_t &= \int_0^t (S_s v_s) dB_s - \int_0^t k_s^L (S_s v_s)^2 ds - \\ &\quad \int_0^t ((S_s v_s) H_s + \rho J_s) ds + \\ &\quad \int_0^t k_s^L (S_s v_s)^2 ds + \int_0^t ((S_s v_s) H_s + \rho J_s) ds \end{aligned}$$

- To choose Q such that S is a local martingale we need

$$k_t^L (S_t v_t)^2 = -(S_t v_t) H_t - \rho J_t.$$

- We make two assumptions:

$$\begin{array}{ll} (A1) & k, H, \text{ and } J \text{ have right continuous paths } a.s. \\ (A2) & Q(\omega : k_0^L > 0) > 0 \end{array}$$

- Assumptions (A1) and (A2) are true in known examples, and they are also true in some specially constructed examples that are appropriate to the modeling of an information infusion to the market
- In the specially constructed examples, it involves some work to show that (A1) actually holds

- The drift in the volatility equation changes too: The new drift of v, which we will call $\hat{b}(v_t)$, satisfies:

$$\hat{b}(v_t) = b(v_t) + k_t^L \mu^2(v_t) + (\rho H_t + J_t)\mu(v_t).$$

- It is this change in the volatility that allows the change from a martingale to a strict local martingale
- Notice that we can no longer represent the drift in deterministic terms as simply functions of the real variable x

- We cannot immediately invoke the results of Lions & Musiela.
- To address this, let us fix $0<\varepsilon^{(1)}<k_0^L$ and $|\rho H_0 + J_0|<\varepsilon^{(2)}$ and define the following random times:

$$\begin{aligned} \tau^k &= \inf\{t : |k_t^L|<\varepsilon^{(1)}\} \\ \tau^{H,J} &= \inf\{t : |\rho H_t + J_t|>\varepsilon^{(2)}\} \end{aligned}$$

- Define the stopping time τ :

$$\tau = (\tau^k \wedge \tau^{H,J}).$$

- Denote by $m : \min(\varepsilon^{(1)}, \varepsilon^{(2)})$ and by $M : \max(\varepsilon^{(1)}, \varepsilon^{(2)})$
- **On the stochastic interval $[0,\tau]$, we have the following lower bound on our drift coefficients:**

$$\hat{b}(v_t) \geq b(v_t) + m\mu^2(v_t) - M\mu(v_t)$$

- **Theorem:** Assume the following conditions

$$\limsup_{x \to +\infty} \frac{\rho\ x\mu(x) + b(x)}{x} < \infty$$

$$\liminf_{x \to +\infty} (\rho\ x\mu(x) + b(x) + m\mu^2(x) - M\mu(x)\phi(x)^{-1} > 0$$

on the functions μ,b are satisfied, and assume that B and W are correlated Brownian motions with correlation $\rho > 0$. Let the process S be the unique strong solution of the SDE

$$\begin{aligned} dS_t &= S_t v_t dB_t \\ dv_t &= \mu(v_t) dW_t + b(v_t) dt \end{aligned}$$

on $(P, \mathbb{F})$, and assume that S is strictly positive.

- The solution S is also the solution of

$$\begin{aligned} dS_t &= S_t v_t dB_t \\ dv_t &= \mu(v_t)dW_t + b(v_t)dt + k_t^L \mu^2(v_t)dt + (\rho H_t + J_t)\mu(v_t)dt \end{aligned}$$

  on $(Q, \mathbb{G})$.
- **Then S is a $(P, \mathbb{F})$ martingale and a $(Q, \mathbb{G})$ strict local martingale on the stochastic interval $[0, \tau]$.**
- Specifically, we have $E[S_t^\tau] < S_0$.

$\square$

- We have similar results in the Lévy driven case for S, as long as v is still driven by a Brownian motion (so that we can still use Feller's test for explosions in the spirit of Lions & Musiela)

- The idea of changing the drift of the stochastic volatility equation v to create a strict local martingale for S originates in a recent paper of **F. Biagini, H. Föllmer, and S. Nedelcu**
- The change in the risk neutral measure also has an impact on the price structure of financial derivatives
- The absence of arbitrage in such a model can be assured in a local sense, via the ideas developed in the PhD thesis of **Roseline Bilina Falafala** (Columbia, 2014)

The End

Thank you for your attention