Semi-Static Completeness and Robust Pricing by Informed Investors

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joint work with Beatrice Acciaio

Enlargement of Filtrations and Financial Applications

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Robust pricing and informed investors

A central problem in robust finance is to prove

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[\Phi] \quad = \quad \inf\left\{c \in \mathbb{R} \ : \ \begin{array}{l}\Phi \text{ can be super-hedged starting} \\ \text{from initial capital } c,\ \mathcal{M}\text{-q.s.}\end{array}\right\}$$

in various settings, where $\mathcal{M}$ is a suitable set of martingale measures.

Beiglböck, Henry-Labordère, Penkner (2013); Galichon, Henry-Labordère, Touzi (2014); Acciaio, Beiglböck, Penkner, Schachermayer (2013); Bouchard, Nutz (2013); Dolinsky, Soner (2014a,2014b); Beiglböck, Cox, Huesmann (2014); Biagini, Bouchard, Kardaras, Nutz (2014); Beiglböck, Nutz, Touzi (2015); Guo, Tan, Touzi (2015); Hou, Obłój (2015); etc.

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Question:

- What can be said about the relation between the super-hedging price and the choice of filtration?
- When passing from $\mathbb{F}$ to $\mathbb{G} \supset \mathbb{F}$, which measures $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ are still relevant for pricing?

Outline

- Setup
- Semi-static completeness and the Jacod-Yor theorem
- Semi-static completeness for continuous price processes
- Pricing by informed investors
- Conclusion

Setup

- $(\Omega, \mathbb{F}, \mathcal{F})$: Filtered measurable space with $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ right-continuous.

  Later we will consider other filtrations.

- $\mathcal{P}$: Any collection of probability measures on $\mathcal{F}_T$ such that $\mathbb{P} \in \mathcal{P}$, $\mathbb{Q} \ll \mathbb{P}$ implies $\mathbb{Q} \in \mathcal{P}$.

- $S = (S_t)_{0 \le t \le T}$: càdlàg $\mathbb{F}$-adapted discounted price process of an asset available for dynamic trading. We assume $S_0 = 0$.

- $\Psi = \{\psi_1, \ldots, \psi_n\}$: a set of $\mathcal{F}_T$-measurable payoffs available for static (buy-and-hold) trading. Today's price of ψ_i is zero for each i.

- A risk-free asset with price $\equiv 1$ is available for dynamic trading.

Martingale measures and extreme points

Calibrated L^2-martingale measures:

$$\mathcal{M}(\mathbb{F}) = \left\{ \mathbb{Q} \in \mathcal{P} : \begin{array}{l} S \text{ is an } \mathbb{F}\text{-martingale, } \mathbb{E}_{\mathbb{Q}}[S_T^2] < \infty, \\ \mathbb{E}_{\mathbb{Q}}[\psi_i \mid \mathcal{F}_0] = 0,\ \mathbb{E}_{\mathbb{Q}}[\psi_i^2] < \infty \text{ for all } i \end{array} \right\}$$

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Extreme points: $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ is called an extreme point if

$$\begin{array}{l} \mathbb{Q} = \lambda \mathbb{Q}^1 + (1-\lambda)\mathbb{Q}^2 \\ \text{for } \mathbb{Q}^i \in \mathcal{M}(\mathbb{F}),\ \lambda \in (0,1) \end{array} \quad \Longrightarrow \quad \mathbb{Q}^1 = \mathbb{Q}^2 = \mathbb{Q}$$

Denote

$$\operatorname{ext} \mathcal{M}(\mathbb{F}) = \{\text{all extreme points of } \mathcal{M}(\mathbb{F})\}$$

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Note: Purely algebraic condition. Independent of any topology we may put on the space of probability measures.

Martingale measures and extreme points

Why do we care about extreme points?

- Consider an $\mathcal{F}_T$-measurable payoff Φ.
- Under suitable continuity and compactness assumptions,

$$\sup_{\mathbb{Q}\in\mathcal{M}(\mathbb{F})} \mathbb{E}_{\mathbb{Q}}[\Phi] = \sup_{\mathbb{Q}\in\operatorname{ext}\mathcal{M}(\mathbb{F})} \mathbb{E}_{\mathbb{Q}}[\Phi]$$

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Furthermore:

- In the classical case $\Psi = \emptyset$ (no static claims), there is a well-known connection between extreme points and completeness.
- An analogous connection exists in the semi-static case.

Semi-static completeness and the Jacod-Yor theorem

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- Suppose $\Psi = \emptyset$ (no static claims).
- For $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, the classical **Jacod-Yor (1977) theorem** yields

$$\mathbb{Q} \in \operatorname{ext} \mathcal{M}(\mathbb{F}) \quad \Longleftrightarrow \quad L^2(\mathcal{F}_T, \mathbb{Q}) = \{x + (H \cdot S)_T \colon H \in L^2(S)\}.$$

- This result can be generalized to the semi-static case.

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Definition. We say that **semi-static completeness** (under $(\mathbb{Q}, \mathbb{F})$) holds if any $X \in L^2(\mathcal{F}_T)$ can be represented as

$$X = x + a_1\psi_1 + \cdots + a_n\psi_n + (H \cdot S)_T$$

for some $x, a_1, \ldots, a_n \in \mathbb{R}$ and $H \in L^2(S)$.

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Theorem (semi-static Jacod-Yor theorem):

$$\mathbb{Q} \in \operatorname{ext} \mathcal{M}(\mathbb{F}) \quad \Longleftrightarrow \quad \text{semi-static completeness holds}$$

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  - Theorem of **Douglas (1964)** yields W dense in L^1
  - Hahn-Banach + theorem of **Yor (1978)** + induction yields

$$\overline{W}^{L^1} \cap L^2 = W$$

Semi-static completeness and the Jacod-Yor theorem

Remarks.

- Infinitely many ψ_i would allow strategies like

$$f(S_T) + (H \cdot S)_T$$

  where $f \in L^2(\mu)$ for a fixed (by the market) marginal law $S_T \sim \mu$.

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- Our inductive proof does not work for such a setup. In fact:

> **Theorem (Acciaio-L.-Schachermayer, 2016).** There exists a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a Brownian motion B and a stopping time T such that $S := B^T$ is bounded and
>
> $$W = \left\{ f(S_T) + (H \cdot S)_T \colon f \in L^2(\mu),\ H \in L^2(S) \right\}$$
>
> is not closed in L^2.

Semi-static completeness and the Jacod-Yor theorem

Can we say more?

- In the classical case ($\Psi = \emptyset$), completeness is a strong property — but still allows for many "unstructured" models.
- For instance, completeness holds if $\mathbb{F}$ is generated by S, and S is a strong solution to a possibly path-dependent SDE of the form

$$dS_t = \sigma(t; S_u : u \leq t) dW_t$$

  where W is a Brownian motion and σ is never zero.

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Question: Should we expect additional structure in the semi-static case?

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Notation: For any martingale N, let

$$\mathcal{S}(N) = \{H \cdot N \colon H \in L^2(N)\}.$$

This is a closed subspace of $\mathcal{H}^2$.

A curious consequence of completeness

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- Let $K \cdot S$ be the orthogonal projection of $\mathbb{E}_{\mathbb{Q}}[\psi \mid \mathcal{F}_t]$ onto $\mathcal{S}(S)$ and define the **unhedgeable part** of ψ:

$$M_t = \mathbb{E}_{\mathbb{Q}}[\psi \mid \mathcal{F}_t] - (K \cdot S)_t$$

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- By semi-static completeness,

$$\mathcal{H}^2 = \mathrm{span}\{1\} \oplus \mathrm{span}\{M\} \oplus \mathcal{S}(S)$$

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- Thus,

$$\mathcal{S}(M) = \mathrm{span}\{M\},$$

  which is one-dimensional!

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Consequence:

- for some $t^* \in (0, T]$ and some $\mathbb{Q}$-atom B of $\mathcal{F}_{t^*-}$,

$$M = M_T \mathbf{1}_{B \times [t^*, T]}$$

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Hence $\mathbb{Q}(B) = 1$.

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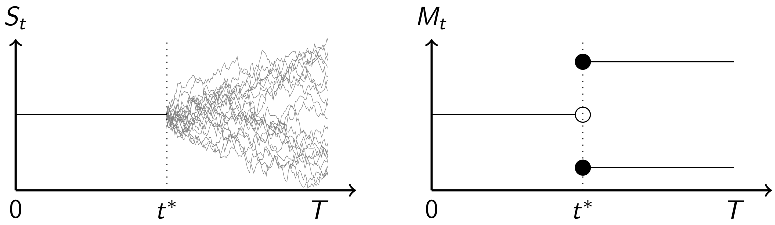
Hence $\mathbb{Q}(B) = 1$. From this we deduce:



Semi-static completeness for continuous price processes

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Fix $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$

For $A \in \mathcal{F}_T$, let $t(A)$ be the first time A is observed:

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Definition (atomic tree). An **atomic tree** is finite collection $\mathbf{T}$ of events in $\mathcal{F}_T$ such that:

- every $A \in \mathbf{T}$ is a non-null $\mathbb{Q}$-atom of $\mathcal{F}_{t(A)}$;
- for every $A, A' \in \mathbf{T}$ with $t(A) < t(A')$, either $A \supseteq A'$ or $A \cap A' = \emptyset$;
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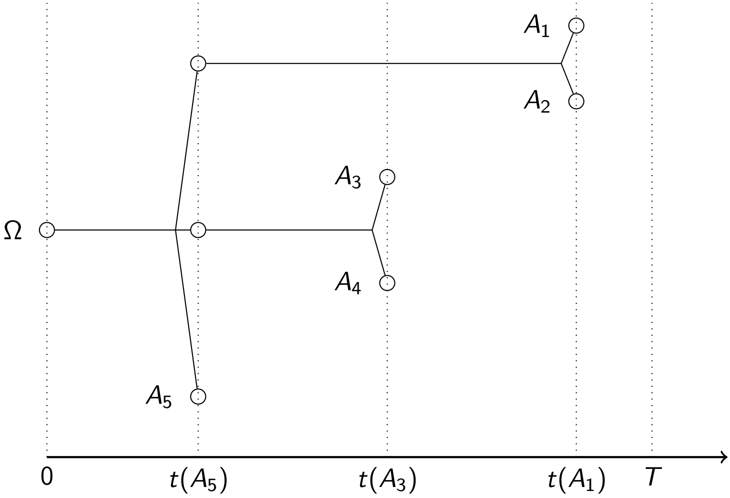
- **leaf:** $A \in \mathbf{T}$ such that there is no $A' \in \mathbf{T}$ with $A' \subsetneq A$.
- $\dim \mathbf{T}$ = number of leaves in $\mathbf{T}$.
- $\mathbf{T}$ is **full** if its leaves form a partition of Ω (up to nullsets), and if A is an atom of $\mathcal{F}_{t(A')-}$ whenever A' is a child of A.

Semi-static completeness for continuous price processes



Semi-static completeness for continuous price processes

Remarks.

- $\sigma(\mathbf{T})$ is well-defined. We have $\sigma(\mathbf{T}) = \mathcal{F}_{\zeta(\mathbf{T})}$ where the stopping time $\zeta(\mathbf{T})$ is the "end" of the tree:

$$\zeta(\mathbf{T}) = \sum_{A \in \mathbf{T} \text{ is a leaf}} t(A)\mathbf{1}_A.$$

- If $\mathbf{T}$ is full, then $\dim \mathbf{T} = \dim L^2(\sigma(\mathbf{T}))$.

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Definition. S is **complete on** $A \times [t, T]$ for given $t \in [0, T]$ and $A \in \mathcal{F}_t$ if any $X \in L^2(\mathcal{F}_T)$ can be dynamically replicated there:

$$X = x + (H \cdot S)_T \quad \text{on} \quad A$$

for some $x \in \mathbb{R}$ and some $H \in L^2(S)$ with $H = 0$ on $[\![0, t]\!]$.

Semi-static completeness for continuous price processes

Recall: $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ is fixed.

Theorem. Assume S is continuous. Semi-static completeness holds if and only if there exists a full atomic tree $\mathbf{T}$ such that

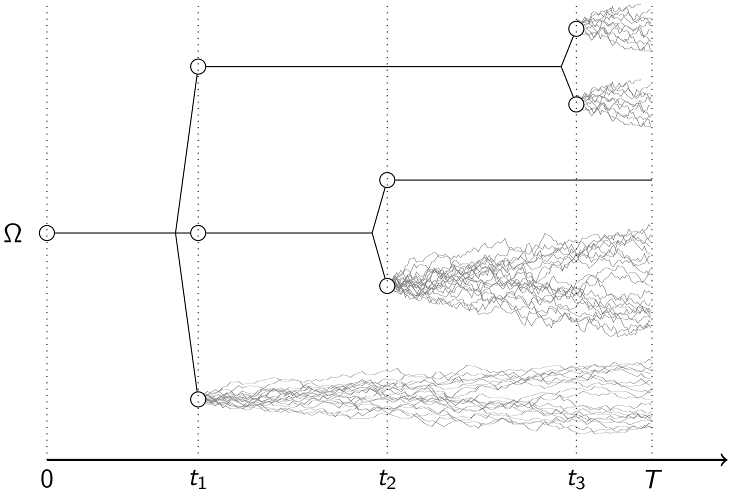
1. S is complete on $A \times [t(A), T]$ for each leaf $A \in \mathbf{T}$,
2. $\mathrm{span}\Big\{\mathbb{E}_{\mathbb{Q}}[\psi_i \mid \sigma(\mathbf{T})] \colon i = 1, \dots, n\Big\}$ is $(\dim \mathbf{T} - 1)$-dim.

In this case, S is constant on $[\![0, \zeta(\mathbf{T})]\!]$ and

$$L^2(\mathcal{F}_T) = \mathrm{span}(1, \psi_1, \dots, \psi_n) + \mathcal{S}(S) = L^2(\sigma(\mathbf{T})) \oplus \mathcal{S}(S).$$

Remark: $\psi_i = \mathbb{E}_{\mathbb{Q}}[\psi_i \mid \sigma(\mathbf{T})] + (H^i \cdot S)_T$ for some H^i.

Semi-static completeness for continuous price processes



The filtration $\mathbb{F}$ under $\mathbb{Q} \in \operatorname{ext} \mathcal{M}(\mathbb{F})$. Each set of lines emanating from the leaves of $\mathbf{T}$ corresponds to a dynamically complete stock price model.

Pricing by informed investors

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Setup:

- S is continuous
- $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$: Right-continuous filtration with

$$\mathcal{F}_t \subseteq \mathcal{G}_t, \qquad 0 \leq t \leq T.$$

- Consider payoff Φ. Robust super-hedging price of informed agent:

$$\sup_{\mathbb{Q} \in \mathcal{M}(\mathbb{G})} \mathbb{E}_{\mathbb{Q}}[\Phi]$$

- As before, we wish to study $\operatorname{ext} \mathcal{M}(\mathbb{G})$.

Question: How are $\operatorname{ext} \mathcal{M}(\mathbb{G})$ and $\operatorname{ext} \mathcal{M}(\mathbb{F})$ related?

Pricing by informed investors

Specification of $\mathbb{G}$:

- Progressive enlargement of $\mathbb{G}$ with $\mathbb{H}$:

$$\mathcal{G}_t = \bigcap_{u>t} \mathcal{F}_u \vee \mathcal{H}_u.$$

  Smallest right-continuous filtration that contains both $\mathbb{F}$ and $\mathbb{H}$.

- $\mathbb{H}$ generated by a collection of single-jump processes $X\mathbf{1}_{[\![\tau, T]\!]}$, where X is a r.v. and τ is a random time, i.e. $[0, T] \cup \{\infty\}$-valued r.v.
- The classical progressive enlargement with random times as well as initial enlargement with random variables are special cases.

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Let σ be the first time S starts moving:

$$\sigma = \inf\{t \in [0, T] \colon S_t \neq 0\}.$$

Theorem. Let $\mathbb{H}$ be generated by finitely many $X_k \mathbf{1}_{[\![\tau_k, T]\!]}$. Assume $\tau_k > \sigma$ on $\{0 < \tau_k < \infty\}$ for all k. Then

$$\operatorname{ext} \mathcal{M}(\mathbb{G}) = \{\mathbb{Q} \colon \mathbb{F} \text{ and } \mathbb{G} \text{ coincide under } \mathbb{Q}, \text{ and } \mathbb{Q} \in \operatorname{ext} \mathcal{M}(\mathbb{F})\}.$$

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This restricts $\mathbb{Q}$ rather severely

- **Example:** If $\tau = \sup\{t \in [0, T] \colon S_t = 1\}$, then must have:
  - Either $\mathbb{Q}(S < 1) = 1$;
  - Or $\mathbb{Q}(S = S^\rho) = 1$ with $\rho = \inf\{t : S_t = 1\}$.
- $\mathbb{Q}$ should also price the ψ_i correctly.
- It can of course happen that $\mathcal{M}(\mathbb{G}) = \emptyset$.

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Extensions to infinitely many $X_i \mathbf{1}_{[\![\tau_i,\infty[\![}$ are possible:

Theorem. Let $\mathbb{H}$ be generated by countably many $X_k \mathbf{1}_{[\![\tau_k,T]\!]}$. Assume $\tau_k > \sigma$ on $\{0 < \tau_k < \infty\}$ $\forall k$, and $|\{k\colon \tau_k(\omega) \leq T\}| < \infty$ for every ω. Then

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Remark. Some condition on $\mathbb{H}$ is needed: Suppose

- $\mathbb{G}$ is generated by Brownian motion W
- $\mathbb{F}$ is generated by $S_t = \int_0^t \operatorname{sgn}(W_s) dW_s$
- Then completeness holds in both $\mathbb{F}$ and $\mathbb{G}$, but they do not coincide.

Conclusions

- Motivated by robust super-hedging price computation, we study extremal calibrated martingale measures
- We obtain:
  - Semi-static version of the Jacod-Yor theorem.
  - Description of semi-statically complete models in terms of dynamically complete models glued together by means of an atomic tree.
  - Application to robust pricing by informed agents: under structural assumptions, informed agents price using only those models that render the additional information uninformative.

Thank you!