Invariance Times

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OVERVIEW: Finding conditions under which the Jeulin-Yor enlargement formula can be compensated by the Girsanov formula of an equivalent change of probability measure.

- The Jeulin-Yor formula, initially proved with projection computations, has long been supposed to be affiliated with the Girsanov formula.
- Yoeurp (1985) gives a formal proof that the Jeulin-Yor formula can be obtained by the Girsanov formula.
- Actually, according to Song (1987, 2013), not only the Jeulin-Yor formula, but most of the formulas in enlargement of filtration, can be retrieved by the so-called local solution method, i.e. by the Girsanov formula.
- However, the methodologies of these papers are not applicable to the study of the above problem, notably because they operate in an enlarged probability space and with a non equivalent probability measure.
- Hence, a new approach is required for that problem.

Financial motivation

- Intensity models of counterparty credit risk with strong (adverse) dependence between the credit risk of a counterparty and the underlying market exposure
  - Wrong way and gap risk modeling
    - Frailty and contagion in the case of a credit derivatives exposure
  - As opposed to factor dependence only in standard Cox (doubly stochastic) models
- When applied to a defaultable asset, a basic no-arbitrage pricing formula explicitly involves the default time θ, whereas it's only the intensity of θ that is observable (through calibration to market data).
- To tackle this issue, Duffie, Schroder, and Skiadas (1996) have established a defaultable asset pricing formula stated in terms of the intensity process (assumed to exist) of θ.
  - From a financial interpretation point of view, their intensity-based formula also shows that credit risk can be valued as a shift in interest rates.

- However, the tractability of the Duffie, Schroder, and Skiadas (1996) formula is subject to a technical no-jump condition at time θ.
- In a progressive enlargement of filtration setup satisfying the restrictive immersion assumption, this no-jump condition is satisfied (cf. Bielecki, Jeanblanc, and Rutkowski (2009))
- The present work extends the progressive enlargement pricing formula from the restrictive immersion setup to the much broader invariance time setup.

Standing notation

- $Y^\tau = Y\mathbb{1}_{[0,\tau)} + Y_\tau \mathbb{1}_{[\tau,+\infty)}$
  $Y^{\tau-} = Y\mathbb{1}_{[0,\tau)} + Y_{\tau-}\mathbb{1}_{[\tau,+\infty)}$ (for any left-limited process Y)
- $\mathcal{S}_{\mathcal{I}}(\mathbb{G},\mathbb{Q}), \mathcal{M}_{\mathcal{I}}(\mathbb{G},\mathbb{Q})$ Semimartingales, local martingales on a predictable interval $\mathcal{I}$ ($= \mathbb{R}_+$ by default)

Y is a semimartingale (local martingale) , $Z = L\centerdot Y$ on $\mathcal{I}$ means that

$$Y^{\tau_n} \text{ is a semimartingale (local martingale), } Z^{\tau_n} = L\centerdot(Y^{\tau_n}) \text{ on } \mathbb{R}_+$$

for each τ_n in at least one (i.e. any) nondecreasing sequence of stopping times such that $\cup[0,\tau_n] = \mathcal{I}$.

BSDEs of counterparty risk

Assume

- a counterparty default time θ totally inaccessible with $\mathbb{G}$ compensator $\gamma \boldsymbol{.} \boldsymbol{\lambda}$,
- a recovery (exposure at default) of the form $\mathbb{1}_{\{\theta<T\}}\mathsf{G}_\theta$, where $T>0$ is a fixed time horizon (maturity) and G is a $\mathbb{G}$ predictable process,
- a $\mathcal{P}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R})$ running (funding) cost $g_t(\omega,x)$

Then the counterparty risk BSDE, which prices the funding cost g until θ and the exposure at default G_θ at θ (if $<T$), can be formulated as the following BSDE for $Z\in\mathcal{S}(\mathbb{G},\mathbb{Q})$:

$$\begin{cases} Z_{T-}\mathbb{1}_{\{T\le\theta\}}=0, \\ \\ Z_t^{\theta\wedge T-}+\int_0^{t\wedge\theta\wedge T}\left(g_s(Z_{s-})+(\mathsf{G}_s-Z_{s-})\gamma_s\right)ds\in\mathcal{M}(\mathbb{G},\mathbb{Q}). \end{cases} \tag{1}$$

- Extendable to BSDEs with $G=G_t(x,u)$, $g=g_t(x,u)$, where additional arguments u correspond to integrands in a stochastic integral representation of the martingale part of Z.

Duffie et al. (1996)'s solution: forget θ in (1) (or "send it to infinity"), obtain a solution $\bar{Z}$ of the resulting (simpler) equation and then set $Z=\bar{Z}^{\theta-}$

- Only yields a solution Z to (1) if $\bar{Z}$ does not jump at θ

Reduction of filtration

Let θ be a $\mathbb{G}$ stopping time (not necessarily with an intensity), $\mathsf{J} = \mathbb{1}_{[0,\theta)}$ and let $\mathbb{F}$ be a subfiltration of $\mathbb{G}$ (both satisfying the usual conditions).

Condition (B)

For any $\mathbb{G}$ predictable process L, there exists an $\mathbb{F}$ predictable process L', which we call the $\mathbb{F}$ predictable reduction of L, such that L' coincides with L until θ, i.e. $\mathsf{J}_- L' = \mathsf{J}_- L$.

Lemma 1 ("If" part = Lemma 1 in Jeulin and Yor (1978))

The filtration $\mathbb{F}$ satisfies the condition (B) if and only if $\mathbb{G}$ is a subfiltration of $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\in\mathbb{R}_+}$, where

$$\overline{\mathcal{F}}_t = \{B \in \mathcal{A} : \exists A \in \mathcal{F}_t, A \cap \{t < \theta\} = B \cap \{t < \theta\}\}.$$

- Even if the filtration $\overline{\mathbb{F}}$ was introduced in Jeulin and Yor (1978) or Chapitre XX in Dellacherie, Maisonneuve, and Meyer (1992) for a classical progressive enlargement setting, where $\mathbb{G}$ is given as $\mathbb{F}$ progressively enlarged by θ, their proofs depend only on the relation $\mathbb{G} \subseteq \overline{\mathbb{F}}$.
- Hence, in view of Lemma 1, all the classical results of progressive enlargement of filtration: "key lemma of credit risk", existence of $\mathbb{F}$ optional reductions (also denoted $'$) coinciding with $\mathbb{G}$ optional processes before θ, etc., are valid under (B).
- In particular, let $\mathsf{S} = {}^o\mathsf{J}$ represent the $\mathbb{F}$ Azéma supermartingale of θ, with Doob-Meyer decomposition $\mathsf{S} = \mathsf{Q} - \mathsf{D}$. The Jeulin-Yor theorem says that for any bounded $(\mathbb{F}, \mathbb{Q})$ martingale X, the process

$$X^{\theta-} - \frac{\mathsf{J}_-}{\mathsf{S}_-} \cdot \langle S, X \rangle^{(\mathbb{F},\mathbb{Q})}$$

is a $\mathbb{G}$ uniformly integrable martingale.

$\rightarrow$ any $\mathbb{F}$ semimartingale stopped at θ is $\mathbb{G}$ semimartingale

The next result addresses the "inverse problem" of knowing when an $\mathbb{F}$ semimartingale X is such that $X^{\theta-}$ is a $\mathbb{G}$ local martingale.

Lemma 2 (Song (2014))

- *If* $Y \in \mathcal{M}(\mathbb{G},\mathbb{Q})$ *with* $\Delta_\theta Y = 0$, *then* $Y' \in \mathcal{S}_{\{\mathsf{S}_->0\}}(\mathbb{F},\mathbb{Q})$ *and* $\mathsf{S}_- \centerdot Y' + [\mathsf{S}, Y'] \in \mathcal{M}_{\{\mathsf{S}_->0\}}(\mathbb{F},\mathbb{Q})$.
- *Conversely, for any* $X \in \mathcal{S}_{\{\mathsf{S}_->0\}}(\mathbb{F},\mathbb{Q})$ *such that* $\mathsf{S}_- \centerdot X + [\mathsf{S}, X] \in \mathcal{M}_{\{\mathsf{S}_->0\}}(\mathbb{F},\mathbb{Q})$, *then* $X^{\theta-} \in \mathcal{M}(\mathbb{G},\mathbb{Q})$.

- The Jeulin formula and Lemma 2 can be viewed as progressive enlargement formal analogs of the predictable and optional Girsanov measure change formulas, the Azéma supermartingale S playing the role of the measure change density from the probability measure $\mathbb{Q}$ to some $\mathbb{Q}$ absolutely continuous probability measure $\mathbb{P}$
  - Predictable Girsanov formula
    "For any bounded $X \in \mathcal{M}(\mathbb{F},\mathbb{Q})$, $X - \frac{1}{\mathsf{S}_-} \centerdot \langle S, X\rangle^{(\mathbb{F},\mathbb{Q})} \in \mathcal{M}(\mathbb{F},\mathbb{P})$"
  - Optional Girsanov formula
    "$X \in \mathcal{M}(\mathbb{F},\mathbb{P})$ iff $X \in \mathcal{S}_{\{\mathsf{S}_->0\}}(\mathbb{F},\mathbb{Q})$ and $\mathsf{S}_- \centerdot X + [\mathsf{S}, X] \in \mathcal{M}_{\{\mathsf{S}_->0\}}(\mathbb{F},\mathbb{Q})$"
- These analogies can be made precise by representing the Azéma supermartingale S as a subdensity

Lemma 3 (Song (2014))

One has the following unique predictable multiplicative decomposition of the Azéma supermartingale S *on* $\{{}^p\mathsf{S} > 0\}$:

$$\mathsf{S} = \mathsf{S}_0 \mathcal{E}(-\frac{1}{\mathsf{S}_-} \centerdot \mathsf{D}) \mathcal{E}(\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q}).$$

The following result is classical:

Lemma 4

Assuming $\mathsf{S}_T > 0$, *two* $\mathbb{F}$ *optional processes that coincide before* θ *coincide on* $[0, T]$, *hence* $\mathbb{F}$ *optional reductions are uniquely defined on* $[0, T]$.

Reduced BSDE

- For any càdlàg process X on $\mathbb{R}_+$ (or any predictable set of interval type), we write

$$\overline{X} = X + (g'_{\cdot}(X_-) + (G' - X_-)\gamma') \centerdot \boldsymbol{\lambda}.$$

- Assuming the BSDE (1) has a solution Z, let $U = Z'$. The martingale term in the BSDE (1) satisfies

$$\begin{aligned} Z_t^{\theta\wedge T-} &+ \int_0^{t\wedge\theta\wedge T} \big(g_s(Z_{s-}) + (G_s - Z_{s-})\big)\gamma_s ds \\ &= U_t^{\theta\wedge T-} + \int_0^{t\wedge\theta\wedge T} \big(g'_s(U_{s-}) + (G'_s - U_{s-})\gamma'_s\big) ds \\ &= \overline{U}_t^{\theta\wedge T-} = (\overline{U}^{T-})_t^{\theta-}. \end{aligned}$$

This suggests to solve the BSDE (1) with Lemma 2.

Namely, we consider the following BSDE for $U \in \mathcal{S}_{\{S_->0\}}(\mathbb{F},\mathbb{Q})$:

$$U_{T-}S_{T-} = 0\,,\;\; S_- \centerdot \overline{U}^{T-} + [S, \overline{U}^{T-}] \in \mathcal{M}_{\{S_->0\}}(\mathbb{F},\mathbb{Q}). \tag{2}$$

Proposition 1

The BSDEs (1) and (2) are equivalent. Specifically:
- *If Z is a solution to the BSDE (1), then $U = Z'$ is a solution to the BSDE (2).*
- *Conversely, if U is a solution to the BSDE (2), then $Z = U^{\theta-}$ is a solution to the BSDE (1).*

"Immersion case" where S is continuous and nondecreasing: The martingale condition in (2) reduces to $\overline{U}^{T-} \in \mathcal{M}_{\{S_->0\}}(\mathbb{F},\mathbb{Q})$.

- Alternatively, suppose

Condition (A)

There exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that, for any $(\mathbb{F}, \mathbb{P})$ local martingale P, $P^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$.

Then any solution $U \in \mathcal{S}_{\{\mathsf{S}_->0\}}(\mathbb{F}, \mathbb{P})$ to

$$U_{T-}\mathsf{S}_{T-} = 0\,,\ \ \overline{U}_t^{T-} \in \mathcal{M}_{\{\mathsf{S}_->0\}}(\mathbb{F}, \mathbb{P}) \tag{3}$$

yields a solution $Z = U^{\theta-}$ to (1).

- Equivalence, under the condition (A), between (3) and (1)?
- Strength of the condition (A)??

Condition (A)

$\leftrightarrow$ The measure change "compensates" the reduction of filtration

- The condition (A) is nonstandard in the enlargement of filtration literature.
- On top of the fact that stopping P at $(\theta-)$ rather than at θ in the condition (A) appears naturally in the above BSDE application, there are (at least) two "a priori" reasons for it:
  - First, as visible in the original proof of the Jeulin-Yor formula (Theorem 1 in Jeulin and Yor (1978)), the bracket $\langle S, P\rangle^{(\mathbb{F},\mathbb{Q})}$ is intrinsically linked with $P^{\theta-}$, rather than with P^{θ}.
  - Second, by optional reduction, $P^{\theta-}$ is uniquely determined by the information of $\mathbb{F}$, which is not the case of P^{θ}.

Definition 1

If the condition (A) is satisfied, we call the random time θ an invariance time and the related probability measure $\mathbb{P}$ an invariance probability measure.

Given $\mathbb{F} \subset \mathbb{G}$, let

$\mathsf{S} = {}^{o}\mathsf{J}$ denote the Azéma supermartingale of θ, i.e. $\mathsf{S}_t = \mathbb{Q}(\theta > t \,|\, \mathcal{F}_t)$, $t > 0$,

$S = \mathsf{Q} - \mathsf{D}$ denote the $(\mathbb{F}, \mathbb{Q})$ canonical Doob-Meyer decomposition of S;

$q = \mathcal{E}(\mathsf{q})$ denote the $(\mathbb{F}, \mathbb{Q})$ martingale density function $\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_{t\wedge T}}$, $t \in \mathbb{R}_+$.

Theorem 1

Assuming the condition (B) on $\mathbb{F}$ and given a constant $T > 0$:
(i) *A probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ is an invariance probability measure if and only if one of the following two equivalent conditions holds:*

$$q = q_0 \mathcal{E}\left(\frac{1}{{}^{p}\mathsf{S}} \centerdot \mathsf{Q}\right) \text{ on } \{{}^{p}\mathsf{S} > 0\} \cap [0, T] \tag{4}$$

$$ {}^{p}\mathsf{S} \centerdot \mathsf{q} = \mathsf{Q} - \mathsf{Q}_0 \text{ on } [0, T] \tag{5}$$

(ii) *The condition (A) holds if and only if*

$$\mathcal{E}\left(\mathbb{1}_{\{{}^{p}\mathsf{S} > 0\}} \frac{1}{{}^{p}\mathsf{S}} \centerdot \mathsf{Q}\right) \text{ is a positive } (\mathbb{F}, \mathbb{Q}) \text{ true martingale on } [0, T]. \tag{6}$$

In this case, an invariance probability measure $\mathbb{P}$ is defined by the $\mathbb{Q}$ density $\mathcal{E}\left(\mathbb{1}_{\{{}^{p}\mathsf{S} > 0\}} \frac{1}{{}^{p}\mathsf{S}} \centerdot \mathsf{Q}\right)_T$.

Proof. (ii) is integrability + (i), which is proven as follows.

- The invariance probability measure property for $\mathbb{P}$ says that

$$\forall P \in \mathcal{M}(\mathbb{F},\mathbb{P}),\ (P^{\theta-})^T = (P^T)^{\theta-} \in \mathcal{M}(\mathbb{G},\mathbb{Q}).$$

- By Lemma 2, this property holds if and only if

$$\forall P \in \mathcal{M}(\mathbb{F},\mathbb{P}),\ \mathsf{S}_- \centerdot P^T + [\mathsf{S}, P^T] \in \mathcal{M}_{\{\mathsf{S}_->0\}}(\mathbb{F},\mathbb{Q}).$$

- Note that $\mathcal{M}(\mathbb{F},\mathbb{P}) = \{Qp;\ Q \in \mathcal{M}(\mathbb{F},\mathbb{Q})\}$, where $p = \frac{1}{q}$.
- By IP and Yoeurp's lemma, the above property is then reduced to

$$\forall Q \in \mathcal{M}(\mathbb{F},\mathbb{Q}),\ Q^T(p^T\mathsf{S} + p_-^T \centerdot \mathsf{D}) \in \mathcal{M}_{\{\mathsf{S}_->0\}}(\mathbb{F},\mathbb{Q}),$$

i.e. (by density)

$$p^T\mathsf{S} + p_-^T \centerdot \mathsf{D} = p_0\mathsf{S}_0 \text{ on } \{\mathsf{S}_- > 0\}, \tag{7}$$

i.e. (noting that $p\mathsf{S} + p_- \centerdot \mathsf{D} = p\mathsf{S} + (p\mathsf{S})_- \frac{1}{\mathsf{S}_-} \centerdot \mathsf{D}$)

$$p\mathsf{S} = p_0\mathsf{S}_0\mathcal{E}(-\frac{1}{\mathsf{S}_-} \centerdot \mathsf{D}) \text{ on } \{\mathsf{S}_- > 0\} \cap [0, T]. \tag{8}$$

Lemma 5 (Cf. Jacod (1979))

We have

$$\{S_- > 0\} \setminus \{{}^p S > 0\} = [\eta], \tag{9}$$

where

$$\eta = \inf\{s > 0; S_{s-} = \Delta_s D > 0\} = \inf\{s \in \{S_- > 0\}; \mathcal{E}(-\frac{1}{S_-} \cdot D)_s = 0\}. \tag{10}$$

- Hence, (8) is trivially satisfied at time η (whenever finite), so that (8) is equivalent to the analogous identity on the "smaller" set $\{{}^p S > 0\} \cap [0, T]$, where it reduces, via the multiplicative decomposition of S in Lemma 3, to (4).
- The equivalence between (4) and (5) is a question of integrability. ∎

- Despite the nature of the problem addressed, neither the Girsanov formula nor the Jeulin-Yor formula is involved in the above proof of Theorem 1: the use of Lemmas 2 and 3 make the use of the Girsanov and the Jeulin-Yor formulas unnecessary.
- However, in so doing, we fail to explain how a Girsanov drift can compensate the Jeulin-Yor drift.
- Given the importance of that matter, we provide in the paper an alternative (longer but "direct") proof of Theorem 1, starting with

$$
\begin{aligned}
&(P - q \centerdot [p,P])^{\theta-} - \mathsf{J}_- \frac{1}{\mathsf{S}_-} \centerdot \langle \mathsf{Q}, P - q \centerdot [p,P] \rangle \\
&\qquad = P^{\theta-} - \left(\mathsf{J} q \centerdot [p,P] + \mathsf{J}_- \frac{1}{\mathsf{S}_-} \centerdot \langle \mathsf{Q}, P - q \centerdot [p,P] \rangle \right) \in \mathcal{M}_{[0,T]}(\mathbb{G}, \mathbb{Q}),
\end{aligned}
$$

  for any $P \in \mathcal{M}(\mathbb{F}, \mathbb{P})$
  - Then project, use density arguments, etc.

Proposition 2

- *If $\mathbb{F} = \mathbb{G}$ and θ has an intensity, then θ cannot be an invariance time unless $\mathbb{Q}(\theta \leq T) = 1$.*
- *Given $\mathbb{F} \subseteq \mathbb{G}$ satisfying (B), $\mathbb{P} = \mathbb{Q}$ is an invariance probability measure for all $T > 0$ if and only if $\mathsf{Q} = \mathsf{S}_0$.*

Proof. 1) In the case where $\mathbb{F} = \mathbb{G}$ and θ has an intensity, we have

$$\mathsf{S} = \mathsf{J}\,,\;\; \mathsf{D} \text{ is continuous }\;,\;\; {}^p\mathsf{S} = \mathsf{J}_-,\;\; \mathsf{Q} = \mathsf{J} + \mathsf{D} \text{ and } \mathsf{Q}_0 = \mathsf{S}_0 = 1.$$

Hence, using the stochastic exponential formula

$$\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q})_t = \mathcal{E}(\mathsf{Q})_t = e^{\mathsf{Q}_t - \mathsf{Q}_0}\prod_{s\leq t}(1+\Delta_s\mathsf{Q})e^{-\Delta_s\mathsf{Q}} = e^{\mathsf{J}_t + \mathsf{D}_t - 1}\mathsf{J}_t = e^{D_t}\mathsf{J}_t,$$

which vanishes at θ on $\{\theta \leq T\}$. Therefore, in view of Theorem 1, the condition (A) cannot hold on $[0, T]$ unless $\mathbb{Q}(\theta \leq T) = 0$.
2) In the case where $\mathbb{P} = \mathbb{Q}$, we have $q = q_0$ on $[0, T]$, i.e. $\mathsf{q} = 0$ on $[0, T]$. Hence, in view of Theorem 1 and (5), $\mathbb{P}$ is an invariance probability measure for all $T > 0$ if and only if Q is constant on $[0, T]$. ∎

Example 1

Let $\mathbb{G}$ be the augmentation of the natural filtration of the jump process at an exponential time θ relative to some probability measure $\mathbb{Q}$.

- For $\mathbb{F} = \mathbb{G}$ (so that the condition (B) holds trivially), Proposition 2 1) shows that the condition (A) does not hold. This can also be recovered directly from the definitions. In fact, for any probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$, each $(\mathbb{F}, \mathbb{P})$ local martingale P is necessarily a $(\mathbb{F}, \mathbb{P})$ stochastic integral against the compensated jump process. Thus, the process $P^{\theta-}$ is absolutely continuous, hence it is not a $(\mathbb{G}, \mathbb{Q})$ local martingale unless it is constant. Therefore, $\mathbb{P}$ is not an invariance probability measure.
- For $\mathbb{F}$ trivial, any $\mathbb{G}$ predictable process coincides with a Borel function before θ, hence the condition (B) is satisfied. The constants are the only $(\mathbb{F} = \{\emptyset, \Omega\}, \mathbb{Q})$ local martingales, so that $\mathbb{P} = \mathbb{Q}$ is an invariance probability measure and θ is an invariance time. Consistent with this conclusion in regard of Theorem 1, S is deterministic (equal to the survival function of θ), Q is constant and $q \equiv 1$, hence (4) is satisfied.

- Assume that $\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}$ is Q integrable on $[0,T]$ with respect to $(\mathbb{F},\mathbb{Q})$.
- Let $\varsigma = \inf\{s>0; \mathsf{S}_s = 0\}$.

Theorem 2

$\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}} \centerdot \mathrm{Q}) > 0$ on $[0,T] \Longleftrightarrow {}^p\mathsf{S}_\varsigma = 0$ on $\{\varsigma \le T\} \Longleftrightarrow \varsigma_{\{\varsigma\le T\}}$ is a predictable stopping time.

Theorem 3

Under the condition (A), if θ has an intensity, then

$$\{\mathsf{S}_- > 0\} = \{{}^p\mathsf{S} > 0\} = \{\mathsf{S} > 0\} \tag{11}$$

and a process P is an $(\mathbb{F},\mathbb{P})$ local martingale on $\{\mathsf{S}_- > 0\}\cap[0,T]$ if and only if $\mathsf{S}_- \centerdot P + [\mathsf{S},P]$ is an $(\mathbb{F},\mathbb{Q})$ local martingale on $\{\mathsf{S}_- > 0\}\cap[0,T]$.

This can be used to establish that, under the assumptions of Theorem 3, the BSDEs (3) and (1) are equivalent .

Let $\mathcal{M}_\circ(\mathbb{G},\mathbb{Q})$ denote the set of the $(\mathbb{G},\mathbb{Q})$ local martingales on $[0,\theta\wedge T]$ without jump at θ and let $\mathcal{M}(\mathbb{F},\mathbb{P})$ denote the set of the $(\mathbb{F},\mathbb{P})$ local martingales on $[0,T]$.

Theorem 4

Under the condition (A), if θ has an intensity, then we have the following bijections inverse to each other:

$$\mathcal{M}(\mathbb{F},\mathbb{P}) \underset{\cdot'}{\overset{\cdot^{\theta-}}{\rightleftarrows}} \mathcal{M}^\circ(\mathbb{G},\mathbb{Q}),$$

where $\cdot'$ stands for the $\mathbb{F}$ optional reduction.

Proof. For any $M\in\mathcal{M}_\circ(\mathbb{G},\mathbb{Q})$, Lemma 2 yields that M' is an $(\mathbb{F},\mathbb{Q})$ semimartingale on $[0,T]$ such that $\mathrm{S}_{-}\centerdot M'+[\mathrm{S},M']$ is an $(\mathbb{F},\mathbb{Q})$ local martingale on $[0,T]$. Therefore, Theorem 3 yields that $M'\in\mathcal{M}(\mathbb{F},\mathbb{P})$. Hence the condition (A) implies that $\mathrm{M}=(M')^{\theta-}\in\mathcal{M}_\circ(\mathbb{G},\mathbb{Q})$. Moreover, M and M coincide before θ and don't jump at θ, hence they agree on $[0,\theta\wedge T]$.
Conversely, for any $P\in\mathcal{M}(\mathbb{F},\mathbb{P})$, the condition (A) yields that $P^{\theta-}\in\mathcal{M}_\circ(\mathbb{G},\mathbb{Q})$. Hence, the argument used for M' above shows that $\mathrm{P}=(P^{\theta-})'\in\mathcal{M}(\mathbb{F},\mathbb{P})$. In addition, $\mathrm{P}=P$ before θ, hence on $[0,T]$ by Lemma 4. ∎

We suppose that the conditions (B) and (A) are satisfied on $[0, T]$, that $\theta > 0$ is a $\mathbb{G}$ totally inaccessible stopping time with compensator given as $d\mathbf{v}_t = \gamma_t dt$, for some $(\mathbb{G}, \mathbb{Q})$ intensity process γ, and that θ has a positive Azéma supermartingale on $[0, T]$, i.e. $\mathsf{S}_T > 0$. We write $\Gamma = \int_0^{\cdot} \gamma'_s ds$.
We denote the $\mathbb{P}$ expectation by $\widetilde{\mathbb{E}}$.

Theorem 5

For any $\mathbb{F}$ stopping time $\sigma \leq T$ and nonnegative $\mathcal{F}_\sigma$ measurable random variable χ,

$$\mathbb{E}[\chi \mathbb{1}_{\{\sigma < \theta\}}] = \widetilde{\mathbb{E}}[\chi e^{-\Gamma_\sigma}].$$

For any nonnegative $\mathbb{F}$ predictable process K,

$$\mathbb{E}[K_\theta \mathbb{1}_{\{\theta \leq T\}}] = \widetilde{\mathbb{E}}[\int_0^T K_s e^{-\Gamma_s} \gamma'_s \, ds].$$

Connection with Collin-Dufresne et al. (2004)

- Assume that θ satisfies all the conditions in Collin-Dufresne et al. (2004, Theorem 1), i.e. is a positive time with absolutely continuous $\mathbb{Q}$ compensator $\mathbf{v} = \int_0^{\cdot\wedge\theta} \frac{1}{S_{s-}} dD_s$ (cf. Jeulin (1980)) on $[0, \theta]$, such that $e^{\mathbf{v}_\theta}$ is $\mathbb{Q}$ integrable.
- Under the conditions, Collin-Dufresne et al. (2004) introduce the "default pricing measure" $\overline{\mathbb{P}}$ with the density process $e^{\mathbf{v}} \mathbb{1}_{[0,\theta)}$.
- Since $\{\theta \leq T\}$ has zero probability under $\overline{\mathbb{P}}$, Collin-Dufresne et al. (2004) are able to propose a reinterpretation of the Duffie et al. (1996) formula under $\overline{\mathbb{P}}$, exempt from no-jump condition.
- However, $\overline{\mathbb{P}}$ is not equivalent to $\mathbb{Q}$, even under the classical immersion setup.

Theorem 6

Under the conditions of Collin-Dufresne et al. (2004, Theorem 1), assuming the condition (B) and $S_T > 0$, then θ is an invariance time and the corresponding invariance probability measure $\mathbb{P}$ is the restriction to $\mathcal{F}_T$ of the default pricing measure $\overline{\mathbb{P}}$.

Proof. Projections, integrability and Theorem 2. ∎

- Hence, under these conditions, the "right" pricing measure for defaultable securities, in terms of which one obtains tractable pricing equations without restrictive immersion or so conditions, is (essentially) the same for any subfiltration $\mathbb{F} \subseteq \mathbb{G}$ satisfying the condition (B).
- Namely, it is the (restriction to $\mathcal{F}_T$ of the) default pricing measure $\overline{\mathbb{P}}$ in Collin-Dufresne et al. (2004).
- This is one more reason that justifies the "invariance" terminology.
- Even though the default pricing measure $\overline{\mathbb{P}}$ is not equivalent to $\mathbb{Q}$, it can be viewed as (the extension of) an equivalent martingale measure $\mathbb{P}$, calling $\mathbb{P}$ a martingale measure in view of its invariance property.
- On top of making the connection with Collin-Dufresne et al. (2004), Theorem 6 provides a mild and tractable invariance time sufficiency condition.

Connection with pseudo-stopping times

- This part gives examples which illustrate how the condition (A) can be satisfied in cases where $(\mathbb{F}, \mathbb{P})$ martingales really jump at θ, as well as the connection between the condition (A) and the notion of pseudo-stopping time in Nikeghbali and Yor (2005).
- Consider a $(0, +\infty)$ valued random time θ. It is an $(\mathbb{F}, \mathbb{Q})$ pseudo-stopping time if and only if X^{θ} is a $(\mathbb{G}, \mathbb{Q})$ uniformly integrable martingale for any bounded $\mathbb{F}$ martingale X (cf. Nikeghbali and Yor (2005)).
- Clearly, if a pseudo-stopping time θ avoids the $\mathbb{F}$ stopping times, then it is an invariance time satisfying the condition (A) for any positive constant T, with invariance probability measure $\mathbb{P} = \mathbb{Q}$.

- More generally, let A denote the $\mathbb{F}$ dual optional projection of $\mathbb{1}_{[\theta,\infty)}$. Nikeghbali and Yor (2005) show that θ is a pseudo-stopping time if and only if $\mathsf{S} = 1 - \mathsf{A}$.
- By comparison, Proposition 2 2) shows that $\mathbb{P} = \mathbb{Q}$ is an invariance probability measure for any positive constant T if and only if $\mathsf{S} = 1 - \mathsf{D}$ (noting that $S_0 = 1$ here, as $\theta > 0$).
- Both conditions coincide if and only if $\mathsf{A} = \mathsf{D}$.
- We recall that in the case where θ is a $\mathbb{G}$ totally inaccessible stopping time, $\mathsf{A} = \mathsf{D}$ if and only if θ avoids the $\mathbb{F}$ stopping times. Hence, as soon as θ does not have the avoidance property, we have two similar, but "orthogonal" in a sense, characterizations.
- The difference is due to the fact that a pseudo-stopping time is defined in terms of stopping at θ, whereas invariance is defined in terms of stopping at $(\theta-)$.
- Having said this regarding the case where $\mathbb{P} = \mathbb{Q}$, we emphasize that, with respect to a pseudo-stopping time that is defined with respect to the fixed probability measure $\mathbb{Q}$, the additional flexibility of invariance times lies in the possibility to consider the martingale property under a changed measure $\mathbb{P}$.
  - In fact, the pseudo-stopping time condition is very restrictive. By contrast Theorem 6 shows that invariance times are the rule rather the exception.

Example 2 (An invariance time intersecting $\mathbb{F}$ stopping times..)

- For $i = 1, 2$, let $\mu_i > 0$ be a finite $\mathbb{F}$ stopping time with bounded compensator $\mathbf{v}_i$. Assuming $\mu_2 > T$, define $\theta = \mathbb{1}_A \mu_1 + \mathbb{1}_{A^c} \mu_2$, which intersects the $\mathbb{F}$ stopping times μ_i, for some $A \in \mathcal{G}_\infty$ independent from $\mathcal{F}_\infty$ such that $\alpha = \mathbb{Q}(A) \in (0, 1)$.
- On $[0, T]$, $\mathsf{S} = \mathbb{1}_{[0,\mu_1)}\alpha + \mathbb{1}_{[0,\mu_2)}(1-\alpha)$, $\mathsf{S}_- \geq 1 - \alpha$, and $\mathbf{v} = \int_0^{\cdot\wedge\theta} \frac{1}{\mathsf{S}_{s-}} d\mathsf{D}_s \leq \frac{1}{1-\alpha}\mathsf{D}$ is bounded. Therefore the conditions of Theorem 6 are fulfilled and θ is an invariance time.
- Easy computations yield

$$\mathsf{A} = (\mathbb{1}_{[\theta,\infty)})^o = \mathbb{1}_{[\mu_1,\infty)}\alpha + \mathbb{1}_{[\mu_2,\infty)}(1-\alpha), \quad \mathsf{A}_\infty \equiv 1,$$

so that, by application of Theorem 1 (3) in Nikeghbali and Yor (2005), θ is also a pseudo-stopping time.

Example 3 (..which is not a pseudo stopping time)

- Now, to obtain an invariance time θ intersecting $\mathbb{F}$ stopping times without being a pseudo-stopping time, one can set

$$\theta = \mathbb{1}_{A_1}\mu_1 + \mathbb{1}_{A_2}\mu_2 + \mathbb{1}_{A_3}\tau,$$

  for a non pseudo-stopping time τ and a partition $A_i, i = 1, 2, 3$, independent from $\mathcal{F}_\infty$ and τ.
- With $\alpha_i = \mathbb{Q}(A_i) > 0$, we have

$$\mathsf{A} = (\mathbb{1}_{[\theta,\infty)})^o = \alpha_1\mathbb{1}_{[\mu_1,\infty)} + \alpha_2\mathbb{1}_{[\mu_2,\infty)} + \alpha_3(\mathbb{1}_{[\tau,\infty)})^o,$$

  where $(\mathbb{1}_{[\tau,\infty)})^o_\infty \neq 1$, hence $\mathsf{A}_\infty \neq 1$, with positive $\mathbb{Q}$ probability. so that, by the converse part in the above mentioned theorem, θ is not a pseudo-stopping time.
- But the Azéma supermartingale of θ is given by

$$\mathsf{S} = \mathbb{1}_{[0,\mu_1]}\alpha_1 + \mathbb{1}_{[0,\mu_2]}\alpha_2 +^o (\mathbb{1}_{[0,\tau)})\alpha_3 \geq \alpha_2 \text{ on } [0, T].$$

  Hence, the other computations above do not change, which shows that θ is an invariance time.

Counterparty risk on credit derivatives

- Copula model of $\theta_0, \theta_1, \ldots, \theta_n$, where $\theta_0 = \theta$ corresponds to the default time of the counterparty of a bank in credit derivatives on names $1, \ldots, n$
- Counterparty risk computations: need make the model dynamic by introduction of a suitable model filtration $\mathbb{G}$
- Can one separate the information that comes from θ_0 from a reference filtration $\mathbb{F}$?

  $\rightarrow$ Reduction of filtration in this sense

  - Not unrelated with, but different from, filtration shrinkage, whereby Föllmer and Protter (2011) project local martingales onto smaller filtrations
- For applications, some kind of martingale invariance property is required, but under minimal assumptions, so that the model stays as flexible

$\rightarrow$ Invariance times

Dynamic copula models

- Dynamic Gaussian copula model
  - $\theta = \theta_0$ is an invariance time
  - (A) achieved with $\mathbb{P} \neq \mathbb{Q}$ and $\mathbb{G}$ equal to the classical progressive enlargement of $\mathbb{F}$ by θ_0
  - "wrong-way risk"
- Dynamic Marshall-Olkin copula (common-shock) model
  - $\theta = \theta_0$ is an invariance time
  - (A) achieved with $\mathbb{P} = \mathbb{Q}$ but $\mathbb{G}$ greater than the classical progressive enlargement of $\mathbb{F}$ by θ
  - "gap risk"

Based on

Crépey, S. and S. Song (2015b).
Invariance times.
Submitted (available on math.maths.univ-evry.fr/crepey).

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ANNOUNCEMENT: Postdoc position

- Starting as soon as possible and ending 31 December 2017.
- The position will be based at Université d'Evry, near Paris, and will involve interactions with the team of Ecole Polytechnique as well as with financial practitioners.
- Candidates willing to develop high-quality research on topics such as financial imperfections, new market structures (central counterparties in particular) and risk more generally are invited to apply.
- A solid mathematical finance profile as well as a strong appeal and ability for applications are expected. There are no teaching duties.
- Gross salary range: €30,000 – €36,000 per annum.
- Candidates holding a PhD (or near completion) should send their CV along with the names of two referees, a research statement and two research papers by email to stephane.crepey@univ-evry.fr. Referee letters should be sent directly to the same email address.
- The review of applications will continue until the position is filled.

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