Arbitrage and utility maximisation in market models with insider

Huy N. Chau, Wolfgang Runggaldier and **Peter Tankov**

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Standard model: No Arbitrage / NFLVR

Filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual hypotheses.

Finite planning horizon $T < \infty$, all processes are constant after T.

One riskless asset $S^0 \equiv 1$, d risky assets S (nonnegative semimartingales).

A strategy H is x-admissible ($H \in \mathcal{A}_x^{\mathbb{F}}$) if H is $\mathbb{F}$-predictable, S-integrable and $(H \cdot S)_t \geq -x$, $\mathbb{P}$-a.s.; H is admissible if H is x-admissible for some $x > 0$.

$\mathcal{C}$ contains the bounded claims which can be superreplicated with admissible strategies at zero cost.

- No Arbitrage (NA) holds if $\mathcal{C} \cap L_+^\infty = \{0\}$.
- No Free Lunch with Vanishing Risk (NFLVR) holds if $\overline{\mathcal{C}} \cap L_+^\infty = \{0\}$, where the bar denotes the closure in the supnorm topology of L^∞.

Yes, but there are markets with arbitrage

The NFLVR assumption may be too strong, especially in insider trading context, and part of math finance literature, focuses on markets with arbitrage.

Natural condition: No Unbounded Profit with Bounded Risk (NUPBR):

$$\lim_{c\to\infty} \sup_{H\in\mathcal{A}_1^{\mathbb{F}}} \mathbb{P}[(H\cdot S)_T > c] = 0$$

- Equivalent to the absence of arbitrage of the 1st kind (NA1): $\xi \in \mathcal{F}_T$ with $\mathbb{P}(\xi \geq 0) = 1$ and $\mathbb{P}(\xi > 0) > 0$ such that for all $x > 0$ there exists $H \in \mathcal{A}_x^{\mathbb{F}}$ with $V_T^{x,H} \geq \xi$.
- “minimal” condition of market viability: Karatzas and Kardaras (2007) show that if NUPBR fails, the utility maximization problem either does not have a solution or has infinitely many.

How to compare arbitrage opportunities?

Definition

Given a claim $f \geq 0$, we define

$$x_*^{\mathbb{F}}(f) := \operatorname{ess\,inf}\left\{x \in \mathcal{F}_0 : \exists H \in \mathcal{A}_x^{\mathbb{F}}, V_T^{x,H} \geq f, \mathbb{P} - a.s.\right\}$$

In general $x_*^{\mathbb{F}}(1) \leq 1$ a.s.

We say that the market admits optimal arbitrage if $\mathbb{P}[x_*^{\mathbb{F}}(1) < 1] > 0$.

If $\mathbb{P}[x_*^{\mathbb{F}}(1) < 1] = 1$, the optimal arbitrage is said to be strong.

(this distinction is important for insider trading when $\mathcal{F}_0$ is not trivial)

Filtration enlargement and arbitrage

- For a $G \in \mathcal{F}$, define the initially enlarged filtration $\mathbb{G}$ as the right-continuous augmentation of the filtration $\mathcal{G}_t^0 := \mathcal{F}_t \vee \sigma(G)$.
- Let γ be the law of G and γ_t be the regular $\mathcal{F}_t$-conditional law of G. Jacod's condition is satisfied for $t \in \mathbb{T}$ when $\gamma_t \ll \gamma$ a.s. for all $t \in \mathbb{T}$.

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- Weaker notions of arbitrage such as NUPBR are particularly important in this context since NFLVR is rarely stable under change of filtration.
- Stability of NUPBR has been studied in Acciaio et al. (2014) in the context of *progressive* filtration enlargement and *initial* filtration enlargement on infinite time horizon under Jacod's condition for $t \in [0, \infty)$.
- Previously many authors have studied initially enlarged models with finite log-utility (which implies NUPBR).

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- Initial enlargement on a finite time horizon $T < \infty$.
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Additional utility of insiders with initial information was studied by many authors including Amendinger et al. (1998); Ankirchner (2005); Ankirchner et al. (2006); Grorud and Pontier (1998).

Preliminaries and assumptions

We assume that $G \in \mathcal{F}_T$ and that Jacod's condition is satisfied for $t \in [0, T)$.

$\Rightarrow$ there exists a conditional density process: a nonnegative $\mathcal{B} \otimes \mathcal{O}(\mathbb{F})$-measurable function $\mathbb{R} \times \Omega \times \mathbb{R}_+ \ni (x, \omega, t) \mapsto p_t^x(\omega) \in [0, \infty)$, càdlàg in t such that

1. for every $t \in [0, T)$, we have $\gamma_t(dx) = p_t^x(\omega)\gamma(dx)$.
2. for each $x \in \mathbb{R}$, the process $(p_t^x(\omega))_{t\in[0,T)}$ is a $(\mathbb{F}, \mathbb{P})$-martingale.

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Standing Assumptions

$\rightarrow$ *For every x, the process p^x does not jump to zero, i.e.*

$$\mathbb{P}[\tau^x < T, p^x_{\tau^x -} > 0] = 0.$$

$\rightarrow$ *The $(\mathbb{F}, \mathbb{P})$-market satisfies NFLVR.*

Problem setting

- We assume that G takes values $(g_i)_{i\geq 1}$ with nonzero probabilities.
- Initial enlargement with a discrete random variable is a classical case studied already in Meyer (1978) and by many other authors.
- Every $\mathbb{F}-$local martingale is a $\mathbb{G}-$semimartingale on $[0, T]$ and it is not necessary to impose Jacod's condition.
- The investment opportunities for the insider can be described with a non-equivalent measure change

$$\frac{d\mathbb{Q}^i}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{\mathbb{P}\{G = g_i \mid \mathcal{F}_t\}}{\mathbb{P}\{G = g_i\}} := p_t^{g_i}$$

- We can use the tools from Chau and Tankov (2015) where arbitrage opportunities under non-equivalent measure changes are studied.

NUPBR, Superhedging and optimal arbitrage

Theorem

Let G be discrete. Then,

- *The $(\mathbb{G},\mathbb{P})$-market satisfies NUPBR.*
- *The superhedging price for a claim $f \geq 0$ in the $(\mathbb{G},\mathbb{P})-$market is given by*

$$x_*^{\mathbb{G}}(f) = \sum_i x_*^{\mathbb{F}}(f\mathbf{1}_{\{G=g_i\}})\mathbf{1}_{\{G=g_i\}} = \sum_i \sup_{\mathbb{Q}\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{Q}}\left[f\mathbf{1}_{\{G=g_i\}}\right]\mathbf{1}_{\{G=g_i\}}$$

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$\Rightarrow$ there is optimal arbitrage for $(\mathbb{G},\mathbb{P})-$investor if and only if $x_*^{\mathbb{F}}(\mathbf{1}_{\{G=g_i\}}) < 1$ for some i.

$\Rightarrow$ In a *complete market* there is always strong optimal arbitrage.

Utility maximization for an insider

Given a concave and strictly increasing utility function $U(\cdot)$, the corresponding portfolio optimization problem is given by

$$u(x) := \sup_{H \in \mathcal{A}_x} \mathbb{E}\left[U(V_T^{x,H})\right]$$

Lemma (Consequence of Karatzas and Kardaras (2007))

Assume that the utility function is strictly increasing, concave, and satisfies $U(+\infty) = +\infty$. If there exists $x > 0$ for which $u(x) < +\infty$, then NUPBR holds.

Utility maximization for an insider: general case

Theorem

Let G be discrete and suppose (in addition to Standing Assumptions) that

(i) *The function $U : (0,\infty) \to \mathbb{R}$ is strictly concave, increasing, continuously differentiable and satisfies the Inada conditions at 0 and ∞.*

(ii) *For every $y \in (0,\infty)$, there exists $Z : Z\mathbb{P} \in ELMM(\mathbb{F},\mathbb{P})$ with $\mathbb{E}^{\mathbb{P}}[V(yZ)] < \infty$, where $V(y) = \sup_x (U(x) - xy)$.*

Then,

$$\sup_{H \in \mathcal{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[U(V_T^{1,H})] = \sum_i \inf_{y>0} \left\{ y + \inf_{Z : Z\mathbb{P} \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[V(yZ_T)\, 1_{G=g_i} \right] \right\}.$$

Utility maximization for an insider: power utility

Corollary

Under the assumptions of the above theorem, let $\gamma \in (0,1)$ *and suppose that there exists* $Z : Z\mathbb{P} \in ELMM(\mathbb{F},\mathbb{P})$ *with* $\mathbb{E}^{\mathbb{P}}[(Z_T)^{-\frac{\gamma}{1-\gamma}}] < \infty$*. Then,*

$$\sup_{H \in \mathcal{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[(V_T^{1,H})^\gamma] = \sum_i \left\{ \inf_{Z : Z\mathbb{P} \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}} \left[(Z_T)^{-\frac{\gamma}{1-\gamma}} 1_{G=g_i} \right] \right\}^{1-\gamma}.$$

Utility maximization for an insider: logarithmic utility

Corollary

Under the assumptions of the above theorem, suppose that there exists $Z : Z\mathbb{P} \in ELMM(\mathbb{F}, \mathbb{P})$ *with* $\mathbb{E}^{\mathbb{P}}[\log Z_T] > -\infty$. *Then*

$$\sup_{H \in \mathcal{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}] = -\sum_i \mathbb{P}[G = g_i] \log \mathbb{P}[G = g_i],$$
$$+ \sum_i \inf_{Z : Z\mathbb{P} \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}} \left[1_{G=g_i} \log \frac{1}{Z_T} \right].$$

This corollary extends the results obtained in Amendinger et al. (1998); Ankirchner (2005) to the case of incomplete markets.

Definitions

- Let N^1 and N^2 be two independent standard Poisson processes. We consider a financial market with risky asset price $S_t = e^{N_t^1 - N_t^2}$ so that

$$dS_t = S_{t-}\left((e-1)dN_t^1 + (e^{-1}-1)dN_t^2\right), \qquad S_0 = 1, \qquad t \in [0, T].$$

  Public information $\mathbb{F}$ is generated by N^1 and N^2.

- Define $N_t := N_t^1 - N_t^2$ and assume that the insider knows the value of N_T, and hence S_T, at $t = 0$, so that $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(N_T) = \mathcal{F}_t \vee \sigma(S_T)$.

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- High-frequency trading context: price moves are one tick only and the informed trader knows the future price because, e.g., she has detected that a large order is being filled.

Preliminaries

- The $(\mathbb{F},\mathbb{P})$-market is incomplete and satisfies the NFLVR condition.
- The density Z of any equivalent local martingale measure is of the form

$$dZ_t = Z_{t-}\left((\alpha_t^1 - 1)(dN_t^1 - dt) + (\alpha_t^2 - 1)(dN_t^2 - dt)\right),$$

where α^1, α^2 are positive integrable processes satisfying $\alpha_t^1 = e^{-1}\alpha_t^2$.

- The intensities of N^1 and N^2 in the $\mathbb{G}$-filtration are given by

$$\lambda_t^{\mathbb{G},1} = \frac{I_{|N_T - N_t - 1|}(2(T-t))}{I_{|N_T - N_t|}(2(T-t))} \quad \text{and} \quad \lambda_t^{\mathbb{G},2} = \frac{I_{|N_T - N_t + 1|}(2(T-t))}{I_{|N_T - N_t|}(2(T-t))},$$

where I is the modified Bessel function of the 1st kind.

Optimal arbitrage

To check whether optimal arbitrage exists, we compute, for every $x \in \mathbb{Z}$,

$$x_*^{\mathbb{F},\mathbb{P}}(1_{N_T=x}) = \sup_{\overline{\mathbb{P}} \in ELMM(\mathbb{F},\mathbb{P})} \overline{\mathbb{P}}[N_T = x].$$

Proposition

If $x \leq 0$,

$$x_*^{\mathbb{F},\mathbb{P}}(1_{N_T=x}) = \sup_{\overline{\mathbb{P}} \in ELMM(\mathbb{F},\mathbb{P})} \overline{\mathbb{P}}[N_T = x] = 1$$

and there is no optimal arbitrage. If $x > 0$,

$$x_*^{\mathbb{F},\mathbb{P}}(1_{N_T=x}) = \sup_{\overline{\mathbb{P}} \in ELMM(\mathbb{F},\mathbb{P})} \overline{\mathbb{P}}[N_T = x] = \frac{1}{e^x}.$$

Optimal strategy: buy and hold $\frac{1}{S_T}$ units of the risky asset.

Log utility of the insider

Proposition

$$\sup_{H\in\mathcal{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}] = \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left(\log\left(\frac{\lambda_t^{\mathbb{G},1}+\lambda_t^{\mathbb{G},2}}{e+1}\right)(\lambda_t^{\mathbb{G},1}+\lambda_t^{\mathbb{G},2}) - \lambda_t^{\mathbb{G},1} + 2\right) dt\right]$$

The supremum is attained by the strategy

$$\pi_t^{\mathbb{G}} = \frac{\lambda_t^{\mathbb{G},1}(e-1)+\lambda_t^{\mathbb{G},2}(e^{-1}-1)}{(e-1)(1-e^{-1})(\lambda_t^{\mathbb{G},1}+\lambda_t^{\mathbb{G},2})}$$

Log utility of the insider: idea of the proof

- Recall that the maximal expected log-utility of the insider is

$$\sup_{H\in\mathcal{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}] = -\sum_{x\in\mathbb{Z}} \mathbb{P}[N_T = x] \log \mathbb{P}[N_T = x]$$
$$- \sum_{x\in\mathbb{Z}} \sup_{Z:Z\mathbb{P}\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{N_T=x} \log Z_T\right].$$

- A specific strategy H gives a lower bound for the log utility.
- A specific ELMM density Z provides an upper bound.
- We will construct H and Z for which the two bounds coincide, thus solving the problem.

Upper bound for log utility via duality

We start from

$$\mathbb{E}^{\mathbb{P}}[1_{N_T=x}\log Z_T] = \mathbb{E}^{\mathbb{P}}\left[1_{N_T=x}\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{2}\int_0^T \log\alpha_t^i dN_t^i - \int_0^T(\alpha_t^i - 1)dt\,\middle|\,\mathcal{G}_0\right]\right].$$

Restricting the discussion to regular strategies and taking the expectation,

$$\mathbb{E}^{\mathbb{P}}[1_{N_T=x}\log Z_T] = \mathbb{E}^{\mathbb{P}}\left[1_{N_T=x}\sum_{i=1}^{2}\int_0^T\left(\lambda_t^{\mathbb{G},i}\log\alpha_t^i - (\alpha_t^i - 1)\right)dt\right]$$

This is maximized by $\alpha_t^1 = \frac{\lambda_t^{\mathbb{G},1}+\lambda_t^{\mathbb{G},2}}{e+1}$ so that

$$\sum_{x\in\mathbb{Z}}\sup_{Z:Z\mathbb{P}\in ELMM(\mathbb{F},\mathbb{P})}\mathbb{E}^{\mathbb{P}}[1_{N_T=x}\log Z_T]$$

$$\geq \mathbb{E}^{\mathbb{P}}\left[\int_0^T\left(\log\left(\frac{\lambda_t^{\mathbb{G},1}+\lambda_t^{\mathbb{G},2}}{e+1}\right)(\lambda_t^{\mathbb{G},1}+\lambda_t^{\mathbb{G},2}) - \lambda_t^{\mathbb{G},1} + 2\right)dt\right].$$

Lower bound for log utility via direct computation

Let $\pi^{\mathbb{G}}$ be a $\mathbb{G}$-predictable strategy $V^{1,\pi^{\mathbb{G}}}$ be the corresponding wealth process

$$\frac{dV_t^{1,\pi^{\mathbb{G}}}}{V_{t-}^{1,\pi^{\mathbb{G}}}} = \pi_t^{\mathbb{G}} \frac{dS_t}{S_{t-}} = \pi_t^{\mathbb{G}} \left((e-1)dN_t^1 + (e^{-1}-1)dN_t^2\right).$$

Using regular strategies and taking the expectation, we get

$$\mathbb{E}^{\mathbb{P}}\left[\log V_T^{1,\pi^{\mathbb{G}}}\right] = \mathbb{E}^{\mathbb{P}} \int_0^T \left(\log\left(1+(e-1)\pi_t^{\mathbb{G}}\right)\lambda_t^{\mathbb{G},1} + \log\left(1+(e^{-1}-1)\pi_t^{\mathbb{G}}\right)\lambda_t^{\mathbb{G},2}\right) dt.$$

This is maximized by

$$\pi_t^{\mathbb{G}} = \frac{\lambda_t^{\mathbb{G},1}(e-1) + \lambda_t^{\mathbb{G},2}(e^{-1}-1)}{(e-1)(1-e^{-1})(\lambda_t^{\mathbb{G},1}+\lambda_t^{\mathbb{G},2})}$$

Lower bound for log utility via direct computation

Substituting the candidate optimal strategy into portfolio dynamics, we get

$$\mathbb{E}^{\mathbb{P}}\left[\log V_T^{1,\pi^{\mathbb{G}}}\right] = -\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left((\lambda_t^{\mathbb{G},1} + \lambda_t^{\mathbb{G},1}) \log \frac{\lambda_t^{\mathbb{G},1} + \lambda_t^{\mathbb{G},2}}{e+1} - \lambda_t^{\mathbb{G},1} + 2\right) dt\right]$$

$$+ \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left(\lambda_t^{\mathbb{G},1} \log \lambda_t^{\mathbb{G},1} + \lambda_t^{\mathbb{G},2} \log(\lambda_t^{\mathbb{G},1}) - \lambda_t^{\mathbb{G},1} - \lambda_t^{\mathbb{G},2} + 2\right) dt\right].$$

To finish the proof, it remains to check that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left(\lambda_t^{\mathbb{G},1} \log \lambda_t^{\mathbb{G},1} + \lambda_t^{\mathbb{G},2} \log(\lambda_t^{\mathbb{G},1}) - \lambda_t^{\mathbb{G},1} - \lambda_t^{\mathbb{G},2} + 2\right) dt \middle| N_T = x\right]$$

$$= -\log \mathbb{P}[N_T = x], \quad \text{for } x \in \mathbb{Z}.$$

Problem setting

- We are now interested in the case of initial enlargement by a $\mathcal{F}_T$-measurable random variable G which is not purely atomic.
- Many authors starting with Pikovsky and Karatzas (1996) and Amendinger, Imkeller and Schweizer (1998) have shown in various settings that the utility of the insider is infinite.
- We provide a simple but general result which explains why (and when) this is the case.

A sufficient condition for arbitrage of the first kind

Proposition (Arbitrage of the first kind)

Assume that

- *The law of G is not purely atomic,*
- *The set* $\{Z_T = \frac{d\overline{\mathbb{P}}}{d\mathbb{P}} : \overline{\mathbb{P}} \in ELMM(\mathbb{F}, \mathbb{P})\}$ *is uniformly integrable.*

Then there exists an arbitrage of the first kind for the insider.

- We may only hope to obtain NUPBR when the $\mathbb{F}$-market is incomplete and the set of ELMM is *large enough*.

Incomplete market with UI martingale densities

We consider a risky asset whose (discounted) price evolves as

$$dS_t = S_{t-}\sigma(t)(\theta dN_t^1 + (1-\theta)dN_t^2 - dt)$$

where $\theta \in (0,1)$, $\sigma(t)$ is continuous and not constant on $[0,T]$ and N^1 and N^2 are independent standard Poisson. Any martingale density has the form

$$Z_T = \mathcal{E}\left(\int_0^T (\phi_t^1 - 1)(dN_t^1 - dt)\right)\mathcal{E}\left(\int_0^T (\phi_t^2 - 1)(dN_t^2 - dt)\right),$$

where ϕ^1 and ϕ^2 satisfy $\theta\phi_t^1 + (1-\theta)\phi_t^2 = 1, \mathbb{P} - a.s.$
Therefore, $0 \leq \phi^1 \leq \frac{1}{\theta}$, $0 \leq \phi^2 \leq \frac{1}{1-\theta}$ so that

$$Z_T = e^{-\int_0^T (\phi_t^1 + \phi_t^2 - 2)dt} \prod_{i=1}^{N_T^1} \phi_{t_i}^1 \prod_{j=1}^{N_T^2} \phi_{t_j}^2 \leq e^{2T} \frac{1}{\theta^{N_T^1}} \frac{1}{(1-\theta)^{N_T^2}}.$$

$\Rightarrow$ the set of martingale densities is uniformly integrable.

Example of a market satisfying NUPBR

- Examples of insider markets with initial information about S_T and finite logarithmic utility are given in Kohatsu-Higa and Yamazato (2011).
- Let $S = S_0 e^{X_t}$ where X is a finite variation square integrable Lévy process with Lévy measure ν.
- If $\nu((0,\infty)) > 0$ and $\nu((-\infty,0)) > 0$ then every admissible strategy is bounded and the expected exponential utility of every admissible portfolio is bounded by a universal constant, so that NUPBR is satisfied.

Superhedging for non discrete G: approximation

- To compute the superhedging price for a general G, one may approximate G with a sequence of discrete random variables G^n such that

$$\sigma(G^n) \subset \sigma(G^{n+1}) \subset \cdots \subset \sigma(G) = \sigma\left(\cup_{n\geq 1}\sigma(G^n)\right)$$

  We then define an increasing sequence of filtrations $(\mathbb{G}^n)$ where $\mathbb{G}^n$ corresponds to the initial enlargement of $\mathbb{F}$ with G^n.
- This procedure was used in Ankirchner (2005) to show convergence of utility functions under the assumption that S is continuous.
- It may be extended to show convergence of superhedging prices.
- Search for alternative assumptions allowing to treat discontinuous semimartingales is in progress.

Conclusion

Types of arbitrage in insider trading models with $G \in \mathcal{F}_T$, $T < \infty$

	Discrete G	Non discrete G
Complete market	NUPBR, Strong optimal arbitrage	Arbitrage of the 1st kind
Incomplete market	NUPBR, optimal arbitrage if $x_*^F(\mathbf{1}_{G=g_i}) < 1$	Arbitrage of the 1st kind if $ELMM(\mathbb{F}, \mathbb{P})$ uniformly integrable, examples where NUPBR holds

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