THE REDUCED DIRAC STRUCTURE OF GENERAL RELATIVITY ON MANIFOLDS WITH CORNERS

ALBERTO S. CATTANEO, FILIPPO FILA-ROBATTINO AND MANUEL TECCHIOLLI

Abstract. In this paper, the corner Poisson structure of four-dimensional Palatini–Cartan gravity is derived. Building on the classical description of gravity on manifolds with boundary, specifically on the boundary constraint algebra, a pre- Dirac structure on the space of corner fields is obtained together with a reduction procedure that yields a maximal Dirac structure, identified as the graph of a Poisson bivector field, on the reduced space of corner fields. It is further shown that this Poisson structure admits an equivalent affine Poisson description, which naturally exhibits the reduced corner theory as a BF-like theory and leads to a $\mathrm{BF}^2\mathrm{V}$ formulation. This provides the basis for a unified framework for the bulk, boundary, and corner structures of Palatini–Cartan gravity.

Contents

1. Introduction 2
Structure of the paper 3
Acknowledgments 4
2. Geometrical background 4
2.1. Stratified spaces 4
2.2. Graded symplectic and Poisson geometry 6
2.3. The $\mathrm{BF}^k\mathrm{V}$ formalism 8
2.4. Generalized geometry 9
2.4.1. Dirac Structures 11
3. Dirac corner structures in classical field theories 11
3.1. Classical Lagrangian field theories on manifolds with corners 11
3.1.1. The boundary structure 12
3.1.2. The corner structure 13
3.2. Examples 17
3.2.1. Yang–Mills theory 17
3.2.2. BF theory 18
4. Review of PC gravity on manifolds with boundary 20
4.1. The Palatini–Cartan theory 20
4.2. Boundary structure of PC gravity 22
5. PC gravity on manifolds with corners 25
5.1. Pre-Dirac corner structure of PC gravity 26
5.2. Maximality and reduction 29
5.2.1. The reduced space of corner fields 31
5.2.2. Bianchi identities and boundary fields revisited 33
5.3. Involutivity and isotropy of the induced (pre-)Dirac structure on P_Γ 35
5.4. The Poisson structure on P_Γ 36
6. The $\mathrm{BF}^2\mathrm{V}$ structure of PC gravity 37
7. Conclusion and Outlook 41
Appendix A. Useful results and notation 42
Appendix B. Long computations 44
B.1. Proof of Proposition 5.7 44
B.2. Proof of Theorem 5.29 48

We acknowledge partial support of the SNF Grant No. 200021 227719 and of the Simons Collaboration on Global Categorical Symmetries. This research was (partly) supported by the NCCR SwissMAP, funded by the Swiss National Science Foundation. This article is based upon work from COST Action 21109 CaLISTA, supported by COST (European Cooperation in Science and Technology)(www.cost.eu), MSCA-2021-SE-01-101086123 CaLIGOLA, and MSCA-DN CaLiForNIA -101119552. FFR acknowledges funding from the EU project Caligola HORIZON-MSCA-2021-SE-01, Project ID: 101086123.

References 52

1. Introduction

The study of gauge theories on manifolds with boundary has revealed a rich interplay between variational principles, symplectic geometry, and homological methods. In particular, the BV–BFV formalism provides a coherent framework in which bulk dynamics, boundary geometry and the corresponding reduced phase space can be treated simultaneously. Starting from a classical gauge theory in the bulk, it is possible to associate a BFV theory to the boundary, encoding the symplectic structure and the constraint algebra arising from gauge invariance. Under suitable assumptions, the degree-zero BFV cohomology reproduces the algebra of observables on the reduced phase space.

There is a canonical way of relating bulk and boundary gauge theories. Starting from a BV theory on a manifold with boundary, the failure of the BV action to satisfy the master equation is encoded by boundary contributions, which naturally give rise to BFV data on the boundary. In favorable situations, this leads to a compatible BV–BFV theory and provides a homological description of the reduced phase space.

For four-dimensional Palatini–Cartan gravity, however, this program encounters substantial difficulties. As shown in [CS19c], the singular nature of the boundary presymplectic structure obstructs a direct implementation of the BV–BFV construction. A different strategy was therefore developed in [CCS21a], where one starts from the classical boundary theory and studies the reduction of a coisotropic submanifold of boundary fields. The resulting reduced phase space is then described cohomologically by a BFV theory, providing a resolution of the corresponding quotient. More recently, it was shown in [CC25] that this BFV theory can also be recovered from the bulk BV description after a suitable BV pushforward procedure. Equivalently, the same structure arises from an AKSZ-type construction associated with the BFV data. In this way, the classical and BV descriptions are ultimately reconciled, despite the failure of the naive BV–BFV construction.

Many situations of physical and mathematical interest, however, naturally involve manifolds with corners. Examples include locality and gluing constructions in classical and quantum field theory, the decomposition of spacetime into finite regions, and the study of observables associated with lower-dimensional strata. In these situations, the boundary itself acquires a non-trivial stratification, and one is naturally led to ask whether the structures appearing at codimension one admit analogous extensions to higher codimensions. This question is naturally formulated within the framework of $\mathrm{BF}^{(k)}\mathrm{V}$ theories, where the compatibility between symplectic and homological structures is expected to persist along the entire stratification.

The corner structure of four-dimensional Palatini–Cartan gravity was investigated in [CC24] from the $\mathrm{BF}^k\mathrm{V}$ perspective. Starting from the BFV theory associated with the boundary, the induced codimension-two structure was shown to carry a local P_∞-algebra and to define a pre-$\mathrm{BF}^2\mathrm{V}$ theory. The resulting corner presymplectic form remains singular, and the codimension-two theory is therefore naturally described before reduction. While this construction captures the full homotopy structure inherited from the BFV theory, it does not lead directly to a strict $\mathrm{BF}^2\mathrm{V}$ theory.

The perspective adopted in the present work is instead inspired by the classical construction of [CCS21a]. Rather than starting from the BFV extension and studying the corner theory induced by it, we begin with the classical boundary constraint algebra and investigate the geometric structure generated at codimension two. Our primary aim is to construct and characterize the reduced classical corner geometry, in direct analogy with the construction of the reduced boundary phase space in [CCS21a]. By performing the relevant reduction at the classical level, we remove the degeneracies responsible for the singular corner geometry and obtain a genuine geometric structure on the reduced space of corner fields. The corresponding $\mathrm{BF}^2\mathrm{V}$ structure is not imposed as an initial target, but emerges naturally from the resulting geometry.

More precisely, starting from the boundary constraint algebra of Palatini–Cartan gravity, we show that the residual codimension-two contributions define a pre-Dirac structure on a suitable space of corner fields. Using techniques from generalized geometry, we identify a reduction procedure that removes the obstruction to maximality and yields a genuine Dirac structure on a reduced corner phase space. We then prove that this Dirac structure is the graph of a Poisson bivector, thereby providing a geometric characterization of the reduced corner degrees of freedom. Since the resulting Poisson bivector naturally determines the corresponding homological data, a strict $\mathrm{BF}^2\mathrm{V}$ structure can then be obtained directly from the reduced corner geometry.

Structure of the paper. The paper is organized as follows. In Section 2, we recall the geometric background used throughout the work. We begin with stratified manifolds and manifolds with corners, in order to fix the distinction between bulk, boundary and corner strata. We then review the symplectic and Poisson language needed to describe reduced phase spaces, with particular emphasis on coisotropic reduction, first-class constraints and moment maps. This is followed by a brief recollection of the $\mathrm{BF}^{(k)}\mathrm{V}$ formalism, which provides the cohomological framework organizing the data attached to strata of increasing codimension. Finally, we review the standard Courant algebroid, Dirac structures, and the two basic classes of Dirac structures arising as graphs of closed two-forms and Poisson bivectors. This section fixes the geometric dictionary used later: symplectic geometry describes the codimension-one phase space, while generalized geometry is the natural language for the codimension-two structure.

In Section 3 we explain the general mechanism by which corner structures arise in classical field theory. Starting from a Lagrangian field theory on a manifold with boundary, the variation of the action produces a presymplectic form on the space of boundary fields. After quotienting the kernel of this presymplectic form, one obtains a symplectic boundary field space. The equations of motion then define a coisotropic submanifold of admissible boundary data; equivalently, they give first-class constraints. On a manifold with boundary, these constraints are represented by Hamiltonian vector fields and their coisotropic reduction gives the usual reduced phase space of Cauchy data. When we deal with a manifold with corners, however, integration by parts on the boundary produces residual one-forms supported on the corner. Thus a boundary constraint no longer determines only a vector field: it determines a generalized section

$$X_i + \beta_i \in T\mathcal{F}_\Gamma \oplus T^*\mathcal{F}_\Gamma.$$

The vector part records the local transformation of corner fields, while the one-form part records the variation of the corner charge (when it is exact) or the flux associated with the global part of the field transformation. We show that these pairs naturally form a generalized distribution in the standard Courant algebroid, and we formulate the two basic conditions required of it: isotropy with respect to the Courant pairing and involutivity with respect to the Dorfman bracket. This gives the abstract notion of a pre-Dirac corner structure. We then describe the reduction by the directions in the kernel of the residual one-forms, which removes the obstruction to maximality and may turn the pre-Dirac structure into a genuine Dirac structure. The general construction is illustrated in two model examples: Yang–Mills theory, where the reduction can be seen explicitly and leads to a Poisson corner structure, and four-dimensional BF theory, where the classical corner structure is already Poisson but the full cohomological description also requires the higher P_∞ data coming from reducibility.

In Section 4 we review the boundary structure of four-dimensional Palatini–Cartan gravity. We recall the bulk fields

$$(e, \omega) \in \Omega^1_{\mathrm{nd}}(M, V) \oplus \mathcal{A}_M$$

and the Palatini–Cartan action. Its variation gives the boundary Noether one-form

$$\alpha_\Sigma = \int_\Sigma \frac{1}{2} e^2\, \delta\omega$$

and hence a degenerate presymplectic form on pre-boundary fields. We recall the first reduction which produces the geometric boundary phase space $(\mathcal{F}_\Sigma, \omega_\Sigma)$. We then describe the constraint surface of admissible Cauchy data in terms of the three families of boundary constraints

$$L_c, \qquad P_\xi, \qquad H_\lambda,$$

corresponding respectively to internal Lorentz transformations, diffeomorphisms tangent to the boundary, and normal deformations. These constraints are first-class and define a coisotropic submanifold of the boundary phase space. We also recall the BFV description of this boundary theory, stressing the role of the constraint algebra and the obstruction that appears when one attempts to descend directly to a strict codimension-two $\mathrm{BF}^2\mathrm{V}$ theory.

Section 5 contains the main construction of the paper. Restricting the boundary constraints to the corner produces four residual one-forms,

$$\mathcal{J}_c, \qquad \mathcal{E}_\zeta, \qquad \mathcal{K}_\eta, \qquad \mathcal{F}_\lambda,$$

where the boundary vector field decomposes at the corner as

$$\xi|_\Gamma = \zeta + \eta\partial_m.$$

These one-forms are paired with the push-forwards of the Hamiltonian vector fields of the corresponding boundary constraints. In this way we obtain a generalized distribution

$$\tilde{D} = \operatorname{span}\big\{ \mathbb{J}_c + \mathcal{J}_c,\, \mathbb{E}_\zeta + \mathcal{E}_\zeta,\, \mathbb{K}_\eta + \mathcal{K}_\eta,\, \mathbb{F}_\lambda + \mathcal{F}_\lambda \big\} \subset T\mathcal{F}_\Gamma \oplus T^*\mathcal{F}_\Gamma.$$

We prove that this distribution is isotropic and involutive; hence it is a pre-Dirac structure on the space of corner fields. We then identify the kernel of the one-form components, namely the directions of the pre-corner fields that are invisible to all corner charge variations. Quotienting by this distribution removes the obstruction to maximality. The reduced corner fields are described in terms of the natural variables

$$E = \frac{1}{2}e^2, \qquad \Omega = e(\omega - \omega_0),$$

together with transverse data fixed by the torsion, Einstein and Bianchi relations. On the resulting reduced corner space P_Γ, the induced Dirac structure is shown to be maximal and, more precisely, to be the graph of a Poisson bivector:

$$D = \operatorname{Graph}(\pi).$$

This provides the reduced Poisson geometry of the Palatini–Cartan corner structure.

Finally, in Section 6, we pass from the reduced Poisson geometry to the strict $\mathrm{BF}^2\mathrm{V}$ description. Since a Poisson bivector can be encoded as a degree-two function satisfying the classical master equation on the shifted cotangent bundle of the reduced corner space, the Poisson structure on P_Γ naturally determines a $\mathrm{BF}^2\mathrm{V}$ theory. We first write the $\mathrm{BF}^2\mathrm{V}$ data associated with the Poisson bivector obtained in Section 5. We then perform a field redefinition which exhibits the same structure in an affine Poisson form. In this form the $\mathrm{BF}^2\mathrm{V}$ action becomes

$$S_\Gamma^{PC} = \int_\Gamma \frac{1}{2}E[c,c] + \mu\, e\, d_{\omega_0} c + \frac{1}{2}\mu^2 F_{\omega_0},$$

where the ghost μ packages the diffeomorphism and normal-deformation parameters through

$$\mu = \iota_\zeta e + \lambda\epsilon_n + \eta\epsilon_m.$$

This affine formulation makes the relation with four-dimensional BF theory transparent and identifies the affine term as the classical source of the central-extension data relevant for the associated corner algebra.

Acknowledgments. The authors would like to thank Francesco Bonechi and Giovanni Canepa for the illuminating remarks.

2. Geometrical background

2.1. **Stratified spaces.**

Definition 2.1 (Stratified manifold[1])**.** A *stratified manifold* is a topological space X equipped with a decomposition

$$X = \bigsqcup_{\alpha \in A} S_\alpha,$$

where each S_α is a (locally closed) smooth manifold of dimension d_α, called a *stratum*, and the following conditions hold:

(1) Each S_α is a smooth manifold.
(2) For any two strata S_α, S_β, if

$$S_\beta \cap \overline{S_\alpha} \neq \varnothing,$$

then $S_\beta \subset \overline{S_\alpha}$ and $\dim S_\beta < \dim S_\alpha$. This is sometimes called the *frontier condition*.
(3) The topology of X is such that each closure $\overline{S_\alpha}$ is a union of strata.

Remark 2.2. The stratification induces a partial order on the index set A defined by

$$\beta \preceq \alpha \quad \Longleftrightarrow \quad S_\beta \subset \overline{S_\alpha},$$

which, by the frontier condition, is well-defined and acyclic.

[1]See [Pfl01] for a detailed treatment.

Example 2.3 (The Circular Cone). Consider the quadratic cone

$$X = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\}.$$

It admits a two-stratum decomposition:

$$S_2 = X \setminus \{(0, 0, 0)\},$$
$$S_0 = \{(0, 0, 0)\}.$$

Here the apex is a zero-dimensional stratum, and the smooth part is two-dimensional.

Example 2.4 (Algebraic Varieties and Whitney Stratification). Let $V \subset \mathbb{C}^n$ be an affine algebraic variety defined by the vanishing of polynomials $f_1, \ldots, f_m$. For each $k = 0, \ldots, n$, set

$$V_k = \big\{\, p \in V : \operatorname{rank}\big(Df(p)\big) = n - k\big\},$$

where $Df(p)$ is the Jacobian matrix of $(f_1, \ldots, f_m)$ at p. Equivalently, V_k is the locus where V is a smooth complex submanifold of complex dimension k (real dimension $2k$).

Then

$$V = \bigsqcup_{k=0}^{n} V_k$$

defines a stratification of V satisfying:

(1) *Frontier condition:* if $V_j \cap \overline{V_k} \neq \emptyset$, then $j < k$ and $V_j \subset \overline{V_k}$.
(2) *Whitney regularity:* for each $j < k$, the pair (V_j, V_k) satisfies Whitney's Conditions (A) and (B), ensuring that near any $p \in V_j$ the tangent spaces to V_k vary continuously and intersect T_pV_j in the expected dimension.

Example 2.5 (Square in $\mathbb{R}^2$). Let

$$X = [0, 1]^2 \ \subset \ \mathbb{R}^2.$$

We stratify X by

$$S_0 = (0, 1)^2,$$
$$S_1 = \big(\{0\} \times (0, 1)\big) \cup \big(\{1\} \times (0, 1)\big) \cup \big((0, 1) \times \{0\}\big) \cup \big((0, 1) \times \{1\}\big),$$
$$S_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

Here S_2 consists of the four *corners* of the square—points where two boundary edges meet. The interior $(0, 1)^2$ is S_0, the open edges are S_1, and the corner points are S_2.

Definition 2.6 (Manifold with corners)**.** A *manifold with corners* is a smooth manifold M of dimension N equipped with a stratification

$$M = \bigsqcup_{k=0}^{K} M^{[k]},$$

for some $K \leq N$, satisfying the following:

(1) Each $M^{[k]}$ is the codimension-k stratum, consisting of points locally diffeomorphic to an open subset of

$$\{x_1 = x_2 = \cdots = x_k = 0\} \subset \mathbb{R}^k \times \mathbb{R}^{N-k},$$

i.e., modeled on open neighborhoods of $[0, \infty)^k \times \mathbb{R}^{N-k}$ with exactly k vanishing coordinates.
(2) Each stratum $M^{[k]}$ is a smooth manifold of dimension $N - k$ (without boundary).
(3) The stratification satisfies the frontier condition:

$$\overline{M^{[k]}} \supset \bigsqcup_{\ell > k} M^{[\ell]}.$$

(4) The manifold M admits a smooth structure modeled on open subsets of $[0, \infty)^k \times \mathbb{R}^{N-k}$ for $k = 0, \ldots, K$, with smooth transition maps between overlapping charts.

Remark 2.7. In the following sections, we will restrict our attention to manifolds with corners admitting only strata of codimension at most 2, that is:

$$M = M^{[0]} \sqcup M^{[1]} \sqcup M^{[2]}.$$

Remark 2.8. Notice that, even though a manifold with boundary is a special case of a manifold with corners, consisting of only two strata, one of codimension-0 (the bulk) and one of codimension-1 (the boundary), the definition of manifold with corners usually refers to stratified structures with higher depth.

2.2. **Graded symplectic and Poisson geometry.** In this section, we briefly recall some notions of classical symplectic and Poisson geometry and describe their interplay with graded geometry.

Definition 2.9 (Symplectic manifold)**.** A symplectic manifold is a pair (F, ϖ), where ϖ is a closed two-form on the smooth manifold F (possibly infinite dimensional) such that the natural map $\mathfrak{X}(F) \to \Omega^1(F)$ sending $X \mapsto \iota_X \varpi$ is an isomorphism (or just injective if F is infinite dimensional). If the non-degeneracy condition is dropped, ϖ is called pre–symplectic.[2]

Definition 2.10 (Poisson manifold)**.** A Poisson structure on F is an $\mathbb{R}$-bilinear antisymmetric bracket $\{\cdot,\cdot\}$ on $C^\infty(F)$ satisfying the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \qquad \forall f, g, h \in C^\infty(F),$$

and the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

These requirements are equivalent to the existence of a Poisson bivector field $\pi \in \Gamma(\wedge^2 TF)$ such that

$$\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g), \quad \text{and} \quad [\pi, \pi] = 0,$$

where $[\cdot,\cdot]$ is the Schouten bracket.

A manifold endowed with a Poisson structure (F, π) is called a Poisson manifold, while $\mathcal{C}^\infty(F)$ is called a Poisson algebra.

In general, a symplectic structure induces a Poisson algebra. Indeed, letting (F, ϖ) be symplectic, if one considers any functions that have a Hamiltonian vector fields—we call them Hamiltonian functions—, i.e. any f such that there exists $X_f \in \mathfrak{X}(F)$ for which $\iota_{X_f}\varpi = \mathrm{d}f$,[3] one can define the Poisson bracket

$$\{f, g\} \coloneqq \iota_{X_f}\iota_{X_g}\varpi.$$

Definition 2.11. Let (F, ϖ) be a symplectic manifold. Then, a submanifold $C \subseteq F$ is called

- *Isotropic* if $T_xC \subseteq T_xC^\perp$,
- *Coisotropic* if $T_xC^\perp \subseteq T_xC$,[4]
- *Lagrangian* if $T_xC = T_xC^\perp$,

for all x in C, where

$$T_xC^\perp \coloneqq \{v \in T_xF \mid \varpi_x(v, w) = 0 \ \forall w \in T_xC\}.$$

Remark 2.12. We are mostly interested in the case of a coisotropic submanifold, as that will be the relevant object describing the phase space of the field theories we will take into consideration. Furthermore, when F is infinite-dimensional, we have to resort to a more algebraic definition, which in the finite-dimensional case is equivalent to the previous one. In particular, letting I be the vanishing ideal of C, i.e.

$$I \coloneqq \{f \in \mathcal{C}^\infty(F) \mid f|_C = 0\},$$

we require[5]

$$\{I, I\} \subseteq I.$$

In general, the coisotropic submanifold C is not symplectic, as the restriction of the symplectic form $\varpi|_C$ is degenerate. In fact, the kernel of ω restricted to C is the orthogonal bundle $T^\perp C = \cup_{x\in C}(T_xC)^\perp \subseteq TC$. Moreover, sections on $T^\perp C$ are involution, so we have a regular and involutive distribution (which, in the finite-dimensional case is automatically integrable by the Frobenius theorem). The goal is then to study the coisotropic reduction $\underline{C} \coloneqq C/T^\perp C$. However, in many cases, the quotient $\underline{C}$ is not smooth, hence the geometrical description of the quotient becomes particularly difficult to compute. One solution is to only describe it algebraically, by finding the algebra of functions on $\underline{C}$. An even better solution is to find a cohomological resolution of the quotient in the symplectic setting.

[2]Strictly speaking, one also requires the rank of the characteristic distribution of ω to be constant.

[3]Notice that, because of the non-degeneracy of ϖ, if X_f exists it is unique. Furthermore, when F is finite dimensional, every function is Hamiltonian.

[4]In the infinite-dimensional case, there si also the strong notion of split lagrangian submanifold: in this case, T_xC is required to be isotopic and to have an isotropic complement for all x.

[5]In Dirac's terminology, C is defined by first-class constraints.

We start by assuming that the vanishing ideal is generated by differentially independent Hamiltonian constraints $\phi_i \in \mathcal{C}^\infty(F)$, which is to say that $C = \{\phi_i = 0\}$. Then, their Hamiltonian vector fields $X_i \in \mathfrak{X}(F)$ are by construction in $\mathrm{Ker}(\varpi|_C)$. Indeed, one immediately sees

$$\iota_{X_i}\varpi|_C = \delta\phi_i|_C = 0.$$

Notice also that $\mathcal{C}^\infty(C) = \mathcal{C}^\infty(F)/I$, as two functions on F whose difference vanishes on C will be the same on C. Furthermore, we can also consider the normalizer

$$N(I) \coloneqq \{f \in \mathcal{C}^\infty(F) \mid \{I, f\} \subseteq I\}$$

and see that $N(I)/I$ is exactly the subalgebra of functions of $\mathcal{C}^\infty(F)/I$ that are invariant with respect to the action of I. In other words,

$$N(I)/I = \{f \in \mathcal{C}^\infty(C) \mid \{I, f\} = 0\} = \{f \in \mathcal{C}^\infty(C) \mid \{\phi_i, f\} = X_i(f) = 0 \text{ for all } i\} \simeq \mathcal{C}^\infty(C)^I \simeq \mathcal{C}^\infty(\underline{C}),$$

as it is clear that the functions invariant under the actions of the Hamiltonian vector fields X_i are constant along the leaves of the characteristic foliation of $\varpi|_C$, hence defining the functions on $\underline{C}$.

Remark 2.13. In the particular case where the constraints ϕ_i are the components $\mu_i = \phi_i$ of a moment map $\mu\colon F \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, and such that $0 \in \mathfrak{g}^*$ is a regular value, it follows by the Marsden–Weinstein theorem [MW74] that the coisotropic reduction $\underline{C} = \mu^{-1}(0)/G$ is smooth and symplectic, with symplectic form simply given by $\pi_*(\varpi_C)$, where $\pi\colon C \to \underline{C}$.

As mentioned before, another solution is to obtain a cohomological resolution of the space of functions on $\underline{C}$, by means of the so-called BFV formalism. Starting from the simpler example of Remark 2.13, we notice that the components of the momentum map satisfy the equivariance condition, i.e.

$$\{\mu_i, \mu_j\} = f_{ij}^k \mu_k, \qquad \text{for } i = 1, \dots, \dim G,$$

where f_{ij}^k are the structure constants of $\mathfrak{g}$. The MW theorem can then be reinterpreted with the language of supergeometry. Specifically, one considers the supermanifold[6] $F \times T^*\mathfrak{g}[1]$, where $\mathfrak{g}[1]$ is just the Lie algebra $\mathfrak{g}$ whose coordinates are now odd (shifted by 1). It is easy to see that $\mathcal{C}^\infty(T^*\mathfrak{g}[1]) \simeq \wedge^\bullet\mathfrak{g}^* \otimes \wedge^\bullet\mathfrak{g}$, and therefore

$$\mathcal{C}^\infty(F \times T^*\mathfrak{g}[1]) \simeq \mathcal{C}^\infty(F) \otimes \wedge^\bullet\mathfrak{g}^* \otimes \wedge^\bullet\mathfrak{g}.$$

On $F \times T^*\mathfrak{g}[1]$ one can then define a differential Q which is the twisting of the Koszul differential and the Chevalley–Eilenberg differential, such that, if we define $(c^i, c_i^\dagger)$ as coordinates of $T^*\mathfrak{g}[1]$, we have in components

$$S = \mu_i c^i - \frac{1}{2} f_{ij}^k c^i c^j c_k^\dagger, \qquad \text{and} \qquad Q = \{S, \cdot\},$$

where S is commonly refered to as BRST charge [**BRS**]. One can then show that $H_Q^0 \simeq \mathcal{C}^\infty(\underline{C})$. The proof relies on the fact that $\mathrm{Ker}(Q_0)$ roughly selects the $\mathfrak{g}$-equivariant functions on M, while $\mathrm{Im}(Q_{-1})$ defines the vanishing ideal.[7] As it turns out, this construction can be generalized to the case where one has constraints $\{\phi_i\}_{i=1,\dots,N}$ which are now just components of a map $\phi\colon F \to W$, where W is just an N-dimensional vector space. Furthermore, we have the coisotropic condition

$$\{\phi_i, \phi_j\} = f_{ij}^k \phi_k, \qquad \text{where} \qquad f_{ij}^k \in \mathcal{C}^\infty(F).$$

The BFV [**Schaetz**; BF83; Sta97] prescription is essentially the same as the BRST one, i.e. one constructs a function $S \in \mathcal{C}^\infty(F \times T^*W[1])$ such that $Q = \{S, \cdot\}$ is a cohomological vector field, and such that its degree-0 cohomology is isomorphic to the algebra of functions (as a Poisson algebra) $\mathcal{C}^\infty(\underline{C}) \simeq H_Q^0(F \times T^*\mathfrak{g}[1])$.[8] In particular, one has

$$S = \phi_i c^i - \frac{1}{2} f_{ij}^k c^i c^j c_k^\dagger + R,$$

where R is determined recursively and contains terms of higher degree in the "antighosts" $c^\dagger$.

[6] See the next section for the relevant definitions.

[7] In reality, the proof requires a few more technical steps. For a complete review of the BRST formalism, we refer to [**Figueroa**].

[8] When $\underline{C}$ is not smooth, we take H_Q^0 as the definition of $\mathcal{C}^\infty(\underline{C})$.

2.3. The BF^{k}V formalism.

Definition 2.14 (Supermanifold)**.** Let M be a smooth p-manifold. A supermanifold of dimension $p \mid q$ with body M is a locally ringed space $\mathcal{F} = (M, \mathcal{O}_\mathcal{F})$ such that:

(1) $\mathcal{O}_\mathcal{F}$ is a sheaf of supercommutative $\mathbb{R}$-algebras on M;
(2) for every point $x \in M$ there exists an open neighborhood $U \subseteq M$ with $x \in U$ and an isomorphism of locally ringed spaces
$$(U, \mathcal{O}_\mathcal{F}|_U) \cong (U, \mathcal{C}^\infty_U \otimes_\mathbb{R} \wedge^\bullet \mathbb{R}^q),$$
where $\mathcal{C}^\infty_U$ denotes the sheaf of smooth real–valued functions on U.

Definition 2.15 (dgs supermanifold)**.** A differentially graded symplectic supermanifold (dgs supermanifold) of degree k is a triple $(\mathcal{F}, \mathcal{Q}, \varpi)$ where:

(1) $\mathcal{F}$ is a $\mathbb{Z}$–graded supermanifold;[9]
(2) $\mathcal{Q} \in \mathfrak{X}(\mathcal{F})_1$ is a vector field of degree $+1$ satisfying
$$[\mathcal{Q}, \mathcal{Q}] = 0,$$
that is, $\mathcal{Q}$ is a cohomological vector field;
(3) $\varpi \in \Omega^2(\mathcal{F})_k$ is symplectic form of degree k on $\mathcal{F}$;
(4) $\mathcal{Q}$ preserves ϖ, i.e.
$$\mathcal{L}_\mathcal{Q} \varpi = 0.$$

Definition 2.16 (BF^{k}V manifold)**.** A BF^{k}V manifold is a $(k-1)$–Hamiltonian manifold, that is a quadruple $(\mathcal{F}, \mathcal{Q}, \varpi, \mathcal{S})$ where:

(1) $(\mathcal{F}, \mathcal{Q}, \varpi)$ is a dgs supermanifold of degree $k-1$;
(2) $\mathcal{S} \in C^\infty(\mathcal{F})_k$ is a function of degree k whose Hamiltonian vector field with respect to ϖ coincides with $\mathcal{Q}$, i.e.
$$\iota_\mathcal{Q} \varpi = \delta \mathcal{S},$$
or equivalently
$$\mathcal{Q}(f) = \{\mathcal{S}, f\}$$
for all $f \in C^\infty(\mathcal{F})$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket of degree $1-k$ induced by ϖ.

In this situation, $\mathcal{S}$ automatically satisfies the classical master equation
$$\{\mathcal{S}, \mathcal{S}\} = 0,$$
which is equivalent to $[\mathcal{Q}, \mathcal{Q}] = 0$.

Remark 2.17. Given Definition 2.16, we can recall the two most common cases: -1–Hamiltonian manifolds, called BV manifolds, and 0–Hamiltonian manifolds, also known as BFV manifolds.

In applications to field theories on manifolds with or corners, it is often too restrictive to require the condition
$$\iota_\mathcal{Q} \varpi = \delta \mathcal{S}$$
as in Definition 2.16. After integrating by parts, the action functional $\mathcal{S}$ may fail to admit a Hamiltonian vector field, as it is in presence of a stratified structure. Moreover, in the most constructions, the form ϖ is no often found to be degenerate. It is therefore convenient to relax these conditions, leading to the following definition.

Definition 2.18 (Relaxed BF^{k}V manifold)**.** A relaxed BF^{k}V manifold is a quadruple $(\mathcal{F}, \mathcal{Q}, \varpi, \mathcal{S})$ such that:

(1) $\mathcal{F}$ is a $\mathbb{Z}$-graded supermanifold;
(2) $\varpi \in \Omega^2(\mathcal{F})_{k-1}$ is a closed 2-form of degree $k-1$ on $\mathcal{F}$ (not necessarily non–degenerate);
(3) $\mathcal{S} \in C^\infty(\mathcal{F})_k$ is a function of degree k;
(4) $\mathcal{Q} \in \mathfrak{X}(\mathcal{F})_1$ is a vector field of degree 1 satisfying
$$[\mathcal{Q}, \mathcal{Q}] = 2\mathcal{Q}^2 = 0.$$

[9]We will usually assume that the $\mathbb{Z}$–grading is compatible with the $\mathbb{Z}_2$–parity, i.e. the parity of a homogeneous element equals its degree modulo 2. This grading is also called the "ghost number", when dealing with superfields in the BF^{k}V formalism.

In particular, we do not require that ϖ be symplectic nor that $\mathcal{Q}$ be the Hamiltonian vector field of $\mathcal{S}$; when these additional conditions hold, one recovers a $\mathrm{BF}^k\mathrm{V}$ manifold in the sense of Definition 2.16.

Remark 2.19. We introduce the "check" 1-form of degree k

$$\check{\alpha} \coloneqq \iota_{\mathcal{Q}}\varpi - \delta\mathcal{S},$$

which measures the failure of the Hamiltonian condition $\iota_{\mathcal{Q}}\varpi = \delta\mathcal{S}$ for a $\mathrm{BF}^k\mathrm{V}$ manifold. We then denote by $\check{\varpi}$ its differential,

$$\check{\varpi} \coloneqq \delta\check{\alpha} = -L_{\mathcal{Q}}\varpi.$$

Notice that, since $\mathcal{Q}$ is odd, Cartan's formula takes the form

$$L_{\mathcal{Q}} = \iota_{\mathcal{Q}}\delta - \delta\iota_{\mathcal{Q}}.$$

The equation $[\mathcal{Q}, \mathcal{Q}] = 0$ then immediately implies

$$L_{\mathcal{Q}}\check{\varpi} = 0.$$

The 2–form

$$\check{\varpi} = \delta\check{\alpha}$$

will play the role of the pre–symplectic (in general degenerate) form induced by higher–codimension corner terms.

Remark 2.20. Notice that one has that $\mathcal{Q}$ is now Hamiltonian with respect to $\check{\varpi}$, and that the Hamiltonian $\mathcal{S}$ can be obtained by computing $\iota_{\mathcal{Q}}\iota_{\mathcal{Q}}\varpi$. We refer to [**CWW**] for more details on this construction.

The idea of this construction is to relate the (relaxed) $\mathrm{BF}^k\mathrm{V}$ structures associated with different corner components by transporting the corresponding data from one stratum to another. In the presence of a stratified structure as in Definition 2.6, our construction is compatible with the codimension of the strata: the integer k in $\mathrm{BF}^k\mathrm{V}$ corresponds to the codimension of the stratum. This reflects the idea of having a relaxed $\mathrm{BF}^k\mathrm{V}$ manifold over each corner component of codimension k. In this context, BV is the theory built on the bulk (codimension 0) and BFV is the one living on the boundary (codimension 1) presented in Section 2.2.

2.4. **Generalized geometry.** Firstly introduced in [Hit03] and later refined in [Gua04] in the complex case, generalized geometry stems from the study of the differential geometry of $TM \oplus T^*M$, for a smooth manifold M. Replacing T with $T \oplus T^*$ allows to very neatly unify complex, symplectic and Poisson geometry within a single structure, paving the way for more general cases. In this paper, we will follow a more algebraic perspective, rather than focusing on the geometric structures of the bundle $TM \oplus T^*M$ (e.g. generalized complex, Kähler and metric structures). Indeed, $TM \oplus T^*M$ can be endowed with the algebraic structure of a Courant algebroid (the standard Courant algebroid), whose definition and properties are displayed in the following section.

Definition 2.21 (Courant algebroid[10])**.** A *Courant algebroid* over a smooth manifold M is a vector bundle $E \to M$ equipped with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$, a bilinear bracket $[\cdot,\cdot]\colon \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, and an anchor bundle map $\rho\colon E \to TM$, satisfying the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

(i) $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$,
(ii) $\rho([e_1, e_2]) = \{\rho(e_1), \rho(e_2)\}$,
(iii) $[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2$,
(iv) $\langle e_1, [e_2, e_3] + [e_3, e_2]\rangle = \rho(e_1)\langle e_2, e_3\rangle$,
(v) $\rho(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle$,

where $\{\cdot,\cdot\}$ is the Lie bracket of vector fields.

Remark 2.22. From the properties above, it follows that $[e_1, e_2]+[e_2, e_1] = \mathcal{D}\langle e_1, e_2\rangle$, where $\mathcal{D}\colon C^\infty(M) \to \Gamma(E)$ is a differential operator defined as $\mathcal{D} = \rho^*\mathrm{d}$, with $\rho^*\colon T^*M \to E$ the co-anchor map[11] of ρ defined by $\langle\rho^*(\alpha), e\rangle = \alpha(\rho(e))$, with $\alpha \in \Gamma(T^*M)$ and $e \in \Gamma(E)$, and d the de Rham differential operator. In other words, the bracket $[\cdot,\cdot]$ is not antisymmetric in general, and its lack of antisymmetry is measured by $\mathcal{D}$ and the pairing. Notice that it would be possible to modify the axioms of Definition 2.21 in order to account for an antisymmetric bracket. In this case, it would no longer satisfy the Jacobi-type identity of property (i), and the differential operator $\mathcal{D}$ would then measure its failure. The non–antisymmetric

[10] See [Mei25], [Roy02] and references therein for more details.
[11] a.k.a. transposed anchor map.

bracket is called the Dorfman bracket, whereas its antisymmetric counterpart is known as the Courant bracket.

Remark 2.23. Notice the difference with a Lie algebroid. A Lie algebroid is a vector bundle $A \to M$ equipped with a Lie bracket $[\cdot,\cdot]$ on its space of sections and an anchor map $\rho\colon A \to TM$ satisfying the Leibniz rule

$$[a, fb] = f[a, b] + (\rho(a)f)\, b, \tag{1}$$

for all $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$. In this setting, the bracket is skew-symmetric and satisfies both the Leibniz and the Jacobi identities.

Definition 2.24. Let M be a smooth manifold. The standard Courant algebroid over M is the vector bundle $TM \oplus T^*M$ endowed with the following structures

$$\begin{aligned} \langle X+\alpha, Y+\beta\rangle &= \iota_X\beta + \iota_Y\alpha, \\ \rho(X+\alpha) &= X, \\ [X+\alpha, Y+\beta]_{\mathrm{D}} &= [X,Y] + \mathrm{L}_X\beta - \iota_Y \mathrm{d}\alpha, \end{aligned}$$

defined for all $X+\alpha, Y+\beta \in \Gamma(TM \oplus T^*M)$, where $[X,Y]$ is the Lie bracket of vector fields and $[-,-]_{\mathrm{D}}$ denotes the Dorfman bracket of Definition 2.21.

Remark 2.25. In this case, the differential operator mentioned in Remark 2.22 is just the de Rham differential, indeed

$$[X+\alpha, Y+\beta]_{\mathrm{D}} - [Y+\beta, X+\alpha]_{\mathrm{D}} = d < X+\alpha, Y+\beta > . \tag{2}$$

In accordance with Remark 2.22, notice that one can also define the bundle by means of the Courant bracket, which would read

$$\begin{aligned} [X+\alpha, Y+\beta]_{\mathrm{C}} :=& [X+\alpha, Y+\beta]_{\mathrm{D}} - \mathrm{d}\langle\alpha, Y\rangle \\ =& [X,Y] + \mathrm{L}_X\beta - \iota_Y\mathrm{d}\alpha - \mathrm{d}\iota_Y\alpha \\ =& [X,Y] + \mathrm{L}_X\beta - \mathrm{L}_Y\alpha. \end{aligned}$$

This bracket is antisymmetric but does not satisfy the Jacobi identity.

From now on, we will drop the subscripts and automatically identify $[-,-]$ with the Dorfman bracket. It is however worth mentioning that the Dorfman bracket is not the unique bracket satisfying the properties of Definition 2.21. Indeed, for any closed 3-form $H \in \Omega^3_{\mathrm{cl}}(M)$, we can define the twisted Dorfman bracket as

$$[X+\alpha, Y+\beta]_H \coloneqq [X+\alpha, Y+\beta] + \iota_X\iota_Y H.$$

Another interesting property of the standard Courant algebroid is that it fits in the following exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\rho^*} TM \oplus T^*M \xrightarrow{\rho} TM \longrightarrow 0.$$

Definition 2.26 (B-transformation)**.** Let B be a smooth 2-form on M, viewed as a linear map $TM \to T^*M$ via $X \mapsto \iota_X B$. A B-transformation is an invertible bundle map of the kind

$$e^B \coloneqq \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \ : \ X+\alpha \longmapsto X+\alpha+\iota_X B,$$

where each entry of the matrix is a map from $T^{(*)} \to T^{(*)}$ depending on the position.

It is a known result ([Gua04]) that automorphisms of the standard Courant algebroid are realized as B-transformations,[12] for closed $B \in \Omega^2_{\mathrm{cl}}$. Indeed one observes that

$$[e^B(X+\alpha), e^B(Y+\beta)] = [X+\alpha, Y+\beta] + \iota_X\iota_Y \mathrm{d}B.$$

[12]To be precise, by compositions of diffeomorphisms and B-transformations.

2.4.1. *Dirac Structures.* We have already observed in Remark 2.22 and in Equation (2) that the failure of the Courant algebroid to be a Lie algebroid is ultimately related to exact terms involving $< -, - >$. Restoring such property amounts to finding a subbundle $D \subset T \oplus T^*$ which is closed under the Dorfman bracket (i.e. involutive) and isotropic (i.e. $\langle \Gamma(D), \Gamma(D) \rangle = 0$).

Definition 2.27 (Dirac structure)**.** A *Dirac structure* on a manifold M is a subbundle $D \subseteq (T \oplus T^*)M$ whose space of sections $\Gamma(D)$ is closed under the bracket $[\cdot, \cdot]$ and that is *maximally isotropic* with respect to the bilinear form $\langle \cdot, \cdot \rangle$, i.e. $\langle X + \alpha, Y + \beta \rangle = 0$ for all $X + \alpha, Y + \beta \in \Gamma(D)$ and D has maximal rank among all isotropic subbundles of $(T \oplus T^*)M$.

The two simplest examples of Dirac structures are just TM and T^*M. Indeed, the Dorfmann bracket reduces to the Lie bracket of vector fields on the former and vanishes on the latter. Being half dimensional with respect to $T \oplus T^*$, they are maximal, and they are trivially isotropic.

By "twisting" these two extremal cases, we obtain the following examples.

Example 2.28 (Dirac structure induced by a pre-symplectic form). Let $\varpi \in \Omega^2(M)$ be a pre-symplectic form, then the subbundle

$$D_\varpi \coloneqq e^\varpi(TM) = \{X + \iota_X \varpi \mid X \in \Gamma(TM)\}$$

is a Dirac structure. Indeed the maximal isotropic subspace $e^\varpi(TM)$ is involutive iff $\mathrm{d}\varpi = 0$, as we have seen for the case of B-transformations.

Example 2.29 (Dirac structure induced by a Poisson bivector field). Let $\pi \in \Gamma(\wedge^2 TM)$ be a Poisson bivector field. Then the subbundle

$$D_\pi \coloneqq e^\pi(T^*M) = \{\iota_\alpha \pi + \alpha \mid \alpha \in \Gamma(T^*M)\}$$

is a Dirac structure. Conversely, if D is Dirac and $D = e^\pi(T^*M)$ for some bivector π, then π is Poisson.

To prove such statement, we start by assuming π is Poisson. It is enough to consider the case where $\alpha = \mathrm{d}f$. For any $f, g \in \mathcal{C}^\infty(M)$ we have

$$\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g), \quad \text{and} \quad \iota_{\mathrm{d}f}\pi = X_f,$$

where X_f is regarded as the Hamiltonian vector field of f. Involutivity is observed by noticing

$$\begin{aligned}[X_f + \mathrm{d}f, X_g + \mathrm{d}g] &= [X_f, X_g] + \mathrm{d}\iota_{X_f}\mathrm{d}g \\ &= X_{\{f,g\}} + \mathrm{d}\{f, g\}.\end{aligned}$$

Conversely, assuming $D = e^\pi(T^*M)$ is Dirac, we see that the Jacobi identity is satisfied iff the Jacobiator $\mathrm{Jac}(A, B, C) \coloneqq [A, [B, C]] + [B, [C, A]] + [C, [A, B]]$ vanishes. In our case we see

$$\begin{aligned}&\mathrm{Jac}(X_f + \mathrm{d}f, X_g + \mathrm{d}g, X_h + \mathrm{d}h) = \\ &= \iota_{\mathrm{d}(\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\})}\pi + \mathrm{d}\left(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}\right) \\ &= 0 \quad \Leftrightarrow \quad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,\end{aligned}$$

which is equivalent to asking that $[\pi, \pi] = 0$.[13]

3. Dirac corner structures in classical field theories

3.1. **Classical Lagrangian field theories on manifolds with corners.** In the previous section we introduced a plethora of geometrical objects, whose application to field theory will be eviscerated in the following one. In particular, one can associate symplectic data to the boundary of spacetime, while on codimension 2 strata the relevant objects fit in the language of generalized geometry, where a Dirac structure naturally emerges as the graph of a Poisson structure. As we shall see, reduction might be needed in both cases.

Furthermore, we have that bulk, boundary and corner data can also in principle be nicely incorporated within the BF^{k}V formalism, as we will see in some simple applications.

We start by considering a spacetime manifold M with corners. A field theory on M is specified by a space of fields F_M, usually defined to be the space of sections $F_M = \Gamma(M, F)$ of a vector bundle F, and by an action functional $S_M \in \mathcal{C}^\infty(F_M)$. In general, one can construct the variational bicomplex [**Anderson**; **Delgado**] of local forms on $M \times F_M$ by pulling back the bicomplex of differential forms $\Omega^{\bullet,\bullet}(J^\infty F)$ along the evaluation map

$$\mathrm{ev} \circ (id \times j^\infty) : M \times F_M \longrightarrow M \times \Gamma(M, J^\infty F) \longrightarrow J^\infty F,$$

[13] In principle, the above equation would only imply that $[\pi, \pi]$ is constant, but one can show that the only solution is provided by $[\pi, \pi] = 0$.

obtaining the following bicomplex

$$\begin{array}{ccc} \Omega^{\bullet,\bullet}_{\mathrm{loc}}(M\times F_M) & \xrightarrow{\ \mathrm{d}\ } & \Omega^{\bullet+1,\bullet}_{\mathrm{loc}}(M\times F_M) \\ \Big\downarrow{\scriptstyle \delta} & \lrcorner & \Big\downarrow{\scriptstyle \delta} \\ \Omega^{\bullet,\bullet+1}_{\mathrm{loc}}(M\times F_M) & \xrightarrow{\ \mathrm{d}\ } & \Omega^{\bullet+1,\bullet+1}_{\mathrm{loc}}(M\times F_M) \end{array}$$

with horizontal differential d given by the de-Rham differential on $\Omega^\bullet(M)$ and the vertical differential δ given by the variational differential on $\Omega^\bullet(F_M)$. Notice that, being $D = \mathrm{d} + \delta$ the total differential, one obtains $\mathrm{d}\delta + \delta\mathrm{d} = 0$ from $D^2 = 0$.

Remark 3.1. For a D-dimensional field theory, the Lagrangian is a local form $L \in \Omega^{D,0}_{\mathrm{loc}}(M \times F_M)$ such that, when integrated along M, it produces a 0-form on F_M which is indeed the action functional $S_M \coloneqq \int_M L$.

3.1.1. *The boundary structure.* If we assume that M has a boundary $\Sigma \coloneqq \partial M$, then the varation of S_M produces a boundary term after integration by parts, due to Stokes' theorem. In particular, we have

$$\delta S_M = el_M - \alpha_M,$$

where $el_M \in \Omega^1(F_M)$ is the Euler-Lagrange 1-form, containing the equations of motion of the theory, and $\alpha_M = \int_{\partial M} a_M$, for some $a_M \in \Omega^{D-1,1}_{\mathrm{loc}}(M \times F_M)$. We can then define the following two-form on F_M

$$\varpi_M \coloneqq \delta\alpha_M,$$

which is automatically closed, hence pre-symplectic. We immediately notice that all the vector fields on F_M that preserve the values of the fields (and their jets) at the boundary are in the kernel of ϖ_M, which is defined as the integral of $\delta a_M \in \Omega^{D-1,2}_{\mathrm{loc}}(M \times F_M)$ on the codimension 1 surface $\partial M = \Sigma$. Assuming that such vector fields form a regular involutive distribution, we can quotient with respect to their flows and obtain, as the leaf space, the space of pre-boundary fields $\tilde{F}_\Sigma$, defining $\tilde{\pi}_\Sigma \colon F_M \to \tilde{F}_\Sigma$ to be the quotient map, and $\tilde{\alpha}_\Sigma$ and $\tilde{\varpi}_\Sigma$ such that

$$\alpha_M = \tilde{\pi}^*_\Sigma(\tilde{\alpha}_\Sigma) \qquad \text{and} \qquad \varpi_M = \tilde{\pi}^*_\Sigma(\tilde{\varpi}_\Sigma).$$

In general, it is possible that $\tilde{\varpi}_\Sigma$ is still degenerate, which is the case for gravity, and in which case a further reduction is needed. Also in this case one has to assume that $\mathrm{Ker}(\tilde{\varpi}_\Sigma)$ defines a regular involutive distribution, such that the leaf space with respect to the characteristic foliation of $\tilde{\varpi}_\Sigma$ is smooth. We denote such space by F_Σ and we call it the 'geometric phase space'. Again, we can define the surjective submersion $\pi_\Sigma \colon F_M \to F_\Sigma$ such that

$$\delta S_M = el_M - \pi^*_\Sigma\alpha_\Sigma \qquad \text{and} \qquad \varpi_M = \pi^*_\Sigma(\varpi_\Sigma).$$

It follows that $(F_\Sigma, \varpi_\Sigma)$ defines a symplectic manifold. However, this space does not yet represent the true physical phase space of the theory, as the latter corresponds to the set of Cauchy data.

Remark 3.2. To clarify this point, recall that the Euler–Lagrange equations naturally separate into two types: the evolution equations, which involve derivatives normal to the boundary, and the constraint equations, which depend only on derivatives tangent to the boundary. The constraints must be imposed on the space of pre–boundary fields, which typically enlarges the kernel of the pre–symplectic form. Performing the corresponding symplectic reduction yields the *reduced phase space*.

More conceptually, consider the cylindrical manifold $M_\epsilon \coloneqq \Sigma \times [0, \epsilon]$ for some small $\epsilon > 0$. Its boundary decomposes as

$$\partial M = (\Sigma \times \{0\}) \sqcup (\Sigma \times \{\epsilon\}).$$

The space $\tilde{L}_{M_\epsilon}$ can then be viewed as a relation[14] between the pre–boundary field spaces $\tilde{F}_\Sigma \simeq \tilde{F}_{\Sigma\times\{0\}}$ and $\tilde{F}_{\Sigma\times\{\epsilon\}}$.

The space of Cauchy data, denoted $\tilde{C}_\Sigma$, consists of those pre–boundary configurations on Σ that can be extended to full solutions of the Euler–Lagrange equations within a small cylindrical neighborhood of Σ, i.e.

$$\tilde{C}_\Sigma \coloneqq \left\{ c \in \tilde{F}_\Sigma \simeq \tilde{F}_{\Sigma\times\{0\}} \;\middle|\; \exists\, u \in \tilde{F}_{\Sigma\times\{\epsilon\}} \text{ such that } (c,u) \in \tilde{L}_{M_\epsilon} \right\}.$$

The induced two–form $\tilde{\varpi}^C_\Sigma$ is in general degenerate on $\tilde{C}_\Sigma$, and taking its quotient yields the reduced phase space $\underline{C}_\Sigma$. Although this space is often non-smooth or singular, in field theory one is primarily

[14]That is, a subset of $A \times B$ for two sets A and B.

interested in the algebra of functions defined on it, namely the algebra of physical observables. Under suitable assumptions, this algebra can be obtained cohomologically through the BFV formalism.

Remark 3.3. In principle one could work with the space of boundary fields F_Σ instead of $\tilde{F}_\Sigma$ and define $C_\Sigma \subset F_\Sigma$, however, one first needs to make sure that the constraints defining C_Σ are invariant under the action of the vector fields inside $\mathrm{Ker}(\tilde{\varpi}_\Sigma)$. For Palatini–Cartan gravity in $D = 4$ we will see that this is not true in general, however it is possible to use the non-invariant constraint to cleverly fix a representative of the fields in the quotient space F_Σ, such that the residual constraints on it are indeed invariant under the action of $\mathrm{Ker}(\tilde{\varpi}_\Sigma)$. For more details on the general construction, we refer to [**Albeto review.**]

3.1.2. *The corner structure.* If one assumes that M has a corner Γ, the boundary structure of the field theory is enriched with extra data. We consider the case in which C_Σ is coisotropic and described by a family of constraints $\{\phi_i\}_{i\in I} \subset C^\infty(F_\Sigma)$, possibly dependent, which are first class:

$$\{\phi_i, \phi_j\} = f_{ij}^k \phi_k, \qquad\qquad f_{ij}^k \in C^\infty(F_\Sigma). \tag{3}$$

We write $\approx$ for equality modulo the ideal generated by the constraints.

Assume first that $\partial\Sigma = \emptyset$. If $\mathbb{X}_i$ denotes the Hamiltonian vector field of ϕ_i, then

$$\iota_{\mathbb{X}_i}\varpi_\Sigma = \delta\phi_i. \tag{4}$$

Hence, on shell,

$$[\mathbb{X}_i, \mathbb{X}_j] \approx f_{ij}^k \mathbb{X}_k. \tag{5}$$

Equivalently, setting

$$\phi_{ij} := \{\phi_i, \phi_j\}, \qquad\qquad \mathbb{X}_{ij} := [\mathbb{X}_i, \mathbb{X}_j],$$

one has

$$\iota_{\mathbb{X}_{ij}}\varpi_\Sigma = \delta\phi_{ij}. \tag{6}$$

We now allow the boundary to have a boundary,

$$\Gamma = \partial\Sigma \neq \emptyset.$$

Then Equation (4) is replaced by

$$\delta\phi_i = \iota_{\mathbb{X}_i}\varpi_\Sigma - \mathcal{X}_i^\Sigma, \tag{7}$$

where $\mathcal{X}_i^\Sigma \in \Omega^1(F_\Sigma)$ is a residual one-form supported at the corner.
The splitting in Equation (7) is not canonical. Indeed, for a corner functional $B_i \in C^\infty(\tilde{F}_\Gamma)$, one may set

$$\widehat{\phi}_i := \phi_i + \tilde{\pi}_\Gamma^* B_i, \qquad\qquad \widehat{\mathcal{X}}_i := \tilde{\mathcal{X}}_i - \delta B_i. \tag{8}$$

Then

$$\delta\widehat{\phi}_i = \iota_{\mathbb{X}_i}\varpi_\Sigma - \tilde{\pi}_\Gamma^*\widehat{\mathcal{X}}_i.$$

Thus, adding an exact corner term to the constraint changes the representative of the residual one-form by an exact one-form.

In what follows we impose the following simplifying assumption.

Assumption 3.4. *When the constraints are written by pairing equations with Lagrange multipliers, these multipliers are not differentiated in the functionals ϕ_i.*

At this point, one can repeat the previous reduction argument. Any vector field on F_Σ which preserves the values of the fields, and of their jets, on Γ leaves the residual forms $\mathcal{X}_i^\Sigma$ invariant. Quotienting by the corresponding characteristic foliation defines the space of pre-corner fields $\tilde{F}_\Gamma$, together with a quotient map

$$\tilde{\pi}_\Gamma : F_\Sigma \longrightarrow \tilde{F}_\Gamma$$

such that

$$\delta\phi_i = \iota_{\mathbb{X}_i}\varpi_\Sigma - \tilde{\pi}_\Gamma^*(\tilde{\mathcal{X}}_i), \qquad\qquad \tilde{\mathcal{X}}_i \in \Omega^1(\tilde{F}_\Gamma). \tag{9}$$

The Hamiltonian vector fields $\mathbb{X}_i$ are projectable along $\tilde{\pi}_\Gamma$; we denote their push-forwards by

$$\tilde{\mathbb{X}}_i \in \mathfrak{X}(\tilde{F}_\Gamma).$$

Therefore each boundary constraint determines a generalized section

$$\tilde{\mathbb{X}}_i + \tilde{\mathcal{X}}_i \in \Gamma((T \oplus T^*)\tilde{F}_\Gamma).$$

As in Section 2.4, $(T\oplus T^*)\tilde{F}_\Gamma$ is the standard Courant algebroid on $\tilde{F}_\Gamma$, with pairing, Dorfman bracket and anchor

$$\begin{aligned}\langle \mathbb{X}+\mathcal{X}, \mathbb{Y}+\mathcal{Y}\rangle &= \iota_{\mathbb{X}}\mathcal{Y}+\iota_{\mathbb{Y}}\mathcal{X},\\ [\mathbb{X}+\mathcal{X}, \mathbb{Y}+\mathcal{Y}] &= [\mathbb{X},\mathbb{Y}]+\mathrm{L}_{\mathbb{X}}\mathcal{Y}-\iota_{\mathbb{Y}}\delta\mathcal{X},\\ \rho(\mathbb{X}+\mathcal{X}) &= \mathbb{X}.\end{aligned}$$

Remark 3.5. In the next sections, in order to readily obtain a BFV description of the boundary theory, we will define the constraints ϕ_i by means of odd Lagrange multipliers.[15] Thus both the Hamiltonian vector fields and the residual one-forms are odd. With this convention, the pairing and the Dorfman bracket are read with the corresponding graded signs:

$$\begin{aligned}\langle \mathbb{X}+\mathcal{X}, \mathbb{Y}+\mathcal{Y}\rangle &= \iota_{\mathbb{X}}\mathcal{Y}+(-1)^{xy}\iota_{\mathbb{Y}}\mathcal{X},\\ [\mathbb{X}+\mathcal{X}, \mathbb{Y}+\mathcal{Y}] &= [\mathbb{X},\mathbb{Y}]+\mathrm{L}_{\mathbb{X}}\mathcal{Y}-(-1)^{xy}\iota_{\mathbb{Y}}\delta\mathcal{X},\end{aligned}$$

where x and y are the parities of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and

$$\mathrm{L}_{\mathbb{X}} := [\iota_{\mathbb{X}},\delta] = \iota_{\mathbb{X}}\delta-(-1)^x\delta\iota_{\mathbb{X}}.$$

Let

$$\tilde{D} := \mathrm{span}_{C^\infty(\tilde{F}_\Gamma)}\{\tilde{\mathbb{X}}_i+\tilde{\mathcal{X}}_i\}_{i\in I}\subset\Gamma((T\oplus T^*)\tilde{F}_\Gamma).$$

The following two criteria express isotropy and involutivity in terms of the original boundary constraints.

Lemma 3.6. *The distribution $\tilde{D}$ is isotropic if and only if*

$$\mathbb{X}_i(\phi_j)+\mathbb{X}_j(\phi_i)=0 \qquad \forall i,j. \tag{10}$$

Proof. Using Equation (9), we have

$$\begin{aligned}\iota_{\mathbb{X}_j}\iota_{\mathbb{X}_i}\varpi_\Sigma &= \mathbb{X}_j(\phi_i)+\iota_{\tilde{\mathbb{X}}_j}\tilde{\mathcal{X}}_i,\\ \iota_{\mathbb{X}_i}\iota_{\mathbb{X}_j}\varpi_\Sigma &= \mathbb{X}_i(\phi_j)+\iota_{\tilde{\mathbb{X}}_i}\tilde{\mathcal{X}}_j.\end{aligned}$$

Adding the two identities and using the skew-symmetry of ϖ_Σ gives

$$\begin{aligned}0 &= \mathbb{X}_j(\phi_i)+\mathbb{X}_i(\phi_j)+\iota_{\tilde{\mathbb{X}}_j}\tilde{\mathcal{X}}_i+\iota_{\tilde{\mathbb{X}}_i}\tilde{\mathcal{X}}_j\\ &= \mathbb{X}_j(\phi_i)+\mathbb{X}_i(\phi_j)+\left\langle\tilde{\mathbb{X}}_i+\tilde{\mathcal{X}}_i,\tilde{\mathbb{X}}_j+\tilde{\mathcal{X}}_j\right\rangle.\end{aligned}$$

Thus the Courant pairing vanishes precisely when Equation (10) holds. □

Lemma 3.7. *Assume that*

$$\mathbb{X}_i(\phi_j)\approx f_{ij}^k\phi_k. \tag{11}$$

Then, $\tilde{D}$ is involutive on shell:

$$[\tilde{\mathbb{X}}_i+\tilde{\mathcal{X}}_i,\tilde{\mathbb{X}}_j+\tilde{\mathcal{X}}_j]\approx f_{ij}^k\left(\tilde{\mathbb{X}}_k+\tilde{\mathcal{X}}_k\right). \tag{12}$$

Proof. From Equation (7),

$$\mathcal{X}_i^\Sigma = \iota_{\mathbb{X}_i}\varpi_\Sigma-\delta\phi_i.$$

Since $\delta\varpi_\Sigma=0$, one has

$$\delta\mathcal{X}_i^\Sigma = \mathrm{L}_{\mathbb{X}_i}\varpi_\Sigma.$$

Therefore

$$\begin{aligned}\mathrm{L}_{\mathbb{X}_i}\mathcal{X}_j^\Sigma &= \mathrm{L}_{\mathbb{X}_i}\iota_{\mathbb{X}_j}\varpi_\Sigma-\delta\big(\mathbb{X}_i(\phi_j)\big),\\ \iota_{\mathbb{X}_j}\delta\mathcal{X}_i^\Sigma &= \iota_{\mathbb{X}_j}\mathrm{L}_{\mathbb{X}_i}\varpi_\Sigma.\end{aligned}$$

Taking the difference gives

$$\mathrm{L}_{\mathbb{X}_i}\mathcal{X}_j^\Sigma-\iota_{\mathbb{X}_j}\delta\mathcal{X}_i^\Sigma = \iota_{[\mathbb{X}_i,\mathbb{X}_j]}\varpi_\Sigma-\delta\big(\mathbb{X}_i(\phi_j)\big). \tag{13}$$

Using Equation (11) and Equation (5), we obtain, on shell,

$$\begin{aligned}\delta\big(\mathbb{X}_i(\phi_j)\big) &\approx f_{ij}^k\delta\phi_k,\\ \iota_{[\mathbb{X}_i,\mathbb{X}_j]}\varpi_\Sigma &\approx f_{ij}^k\iota_{\mathbb{X}_k}\varpi_\Sigma = f_{ij}^k\left(\delta\phi_k+\mathcal{X}_k^\Sigma\right).\end{aligned}$$

[15] Identified with the ghosts of the theory.

Substitution in Equation (13) yields

$$\mathrm{L}_{\mathbb{X}_i}\mathcal{X}_j^\Sigma - \iota_{\mathbb{X}_j}\delta\mathcal{X}_i^\Sigma \approx f_{ij}^k\mathcal{X}_k^\Sigma.$$

Together with

$$[\mathbb{X}_i, \mathbb{X}_j] \approx f_{ij}^k\mathbb{X}_k,$$

this is precisely the on-shell closure of the corresponding Dorfman bracket. Since the objects involved are projectable to the pre-corner field space, the identity descends to $\tilde{F}_\Gamma$. □

The condition Equation (12) is not automatic on the full space $\tilde{F}_\Gamma$; in general, the corner fields must be restricted by the constraints inherited from the boundary equations. Such constraints are identified by considering the infinitesimal cylinder

$$\Sigma_\epsilon := \Gamma \times [0, \epsilon], \qquad \epsilon > 0.$$

On Σ_ϵ, let

$$C_{\Sigma_\epsilon} \subset F_{\Sigma_\epsilon}$$

be the coisotropic submanifold of boundary fields. We define F_Γ as the set of field configurations at $\Gamma \times \{0\}$ of fields in C_{Σ_ϵ}. In other words, F_Γ is the space of corner fields that extend to boundary conditions[16] in an infinitesimal neighbourhood of Γ.

By abuse of notation, we still denote by $\tilde{\mathbb{X}}_i + \tilde{\mathcal{X}}_i$ the sections of $(T\oplus T^*)F_\Gamma$ induced by the corresponding sections on $\tilde{F}_\Gamma$. We then impose the following assumption.

Assumption 3.8. *The distribution*

$$\tilde{D} := \mathrm{span}_{C^\infty(F_\Gamma)}\{\tilde{\mathbb{X}}_i + \tilde{\mathcal{X}}_i\}_{i\in I} \subset \Gamma((T \oplus T^*)F_\Gamma)$$

is involutive and isotropic with respect to the Dorfman bracket and the Courant pairing. We call such data a pre-Dirac structure.

We will see in Section 5.1 that this assumption is satisfied for Palatini–Cartan gravity.

It remains to identify the reduction which removes the obstruction to maximality. Let

$$\mathrm{pr}_{T^*} : (T \oplus T^*)F_\Gamma \longrightarrow T^*F_\Gamma$$

be the projection onto the cotangent component. We define the kernel of $\tilde{D}$ by

$$K := \Big\{\mathbb{Y} \in \mathfrak{X}(F_\Gamma) \ : \ \iota_{\mathbb{Y}}\mathcal{X} = 0 \quad \forall\, \mathcal{X} \in \Gamma(\mathrm{pr}_{T^*}\tilde{D})\Big\}. \tag{14}$$

Equivalently, since $\mathrm{pr}_{T^*}\tilde{D}$ is generated by the residual one-forms,

$$K = \Big\{\mathbb{Y} \in \mathfrak{X}(F_\Gamma) \ : \ \iota_{\mathbb{Y}}\tilde{\mathcal{X}}_i = 0 \quad \forall i\Big\}.$$

Lemma 3.9. *The distribution K is involutive if and only if*

$$\iota_{\mathbb{Y}}\iota_{\mathbb{Y}'}\delta\mathcal{X} = 0 \qquad \forall\, \mathbb{Y}, \mathbb{Y}' \in K, \quad \forall\, \mathcal{X} \in \Gamma(\mathrm{pr}_{T^*}\tilde{D}). \tag{15}$$

Proof. Let $\mathbb{Y}, \mathbb{Y}' \in K$. Then $[\mathbb{Y}, \mathbb{Y}'] \in K$ if and only if

$$\iota_{[\mathbb{Y},\mathbb{Y}']}\mathcal{X} = 0 \qquad \forall\, \mathcal{X} \in \Gamma(\mathrm{pr}_{T^*}\tilde{D}).$$

By Cartan calculus,

$$\begin{aligned}\iota_{[\mathbb{Y},\mathbb{Y}']}\mathcal{X} &= [\mathrm{L}_{\mathbb{Y}}, \iota_{\mathbb{Y}'}]\mathcal{X} \\ &= \mathrm{L}_{\mathbb{Y}}(\iota_{\mathbb{Y}'}\mathcal{X}) - \iota_{\mathbb{Y}'}\mathrm{L}_{\mathbb{Y}}\mathcal{X}.\end{aligned}$$

The first term vanishes since $\mathbb{Y}' \in K$. Moreover,

$$\mathrm{L}_{\mathbb{Y}}\mathcal{X} = \iota_{\mathbb{Y}}\delta\mathcal{X} + \delta(\iota_{\mathbb{Y}}\mathcal{X}) = \iota_{\mathbb{Y}}\delta\mathcal{X},$$

because $\mathbb{Y} \in K$. Therefore

$$\iota_{[\mathbb{Y},\mathbb{Y}']}\mathcal{X} = -\iota_{\mathbb{Y}'}\iota_{\mathbb{Y}}\delta\mathcal{X}.$$

This proves the claim. □

Definition 3.10. A partial kernel of $\tilde{D}$ is a regular involutive subdistribution

$$\widehat{K} \subset K.$$

[16] That is, to Cauchy data in C_{Σ_ϵ}.

The distribution used above to pass from F_Σ to $\tilde{F}_\Gamma$, consisting of vector fields which preserve the values of fields and jets at the corner, is an example of such a partial kernel at the pre-corner level.

Another canonical example is

$$K_{\tilde{D}} := \Big\{ \mathbb{Y} \in \mathfrak{X}(F_\Gamma) \ : \ \mathbb{Y} + 0 \in \Gamma(\tilde{D}) \Big\}. \tag{16}$$

Lemma 3.11. *The distribution $K_{\tilde{D}}$ is a partial kernel.*

Proof. If $\mathbb{Y} \in K_{\tilde{D}}$, then $\mathbb{Y} + 0 \in \Gamma(\tilde{D})$. Since $\tilde{D}$ is isotropic, for every $\mathbb{X} + \mathcal{X} \in \Gamma(\tilde{D})$ one has

$$0 = \langle \mathbb{X} + \mathcal{X}, \mathbb{Y} + 0 \rangle = \iota_{\mathbb{Y}} \mathcal{X}.$$

Thus $\mathbb{Y} \in K$. Moreover, if $\mathbb{Y}, \mathbb{Y}' \in K_{\tilde{D}}$, then

$$[\mathbb{Y} + 0, \mathbb{Y}' + 0] = [\mathbb{Y}, \mathbb{Y}'] + 0.$$

Since $\tilde{D}$ is involutive, $[\mathbb{Y}, \mathbb{Y}'] + 0 \in \Gamma(\tilde{D})$, hence $[\mathbb{Y}, \mathbb{Y}'] \in K_{\tilde{D}}$. □

Remark 3.12. For $\mathbb{X} + \mathcal{X} \in \Gamma(\tilde{D})$ and $\mathbb{Y} \in K_{\tilde{D}}$, the Dorfman bracket gives

$$[\mathbb{X} + \mathcal{X}, \mathbb{Y} + 0] = [\mathbb{X}, \mathbb{Y}] - \iota_{\mathbb{Y}} \delta \mathcal{X}. \tag{17}$$

Thus

$$\iota_{\mathbb{Y}} \delta \mathcal{X} = 0 \qquad \Longleftrightarrow \qquad [\mathbb{X}, \mathbb{Y}] \in K_{\tilde{D}}.$$

This is the condition for $\mathbb{X}$ to be projectable along $K_{\tilde{D}}$.

A stronger partial kernel is

$$\begin{aligned} K_f &:= \Big\{ \mathbb{Y} \in K \ : \ \iota_{\mathbb{Y}} \delta \mathcal{X} = 0 \quad \forall\, \mathcal{X} \in \Gamma(\mathrm{pr}_{T^*} \tilde{D}) \Big\} \\ &= \Big\{ \mathbb{Y} \in \mathfrak{X}(F_\Gamma) \ : \ \iota_{\mathbb{Y}} \mathcal{X} = 0 \text{ and } \iota_{\mathbb{Y}} \delta \mathcal{X} = 0 \quad \forall\, \mathcal{X} \in \Gamma(\mathrm{pr}_{T^*} \tilde{D}) \Big\}. \end{aligned} \tag{18}$$

We shall use the following equivalence relation between generalized distributions. Two pre-Dirac distributions are considered equivalent if either they are related by a diffeomorphism or by a B-transformation, or one is obtained from the other by reduction along a regular partial kernel. Thus, if

$$\widehat{F} = F_\Gamma / \widehat{K}$$

for a regular partial kernel $\widehat{K} \subset K$, and if the sections of $\tilde{D}$ are projectable, the reduced distribution $\widehat{D}$ on $\widehat{F}$ is equivalent to $\tilde{D}$. In the special case $\widehat{K} = K_{\tilde{D}}$, this is a Morita-type equivalence.

In particular, when K itself is regular and involutive, one can take

$$P_\Gamma := F_\Gamma / K.$$

The induced distribution is

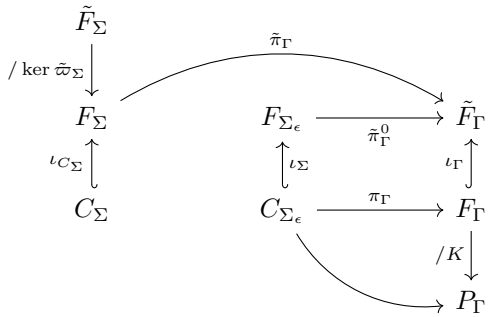
$$D := \mathrm{span}_{C^\infty(P_\Gamma)} \{ \mathbb{X}_i + \mathcal{X}_i \}_{i \in I} \subset \Gamma((T \oplus T^*) P_\Gamma). \tag{19}$$

If involutivity is preserved and D is maximally isotropic, then D is a Dirac structure on P_Γ. In the case in which its cotangent projection is all $T^* P_\Gamma$, it is the graph of a Poisson bivector

$$\Pi \in \Gamma(\wedge^2 T P_\Gamma).$$

Definition 3.13. We say that the field theory is good on Γ if its pre-Dirac corner structure is equivalent, in the above sense, to a Poisson structure on a reduced corner field space.

To summarize the discussion, it is useful to keep in mind the following diagram:



Remark 3.14. Given a Poisson structure $\Pi \in \Gamma(\wedge^2 TP_\Gamma)$, one automatically obtains a $\mathrm{BF}^2\mathrm{V}$ structure on $T^*[1]P_\Gamma$, since Π can equivalently be regarded as a degree-two functional

$$\mathcal{S}_\Gamma \in C^\infty(T^*[1]P_\Gamma)$$

satisfying the classical master equation

$$\{\mathcal{S}_\Gamma, \mathcal{S}_\Gamma\} = 0,$$

which is equivalent to

$$[\Pi, \Pi] = 0.$$

Remark 3.15. In some cases, such as BF theory, ghost-for-ghost fields affect the corner structure. The procedure described above may still provide a Dirac structure given by the graph of a Poisson bivector, but this does not necessarily capture the full homological nature of the theory. Reducibility data require a higher Poisson structure, or P_∞ structure, which in the cases considered here is concentrated in degrees one and two.

In other words, the ordinary Poisson bivector Π is replaced by

$$\Pi = \Pi_1 + \Pi_2,$$

where Π_1 is a degree-one vector field and Π_2 is a bivector. These data satisfy the Maurer–Cartan equation

$$[\Pi_1 + \Pi_2, \Pi_1 + \Pi_2] = 0,$$

where $[\cdot,\cdot]$ denotes the Schouten bracket. Equivalently,

$$[\Pi_1, \Pi_1] = 0, \qquad [\Pi_1, \Pi_2] = 0, \qquad [\Pi_2, \Pi_2] = 0.$$

The first identity says that Π_1 is cohomological, the third is the Poisson condition for Π_2, and the second says that Π_2 is invariant under the differential generated by Π_1. Thus the passage from a Poisson bivector to $\Pi_1 + \Pi_2$ incorporates residual reducibility directly into the cohomological corner description. In practice, this amounts to a Dirac structure with support; see [**BCCS**; CC24].

3.2. **Examples.** We now provide a couple of straightforward examples of field theories presenting corner structures that can be described within the $\mathrm{BF}^2\mathrm{V}$ formalism. In the following, we let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$ and M be a $D = d+1$ manifold.

3.2.1. *Yang–Mills theory.* We work in the so-called first order formalism, where only first order derivatives appear in the Lagrangian. The space of fields of the theory is given by $F_M^{\mathrm{YM}} \coloneqq \Omega^1(M,\mathfrak{g}) \oplus \Omega^{d-1}(M,\mathfrak{g}) \ni (A,B)$, and the action reads

$$S_M^{\mathrm{YM}} = \int_M \mathrm{Tr}(B \wedge F_A) - \frac{1}{2}\mathrm{Tr}(B \wedge *B),$$

where $*$ is the Hodge dual. From now on we will omit the $\wedge$ symbol and the Tr, as they will be assumed in the relevant contexts.

The variation of the action produces EoMs and a boundary term

$$\delta S_M^{\mathrm{YM}} = \int_M \mathrm{d}_A B\delta A + (B - *F_A)\delta B - \int_{\partial M} B\delta A.$$

Setting $\Sigma = \partial M$, we define $F_\Sigma^{\mathrm{YM}} = \Omega^1(\Sigma,\mathfrak{g}) \oplus \Omega^{d-1}(\Sigma,\mathfrak{g}) \ni (A,B)$ as the space of boundary fields, on which we have the symplectic form

$$\varpi_\Sigma^{\mathrm{YM}} = \int_\Sigma \delta B\delta A.$$

The phase space C_Σ^{YM} is defined as the zero locus of the following constraint, which turns out to be first class

$$L_c = \int_\Sigma c\mathrm{d}_A B, \qquad \{L_c, L_c\} = \frac{1}{2}L_{[c,c]}, \qquad \text{for } c \in \Omega^0[1](\Sigma,\mathfrak{g}),$$

having chosen an odd Lagrange multiplier c. The Hamiltonian vector field $\mathbb{L}_c \coloneqq \int_\Sigma \mathbb{L}_A \frac{\delta}{\delta A} + \mathbb{L}_B \frac{\delta}{\delta B}$ of L_c gives the infinitesimal gauge transformations as $\mathbb{L}_A = \mathrm{d}_A c$ and $\mathbb{L}_B = [c,B]$.

In the presence of corners, we have an extra term in the variation of L_c, i.e.

$$\delta L_c = \iota_{\mathbb{L}_c}\varpi_\Sigma^{\mathrm{YM}} - \int_\Gamma c\delta B.$$

Defining $\mathcal{J}_c \coloneqq \int_\Gamma c\delta B$ as a functional on the space of pre-corner fields $\tilde{F}^{\mathrm{YM}}_\Gamma \coloneqq \Omega^1(\Gamma,\mathfrak{g}) \oplus \Omega^{d-1}(\Gamma,\mathfrak{g})$, we can furthermore define

$$\mathbb{J}_c = \int_\Gamma \mathrm{d}_A c \frac{\delta}{\delta A} + [c,B]\frac{\delta}{\delta B} \in \mathfrak{X}(\tilde{F}^{\mathrm{YM}}_\Gamma)$$

as the pushforward of $\mathbb{L}_c$ on $\tilde{F}^{\mathrm{YM}}_\Gamma$ via the projection map $\tilde{\pi}_\Gamma \colon F^{\mathrm{YM}}_\Sigma \to \tilde{F}^{\mathrm{YM}}_\Gamma$. Then the distribution $\tilde{D}^{\mathrm{YM}}_\Gamma \coloneqq \mathrm{span}(\mathbb{J}_c + \mathcal{J}_c)$ is involutive and isotropic, as can easily be checked by noticing

$$\langle \mathbb{J}_c + \mathcal{J}_c, \mathbb{J}_{c'} + \mathcal{J}_{c'} \rangle = \mathcal{J}_{[c,c']} - \mathcal{J}_{[c',c]} = 0,$$
$$[\mathbb{J}_c + \mathcal{J}_c, \mathbb{J}_c + \mathcal{J}_c] = \frac{1}{2}\mathbb{J}_{[c,c]} + \frac{1}{2}\mathcal{J}_{[c,c]}.$$

Remark 3.16. Following the previous section, we would have to work on the restriction to the corners of the space of Cauchy data. However, in the absence of transversal jet fields, we see that the constraint $\mathrm{d}_A B = 0$ does not extend to the $d-1$-dimensional corner, it being a $d-$form. Therefore, in the notation we provided above, we can identify F^{YM}_Γ and $\tilde{F}^{\mathrm{YM}}_\Gamma$.

In order to achieve the maximal isotropicity property, we quotient out with respect to the distribution defined by the kernel of $\mathcal{J}_c$, given by

$$\mathrm{Ker}(\mathcal{J}_c) = \left\{ \int_\Gamma \mathbb{X}_A \frac{\delta}{\delta A} + \mathbb{X}_B \frac{\delta}{\delta B} \in \mathfrak{X}(\tilde{F}^{\mathrm{YM}}_\Gamma) \;\middle|\; \mathbb{X}_B = 0 \right\} \simeq \Omega^1(\Gamma,\mathfrak{g}).$$

Therefore we obtain that the space of classical corner fields is simply given by $P^{\mathrm{YM}}_\Gamma = \tilde{F}^{\mathrm{YM}}_\Gamma / \Omega^1(\Gamma,\mathfrak{g}) \simeq \Omega^{d-1}(\Gamma,\mathfrak{g})$, on which one can define the Poisson bivector field

$$\pi^{\mathrm{YM}}_\Gamma = \int_\Gamma \frac{1}{2} B \left[\frac{\delta}{\delta B}, \frac{\delta}{\delta B} \right] \quad \text{such that } D^{\mathrm{YM}}_\Gamma = \mathrm{Graph}(\pi^{\mathrm{YM}}_\Gamma),$$

such that $\iota_{\mathcal{J}_c}\pi = \mathbb{J}_c$, proving that D^{YM}_Γ is a Dirac structure.

Notice also that the Poisson bivector field π^{YM}_Γ is the same as a degree-2 function on $T^*[1]P^{\mathrm{YM}}_\Gamma$ satisfying the CME. In particular, one has

$$\mathcal{S}^{\mathrm{YM}}_\Gamma = \int_\Gamma \frac{1}{2} B[c,c], \tag{20}$$

where $(B,c) \in \mathcal{F}^{\mathrm{YM}}_\Gamma = \Omega^{d-1}(\Gamma,\mathfrak{g}) \oplus \Omega^0[1](\Gamma,\mathfrak{g})$, after having identified $\mathfrak{g}^* \simeq \mathfrak{g}$ and $(\Omega^{d-1}(\Gamma))^* \simeq \Omega^0(\Gamma)$. Furthermore, on $\mathcal{F}^{\mathrm{YM}}_\Gamma$ one can define the canonical symplectic form

$$\varpi^{\mathrm{YM}}_\Gamma = \int_\Gamma \delta B \delta c. \tag{21}$$

Remark 3.17. For the sake of completeness, we remark that the same result could be achieved by keeping the transversal jets at all orders in the space of pre-corner fields. In the end, however, such jets are all quotiented out to obtain the maximality property. Ww will see in the next chapters that, for example in the case of PC gravity, it is more convenient to provide an initial description which takes the transversal jets into consideration.

Remark 3.18. It is also interesting to notice that the polarization found in this example, obtained from studying the classical corner structure of YM theory, is not the unique choice in the context of the $\mathrm{BF}^2\mathrm{V}$ formalism. Indeed, it was shown in [CC24] that different choices of polarizations lead to the arising of non-trivial P_∞ structures, while preserving the underlying $\mathrm{BF}^2\mathrm{V}$ data.

3.2.2. *BF theory.* With the same assumptions as in the previous example, the space of fields is the same as in first-order Yang–Mills theory, i.e. a connection 1-form A, modeled over the space $\Omega^1(M,\mathfrak{g})$ and a Lie algebra-valued $(d-1)$–form $B \in \Omega^{d-1}(M,\mathfrak{g})$. We denote such space by $F^{BF}_M \coloneqq \Omega^1(M,\mathfrak{g}) \oplus \Omega^{d-1}(M,\mathfrak{g})$. The action reads

$$S^{BF}_M = \int_M BF_A + \Lambda P(B),$$

where Λ is reminiscent of the cosmological constant in gravity and P is an invariant polynomial of degree k such that $k(d-1) = 2$.[17] The variation of the action gives

$$\delta S^{\mathrm{BF}}_M = \int_M \delta B(F_A + \Lambda P'(B)) + \delta A \mathrm{d}_A B - \int_\Sigma B\delta A.$$

[17] P is trivial for any dimension where $d \neq 2,3$.

Again, the space of boundary fields is given by $F^{BF}_\Sigma = \Omega^1(\Sigma,\mathfrak{g}) \oplus \Omega^{d-1}(\Sigma,\mathfrak{g})$, with symplectic form

$$\varpi^{\mathrm{BF}}_\Sigma = \int_\Sigma \delta A \delta B.$$

For the sake of convenience, we now specify the analysis of BF theory to the case $D = 4$. In this setting, there are two constraints on F^{BF}_Σ of the form[18]

$$L_c = \int_\Sigma c \mathrm{d}_A B, \qquad \text{and} \qquad M_\tau = \int_\Sigma \tau(F_A + \Lambda B),$$

where $c \in \Omega^0[1](\Sigma,\mathfrak{g})$ and $\tau \in \Omega^1[1](\Sigma,\mathfrak{g})$ will be indentified with the ghost fields. As expected, such constraints are in involution, which makes the phase space C^{BF}_Σ a coisotropic submanifold. Specifically, one has

$$\mathbb{L}_c = \int_\Sigma \mathrm{d}_A c \frac{\delta}{\delta A} + [c,B]\frac{\delta}{\delta B}, \qquad \mathbb{M}_\tau = \int_\Sigma \Lambda\tau \frac{\delta}{\delta A} + \mathrm{d}_A \tau \frac{\delta}{\delta B},$$

and

$$\{L_c, L_c\} = \frac{1}{2} L_{[c,c]} \qquad \{L_c, M_\tau\} = -L_{[c,\tau]} \qquad \{M_\tau, M_\tau\} = 0.$$

When corner terms are not neglected, we obtain

$$\delta M_\tau = \iota_{\mathbb{M}_\tau} \varpi^{BF}_\Sigma - \int_\Gamma \tau \delta A, \qquad \text{and} \qquad \delta L_c = \iota_{\mathbb{L}_c} \varpi^{BF}_\Sigma - \int_\Gamma c \delta B.$$

Defining $\mathcal{J}_c \coloneqq \int_\Gamma c\delta B$ and $\mathcal{V}_\tau \coloneqq \tau \delta A$ as one-forms on the space of corner fields $\tilde{F}^{BF}_\Gamma \coloneqq \Omega^1(\Gamma,\mathfrak{g}) \oplus \Omega^2(\Gamma,\mathfrak{g}) \ni (A,B)$, we can furthermore push the Hamiltonian vector fields forward to F^{BF}_Σ along the projection $\tilde{\pi}^\Gamma \colon F^{BF}_\Sigma \to \tilde{F}^{BF}_\Gamma$, defining

$$\mathbb{J}_c \coloneqq \pi^\Gamma_* \mathbb{L}_c, \qquad \text{and} \qquad \mathbb{V}_\tau \coloneqq \pi^\Gamma_* \mathbb{M}_\tau.$$

It follows that the distribution $\tilde{D}^{BF} \coloneqq \mathrm{span}_{\mathcal{C}^\infty(\tilde{F}^{BF}_\Gamma)}(\mathbb{J}_c + \mathcal{J}_c, \mathbb{V}_\tau + \mathcal{V}_\tau)$ is isotropic and involutive, as

$$\langle \mathbb{J}_c + \mathcal{J}_c, \mathbb{J}_{c'} + \mathcal{J}_{c'} \rangle = 0, \qquad \langle \mathbb{J}_c + \mathcal{J}_c, \mathbb{V}_\tau + \mathcal{V}_\tau \rangle = 0, \quad \langle \mathbb{V}_\tau + \mathcal{V}_\tau, \mathbb{V}_{\tau'} + \mathcal{V}_{\tau'} \rangle = 0$$

$$[\mathbb{J}_c + \mathcal{J}_c, \mathbb{J}_c + \mathcal{J}_c] = \frac{1}{2}\left(\mathbb{J}_{[c,c]} + \mathcal{J}[c,c]\right), \quad [\mathbb{J}_c + \mathcal{J}_c, \mathbb{V}_\tau + \mathcal{V}_\tau] = -\mathbb{J}_{[c,\tau]} - \mathcal{J}_{[c,\tau]}, \quad [\mathbb{V}_\tau + \mathcal{V}_\tau, \mathbb{V}_\tau + \mathcal{V}_\tau] = 0.$$

Furthermore, $\tilde{D}^{BF}$ is automatically the graph of the following Poisson structure

$$\pi^{BF}_2 = \int_\Gamma \frac{1}{2} B \left[\frac{\delta}{\delta B}, \frac{\delta}{\delta B}\right] + \frac{\delta}{\delta A} \mathrm{d}_A \frac{\delta}{\delta B} + \Lambda \frac{\delta}{\delta A}\frac{\delta}{\delta A}.$$

Remark 3.19. It is interesting to notice that in the case of BF one can in principle obtain a Dirac structure without the need of restricting to the submanifold F^{BF}_Γ defined as those corner field configurations that extend to boundary conditions in a neighborhood of Γ (or, in other words, the restriction of the Cauchy data to the corners). Furthermore, it is not even necessary to reduce our space of (pre-)corner fields, as it already admits the existence of a Poisson structure. This is confirmed by the fact that the kernel of the 1-forms over $\tilde{F}^{BF}_\Gamma$ generated by $\mathcal{J}_c$ and $\mathcal{V}_\tau$ is trivial, indeed

$$\mathrm{Ker}(\mathcal{J}_c + \mathcal{V}_\tau) = \left\{ \mathbb{X} = \int_\Gamma \mathbb{X}_A \frac{\delta}{\delta A} + \mathbb{X}_B \frac{\delta}{\delta B} \;\middle|\; \iota_{\mathbb{X}} \mathcal{J}_c = \iota_{\mathbb{X}} \mathcal{V}_\tau = 0 \right\} = \{0\}.$$

However, we have not yet taken into account the presence of a ghost for ghost field. Indeed, considering the particular case where $\tau = \mathrm{d}_A \phi$, we have, on the boundary

$$M_{\mathrm{d}_A\phi} = \int_\Sigma \mathrm{d}_A\phi(F_A + \Lambda B) = \int_\Sigma \Lambda\phi \mathrm{d}_A B = L_{\Lambda\phi},$$

which tells us that the gauge symmetry generated by the ghost τ is reducible. To do so, one needs to introduce another ghost, in fact a ghost for ghost field $\phi \in \Omega^0[2](\Sigma,\mathfrak{g})$, and introduce gauge transformations for c and τ as follows

$$c \longmapsto c - \Lambda\phi, \qquad \tau \longmapsto \tau + \mathrm{d}_A\phi.$$

[18]Here we choose $P(B) = \frac{1}{2}B^2$

Furthermore, one needs to assume that ϕ transforms as $\phi \mapsto \phi + [c, \phi]$.[19] This is where classical generalised geometry fails to include such contributions. Specifically, at the corner level, one needs to introduce a P_∞ structure. Such case is best analysed in the BF^{2}V formalism. The space of BF^{2}V fields is $\mathcal{F}_\Gamma^{BF} = \tilde{F}_\Gamma^{BF} \times (\Omega^0[1](\Gamma, \mathfrak{g}) \oplus \Omega^0[2](\Gamma, \mathfrak{g}) \oplus T^*(\Omega^1[1](\Gamma, \mathfrak{g})))$, which can be regrouped into superfields

$$\tilde{A} \coloneqq c + A + B^\dagger, \qquad \text{and} \qquad \tilde{B} \coloneqq \phi + \tau + B,$$

with symplectic form and action

$$\begin{aligned}
\varpi_\Gamma^{BF} &= \int_\Gamma \delta\tilde{A}\delta\tilde{B} \\
\mathcal{S}_\Gamma^{BF} &= \int_\Gamma \tilde{B}F_{\tilde{A}} + \frac{1}{2}\Lambda\tilde{B}\tilde{B} \\
&= \int_\Gamma \frac{1}{2}B[c,c] + \tau \mathrm{d}_A c + \phi(F_A + [c, B^\dagger]) + \Lambda\left(\frac{1}{2}\tau^2 + B\phi\right).
\end{aligned}$$

Remark 3.20. Notice that, if we set $B^\dagger = 0$ and $\phi = 0$, we are in the case above, where the action $\mathcal{S}_\Gamma^{BF}$ is equivalent to the Poisson structure π_2^{BF}.

One way to interpret the full $\mathcal{S}_\Gamma^{BF}$ is as a functional over the tangent space (shifted by one) to the space $\mathcal{N} \coloneqq \tilde{F}_\Gamma^{BF} \oplus \Omega^2[-1](\Gamma, \mathfrak{g}) \ni (A, B, B^\dagger)$.[20] In this case, $\mathcal{S}_\Gamma^{BF}$ is not of homogeneous degree 2 in the fiber of $T[1]\mathcal{N}$, but there is a contribution which is only linear in the fiber, hence defining a vector field on $\mathcal{N}$. Indeed, $\mathcal{S}_\Gamma^{BF}$ corresponds to a P_∞ structure of the kind $\pi = \pi_1 + \pi_2$, with

$$\begin{aligned}
\pi_1 &= \int_\Gamma (F_A + \Lambda B)\frac{\delta}{\delta B^\dagger}, \\
\pi_2 &= \pi_2^{BF} + \int_\Gamma B^\dagger \left[\frac{\delta}{\delta B^\dagger}, \frac{\delta}{\delta B}\right].
\end{aligned}$$

The CME guarantees that π_1 is cohomological, π_2 is a Poisson structure which is invariant under π_1. Furthermore, the degree-0 cohomology of π_1 amounts to the quotient of the classical functionals on $\tilde{F}_\Gamma^{BF}$ by the ideal generated by the constraint $F_A + \Lambda B$. In other words, it imposes the restriction to the Poisson submanifold $F_\Gamma^{BF} \coloneqq \{(A, B) \in \tilde{F}_\Gamma^{BF} \mid F_A + \Lambda B = 0\}$.

Remark 3.21. Notice that, considering the space F_Γ^{BF} exactly amounts to restricting ourselves to the field configurations on the corner which extend to Cauchy data on a neighborhood of the corner. Indeed, in the absence of the transversal jets, only the constraint $F_A + \Lambda B = 0$ survives, since $\mathrm{d}_A B = 0$ is a three-form, which trivially vanishes on the corners. What we have described is then known as a Dirac structure with support, whenever one considers the full homological description provided by the P_∞ structure $\pi = \pi_1 + \pi_2$.

In the remainder of the paper we will see how, for Palatini–Cartan gravity, both the reduction step and the restriction to a constrained set are necessary to fully describe the Dirac structure at the corners.

4. Review of PC gravity on manifolds with boundary

In this section we provide a quick summary of the results in [CCS21a].

4.1. **The Palatini–Cartan theory.** General Relativity is traditionally formulated ([Ein16]) using a (pseudo)Riemannian metric and its associated Levi-Civita connection. This leads to Einstein's field equations, expressed in terms of the Ricci tensor $R_{\mu\nu}$, which depends on the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ and, consequently, on the metric $g_{\mu\nu}$. The Ricci scalar R and the metric $g_{\mu\nu}$ further appear in the Einstein–Hilbert action functional:

$$S = \int R\sqrt{-g}\,\mathrm{d}^4x. \tag{22}$$

[19] In order to implement this in the BFV formalism, we define $\mathcal{F}_\Sigma^{BF} = F_\Sigma^{BF} \times T^*(\Omega^0[1](\Sigma, \mathfrak{g}) \oplus \Omega^1[1](\Sigma, \mathfrak{g}) \oplus \Omega^0[2](\Sigma, \mathfrak{g}))$, with symplectic form and action given by

$$\begin{aligned}
\varpi_\Sigma^{BF} &= \int_\Sigma \delta A\delta B + \delta A^\dagger\delta c + \delta B^\dagger\delta\tau + \delta\tau^\dagger\delta\phi; \\
\mathcal{S}_\Sigma^{BF} &= \int_\Sigma \frac{1}{2}A^\dagger[c,c] + c\mathrm{d}_A B + \tau(F_A + \Lambda B) + \phi([c,\tau^\dagger] - \mathrm{d}_A B^\dagger) + \Lambda(B\tau - A^\dagger\phi).
\end{aligned}$$

[20] In particular, the conjugate fields to A, B and $B^\dagger$ are identified with their dual, respectively given by τ, c and ϕ.

Alternatively, one can consider the same action Equation (22), but treat the connection ∇ as an independent variable, so that $S = S[g, \nabla]$. This approach allows Γ to represent the coefficients of a generic affine connection, which may, in general, have torsion. If we impose metric compatibility, the variational principle ensures the torsion-free condition (and vice versa). This formulation is known as the Palatini formalism ([Pal19a]).

By means of the coframe formalism we can cast the theory into a different setting. Namely, we are able to express the affine and the metric structure of the theory through a coframe field (a.k.a. a vielbein) e and a principal connection ω, letting general relativity resemble, from a geometrical perspective, an $\mathrm{SU}(N)$ gauge theory. This approach is known as the coframe or Palatini–Cartan formulation.

We start by considering a principal $SO(D-1,1)$ bundle P over a D-dimensional manifold M. Furthermore, we fix a D-dimensional vector space V with a reference metric η with Lorentzian signature. Then, considering ρ to be the fundamental representation of the Lorentz group, we define the associated vector bundle

$$\mathcal{V} \coloneqq P \times_\rho V,$$

which takes the name of "Minkowski bundle" (a.k.a. "fake tangent bundle"). In order to define a metric on M, which will ultimately be the dynamical object of the theory, we resort to coframes (or vielbein), defined to be non-degenerate linear isomorphisms of the kind

$$e : TM \longrightarrow \mathcal{V}.$$

One can then define a metric g on M as $g = e^*(\eta)$ or, equivalently, as $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. The other fundamental dynamical field is a principal connection $\omega \in \mathcal{A}_M$, defined on M. In general, given a reference connection ω_0, we have

$$\omega - \omega_0 \in \Omega^1(M, \mathfrak{so}(D-1,1)).$$

Since the Lie algebra of $SO(D-1,1)$ can be identified with $D \times D$ antisymmetric matrices, we model the space of connections by $\Omega^1(M, \wedge^2\mathcal{V})$.

From now on, we will adopt the following notation

$$\Omega^{i,j} \coloneqq \Omega^i(M, \wedge^j \mathcal{V}).$$

Then we are ready to give the definition of PC theory.

Definition 4.1. The classical Palatini–Cartan theory is the assignment of the pair (F_M, S_M) to every pseudo Riemannian D-dimensional manifold admitting a Lorentzian structure,[21] with space of fields given by

$$F_M = \Omega^1_{\mathrm{n.d.}}(M, \mathcal{V}) \oplus \mathcal{A}_M \ni (e, \omega),$$

and action functional (omitting the wedge product symbol)

$$S_M = \int_M \frac{1}{(D-2)!} e^{D-2} F_\omega + \frac{1}{D!} \Lambda e^D, \tag{23}$$

where $\Lambda \in \mathbb{R}$ represents the cosmological constant and $e^k \coloneqq e \wedge \cdots \wedge e$.

Remark 4.2. The integrand in Equation (23) (i.e., the Lagrangian) is a form in $\Omega^{D,D}_M = \Omega^D(M, \wedge^D\mathcal{V})$, which can canonically be identified with a density [CCS21a]. For simplicity, in the following we assume M to be orientable. A choice of orientation is reflected by a "trace" map $\mathrm{Tr}\colon \Lambda^D \to \mathbb{R}$, which applied to the Lagrangian density yields a top form. We may then view the action as the integral of this top form on M with the given orientation.

The Euler–Lagrange equations coming from the action principle $\delta S_M = 0$ are, respectively for the variations in e and ω

$$\frac{1}{(N-3)!} e^{D-3} F_\omega - \frac{1}{(D-1)!} \Lambda e^{D-1} = 0 \tag{24}$$

$$e^{D-3} \mathrm{d}_\omega e = 0, \tag{25}$$

[21] Note that any particular choice of the Lorentzian structure on V is immaterial, since a change in V would just isomorphically reflect to the space of fields without changing S_M.

which, in $D = 4$, reduce to

$$eF_\omega - \frac{1}{3!}\Lambda e^3 = 0 \tag{26}$$

$$e\mathrm{d}_\omega e = 0. \tag{27}$$

By injectivity of the map $e \wedge \cdot$ on $\Omega^{2,1}$ (see [CCS21a]), Equation (27) is equivalent to[22]

$$\mathrm{d}_\omega e = 0, \tag{28}$$

which is the torsion-free condition. Therefore, this is the equation that identifies the Levi-Civita connection for the metric induced by e. It is a matter of computations to see that the Palatini–Cartan action is classically equivalent (on-shell) to the Einstein–Hilbert one.

4.2. **Boundary structure of PC gravity.** In this section, we will assume that our stratified structure, with respect of Definition 2.6, has only two strata, i.e. we will work on a manifold with boundary. Moreover, from now on, we will specify to the case $D = 4$. The boundary structure is then obtained by applying the KT construction [KT79] outlined in the previous section. See [**cemp**; CCS21a]

Remark 4.3. Given $i\colon : \Sigma \hookrightarrow M$ the boundary inclusion. We will use the following definition for bundle valued differential forms on the boundary

$$\Omega^{i,j}_\Sigma \coloneqq \Omega^i(\Sigma, \textstyle\bigwedge^j i^*V).$$

We notice that the integration by parts in the variation of the action S_M gives rise to a boundary term

$$\tilde{\alpha}_\Sigma = \int_\Sigma \frac{1}{2} e^2 \delta\omega, \tag{29}$$

which we call the Noether 1-form. Such object is defined on the space of pre-boundary fields $\tilde{F}_\Sigma$, given by the restriction of the fields on the boundary, their transversal components and their transversal jets (to all orders).[23] Specifically, considering an infinitesimal cylindrical neighborhood $M_\epsilon \coloneqq [0, \epsilon] \times \Sigma$ of Σ, we have that a field $\Phi \in \Omega^{\bullet,\bullet}_{M_\epsilon}$ splits as

$$\Phi = \phi + \phi_n dx^n,$$

where x^n is the coordinate along the interval $[0, \epsilon]$, and

$$\phi \in \mathcal{C}^\infty([0,\epsilon]) \otimes \Omega^{\bullet,\bullet}_\Sigma, \qquad \phi_n dx^n \in \Omega^1([0,\epsilon]) \otimes \Omega^{\bullet-1,\bullet}_\Sigma.$$

The space of pre-boundary fields is then obtained by considering the restriction of the fields, their transversal components and their jets at $\Sigma \times \{0\}$. In practice, we have

$$\tilde{F}_\Sigma \coloneqq (\Omega^{1,1}_{\Sigma,\mathrm{n.d.}} \oplus \Omega^{0,1}_\Sigma) \oplus (\mathcal{A}_\Sigma \oplus \Omega^{0,2}_\Sigma) \oplus \mathfrak{J}_n(\Omega^{1,1}_\Sigma \oplus \mathcal{A}_\Sigma) \ni (e, e_n, \omega, \omega_n, \partial^k_n e, \partial^k_n \omega),$$

where $\mathfrak{J}_n(\Omega^{1,1}_\Sigma \oplus \mathcal{A}_\Sigma)$ denotes the space of transversal jets (to all orders) to e and ω.

Remark 4.4. Notice that we denote fields on the boundary with the exact same notation of the ones in the bulk. Furthermore, we assume the coframe field e restricted to the boundary to give rise to a non-degenerate boundary metric g^Σ, which is therefore either space-like or time-like.

Lemma 4.5. *The 2-form on $\widetilde{F}_\Sigma$ defined via*

$$\tilde{\varpi}_\Sigma = \delta\tilde{\alpha}_\Sigma = \int_\Sigma e\delta e\delta\omega \tag{30}$$

is closed and degenerate. Furthermore, $\mathrm{Ker}(\tilde{\varpi}_\Sigma)$ *defines an involutive regular distribution.*

Proof. The two-form is closed because it is exact. The kernel of the pre-symplectic form is not empty, indeed, it splits as

$$\mathrm{Ker}(\tilde{\varpi}_\Sigma) = \mathfrak{A} \oplus \mathfrak{X}\left(\Omega^{0,1}_\Sigma \oplus \Omega^{0,2}_\Sigma \oplus \mathfrak{J}_n(\Omega^{1,1}_\Sigma \oplus \mathcal{A}_\Sigma)\right),$$

since all the vector fields $\mathbb{Y}$ tangent to the transversal fields and jets are automatically satisfy $\iota_\mathbb{Y}\tilde{\varpi}_\Sigma = 0$. The distribution $\mathfrak{A} \subset \mathfrak{X}\left(\Omega^{1,1}_{\Sigma,\mathrm{n.d.}} \oplus \mathcal{A}_\Sigma\right)$ is such that, given

$$\mathbb{X} = \int_\Sigma \mathbb{X}_e \frac{\delta}{\delta e} + \mathbb{X}_\omega \frac{\delta}{\delta\omega} \in \mathfrak{X}\left(\Omega^{1,1}_{\Sigma,\mathrm{n.d.}} \oplus \mathcal{A}_\Sigma\right),$$

[22] This generalizes to a generic D.

[23] This description differs from the description given in [CCS21a], where only the restriction of the fields to the boundary was taken into consideration.

we get

$$\iota_{\mathbb{X}}\tilde{\varpi}_\Sigma = \int_\Sigma e(\mathbb{X}_\omega \delta e + \mathbb{X}_e \delta\omega) = 0.$$

Since we already mentioned that $e \wedge \cdot$ is injective on $\Omega^{1,1}_\Sigma$ but not on $\Omega^{1,2}_\Sigma$, it follows that $\mathbb{X}_e = 0$ and $e\mathbb{X}_\omega = 0$, namely the flow of $\mathbb{X}$ acts on ω as $\omega \mapsto \omega + v$ with $v \in \Omega^{1,2}_\Sigma$ such that $ev = 0$. The distribution is clearly regular. It is involutive because it is the kernel of a closed 2-form. □

Before performing the quotient with respect to the above distribution, we give the definition of some maps.

Definition 4.6. Considering $e \in \Omega^{1,1}_\Sigma$, we define the following:

$$\begin{aligned} W_k^{\Sigma,(i,j)} \colon \Omega^{i,j}_\Sigma &\longrightarrow \Omega^{i+k,j+k}_\Sigma \\ \alpha &\longmapsto e^k \wedge \alpha \end{aligned}$$

$$\begin{aligned} \varrho^{(i,j)} \colon \Omega^{i,j}_\Sigma &\longrightarrow \Omega^{i+1,j-1}_\Sigma \\ \alpha &\longmapsto [e, \alpha] \end{aligned}$$

Then the geometric phase space for the Palatini–Cartan theory is the symplectic manifold $(F_\Sigma, \varpi_\Sigma)$ obtained by a symplectic reduction of the space of pre-boundary fields $\widetilde{F}_\Sigma$. The symplectic form ϖ is the unique 2-form on F_Σ such that $p^*\varpi_\Sigma = \tilde{\varpi}_\Sigma$, where $p \colon \widetilde{F}_\Sigma \to F_\Sigma$ is the canonical projection, and it is given by

$$\varpi_\Sigma = \int_\Sigma e\delta e\delta[\omega], \tag{31}$$

where

$$\omega' \sim \omega \quad \Longleftrightarrow \quad \omega' - \omega \in \mathrm{Ker} W_1^{\Sigma,(1,2)}. \tag{32}$$

We refer to this equivalence class as $\mathcal{A}_{\mathrm{red}}(\Sigma)$.

As mentioned in Remark 3.2, we interpret the field equations containing non-transversal derivatives to the boundary as local functionals (constraints) on the space of pre-boundary fields of the theory. Such constraints are given by

$$\mathrm{d}_\omega e = 0, \qquad eF_\omega + \frac{1}{3!}\Lambda e^3 = 0, \tag{33}$$

while the transversal equations are

$$\mathrm{d}_\omega e_n + \mathrm{d}_{\omega_n} e = 0 \tag{34}$$

$$e_n F_\omega - e F_{\omega_n} = 0, \tag{35}$$

which are evolution equations.[24]

We notice immediately that the constraints are not well defined on F_Σ, as the torsion equation is not invariant under the action of $\mathrm{Ker}(\tilde{\varpi}_\Sigma)$.[25] However, given that $\mathrm{Ker}(\tilde{\varpi}_\Sigma) = \mathfrak{A} \oplus \mathfrak{E}$, with the identification $\mathfrak{E} \coloneqq \mathfrak{X}\left(\Omega^{0,1}_\Sigma \oplus \Omega^{0,2}_\Sigma \oplus \mathfrak{J}_n(\Omega^{1,1}_\Sigma \oplus \mathcal{A}_\Sigma)\right)$, we can perform a partial reduction $\tilde{F}_\Sigma/\mathfrak{E}$.

Lemma 4.7. *$\tilde{F}_\Sigma/\mathfrak{E}$ is diffeomorphic to the space*

$$\tilde{F}^n_\Sigma \coloneqq \{(e, \omega, \epsilon_n) \in \Omega^{1,1}_{\Sigma,\mathrm{n.d.}} \oplus \mathcal{A}_\Sigma \oplus \Omega^{0,1}_\Sigma \mid (e, \epsilon_n) \text{ forms a basis of } \mathcal{V} \text{ and } \delta\epsilon_n = 0\} \simeq \Omega^{1,1}_{\Sigma,\mathrm{n.d.}} \oplus \mathcal{A}_\Sigma$$

Proof. Clearly, quotienting by $\mathfrak{E}$ removes all the transversal fields and jets at all orders. In particular, since the only expressions containing $\omega_n, \partial^k_n e$ and $\partial^k_n \omega$ are evolution equations —which we do not take into consideration when defining the phase space— we can just discard them.

The field e_n is different, since we require $e + e_n dx^n$ to define a coframe field on M_{Σ_ϵ}, i.e. to be such that (e, e_n) is a (local) basis for $\mathcal{V}$. In the quotient space, we can then choose a representative $\epsilon_n \in [e_n]$ to be a nowhere vanishing element of $\Omega^{0,1}_\Sigma$ such that the above requirement is satisfied.

In principle, such representative depends on e. However, for any $e \in \Omega^{1,1}_{\Sigma,\mathrm{n.d.}}$, there exists an open $\mathcal{U} \subset \Omega^{1,1}_{\Sigma,\mathrm{n.d.}}$ such that $\delta\epsilon_n = 0$ on $\mathcal{U}$. The situation is further simplified if Σ is assumed to be space-like, as the choice of any time-like ϵ_n defines a globally (with respect to the full $\Omega^{1,1}_{\Sigma,\mathrm{n.d.}}$) constant section. □

[24] We will see in section 5.2.2 how they can be used to fix the representatives of the equivalence classes of the transversal jets of order one.

[25] Specifically, $d_\omega e = 0$ is not invariant under $\omega \mapsto \omega + v$.

Remark 4.8. The expression $\frac{1}{3!}\epsilon_n e^3$ is a nowhere vanishing element of $\Omega^{3,4}_\Sigma$ and can be identified with a volume form on Σ, after using the identification $\wedge^4\mathcal{V} \simeq \mathcal{C}^\infty(\Sigma)$.

The partial reduction has not solved the problem of the torsion constraint $\mathrm{d}_\omega e = 0$ not being invariant under the action of the residual vector fields in the kernel of $\tilde{\varpi}_\Sigma$, which is identified with the induced distribution on $\tilde{F}^n_\Sigma$ denoted (by abuse of notation) by $\mathfrak{U} \subset \mathfrak{X}(\tilde{F}^n_\Sigma)$. However notice that, due to the injectivity of the map $W^{1,2}_e \colon \alpha \to e \wedge \alpha$, $\mathrm{d}_\omega e = 0$ is equivalent to $e\mathrm{d}_\omega e = 0$ in the bulk. On the contrary, when moving to the boundary such injectivity property is lost, and therefore one needs to complement $e\mathrm{d}_\omega e = 0$ —which is invariant under the action of $\mathrm{Ker}(\tilde{\varpi})_\Sigma$—[26] with an appropriate extra constraint, ensuring that $\mathrm{d}_\omega e = 0$ is satisfied on the boundary.

The following lemma allows to rewrite $\mathrm{d}_\omega e = 0$ as an invariant constraint and an extra equation, which we call structural constraint for reasons that will soon become apparent.

Lemma 4.9. *Assuming the boundary metric is non-degenerate, given $\alpha \in \Omega^{2,1}_\Sigma$, we have that*

$$\alpha = 0 \qquad \Leftrightarrow \qquad \begin{cases} e\alpha = 0 \\ \epsilon_n \alpha \in \operatorname{Im} W_1^{\Sigma,(1,1)} \end{cases}$$

which implies, setting $\alpha = \mathrm{d}_\omega e$, that

$$\mathrm{d}_\omega e = 0 \qquad \Leftrightarrow \qquad \begin{cases} e\mathrm{d}_\omega e = 0 \\ \epsilon_n \mathrm{d}_\omega e = e\sigma \end{cases}$$

for some $\sigma \in \Omega^{1,1}_\Sigma$.

Remark 4.10. A quick computation shows that the transversal component of $e\mathrm{d}_\omega e = 0$ is given by

$$e_n \mathrm{d}_\omega e = e(\mathrm{d}_\omega e_n + \mathrm{d}_{\omega_n} e) = e(\mathrm{d}_\omega e_n + \partial_n e + [\omega_n, e]),$$

which essentially reproduces the extra constraint, after setting $\sigma = \sigma(e, e_n, \omega, \omega_n, \partial_n e)$. Working on the quotient space $\tilde{F}^n_\Sigma$, the dependence on ω_n and $\partial_n e$ are lost and e_n is replaced by ϵ_n.

Remark 4.11. If the boundary metric is degenerate, one needs an extra constrain to be imposed in order to obtain $\mathrm{d}_\omega e = 0$ on the boundary. We refer to [CCT21] for more details on this specific instance. In the following, we will assume $e \in \Omega^{1,1}_\Sigma$ to define a non-degenerate boundary metric whenever unspecified.

The following theorem allows to make sense of the structural constraint as a way to fix a representative of $[\omega] \in \mathcal{A}_{\mathrm{red}}$. In other words, ensuring the equivalence of $\mathrm{d}_\omega e = 0$ and $e\mathrm{d}_\omega e = 0$ on the boundary, is enough to determine uniquely the representatives of the equivalence classes defined in Equation (32).

Theorem 4.12 ([CCS21a])**.** *Given $\omega' \in \mathcal{A}_\Sigma$, there exists a unique decomposition*

$$\omega' = \omega + v$$

such that

$$ev = 0, \quad \text{and} \quad \epsilon_n \mathrm{d}_\omega e = e\sigma \tag{36}$$

Furthermore, the space

$$F^n_\Sigma := \{(e, \omega) \in \Omega^{1,1}_{\epsilon_n} \oplus \Omega^{1,2}_\Sigma \mid \epsilon_n \mathrm{d}_\omega e = e\sigma\} \subset \tilde{F}_\Sigma$$

is diffeomorphic to $\tilde{F}^n_\Sigma/\mathfrak{U} \simeq \tilde{F}_\Sigma/(\mathrm{Ker}(\tilde{\varpi}_\Sigma)) \simeq F_\Sigma$. We use such diffeomorphism to endow F^n_Σ with a symplectic form.

We now display the constraints of the theory, which we define as local functionals on F_Σ by means of Lagrange multiplier. The vanishing locus C_Σ of these functionals is then the phase space of the theory.

[26] The simple computation is given by $e\mathrm{d}_{\omega+v} e = e\mathrm{d}_\omega e + e[v, e] = e\mathrm{d}_\omega e + [ev, e] = e\mathrm{d}_\omega e$, using $[e, e] = 0$.

Definition 4.13. Let $c \in \Omega^{0,2}_\Sigma[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$ and $\lambda \in C^\infty(\Sigma)[1]$. Then, we define the following functionals

$$L_c = \int_\Sigma c e \mathrm{d}_\omega e$$
$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \iota_\xi(\omega-\omega_0)e\mathrm{d}_\omega e$$
$$H_\lambda = \int_\Sigma \lambda\epsilon_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3\Big)$$

We refer to these as the constraints of the Palatini–Cartan theory.

Remark 4.14. We have taken the Lagrange multipliers[27] [28] as odd (shifted by one) fields as they will represent ghosts within the BFV formalism.

We are now able to determine the algebra of the constraints of the theory.

Theorem 4.15 ([CCS21a])**.** *The phase space C_Σ is a coisotropic submanifold of F_Σ. In particular, the Poisson brackets of the constraints of Definition 4.13 read*

$$\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]} \qquad \{P_\xi, P_\xi\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}$$
$$\{L_c, P_\xi\} = L_{\mathrm{L}^{\omega_0}_\xi c} \qquad \{H_\lambda, H_\lambda\} = 0$$
$$\{L_c, H_\lambda\} = -P_{X^{(i)}} + L_{X^{(i)}(\omega-\omega_0)_i} - H_{X^{(n)}}$$
$$\{P_\xi, H_\lambda\} = P_{Y^{(i)}} - L_{Y^{(i)}(\omega-\omega_0)_i} + H_{Y^{(n)}}$$

with $X = [c, \lambda\epsilon_n]$ and $Y = \mathrm{L}^{\omega_0}_\xi(\lambda\epsilon_n)$ and where the superscripts (i) and (n) describe their components with respect to $\{e_i, \epsilon_n\}$.

For the sake of completeness, we provide the expression of the Hamiltonian vector fields of the constraints:

$$\mathbb{L}_e = [c, e] \qquad \mathbb{L}_\omega = \mathrm{d}_\omega c + \mathbb{V}_L$$
$$\mathbb{P}_e = -\mathrm{L}^{\omega_0}_\xi e \qquad \mathbb{P}_\omega = -\mathrm{L}^{\omega_0}_\xi(\omega-\omega_0) - \iota_\xi F_{\omega_0} + \mathbb{V}_P$$
$$\mathbb{H}_e = \mathrm{d}_\omega(\lambda\epsilon_n) + \lambda\sigma \qquad e\mathbb{H}_\omega = \lambda\epsilon_n F_\omega + \frac{1}{2}\Lambda\lambda\epsilon_n e^2.$$

where we stress that the component $\mathbb{X}_\omega$ of the above vector fields is defined up to elements in $\ker(W_1^{\Sigma,(1,2)})$.[29]

5. PC gravity on manifolds with corners

In the following, for simplicity we will assume $\Lambda = 0$, as that will have hardly any impact on the computations. The geometric set-up we are working with is that of a stratified manifold—more precisely, a manifold with corners as defined in Definition 2.6 and specified in Remark 2.7. From now on, we will refer to the codimension-1 and codimension-2 strata as, respectively, the boundary and the corner of the manifold.

We start by considering a neighborhood of the corner $\Sigma_\epsilon \coloneqq \Gamma \times [0,\epsilon]$, for an arbitrarily small $\epsilon > 0$. Setting $\{x^m\}$ to be the coordinate along $[0,\epsilon]$, any field of the form $\Phi \in \Omega^{\bullet,\bullet}(\Sigma_\epsilon)$ is split as

$$\Phi \mapsto \phi + \phi_m \mathrm{d}x^m,$$

with $\phi \in \mathcal{C}^\infty([0,\epsilon]) \otimes \Omega^{\bullet,\bullet}_\Gamma$ and $\phi_m \in \Omega^1([0,\epsilon]) \otimes \Omega^{\bullet-1,\bullet}_\Gamma$.

In this section, given $i\colon \Sigma \to M$ and $j\colon \Gamma \to \Sigma$, we will use the following notation:

$$\Omega^{i,j}_\Gamma \coloneqq \Omega^i(\Gamma, \bigwedge^j (j^*i^*\mathcal{V})), \qquad W^{\Gamma,(i,j)}_k\colon \Omega^{i,j}_\Gamma \longrightarrow \Omega^{i+1,j+1}_\Gamma, \quad \alpha \longmapsto e\wedge\alpha.$$

[27] In principle, one would need just one Lagrange multiplier $\mu \in \Omega^{0,1}_\Sigma[1]$ to impose the constraint $eF_\omega = 0$; however, we can split $\mu = \iota_\xi e + \lambda\epsilon_n$. Indeed the vector field ξ and the function λ will respectively represent the ghost fields associated to diffeomorphisms tangent to Σ and normal to Σ.

[28] Also notice that the constraint P_ξ has been complemented with the constraint $L_{\iota_\xi(\omega-\omega_0)}$, which does not change the vanishing locus of the constraint set but will simplify the computation of the Hamltonian vector field associated to it.

[29] Indeed, one can think of the physical fields in the geometric phase space to be $\{e, e(\omega-\omega_0)\}$, where the new field $\Omega \coloneqq e(\omega-\omega_0) \in \Omega^{2,3}_\Sigma$ only depends on the equivalence class of $\omega - \omega_0 \in \Omega^{1,2}_\Sigma/\mathrm{Ker}(W_e^{\Sigma(1,2)})$.

By considering the restriction at $\Gamma \times \{0\}$ of F_{Σ_ϵ}, we obtain the space of pre-corner fields

$$\tilde{F}_\Gamma \coloneqq \Omega^{1,1}_\Gamma \oplus \Omega^{0,1}_\Gamma \oplus \mathcal{A}_\Gamma \oplus \Omega^{0,2}_\Gamma \oplus \mathfrak{J}_m(\Omega^{1,1}_\Gamma \oplus \mathcal{A}_\Gamma) \ni (e, e_m, \omega, \omega_m, \partial_m^k e, \partial_m^k \omega),$$

where $e \in \Omega^{1,1}_\Gamma$ and $e_m \in \Omega^{0,1}_\Gamma$ are such that $\{e(\partial_1), e(\partial_2), e_m, \epsilon_n\}$ is a basis of $j^*i^*\mathcal{V}$ where $\{\partial_1, \partial_2\}$ is a local basis of $T\Gamma$. Notice, however, that e_m is not fixed, conversely to the case of ϵ_n. Moreover, we denoted by $\mathfrak{J}_m(\Omega^{1,1}_\Gamma \oplus \mathcal{A}_\Gamma)$ the space of transversal jets (to all orders) of e and ω.

Furthermore, we need impose the constraints which are induced from the ones on the geometric phase F_{Σ_ϵ}, i.e.

$$\epsilon_n \mathrm{d}_\omega e = e\sigma, \qquad \text{and} \qquad \epsilon_n(\mathrm{d}_{\omega_m} e - \mathrm{d}_\omega e_m) = e_m\sigma + e\sigma_m.$$

Remark 5.1. In Section 5.2.1 we will see that the transversal jets only play a marginal role in the corner structure, as their value will be fixed by constraints, or set to zero identically. The reason to keep them into consideration is simply to be able to rigorously use the Leibniz rule with respect to the operator ∂_m.

Remark 5.2. The set $\tilde{F}_\Gamma$ can also be interpreted as the quotient of $\tilde{F}_{\Sigma_\epsilon}$ by the distribution of vector fields that preserve the values of the fields and their jets at $\Gamma \times \{0\}$, from which we obtain the quotient map

$$\tilde{\pi}_\Gamma \colon F_{\Sigma_\epsilon} \longrightarrow \tilde{F}_\Gamma.$$

5.1. **Pre-Dirac corner structure of PC gravity.** Referring to the discussion of Section 3.1.2, we have that, in presence of corners, the defining equation of the Hamiltonian vector fields of the constraints ϕ_α is modified as follows

$$\delta\phi_\alpha = \iota_{\mathbb{X}_\alpha}\varpi_\Sigma - \tilde{\pi}^*_\Gamma(\mathcal{X}_\alpha), \tag{37}$$

where $\phi_\alpha = \{L_c, P_\xi, H_\lambda\}$. Notice also that the ghost ξ splits on the corner as $\xi \mapsto \zeta + \eta\partial_m$, with $\zeta \in \mathfrak{X}(\Gamma)[1]$ and $\eta \in \mathcal{C}^\infty(\Gamma)[1]$. Applying Equation (37), we obtain four (1-shifted) 1-forms on $\tilde{F}_\Gamma$, given by

$$\begin{aligned}
\mathcal{J}_c &= \int_\Gamma ce\delta e,\\
\mathcal{E}_\zeta &= \int_\Gamma \iota_\zeta ee\delta\omega + \iota_\zeta(\omega - \omega_0)e\delta e,\\
\mathcal{K}_\eta &= \int_\Gamma \eta e_m e\delta\omega + \eta(\omega - \omega_0)_m e\delta e,\\
\mathcal{F}_\lambda &= \int_\Gamma \lambda\epsilon_n e\delta\omega,
\end{aligned} \tag{38}$$

to each of which we pair the corresponding odd vector field, given by the push forward of the Hamiltonian vector fields of the constraints $(\tilde{\pi}_\Gamma)_*(\mathbb{X}_\alpha)$. Explicitly, after some manipulations, we have

$$\begin{aligned}
&\mathbb{J}_{c,e} = [c, e] && \mathbb{J}_{c,e_m} = [c, e_m]\\
&\mathbb{J}_{c,\omega} = \mathrm{d}_\omega c && \mathbb{J}_{c,\omega_m} = \mathrm{d}_{\omega_m} c\\
&\mathbb{E}_{\zeta,e} = -L^{\omega_0}_\zeta e && \mathbb{E}_{\zeta,e_m} = L^{\omega_0}_\zeta e_m + \iota_{\partial_m\zeta} e\\
&\mathbb{E}_{\zeta,\omega} = -\iota_\zeta F_{\omega_0} - L^{\omega_0}_\zeta(\omega - \omega_0) && \mathbb{E}_{\zeta,\omega_m} = L^{\omega_0}_\zeta \omega_m + \iota_{\partial_m\zeta}(\omega - \omega_0) - \iota_{\partial_m\zeta}\omega_0\\
&\mathbb{F}_{\lambda,e} = \mathrm{d}_\omega(\lambda\epsilon_n) + \lambda\sigma && \mathbb{F}_{\lambda,e_m} = \mathrm{d}_{\omega_m}(\lambda\epsilon_n) + \lambda\sigma_m\\
&e\mathbb{F}_{\lambda,\omega} = \lambda\epsilon_n F_\omega && e\mathbb{F}_{\lambda,\omega_m} + e_m\mathbb{F}_{\lambda,\omega} = \lambda\epsilon_n(\partial_m\omega - \mathrm{d}_\omega\omega_m)\\
&\mathbb{K}_{\eta,e} = -\eta\mathrm{d}_{\omega^0_m} e + d\eta e_m && \mathbb{K}_{\eta,e_m} = \mathrm{d}_{\omega^0_m}(\eta e_m)\\
&\mathbb{K}_{\eta,\omega} = -\eta\mathrm{d}_{\omega^0_m}\omega + d\eta(\omega - \omega^0)_m + \eta\mathrm{d}\omega^0_m && \mathbb{K}_{\eta,\omega_m} = \mathrm{d}_{\omega^0_m}(\eta(\omega - \omega^0)_m).
\end{aligned} \tag{39}$$

Definition 5.3. Considering again the infinitesimal neighborhood $\Sigma_\epsilon = \Gamma \times [0, \epsilon]$ of the corner, the **space of corner fields** F_Γ is defined to be the restriction at $\Gamma \times \{0\}$ of the coisotropic submanifold $C_{\Sigma_\epsilon} \subset F_{\Sigma_\epsilon}$. In particular, F_Γ can be realized as a submanifold of $\tilde{F}_\Gamma$ by imposing the following constraints, descending from the definition of C_{Σ_ϵ}:

$$e_m \mathrm{d}_\omega e = e\mathrm{d}_\omega e_m - e\mathrm{d}_{\omega_m} e \tag{40}$$

$$e_m F_\omega - eF_{\omega_m} = 0. \tag{41}$$

Additionally, F_Γ is subject to the structural constraints mentioned above

$$\epsilon_n \mathrm{d}_\omega e = e\sigma, \tag{42}$$

$$\epsilon_n(\mathrm{d}_{\omega_m} e + \mathrm{d}_\omega e_m) = e_m\sigma + e\sigma_m \tag{43}$$

which the restriction to the structural constraint Equation (36) in the boundary. We therefore obtain a surjective submersion $\pi_\Gamma \colon F_\Sigma \to F_\Gamma$ by post-composing the projection $\tilde{\pi}_\Gamma$ with the restriction map $\tilde{F}_\Gamma \to F_\Gamma$.

Remark 5.4. In the following, we will assume the metric $g^\Gamma \coloneqq e^*(\eta)$ induced on the corners is either time-like or space-like, i.e. such that it is non-degenerate.

Remark 5.5. Working on F_Γ is the same as working on-shell. Indeed, this can be understood by regrouping the constraints as $\epsilon_m F_\omega - e F_{\omega_m} = 0$ and

$$\begin{cases} \epsilon_n \mathrm{d}_\omega e = e\sigma \\ e_m \mathrm{d}_\omega e = e(\mathrm{d}_\omega e_m - \mathrm{d}_{\omega_m} e) \\ \epsilon_n(\mathrm{d}_\omega e_m - \mathrm{d}_{\omega_m} e) + e_m \sigma = e\sigma_m, \end{cases}$$

which, by applying Lemma A.3 with $\alpha = \mathrm{d}_\omega e$ and $\rho = \mathrm{d}_\omega e_m - \mathrm{d}_{\omega_m} e$, uniquely implies

$$\mathrm{d}_\omega e = 0 \tag{44}$$

$$\mathrm{d}_{\omega_m} e - \mathrm{d}_\omega e_m = 0. \tag{45}$$

This is not a surprise, since on the boundary Σ, thanks to Lemma A.2, imposing $e\mathrm{d}_\omega e = 0$ and $\epsilon_n \mathrm{d}_\omega e = e\sigma$ implies $\mathrm{d}_\omega e = 0$. The same condition then must descend to the corner, along with its transverse component.

Definition 5.6. Letting $\mathcal{X}_\alpha = \{\mathcal{J}_c, \mathcal{E}_\zeta, \mathcal{K}_\eta, \mathcal{F}_\lambda\} \in \Omega^1(F_\Gamma)[1]$ and $\mathbb{X}_\alpha = \{\mathbb{J}_c, \mathbb{E}_\zeta, \mathbb{K}_\eta, \mathbb{F}_\lambda\} \in \mathfrak{X}[1](F_\Gamma)$, we define D to be the distribution

$$D \coloneqq \mathrm{span}_{\mathcal{C}^\infty(F_\Gamma)}\{\mathbb{X}_\alpha + \mathcal{X}_\alpha\}_{\alpha=\{c,\zeta,\eta,\lambda\}} \subset \Gamma((T \oplus T^*)[1]F_\Gamma)$$

Proposition 5.7. *The distribution D is involutive and isotropic, i.e.*

$$[D, D] \subset D \qquad \text{and} \qquad \langle D, D\rangle = 0,$$

where $[\cdot,\cdot]$ is the Dorfmann bracket and $\langle\cdot,\cdot\rangle$ is the inner product on the standard Courant algebroid.

Proof. We limit ourselves to show the isotropicity, leaving the involutivity for Appendix B.1. First notice that, since we're working with odd quantities, any element $\mathbb{X}_\alpha + \mathcal{X}_\alpha$ is null-like, i.e. such that

$$\langle \mathbb{X}_\alpha + \mathcal{X}_\alpha, \mathbb{X}_\alpha + \mathcal{X}_\alpha\rangle = 0.$$

For the sake of completeness, however, we consider couples of parameters of the same kind and show the isotropicity. In particular, we start with

$$\langle \mathbb{J}_c + \mathcal{J}_c, \mathbb{J}_{c'} + \mathcal{J}_{c'}\rangle = \int_\Gamma c' e[c, e] - ce[c', e] = \int_\Gamma c'\left[c, \frac{e^2}{2}\right] - c\left[c', \frac{e^2}{2}\right] = 0,$$

having used graded Leibniz.

$$\begin{aligned}
\langle \mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{E}_{\zeta'} + \mathcal{E}_{\zeta'}\rangle &= \int_\Gamma \iota_{\zeta'} \frac{e^2}{2}(-\iota_\zeta F_{\omega_0} - \mathrm{L}_\zeta^{\omega_0}(\omega - \omega_0)) - \iota_{\zeta'}(\omega - \omega_0)\mathrm{L}_\zeta^{\omega_0}\frac{e^2}{2} \\
&\quad - \iota_\zeta \frac{e^2}{2}(-\iota_{\zeta'} F_{\omega_0} - \mathrm{L}_{\zeta'}^{\omega_0}(\omega - \omega_0)) + \iota_\zeta(\omega - \omega_0)\mathrm{L}_{\zeta'}^{\omega_0}\frac{e^2}{2} \\
&= \int_\Gamma \frac{e^2}{2}\left(\iota_{\zeta'}\iota_\zeta F_{\omega_0} + \iota_{\zeta'}\mathrm{L}_\zeta^{\omega_0}(\omega - \omega_0) - \mathrm{L}_\zeta^{\omega_0}\iota_{\zeta'}(\omega - \omega_0)\right) \\
&\quad - \frac{e^2}{2}\left(\iota_\zeta\iota_{\zeta'} F_{\omega_0} + \iota_\zeta \mathrm{L}_{\zeta'}^{\omega_0}(\omega - \omega_0) - \mathrm{L}_{\zeta'}^{\omega_0}\iota_\zeta(\omega - \omega_0)\right) \\
&= \int_\Gamma \frac{e^2}{2}\left([\iota_{\zeta'}, \mathrm{L}_\zeta^{\omega_0}](\omega - \omega_0) - [\iota_\zeta, \mathrm{L}_{\zeta'}^{\omega_0}](\omega - \omega_0)\right) \\
&= \int_\Gamma \frac{e^2}{2}\left(\iota_{[\zeta',\zeta]}(\omega - \omega_0) - \iota_{[\zeta,\zeta']}(\omega - \omega_0)\right) = 0,
\end{aligned}$$

having used that $\iota_\zeta\iota_{\zeta'} = \iota_{\zeta'}\iota_\zeta$ and $[\zeta, \zeta'] = [\zeta', \zeta]$.

$$\begin{aligned}
\langle \mathbb{K}_\eta + \mathcal{K}_\eta, \mathbb{K}_{\eta'} + \mathcal{K}_{\eta'}\rangle &= \int_\Gamma \eta'\epsilon_m e(-\eta \mathrm{d}_{\omega_m^0}\omega + \mathrm{d}\eta(\omega - \omega_0)_m + \eta \mathrm{d}\omega_m^0) + \eta'(\omega - \omega_0)_m \mathrm{d}_\omega(\eta\epsilon_m e) \\
&\quad - \eta\epsilon_m e(-\eta' \mathrm{d}_{\omega_m^0}\omega + \mathrm{d}\eta'(\omega - \omega_0)_m + \eta' \mathrm{d}\omega_m^0) - \eta(\omega - \omega_0)_m \mathrm{d}_\omega(\eta'\epsilon_m e) \\
&= \int_\Gamma \eta'\epsilon_m e \mathrm{d}\eta(\omega - \omega_0)m + \eta'(\omega - \omega_0)_m \mathrm{d}\eta\epsilon_m e - (\eta \leftrightarrow \eta') = 0,
\end{aligned}$$

$$\langle \mathbb{F}_\lambda + \mathcal{F}_\lambda, \mathbb{F}_{\lambda'} + \mathcal{F}_{\lambda'} \rangle = \int_\Gamma \lambda' \epsilon_n \lambda \epsilon_n F_\omega - \lambda \epsilon_n \lambda' \epsilon_n F_\omega = 0.$$

We are now left to show the isotropicity in the off-diagonal entries.

$$\begin{aligned}
\langle \mathbb{J}_c + \mathcal{J}_c, \mathbb{E}_\zeta + \mathcal{E}_\zeta \rangle &= \iota_{\mathbb{J}_c} \int_\Gamma \iota_\zeta \frac{e^2}{2} \delta\omega + \iota_\zeta(\omega - \omega_0) e \delta e - \iota_{\mathbb{E}_\zeta} \int_\Gamma c e \delta e \\
&= \int_\Gamma \iota_\zeta \frac{e^2}{2} \mathrm{d}_\omega c + \iota_\zeta(\omega - \omega_0) e [c, e] + c e L_\zeta^{\omega_0} e \\
&= \int_\Gamma - \frac{e^2}{2} \iota_\zeta \mathrm{d}_\omega c + \frac{e^2}{2} [\iota_\zeta(\omega - \omega_0), c] + \frac{e^2}{2} L_\zeta^{\omega_0} c \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\langle \mathbb{J}_c + \mathcal{J}_c, \mathbb{K}_\eta + \mathcal{K}_\eta \rangle &= \iota_{\mathbb{J}_c} \int_\Gamma \eta e_m e \delta\omega + \eta(\omega - \omega_0)_m e \delta e - \iota_{\mathbb{K}_\eta} \int_\Gamma c e \delta e \\
&= \int_\Gamma \eta e_m e \mathrm{d}_\omega c + \eta(\omega - \omega_0)_m [c, \frac{e^2}{2}] + c e (\eta \mathrm{d}_{\omega_m^0} e - \mathrm{d}\eta e_m) \\
&= \int_\Gamma \mathrm{d}\eta e_m e c - \eta \mathrm{d}_\omega (e_m e) c + c [\eta(\omega - \omega_0)_m, \frac{e^2}{2}] \\
&\qquad + c \eta \mathrm{d}_{\omega_m^0} \frac{e^2}{2} - c e \mathrm{d}\eta e_m \\
&= \int_\Gamma c \eta \mathrm{d}_\omega (e_m e) + c \eta e \mathrm{d}_{\omega_m} e \\
&= 0,
\end{aligned}$$

where we imposed Equation (44) and Equation (45),

$$\begin{aligned}
\langle \mathbb{J}_c + \mathcal{J}_c, \mathbb{F}_\lambda + \mathcal{F}_\lambda \rangle &= \iota_{\mathbb{J}_c} \int_\Gamma \lambda \epsilon_n e \delta\omega - \iota_{\mathbb{F}_\lambda} \int_\Gamma c e \delta e \\
&= \int_\Gamma \lambda \epsilon_n e \mathrm{d}_\omega c - c \mathrm{d}_\omega (\lambda \epsilon_n e) \\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
\langle \mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{K}_\eta + \mathcal{K}_\eta \rangle &= \iota_{\mathbb{E}_\zeta} \int_\Gamma \eta e_m e \delta\omega + \eta(\omega-\omega_0)_m e \delta \iota_{\mathbb{K}_\eta} \int_\Gamma \iota_\zeta \frac{e^2}{2} \delta\omega + \iota_\zeta(\omega-\omega_0) e \delta e \\
&= \int_\Gamma \eta e_m e (\iota_\zeta F_\omega + \mathrm{d}_\omega \iota_\zeta(\omega-\omega_0)) - \eta(\omega-\omega_0)_m L_\zeta^{\omega_0} \frac{e^2}{2} - \iota_\zeta e e F_{\omega_m} \\
&\qquad - \iota_\zeta e e \mathrm{d}_\omega(\eta(\omega-\omega_0)_m) + \iota_\zeta(\omega-\omega_0) e \eta \mathrm{d}_{\omega_m^0} e - \iota_\zeta(\omega-\omega_0) e \mathrm{d}\eta e_m \\
&= \int_\Gamma \eta e_m \iota_\zeta e F_\omega - \mathrm{d}\eta e_m e \iota_\zeta(\omega-\omega_0) + \eta \mathrm{d}_\omega(e_m e) \iota_\zeta(\omega-\omega_0) \\
&\qquad + \eta(\omega-\omega_0)_m \mathrm{d}_{\omega_0} \iota_\zeta \frac{e^2}{2} - \iota_\zeta e e F_{\omega_m} - \eta \mathrm{d}_\omega \iota_\zeta \frac{e^2}{2}(\omega-\omega_0)_m \\
&\qquad + \iota_\zeta(\omega-\omega_0) e \eta \mathrm{d}_{\omega_m^0} e - \iota_\zeta(\omega-\omega_0) e \mathrm{d}\eta e_m \\
&= \int_\Gamma \eta e_m \iota_\zeta e F_\omega + \eta e \mathrm{d}_{\omega_m} e \iota_\zeta(\omega-\omega_0) - \iota_\zeta e e F_{\omega_m} \\
&\qquad - \eta(\omega-\omega_0)_m [\omega-\omega_0, \iota_\zeta \frac{e^2}{2}] + \iota_\zeta(\omega-\omega_0) e \eta \mathrm{d}_{\omega_m^0} e \\
&= \int_\Gamma \eta \iota_\zeta e (e_m F_\omega - e F_{\omega_m}) - \eta e \mathrm{d}_{\omega_m} e \iota_\zeta(\omega-\omega_0) \\
&\qquad + \eta \iota_\zeta(\omega-\omega_0)([(\omega-\omega_0)_m, \frac{e^2}{2}] - \mathrm{d}_{\omega_m^0} \frac{e^2}{2}) \\
&= 0,
\end{aligned}
$$

where we used Equation (44), Equation (45) and Equation (41),

$$
\begin{aligned}
\langle \mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{F}_\lambda + \mathcal{F}_\lambda \rangle &= \iota_{\mathbb{E}_\zeta} \int_\Gamma \lambda \epsilon_n e \delta\omega - \iota_{\mathbb{F}_\lambda} \int_\Gamma \iota_\zeta \frac{e^2}{2} \delta\omega + \iota_\zeta(\omega-\omega_0) e \delta e \\
&= \int_\Gamma \lambda \epsilon_n e (-\iota_\zeta F_\omega + \mathrm{d}\omega \iota_\zeta(\omega-\omega_0)) - \iota_\zeta e \lambda \epsilon_n F_\omega - \iota_\zeta(\omega-\omega_0) \mathrm{d}_\omega(\lambda \epsilon_n e) \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\langle \mathbb{F}_\lambda + \mathcal{F}_\lambda, \mathbb{K}_\eta + \mathcal{K}_\eta \rangle &= \iota_{\mathbb{F}_\lambda} \int_\Gamma \eta e_m e \delta\omega + \eta(\omega-\omega_0)_m e \delta e - \iota_{\mathbb{K}_\eta} \int_\Gamma \lambda \epsilon_n e \delta\omega \\
&= \int_\Gamma \eta e_m \lambda \epsilon_n F_\omega + \eta(\omega-\omega_0)_m \mathrm{d}_\omega(\lambda \epsilon_n e) - \lambda \epsilon_n e \eta F_{\omega_m} - \lambda \epsilon_n e \mathrm{d}_\omega(\eta(\omega-\omega_0)_m) \\
&= \int_\Gamma \eta \lambda \epsilon_n (e_m F_\omega - e F_{\omega_m}) \\
&= 0,
\end{aligned}
$$

where we implemented again Equation (41), which concludes the proof of the isotropy.

□

5.2. **Maximality and reduction.** In order to obtain a genuine Dirac structure, which equates to imposing the maximality condition, we follow the prescription described in Section 3.1.2. Given that F_Γ is the submanifold of $\tilde{F}_\Gamma$ on which constraints Equation (44), Equation (45) and Equation (41) are imposed, it is not easy to work with TF_Γ. Therefore, we quotient $\tilde{F}_\Gamma$ with respect to the distribution $\tilde{\mathfrak{W}} := \mathrm{Ker}(\sum_\alpha \mathcal{X}_\alpha) \subset \Gamma(T\tilde{F}_\Gamma)$ and then apply the constraints on the quotient.[30]

[30] As we will see, this will not spoil the involutivity and isotropy property.

A quick computation[31] shows that $\tilde{\mathfrak{W}}$ is given by

$$\left\{ \int_\Gamma \mathbb{X}_e \frac{\delta}{\delta e} + \mathbb{X}_{e_m} \frac{\delta}{\delta e_m} + \mathbb{X}_\omega \frac{\delta}{\delta \omega} + \mathbb{X}_{\omega_m} \frac{\delta}{\delta \omega_m} + \sum_{k\geq 1} \left(\mathbb{X}_{\partial_m^k e} \frac{\delta}{\delta(\partial_m^k e)} + \mathbb{X}_{\partial_m^k \omega} \frac{\delta}{\delta(\partial_m^k \omega)} \right) \;\middle|\; e\mathbb{X}_e = 0,\ e\mathbb{X}_\omega = 0 \right\}.$$

Notice that we can split $\tilde{\mathfrak{W}} \simeq \tilde{\mathcal{B}} \oplus \mathfrak{X}(\Omega_\Gamma^{0,1} \oplus \Omega_\Gamma^{0,2} \oplus \mathfrak{J}_m(\Omega_\Gamma^{1,1} \oplus \mathcal{A}_\Gamma))$ with

$$\tilde{\mathcal{B}} = \left\{ \mathbb{X} = \int_\Gamma \mathbb{X}_e \frac{\delta}{\delta e} + \mathbb{X}_\omega \frac{\delta}{\delta \omega} \;\middle|\; e\mathbb{X}_e = 0,\ e\mathbb{X}_\omega = 0. \right\}.$$

Remark 5.8. The above splitting will imply that, in the quotient, we can discard the transversal fields e_m and ω_m and the jets to all orders $\partial_m^k e$ and $\partial_m^k \omega$ as dynamical fields, fixing their values in a way compatible with the constraints.

Proposition 5.9. *The distribution $\tilde{\mathfrak{W}}$ is regular and involutive.*

Proof. The regularity condition is immediately verified by noticing that $\dim \tilde{\mathcal{B}} = \dim \ker(W_1^{\Gamma,(1,1)}) + \dim \ker(W_1^{\Gamma,(1,2)})$, which is constant.

Regarding involutivity, we first notice that $\mathfrak{X}(\Omega_\Gamma^{0,1} \oplus \Omega_\Gamma^{0,2} \oplus \mathfrak{J}_m(\Omega_\Gamma^{1,1} \oplus \mathcal{A}_\Gamma))$ is involutive by construction, while $[\tilde{\mathcal{B}}, \mathfrak{X}(\Omega_\Gamma^{0,1} \oplus \Omega_\Gamma^{0,2} \oplus \mathfrak{J}_m(\Omega_\Gamma^{1,1} \oplus \mathcal{A}_\Gamma))] = 0$ trivially. We are then left with showing that $\tilde{\mathcal{B}}$ is involutive. To do so, consider the following identity

$$\iota_{[\mathbb{X},\mathbb{Y}]} \mathcal{X}_\alpha = \mathrm{L}_\mathbb{X} \iota_\mathbb{Y} \mathcal{X}_\alpha - \iota_\mathbb{Y} \mathrm{L}_\mathbb{X} \mathcal{X}_\alpha,$$

where $\mathrm{L}_\mathbb{X} = \iota_\mathbb{X} \delta + \delta \iota_\mathbb{X}$. Letting $\mathbb{X}, \mathbb{Y} \in \tilde{\mathcal{B}}$, i.e. such that for all $\alpha = c, \zeta, \eta, \lambda$, $\iota_\mathbb{X} \mathcal{X}_\alpha = \iota_\mathbb{Y} \mathcal{X}_\alpha = 0$, we have

$$\begin{aligned} -\iota_{[\mathbb{X},\mathbb{Y}]} \mathcal{X}_\alpha =& \iota_\mathbb{Y} \mathrm{L}_\mathbb{X} \mathcal{X}_\alpha \\ =& \iota_\mathbb{Y} \iota_\mathbb{X} \delta \mathcal{X}_\alpha + \iota_\mathbb{Y} \delta \iota_\mathbb{X} \mathcal{X}_\alpha \\ =& \iota_\mathbb{Y} \iota_\mathbb{X} \delta \mathcal{X}_\alpha. \end{aligned}$$

For $\alpha = c, \zeta$, we notice that $\mathcal{J}_c$ and $\mathcal{E}_\zeta$ are δ-exact, hence obtaining $\iota_{[\mathbb{X},\mathbb{Y}]} \mathcal{J}_c = \iota_{[\mathbb{X},\mathbb{Y}]} \mathcal{E}_\zeta = 0$. We then just need to show $\iota_{[\mathbb{X},\mathbb{Y}]} \mathcal{K}_\eta = \iota_{[\mathbb{X},\mathbb{Y}]} \mathcal{F}_\lambda = 0$. We have

$$\begin{aligned} \iota_{[\mathbb{X},\mathbb{Y}]} \mathcal{K}_\eta &= \int_\Gamma \eta \mathbb{X}_{e_m} e\mathbb{Y}_\omega - \eta \mathbb{Y}_{e_m} e\mathbb{X}_\omega - \eta e_m (\mathbb{X}_e \mathbb{Y}_\omega - \mathbb{Y}_e \mathbb{X}_\omega) - \eta \mathbb{X}_{\omega_m} e\mathbb{Y}_e + \eta \mathbb{Y}_{\omega_m} e\mathbb{X}_e \\ &= \int_\Gamma \eta e_m e (\mathrm{L}_\mathbb{X} \mathrm{L}_\mathbb{Y} (\omega - \omega_0) - \mathrm{L}_\mathbb{Y} \mathrm{L}_\mathbb{X} (\omega - \omega_0)) \\ &= \int_\Gamma \eta e_m e \mathrm{L}_{[\mathbb{X},\mathbb{Y}]} (\omega - \omega_0) \\ &= \int_\Gamma \eta e_m e [\mathbb{X}, \mathbb{Y}]_\omega, \end{aligned}$$

having used $e\mathbb{X}_e = e\mathbb{Y}_e = 0$, $e\mathbb{X}_\omega = e\mathbb{Y}_\omega = 0$, the fact that $\mathbb{X}_\Phi = \mathrm{L}_\mathbb{X} \Phi$ for any $\Phi \in \tilde{F}_\Gamma$ and the identity

$$\mathrm{L}_\mathbb{X} \mathrm{L}_\mathbb{Y} - \mathrm{L}_\mathbb{Y} \mathrm{L}_\mathbb{X} = \mathrm{L}_{[\mathbb{X},\mathbb{Y}]}.$$

It is a quick computation to see that

$$[\mathbb{X}, \mathbb{Y}]_\omega = \left(\mathbb{X}_e \frac{\delta \mathbb{Y}_\omega}{\delta e} + \mathbb{X}_\omega \frac{\delta \mathbb{Y}_\omega}{\delta \omega} \right) - \left(\mathbb{Y}_e \frac{\delta \mathbb{X}_\omega}{\delta e} + \mathbb{Y}_\omega \frac{\delta \mathbb{X}_\omega}{\delta \omega} \right),$$

hence verifying $e[\mathbb{X}, \mathbb{Y}]_\omega = 0$. Lastly, we are left with

$$\iota_{[\mathbb{X},\mathbb{Y}]} \mathcal{F}_\lambda = \int_\Gamma \lambda \epsilon_n e [\mathbb{X}, \mathbb{Y}]_\omega = 0,$$

showing involutivity. □

[31] To be precise, $\tilde{\mathfrak{W}}$ is given by those vector fields $\mathbb{X} \in \mathfrak{X}(F_\Gamma)$ such that

$$\int_\Gamma c e \mathbb{X}_e = 0 \qquad \int_\Gamma \iota_\zeta e e \mathbb{X}_\omega + \iota_\zeta (\omega - \omega_0) e \mathbb{X}_e = 0$$
$$\int_\Gamma \lambda \epsilon_n e \mathbb{X}_\omega = 0 \qquad \int_\Gamma \eta e_m e \mathbb{X}_\omega + \eta (\omega - \omega_0)_m e \mathbb{X}_e = 0.$$

From which we obtain $e\mathbb{X}_e = 0$ and $(\iota_\zeta e + \eta e_m + \lambda \epsilon_n) e\mathbb{X}_\omega = \mu e \mathbb{X}_\omega = 0$, for all $\mu \in \Omega_\Gamma^{0,1}$, implying $e\mathbb{X}_\omega = 0$.

5.2.1. *The reduced space of corner fields.* The quotient of $\tilde{F}_\Gamma$ with respect to $\tilde{\mathfrak{W}}$ is given by

$$\tilde{P}_\Gamma \coloneqq \tilde{F}_\Gamma/\tilde{\mathfrak{W}} \simeq (\Omega^{1,1}_\Gamma \oplus \mathcal{A}_\Gamma)/\tilde{\mathcal{B}}.$$

This is achieved by fixing the transversal fields $e_m \in \Omega^{0,1}_\Gamma$ and $\omega_m \in \Omega^{0,2}_\Gamma$ and the transversal jets $\partial^k_m e \in \Omega^{1,1}_\Gamma$ and $\partial^k_m \omega \in \mathcal{A}_\Gamma$. We will see how some of these fields will be fixed by imposing the structural constraints on the reduced space of corner fields. For the moment, we choose a particular e_m that is convenient for future calculations.

Definition 5.10. We define $\epsilon_m \in \Omega^{0,1}_\Gamma$ to be the unique section of $\mathcal{V}$ which satisfies the following conditions

$$\begin{aligned} &[e, \epsilon_m] = 0 \\ &[\epsilon_n, \epsilon_m] = 0 \\ &[\epsilon_m, \epsilon_m] = \nu, \end{aligned} \tag{46}$$

with $\nu = \pm 1$ depending on the signature of Γ and Σ.

Remark 5.11. The above definition is well given, as ϵ_m has four components which are point-wise uniquely determined by the above four equations. They amount to requesting ϵ_m to be linearly independent from e and ϵ_n, while also being normalized to ± 1. It follows that $\epsilon_m \in \Omega^{0,1}_\Gamma$ is such that $\{e_i, \epsilon_m, \epsilon_n\}$ form a local basis of $\mathcal{V}$.

Proposition 5.12. *We can fix $\epsilon_m \in \Omega^{0,1}_\Gamma$ as a representative of $[e_m]$ in $\tilde{P}_\Gamma$.*

Remark 5.13. It is important to notice that ϵ_m depends on e, and therefore $\delta\epsilon_m \neq 0$.

Remark 5.14. We can equivalently define ϵ_m by taking the hodge dual of $E\epsilon_n$ (both in $\wedge^\bullet V$ and $\wedge^\bullet T^*M$). Indeed, assuming $\{v_a\}$ is an orthonormal basis of V, i.e. such that $[v_a, v_b] = \eta_{ab}$, one has $v_a v_b v_c v_d = \frac{1}{4!}\epsilon_{abcd}\mathrm{Vol}_V$, hence we can define

$$\epsilon^a_m \coloneqq \frac{\varsigma}{2 \cdot 4!}\epsilon^{ij}\eta^{ab}\epsilon_{bcdf}e^c_i e^d_j \epsilon^f_n, \tag{47}$$

where ς is a normalisation factor defined such that $E\epsilon_m\epsilon_n = \mathrm{Vol}_\Gamma \mathrm{Vol}_V$.[32]

We can now move on to describe the quotient $(\Omega^{1,1}_\Gamma \oplus \mathcal{A}_\Gamma)/\tilde{\mathcal{B}}$, starting from the coframes $[e]$. Denoting the reduced space of corner vielbeins by $\Omega^{1,1}_{\Gamma,r}$, we have

$$\Omega^{1,1}_{\Gamma,r} \coloneqq \{[e] \subset \Omega^{1,1}_\Gamma \mid e' \sim e \text{ iff } \exists\, t \geq 0 \text{ s.t. } e' = \Psi^t_{\mathbb{X}}(e), \text{ for some } \mathbb{X} \in \tilde{\mathfrak{W}}\ \},$$

where $\Psi^t_{\mathbb{X}}$ is the flow of $\mathbb{X}$ at time t. Then, on an open $\mathcal{U}_\Gamma \subset \Omega^{1,1}_{\Gamma,r}$, one can identify $\tilde{P}_\Gamma$ with the fiber bundle $\tilde{P}_\Gamma \longrightarrow \Omega^{1,1}_{\Gamma,r}$ such that locally

$$\tilde{P}_\Gamma\big|_{\mathcal{U}_\Gamma} \simeq \mathcal{U}_\Gamma \times \mathcal{A}_{\Gamma,r},$$

where $\mathcal{A}_{\Gamma,r} \coloneqq \{[\omega] \subset \mathcal{A}_\Gamma \mid \omega' \sim \omega \text{ iff } \omega' - \omega = v, \text{ s.t. } ev = 0\}$.

The difficulty of working with quotient spaces is evident. Therefore, it is convenient to find a more manageable description of such spaces. In particular, we can overcome the problem of dealing with $\Omega^{1,1}_{\Gamma,r}$ with two equivalent strategies: fix a representative of $[e]$ by working in a specific submanifold of $\Omega^{1,1}_\Gamma$ that is in 1-to-1 correspondence with $\Omega^{1,1}_{\Gamma,r}$, or establish a different bijection between $\Omega^{1,1}_{\Gamma,r}$ and another more convenient space.

The latter option has been explored in [CC24] and is realized by considering the field redefinition $E \coloneqq \frac{e^2}{2} \in \Omega^{2,2}_\Gamma$. It is easy to prove that E only depends on the equivalence class $[e]$. Indeed, considering the infinitesimal transformation $e \mapsto e + \alpha$, with $e\alpha = 0$ and $\alpha^2 = 0$,[33] we see

$$E \longmapsto \frac{1}{2}(e+\alpha)^2 = \frac{1}{2}e^2 + e\alpha + \frac{1}{2}\alpha^2 = E.$$

It was established in [CC24] that any element $B \in \Omega^{2,2}_\Gamma$ is of the form $B = \frac{1}{2}a^2$ for some $a \in \Omega^{1,1}_\Gamma$ if and only if $\mathrm{Pf}(B) = 0$, where Pf indicates the Pfaffian. Denoting by $\Omega^{2,2}_{\Gamma,0}$ the subset of $\Omega^{2,2}_\Gamma$ of elements with vanishing Pfaffian, one obtains the isomorphism

$$\Omega^{1,1}_{\Gamma,r} \simeq \Omega^{2,2}_{\Gamma,0}.$$

Remark 5.15. The non-degeneracy assumption of the vielbein e is reformulated by the open condition

$$\epsilon_m\epsilon_n E \neq 0.$$

[32] With such definition, one automatically obtains that $[e, \epsilon_m] = 0$ and $[\epsilon_n, \epsilon_m] = 0$.

[33] One can think of α as $\alpha = \mathbb{X}_e dt$, then $\alpha^2 \propto dt^2 = 0$.

Alternatively, one can find a representative of $[e]$ thanks to the following.

Theorem 5.16. *Given $e^0 \in \Omega^{1,1}_\Gamma$ as a reference vielbein, for all $\tilde{e} \in \Omega^{1,1}_\Gamma$ such that $\epsilon_m\epsilon_n e(e^0)^2$ induces the same orientation as $\epsilon_m\epsilon_n\tilde{e}^2$,*[34] *there exists a unique $e \in [\tilde{e}]$ such that*

$$\epsilon_n\epsilon_m e \in \operatorname{Im} W^{\Gamma,(0,2)}_{e^0}. \tag{48}$$

Proof. By Lemma A.7, we have that in general

$$\epsilon_m\epsilon_n\tilde{e} = e^0\alpha + \epsilon_m\epsilon_n\beta, \qquad \text{s.t.} \qquad \begin{cases}\epsilon_m\epsilon_n\alpha = 0\\ e^0\beta = 0\end{cases}$$

Therefore, defining $e \coloneqq \tilde{e} - \beta$ uniquely defines a representative of the equivalence class

$$[\tilde{e}]_0 \coloneqq \{e \in \Omega^{1,1}_\Gamma \mid e^0(\tilde{e} - e) = 0\}.$$

However, it is immediate to see that the two spaces $\Omega^{1,1}_{\Gamma,r_0} \coloneqq \{[\tilde{e}]_0 \subset \Omega^{1,1}_\Gamma\}$ and $\Omega^{1,1}_{\Gamma,r}$ are isomophic. Therefore, fixing a representative of $[\tilde{e}]_0$ is equivalent to fixing a representative of $[e]$. □

Remark 5.17. To have a more constructive perspective on the last theorem, one can alternatively notice that it is possible to write

$$e = \iota_\chi e^0 + \alpha_m\epsilon_m + \alpha_n\epsilon_n,$$

with $\chi \in \Omega^{1,0}_\Gamma \otimes \mathfrak{X}(\Gamma)$ and $\alpha_m, \alpha_n \in \Omega^{0,0}_\Gamma$. Then the constraint $\epsilon_m\epsilon_n e = e^0\alpha$ implies

$$\iota_\chi e^0\epsilon_m\epsilon_n = e^0\alpha,$$

which, after retracing the steps in the proof of A.7, is solved by $\iota_\chi e^0 = fe^0$ for some $f \in \Omega^{0,0}_\Gamma$. Therefore, Equation (48) imposes 3 equations, which ultimately fix the representative in the 5-dimensional space $\Omega^{1,1}_{\Gamma,r}$.

Additionally, from the proof of proposition 62 in [CC24], we have that the isomorphism $\Omega^{1,1}_{\Gamma,r} \simeq \Omega^{2,2}_{\Gamma,0}$ is ensured by requiring that $E = \frac{1}{2}e^2$ is such that $e = fe^0 + \alpha_m\epsilon_m + \alpha_n\epsilon_n$, which is exactly what we find by imposing Equation (48).

We are still left with the task of describing $\mathcal{A}_{\Gamma,r}$. Again, we can obtain an equivalent description by introducing a reference connection ω_0,[35] and by considering the field redefinition

$$\Omega \coloneqq e(\omega - \omega_0).$$

Indeed the map $W^{\Gamma,(1,2)}_1 \colon (\omega - \omega_0) \mapsto e(\omega - \omega_0)$ is only sensitive to the equivalence class $[\omega - \omega_0]$. Furthermore, due to the surjectivity of $W^{\Gamma,(1,2)}_1$, we have that $W^{\Gamma,(1,2)}_1$ establishes an isomorphism

$$\Omega^{1,2}_{\Gamma,r} \simeq \Omega^{2,3}_\Gamma,$$

where $\Omega^{1,2}_{\Gamma,r} \coloneqq \{[\omega - \omega_0] \subset \Omega^{1,2}_\Gamma \mid \omega \sim \omega', \iff \omega - \omega' \in \operatorname{Ker}(W^{\Gamma,(1,2)}_1)\}$.

Remark 5.18. Fixing the representative of $[e] \in \Omega^{1,1}_{\Gamma,r}$ ensures that the maps $W^{\Gamma,(i,j)}_e$ are well defined on the quotient space $\tilde{P}_\Gamma$, guaranteeing that Ω is well-defined.

Equivalently, we can obtain a description of $\Omega^{1,2}_{\Gamma,r}$ by the following.

Theorem 5.19. *For any $\omega' \in \mathcal{A}_\Gamma$, there exist unique $r \in \Omega^{0,1}_\Gamma$, $b \in \Omega^{1,0}_\Gamma$ and $v \in \Omega^{1,2}_\Gamma$ such that*

$$\omega' - \omega_0 = v + er + \epsilon_m\epsilon_n b, \quad \text{s.t.} \quad \begin{cases}[e, r] = \varrho^{0,1}_\Gamma r = 0\\ ev = 0\end{cases}$$

Therefore, the constraint

$$\omega - \omega_0 = er + \epsilon_m\epsilon_n b \tag{49}$$

uniquely fixes a representative of $[\omega - \omega_0]$.

[34] In particular, one still requires $\{e_0, \epsilon_m, \epsilon_n\}$ to define a local basis of $\mathcal{V}$.

[35] This enables us to consider $\Omega^{1,2}_\Gamma$ as our model space for $\mathcal{A}_\Gamma$, as $(\omega - \omega_0) \in \Omega^{1,2}_\Gamma$.

Proof. We want to prove that

$$\Omega^{1,2}_\Gamma = \ker W_1^{\Gamma,(1,2)} \oplus e \ker(\varrho^{0,1}_\Gamma) \oplus \epsilon_m\epsilon_n \Omega^{1,0}_\Gamma.$$

We start by checking that the dimensions of the subspaces add up to the dimension of $\Omega^{1,2}_\Gamma$. In particular, we know by [CC24] that $\dim(\ker(W_1^{\Gamma,(1,2)})) = 8$, while it is easy to check that $[e, r] = 0$ is a set of two independent equations, hence $\dim(\ker(\varrho^{0,1}_\Gamma)) = 2$. One also trivially has $\dim \Omega^{1,0}_\Gamma = 2$. We observe

$$\dim \Omega^{1,2}_\Gamma = 12 = \dim\ker W_1^{\Gamma,(1,2)} + \dim\ker \Omega^{0,1}_\Gamma + \dim \Omega^{1,0}_\Gamma.$$

Secondly, from Theorem A.1 one sees that $W_1^{\Gamma,(0,1)}$ is injective. In the same way, a simple computations shows that $W^{1,0}_{mn}$ is also injective. We are left to show that $v + er + \epsilon_m\epsilon_n b = 0$ implies $v = 0$, $r = 0$ and $b = 0$. That is to say the three terms are linearly independent.

By separately multiplying by ϵ_m and ϵ_n we obtain

$$\epsilon_n v = \epsilon_n e r, \qquad \epsilon_m v = \epsilon_m e r,$$

Therefore, by defining $v_n = \epsilon_n r$ and $v_m = \epsilon_n r$, since $ev = 0$, we can apply Lemma A.6 to show that $v = 0$.

We are then left with $er = \epsilon_n\epsilon_m b$, from which we infer

$$\begin{cases} e^2\epsilon_n r = 0 \\ e^2\epsilon_m r = 0 \\ e\epsilon_n\epsilon_m r = 0 \end{cases}$$

then, from A.4, it follows $r = 0$. Lastly, $\epsilon_n\epsilon_m b = 0$ is uniquely solved by $b = 0$ due to injectivity of $W^{1,0}_{mn}$. □

Given $\omega - \omega_0 = er + \epsilon_m\epsilon_n b$, one would be tempted to fix

$$(\omega - \omega_0)_m = \epsilon_m(r + \epsilon_n b_m),$$

however, the following choice will be more suitable for the computations to come.

Definition 5.20. We fix the representative of $[\omega_m]$ by requiring

$$(\omega - \omega_0)_m = 0. \tag{50}$$

Finally, we can define the reduced space of corner fields P_Γ by imposing the constraints Equation (44), Equation (45) and Equation (41).

Proposition 5.21. *The constraint Equation* (45) *fixes the values of the transversal jet* $\partial_m e$.

Remark 5.22. In principle, one cannot integrate by parts with respect to ∂_m, as $\partial_m e$ and $\partial_m\omega$ are to be treated as fields. However, this operation becomes rigorous when considering the whole space of jets to all orders.

Proof. Wee see that Equation (45) is such that

$$\partial_m e = \mathrm{d}_\omega \epsilon_m - [\omega_m, e],$$

hence fixing $\partial_m e$. □

Remark 5.23. In the following subsection, we will see how to fix the transversal jet $\partial_m\omega$.

Lastly, since the higher order transversal jets never appear in any computations, we simply fix them by requiring that the vanish in the quotient space $\tilde{F}_\Gamma/\tilde{\mathfrak{W}}$. The last constraint fixes some components of ω, by requiring $\mathrm{d}_\omega e = 0$, but not all of them.

5.2.2. *Bianchi identities and boundary fields revisited.* Bianchi identities in the bulk simply read

$$\mathrm{d}_\omega F_\omega = 0 \qquad \text{and} \qquad \mathrm{d}^2_\omega(-) = [F_\omega, -].$$

The question arises on how such equations "propagate" to higher codimension strata.[36] In order to answer the question fully, one needs to consider the transversal fields and jets to the boundary Σ. In a way similar to the one presented above, it is possible to fix the values of these extra fields such that the resulting theory is equivalent to the one we presented in Section 4.2, when restricting to the coisotropic

[36]In the following, we will denote bulk, boundary and corner fields with the same notation. They will be distinguishable by the context.

submanifold C_Σ. In particular, considering $(e_n, \omega_n) \in \Omega^{0,1}_\Sigma \oplus \Omega^{(0,2)}_\Sigma$ and the transversal jets $\partial^k_n e$ and $\partial^i_n \omega$ of the fields at all orders, one can, for example, rewrite the condition $\mathrm{d}_\omega e = 0$ as

$$e\mathrm{d}_\omega e = 0, \tag{51}$$

$$\epsilon_n \mathrm{d}_\omega e = e(\mathrm{d}_{\omega_n} e - \mathrm{d}_\omega \epsilon_n), \tag{52}$$

having fixed the value of e_n as $e_n = \epsilon_n$. As already mentioned in Remark 4.10, notice here that Equation (52) is a reinterpretation of the structural constraint Equation (36) which includes the transversal jet $\partial_n e$. Furthermore, if we restrict ourselves to the submanifold where the full torsion constraint $\mathrm{d}_\omega e = 0$ is imposed, Equation (52) reduces to

$$\mathrm{d}_{\omega_n} e = \mathrm{d}_\omega \epsilon_n,$$

which fixes the value of the transversal jet $\partial_n e$.

On the other hand, when considering the transversal fields and jets, Einstein's equation in the bulk gives rise to an additional constraint descending to the boundary:

$$\epsilon_n F_\omega = e F_{\omega_n}. \tag{53}$$

At first, one could think that this equation should somehow fix some components the curvature F_ω, however, one can always find an element $\gamma \in \Omega^{1,2}_\Sigma$ such that $\epsilon_n F_\omega = e\gamma$, as $W_e^{\Sigma,(1,2)}$ is surjective. We can then use Equation (53) to fix some components of F_{ω_n} (and henceforth of $\partial_n \omega$), leaving out the ones which lie in the kernel of $W_e^{\Sigma,1,2}$.

Taking the covariant derivative of the torsionless condition, the second Bianchi identity then simply implies

$$[F_\omega, e] = 0.$$

We stress that such equation holds only on the coisotropic submanifold which we called C_Σ, where $\mathrm{d}_\omega e = 0$ has been imposed.

The exterior covariant derivative of Equation (52) yields

$$\mathrm{d}_\omega \epsilon_n \mathrm{d}_\omega e - \epsilon_n [F_\omega, e] = \mathrm{d}_\omega e(\mathrm{d}_{\omega_n} e - \mathrm{d}_\omega \epsilon_n) + e(\mathrm{d}_{\omega_n} \mathrm{d}_\omega e + [F_{\omega_n}, e] - [F_\omega, \epsilon_n]),$$

which, applying the on-shell condition and the second Bianchi identity, implies

$$e([F_{\omega_n}, e] - [F_\omega, \epsilon_n]) = 0. \tag{54}$$

This suggest considering the following equation, which we will refer to as **n-transversal Bianchi identity**:

$$[e, F_{\omega_n}] = [\epsilon_n, F_\omega]. \tag{55}$$

Proposition 5.24. *The n-transversal Bianchi identity Equation* (55) *and the n-transversal Einstein equation Equation* (53) *uniquely fix F_{ω_n}, hence the transversal jet $\partial_n \omega$ to the boundary.*

Proof. We give some hints of proof here and leave the full one for the reader.

We start by noticing that $e([F_{\omega_n}, e] - [F_\omega, \epsilon_n]) = 0$ and $[F_{\omega_n}, e] - [F_\omega, \epsilon_n] = 0$ are not equivalent. In particular, the latter comprises 12 equations while the former only 6. However, we argue that equation Equation (54) holds trivially on C_Σ. Indeed, using the fact that $[e, e] = 0$, Equation (53) and the second Bianchi identity $[e, F_\omega] = 0$, one can rewrite Equation (54) as

$$[\epsilon_n, eF_\omega] = 0.$$

Therefore, on C_Σ Equation (55) only imposes 6 equations (lying in the kernel of $W_e^{\Sigma,(1,2)}$), which, along with the 12 from Equation (53), uniquely fix the 18 components of $\partial_n \omega$.[37] □

Now we can repeat the same arguments for the corner fields. We summarise the result in the following table, setting $\sigma \coloneqq \mathrm{d}_{\omega_n} e - \mathrm{d}_\omega \epsilon_n$.

[37] Strictly speaking, the above equations fix F_{ω_n}, but $F_{\omega_n} = -\partial_n \omega + \mathrm{d}_\omega(\omega_n)$.

M	Σ	Γ
$\mathrm{d}_\omega e = 0$	$e\mathrm{d}_\omega e = 0,$ $\epsilon_n \mathrm{d}_\omega e = e(\mathrm{d}_{\omega_n} e - \mathrm{d}_\omega \epsilon_n)$	$\epsilon_m \mathrm{d}_\omega e = e(\mathrm{d}_{\omega_m} e - \mathrm{d}_\omega \epsilon_m),$ $\epsilon_n \mathrm{d}_\omega e = e(\mathrm{d}_{\omega_n} e - \mathrm{d}_\omega \epsilon_n),$ $\epsilon_n(\mathrm{d}_\omega \epsilon_m - \mathrm{d}_{\omega_m} e) = \epsilon_m \sigma + e\sigma_m$
$[e, F_\omega] = 0$	$[e, F_\omega] = 0,$ $[\epsilon_n, F_\omega] = [e, F_{\omega_n}]$	$[\epsilon_n, F_\omega] = [e, F_{\omega_n}],$ $[\epsilon_m, F_\omega] = [e, F_{\omega_m}],$ $[\epsilon_n, F_{\omega_m}] = [\epsilon_m, F_{\omega_n}]$
$eF_\omega = 0$	$eF_\omega = 0,$ $\epsilon_n F_\omega = eF_{\omega_n}$	$\epsilon_n F_\omega = eF_{\omega_n},$ $\epsilon_m F_\omega = eF_{\omega_m},$ $\epsilon_n F_{\omega_m} = \epsilon_m F_{\omega_n}$

TABLE 1. Torsion, Einstein and second Bianchi in various codimensions

Proposition 5.25. *The transversal jet $\partial_m \omega$ is uniquely fixed (in the reduced space of corner fields P_Γ) by requiring*

$$\epsilon_m F_\omega = eF_{\omega_m}, \qquad \epsilon_n F_{\omega_m} = \epsilon_m F_{\omega_n},$$
$$[\epsilon_m, F_\omega] = [e, F_{\omega_m}], \qquad [\epsilon_n, F_{\omega_m}] = [\epsilon_m, F_{\omega_n}].$$

Proof. The proof follows the same steps as the proof of Proposition 5.24 on the manifold $\Gamma \times [0, \epsilon] =: \Sigma_\epsilon$, by redefining

$$e|_{\Sigma_\epsilon} = e + \epsilon_m \mathrm{d}x^m,$$
$$F_\omega|_{\Sigma_\epsilon} = F_\omega + F_{\omega_m} \mathrm{d}x^m,$$

where x^m is interpreted as the coordinate along the infinitesimal interval $[0, \epsilon]$. □

To summarize the above discussion, we will consider the following definition.

Theorem 5.26. *The reduced space of corner fields is given by the submanifold P_Γ of $\tilde{P}_\Gamma$, on which the constraint $\mathrm{d}_\omega e = 0$ is imposed. Explicitly P_Γ is diffeomorphic to the following space*

$$\{(E, \Omega, \epsilon_m, \omega_m, \partial_m e, \partial_m \omega) \mid \text{s.t. } \textit{Equation } (46), \textit{Equation } (48), \textit{Equation } (49), \textit{Equation } (50) \textit{ and } 3^{\text{rd}} \textit{ column of Table 5.2}$$

while all higher transversal jets are identically set to zero.

5.3. **Involutivity and isotropy of the induced (pre-)Dirac structure on P_Γ.** Having defined the reduced space of corner fields P_Γ, we can induce vector fields $\mathbb{X}_\alpha$ and one forms χ_α from Equation (39) and Equation (37). In particular, this amounts to rewriting the vector fields and one forms in terms of the dynamical fields E and Ω in P_Γ. For example, for $\alpha = c$ we automatically have

$$\mathcal{J}_c = \int_\Gamma c\delta E.$$

For $\alpha = \zeta$, we see

$$\begin{aligned}\mathcal{E}_\zeta = \delta \int_\Gamma -e\iota_\zeta \Omega &= \int (\iota_\zeta e(\omega - \omega_0) + e\iota_\zeta(\omega - \omega_0))\delta e + \iota_\zeta e \delta\Omega \\ &= \int_\Gamma \iota_\zeta e\delta\Omega + \iota_\zeta(\omega - \omega_0)\delta E + \iota_\zeta e(\omega - \omega_0)\delta e.\end{aligned}$$

Notice however how, by Theorem 5.19, we can rewrite $\iota_\zeta e(\omega - \omega_0)\delta e = \iota_\zeta e(er + \epsilon_m \epsilon_n b)\delta e$. We notice that $\iota_\zeta e \epsilon_m \epsilon_n b \in \Omega^{1,3}_\Gamma$, therefore we can apply lemma A.7 to see

$$\iota_\zeta e \epsilon_m \epsilon_n b = e\alpha_\zeta + \epsilon_m \epsilon_n \beta_\zeta,$$

for some uniquely defined $\alpha_\zeta \in \Omega^{0,2}_\Gamma$ and $\beta_\zeta \in \ker(W_e^{\Gamma(1,1)})$. We then see

$$\begin{aligned}\epsilon_m\epsilon_n\beta_\zeta\delta e &= \epsilon_m\epsilon_n\delta(e\beta_\zeta) - \epsilon_m\epsilon_n e\delta\beta_\zeta \\ &= -\epsilon_m\epsilon_n e\left(\frac{\delta\beta_\zeta}{\delta e}\delta e + \frac{\delta\beta_\zeta}{\delta\omega}\delta\omega\right) \\ &= -\epsilon_m\epsilon_n e\frac{\delta\beta_\zeta}{\delta e}\delta e - \epsilon_m\epsilon_n\left(\frac{\delta(e\beta_\zeta)}{\delta\omega} - \frac{\delta e}{\delta\omega}\beta_\zeta\right)\delta\omega \\ &= -\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}\delta E\end{aligned}$$

We obtain

$$\mathcal{E}_\zeta = \int_\Gamma \iota_\zeta e\delta\Omega + \left(\iota_\zeta(\omega-\omega_0) + \iota_\zeta e r + \alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}\right)\delta E.$$

Similarly, we have

$$\mathcal{K}_\eta = \int \eta\epsilon_m\delta\Omega + \eta\epsilon_m r\delta E.$$

Lastly, we get

$$\mathcal{F}_\lambda = \int_\Gamma \lambda\epsilon_n\delta\Omega + \lambda\epsilon_n r\delta E.$$

The vector fields $\mathbb{X}_\alpha$ also change according to the redefinitions of the fields. Specifically, using constraints again the transversal torsionless, Einstein and Bianchi identities, we obtain

$$\begin{aligned}
&\mathbb{J}_{c,E} = [c,E] && \mathbb{J}_{c,\Omega} = e\mathrm{d}_{\omega_0}c + [c,\Omega] \\
&\mathbb{E}_{\zeta,E} = -\mathrm{L}^{\omega_0}_\zeta E && \mathbb{E}_{\zeta,\Omega} = -e\iota_\zeta F_\omega + \mathrm{d}_\omega\iota_\zeta\Omega - [\iota_\zeta(\omega-\omega_0),e](\omega-\omega_0) \\
&\mathbb{F}_{\lambda,E} = \mathrm{d}_\omega(\lambda\epsilon_n e) && \mathbb{F}_{\lambda,\Omega} = \lambda\epsilon_n F_\omega + \mathrm{d}_\omega(\lambda\epsilon_n)(\omega-\omega_0) \\
&\mathbb{K}_{\eta,E} = \mathrm{d}_\omega(\eta\epsilon_m e) && \mathbb{K}_{\eta,\Omega} = -\mathrm{d}_\omega(\eta\Omega_m) - \eta\epsilon_m\mathrm{d}_\omega(\omega-\omega_0) + \eta\epsilon_m F_\omega
\end{aligned} \tag{56}$$

Remark 5.27. Regarding the vector fields along ω_m, we have naively $\mathbb{X}_{\omega_m} = \mathrm{L}_{\mathbb{X}}(\omega-\omega_0)_m = 0$. However, we can define $\Omega_m \coloneqq \epsilon_m(\omega-\omega_0) + e(\omega-\omega_0)_m = \epsilon_m(\omega-\omega_0)$, and see that the vector fields along the transversal fields Ω_m and ϵ_m are

$$\begin{aligned}
&\mathbb{J}_{c,\epsilon_m} = [c,\epsilon_m] && \mathbb{J}_{c,\Omega_m} = e\mathrm{d}_{\omega^0_m}c + \epsilon_m\mathrm{d}_{\omega_0}e + [c,\Omega_m] \\
&\mathbb{E}_{\zeta,e_m} = \mathrm{L}^{\omega_0}_\zeta e_m + \iota_{\partial_m\zeta}e && \mathbb{E}_{\lambda,\Omega_m} = \mathrm{d}_\omega(\iota_\zeta\Omega_m)) + \mathrm{d}_{\omega_m}(\iota_\zeta\Omega) - \iota_\zeta e F_{\omega_m} - [\iota_\zeta(\omega-\omega_0),\epsilon_m](\omega-\omega_0) \\
&\mathbb{F}_{\lambda,e_m} = \mathrm{d}_{\omega_m}(\lambda\epsilon_n) && \mathbb{F}_{\lambda,\Omega_m} = \mathrm{d}_{\omega_m}(\lambda\epsilon_n)(\omega-\omega_0) + \lambda\epsilon_n F_{\omega_m} \\
&\mathbb{K}_{\eta,\epsilon_m} = \mathrm{d}_{\omega^0_m}(\eta\epsilon_m) && \mathbb{K}_{\eta,\Omega_m} = \eta\epsilon_m F_{\omega_m} + \mathrm{d}_{\omega_m}(\eta\Omega_m) - \eta\epsilon_m\mathrm{d}_{\omega_m}(\omega-\omega_0).
\end{aligned}$$

Proposition 5.28. *Let D be the distribution defined by*

$$D \coloneqq \mathrm{span}_{\mathcal{C}^\infty(P_\Gamma)}(\mathbb{X}_\alpha + \mathcal{X}_\alpha) \subset \Gamma((T\oplus T^*)[1]P_\Gamma), \qquad \text{with } \alpha = c,\zeta,\lambda,\eta. \tag{57}$$

Then, D is involutive and isotropic.

Proof. The proof follows, mutatis mutandis, from the proof of proposition 5.7. □

5.4. The Poisson structure on P_Γ.

Theorem 5.29. *Let D be the distribution defined by Equation* (57)*. Then D is a Dirac structure, obtained as the graph of the Poisson structure (P_Γ,π), where $\pi \in \Gamma(\wedge^2 TP_\Gamma)$ is the Poisson bivector field given by*

$$\begin{aligned}\pi = \int_\Gamma &\frac{1}{2}E\left[\frac{\delta}{\delta E},\frac{\delta}{\delta E}\right] - \frac{\delta}{\delta\Omega}\left(e\mathrm{d}_{\omega_0}\frac{\delta}{\delta E} + \left[\Omega,\frac{\delta}{\delta E}\right]\right) + \left[\frac{\delta}{\delta E},\epsilon_m\right]\frac{\delta}{\delta\epsilon_m} \\ &+ \frac{\delta}{\delta\Omega_m}\left(e\mathrm{d}_{\omega^0_m}\frac{\delta}{\delta E} + \left[\Omega_m,\frac{\delta}{\delta E}\right] + \epsilon_m\mathrm{d}_{\omega_0}\frac{\delta}{\delta E}\right) + \frac{1}{2}(F_\omega+\Theta)\frac{\delta}{\delta\Omega}\frac{\delta}{\delta\Omega} \\ &+ (F_{\omega_m}+\Theta_m)\frac{\delta}{\delta\Omega}\frac{\delta}{\delta\Omega_m} + \left(\mathrm{d}_{\omega_m}\frac{\delta}{\delta\Omega} - \left[\frac{\delta}{\delta\Omega}r,\epsilon_m\right]\right)\frac{\delta}{\delta\epsilon_m} \\ &- W_e^{-1}\left[\frac{\delta}{\delta\Omega}\epsilon_m\epsilon_n b,\epsilon_m\right]\frac{\delta}{\delta\epsilon_m},\end{aligned} \tag{58}$$

where

$$\Theta = -[e, \omega - \omega_0]r - \mathrm{d}_\omega(\omega - \omega_0), \qquad \Theta_m = -[\epsilon_m, \omega - \omega_0]r - \mathrm{d}_\omega(\omega - \omega_0)_m - \mathrm{d}_{\omega_m}(\omega - \omega_0),$$

and having defined $\Omega_m \coloneqq \epsilon_m(\omega - \omega_0)$.

Proof. We work with representatives of the equivalence classes fixed by Definition 5.20, where it holds that $\Theta_m = -[\epsilon_m, \omega - \omega_0]r - \mathrm{d}_{\omega_m}(\omega - \omega_0)$. Since we have already shown that D is both involutive and isotropic, if we can find a bivector $\pi \in \Gamma(\wedge^2 P_\Gamma)$ such that

$$D = e^\pi(\Omega^1[1](P_\Gamma)) = \{\iota_{\mathcal{X}}\pi + \mathcal{X} \mid \mathcal{X} \in \Omega^1[1](P_\Gamma)\},$$

we are guaranteed to have maximality as well, from which it follows that D is Dirac. Moreover, by Example 2.29 we have, as a consequence of involutivity, that $[\pi, \pi] = 0$, showing that π is Poisson.

In order to show that $D = e^\pi(\Omega^1[1](P_\Gamma))$, we start by noticing that 1-shifted one–forms on P_Γ are generated by

$$A \coloneqq \int_\Gamma a\delta E, \qquad \text{and} \qquad B \coloneqq \int_\Gamma b\delta\Omega,$$

with $a \in \Gamma[1](P_\Gamma, \Omega^{0,2}_\Gamma)$ and $b \in \Gamma[1](P_\Gamma, \Omega^{0,1}_\Gamma)$. However, we can always decompose

$$b = \iota_\varsigma e + \nu\epsilon_n + \tau\epsilon_m,$$

with $\nu, \tau \in \mathcal{C}^\infty[1](P_\Gamma)$ and $\varsigma \in \Gamma[1](P_\Gamma, \mathfrak{X}(\Gamma))$, obtaining

$$B = \mathcal{E}_\varsigma - \mathcal{J}_{\iota_\varsigma(\omega-\omega_0)} - \mathcal{J}_{\iota_\varsigma e r} - \mathcal{J}_{W_e^{-1}(\iota_\varsigma e \epsilon_n \epsilon_m b)} + \mathcal{F}_\nu + \mathcal{K}_\tau,$$

which, after noticing that $A = \mathcal{J}_a$, implies that

$$\Omega^1(P_\Gamma) = \mathrm{span}_{\mathcal{C}^\infty(P_\Gamma)}(\mathcal{X}_\alpha) \qquad \text{with } \alpha = a, \varsigma, \nu, \tau.$$

It follows that it suffices to check that $\pi(\mathcal{X}_\alpha) = \iota_{\mathcal{X}_\alpha}\pi = \mathbb{X}_\alpha$ for all α's. We leave this for appendix B.2. □

6. The BF²V structure of PC gravity

Given a poisson structure on a manifold P, a BF²V structure is automatically obtained by promoting the Poisson bivector to a function on the 1-shifted tangent bundle of P. In the Palatini–Cartan case, such Poisson bivector depends, in principle, only on the fields E and Ω in the reduced space of corner fields. However, the strict dependency on them is not manifest in Equation (58), as it contains terms proportional to $\frac{\delta}{\delta\Omega_m}$ and $\frac{\delta}{\delta\epsilon_m}$, which have not been expressed explicitly in terms of the two dynamical fields E and Ω.

In order to obtain a BF²V action of PC gravity, we set $\mathcal{F}_\Gamma \coloneqq T[1]P_\Gamma$. Since P_Γ is decorated with the transversal fields Ω_m and ϵ_m —which have been fixed to obtain P_Γ as a submanifold of the original space of corner fields F_Γ— the terms proportional to $\frac{\delta}{\delta\Omega_m}$ and $\frac{\delta}{\delta\epsilon_m}$ in the BF²V action require introducing the corresponding conjugate fields (in the fiber of $T[1]$) and fix them such that they can only depend on E, Ω and their conjugate fields.

Remark 6.1. While $\Omega \in \Omega^{2,3}_\Gamma$ is not subject to any constraint, $E \in \Omega^{2,2}_\Gamma$ is required to have a vanishing Pfaffian. Such constraint will be reflected also on its conjugate field in $T[1]\Omega^{2,2}_\Gamma$.

We introduce the following

$$\mathcal{F}_\Gamma \coloneqq \{\ (E, \Omega, \epsilon_m, \Omega_m, c, \mu, \nu^\dagger, f^\dagger) \mid c \in \Omega^{0,2}_{\Gamma,r},\ \mu \in \Omega^{0,1}_\Gamma,\ f^\dagger, \nu^\dagger \text{ fixed}\}.$$

We will shortly see how $f^\dagger \in \Omega^{2,3}_\Gamma[1]$ and $\nu^\dagger \in \Omega^{1,2}_\Gamma$ are fixed in terms of (E, Ω, c, μ). First, we have to define $\Omega^{0,2}_{\Gamma,r}$ as the fiber of $T[1]\Omega^{1,1}_{\Gamma,r}$. In order to do so, we consider the symplectic form on $T[1](\Omega^{1,1}_\Gamma \oplus \Omega^{2,2}_\Gamma)$

$$\tilde{\varpi}_\Gamma \coloneqq \int_\Gamma \delta E\delta c + \delta\Omega\delta\mu. \tag{59}$$

After setting $E = \frac{e^2}{2}$, we notice that that $\tilde{\varpi}_\Gamma$ becomes pre-symplectic, indeed

$$\mathrm{Ker}(\tilde{\varpi}_\Gamma) = \left\{ \int_\Gamma \mathbb{X}_c \frac{\delta}{\delta c} \in \mathfrak{X}(T[1](\Omega^{1,1}_\Gamma \oplus \Omega^{2,2}_\Gamma)) \mid e\mathbb{X}_c = 0 \right\}.$$

Therefore, in order to have a symplectic form on $T[1]P_\Gamma$, we need to consider only those elements $[c]$ living in the quotient $\Omega^{0,2}_{\Gamma,r}[1] \coloneqq \Omega^{2,2}_\Gamma[1]/\mathrm{Ker}(W_e^{\Gamma,(0,2)})$.

Lemma 6.2. *There exists a unique representative of $c \in [c] \in \Omega^{0,2}_{\Gamma,r}[1]$ such that*

$$\epsilon_m \epsilon_n c = 0. \tag{60}$$

Proof. We know, essentially by construction, that $\mathrm{Im}(W_e^{\Gamma,(0,2)})[1] \simeq \Omega^{2,2}_\Gamma[1]/\mathrm{Ker}(W_e^{\Gamma,(0,2)})$. Indeed, defining $C \coloneqq ec \in \mathrm{Im}(W_e^{\Gamma,(0,2)})[1]$, have $C = ec'$ if and only if $c' - c \in \mathrm{Ker}(W_e^{\Gamma,(0,2)})$. However, by Lemma A.7, any $K \in \Omega^{1,3}_\Gamma$ splits uniquely as

$$K = e\alpha + \epsilon_m \epsilon_n \beta \qquad \text{s.t.} \quad \epsilon_m \epsilon_n \alpha = 0 \quad \text{and} \quad e\beta = 0.$$

This implies that for any $C \in \mathrm{Im}(W_e^{\Gamma,(0,2)})[1] \subset \Omega^{1,3}_\Gamma[1]$ there exists a unique $c \in \Omega^{0,2}_\Gamma[1]$ such that

$$C = ec \qquad \text{and} \qquad \epsilon_m \epsilon_n c = 0.$$

□

Naively, the BF^2V action reads

$$\begin{aligned}\mathcal{S}_\Gamma = \int_\Gamma & \frac{1}{2} E[c,c] - \mu(e\mathrm{d}_{\omega_0} c + [\Omega, c]) + [c, \epsilon_m] f^\dagger \\ & + \nu^\dagger (e\mathrm{d}_{\omega_m^0} c + [\Omega_m, c] + \epsilon_m \mathrm{d}_{\omega_0} c) + \frac{1}{2}(F_\omega + \Theta)\mu^2 \\ & + (F_{\omega_m} + \Theta_m)\mu\nu^\dagger + (\mathrm{d}_{\omega_m}\mu - [\mu r, \epsilon_m]) f^\dagger \\ & - W_e^{-1}[\mu \epsilon_m \epsilon_n b, \epsilon_m] f^\dagger,\end{aligned} \tag{61}$$

while the symplectic form is

$$\varpi_\Gamma = \int_\Gamma \delta E \delta c + \delta\Omega\delta\mu + \delta\Omega_m \delta\nu^\dagger + \delta\epsilon_m \delta f^\dagger., \tag{62}$$

stressing that $\delta\Omega_m\delta\nu^\dagger + \delta\epsilon_m\delta f^\dagger$ is going to depend just on the other fields.

The above expressions are quite cumbersome. Specifically, we would like to find a symplectomorphism that symplifies both the action and the symplectic form. We start by noticing that, due to our own choice, we have $\Omega_m = \epsilon_m(\omega - \omega_0)$. Hence, we obtain

$$\begin{aligned}\delta\nu^\dagger\delta\Omega_m &= \delta\nu^\dagger(\delta\epsilon_m(\omega - \omega_0) - \epsilon_m\delta(\omega - \omega_0)) \\ &= \delta(\nu^\dagger(\omega - \omega_0))\delta\epsilon_m - \delta(\nu^\dagger\epsilon_m)\delta\omega.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\delta\mu\delta\Omega &= \delta\mu(\delta e(\omega - \omega_0) + e\delta(\omega - \omega_0)) \\ &= -\delta(\mu(\omega - \omega_0))\delta e + \delta M \delta\omega,\end{aligned}$$

having defined $M \coloneqq e\mu$. The symplectic form is then

$$\varpi_\Gamma = \int_\Gamma \delta(ec - \mu(\omega - \omega_0)) + \delta(M - \nu^\dagger\epsilon_m)\delta\omega + \delta(f^\dagger + \nu^\dagger(\omega - \omega_0))\delta\epsilon_m.$$

This suggests fixing $f^\dagger = -\nu^\dagger(\omega - \omega_0)$, while $\nu^\dagger = e\gamma^\dagger$, with $\gamma^\dagger = [\epsilon_m, \mu]$. Furthermore, consider the field redefinition $\Psi\colon T[1]P_\Gamma \to T[1]P_\Gamma$, such that it acts as the identity on all the fields except μ and c, where

$$\begin{aligned}\Psi^*(\mu) &= \mu + \epsilon_m\gamma^\dagger \\ \Psi^*(c) &= c - W_e^{-1}(\mu(\omega - \omega_0)).\end{aligned}$$

Remark 6.3. Notice that $\mu \in \Omega^{0,1}_\Gamma$, can be rewritten as $\mu = \iota_\zeta e + \lambda\epsilon_n + \eta\epsilon_m$. This implies that the ghosts are the conjugate momenta to the fields in the $\mathrm{BF}^2\mathrm{V}$ structure of PC gravity.

Proposition 6.4. *With the above (re-)definition of the fields, the data*

$$\varpi'_\Gamma = \Psi^*(\varpi_\Gamma) = \int_\Gamma \delta c \delta E + \delta\mu e \delta\omega + \mu\delta e\delta\omega, \tag{63}$$

and

$$\mathcal{S}'_\Gamma = \Psi^*(\mathcal{S}_\Gamma) = \int_\Gamma \frac{1}{2} E[c,c] + \mu e\mathrm{d}_\omega c + \frac{1}{2}\mu^2 F_\omega \tag{64}$$

define a $BF^2\,V$ structure on $\mathcal{F}_\Gamma$.

Proof. Since the full computation of the pullback of $\mathcal{S}_\Gamma$ along Ψ is quite involved, we prefer to omit it and to prove that the resulting data is that of a genuine $\mathrm{BF}^2\mathrm{V}$ structure on $\mathcal{F}_\Gamma$ by means of the classical master equation.

We start by finding the Hamiltonian vector field $\mathbb{Q}_\Gamma$ of $\mathcal{S}'_\Gamma$, which, after some manipulations, is simply obtained as

$$\begin{aligned} \mathbb{Q}_{\Gamma,e} &= [c,e] + \mathrm{d}_\omega \mu & e\mathbb{Q}_{\Gamma,\omega} &= e\mathrm{d}_\omega c + \mu F_\omega \\ e\mathbb{Q}_{\Gamma,c} &= \frac{1}{2} e[c,c] - \mu W_e^{-1}(\mu F_\omega) & \mathbb{Q}_{\Gamma,\mu} &= [c,\mu]. \end{aligned} \tag{65}$$

We remark that $\mathbb{Q}_{\Gamma,e}$ is defined up to terms in $\ker(W_e^{\Gamma(1,1)})$. The same is true for $\mathbb{Q}_{\Gamma,c}$ and $\mathbb{Q}_{\Gamma,\omega}$. It is interesting to consider the expression μF_ω. Indeed, using Einstein's equation and its transversal components, summarized in Table 5.2.2, it can be rewritten as

$$\begin{aligned} \mu F_\omega =& (\iota_\zeta e + \lambda\epsilon_n + \eta\epsilon_m) F_\omega \\ =& e(-\iota_\zeta F_\omega + \lambda F_{\omega_n} + \eta F_{\omega_m}), \end{aligned}$$

hence

$$\begin{aligned} \mathbb{Q}_{\Gamma,e} &= [c,e] + \mathrm{d}_\omega \mu & \mathbb{Q}_{\Gamma,\omega} &= \mathrm{d}_\omega c - \iota_\zeta F_\omega + \lambda F_{\omega_n} + \eta F_{\omega_m} \\ \mathbb{Q}_{\Gamma,\mu} &= [c,\mu] & e\mathbb{Q}_{\Gamma,c} &= \frac{1}{2} e[c,c] + \mu(\iota_\zeta F_\omega - \lambda F_{\omega_n} - \eta F_{\omega_m}). \end{aligned}$$

For the following computations, it is also convenient to compute $W_e^{-2}(\mu^2 F_\omega) = W_e^{-1}(\mu W_e^{-1}(\mu F_\omega))$. We find, using the transversal Bianchi identities from table 5.2.2,

$$\begin{aligned} \mu^2 F_\omega =& \iota_\zeta e \iota_\zeta e F_\omega + 2\iota_\zeta e(\lambda\epsilon_n + \eta\epsilon_m) F_\omega + 2\eta\lambda\epsilon_n\epsilon_m F_\omega \\ =& -\iota_\zeta E \iota_\zeta F_\omega - 2e\lambda\epsilon_n \iota_\zeta F_\omega - 2e\eta\epsilon_m \iota_\zeta F_\omega \\ =& E\iota_\zeta\iota_\zeta F_\omega - 2e(\lambda\iota_\zeta(eF_{\omega_n}) + \eta\iota_\zeta(eF_{\omega_m})) \\ =& E\iota_\zeta\iota_\zeta F_\omega - 2E(\lambda\iota_\zeta F_{\omega_n} + \eta\iota_\zeta F_{\omega_m}). \end{aligned}$$

having used $2\epsilon_n\epsilon_m F_\omega = \epsilon_n F_{\omega_m} - \epsilon_m F_{\omega_n} = 0$. We can then infer

$$W_e^{-2}(\mu^2 F_\omega) = \frac{1}{2}\iota_\zeta\iota_\zeta F_\omega - \lambda\iota_\zeta F_{\omega_n} - \eta\iota_\zeta F_{\omega_m}. \tag{66}$$

The classical master equation, in absence of vertices, is equivalent to the cohomological property of the Hamiltonian vector field $\mathbb{Q}_\Gamma$. Hence, we just need to check $\mathbb{Q}_\Gamma^2 = 0$. We have

$$\begin{aligned} \mathbb{Q}^2_{\Gamma,\mu} =& \frac{1}{2}[[c,c],\mu] - [W_e^{-2}(\mu^2 F_\omega),\mu] - e[c,[c,\mu]] \\ =& -[W_e^{-2}(\mu^2 F_\omega),\mu] \\ =& [\iota_\zeta e + \lambda\epsilon_n + \eta\epsilon_m, \frac{1}{2}\iota_\zeta\iota_\zeta F_\omega - \lambda\iota_\zeta F_{\omega_n} - \eta\iota_\zeta F_{\omega_m}] \\ =& \frac{1}{2}\iota_\zeta\iota_\zeta\iota_\zeta[e,F_\omega] - \frac{1}{2}\iota_\zeta\iota_\zeta[e,\lambda F_{\omega_n} - \eta F_{\omega_m}] \\ &+ [\lambda\epsilon_n, \frac{1}{2}\iota_\zeta\iota_\zeta F_\omega - \lambda\iota_\zeta F_{\omega_n} - \eta\iota_\zeta F_{\omega_m}] \\ &+ [\eta\epsilon_m, \frac{1}{2}\iota_\zeta\iota_\zeta F_\omega - \lambda\iota_\zeta F_{\omega_n} - \eta\iota_\zeta F_{\omega_m}] \\ =& -\frac{1}{2}\iota_\zeta\iota_\zeta(\lambda[\epsilon_n, F_\omega] + \eta[\epsilon_m, F_\omega]) \\ &+ [\lambda\epsilon_n, \frac{1}{2}\iota_\zeta\iota_\zeta F_\omega] + \lambda\eta\iota_\zeta[\epsilon_n, F_{\omega_m}] \\ &+ [\eta\epsilon_m, \frac{1}{2}\iota_\zeta\iota_\zeta F_\omega] - \lambda\eta\iota_\zeta[\epsilon_m, F_{\omega_n}] \\ =& 0, \end{aligned}$$

having used the graded Jacobi identity and the transversal Bianchi identities. In a similar way, one can also show that

$$[F_\omega, \mu] = [W_e^{-1}(\mu F_\omega), e] \tag{67}$$

$$[W_e^{-1}(\mu F_\omega), \mu] = [W_e^{-2}(\mu^2 F_\omega), e]. \tag{68}$$

$$\begin{aligned}
e\mathbb{Q}^2_{\Gamma,\omega} =&\mathbb{Q}_\Gamma(e\mathbb{Q}_{\Gamma\omega}) - (\mathbb{Q}_{\Gamma,e})(\mathbb{Q}_{\Gamma,\omega})\\
=&e[\mathrm{d}_\omega c, c] + e[W_e^{-1}(\mu F_\omega), c] - \frac{1}{2}e\mathrm{d}_\omega[c,c] + e\mathrm{d}_\omega(W_e^{-2}(\mu^2 F_\omega)) + [c,\mu]F_\omega\\
&- \mu\mathrm{d}_\omega(\mathrm{d}_\omega c + W_e^{-1}(\mu F_\omega)) - [c,e]W_e^{-1}(\mu F_\omega) - \mathrm{d}_\omega\mu W_e^{-1}(\mu F_\omega)\\
=&[\mu F_\omega, c] + [c,e]W_e^{-1}(\mu F_\omega) - \mathrm{d}_\omega(\mu W_e^{-1}(\mu F_\omega)) + [c,\mu]F_\omega\\
&- \mu[F_\omega, c] - \mu\mathrm{d}_\omega(W_e^{-1}(\mu F_\omega)) - [c,e]W_e^{-1}(\mu F_\omega) - \mathrm{d}_\omega\mu W_e^{-1}(\mu F_\omega)\\
=&0,
\end{aligned}$$

$$\begin{aligned}
\mathbb{Q}^2_{\Gamma,e} =&\frac{1}{2}[[c,c],e] - [W_e^{-2}(\mu^2 F_\omega), e] - [c,[c,e]] - [c,\mathrm{d}_\omega\mu]\\
&+ [\mathrm{d}_\omega c, \mu] + [W_e^{-1}(\mu F_\omega),\mu] - \mathrm{d}_\omega[c,\mu]\\
=&-[W_e^{-2}(\mu^2 F_\omega), e] + [W_e^{-1}(\mu F_\omega),\mu]\\
\overset{\textit{Equation (68)}}{=}&0,
\end{aligned}$$

Lastly, we need to check that the remaining constraint on the fields (which is not a "structural constraint" used to fix the values of the transversal fields and jets) is invariant under the action of $\mathbb{Q}_\Gamma$. One finds

$$\begin{aligned}
\mathbb{Q}_\Gamma(\mathrm{d}_\omega e) &= [\mathrm{d}_\omega c + W_e^{-1}(\mu F_\omega), e] - \mathrm{d}_\omega([c,e] + \mathrm{d}_\omega\mu)\\
&= [W_e^{-1}(\mu F_\omega), e] - [F_\omega,\mu] + [c,\mathrm{d}_\omega e]\\
&\overset{\textit{Equation (67)}}{=} [c,\mathrm{d}_\omega e],
\end{aligned}$$

which vanishes on-shell. Similarly, one can check the transversal Bianchi identities to be preserved by the action of $\mathbb{Q}_\Gamma$. □

Remark 6.5. As one can check by quickly inspecting Equation (64), the BF^{2}V action contains quadratic terms in the classical fields e and ω. However, the wish is to be able to rewrite it—through yet another field redefinition—such that it only contains linear terms in the classical fields. The BF^{2}V structure would then correspond to a linear (or an affine) Poisson structure on the original space of reduced corner fields P_Γ.

Proposition 6.6. *Let $\Phi\colon T[1]P_\Gamma \to T[1]P_\Gamma$ be the diffeomorphism acting as the identity on all the fields except on c, where*

$$\Phi^*(c) = c - \iota_\zeta(\omega - \omega_0).$$

Then, we have

$$\mathcal{S}^{PC}_\Gamma \coloneqq \Phi^*(S'_\Gamma) = \int_\Gamma \frac{1}{2}E[c,c] + \mu e\mathrm{d}_{\omega_0}c + \frac{1}{2}\mu^2 F_{\omega_0}, \tag{69}$$

while the symplectic form becomes

$$\varpi^{PC}_\Gamma \coloneqq \Phi^*(\varpi'_\Gamma) = \int_\Gamma \delta E\delta(c - \iota_\zeta(\omega - \omega_0)) + \delta\mu e\delta\omega + \mu\delta e\delta\omega. \tag{70}$$

Proof. The proof is just the result of a quite long but simple computation, which we omit for the sake of readability. □

Remark 6.7. The above BF^{2}V data correspond to the following affine Poisson structure

$$\pi^{PC} \coloneqq \int_\Gamma \frac{1}{2}E\big[\frac{\delta}{\delta E}, \frac{\delta}{\delta E}\big] + \frac{\delta}{\delta\Omega}e\mathrm{d}_{\omega_0}\frac{\delta}{\delta E} + \frac{1}{2}\frac{\delta}{\delta\Omega}\frac{\delta}{\delta\Omega}F_{\omega_0}. \tag{71}$$

Furthermore, notice that such action closely resembles the one found in the BF theory of Example 3.2.2, after setting $F_{\omega_0} = 0$. Specifically, one can recover the BF^{2}V structure of PC gravity as a particular BF theory for the Lie algebra $\mathfrak{so}(3,1)$, restricted to the submanifold defined by the following field parametrisation

$$\begin{aligned}
&c = c && A = \omega && B = E,\\
&\phi = \frac{1}{2}\mu^2 && \tau = e\mu && B^\dagger = 0.
\end{aligned}$$

7. Conclusion and Outlook

In this paper, we investigated the corner structure of four-dimensional Palatini–Cartan gravity on manifolds with corners from a classical perspective. Starting from the boundary constraint algebra, we associated to the codimension-two strata a pre-Dirac structure on the space of corner fields. Using techniques from generalized geometry, we identified a natural reduction procedure that removes the degeneracies responsible for the failure of maximality and yields a genuine Dirac structure on a reduced corner phase space. We further showed that this Dirac structure can be identified with the graph of a Poisson bivector, thereby providing a geometric characterization of the reduced corner degrees of freedom.

The resulting Poisson geometry naturally gives rise to a strict $\mathrm{BF}^2\mathrm{V}$ structure. In this way, the corner theory is obtained directly from the reduced classical geometry, in close analogy with the construction of the BFV theory for Palatini–Cartan gravity developed in [**CCS21**]. This should be contrasted with the BFV-induced construction of [**CC23**], where the codimension-two theory is obtained by descending from the boundary BFV data and is naturally described by a local P_∞-algebra. There, the induced presymplectic structure remains singular and the resulting theory is pre-$\mathrm{BF}^2\mathrm{V}$ in nature. By performing the relevant reduction before introducing antifields, the construction presented here removes this obstruction and yields a genuine $\mathrm{BF}^2\mathrm{V}$ theory associated with the reduced corner phase space.

A second outcome of our analysis is the identification of an equivalent affine Poisson description of the reduced corner theory. This reformulation reveals a close relation with the corner structure of four-dimensional BF theory and places the gravitational corner degrees of freedom in a setting that is particularly well suited for quantization. Indeed, affine Poisson structures naturally determine central extensions of Lie algebras through the Kirillov–Kostant construction. Consequently, part of the quantization procedure is already encoded in the classical geometry itself: the affine term determines the extension data that govern the corresponding quantum corner algebra.

This observation suggests a possible connection with the quantization program developed for corners in four-dimensional BF theory. In that setting, the affine Poisson structure gives rise to a centrally extended infinite-dimensional Lie algebra whose universal enveloping algebra defines the free corner algebra. The physical corner algebra is subsequently obtained by imposing the quantum counterparts of the classical constraints. While this construction successfully produces a rich class of representations of the free corner algebra, the passage to the physical algebra turns out to be considerably more subtle. In the non-abelian case, the constraint ideal acts non-trivially on the available modules, and the resulting representations descend only trivially to the physical corner algebra. Thus, the main challenge lies not in the construction of the free quantum corner algebra itself, but rather in understanding the physical quotient and its representation theory.

The affine Poisson structure obtained in the present work suggests that analogous questions can now be formulated for gravitational corners. In particular, one may hope to associate to the reduced corner theory a gravitational corner algebra obtained from the quantization of the corresponding central extension and to study its representations as candidate quantum corner state spaces. Whether non-trivial physical representations survive the implementation of the gravitational constraints remains a completely open question. Nevertheless, the framework developed here provides a concrete geometric and algebraic setting in which such problems can be systematically investigated.

Another natural direction concerns the relation between the $\mathrm{BF}^2\mathrm{V}$ structure constructed here and the BFV theory of Palatini–Cartan gravity. Since our approach follows the philosophy of [**CCS21**], it is natural to ask whether there exists a boundary BFV theory whose codimension-two descendant automatically reproduces the strict $\mathrm{BF}^2\mathrm{V}$ structure obtained in this paper. Such a construction would provide a genuine $\mathrm{BF}^{(k)}\mathrm{V}$ description of Palatini–Cartan gravity compatible with the reduced corner geometry.

A particularly intriguing possibility is that such a BFV theory may be related to the one constructed in [**CCS21**] through a suitable partial coisotropic reduction. If so, the reduced corner theory described here could emerge naturally from a reduced BFV structure already at codimension one. Clarifying the relation between these constructions would provide a better understanding of the interplay between reduction and the $\mathrm{BF}^{(k)}\mathrm{V}$ descent procedure and could ultimately lead to a unified description of bulk, boundary, and corner structures in Palatini–Cartan gravity.

More broadly, the appearance of Dirac and affine Poisson geometry at codimension two suggests that generalized geometric structures may play a fundamental role in the organization of higher-codimensional data in gauge theories. It would therefore be interesting to investigate whether similar mechanisms arise in other gauge theories admitting $\mathrm{BF}^{(k)}\mathrm{V}$ descriptions, and whether the affine Poisson structures obtained in this way admit a systematic quantization compatible with gluing and locality principles. We hope that the present work provides a first step in this direction.

Appendix A. Useful results and notation

The following section contains lemmas and identities useful throughout the paper. They are regrouped here for convenience. Considering inclusions $i\colon \Sigma \hookrightarrow M$ and $j\colon \Gamma \hookrightarrow \Sigma$, by abuse of notation we will denote by $\mathcal{V}$ also the induced vector bundles on the boudnary and corners $i^*\mathcal{V}$ and $(i\circ j)^*\mathcal{V}$. First, we recall the definitions

$$\Omega^{i,j} \coloneqq \Omega^i(M, \wedge^j V), \qquad \Omega^{i,j}_\Sigma \coloneqq \Omega^i(\Sigma, \wedge^j V), \qquad \Omega^{i,j}_\Gamma \coloneqq \Omega^i(\Gamma, \wedge^j V), \tag{72}$$

and maps $W^{i,j}_k \coloneqq e^k \wedge -$, where e is understood to be the vielbein on M, Σ or Γ depending on the context,

$$W^{i,j}_k \,:\, \Omega^{i,j} \longrightarrow \Omega^{i+k,j+k} \qquad W^{\Sigma(i,j)}_k \colon\, \Omega^{i,j}_\Sigma \longrightarrow \Omega^{i+k,j+k}_\Sigma \qquad W^{\Gamma(i,j)}_k \colon\, \Omega^{i,j}_\Gamma \longrightarrow \Omega^{i+k,j+k}_\Gamma.$$

We also define $\varrho^{i,j}_{(\Gamma,\Sigma)} \,:\, \Omega^{i,j}_{(\Gamma,\Sigma)} \to \Omega^{i+1,j-1}_{(\Gamma,\Sigma)} \,:\, \alpha \mapsto [e,\alpha]$.

Additionally, defining $\epsilon_m \in \Omega^{0,1}_\Gamma$ such that $\{e_i, \epsilon_m, \epsilon_n\}$ is a local basis of $\mathcal{V}$, with e_i is such that $e = e_i \mathrm{d}x^i$ is the corner vielbein, we have the following map

$$W^{i,j}_{mn}\colon \Omega^{i,j}_\Gamma \longrightarrow \Omega^{i,j+2}_\Gamma \tag{73}$$

$$\alpha \longmapsto \epsilon_m\epsilon_n\alpha. \tag{74}$$

With the above definitions, we have the following theorem

Theorem A.1. *[Can24] Letting l be the codimension of the manifold we're working on, i.e. $l(M) = 0$, $l(\Sigma) = 1$ and $l(\Gamma) = 2$. Then $W^{l,(i,j)}_k$ is surjective if and only if*

$$i + j \geq 4 - k$$

and injective if and only if

$$i + j \leq 4 - l - k.$$

In the following, we will make extensive use of the following manifolds

$$\Sigma_m \coloneqq \Gamma \times [0,1] \ni (x, x^m), \qquad \text{and} \qquad M_{mn} \coloneqq \Gamma \times [0,1]^2 \ni (x, x^m, x^n).$$

Lemma A.2. *[CCS21a] Let $\alpha \in \Omega^{2,1}_\Sigma$. Then*

$$\alpha = 0 \qquad \Longleftrightarrow \qquad \begin{cases} e\alpha = 0 \\ \epsilon_n\alpha \in \operatorname{Im} W^{\Sigma,(1,1)}_1 \end{cases} \tag{75}$$

where the last condition implies $\epsilon_n\alpha = e\sigma$ for some $\sigma \in \Omega^{1,1}_\Sigma$.

Lemma A.3. *Let $\alpha \in \Omega^{2,1}_\Gamma$. Then*

$$\alpha = 0 \qquad \Longleftrightarrow \qquad \begin{cases} \epsilon_n\alpha = e\sigma \\ \epsilon_m\alpha = e\rho \\ \epsilon_n\rho + \epsilon_m\sigma \in \operatorname{Im} W^{\Gamma,(0,1)}_1 \end{cases}$$

for some $\sigma, \rho \in \Omega^{1,1}_\Gamma$. Additionally, we also obtain $\sigma = \rho = 0$

Proof. Consider $M_{mn} \coloneqq \Gamma \times [0,1]^2 \ni (x, x^m, x^n)$, and define $E \coloneqq e + \epsilon_n \mathrm{d}x^n + \epsilon_m \mathrm{d}x^m$ vielbein on M_{mn}. Let $\Omega^{2,1} \ni A = \alpha + \sigma \mathrm{d}x^n + \rho \mathrm{d}x^m + \gamma \mathrm{d}x^m\mathrm{d}x^n$, with $\alpha \in \Omega^{2,1}_\Gamma$, $\sigma, \rho \in \Omega^{1,1}_\Gamma$ and $\gamma \in \Omega^{0,1}_\Gamma$. By Theorem A.1 we have that

$$A = 0 \;\Leftrightarrow\; E \wedge A = 0.$$

Expanding the latter equation, we obtain 4 equations

$$\begin{cases} e\alpha = 0 \\ \epsilon_n\alpha = e\sigma \\ \epsilon_m\alpha = e\rho \\ \epsilon_n\rho + \epsilon_m\sigma = e\gamma. \end{cases}$$

Notice that $e\alpha = 0$ identically as $\Omega^3(\Gamma) = \{0\}$. Since the above system of equations is equivalent to $A = 0$, this shows the lemma. □

Lemma A.4. *Let $r \in \Omega^{0,1}_\Gamma$. Then*

$$r = 0 \qquad \Longleftrightarrow \qquad \begin{cases} \epsilon_n e^2 r = 0 \\ \epsilon_m e^2 r = 0 \\ \epsilon_n \epsilon_m e r = 0 \end{cases}$$

Proof. Considering M_{mn} and $E \coloneqq e + \epsilon_n \mathrm{d}x^n + \epsilon_m \mathrm{d}x^m$ as before, letting $\Omega^{0,1} \ni R = r$, we know from Theorem A.1 that $E^3 \wedge R = 0$ if and only if $R = 0$. Expanding in each component of E, we obtain the desired system of equations. □

Lemma A.5. *Let $\beta \in \Omega^{1,1}_\Gamma$. Then*

$$\beta = 0 \qquad \Longleftrightarrow \qquad \begin{cases} \epsilon_n e\beta = \frac{e^2}{2}\beta_n \\ \epsilon_m e\beta = \frac{e^2}{2}\beta_m \\ \epsilon_n \epsilon_m \beta = e(\epsilon_n \beta_m + \epsilon_m \beta_m) \end{cases}$$

for some $\beta_n, \beta_m \in \Omega^{0,1}_\Gamma$.

Proof. Consider again $E \coloneqq e + \epsilon_n \mathrm{d}x^n + \epsilon_m \mathrm{d}x^m$ coframe on M_{mn} and let $\Omega^{1,1} \ni B \coloneqq \beta + \beta_m \mathrm{d}x^m + \beta_n \mathrm{d}x^n$. From Theorem A.1, we know $E^2 \wedge B = 0$ if and only if $B = 0$, hence $\beta = 0$ if and only if

$$\begin{cases} e^2\beta = 0 \\ \epsilon_n e\beta = \frac{e^2}{2}\beta_n \\ \epsilon_m e\beta = \frac{e^2}{2}\beta_m \\ \epsilon_n \epsilon_m \beta = e(\epsilon_n \beta_m + \epsilon_m \beta_m), \end{cases}$$

where $e^2\beta = 0$ identically. We therefore obtain the statement of the lemma. □

Lemma A.6. *Let $v \in \Omega^{1,2}_\Gamma$. Then*

$$v = 0 \qquad \Longleftrightarrow \qquad \begin{cases} ev = 0 \\ \epsilon_n v = e v_n \\ \epsilon_m v = e v_m \\ \epsilon_n v_m - \epsilon_m v_n = 0 \end{cases}$$

for some $v_m, v_n \in \Omega^{0,2}_\Gamma$.

Proof. Consider again $E \coloneqq e + \epsilon_n \mathrm{d}x^n + \epsilon_m \mathrm{d}x^m$ coframe on M_{mn} and let $\Omega^{1,2} \ni V = v + v_m \mathrm{d}x^m + v_n \mathrm{d}x^n$. By theorem A.1 $E \wedge V = 0$ if and only if $V = 0$. Hence, expanding the former equation,

$$v = 0 \qquad \Longleftrightarrow \qquad \begin{cases} ev = 0 \\ \epsilon_n v = e v_n \\ \epsilon_m v = e v_m \\ \epsilon_n v_m - \epsilon_m v_n = 0 \end{cases}$$

which shows the lemma. □

Lemma A.7. *For any $\Theta \in \Omega^{1,3}_\Gamma$, there exists a unique decomposition*

$$\Theta = e\alpha + \epsilon_m \epsilon_n \beta, \quad \text{with} \quad \alpha \in \Omega^{0,2}_\Gamma,\ \beta \in \Omega^{1,1}$$

such that

$$\epsilon_n \epsilon_m \alpha = 0 \qquad \text{and} \qquad e\beta = 0.$$

Proof. We have to show that $\Omega^{1,3}_\Gamma = e \ker(W^{0,2}_{mn}) \oplus \epsilon_m \epsilon_n \ker(W_1^{\Gamma,(1,1)})$. It was proved in [CC24] that $\dim(\ker(W_1^{\Gamma,(1,1)})) = 3$.[38] One can also check that $\dim \ker(W^{0,2}_{mn}) = 5$, as $\epsilon_m \epsilon_n \alpha = 0 \in \Omega^{0,4}_\Gamma$ fixes 1 component of α, and $\dim \Omega^{0,2}_\Gamma = 6$. It follows that $\dim \Omega^{1,3}_\Gamma = 8 = \dim \ker(W^{0,2}_{mn}) + \dim(\ker(W_1^{\Gamma,(1,1)}))$.

Now we prove that

$$\mathrm{Im}\left(W_1^{\Gamma,(0,2)}|_{\ker(W^{0,2}_{mn})}\right) \cap \mathrm{Im}\left(W^{1,1}_{mn}|_{\ker(W_1^{\Gamma,(1,1)})}\right) = \{0\}.$$

Consider $\beta \in \ker(W_1^{\Gamma,(1,1)})$ and $\alpha \in \ker(W^{0,2}_{mn}$. By contradiction, assume

$$e\alpha = \epsilon_m \epsilon_n \beta.$$

[38] By dimension, we mean the dimension of the fiber of $\Omega^{i,j}$ at a point $x \in \Gamma$.

In general, one can write $\alpha = \frac{1}{2}\chi^{ij}e_ie_j + \chi^i_m\epsilon_m e_i + \chi^i_n\epsilon_n e_i + \chi\epsilon_n\epsilon_m$, with

$$\chi^{ij} = \epsilon^{ij}\chi^{12}, \text{ for } \chi^{12} \in \mathcal{C}^\infty(\Gamma), \quad \chi_m, \chi_n \in \mathfrak{X}(\Gamma), \quad \chi \in \mathcal{C}^\infty(\Gamma).$$

From $\epsilon_n\epsilon_m\alpha = 0$ it follows $\alpha^{ij} = 0$, hence obtaining

$$e\alpha = \iota_{\chi_m}\frac{e^2}{2}\epsilon_m + \iota_{\chi_n}\frac{e^2}{2}\epsilon_n + \chi e\epsilon_m\epsilon_n.$$

Imposing $e\alpha = \epsilon_m\epsilon_n\beta$ implies

$$\begin{cases}\chi_m = \chi_n = 0\\ \epsilon_m\epsilon_n\beta = e\epsilon_m\epsilon_n\chi.\end{cases}$$

But now, since by assumption g^Γ is non-degenerate, then $e^2\epsilon_m\epsilon_n = f\mathrm{Vol}_g$, for some non-vanishing $f \in \mathcal{C}^\infty(\Gamma)$. From $e\beta = 0$, one can infer $\epsilon_m\epsilon_n e^2\chi = 0$, which is uniquely solved by $\chi = 0$, proving $\alpha = 0$. Additionally, the following equations are also satisfied

$$\begin{cases}\epsilon_m e\beta = 0\\ \epsilon_n e\beta = 0\\ \epsilon_m\epsilon_n\beta = 0\end{cases}$$

implying, by Lemma A.5, that $\beta = 0$.

Lastly, we are only left with checking that $W_1^{\Gamma,(0,2)}|_{\ker W^{0,2}_{mn}}$ and $W^{1,1}_{mn}|_{\ker(W_1^{\Gamma,(1,1)})}$ are injective. This is ensured by noticing that, repeating the above proof, one has

$$\begin{cases}e\alpha = 0\\ \epsilon_m\epsilon_n\alpha = 0\end{cases} \Rightarrow \alpha = 0, \quad \text{and} \quad \begin{cases}e\beta = 0\\ \epsilon_m\epsilon_n\beta = 0\end{cases} \Rightarrow \beta = 0.$$

□

Appendix B. Long computations

B.1. **Proof of Proposition 5.7.**

Proof. First of all notice that the isotropy of D guarantees that $[-,-]$ is (graded)skew-symmetric. In particular, assuming $\mathbb{X}+\mathcal{X}, \mathbb{Y}+\mathcal{Y} \in D$, one has[39]

$$[\mathbb{X}+\mathcal{X}, \mathbb{Y}+\mathcal{Y}] + [\mathbb{Y}+\mathcal{Y}, \mathbb{X}+\mathcal{Y}] = \delta\langle\mathbb{X}+\mathcal{X}, \mathbb{Y}+\mathcal{Y}\rangle = 0$$

Secondly, because of the odd parity of $\mathbb{X}_\alpha + \mathcal{X}_\alpha$, unlike in the usual case one has

$$[\mathbb{X}_\alpha + \mathcal{X}_\alpha, \mathbb{X}_\alpha + \mathcal{X}_\alpha] \neq 0.$$

We have

$$\begin{aligned}[\mathbb{J}_c + \mathcal{J}_c, \mathbb{J}_c + \mathcal{J}_c] &= -\frac{1}{2}\mathbb{J}_{[c,c]} + \frac{1}{2}\cancel{\iota_{\mathbb{J}_c}\delta\mathcal{J}_c} - \frac{1}{2}\delta(\iota_{\mathbb{J}_c}\mathcal{J}_c) + \frac{1}{2}\cancel{\iota_{\mathbb{J}_c}\delta\mathcal{J}_c}\\ &= -\frac{1}{2}\mathbb{J}_{[c,c]} - \frac{1}{2}\int_\Gamma c[c, e\delta e] = -\frac{1}{2}\mathbb{J}_{[c,c]} - \frac{1}{2}\int_\Gamma [c,c]e\delta e\\ &= -\frac{1}{2}\mathbb{J}_{[c,c]} - \frac{1}{2}\mathcal{J}_{[c,c]},\end{aligned}$$

where we used the exactness property of $\mathcal{J}_c$,

$$\begin{aligned}[\mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{J}_c + \mathcal{J}_c] &= \mathbb{J}_{L^{\omega_0}_\zeta c} - \delta(\iota_{\mathbb{E}_\zeta}\mathcal{J}_c)\\ &= \mathbb{J}_{L^{\omega_0}_\zeta c} + \delta\int_\Gamma ceL^{\omega_0}_\zeta e\\ &= \mathbb{J}_{L^{\omega_0}_\zeta c} + \delta\int_\Gamma \frac{e^2}{2}L^{\omega_0}_\zeta c\\ &= \mathbb{J}_{L^{\omega_0}_\zeta c} + \mathcal{J}_{L^{\omega_0}_\zeta c},\end{aligned}$$

[39] In the following, choosing whether to compute $[\mathbb{X}+\mathcal{X}, \mathbb{Y}+\mathcal{Y}]$ or $[\mathbb{Y}+\mathcal{Y}, \mathbb{X}+\mathcal{X}]$ is based on computational convenience.

$$
\begin{aligned}
[\mathbb{J}_c + \mathcal{J}_c, \mathbb{K}_\eta + \mathcal{K}_\eta] &= \mathbb{J}_{\eta \mathrm{d}_{\omega_m^0} c} + \iota_{\mathbb{J}_c} \delta \mathcal{K}_\eta - \delta(\iota_{\mathbb{J}_c} \mathcal{K}_\eta) \\
&= \mathbb{J}_{\eta \mathrm{d}_{\omega_m^0} c} + \iota_{\mathbb{J}_c} \int_\Gamma - \eta \delta e_m e \delta\omega + \eta e_m \delta e \delta\omega - \eta e \delta\omega_m \delta e \\
&\quad - \delta(\int_\Gamma \eta e_m e \mathrm{d}_\omega c - \eta e (\omega - \omega_0)_m [c, e]) \\
&= \mathbb{J}_{\eta \mathrm{d}_{\omega_m^0} c} + \int_\Gamma \eta e [c, e_m] \delta\omega - \eta \delta e_m e \mathrm{d}_\omega c + \eta e_m [c, e] \delta\omega \\
&\quad + \eta e_m \delta e \mathrm{d}_\omega c + \eta e \mathrm{d}_{\omega_m} c \delta e + \eta e \delta\omega_m [c, e] \\
&\quad + \eta \delta e_m e \mathrm{d}_\omega c - \eta e_m \delta e \mathrm{d}_\omega c - \eta e_m [\delta\omega, c] \\
&\quad + \eta (\omega - \omega_0)_m [c, e \delta e] - \eta \delta\omega_m [c, \frac{e^2}{2}] \\
&= \mathbb{J}_{\eta \mathrm{d}_{\omega_m^0} c} + \int_\Gamma \eta \mathrm{d}_{\omega_m^0} c e \delta e \\
&= \mathbb{J}_{\eta \mathrm{d}_{\omega_m^0} c} + \mathcal{J}_{\eta \mathrm{d}_{\omega_m^0} c},
\end{aligned}
$$

$$
\begin{aligned}
[\mathbb{F}_\lambda + \mathcal{F}_\lambda, \mathbb{J}_c + \mathcal{J}_c] &= -\mathbb{E}_{[c,\lambda\epsilon_n]^i} - \mathbb{R}_{[c,\lambda\epsilon_n]^m} - \mathbb{F}_{[c,\lambda\epsilon_n]^n} + \mathbb{J}_{[c,\lambda\epsilon_n]^i (\omega - \omega_0)_i} \\
&\quad + \mathbb{J}_{[c,\lambda\epsilon_n]^m (\omega - \omega_0)_m} - \delta(\iota_{\mathbb{F}_\lambda} \mathcal{J}_c) + \iota_{\mathbb{J}_c} \delta \mathcal{F}_\lambda \\
&= -\mathbb{E}_{[c,\lambda\epsilon_n]^i} - \mathbb{R}_{[c,\lambda\epsilon_n]^m} - \mathbb{F}_{[c,\lambda\epsilon_n]^n} + \mathbb{J}_{[c,\lambda\epsilon_n]^i (\omega - \omega_0)_i} \\
&\quad + \mathbb{J}_{[c,\lambda\epsilon_n]^m (\omega - \omega_0)_m} - \delta \int_\Gamma c e (\mathrm{d}_\omega(\lambda\epsilon_n) + \lambda\sigma) + \iota_{\mathbb{J}_c} \int_\Gamma \lambda\epsilon_n \delta e \delta\omega \\
&= -\mathbb{E}_{[c,\lambda\epsilon_n]^i} - \mathbb{R}_{[c,\lambda\epsilon_n]^m} - \mathbb{F}_{[c,\lambda\epsilon_n]^n} + \mathbb{J}_{[c,\lambda\epsilon_n]^i (\omega - \omega_0)_i} + \mathbb{J}_{[c,\lambda\epsilon_n]^m (\omega - \omega_0)_m} \\
&\quad - \delta \int_\Gamma \mathrm{d}_\omega c (e \lambda\epsilon_n) - \int_\Gamma \lambda\epsilon_n [c, e] \delta\omega - \lambda\epsilon_n \mathrm{d}_\omega c \delta e \\
&= -\mathbb{E}_{[c,\lambda\epsilon_n]^i} - \mathbb{R}_{[c,\lambda\epsilon_n]^m} - \mathbb{F}_{[c,\lambda\epsilon_n]^n} + \mathbb{J}_{[c,\lambda\epsilon_n]^i (\omega - \omega_0)_i} + \mathbb{J}_{[c,\lambda\epsilon_n]^m (\omega - \omega_0)_m} \\
&\quad - \int_\Gamma [\delta\omega, c] \lambda\epsilon_n e + \lambda\epsilon_n \mathrm{d}_\omega c \delta e + \lambda\epsilon_n [c, e] \delta\omega + \lambda\epsilon_n \mathrm{d}_\omega c \delta e \\
&= -\mathbb{E}_{[c,\lambda\epsilon_n]^i} - \mathbb{R}_{[c,\lambda\epsilon_n]^m} - \mathbb{F}_{[c,\lambda\epsilon_n]^n} + \mathbb{J}_{[c,\lambda\epsilon_n]^i (\omega - \omega_0)_i} + \mathbb{J}_{[c,\lambda\epsilon_n]^m (\omega - \omega_0)_m} \\
&\quad - \mathcal{E}_{[c,\lambda\epsilon_n]^i} - \mathcal{R}_{[c,\lambda\epsilon_n]^m} - \mathcal{F}_{[c,\lambda\epsilon_n]^n} + \mathcal{J}_{[c,\lambda\epsilon_n]^i (\omega - \omega_0)_i} + \mathcal{J}_{[c,\lambda\epsilon_n]^m (\omega - \omega_0)_m},
\end{aligned}
$$

$$
\begin{aligned}
[\mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{E}_\zeta + \mathcal{E}_\zeta] &= \frac{1}{2} \mathbb{E}_{[\zeta,\zeta]} - \frac{1}{2} \mathbb{J}_{\iota_\zeta \iota_\zeta F_{\omega_0}} - \frac{1}{2} \delta(\iota_{\mathbb{E}_\zeta} \mathcal{E}_\zeta) \\
&= \frac{1}{2} \mathbb{E}_{[\zeta,\zeta]} - \frac{1}{2} \mathbb{J}_{\iota_\zeta \iota_\zeta F_{\omega_0}} + \frac{1}{2} \delta \int_\Gamma \iota_\zeta \frac{e^2}{2} (\iota_\zeta F_{\omega_0} + L_\zeta^{\omega_0} (\omega - \omega_0)) + \iota_\zeta (\omega - \omega_0) L_\zeta^{\omega_0} \frac{e^2}{2} \\
&= -\frac{1}{2} \delta \int_\Gamma \frac{e^2}{2} \iota_\zeta \iota_\zeta F_{\omega_0} + \frac{e^2}{2} \iota_\zeta L_\zeta^{\omega_0} (\omega - \omega_0) - \frac{e^2}{2} (\iota_{[\zeta,\zeta]} (\omega - \omega_0) + \iota_\zeta L_\zeta^{\omega_0} (\omega - \omega_0)) \\
&= \frac{1}{2} \mathbb{E}_{[\zeta,\zeta]} + \frac{1}{2} \mathcal{E}_{[\zeta,\zeta]} - \frac{1}{2} \mathbb{J}_{\iota_\zeta \iota_\zeta F_{\omega_0}} - \frac{1}{2} \mathcal{J}_{\iota_\zeta \iota_\zeta F_{\omega_0}},
\end{aligned}
$$

$$
\begin{aligned}
[\mathbb{F}_\lambda + \mathcal{F}_\lambda, \mathbb{F}_\lambda + \mathcal{F}_\lambda] &= \iota_{\mathbb{F}_\lambda} \delta \mathcal{F}_\lambda - \frac{1}{2} \delta(\iota_{\mathbb{F}_\lambda} \mathcal{F}_\lambda) \\
&= \iota_{\mathbb{F}_\lambda} \int_\Gamma \lambda\epsilon_n \delta e \delta\omega - \frac{1}{2} \delta \int_\Gamma \lambda\epsilon_n \lambda\epsilon_n F_\omega \\
&= \int_\Gamma \lambda\epsilon_n (\mathrm{d}_\omega(\lambda\epsilon_n) + \lambda\sigma) \delta\omega + \lambda\epsilon_n \mathbb{F}_{\lambda,\omega} \delta e \\
&= \int_\Gamma \lambda\epsilon_n (\mathrm{d}_\omega \lambda\epsilon_n - \lambda \mathrm{d}_\omega \epsilon_n) \delta\omega \\
&= 0,
\end{aligned}
$$

where we used $\lambda^2 = \epsilon_n^2 = 0$ and the fact that $\mathbb{F}_{\lambda,\omega}$ is proportional to λ,

$$
\begin{aligned}
[\mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{K}_\eta + \mathcal{K}_\eta] &= \mathbb{K}_{\iota_\zeta \mathrm{d}\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} + \iota_{\mathbb{E}_\eta}\delta\mathcal{K}_\eta - \delta(\iota_{\mathbb{E}_\zeta}\mathcal{K}_\eta) \\
&= \mathbb{K}_{\iota_\zeta \mathrm{d}\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta}\iota_{\mathbb{E}_\zeta} + \int_\Gamma -\eta\delta e_m e\delta\omega + \eta e_m \delta e\delta\omega - \eta e\delta\omega_m\delta e \\
&\quad - \delta\int_\Gamma \eta e_m e(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0)) + \eta e(\omega-\omega_0)_m L_\zeta^{\omega_0} e \\
&= \mathbb{K}_{\iota_\zeta \mathrm{d}\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} + \int_\Gamma \eta L_\zeta^{\omega_0} e_m e\delta\omega - \eta\delta e_m e(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0)) \\
&\quad - \eta e_m L_\zeta^{\omega_0} e\delta\omega + \eta e_m(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0))\delta\omega - \eta e\delta\omega_m L_\zeta^{\omega_0} e \\
&\quad - \eta e(-L_\zeta^{\omega_0}\omega_m - \iota_{\partial_m\zeta}(\omega-\omega_0) + \iota_\zeta\partial_m\omega_0)\delta e \\
&\quad + \eta\delta e_m e(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0)) - \eta e_m\delta e(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0)) \\
&\quad + \eta(\omega-\omega_0)_m L_\zeta^{\omega_0}(e\delta e) - \eta\delta\omega_m L_\zeta^{\omega_0}(\frac{e^2}{2}) - \eta e_m e L_\zeta^{\omega_0}\delta\omega + \eta\iota_{\partial_m\zeta}\frac{e^2}{2}\delta\omega \\
&= \mathbb{K}_{\iota_\zeta \mathrm{d}\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} + \int_\Gamma \iota_\zeta \mathrm{d}\eta e_m e\delta\omega + \iota_\zeta d\eta e(\omega-\omega_0)_m\delta e \\
&\quad - \eta L_\zeta^{\omega_0}(\omega-\omega_0)_m e\delta e + \eta e(L_\zeta^{\omega_0}\omega_m + \iota_{\partial_m\zeta}(\omega-\omega_0) + \iota_\zeta\partial_m(\omega-\omega_0)\delta e \\
&= \mathbb{K}_{\iota_\zeta \mathrm{d}\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} + \mathcal{K}_{\iota_\zeta \mathrm{d}\eta} + \eta\iota_{\partial_m\zeta}\frac{e^2}{2}\delta\omega \\
&\quad + \int_\Gamma \eta e(\iota_\zeta \mathrm{d}\omega_m^0 + \iota_{\partial_m\zeta}(\omega-\omega_0) + \iota_\zeta\partial_m\omega_0)\delta e \\
&= \mathbb{K}_{\iota_\zeta \mathrm{d}\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} + \mathcal{K}_{\iota_\zeta \mathrm{d}\eta} + \mathcal{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \int_\Gamma \eta(\iota_{\partial_m\zeta}\frac{e^2}{2}\delta\omega + \iota_{\partial_m\zeta}(\omega-\omega_0)\frac{\delta e^2}{2}) \\
&= \mathbb{K}_{\iota_\zeta \mathrm{d}\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} + \mathcal{K}_{\iota_\zeta \mathrm{d}\eta} + \mathcal{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathcal{E}_{\eta\partial_m\zeta}
\end{aligned}
$$

$$
\begin{aligned}
[\mathbb{K}_\eta + \mathcal{K}_\eta, \mathbb{K}_\eta + \mathcal{K}_\eta] &= \mathbb{K}_{\eta\partial_m\eta} + \iota_{\mathbb{K}_\eta}\delta\mathcal{K}_\eta - \frac{1}{2}\delta(\iota_{\mathbb{K}_\eta}\mathcal{K}_\eta) \\
&= \mathbb{K}_{\eta\partial_m\eta} + \iota_{\mathbb{K}_\eta}\int_\Gamma -\eta\delta e_m e\delta\omega + \eta e_m\delta e\delta\omega - \eta e\delta\omega_m\delta e \\
&\quad + \frac{1}{2}\delta\int_\Gamma \eta(e e_m\delta\omega + (\omega-\omega_0)_m e\delta e \\
&= \mathbb{K}_{\eta\partial_m\eta} + \int_\Gamma \eta\partial_m\eta e_m e\delta\omega + \eta\delta e_m e\mathrm{d}\eta(\omega-\omega_0)_m + \eta e_m\delta e\mathrm{d}\eta(\omega-\omega_0)_m \\
&\quad + \eta e\partial_m\eta(\omega-\omega_0)_m\delta e - \eta e d\eta e_m\delta\omega_m + \frac{1}{2}\delta\int_\Gamma \eta e e_m d\eta(\omega-\omega_0)_m + \eta(\omega-\omega_0)_m e d\eta e_m \\
&= \mathbb{K}_{\eta\partial_m\eta} + \int_\Gamma \eta\partial_m\eta(e_m e\delta\omega + e(\omega-\omega_0)_m\delta e) + \eta\mathrm{d}\eta\delta(e_m e(\omega-\omega_0)_m) \\
&\quad + \delta\int_\Gamma \eta\mathrm{d}\eta e_m e(\omega-\omega_0)_m \\
&= \mathbb{K}_{\eta\partial_m\eta} + \int_\Gamma \eta\partial_m\eta(e_m e\delta\omega + e(\omega-\omega_0)_m\delta e) \\
&= \mathbb{K}_{\eta\partial_m\eta} + \mathcal{K}_{\eta\partial_m\eta},
\end{aligned}
$$

$$
\begin{aligned}
[\mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{F}_\lambda + \mathcal{F}_\lambda] &= \mathbb{E}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i} + \mathbb{F}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^n} + \mathbb{R}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i(\omega-\omega_0)_i} \\
&\quad - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m(\omega-\omega_0)_m} + \iota_{\mathbb{E}_\zeta}\delta\mathcal{F}_\lambda - \delta(\iota_{\mathbb{E}_\zeta}\mathcal{F}_\lambda) \\
&= \mathbb{E}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i} + \mathbb{F}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^n} + \mathbb{R}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m(\omega-\omega_0)_m} \\
&\quad + \iota_{\mathbb{E}_\zeta}\int_\Gamma \lambda\epsilon_n\delta e\delta\omega + \delta\int_\Gamma \lambda\epsilon_n e(L_\zeta^{\omega_0}(\omega-\omega_0) + \iota_\zeta F_{\omega_0}) \\
&= \mathbb{E}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i} + \mathbb{F}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^n} + \mathbb{R}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m(\omega-\omega_0)_m} \\
&\quad - \int_\Gamma \lambda\epsilon_n L_\zeta^{\omega_0} e\delta\omega + \lambda\epsilon_n(\iota_\zeta F_{\omega_0} + L_\zeta^{\omega_0}(\omega-\omega_0))\delta e \\
&\quad + \lambda\epsilon_n(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0))\delta e + \lambda\epsilon_n e L_\zeta^{\omega_0}\delta\omega \\
&= \mathbb{E}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i} + \mathbb{F}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^n} + \mathbb{R}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m} Z - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m(\omega-\omega_0)_m} \\
&\quad + \int_\Gamma (L_\zeta^{\omega_0}(\lambda\epsilon_n)^i e_i + L_\zeta^{\omega_0}(\lambda\epsilon_n)^n\epsilon_n + L_\zeta^{\omega_0}(\lambda\epsilon_n)^m e_m)e\delta\omega \\
&= \mathbb{E}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i} + \mathbb{F}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^n} + \mathbb{R}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m(\omega-\omega_0)_m} \\
&\quad + \mathcal{E}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i} + \mathcal{F}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^n} + \mathcal{R}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m} - \mathcal{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathcal{J}_{L_\zeta^{\omega_0}(\lambda\epsilon_n)^m(\omega-\omega_0)_m},
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathbb{F}_\lambda + \mathcal{F}_\lambda, \mathbb{K}_\eta + \mathcal{K}_\eta] &= \mathbb{E}_{\eta\partial_m(\lambda\epsilon_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda\epsilon_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda\epsilon_n)^m} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^i(\omega-\omega_0)_i} \\
&\quad - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^m(\omega-\omega_0)_m} + \iota_{\mathbb{F}_\lambda}\delta\mathcal{K}_\eta - \delta(\iota_{\mathbb{F}_\lambda}\mathcal{K}_\eta) + \iota_{\mathbb{K}_\eta}\delta\mathcal{F}_\lambda \\
&= \mathbb{E}_{\eta\partial_m(\lambda\epsilon_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda\epsilon_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda\epsilon_n)^m} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^m(\omega-\omega_0)_m} \\
&\quad + \iota_{\mathbb{F}_\lambda}\int_\Gamma - \eta\delta e e_m\delta\omega - \eta\delta e_m e\delta\omega - \eta e\delta\omega_m\delta e - \delta\int_\Gamma \eta e_m\lambda\epsilon_n F_\omega \\
&\quad - \delta\int_\Gamma - \eta(\omega-\omega_0)_m \mathrm{d}_\omega(\lambda\epsilon_n e) + \iota_{\mathbb{K}_\eta}\int_\Gamma \lambda\epsilon_n\delta e\delta\omega \\
&= \mathbb{E}_{\eta\partial_m(\lambda\epsilon_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda\epsilon_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda\epsilon_n)^m} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^m(\omega-\omega_0)_m} \\
&\quad + \int_\Gamma \eta(\mathrm{d}_\omega(\lambda\epsilon_n) + \lambda\sigma)e_m\delta\omega + \eta e_m\mathbb{F}_\omega\delta + \eta(\mathrm{d}_{\omega_m}(\lambda\epsilon_n) + \lambda\sigma)e\delta\omega - \eta\delta e_m\lambda\epsilon_n F_\omega \\
&\quad + \eta e\mathbb{F}_\omega\delta e - \eta e(\mathrm{d}_\omega(\lambda\epsilon_n) + \lambda\sigma)\delta\omega_m + \int_\Gamma \eta\delta e_m\lambda\epsilon_n F_\omega + \eta e_m\lambda\epsilon_n\mathrm{d}_\omega\delta\omega \\
&\quad - \eta\delta\omega_m\mathrm{d}_\omega(\lambda\epsilon_n e) - \eta(\omega-\omega_0)_m[\delta\omega, \lambda\epsilon_n e] + \eta(\omega-\omega_0)_m\mathrm{d}_\omega(\lambda\epsilon_n\delta e) \\
&\quad + \int_\Gamma \lambda\epsilon_n(-\eta\mathrm{d}_{\omega_m^0} e - d\eta e_m)\delta\omega + \lambda\epsilon_n(-\eta\mathrm{d}_{\omega_m^0}\omega + d\eta(\omega-\omega_0)_m)\delta e \\
&= \mathbb{E}_{\eta\partial_m(\lambda\epsilon_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda\epsilon_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda\epsilon_n)^m} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^m(\omega-\omega_0)_m} \\
&\quad + \int_\Gamma \eta(\mathrm{d}_\omega(\lambda\epsilon_n) + \lambda\sigma)e_m\delta\omega + \eta\lambda\epsilon_n(\mathrm{d}_\omega\omega_m - \partial_m\omega)\delta e \\
&\quad + \eta(\mathrm{d}_{\omega_m}(\lambda\epsilon_n) + \lambda\sigma_m)e\delta\omega - \eta\delta e_m\lambda\epsilon_n F_\omega \\
&\quad - \eta\mathrm{d}_\omega(\lambda\epsilon_n e)\delta\omega_m + \int_\Gamma \eta\lambda\epsilon_n F_\omega\delta e_m - \mathrm{d}_\omega(\eta e_m\lambda\epsilon_n)\delta\omega \\
&\quad + \eta\delta\omega[(\omega-\omega_0)_m\lambda\epsilon_n e] + \mathrm{d}_\omega(\eta(\omega-\omega_0)_m)\lambda\epsilon_n\delta e \\
&\quad + \int_\Gamma \lambda\epsilon_n(-\eta\mathrm{d}_{\omega_m^0} e - d\eta e_m)\delta\omega + \lambda\epsilon_n(-\eta\mathrm{d}_{\omega_m^0}\omega + d\eta(\omega-\omega_0)_m)\delta e
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}_{\eta\partial_m(\lambda\epsilon_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda\epsilon_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda\epsilon_n)^m} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^m(\omega-\omega_0)_m} \\
&\quad + \int_\Gamma \eta(\mathrm{d}_\omega(\lambda\epsilon_n) + \lambda\sigma)e_m\delta\omega + \eta\lambda\epsilon_n(\mathrm{d}_\omega\omega_m - \partial_m\omega)\delta e \\
&\quad + \eta(\mathrm{d}_{\omega_m}(\lambda\epsilon_n) + \lambda\sigma_m)e\delta\omega - \eta\delta e_m\lambda\epsilon_n F_\omega - \eta\mathrm{d}_\omega(\lambda\epsilon_n e)\delta\omega_m \\
&\quad + \int_\Gamma \mathrm{d}\eta e_m\lambda\epsilon_n\delta\omega + \eta\lambda(\epsilon_n\mathrm{d}_{\omega_m}ee\sigma_m)\delta\omega - \eta e_m\mathrm{d}_\omega(\lambda\epsilon_n)\delta\omega \\
&\quad + \eta\delta\omega[(\omega-\omega_0)_m, \lambda\epsilon_n e] + d\eta(\omega-\omega_0)_m\lambda\epsilon_n\delta e \\
&\quad - \eta\mathrm{d}_\omega(\omega-\omega_0)_m\lambda\epsilon_n\delta e + \int_\Gamma \lambda\epsilon_n(-\eta\mathrm{d}_{\omega_m^0}e - d\eta e_m)\delta\omega \\
&\quad + \lambda\epsilon_n(-\eta\mathrm{d}_{\omega_m^0}\omega + d\eta(\omega-\omega_0)_m)\delta e \\
&= \mathbb{E}_{\eta\partial_m(\lambda\epsilon_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda\epsilon_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda\epsilon_n)^m} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^m(\omega-\omega_0)_m} \\
&\quad + \int_\Gamma \eta\lambda\epsilon_n(\mathrm{d}_\omega\omega_m - \partial_m\omega)\delta e + \eta\partial_m(\lambda\epsilon_n)e\delta\omega - \eta\lambda\epsilon_n\mathrm{d}_{\omega_m^0}e\delta\omega \\
&\quad - \eta\lambda\epsilon_n\mathrm{d}_{\omega_m^0}\omega\delta e + \eta\lambda\epsilon_n\mathrm{d}_{\omega_m}e\delta\omega + \eta\delta\omega[(\omega-\omega_0)_m, \lambda\epsilon_n e] - \eta\mathrm{d}_\omega\omega_m\lambda\epsilon_n\delta e \\
&\quad + \eta\mathrm{d}_\omega\omega_m^0\lambda\epsilon_n\delta e \\
&= \mathbb{E}_{\eta\partial_m(\lambda\epsilon_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda\epsilon_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda\epsilon_n)^m} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda\epsilon_n)^m(\omega-\omega_0)_m} \\
&\quad + \mathcal{E}_{\eta\partial_m(\lambda\epsilon_n)^i} + \mathcal{F}_{\eta\partial_m(\lambda\epsilon_n)^n} + \mathcal{K}_{\eta\partial_m(\lambda\epsilon_n)^m} - \mathcal{J}_{\eta\partial_m(\lambda\epsilon_n)^i(\omega-\omega_0)_i} - \mathcal{J}_{\eta\partial_m(\lambda\epsilon_n)^m(\omega-\omega_0)_m}.
\end{aligned}
$$

□

B.2. **Proof of Theorem 5.29.**

Proof. We have

$$
\pi(\mathcal{J}_c) = \int_\Gamma [c, E]\frac{\delta}{\delta E} + (e\mathrm{d}_{\omega_0}c + [c, \Omega])\frac{\delta}{\delta\Omega} + [c, \epsilon_m]\frac{\delta}{\delta\epsilon_m} + (e\mathrm{d}_{\omega_0}c + \epsilon_m\mathrm{d}_{\omega_0}c + [c, \Omega_m])\frac{\delta}{\delta\Omega_m} = \mathbb{J}_c
$$

$$
\begin{aligned}
\pi(\mathcal{F}_\lambda) = \int_\Gamma &[\lambda\epsilon_n r, E]\frac{\delta}{\delta E} + \frac{\delta}{\delta\Omega}(e\mathrm{d}_{\omega_0}(\lambda\epsilon_n r) + [e(\omega-\omega_0), \lambda\epsilon_n r]) \\
&- \lambda\epsilon_n(e\mathrm{d}_{\omega_0}\frac{\delta}{\delta E} - [e(\omega-\omega_0), \frac{\delta}{\delta E}]) + \lambda\epsilon_n(F_\omega + \Theta)\frac{\delta}{\delta\Omega} \\
&+ (e\mathrm{d}_{\omega_m^0}(\lambda\epsilon_n r) + [\Omega_m, \lambda\epsilon_n r] + \epsilon_m\mathrm{d}_\omega(\lambda\epsilon_n r))\frac{\delta}{\delta\Omega_m} \\
&+ (\lambda\epsilon_n F_{\omega_m} + \Theta_m)\frac{\delta}{\delta\Omega_m} + [\lambda\epsilon_n r, \epsilon_m]\frac{\delta}{\delta\epsilon_m} + \mathrm{d}_\omega(\lambda\epsilon_n)\frac{\delta}{\delta\epsilon_m} \\
&- [\lambda\epsilon_n r, \epsilon_m]\frac{\delta}{\delta\epsilon_m}.
\end{aligned}
$$

We now notice that the first and the third terms become

$$
\begin{aligned}
[\lambda\epsilon_n r, E] + \mathrm{d}_{\omega_0}(\lambda\epsilon_n e) + [e(\omega-\omega_0), \lambda\epsilon_n] &= [\lambda\epsilon_n(\omega-\omega_0), e] + \mathrm{d}_{\omega_0}(\lambda\epsilon_n e) \\
&\quad - (\omega-\omega_0)[e, \lambda\epsilon_n] + e[\omega-\omega_0, \lambda\epsilon_n] \\
&= -\lambda\epsilon_n[\omega-\omega_0, e] + [\lambda\epsilon_n, e](\omega-\omega_0) + \mathrm{d}_{\omega_0}(\lambda\epsilon_n e) \\
&\quad - (\omega-\omega_0)[e, \lambda\epsilon_n] + e[\omega-\omega_0, \lambda\epsilon_n] \\
&= \mathrm{d}_{\omega_0}(\lambda\epsilon_n e) + [\omega-\omega_0, \lambda\epsilon_n e] = \mathrm{d}_\omega(\lambda\epsilon_n e),
\end{aligned}
$$

whereas the second and the fourth reduce to

$$\begin{aligned}
&e\mathrm{d}_{\omega_0}(\lambda\epsilon_n r) + [e(\omega-\omega_0), \lambda\epsilon_n r] + \lambda\epsilon_n(F_\omega + \Theta) = \\
&\quad = e\mathrm{d}_{\omega_0}(\lambda\epsilon_n) r + \lambda\epsilon_n e\mathrm{d}_{\omega_0} r - (\omega-\omega_0)[e, \lambda\epsilon_n r] \\
&\qquad + e[\omega-\omega_0, \lambda\epsilon_n] r + \lambda\epsilon_n e[\omega-\omega_0, r] + \lambda\epsilon_n(F_\omega+\Theta) \\
&\quad = \mathrm{d}_\omega(\lambda\epsilon_n)(\omega-\omega_0) - \mathrm{d}_\omega(\lambda\epsilon_n)\epsilon_m\epsilon_n b + \lambda\epsilon_n e\mathrm{d}_\omega r \\
&\qquad + [e, \lambda\epsilon_n r(\omega-\omega_0)] + \lambda\epsilon_n[e, \omega-\omega_0] r + \lambda\epsilon_n(F_\omega+\Theta) \\
&\quad = \mathrm{d}_\omega(\lambda\epsilon_n)(\omega-\omega_0) + \lambda\epsilon_n F_\omega + \lambda\epsilon_n \mathrm{d}_\omega(\omega-\omega_0) \\
&\qquad + \lambda\epsilon_n[e, \omega-\omega_0] r + \lambda\epsilon_n\Theta,
\end{aligned}$$

which produces exactly $\mathbb{F}_\Omega$ by substituting

$$\Theta = -[e, \omega-\omega_0] r - \mathrm{d}_\omega(\omega-\omega_0).$$

Regrouping all the terms proportional to $\frac{\delta}{\delta\Omega_m}$, we obtain the expected result after using

$$\Theta_m = -[\epsilon_m, \omega-\omega_0] r - \mathrm{d}_{\omega_m}(\omega-\omega_0),$$

and after confronting the expression with $\mathbb{F}_{\Omega_m}$ in Equation (56).

$$\begin{aligned}
\pi(\mathcal{K}_\eta) = \int_\Gamma &\left(\underline{[\eta\epsilon_m r, E]}_1 + \underline{\eta[(\omega-\omega_0), E]}_2\right)\frac{\delta}{\delta E} - \eta\epsilon_m\left(e\mathrm{d}_{\omega_0}\frac{\delta}{\delta E} + [\Omega, \frac{\delta}{\delta E}]\right) \\
&+ \frac{\delta}{\delta\Omega}\left(e\mathrm{d}_{\omega_0}\eta\epsilon_m r + [\Omega, \eta\epsilon_m r]\right) + \underline{[\eta\epsilon_m r}_3, \epsilon_m]\frac{\delta}{\delta\epsilon_m} + \frac{\delta}{\delta\Omega_m}(e\mathrm{d}_{\omega_m^0}\eta\epsilon_m r) \\
&+ \frac{\delta}{\delta\Omega_m}\left(\epsilon_m\mathrm{d}_\omega\eta\epsilon_m r + [\Omega_m, \eta\epsilon_m r]\right) + \eta\epsilon_m(F_\omega+\Theta)\frac{\delta}{\delta\Omega} \\
&+ \eta\epsilon_m(F_{\omega_m}+\Theta_m)\frac{\delta}{\delta\Omega_m} + \left(\mathrm{d}_{\omega_m}(\eta\epsilon_m) - \underline{[\eta\epsilon_m r, \epsilon_m]}_4\right)\frac{\delta}{\delta\epsilon_m}.
\end{aligned} \tag{76}$$

We first notice that the terms (76.[1]) = (76.[2]) = 0, because $[e, \epsilon_m] = [e, r] = 0$, and (76.[3]) + (76.[4]) = 0. By inspection, one finds that the terms along $\frac{\delta}{\delta\epsilon_m}$ automatically yield $\mathbb{K}_{\eta,\epsilon_m}$. Similarly, regrouping the terms along $\frac{\delta}{\delta E}$, recalling $[e, \omega-\omega_0] = [e, er] + [e, \epsilon_m\epsilon_n b] = 0$, we obtain

$$\begin{aligned}
\mathrm{d}_{\omega_0}(\eta\epsilon_m e) + [e(\omega-\omega_0), \eta\epsilon_m] =& \mathrm{d}_\omega(\eta\epsilon_m e) - [\omega-\omega_0, \eta\epsilon_m e] + [e(\omega-\omega_0), \eta\epsilon_m] \\
=& \mathrm{d}_\omega(\eta\epsilon_m e) = \mathbb{K}_{\eta,E}.
\end{aligned}$$

The terms along $\frac{\delta}{\delta\Omega}$ are

$$\begin{aligned}
&e\mathrm{d}_{\omega_0}\eta\epsilon_m r + [\Omega, \eta\epsilon_m r] + \eta\epsilon_m(F_\omega+\Theta) = \\
&=\mathrm{d}\eta\epsilon_m(\omega-\omega_0) - \eta\mathrm{d}_{\omega_0}\epsilon_m(\omega-\omega_0-\epsilon_m\epsilon_n b) + \eta\epsilon_m e\mathrm{d}_{\omega_0} r \\
&\quad + \eta\epsilon_m F_\omega - \eta\epsilon_m([e, \omega-\omega_0] r + \mathrm{d}_\omega(\omega-\omega_0)) \\
&=\mathbb{K}_{\eta,\Omega} - \eta\mathrm{d}_\omega\epsilon_m(\omega-\omega_0) - \eta\mathrm{d}_{\omega_0}\epsilon_m((\omega-\omega_0) - \underline{\epsilon_m\epsilon_n b}_1) \\
&\quad - \underline{\eta\epsilon_m[\omega-\omega_0, er]}_2 + \eta\epsilon_m\mathrm{d}_\omega(\omega-\omega_0) - \underline{\eta\epsilon_m\mathrm{d}_\omega(\epsilon_m\epsilon_n b)}_3 \\
&\quad + \eta[\omega-\omega_0, \epsilon_m](\omega-\omega_0-\underline{\epsilon_m\epsilon_n b}_4) - \underline{\eta\epsilon_m[\omega-\omega_0, r]e}_5 \\
&=\mathbb{K}_{\eta,\Omega} + \underline{\eta\mathrm{d}_\omega(\epsilon_m(\omega-\omega_0))}_6 - \underline{\eta\mathrm{d}_\omega\epsilon_m\omega-\omega_0}_7 + \underline{\eta\epsilon_m\mathrm{d}_\omega(\omega-\omega_0)}_8 \\
&=\mathbb{K}_{\eta,\Omega},
\end{aligned} \tag{77}$$

having noticed that (77.[2]) + (77.[5]) = (77.[1]) + (77.[3]) + (77.[4]) = (77.[6]) + (77.[7]) + (77.[8]) = 0. From the terms along $\frac{\delta}{\delta\Omega_m}$, we obtain

$$\begin{aligned}
&e\mathrm{d}_{\omega_m^0}\eta\epsilon_m r + \epsilon_m\mathrm{d}_\omega\eta\epsilon_m r + \underline{[\Omega_m, \eta\epsilon_m r]}_1 + \eta\epsilon_m(F_{\omega_m}+\Theta_m) \\
&= e\mathrm{d}_{\omega_m}\eta\epsilon_m r + \mathrm{d}_\omega\epsilon_m\eta\epsilon_m r + \eta\epsilon_m F_{\omega_m} \\
&\quad - \eta\epsilon_m([\epsilon_m, \omega-\omega_0] r + \mathrm{d}_\omega(\omega-\omega_0)_m + \mathrm{d}_{\omega_m}(\omega-\omega_0)) \\
&= \mathbb{K}_{\eta,\Omega_m},
\end{aligned} \tag{78}$$

having noticed (78.[1]) = 0.

Lastly, we are left with showing that $\pi(\mathcal{E}_\zeta) = \mathbb{E}_\zeta$. We have

$$\begin{aligned}
\pi(\mathcal{E}_\zeta) = \int_\Gamma & \left([\iota_\zeta(\omega-\omega_0)+\iota_\zeta e r, E]\right)\frac{\delta}{\delta E} - \iota_\zeta e\left(e\mathrm{d}_{\omega_0}\frac{\delta}{\delta E} + [\Omega, \frac{\delta}{\delta E}]\right) \\
&+ \frac{\delta}{\delta\Omega}\left(e\mathrm{d}_{\omega_0}(\iota_\zeta(\omega-\omega_0)+\iota_\zeta e r) + [\Omega, \iota_\zeta(\omega-\omega_0)+\iota_\zeta e r]\right) \\
&+ \frac{\delta}{\delta\Omega}\left(e\mathrm{d}_\omega\left(\alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}\right) - (\omega-\omega_0)[e, \alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}]\right) \\
&+ \left(\underline{[\iota_\zeta e r}_1 + \iota_\zeta(\omega-\omega_0), \epsilon_m] + \underline{[\alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}, \epsilon_m]}_2\right)\frac{\delta}{\delta\epsilon_m} \\
&+ \frac{\delta}{\delta\Omega_m}\left(e\mathrm{d}_{\omega_m^0}(\iota_\zeta(\omega-\omega_0+\iota_\zeta e r+\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}) + \epsilon_m\mathrm{d}_{\omega_0}(\iota_\zeta(\omega-\omega_0+\iota_\zeta e r)\right) \\
&+ \frac{\delta}{\delta\Omega_m}\left(\epsilon_m\mathrm{d}_{\omega_0}(\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}) + [\Omega_m, \iota_\zeta(\omega-\omega_0+\iota_\zeta e r+\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}]\right) \\
&+ \iota_\zeta e(F_\omega+\Theta)\frac{\delta}{\delta\Omega} + \iota_\zeta e(F_{\omega_m}+\Theta_m)\frac{\delta}{\delta\Omega_m} + \left(\mathrm{d}_{\omega_m}(\iota_\zeta e) - \underline{[\iota_\zeta e r, \epsilon_m]}_3\right)\frac{\delta}{\delta\epsilon_m} \\
&- \underline{W_e^{-1}\left([\iota_\zeta e\epsilon_n\epsilon_m b, \epsilon_m]\right)}_4\frac{\delta}{\delta\epsilon_m}.
\end{aligned} \tag{79}$$

We immediately see that (79.[1]) + (79.[3]) = 0. Similarly, we see that (79.[2]) is such that

$$e[\alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}, \epsilon_m] = [e\left(\alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}\right), \epsilon_m] = [\iota_\zeta e\epsilon_n\epsilon_m b, \epsilon_m],$$

and therefore (79.[2]) + (79.[4]) = 0. Then, along $\frac{\delta}{\delta E}$, we have

$$\begin{aligned}
&[\iota_\zeta(\omega-\omega_0)+\iota_\zeta e r, E] + [\alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}, E] - L_\zeta^{\omega_0}E + [e(\omega-\omega_0), \iota_\zeta e] \\
&= [\iota_\zeta(\omega-\omega_0), E] - [\iota_\zeta e, e]\epsilon_m\epsilon_n b - [e, e(\alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}] - L_\zeta^{\omega_0}E + e[\omega-\omega_0, \iota_\zeta e] \\
&= -[\omega-\omega_0, e]\iota_\zeta e + [e, \iota_\zeta e]\epsilon_m\epsilon_n b - [e, \iota_\zeta e\epsilon_m\epsilon_n b] - L_\zeta^{\omega_0}E \\
&= [e, er+\epsilon_m\epsilon_n b]\iota_\zeta e + [e, \iota_\zeta e]\epsilon_m\epsilon_n b - [e, \iota_\zeta e\epsilon_m\epsilon_n b] - L_\zeta^{\omega_0}E \\
&= -L_\zeta^{\omega_0}E,
\end{aligned}$$

where we simply developed the brackets and simplified the resulting terms.

Moreover, along $\frac{\delta}{\delta\Omega}$, we have

$$\begin{aligned}
&e\mathrm{d}_\omega(\iota_\zeta(\omega-\omega_0)) + e\mathrm{d}_\omega(\iota_\zeta e r) - (\omega-\omega_0)[e, \iota_\zeta(\omega-\omega_0)+\iota_\zeta e r] \\
&\quad + e\mathrm{d}_\omega(\alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}) - (\omega-\omega_0)[e, \alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}] + \iota_\zeta e(F_\omega+\Theta) \\
&= \mathrm{d}_\omega(e\iota_\zeta(\omega-\omega_0)) + \mathrm{d}_\omega(\iota_\zeta e(\omega-\omega_0)) - [\iota_\zeta(\omega-\omega_0), e](\omega-\omega_0) \\
&\quad - [e, \iota_\zeta e](er+\epsilon_m\epsilon_n b)r - (\omega-\omega_0)[e, \alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}] \\
&\quad - e\iota_\zeta F_\omega - \iota_\zeta e[e, \omega-\omega_0]r - \iota_\zeta e\mathrm{d}_\omega(\omega-\omega_0) \\
&= \mathrm{d}_\omega\iota_\zeta\Omega - [\iota_\zeta(\omega-\omega_0), e](\omega-\omega_0) - [e, \iota_\zeta e]\epsilon_m\epsilon_n br \\
&\quad - (er+\epsilon_m\epsilon_n b)[e, \alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}] - e\iota_\zeta F_\omega \\
&\quad - \iota_\zeta e[e, \epsilon_m\epsilon_n b]r - \iota_\zeta e\mathrm{d}_\omega(\omega-\omega_0) \\
&= E_\Omega + \epsilon_m\epsilon_n b[e, \alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}],
\end{aligned}$$

where

$$\begin{aligned}
\epsilon_m\epsilon_n b[e, \alpha_\zeta - \epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}] &= \epsilon_m\epsilon_n b W_e^{-1}([e, \iota_\zeta e\epsilon_m\epsilon_n b]) \\
&= \epsilon_m\epsilon_n b W_e^{-1}([e, \iota_\zeta e\epsilon_m\epsilon_n]b) \\
&= b W_e^{-1}(\epsilon_m\epsilon_n[e, \iota_\zeta e\epsilon_m\epsilon_n]b) \\
&= 0.
\end{aligned}$$

Lastly, along $\frac{\delta}{\delta\Omega_m}$, we have

$$\begin{aligned}
&e\mathrm{d}_{\omega_m^0}(\iota_\zeta(\omega-\omega_0)+\iota_\zeta er+\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e})+\epsilon_m\mathrm{d}_\omega(\iota_\zeta(\omega-\omega_0)+\iota_\zeta er+\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e})\\
&+[\Omega_m,\iota_\zeta(\omega-\omega_0)+\iota_\zeta er+\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}]+\iota_\zeta e(F_{\omega_m}+\Theta_m)\\
=\;&e\mathrm{d}_{\omega_m}(\iota_\zeta(\omega-\omega_0)+\iota_\zeta er+\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e})+\epsilon_m\mathrm{d}_\omega(\iota_\zeta(\omega-\omega_0)+\iota_\zeta er+\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e})\\
&+(\omega-\omega_0)[\epsilon_m,\iota_\zeta(\omega-\omega_0)+\iota_\zeta er+\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}]-e\iota_\zeta F_{\omega_m}\\
&-\iota_\zeta e\mathrm{d}_{\omega_m}(\omega-\omega_0)-\iota_\zeta e[\epsilon_m,\omega-\omega_0]r\\
=\;&\mathrm{d}_{\omega_m}(e\iota_\zeta(\omega-\omega_0)+\iota_\zeta e(\omega-\omega_0-\epsilon_m\epsilon_n b)+\iota_\zeta e\epsilon_m\epsilon_n b)\\
&-\mathrm{d}_\omega\epsilon_m(\iota_\zeta(\omega-\omega_0)+\iota_\zeta er)-\mathrm{d}_\omega\epsilon_m(\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e})\\
&-\mathrm{d}_\omega(\epsilon_m\iota_\zeta(\omega-\omega_0)+\iota_\zeta e\epsilon_m r+\epsilon_m\alpha_\zeta-\epsilon_m^2\epsilon_n\frac{\delta\beta_\zeta}{\delta e})\\
&+\mathrm{d}_\omega\epsilon_m(\iota_\zeta(\omega-\omega_0)+\iota_\zeta er+\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e})-[\iota_\zeta(\omega-\omega_0),\epsilon_m](\omega-\omega_0)\\
&-[\epsilon_m,(\omega-\omega_0)\iota_\zeta er]+\iota_\zeta e[\epsilon_m,\omega-\omega_0]r+er[\epsilon_m,\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}]\\
&+\epsilon_m\epsilon_n b[\epsilon_m,\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}]-e\iota_\zeta F_{\omega_m}\\
&+\epsilon_m r[e,\iota_\zeta(\omega-\omega_0)+\iota_\zeta er+\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}]\\
&-\iota_\zeta e(\mathrm{d}_{\omega_m}(\omega-\omega_0))-\iota_\zeta e[\epsilon_m,\omega-\omega_0]r,
\end{aligned}$$

which, after simplification, reduces to

$$\begin{aligned}
&=\mathrm{d}_{\omega_m}\iota_\zeta\Omega+\mathrm{d}_\omega\iota_\zeta\Omega_m+\mathrm{d}_\omega(\epsilon_m W_e^{-1}(\iota_\zeta e\epsilon_m\epsilon_n b))-[\iota_\zeta(\omega-\omega_0),\epsilon_m](\omega-\omega_0)\\
&\quad-[\epsilon_m,\epsilon_m\epsilon_n b\iota_\zeta er]+[\epsilon_m,\iota_\zeta e\epsilon_m\epsilon_n b]r-e\iota_\zeta F_{\omega_m}-\iota_\zeta e\mathrm{d}_{\omega_m}(\omega-\omega_0),
\end{aligned}$$

where we used the identities

$$\begin{aligned}
\epsilon_m\epsilon_n b[\epsilon_m,\alpha_\zeta-\epsilon_m\epsilon_n\frac{\delta\beta_\zeta}{\delta e}]&=\epsilon_m\epsilon_n bW_e^{-1}([\epsilon_m,\iota_\zeta e\epsilon_m\epsilon_n b])\\
&=bW_e^{-1}([\epsilon_m,\iota_\zeta e\epsilon_m\epsilon_n b]\epsilon_m\epsilon_n)=0
\end{aligned}$$

and

$$\epsilon_m W_e^{-1}(\iota_\zeta e\epsilon_m\epsilon_n b)=W_e^{-1}(\iota_\zeta e\epsilon_m^2\epsilon_n b)=0.$$

The remaining terms in $\pi(\mathcal{E}_\zeta)$ form exactly $\mathbb{E}_{\zeta,\epsilon_m}$, which concludes the proof. □

References

[Ale+97] M. Aleksandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky. "The geometry of the master equation and topological quantum field theory". *Int. J. Mod. Phys. A* (12 1997), pp. 1405–1430. DOI: 10.1142/S0217751X97001031.

[AS15] S. Alexandrov and S. Speziale. "First order gravity on the light front". *Physical Review D* 91.6 (Mar. 2015). ISSN: 1550-2368. DOI: 10.1103/physrevd.91.064043.

[AG14] K. Alkalaev and M. Grigoriev. "Frame-like Lagrangians and presymplectic AKSZ-type sigma models". *International Journal of Modern Physics A* 29.18 (2014), p. 1450103. DOI: 10.1142/S0217751X14501036.

[Ans16] D. Anselmi. "Some reference formulas for the generating functions of canonical transformations". *European Physical Journal C* 76, 49 (Feb. 2016), p. 49. DOI: 10.1140/epjc/s10052-015-3874-y.

[ADM59] R. Arnowitt, S. Deser, and C. W. Misner. "Dynamical Structure and Definition of Energy in General Relativity". *Phys. Rev.* 116 (5 Dec. 1959), pp. 1322–1330. DOI: 10.1103/PhysRev.116.1322.

[Ash86] A. Ashtekar. "New variables for classical and quantum gravity". *Physical Review Letters* 57 (Nov. 1986), pp. 2244–2247. DOI: 10.1103/PhysRevLett.57.2244.

[BM94] J. Baez and J. P. Muniain. *Gauge Fields, Knots and Gravity*. World Scientific Publishing Company, Oct. 1994.

[BBH95] G. Barnich, F. Brandt, and M. Henneaux. "Local BRST cohomology in the antifield formalism. I. General theorems". *Comm. Math. Phys.* 174.1 (1995), pp. 57–91.

[BG11] G. Barnich and M. Grigoriev. "First order parent formulation for generic gauge field theories". *Journal of High Energy Physics* 2011.1 (2011), p. 122. DOI: 10.1007/JHEP01(2011)122.

[BG05] G. Barnich and M. Grigoriev. "Hamiltonian BRST and Batalin-Vilkovisky Formalisms for Second Quantization of Gauge Theories". *Communications in Mathematical Physics* 254.3 (2005), pp. 581–601. DOI: 10.1007/s00220-004-1275-4.

[BF83] I. A. Batalin and E. S. Fradkin. "A generalized canonical formalism and quantization of reducible gauge theories". *Physics Letters B* 122.2 (Mar. 1983), pp. 157–164. DOI: 10.1016/0370-2693(83)90784-0.

[BV81] I. A. Batalin and G. A. Vilkovisky. "Gauge algebra and quantization". *Physics Letters B* 102.1 (June 1981), pp. 27–31. DOI: 10.1016/0370-2693(81)90205-7.

[BV77] I. A. Batalin and G. A. Vilkovisky. "Relativistic S-matrix of dynamical systems with boson and fermion constraints". *Physics Letters B* 69.3 (Aug. 1977), pp. 309–312. DOI: 10.1016/0370-2693(77)90553-6.

[BFW13] C. Blohmann, M. C. B. Fernandes, and A. Weinstein. "Groupoid symmetry and constraints in general relativity". *Communications in Contemporary Mathematics* 15.01 (Jan. 2013), p. 1250061. ISSN: 1793-6683. DOI: 10.1142/s0219199712500617.

[BW18] C. Blohmann and A. Weinstein. "Hamiltonian Lie algebroids" (2018). arXiv: 1811.11109 [math.SG].

[BC18] Y. Bonder and C. Corral. "Unimodular Einstein–Cartan gravity: Dynamics and conservation laws". *Phys. Rev. D* 97.8 (2018), p. 084001. DOI: 10.1103/PhysRevD.97.084001.

[BNB19] S. Bravo Medina, M. Nowakowski, and D. Batic. "Einstein-Cartan Cosmologies". *Ann. Phys.* 400 (2019), p. 64.

[Brz09] T. Brzeziński. "On synthetic interpretation of quantum principal bundles" (2009). arXiv: 0912.0213 [math.QA].

[Bur] M. Burkardt. "Light Front Quantization". *Advances in Nuclear Physics* (), pp. 1–74. DOI: 10.1007/0-306-47067-5_1.

[CLR19] F. Cabral, F. S. N. Lobo, and D. Rubiera-Garcia. "Einstein-Cartan-Dirac gravity with $U(1)$ symmetry breaking" (2019). arXiv: 1902.02222 [gr-qc].

[Can24] G. Canepa. "On the properties of coframes" (Oct. 2024). arXiv: 2410.17682 [math-ph].

[CC25] G. Canepa and A. S. Cattaneo. *BV Pushforward of Palatini-Cartan gravity*. 2025. arXiv: 2507.06279 [math-ph]. URL: https://arxiv.org/abs/2507.06279.

[CC24] G. Canepa and A. S. Cattaneo. "Corner Structure of Four-Dimensional General Relativity in the Coframe Formalism". *Annales Henri Poincaré* 25.5 (2024), pp. 2585–2639. DOI: 10.1007/s00023-023-01360-8.

[CCF22] G. Canepa, A. S. Cattaneo, and F. Fila-Robattino. "Boundary structure of gauge and matter fields coupled to gravity" (June 2022). arXiv: 2206.14680 [math-ph].

[Can+26] G. Canepa, A. S. Cattaneo, F. Fila-Robattino, and T. Leupp. *Corner Quantization of 4D BF Theory*. 2026. arXiv: 2605.30006 [math-ph]. URL: https://arxiv.org/abs/2605.30006.

[Can+23] G. Canepa, A. S. Cattaneo, F. Fila-Robattino, and M. Tecchiolli. "Boundary structure of the standard model coupled to gravity" (July 2023). arXiv: 2307.14955 [math-ph].

[CCS21a] G. Canepa, A. S. Cattaneo, and M. Schiavina. "Boundary structure of general relativity in tetrad variables". *Adv. Theor. Math. Phys.* 25.2 (2021), pp. 327–377. DOI: 10.4310/ATMP.2021.v25.n2.a3.

[CCS21b] G. Canepa, A. S. Cattaneo, and M. Schiavina. "General Relativity and the AKSZ Construction". *Commun. Math. Phys.* 385.3 (2021), pp. 1571–1614. DOI: 10.1007/s00220-021-04127-6.

[CCT21] G. Canepa, A. S. Cattaneo, and M. Tecchiolli. "Gravitational Constraints on a Lightlike Boundary". *Annales Henri Poincare* 22.9 (2021), pp. 3149–3198. DOI: 10.1007/s00023-021-01038-z.

[CS19a] G. Canepa and M. Schiavina. "Fully extended BV-BFV description of General Relativity in three dimensions" (2019). arXiv: 1905.09333 [math-ph].

[Car98] S. Carlip. "Quantum Gravity in 2+1 Dimensions". *Cambridge Monographs on Mathematical Physics, Cambridge University Press* (1998).

[Car22] E. Cartan. "Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion." *C. R. Acad. Sci.* 174 (1922), pp. 593–595.

[CS19b] A. S. Cattaneo and M. Schiavina. "The reduced phase space of Palatini-Cartan-Holst theory". *Annales Henri Poincare* 20.2 (2019), p. 445. DOI: 10.1007/s00023-018-0733-z.

[Cat23] A. S. Cattaneo. "Phase space for gravity with boundaries" (July 2023). arXiv: 2307.04666 [math-ph].

[CDW11] A. S. Cattaneo, B. Dherin, and A. Weinstein. "Symplectic Microgeometry II: Generating functions" (Mar. 2011). arXiv: 1103.0672 [math.SG].

[CMR20] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. "A Cellular Topological Field Theory". *Communications in Mathematical Physics* 374.2 (2020), pp. 1229–1320. URL: https://doi.org/10.1007/s00220-020-03687-3.

[CMR11] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. "Classical and quantum Lagrangian field theories with boundary". *PoS* CORFU2011 (2011), p. 044. DOI: 10.22323/1.155.0044.

[CMR14] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. "Classical BV Theories on Manifolds with Boundary". *Communications in Mathematical Physics* 332.2 (2014), pp. 535–603. ISSN: 1432-0916. DOI: 10.1007/s00220-014-2145-3.

[CMR18] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. "Perturbative Quantum Gauge Theories on Manifolds with Boundary". *Communications in Mathematical Physics* 357.2 (Jan. 2018), pp. 631–730. DOI: 10.1007/s00220-017-3031-6.

[CMW17] A. S. Cattaneo, P. Mnev, and K. Wernli. "Split Chern–Simons Theory in the BV-BFV Formalism". *9th Summer School on Geometric, Algebraic and Topological Methods for Quantum Field Theory*. Mathematical Physics Studies. 2017, pp. 293–324. DOI: 10.1007/978-3-319-65427-0_9.

[CMW19a] A. S. Cattaneo, P. Mnev, and K. Wernli. "Theta Invariants of lens spaces via the BV-BFV formalism" (2019). arXiv: 1810.06663 [math.AT].

[CMW19b] A. S. Cattaneo, N. Moshayedi, and K. Wernli. "Globalization for Perturbative Quantization of Nonlinear Split AKSZ Sigma Models on Manifolds with Boundary". *Communications in Mathematical Physics* 372.1 (2019), pp. 213–260. URL: https://doi.org/10.1007/s00220-019-03591-5.

[CS19c] A. S. Cattaneo and M. Schiavina. "BV-BFV approach to General Relativity: Palatini–Cartan–Holst action". *Adv. Theor. Math. Phys.* 23.8 (2019), pp. 2025–2059. DOI: 10.4310/ATMP.2019.v23.n8.a3.

[CS19d] A. S. Cattaneo and M. Schiavina. "The reduced phase space of Palatini-Cartan-Holst theory". *Annales Henri Poincare* 20.2 (2019), pp. 445–480. DOI: 10.1007/s00023-018-0733-z.

[CL18] S. Chakrabarty and A. Lahiri. "Different types of torsion and their effect on the dynamics of fields". *Eur. Phys. J. Plus* 133.6 (2018), p. 242. DOI: 10.1140/epjp/i2018-12070-6.

[CG16] K. Costello and O. Gwilliam. *Factorization Algebras in Quantum Field Theory*. Vol. 1. Cambridge University Press, 2016. DOI: 10.1017/9781316678626.

[Cou90] T. Courant. "Dirac manifolds". *Transactions of the American Mathematical Society* 319.2 (1990), pp. 631–661.

[CHv95] O. Coussaert, M. Henneaux, and P. van Driel. "The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant". *Classical and Quantum Gravity* 12 (Dec. 1995), pp. 2961–2966. DOI: 10.1088/0264-9381/12/12/012.

[dLV06] R. d'Inverno, P. Lambert, and J. Vickers. "Hamiltonian analysis of the double null 2+2 decomposition of Ashtekar variables". *Class. Quant. Grav.* 23 (2006), pp. 3747–3762. DOI: 10.1088/0264-9381/23/11/005.

[DS17] E. De Paoli and S. Speziale. "Sachs' free data in real connection variables". *JHEP* 11 (2017), p. 205. DOI: 10.1007/JHEP11(2017)205.

[DeW67] B. S. DeWitt. "Quantum Theory of Gravity. I. The Canonical Theory". *Phys. Rev.* 160 (5 Aug. 1967), pp. 1113–1148. DOI: 10.1103/PhysRev.160.1113.

[DC19] C. F. Diether and J. Christian. "Existence of Matter as a Proof of the Existence of Gravitational Torsion". 10 (2019). arXiv: 1903.10902 [gr-qc].

[Dir50] P. Dirac. "Generalized Hamiltonian Dynamics". *Canadian Journal of Mathematics* 2 (1950), pp. 4129–4148.

[Dir58] P. A. M. Dirac. "Generalized Hamiltonian Dynamics". *Proceedings of the Royal Society of London. Series A, Math. and Phys. Sciences* 246.1246 (1958), pp. 326–332.

[DGH90] A. Dresse, P. Gregoire, and M. Henneaux. "Path-integral equivalence between the extended and non-extended hamiltonian formalisms". *Physics Letters B* 245.2 (1990), pp. 192–196.

[Ein16] A. Einstein. "Die Grundlage der allgemeinen Relativitätstheorie". *Annalen der Physik* 49 (1916), pp. 769–822. DOI: 10.1002/andp.19163540702.

[EKT87] D. Evens, G. Kunstatter, and C. Torre. "Hamiltonian analysis of linearised gravity on a null plane". *Classical and Quantum Gravity* 4.6 (Nov. 1987), pp. 1503–1508. DOI: 10.1088/0264-9381/4/6/009.

[Fab15] L. Fabbri. "A discussion on the most general torsion-gravity with electrodynamics for Dirac spinor matter fields". *Int. J. Geom. Meth. Mod. Phys.* 12.09 (2015), p. 1550099. DOI: 10.1142/S0219887815500991.

[Fab] L. Fabbri. *Foundations Quadrilogy*. arXiv: 1703.02287 [gr-qc].

[Fab17] L. Fabbri. "Singularity-free spinors in gravity with propagating torsion". *Mod. Phys. Lett. A* 32.40 (2017), p. 1750221. DOI: 10.1142/S0217732317502212.

[FdR18] L. Fabbri and R. da Rocha. "Torsion Axial Vector and Yvon-Takabayashi Angle: Zitterbewegung, Chirality and all that". *Eur. Phys. J. C* 78.3 (2018), p. 207. DOI: 10.1140/epjc/s10052-018-5706-3.

[FT18] L. Fabbri and M. Tecchiolli. "Restrictions on torsion-spinor field theory" (2018). arXiv: 1811.12060 [physics.gen-ph].

[Fat18] L. Fatibene. *Relativistic Theories, Gravitational Theories and General Relativity*. 2018. URL: http://www.fatibene.org/book.html.

[Fec11] M. Fecko. *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press, 2011.

[Fig06] J. Figueroa–O'Farrill. *Connections on principal fibre bundles*. 2006.

[FP90] R. Floreanini and R. Percacci. "Palatini formalism and new canonical variables for GL(4) invariant gravity". *Class. Quant. Grav.* 7 (1990), pp. 1805–1818. DOI: 10.1088/0264-9381/7/10/011.

[FR12] K. Fredenhagen and K. Rejzner. "Batalin-Vilkovisky Formalism in Perturbative Algebraic Quantum Field Theory". *Communications in Mathematical Physics* 317.3 (Nov. 2012), pp. 697–725. ISSN: 1432-0916. DOI: 10.1007/s00220-012-1601-1.

[GP18] E. Getzler and S. Pohorence. "Covariance of the classical Brink-Schwarz superparticle" (June 2018). arXiv: 1806.09021 [math-ph].

[GW13] S. Gielen and D. K. Wise. "Lifting General Relativity to Observer Space". *J. Math. Phys.* 54 (2013), p. 052501. DOI: 10.1063/1.4802878.

[GL14] H. Gies and S. Lippoldt. "Fermions in gravity with local spin-base invariance". *Phys. Rev. D* 89.6 (2014), p. 064040. DOI: 10.1103/PhysRevD.89.064040.

[GM88] M. Goresky and R. MacPherson. *Stratified Morse Theory*. Springer, 1988.

[GD00] M. Grigoriev and P. Damgaard. "Superfield BRST charge and the master action". *Physics Letters B* 474.3 (2000), pp. 323–330.

[Gri11] M. Grigoriev. "Parent formulation at the Lagrangian level". *Journal of High Energy Physics* 2011.7 (2011), p. 61. DOI: 10.1007/JHEP07(2011)061.

[Gri16] M. Grigoriev. *Presymplectic structures and intrinsic Lagrangians*. 2016. arXiv: 1606.07532 [hep-th].

[Gua04] M. Gualtieri. "Generalized Complex Geometry". PhD Thesis, supervised by Nigel Hitchin. PhD thesis. University of Oxford, 2004. URL: https://arxiv.org/abs/math/0401221.

[Heh+76a] F. W. Hehl, P. V. D. Heyde, G. D. Kerlick, and J. M. Nester. "General Relativity with Spin and Torsion: Foundations and Prospects". *Rev. Mod. Phys.* 48 (1976), p. 393. DOI: 10.1103/RevModPhys.48.393.

[HO07] F. W. Hehl and Y. N. Obukhov. "Elie Cartan's torsion in geometry and in field theory, an essay". 32 (2007). arXiv: 0711.1535 [gr-qc].

[Heh+76b] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester. "General relativity with spin and torsion: Foundations and prospects". *Rev. Mod. Phys.* 48.3 (July 1976), pp. 393–416. DOI: 10.1103/RevModPhys.48.393.

[HT92] M. Henneaux and C. Teitelboim. *Quantization of gauge systems*. 1992. ISBN: 978-0-691-03769-1.

[Hen90] M. Henneaux. "Elimination of the auxiliary fields in the antifield formalism". *Physics Letters B* 238.2 (1990), pp. 299–304.

[Hit03] N. Hitchin. "Generalized Calabi–Yau manifolds". *The Quarterly Journal of Mathematics* 54.3 (2003), pp. 281–308. DOI: 10.1093/qjmath/54.3.281.

[Hol96] S. Holst. "Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action". *Phys. Rev. D* 53 (10 May 1996), pp. 5966–5969. DOI: 10.1103/PhysRevD.53.5966.

[IJ19] S. Inglis and P. Jarvis. "The self-coupled Einstein–Cartan–Dirac equations in terms of Dirac bilinears". *J. Phys. A* 52.4 (2019), p. 045301. DOI: 10.1088/1751-8121/aaf4e0.

[IM19] R. Iraso and P. Mnev. "Two-Dimensional Yang–Mills Theory on Surfaces with Corners in Batalin–Vilkovisky Formalism". *Communications in Mathematical Physics* 370.2 (2019), pp. 637–702. URL: https://doi.org/10.1007/s00220-019-03392-w.

[Jia] Y. Jiang. "Angular momentum conservation law in Einstein-Cartan space-time" (). eprint: gr-qc/0010047.

[K S16] K. Schwarzschild. "On the gravitational field of a mass point according to Einstein's theory" (Jan. 1916).

[KT79] J. Kijowski and W. M. Tulczyjew. "A Symplectic Framework for Field Theories". *Lect. Notes Phys.* 107 (1979).

[KN69] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, Vol. 2*. New York: Wiley and Sons, 1969.

[KS87] B. Kostant and S. Sternberg. "Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras". *Annals of Physics* 176.1 (1987), p. 49. ISSN: 0003-4916. DOI: 10.1016/0003-4916(87)90178-3.

[Kra+19] D. Kranas, C. G. Tsagas, J. D. Barrow, and D. Iosifidis. "Friedmann-like universes with torsion". *Eur. Phys. J. C* 79 (2019), p. 341.

[Lot] J. Lott. *The Geometry of Supergravity Torsion Constraints*. arXiv: math/0108125.

[Mac87] K. Mackenzie. *Lie Groupoids and Lie Algebroids in Differential Geometry*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1987. DOI: 10.1017/CBO9780511661839.

[MZ91] A. Mardones and J. Zanelli. "Lovelock-Cartan theory of gravity". *Class. Quantum Grav.* 8 (1991), p. 1545.

[MW74] J. Marsden and A. Weinstein. "Reduction of symplectic manifolds with symmetry". *Reports on Mathematical Physics* 5.1 (1974), pp. 121–130. ISSN: 0034-4877. DOI: 10.1016/0034-4877(74)90021-4.

[Mat75] J. Mather. *Notes on Topological Stability*. Tech. rep. Harvard University, 1975.

[MZ17] M. R. Mehdizadeh and A. H. Ziaie. "Einstein-Cartan Wormhole Solutions". *Phys. Rev. D* 95 (2017), p. 064049.

[Mei25] E. Meinrenken. "Introduction to Moduli Spaces and Dirac Geometry". *arXiv preprint arXiv:2506.04150* (2025). DOI: 10.48550/arXiv.2506.04150.

[Mel] R. B. Melrose. *Differential Analysis on Manifolds with Corners*. Lecture notes. URL: https://math.mit.edu/~rbm/18.158/daomwc.1/daomwc.1.pdf.

[MKS93] P. W. Michor, I. Kolar, and J. Slovak. *Natural Operations in Differential Geometry*. Springer, 1993.

[Mne17] P. Mnev. "Lectures on Batalin-Vilkovisky formalism and its applications in topological quantum field theory" (July 2017). arXiv: 1707.08096 [math-ph].

[MSW19] P. Mnev, M. Schiavina, and K. Wernli. "Towards holography in the BV-BFV setting". *Annales Henri Poincare* 21.3 (2019), pp. 993–1044. DOI: 10.1007/s00023-019-00862-8.

[Mon+20] M. Montesinos, R. Escobedo, J. Romero, and M. Celada. "Canonical analysis of n-dimensional Palatini action without second-class constraints". *Phys. Rev. D* 101 (2 Jan. 2020), p. 024042. DOI: 10.1103/PhysRevD.101.024042.

[MRC19] M. Montesinos, J. Romero, and M. Celada. "Canonical analysis of Holst action with first-class constraints only" (2019). arXiv: 1911.09690 [gr-qc].

[Mor01] S. Morita. *The Geometry of Differential Forms*. Vol. 201. American Mathematical Society, 2001.

[NY99] J. M. Nester and H. J. Yo. "Symmetric teleparallel general relativity". 37 (1999). eprint: gr-qc/9809049.

[Pal19a] A. Palatini. "Deduzione invariantiva delle equazioni gravitazionali dal principio di Hamilton". *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 43.1 (1919), pp. 203–212. ISSN: 0009-725X. DOI: 10.1007/BF03014670.

[Pal19b] A. Palatini. "Deduzione invariantiva delle equazioni gravitazionali dal principio di Hamilton". *Rendiconti del Circolo Matematico di Palermo* 43 (1919), pp. 203–212. DOI: 10.1007/BF03014670.

[Pen80] R. Penrose. "Golden Oldie: Null Hypersurface Initial Data for Classical Fields of Arbitrary Spin and for General Relativity". *Gen Relat Gravit* 12 (1980), pp. 225–264. DOI: 10.1007/BF00756234.

[PTB17] K. Pesmatsiou, C. G. Tsagas, and J. D. Barrow. "Kinematics of the Einstein-Cartan universes". *Phys. Rev. D* 95 (2017), p. 104007.

[Pfl01] M. J. Pflaum. *Analytic and Geometric Study of Stratified Spaces*. Vol. 1768. Lecture Notes in Mathematics. Springer, 2001.

[Pop10] N. J. Poplawski. "Cosmology with Torsion: An alternative to cosmic inflation". *Phys. Lett. B* 694 (2010), p. 181.

[Pop16] N. J. Poplawski. "Universe in a Black Hole with Spin and Torsion". *ApJ* 832 (2016), p. 96.

[Rei18] M. P. Reisenberger. "The Poisson brackets of free null initial data for vacuum general relativity". *Classical and Quantum Gravity* 35.18 (Aug. 2018), p. 185012. ISSN: 1361-6382. DOI: 10.1088/1361-6382/aad569.

[Rei13] M. P. Reisenberger. "The symplectic 2-form for gravity in terms of free null initial data". *Classical and Quantum Gravity* 30.15 (July 2013), p. 155022. ISSN: 1361-6382. DOI: 10.1088/0264-9381/30/15/155022.

[Rom93] J. D. Romano. "Geometrodynamics versus connection dynamics (in the context of (2+1) and (3+1) gravity)". *Gen. Rel. Grav.* 25 (1993), p. 759. DOI: 10.1007/BF00758384.

[Roy02] D. Roytenberg. "On the structure of graded symplectic supermanifolds and Courant algebroids". *arXiv:math/0203110 [math.SG]* (2002).

[Sac62] R. K. Sachs. "On the Characteristic Initial Value Problem in Gravitational Theory". *Journal of Mathematical Physics* 3.5 (1962), pp. 908–914. DOI: 10.1063/1.1724305.

[Šev17] P. Ševera. *Letters to Alan Weinstein about Courant algebroids*. arXiv preprint. arXiv:1707.00265. 2017. URL: https://arxiv.org/abs/1707.00265.

[Sha97] R. W. Sharpe. *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*. New York: Springer, 1997.

[Sta97] J. Stasheff. "Homological reduction of constrained Poisson algebras". *J. Differential Geom.* 45.1 (1997), pp. 221–240. DOI: 10.4310/jdg/1214459757.

[Str12] N. Straumann. *General Relativity*. Springer, 2012.

[Tec19a] M. Tecchiolli. "Algebra of Constraints for the Linearized Palatini-Cartan Theory on a Light-Like Boundary". Master thesis. Nov. 2019. URL: http://user.math.uzh.ch/cattaneo/tecchiolli.pdf.

[Tec19b] M. Tecchiolli. "On the Mathematics of Coframe Formalism and Einstein–Cartan Theory—A Brief Review". *Universe* 5.10 (Sept. 2019), p. 206. ISSN: 2218-1997. DOI: 10.3390/universe5100206.

[Thi07] T. Thiemann. *Modern Canonical Quantum General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007. DOI: 10.1017/CBO9780511755682.

[Tor86] C. Torre. "Null Surface Geometrodynamics". *Class. Quant. Grav.* 3 (1986), p. 773. DOI: 10.1088/0264-9381/3/5/008.

[Ver84] A. Verona. *Stratified Mappings—Structure and Triangulability*. Springer, 1984.

[Wer19] K. Wernli. *Lecture Notes on Spin Geometry*. 2019. arXiv: 1911.09766 [math.DG].

[Wis09] D. K. Wise. "Symmetric Space Cartan Connections and Gravity in Three and Four Dimensions". *SIGMA* 5, 080 (Aug. 2009), p. 080. DOI: 10.3842/SIGMA.2009.080.

[Wit89] E. Witten. "2+1 dimensional gravity as an exactly soluble system". *Nuclear Physics B* (311 1988/89), pp. 46–78.

[Wit88] E. Witten. "Topological sigma models". *Comm. Math. Phys.* 118.3 (1988), pp. 411–449.