FROM L_∞ ALGEBROIDS TO L_∞ SPACES: PART I

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Abstract. We develop the notion of L_∞ spaces over dg manifolds. We establish an equivalence between the fibered category of transitive L_∞ algebroids over dg manifolds and the split fibered category of L_∞ spaces over dg manifolds, which detects weak equivalences. Moreover, we construct a faithful functor from L_∞ algebroids to L_∞ spaces which detects weak equivalences.

Contents

1. Introduction 1
2. Dg vector bundles over dg manifolds 4
3. L_∞ algebroids over dg manifolds 11
4. L_∞ spaces over dg manifolds 19
5. The fibrant replacement functor 25
Appendix A. Differential operators and jets 33
Appendix B. De Rham complexes of D-modules 40
References 45

1. Introduction

L_∞ spaces were introduced by Costello as an approach to derived stacks motivated by problems in quantum field theory [Cos10, Cos11a]. In combination with the Batalin–Vilkovisky formalism, L_∞ spaces provide a convenient framework for describing quantization in the formal neighborhood of a family of classical solutions parameterized by a smooth manifold [Cos11b]. Applications to non-linear sigma models recover classical invariants of the target manifold, such as the $\hat{A}$-genus [GG14] and the Witten genus [Cos10, Cos11a]. In this case, the relevant family of classical solutions consists of constant maps, parameterized by the target manifold.

In many quantum field theory applications, however, the family of classical solutions of interest is not parameterized by a smooth manifold, but rather by (a singular truncation of) a derived manifold. Such situations lie outside the scope of Costello's original formulation, since L_∞ spaces over smooth manifolds encode derived structure only through nilpotent directions [GG15]. The present paper addresses this limitation by adapting Costello's notion of L_∞ spaces to the setting of derived manifolds, or more precisely, to that of dg manifolds [BLX20, Car23].

L_∞ algebroids serve as a homotopical generalization of dg Lie algebroids in which the differential l_1, the binary Lie bracket l_2, and the unary anchor map ρ_1 are extended into a hierarchy of higher multilinear operations $\{l_n\}_{n\geq 1}$ and $\{\rho_n\}_{n\geq 1}$ satisfying higher coherent

identities, in the same way that L_∞ algebras serve as a homotopical generalization of dg Lie algebras [LS93, LM95]. They appear in geometry as models of singular foliations and higher Dirac structures [Sev01, Zam12, LLS20], in string theory as geometric formulations of higher gauge field content [SSS12], and in AKSZ theories as target spaces [ASZK97]. L_∞ algebroids also generalize Lie n-algebroids [BP13, SZ17]. The latter form a special class of L_∞ algebroids whose underlying graded vector bundles are concentrated in degrees $1-n, \ldots, 0$. More generally, L_∞ algebroids can be defined over a dg manifold $\mathcal{M}$ by allowing a zeroth anchor map ρ_0, which coincides with the cohomological vector field on $\mathcal{M}$ [Bru11, BCSX20].

There exists a close parallel between L_∞ spaces and Lie theoretic structures in higher differential geometry, as revealed by a result of Grady and Gwilliam [GG20]. They show that to any dg Lie algebroid one can associate an L_∞ space in a functorial manner compatible with weak equivalences. It is therefore natural to expect that a theory of L_∞ spaces over dg manifolds should develop in correspondence with the existing notion of L_∞ algebroids over dg manifolds. The main results of our paper confirm this expectation.

1.1. **Main results.** To study L_∞ algebroids and L_∞ spaces over dg manifolds in a unified way, we introduce the intermediate notion of *quasi-dg manifolds* from a commutative algebraic perspective. Roughly speaking, a quasi-dg manifold is a ringed space $\mathcal{C} = (C, \mathcal{O}_\mathcal{C})$ whose structure sheaf $\mathcal{O}_\mathcal{C}$ is a sheaf of commutative differential graded algebras, such that, modulo a dg ideal $\mathcal{I}_\mathcal{C} \subset \mathcal{O}_\mathcal{C}$, $\mathcal{C}$ reduces to an ordinary dg manifold. Using Chevalley–Eilenberg type functors, we establish the following equivalences between categories:

(1) $\mathbf{L}_\infty\mathbf{Algd} \cong \mathbf{QdgM}^{\mathrm{bp}}$ between the category of L_∞ algebroids over dg manifolds and that of quasi-dg manifolds with base-preserving morphisms;
(2) $\mathbf{L}_\infty\mathbf{Algd}_{\mathrm{fib}} \cong \mathbf{QdgM}^{\mathrm{bp}}_{\mathrm{fib}}$ between the category of transitive L_∞ algebroids over dg manifolds and that of fibrant quasi-dg manifolds with base-preserving morphisms;
(3) $\mathbf{L}_\infty\mathbf{Sp} \cong \mathbf{QdgM}^{\mathrm{bp}}_{\mathrm{fib}}$ between the category of L_∞ spaces over dg manifolds and that of fibrant quasi-dg manifolds with base-preserving morphisms.

Moreover, we prove that all these equivalences detect weak equivalences. Combining the last two, we obtain our first main result:

Theorem 1.1 (Corollary 4.13)**.** *There is an equivalence between the fibered category of transitive L_∞ algebroids over dg manifolds and the split fibered category of L_∞ spaces over dg manifolds, which detects weak equivalences.*

Another advantage of working with quasi-dg manifolds is that they naturally admit the language of infinite jets and D-modules. We provide a thorough review and adaptation of these notions on graded manifolds in the appendices. With these tools in hand, we introduce the notion of the *jet space* of a quasi-dg manifold, which gives rise to a functor

$$\mathbf{J}^\infty : \mathbf{QdgM} \longrightarrow \mathbf{QdgM}_{\mathrm{fib}},$$

called the *jet space functor*. Using the infinite jet prolongation j^∞, we construct a natural transformation from $\mathbf{J}^\infty$ to the identity functor. Since j^∞, viewed as a morphism of quasi-dg manifolds, is a weak equivalence, it follows that the jet space functor detects weak equivalences. Combined with the previous equivalences of categories, this yields our second main result:

Theorem 1.2 (Corollary 5.22)**.** *One can construct a faithful functor*

$$\mathbf{Fib} : \mathbf{L}_\infty\mathbf{Algd} \longrightarrow \mathbf{L}_\infty\mathbf{Sp} \cong \mathbf{L}_\infty\mathbf{Algd}_{\mathrm{fib}},$$

which detects weak equivalences.

We refer to **Fib** as a *fibrant replacement functor*. The terminology is due to the fact that L_∞ spaces over a fixed dg manifold form a category of fibrant objects. We prove this result in a companion paper [Jia26].

1.2. **Future work and applications.** As the title suggests, the present paper is the first in a series. In subsequent work, we will apply the fibrant replacement functor to representations of L_∞ algebroids, their characteristic classes, and shifted symplectic structures. One reason for this is that these structures become more transparent and computationally tractable in the setting of L_∞ spaces. For later applications in field theory, we will also study the Maurer–Cartan functor of L_∞ spaces over dg manifolds and its homotopical properties.

One of the main motivations for our work comes from applications to the problem of globalizing partition functions and observables over moduli spaces of classical solutions within the Batalin–Vilkovisky formalism. This problem has been studied, in a language different from that of L_∞ spaces but closely related to it, in the context of AKSZ sigma models, where one considers moduli spaces of constant maps [CF07, BCM12, GLL17, CMW19], as well as in two-dimensional Yang–Mills theory and global Chern–Simons theory, where one considers the smooth locus of moduli spaces of gauge-equivalence classes of flat connections [IM19, MW25]. The framework developed in the present paper, together with its sequels, is intended to provide an approach to the globalization problem in the presence of singular moduli spaces.

The two main classes of field theories we have in mind for applications are cohomological field theories (see [JJ26] for an exposition formulated in a way suitable for the Batalin–Vilkovisky formalism) and AKSZ theories with cotangent derived targets, both of which admit one-loop exact quantization. The latter is technically more tractable and may be viewed as a broad generalization of the existing results in the literature; it is expected to produce interesting invariants of derived manifolds [Jia].

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1.4. **Notation.** We work throughout in the real smooth setting.

cdga: Commutative differential graded algebra
dg manifold: Differential graded manifold
dg vector bundle: Differential graded vector bundle
$\mathcal{M}$**:** Dg manifold (or graded manifold)
$\mathcal{O}_\mathcal{M}$**:** Structure sheaf of $\mathcal{M}$
$\mathcal{T}_\mathcal{M}$**:** Tangent sheaf of $\mathcal{M}$

$\widehat{\Omega}_{\mathcal{M}}$: Sheaf of completed differential forms on $\mathcal{M}$
$\mathcal{D}_{\mathcal{M}}$: Sheaf of differential operators on $\mathcal{M}$
$\mathcal{J}^{\infty}_{\mathcal{M}}$: Sheaf of infinite jets on $\mathcal{M}$

2. Dg vector bundles over dg manifolds

In this section, we review dg manifolds and dg vector bundles over them. In particular, we prove a Whitehead-type theorem for dg vector bundles.

2.1. **Differential graded manifolds.** Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ be a graded vector space of finite total rank. Suppose that V is concentrated in non-negatively degrees, i.e., $V_i = 0$ for all $i < 0$. We denote by $(V_0, \mathcal{O}_V)$ the ringed space whose structure sheaf assigns each open $U \subset V_0$ the graded commutative algebra

$$\mathcal{O}_V(U) = C^{\infty}(U) \otimes_{\mathrm{Sym}(V_0^{\vee})} \mathrm{Sym}(V^{\vee}),$$

where $V^{\vee}$ denotes the dual graded vector space of V, and $\mathrm{Sym}(V^{\vee})$ denotes the free graded commutative algebra generated by $V^{\vee}$.

Definition 2.1. *A* (non-positively) graded manifold *is a ringed space $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$, where M is a smooth manifold and $\mathcal{O}_{\mathcal{M}}$ is a sheaf of graded commutative algebras, which is locally isomorphic to $(V_0, \mathcal{O}_V)$ for some graded vector space V as above.*

Let $\mathcal{M}$ be a graded manifold. For each open $U \subset M$, define

$$\mathcal{T}_{\mathcal{M}}(U) := \mathrm{Der}(\mathcal{O}_{\mathcal{M}}(U)),$$

where $\mathrm{Der}(\cdot)$ denotes the graded Lie algebra of derivations of a graded commutative algebra. The assignment $\mathcal{T}_{\mathcal{M}}$ defines a sheaf, called the *tangent sheaf* of $\mathcal{M}$. Global sections of $\mathcal{T}_{\mathcal{M}}$ are called *vector fields* on $\mathcal{M}$.

Definition 2.2. *A* dg manifold *is a graded manifold $\mathcal{M}$ together with a degree* 1 *vector field $Q_{\mathcal{M}}$ over $\mathcal{M}$ satisfying $[Q_{\mathcal{M}}, Q_{\mathcal{M}}] = 0$.*

The vector field $Q_{\mathcal{M}}$ defines a differential on $\mathcal{O}_{\mathcal{M}}$, making it into a sheaf of commutative differential graded algebras (cdgas).

Definition 2.3. *A* morphism *of dg manifolds $\mathcal{M} \to \mathcal{N}$ is a morphism of ringed spaces*

$$(f, f^{\sharp}) : (M, \mathcal{O}_{\mathcal{M}}) \longrightarrow (N, \mathcal{O}_{\mathcal{N}}),$$

where $f : M \to N$ is smooth, $f^{\sharp} : \mathcal{O}_{\mathcal{N}} \to f_*\mathcal{O}_{\mathcal{M}}$ is a morphism of sheaves of cdgas. Here f_* denotes the direct image functor associated to f.

By a slight abuse of notation, we denote a dg manifold morphism $(f, f^{\sharp})$ simply by f.

Remark 2.4. *By Batchelor's theorem, dg manifolds are essentially curved L_{∞} bundles in disguise. They form a category of fibrant objects* [BLX20]*: a morphism $f : \mathcal{M} \to \mathcal{N}$ is called*

- *a* weak equivalence *if $f_N^{\sharp} : \mathcal{O}_{\mathcal{N}}(N) \to \mathcal{O}_{\mathcal{M}}(M)$ is a quasi-isomorphism;*
- *a* fibration *if it is a submersion of graded manifolds.*

Formally inverting weak equivalences in this category produces an ∞-category equivalent to the ∞-category of derived manifolds [Car23].

2.2. **Differential graded vector bundles.** Let $\mathcal{M}$ be a graded manifold.

Definition 2.5. *A* graded vector bundle *over $\mathcal{M}$ is a locally free graded (left) $\mathcal{O}_{\mathcal{M}}$-module of finite total rank.*

The following result for graded vector bundles over $\mathcal{M}$ is not trivial, despite first appearances, since $\mathcal{M}$ is not a locally ringed space.

Proposition 2.6. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be graded vector bundles over $\mathcal{M}$. Let $\rho : \mathcal{E}_1 \to \mathcal{E}_2$ be a surjective morphism of graded $\mathcal{O}_{\mathcal{M}}$-modules. Then* $\ker \rho$ *is a graded vector bundle over $\mathcal{M}$.*

We begin by stating a super version of the Quillen–Suslin theorem needed for the proof, which we were unable to locate in the literature.

Lemma 2.7 (Super Quillen–Suslin theorem)**.** *Every finitely generated projective module over the super polynomial ring $\mathbb{R}[x_1, \dots, x_n, \theta_1, \dots, \theta_m]$ is free, where $x_1, \dots, x_n$ denote the even variables and $\theta_1, \dots, \theta_m$ denote the odd variables.*

Proof. For convenience, we denote the exterior algebra $\mathbb{R}[\theta_1, \dots, \theta_m]$ by Λ_m. The super polynomial ring can then be identified with the tensor product

$$R := \mathbb{R}[x_1, \dots, x_n] \otimes_{\mathbb{R}} \Lambda_m.$$

Let $I \subset R$ denote the nilpotent ideal $\mathbb{R}[x_1, \dots, x_n] \otimes \Lambda_m^{>0}$, where $\Lambda_m^{>0}$ is the maximal ideal of Λ_m generated by the odd variables. Then

$$R/I \cong \mathbb{R}[x_1, \dots, x_n].$$

Let M be a finitely generated projective R-module. Then $M/IM \cong M \otimes_R R/I$ is a finitely generated projective R/I-module. The classical Quillen–Suslin theorem then implies that M/IM is a free R/I-module [Qui76, Sus76].

Since I is nilpotent, $I \subset \operatorname{rad}(R)$. A lemma of Bass (see Lemma 2.4 in [Bas61]) states that for projective R-modules P and P',

$$P \cong P' \quad \text{if and only if} \quad P/IP \cong P'/IP'.$$

In particular, a projective R-module P is free if and only if P/IP is a free R/I-module. Applying this criterion to M completes the proof. □

To apply the super Quillen–Suslin theorem, we work in the geometric language of graded vector bundles. Using Batchelor's theorem, we choose a positively graded vector bundle E over M such that

$$\mathcal{O}_{\mathcal{M}} \cong \operatorname{Sym}(E^\vee),$$

where $\operatorname{Sym}(E^\vee)$ is understood as the corresponding locally free sheaf on M.

Lemma 2.8. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be graded vector bundles over $\mathcal{M}$. Then there exist graded vector bundles E_1 and E_2 over M such that*

$$\mathcal{E}_i \cong \operatorname{Sym}(E^\vee) \otimes E_i, \quad i = 1, 2. \tag{2.1}$$

Moreover,

$$\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2) \cong \operatorname{Hom}\big(\operatorname{Sym}(E^\vee) \otimes E_1,\, \operatorname{Sym}(E^\vee) \otimes E_2\big),$$

where the first Hom *denotes the set of morphisms of graded $\mathcal{O}_{\mathcal{M}}$-modules over $\mathcal{M}$, and the second* Hom *denotes the set of morphisms of graded* $\operatorname{Sym}(E^\vee)$*-modules over M.*

Proof. The graded vector bundles E_i are obtained by restricting $\mathcal{E}_i$ to M. The isomorphisms (2.1) then follow from homotopy invariance of graded vector bundles over split graded manifolds; see Proposition A.6 in [Seo25].

We can then identify a graded $\mathcal{O}_{\mathcal{M}}$-module morphism $\rho : \mathcal{E}_1 \to \mathcal{E}_2$ with a family of C^∞-linear sheaf morphisms

$$\rho^n : E_1 \longrightarrow \mathrm{Sym}^n(E^\vee) \otimes E_2, \qquad n \geq 0,$$

where all bundles are understood as the corresponding locally free sheaves on M. Since each E_i has finite total rank, we can further identify these sheaf morphisms with morphisms of graded vector bundles on M

$$\underline{\rho}^n : E_1 \longrightarrow \mathrm{Sym}^n(E^\vee) \otimes E_2, \qquad n \geq 0.$$

Consequently, ρ can be identified with a morphism of graded vector bundles on M

$$\underline{\rho} : \mathrm{Sym}(E^\vee) \otimes E_1 \longrightarrow \mathrm{Sym}(E^\vee) \otimes E_2,$$

which is $\mathrm{Sym}(E^\vee)$-linear. □

We now return to the proof of Proposition 2.6. It follows from Lemma 2.8 that the short exact sequence of graded $\mathcal{O}_{\mathcal{M}}$-modules

$$0 \longrightarrow \ker \rho \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow 0.$$

can be identified with the short exact sequence of graded $\mathrm{Sym}(E^\vee)$-modules

$$0 \longrightarrow \ker \underline{\rho} \longrightarrow \mathrm{Sym}(E^\vee) \otimes E_1 \longrightarrow \mathrm{Sym}(E^\vee) \otimes E_2 \longrightarrow 0.$$

Thus, $\ker \underline{\rho}$ is a graded vector bundle on M equipped with an induced graded $\mathrm{Sym}(E^\vee)$-module structure from $\mathrm{Sym}(E^\vee) \otimes E_1$. For each $x \in M$, the fiber $(\ker \underline{\rho})_x$ is a direct summand of the free graded $\mathrm{Sym}(E_x^\vee)$-module $\mathrm{Sym}(E_x^\vee) \otimes_{\mathbb{R}} (E_1)_x$ of finite total rank, and hence finitely generated and projective over the super polynomial ring $\mathrm{Sym}(E_x^\vee)$. By Lemma 2.7, $(\ker \underline{\rho})_x$ is a free graded $\mathrm{Sym}(E_x^\vee)$-module of finite total rank.

Let $\{e_{x,i}\}_{i=1}^p$ be a basis of $(\ker \underline{\rho})_x$ over $\mathrm{Sym}(E_x^\vee)$, where p denotes the total rank of $(\ker \underline{\rho})_x$. Since $\ker \underline{\rho}$ is a graded vector bundle over M, we may extend $\{e_{x,i}\}_{i=1}^p$ to sections $\{e_i\}_{i=1}^p$ of $\ker \underline{\rho}|_U$ for some open neighborhood $U \ni x$ satisfying

$$e_i(x) = e_{x,i}, \qquad i = 1, \dots p.$$

For convenience, set

$$R := \mathrm{Sym}(E^\vee|_U), \qquad P := \ker \underline{\rho}|_U.$$

Then $\{e_i\}_{i=1}^p$ induces a graded R-module morphism

$$\phi : \bigoplus_{i=1}^{p} R[-d_i] \longrightarrow P,$$

where $d_i = \deg(e_i)$, $i = 1, \dots, p$. After shrinking U further if necessary, choose a local frame $\{e_i\}_{i=p+1}^{p+q}$ of the graded R-module $R \otimes E_2|_U$, where q is the total rank of E_2. This induces a graded R-module isomorphism

$$P \oplus \bigoplus_{i=p+1}^{p+q} R[-d_i] \cong P \oplus (R \otimes E_2|_U) \cong R \otimes E_1|_U \cong \bigoplus_{i=1}^{p+q} R[-d_i],$$

where $d_i = \deg(e_i)$, $i = p+1, \dots, p+q$.

Now define

$$\widetilde{\phi} : \bigoplus_{i=1}^{p+q} R[-d_i] \cong \bigoplus_{i=1}^{p} R[-d_i] \oplus \bigoplus_{i=p+1}^{p+q} R[-d_i] \xrightarrow{(\phi,\mathrm{id})} P \oplus \bigoplus_{i=p+1}^{p+q} R[-d_i] \cong \bigoplus_{i=1}^{p+q} R[-d_i].$$

By construction, the fiber map $\widetilde{\phi}_x$ is invertible. Hence, after shrinking U once more if necessary, the determinant $\det(\widetilde{\phi}_y) \in R_y$ is invertible for all $y \in U$, and therefore $\widetilde{\phi}$ is an isomorphism over U. It follows that ϕ is also an isomorphism over U. This completes the proof of Proposition 2.6.

Lemma 2.9. *The tensor product $\mathcal{E}_1 \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{E}_2$ of two graded vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ is again a graded vector bundle. Moreover,*

$$\big(\mathcal{E}_1 \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{E}_2\big)(M) \cong \mathcal{E}_1(M) \otimes_{\mathcal{O}_\mathcal{M}(M)} \mathcal{E}_2(M).$$

Proof. The first statement is obvious. For the second, recall that on smooth manifolds the global sections functor on vector bundles commutes with tensor products; see Theorem 12.39 of [Nes20]. The claim then follows from the identifications in (2.1). □

Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ be a graded manifold morphism. Let $\mathcal{E}_2$ be a graded vector bundle over $\mathcal{M}_2$. We call the graded $\mathcal{O}_{\mathcal{M}_1}$-module

$$f^*\mathcal{E}_2 := \mathcal{O}_{\mathcal{M}_1} \otimes_{f^{-1}\mathcal{O}_{\mathcal{M}_2}} f^{-1}\mathcal{E}_2$$

the *pullback* of $\mathcal{E}_2$ along f, where f^{-1} denote the inverse image functor associated to f.

Lemma 2.10. *$f^*\mathcal{E}_2$ is a graded vector bundle over $\mathcal{M}_1$. Moreover,*

$$f^*\mathcal{E}_2(M_1) \cong \mathcal{O}_{\mathcal{M}_1}(M_1) \otimes_{\mathcal{O}_{\mathcal{M}_2}(M_2)} \mathcal{E}_2(M_2). \tag{2.2}$$

Proof. The first statement is obvious. The second holds in the setting of smooth manifolds; see Theorem 12.43 of [Nes20]. The result then follows from the identifications in (2.1). □

Remark 2.11. *Using* (2.2)*, one obtains a canonical identification*

$$f^*\mathcal{T}_{\mathcal{M}_2}(M_1) \cong \mathrm{Der}\left(\mathcal{O}_{\mathcal{M}_2}(M_2), \mathcal{O}_{\mathcal{M}_1}(M_1)\right), \tag{2.3}$$

where $\mathrm{Der}(\mathcal{O}_{\mathcal{M}_2}(M_2), \mathcal{O}_{\mathcal{M}_1}(M_1))$ *denotes the space of vector fields along f, i.e. $\mathbb{R}$-linear maps*

$$X : \mathcal{O}_{\mathcal{M}_2}(M_2) \to \mathcal{O}_{\mathcal{M}_1}(M_1)$$

satisfying the Leibniz rule

$$X(gh) = X(g)\, f^\sharp_{M_2}(h) + (-1)^{|X||g|} f^\sharp_{M_2}(g)\, X(h),$$

for all $g, h \in \mathcal{O}_{\mathcal{M}_2}(M_2)$. See [Seo25].

From now on, let $\mathcal{M}$ be a dg manifold.

Definition 2.12. *A* dg vector bundle *over $\mathcal{M}$ is a graded vector bundle $\mathcal{E}$ equipped with a differential $d_\mathcal{E}$ making $\mathcal{E}$ into a dg module over $\mathcal{O}_\mathcal{M}$.*

We denote a dg vector bundle by $\mathcal{E}/\mathcal{M}$, or simply by $\mathcal{E}$ when its base is clear from context.

Example 2.13. *The tangent sheaf $\mathcal{T}_\mathcal{M}$ is naturally a dg vector bundle, with differential given by the adjoint action of the cohomological vector field:*

$$d_{\mathcal{T}_\mathcal{M}} := [Q_\mathcal{M}, \cdot].$$

Given a dg manifold morphism $f\colon \mathcal{M}_1 \to \mathcal{M}_2$ and a dg vector bundle $\mathcal{E}_2$ over $\mathcal{M}_2$, the pullback graded vector bundle $f^*\mathcal{E}_2$ carries a canonical dg vector bundle structure. Its differential is given by

$$d_{f^*\mathcal{E}_2}(g \otimes e) = f^* d_{\mathcal{E}_2}(g \otimes e) = Q_{\mathcal{M}_1}(g) \otimes e + (-1)^{|g|} g \otimes f^{-1} d_{\mathcal{E}_2}(e),$$

for all $g \in \mathcal{O}_{\mathcal{M}_1}$ and $e \in f^{-1}\mathcal{E}_2$.

Definition 2.14. *A* morphism *between dg vector bundles $\mathcal{E}_1/\mathcal{M}_1 \to \mathcal{E}_2/\mathcal{M}_2$ consists of a pair $\phi = (f, \phi^\sharp)$, where $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a dg manifold morphism and*

$$\phi^\sharp : \mathcal{E}_1 \longrightarrow f^*\mathcal{E}_2$$

is a morphism of dg $\mathcal{O}_{\mathcal{M}_1}$-modules. Such ϕ is called base-fixing *if $f = \mathrm{id}$.*

Example 2.15. *Given any dg manifold morphism $f : \mathcal{M}_1 \to \mathcal{M}_2$, there exists a canonical dg vector bundle morphism $Tf : \mathcal{T}_{\mathcal{M}_1} \to \mathcal{T}_{\mathcal{M}_2}$, with $(Tf)^\sharp : \mathcal{T}_{\mathcal{M}_1} \to f^*\mathcal{T}_{\mathcal{M}_2}$ defined by*

$$(Tf)^\sharp(X)(\cdot) = X(f^\sharp(\cdot)),$$

where we use the identification (2.3)*.*

Example 2.16. *We define the following dg vector bundles over $\mathcal{M}$ associated to a dg vector bundle $\mathcal{E}/\mathcal{M}$:*

- *the* suspension*:*

  $$\mathcal{E}[1] := \bigoplus_{i\in\mathbb{Z}} \mathcal{E}_{i+1},$$

  equipped with the differential

  $$d_{\mathcal{E}[1]} := -d_{\mathcal{E}};$$

- *the* dual*:*

  $$\mathcal{E}^\vee := \mathrm{Hom}_{\mathcal{O}_\mathcal{M}}(\mathcal{E}, \mathcal{O}_\mathcal{M}),$$

  where $\mathrm{Hom}_{\mathcal{O}_\mathcal{M}}$ *denote the internal Hom in the category of graded (left) $\mathcal{O}_\mathcal{M}$-modules, equipped with the differential*

  $$d_{\mathcal{E}^\vee}(\alpha)(X) = Q_\mathcal{M}\big(\alpha(X)\big) - (-1)^{|\alpha|}\alpha\big(d_\mathcal{E}(X)\big);$$

- *the n-th* tensor power*:*

  $$\mathcal{E}^{\otimes n} := \underbrace{\mathcal{E} \otimes_{\mathcal{O}_\mathcal{M}} \cdots \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{E}}_{n \text{ times}},$$

  equipped with the differential

  $$d_{\mathcal{E}^{\otimes n}}(X_1 \otimes \cdots \otimes X_n) = \sum_{i=1}^{n} (-1)^{|X_1|+\cdots+|X_{i-1}|} X_1 \otimes \cdots \otimes d_\mathcal{E}(X_i) \otimes \cdots \otimes X_n;$$

- *the n-th* symmetric power*:*

  $$\mathrm{Sym}^n_{\mathcal{O}_\mathcal{M}}(\mathcal{E}) := \{X \in \mathcal{E}^{\otimes n} \mid \sigma(X) = X \text{ for all } \sigma \in \Sigma_n\},$$

  where Σ_n denotes the symmetric group on n letters, equipped with the differential

  $$d_{\mathrm{Sym}^n_{\mathcal{O}_\mathcal{M}}(\mathcal{E})} := d_{\mathcal{E}^{\otimes n}}\big|_{\mathrm{Sym}^n_{\mathcal{O}_\mathcal{M}}(\mathcal{E})}.$$

All of these constructions are functorial in the category of dg vector bundles over $\mathcal{M}$.

Definition 2.17. *A dg vector bundle morphism $\phi : \mathcal{E}_1/\mathcal{M}_1 \to \mathcal{E}_2/\mathcal{M}_2$ is a* weak equivalence *if f is a weak equivalence of dg manifolds and*

$$\phi^\sharp_{M_1} : \mathcal{E}_1(M_1) \to f^*\mathcal{E}_2(M_1)$$

is a quasi-isomorphism.

A dg vector bundle $\mathcal{E}/\mathcal{M}$ is called *weakly contractible* if the canonical morphism from the zero dg vector bundle over $\mathcal{M}$ to $\mathcal{E}$ is a weak equivalence; equivalently, the dg $\mathcal{O}_\mathcal{M}(M)$-module $\mathcal{E}(M)$ is acyclic.

Example 2.18. *Let $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ be a base-fixing morphism of dg vector bundles. The* mapping cone *of ϕ is the dg vector bundle*

$$C(\phi) := \left(\mathcal{E}_1[1] \oplus \mathcal{E}_2, \quad \begin{pmatrix} d_{\mathcal{E}_1[1]} & 0 \\ \phi^\sharp & d_{\mathcal{E}_2} \end{pmatrix}\right).$$

A standard computation shows that ϕ is a weak equivalence if and only if $C(\phi)$ is weakly contractible.

A dg vector bundle $\mathcal{E}/\mathcal{M}$ is called *contractible* if there exists an $\mathcal{O}_\mathcal{M}$-linear map $h : \mathcal{E} \to \mathcal{E}$ of degree -1 satisfying

$$d_\mathcal{E} h + h d_\mathcal{E} = \mathrm{id}_\mathcal{E}.$$

We refer to such h as a *(homotopy) contraction.*

Proposition 2.19. *Every weakly contractible dg vector bundle is contractible.*

To prove this proposition, we need the following useful lemma.

Lemma 2.20 (Lemma 2.3 in [Seo25])**.** *Let $\mathcal{E}/\mathcal{M}$ be a dg vector bundle. Then $\mathcal{E}(M)$ is a homotopy flat dg $\mathcal{O}_\mathcal{M}(M)$-module; that is, for any acyclic dg $\mathcal{O}_\mathcal{M}(M)$-module A, the tensor product $A \otimes_{\mathcal{O}_\mathcal{M}(M)} \mathcal{E}(M)$ is again acyclic.*

Remark 2.21. *Using the mapping cone argument, one sees that tensoring with $\mathcal{E}(M)$ preserves quasi-isomorphisms of dg $\mathcal{O}_\mathcal{M}(M)$-modules.*

Sketch of proof. The graded commutative algebra $\mathcal{O}_\mathcal{M}(M)$ is canonically a $C^\infty(M)$-algebra. Recall that there exists a graded vector bundle $E = \bigoplus_{i\in\mathbb{Z}} E_i$ on M such that

$$\mathcal{E}(M) \cong \mathcal{O}_\mathcal{M}(M) \otimes_{C^\infty(M)} \Gamma(E).$$

This allows us to define a complete descending filtration on $\mathcal{E}(M)$ by

$$F^p\mathcal{E}(M) := \bigoplus_{i\geq p} \mathcal{O}_\mathcal{M}(M) \otimes_{C^\infty(M)} \Gamma(E_i).$$

Since $\mathcal{O}_\mathcal{M}(M)$ is non-positively graded, the differential $d_\mathcal{E}$ preserves this filtration, and $\mathrm{Gr}^p d_\mathcal{E} = Q_\mathcal{M} \otimes_{C^\infty(M)} 1$ for all p. It follows that $\mathrm{Gr}^p\big(A \otimes_{\mathcal{O}_\mathcal{M}(M)} \mathcal{E}(M)\big) \cong A \otimes_{C^\infty(M)} \Gamma(E_p)$ is acyclic for all p, and hence $A \otimes_{\mathcal{O}_\mathcal{M}(M)} \mathcal{E}(M)$ is acyclic. □

We now return to the proof of Proposition 2.19. Let $\mathcal{E}/\mathcal{M}$ be a dg vector bundle. The graded vector bundle

$$\mathrm{End}(\mathcal{E}) := \mathrm{Hom}_{\mathcal{O}_\mathcal{M}}(\mathcal{E}, \mathcal{E})$$

carries a canonical dg vector bundle structure, with differential

$$d_{\mathrm{End}(\mathcal{E})}(h) = [d_\mathcal{E}, h], \qquad h \in \mathrm{End}(\mathcal{E}).$$

One can see that $\mathcal{E}$ is contractible if and only if $\mathrm{id}_{\mathcal{E}} \in \mathrm{End}(\mathcal{E})$ is exact. Now assume that $\mathcal{E}$ is weakly contractible. Note that

$$\mathrm{End}(\mathcal{E}) \cong \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{E}^{\vee}.$$

By Lemma 2.9, there is an isomorphism

$$\mathrm{End}(\mathcal{E})(M) \cong \mathcal{E}(M) \otimes_{\mathcal{O}_{\mathcal{M}}(M)} \mathcal{E}^{\vee}(M).$$

The latter complex is acyclic by Lemma 2.20. It follows that $\mathrm{id}_{\mathcal{E}}$ must be exact. This completes the proof of Proposition 2.19.

We conclude this section with three lemmas that will be used later.

Lemma 2.22. *Let $f\colon \mathcal{M}_1 \to \mathcal{M}_2$ be a weak equivalence of dg manifolds. Let $\mathcal{E}_i/\mathcal{M}_i$, $i = 1, 2$, be dg vector bundles. Under the natural isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{M}_1}}(f^*\mathcal{E}_2, \mathcal{E}_1) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{M}_2}}(\mathcal{E}_2, f_*\mathcal{E}_1),$$

a morphism $f^*\mathcal{E}_2 \to \mathcal{E}_1$ is a weak equivalence of dg vector bundles if and only if the induced map on global sections

$$\mathcal{E}_2(M_2) \longrightarrow \mathcal{E}_1(M_1)$$

is a quasi-isomorphism.

Proof. Since $f^\sharp : \mathcal{O}_{\mathcal{M}_2}(M_2) \to \mathcal{O}_{\mathcal{M}_1}(M_1)$ is a quasi-isomorphism, it follows from Lemmas 2.10 and 2.20 that the canonical map

$$\mathcal{E}_2(M_2) \cong \mathcal{E}_2(M_2) \otimes_{\mathcal{O}_{\mathcal{M}_2}(M_2)} \mathcal{O}_{\mathcal{M}_2}(M_2) \longrightarrow \mathcal{E}_2(M_2) \otimes_{\mathcal{O}_{\mathcal{M}_2}(M_2)} \mathcal{O}_{\mathcal{M}_1}(M_1) \cong f^*\mathcal{E}_2(M_1)$$

is a quasi-isomorphism. The claim then follows from the fact that $\mathcal{E}_2(M_2) \to \mathcal{E}_1(M_1)$ is obtained as the composition

$$\mathcal{E}_2(M_2) \longrightarrow f^*\mathcal{E}_2(M_1) \longrightarrow \mathcal{E}_1(M_1),$$

and the 2-out-of-3 property of quasi-isomorphisms. □

Lemma 2.23. *The operations listed in Example 2.16 preserve base-fixing weak equivalences.*

Proof. For the suspension, the claim is immediate. For the tensor power (and consequently for the symmetric power construction), the result follows directly from Lemma 2.20.

More precisely, given a base-fixing weak equivalence $\phi : \mathcal{E}_1 \to \mathcal{E}_2$, the induced morphism $\phi \otimes \phi : \mathcal{E}_1 \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{E}_1 \to \mathcal{E}_2 \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{E}_2$ can be factored as

$$\mathcal{E}_1 \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{E}_1 \xrightarrow{\mathrm{id}_{\mathcal{E}_1} \otimes \phi} \mathcal{E}_1 \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{E}_2 \xrightarrow{\phi \otimes \mathrm{id}_{\mathcal{E}_2}} \mathcal{E}_2 \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{E}_2.$$

Since each factor is a weak equivalence, their composition is a weak equivalence. By induction, the induced morphism $\mathcal{E}_1^{\otimes n} \to \mathcal{E}_2^{\otimes n}$ is a weak equivalence for all n. Consequently, the claim also holds for the symmetric power, since the symmetrization projector realizes the symmetric power as a direct summand of the tensor power, compatible with the induced morphisms.

We now turn to the dual. Since the dual commutes with the mapping cone (up to suspension), it suffices to show that it preserves weak contractibility. By Proposition 2.19, every weakly contractible dg vector bundle $\mathcal{E}$ admits a contraction $h : \mathcal{E} \to \mathcal{E}$. One readily verifies that its dual map $h^\vee : \mathcal{E}^\vee \to \mathcal{E}^\vee$ satisfies

$$d_{\mathcal{E}^\vee} h^\vee + h^\vee d_{\mathcal{E}^\vee} = \mathrm{id}_{\mathcal{E}^\vee}.$$

Therefore, $\mathcal{E}^\vee$ is contractible, hence weakly contractible. □

Lemma 2.24 (Section 3 in [Seo25]). *If $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a weak equivalence of dg manifolds, then $Tf : \mathcal{T}_{\mathcal{M}_1} \to \mathcal{T}_{\mathcal{M}_2}$ is a weak equivalence of dg vector bundles.*

3. L_∞ algebroids over dg manifolds

In this section, we review the notion of L_∞ algebroids over dg manifolds from the Lie algebraic picture. We then introduce a commutative algebraic picture and prove its equivalence to the Lie algebraic one.

3.1. **Lie algebraic perspective.** Let $\mathcal{M}$ be a graded manifold and $\mathcal{L}$ a graded vector bundle over $\mathcal{M}$. Set

$$\Omega^\bullet_{\mathcal{L}} := \mathrm{Sym}^\bullet_{\mathcal{O}_\mathcal{M}}\big(\mathcal{L}[1]^\vee\big).$$

This graded commutative algebra carries a canonical descending filtration

$$F^p\Omega_{\mathcal{L}} := \bigoplus_{n\geq p} \Omega^n_{\mathcal{L}}, \qquad p \in \mathbb{Z}.$$

Its completion with respect to this filtration is

$$\widehat{\Omega}_{\mathcal{L}} := \widehat{\mathrm{Sym}}_{\mathcal{O}_\mathcal{M}}\big(\mathcal{L}[1]^\vee\big),$$

where $\widehat{\mathrm{Sym}}_{\mathcal{O}_\mathcal{M}}$ denotes the completed symmetric power. When $\mathcal{L} = \mathcal{T}_\mathcal{M}$, we write $\widehat{\Omega}_{\mathcal{T}_\mathcal{M}}$ simply as $\widehat{\Omega}_\mathcal{M}$. In this case, $\widehat{\Omega}_\mathcal{M}$ is the sheaf of completed differential forms on $\mathcal{M}$.

Definition 3.1. *Let $\mathcal{M}$ be a dg manifold. An L_∞* algebroid over $\mathcal{M}$ *is a graded vector bundle $\mathcal{L}$ over $\mathcal{M}$, together with a filtration preserving differential $D_\mathcal{L}$ on $\widehat{\Omega}_\mathcal{L}$, whose associated graded operator satisfies*

$$\mathrm{Gr}D_\mathcal{L}|_{\mathcal{O}_\mathcal{M}} = Q_\mathcal{M}.$$

The filtered cdga $\mathrm{CE}(\mathcal{L}) := (\widehat{\Omega}_\mathcal{L}, D_\mathcal{L})$ *is called the* Chevalley–Eilenberg algebra *of $\mathcal{L}$.*

By a slight abuse of notation, we denote an L_∞ algebroid by $\mathcal{L}/\mathcal{M}$, or simply by $\mathcal{L}$ when the base $\mathcal{M}$ is clear from context.

The differential $D_\mathcal{L}$ determines

- a sequence $\mathcal{O}_\mathcal{M}$-linear maps of degree 0
$$\rho_n : \mathrm{Sym}^n_{\mathcal{O}_\mathcal{M}}(\mathcal{L}[1]) \longrightarrow \mathcal{T}_\mathcal{M}[1], \qquad n \geq 0,$$
called the *multi-anchors* of $\mathcal{L}$, with $\rho_0 = Q_\mathcal{M}$;
- a sequence $\mathbb{R}$-linear maps of degree 1
$$l_n : \mathrm{Sym}^n_{\mathcal{O}_\mathcal{M}}(\mathcal{L}[1]) \longrightarrow \mathcal{L}[1], \qquad n \geq 1,$$
called the *multi-brackets* of $\mathcal{L}$.

More precisely, these maps are determined by the action of $D_\mathcal{L}$ on the generators of $\widehat{\Omega}_\mathcal{L}$:

$$\Omega^0_\mathcal{L} \longrightarrow \widehat{\Omega}_\mathcal{L}, \quad f \mapsto D_\mathcal{L} f = \sum_{n\geq 0} \rho_n^\vee(df),$$

$$\Omega^1_\mathcal{L} \longrightarrow \widehat{\Omega}_\mathcal{L}, \quad \alpha \mapsto D_\mathcal{L}\alpha = \sum_{n\geq 1} l_n^\vee(\alpha),$$

where $\rho_n^\vee : \Omega^1_{\mathcal{M}} \to \Omega^n_{\mathcal{L}}$ is specified by

$$\rho_n^\vee(\alpha)(X_1,\dots,X_n) + (-1)^{|\alpha|}\alpha\left(\rho_n(X_1,\dots,X_n)\right) = 0,$$

and $l_n^\vee : \Omega^1_{\mathcal{L}} \to \Omega^n_{\mathcal{L}}$ is specified by

$$\begin{aligned} &l_n^\vee(\alpha)(X_1,\dots,X_n) + (-1)^{|\alpha|}\alpha\left(l_n(X_1,\dots,X_n)\right) \\ &= \sum_{i=1}^{n}(-1)^{|X_i|(|X_0|+\cdots+|X_{i-1}|)} d(\alpha(X_i))\left(\rho_{n-1}(X_1,\dots,\widehat{X_i},\dots,X_n)\right). \end{aligned}$$

They are characterized by the following conditions:

- The failure of $\mathcal{O}_{\mathcal{M}}$-linearity of the n-th multi-bracket l_n is measured by the $(n-1)$-th multi-anchor ρ_{n-1}:

$$l_n(X_1,\dots,fX_n) = \rho_{n-1}(X_1,\dots,X_{n-1})(f)X_n + (-1)^{|f|(\sum_{i=1}^{n-1}|X_i|+1)} f l_n(X_1,\dots X_n).$$

  For $n=1$, one has

$$l_1(fX) = Q_{\mathcal{M}}(f)X + (-1)^{|f|} f l_1(X).$$

- The multi-brackets $\{l_n\}_{n\geq 1}$ satisfy the *strong homotopy Jacobi identities*:

$$\sum_{p+q=n}\sum_{\sigma\in\Sigma_n}\frac{\epsilon(\sigma)}{p!q!} l_{q+1}(l_p(X_{\sigma(1)},\dots),\dots,X_{\sigma(n)}) = 0.$$

  For $n=1$, ones

$$l_1 \circ l_1 = 0.$$

  Thus, equipped with l_1, $\mathcal{L}$ becomes a dg vector bundle over $\mathcal{M}$.
- The multi-anchors $\{\rho_n\}_{n\geq 0}$ intertwine the multi-brackets l_n on $\mathcal{L}[1]$ with the Lie bracket $[\cdot,\cdot]$ on $\mathcal{T}_{\mathcal{M}}[1]$:

$$\begin{aligned} &\sum_{p+q=n}\sum_{\sigma\in\Sigma_n}\frac{\epsilon(\sigma)}{p!q!}\rho_{q+1}\big(l_p(X_{\sigma(1)},\dots),\dots,X_{\sigma(n)}\big) \\ &= \frac{1}{2}\sum_{p+q=n}\sum_{\sigma\in\Sigma_n}\frac{\epsilon(\sigma)}{p!q!}\big[\rho_p(X_{\sigma(1)},\dots),\rho_q(\dots,X_{\sigma(n)})\big], \end{aligned}$$

  where $\epsilon(\sigma)$ is the Koszul sign of the permutation σ. For $n=0$, this reduces to $[Q_{\mathcal{M}},Q_{\mathcal{M}}]=0$. For $n=1$, one has

$$\rho_1(l_1(X)) = [Q_{\mathcal{M}},\rho_1(X)].$$

  In other words, $\rho_1 : \mathcal{L} \to \mathcal{T}_{\mathcal{M}}$ is a dg vector bundle morphism.

In fact, the first condition follows from the fact that $D_{\mathcal{L}}$ is a derivation, while the second and third follow from the vanishing of $D^2_{\mathcal{L}}$ on $\Omega^1_{\mathcal{L}}$ and $\Omega^0_{\mathcal{L}}$, respectively.

Example 3.2. *A finite dimensional L_∞ algebra can be seen as an L_∞ algebroid over a point.*

Example 3.3. *Let $\mathcal{M}$ be a dg manifold. There are two canonical L_∞ algebroids over $\mathcal{M}$:*

- *The zero vector bundle $0_{\mathcal{M}}$, equipped with zero multi-brackets and zero multi-anchors, defines an L_∞ algebroid, called the* zero algebroid *on $\mathcal{M}$. Its Chevalley–Eilenberg algebra is given by $(\mathcal{O}_{\mathcal{M}}, Q_{\mathcal{M}})$.*

- *Recall that the de Rham differential d_{dR} is the unique differential on $\widehat{\Omega}_{\mathcal{M}}$ extending*
$$\Omega^0_{\mathcal{M}} \longrightarrow \Omega^1_{\mathcal{M}}, \qquad f \mapsto df,$$
where $df(X) = (-1)^{|X||f|}X(f)$ for all vector fields X. The Lie derivative L_X is the unique derivation on $\widehat{\Omega}_{\mathcal{M}}$ extending
$$\Omega^0_{\mathcal{M}} \longrightarrow \Omega^0_{\mathcal{M}}, \qquad f \mapsto X(f).$$
Now consider the total differential
$$D_{\mathcal{M}} := d_{dR} + L_{Q_{\mathcal{M}}}$$
on $\widehat{\Omega}_{\mathcal{M}}$. Obviously, $D_{\mathcal{M}}$ preserves the filtration on $\widehat{\Omega}_{\mathcal{M}}$. Moreover,
$$\mathrm{Gr}D_{\mathcal{L}}|_{\mathcal{O}_{\mathcal{M}}} = Q_{\mathcal{M}}.$$
Thus, $\mathcal{T}_{\mathcal{M}}$ is an L_∞ algebroid, called the tangent algebroid *of $\mathcal{M}$. In terms of multi-anchors and multi-brackets, we have*
$$\rho_0 = Q_{\mathcal{M}}, \quad \rho_1 = \mathrm{id}_{\mathcal{T}_{\mathcal{M}}}, \quad l_1 = [Q_{\mathcal{M}}, \cdot], \quad l_2 = [\cdot,\cdot],$$
and $\rho_n = l_{n+1} = 0$ for all $n \geq 2$.

Definition 3.4. *An L_∞ algebroid is called* strict *if $\rho_n = 0$ for all $n \geq 2$. A strict L_∞ algebroid is called an L_∞* algebra bundle *if, in addition, $\rho_1 = 0$.*

The compatibility conditions between the multi-brackets $\{l_n\}_{n=1}^\infty$ and the unary anchor ρ_1 of a strict L_∞ algebroid $\mathcal{L}/\mathcal{M}$ simplify considerably due to the vanishing of the higher multi-anchors. One has
$$\rho_1(l_1(X)) = [Q_{\mathcal{M}}, \rho_1(X)], \quad \rho_1\big(l_2(X_1,X_2)\big) = \big[\rho_1(X_1), \rho_1(X_2)\big],$$
and for all $n \geq 3$,
$$\rho_1\big(l_n(X_1,\ldots,X_n)\big) = 0.$$

Definition 3.5. *A* (base-preserving) morphism *between two L_∞ algebroids $\mathcal{L}_1/\mathcal{M}_1$ and $\mathcal{L}_2/\mathcal{M}_2$ consists of a pair $\Phi = (f, \Phi^\sharp)$ where*
$$f = (f, f^\sharp) : \mathcal{M}_1 \to \mathcal{M}_2;$$
is a dg manifold morphism, and
$$\Phi^\sharp : \mathrm{CE}(\mathcal{L}_2) \longrightarrow f_*\mathrm{CE}(\mathcal{L}_1)$$
is a filtered cdga morphism satisfying
$$\Phi^\sharp|_{\mathcal{O}_{\mathcal{M}_2}} = f^\sharp.$$
Such Φ is called base-fixing *if $f = \mathrm{id}$.*

The filtered cdga morphism $\Phi^\sharp$ is uniquely determined by a sequence $\mathcal{O}_{\mathcal{M}_1}$-linear maps of degree 0
$$\phi^\sharp_n : \mathrm{Sym}^n_{\mathcal{O}_{\mathcal{M}_1}}(\mathcal{L}_1[1]) \longrightarrow f^*\mathcal{L}_2[1], \quad n \geq 1.$$
Here, we use the canonical identifications
$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_{\mathcal{M}_2}}\big(\mathcal{L}_2[1]^\vee, f_*\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{M}_1}}(\mathcal{L}_1[1]^\vee)\big) &\cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{M}_1}}\big(f^*\mathcal{L}_2[1]^\vee, \mathrm{Sym}^n_{\mathcal{O}_{\mathcal{M}_1}}(\mathcal{L}_1[1]^\vee)\big)\\ &\cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{M}_1}}\big(\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{M}_1}}(\mathcal{L}_1[1]), f^*\mathcal{L}_2[1]\big).\end{aligned}$$

The second isomorphism follows from the fact that pullback commutes with duals for graded vector bundles, together with the canonical identification of a graded vector bundle with its double dual. The latter can be proved using Proposition 2.6.

One can easily check that the condition

$$\Phi^\sharp \circ D_{\mathcal{L}_2}|_{\Omega^0_{\mathcal{L}_2}} = f_* D_{\mathcal{L}_1} \circ \Phi^\sharp|_{\Omega^0_{\mathcal{L}_2}}$$

is equivalent to the compatibility of $\{\phi_n^\sharp\}_{n=1}^\infty$ with the multi-anchors on $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$Tf^\sharp(\rho_n(X_1,\dots,X_n)) = \sum_{k_1+\cdots+k_j=n} \sum_{\sigma\in\Sigma_n} \frac{\epsilon(\sigma)}{j!k_1!\cdots k_j!} f^*\rho_j\big(\phi^\sharp_{k_1}(X_{\sigma(1)},\dots),\dots,\phi^\sharp_{k_j}(\dots,X_{\sigma(n)})\big).$$

For $n=0$, this reduces to $Tf^\sharp(Q_{\mathcal{M}_1}) = f^*Q_{\mathcal{M}_2}$. For $n=1$, one has

$$Tf^\sharp(\rho_1(X)) = f^*\rho_1(\phi_1^\sharp(X)).$$

The condition

$$\Phi^\sharp \circ D_{\mathcal{L}_2}|_{\Omega^1_{\mathcal{L}_2}} = f_* D_{\mathcal{L}_1} \circ \Phi^\sharp|_{\Omega^1_{\mathcal{L}_2}}$$

is more delicate to interpret. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are both L_∞ algebra bundles, or if $f=\mathrm{id}$, then it is equivalent to the compatibility of $\{\phi_n^\sharp\}_{n=1}^\infty$ with the multi-brackets on $\mathcal{L}_1$ and $\mathcal{L}_2$, namely

$$\begin{aligned}&\sum_{p+q=n}\sum_{\sigma\in\Sigma_n}\frac{\epsilon(\sigma)}{p!\,q!}\,\phi^\sharp_{q+1}\big(l_p(X_{\sigma(1)},\dots),\dots,X_{\sigma(n)}\big)\\ &=\sum_{k_1+\cdots+k_j=n}\sum_{\sigma\in\Sigma_n}\frac{\epsilon(\sigma)}{j!\,k_1!\cdots k_j!}\,f^*l_j\big(\phi^\sharp_{k_1}(X_{\sigma(1)},\dots),\dots,\phi^\sharp_{k_j}(\dots,X_{\sigma(n)})\big).\end{aligned}$$

For $n=1$, one has

$$\phi_1^\sharp\big(l_1(X)\big) = f^*l_1\big(\phi_1^\sharp(X)\big).$$

In general, however, for $f\neq\mathrm{id}$ and $n>1$, the pullback multi-brackets f^*l_n are not well defined due to the presence of multi-anchors.

Definition 3.6. *An L_∞ algebroid morphism $\Phi:\mathcal{L}_1/\mathcal{M}_1\to\mathcal{L}_2/\mathcal{M}_2$ is a* weak equivalence *if the linear term $\phi_1=(f,\phi_1^\sharp)$ is a weak equivalence of the underlying dg vector bundles.*

Example 3.7. *Let $f:\mathcal{M}_1\to\mathcal{M}_2$ be a morphism of dg manifolds. There is a canonical $\mathcal{O}_{\mathcal{M}_2}$-linear map*

$$f^\sharp_{dR}:\widehat{\Omega}_{\mathcal{M}_2}\longrightarrow f_*\widehat{\Omega}_{\mathcal{M}_1},$$

defined by

$$f^\sharp_{dR}(g)=f^\sharp(g),\qquad f^\sharp_{dR}(dg)=f_*d(f^\sharp(g))$$

for $g\in\mathcal{O}_{\mathcal{M}_2}$. One can easily check that $f^\sharp_{dR}$ defines a filtered cdga morphism from $(\widehat{\Omega}_{\mathcal{M}_2},D_{\mathcal{M}_2})$ to $f_*(\widehat{\Omega}_{\mathcal{M}_1},D_{\mathcal{M}_1})$. It follows that $f_{dR}:=(f,f^\sharp_{dR})$ defines an L_∞ algebroid morphism between the tangent algebroids.

By Lemma 2.24, f_{dR} is a weak equivalence of tangent algebroids whenever f is a weak equivalence of dg manifolds.

Definition 3.8. *An L_∞ algebroid morphism $\Phi:\mathcal{L}_1/\mathcal{M}_1\to\mathcal{L}_2/\mathcal{M}_2$ is called* strict *if $\phi_n^\sharp=0$ for all $n\geq 2$.*

The multi-anchors $\{\rho_n\}_{n\geq 1}$ of an L_∞ algebroid $\mathcal{L}/\mathcal{M}$ defines a base-fixing L_∞ algebroid morphism from $\mathcal{L}$ to the tangent algebroid $\mathcal{T}_\mathcal{M}$, which is strict if and only if $\mathcal{L}$ is strict. In other words, there is a canonical morphism

$$\mathrm{CE}(\mathcal{T}_\mathcal{M}) = (\widehat{\Omega}_\mathcal{M}, D_\mathcal{M}) \longrightarrow \mathrm{CE}(\mathcal{L}) = (\widehat{\Omega}_\mathcal{L}, D_\mathcal{L})$$

between filtered cdgas.

Definition 3.9. *An L_∞ algebroid $\mathcal{L}/\mathcal{M}$ is called* transitive (or fibrant) *if its unary anchor*

$$\rho_1 : \mathcal{L} \longrightarrow \mathcal{T}_\mathcal{M}$$

is surjective.

Lemma 3.10. *Every transitive L_∞ algebroid is strict up to an isomorphism.*

The proof follows by adapting a standard argument from the L_∞ algebra literature.

Proof. Let $\mathcal{L}/\mathcal{M}$ be a transitive L_∞ algebroid. We need to construct an $\mathcal{O}_\mathcal{M}$-linear automorphism $\Phi^\sharp$ of the filtered graded commutative algebra $\widehat{\Omega}_\mathcal{L}$ such that

$$(\Phi^\sharp \circ D_\mathcal{L} \circ (\Phi^\sharp)^{-1})(f) \in \Omega^0_\mathcal{L} \oplus \Omega^1_\mathcal{L}$$

for all $f \in \mathcal{O}_\mathcal{M}$.

For $n \geq m \geq 1$, let ϕ^m_n denote the composition

$$\Omega^m_\mathcal{L} \xrightarrow{\Phi^\sharp|_{\Omega^m_\mathcal{L}}} \widehat{\Omega}_\mathcal{L} \longrightarrow \Omega^n_\mathcal{L}.$$

Recall that $D_\mathcal{L}(f) = Q_\mathcal{M}(f) + \sum_{m\geq 1} \rho^\vee_m(df)$. Then

$$\Phi^\sharp \circ D_\mathcal{L}(f) = Q_\mathcal{M}(f) + \sum_{m\geq 1} \sum_{n\geq m} \phi^m_n\big(\rho^\vee_m(df)\big).$$

Note that each ϕ^m_n is uniquely determined by $\phi^1_1, \ldots, \phi^1_{n+1-m}$. We are therefore led to solve the following equations for ϕ^1_k's

$$\sum_{m=1}^{n} \phi^m_n(\phi^1_1, \ldots, \phi^1_{n+1-m}) \circ \rho^\vee_m = 0, \qquad n \geq 2.$$

We impose the initial condition $\phi^1_1 = \mathrm{Id}_{\Omega^1_\mathcal{L}}$, which implies $\phi^n_n = \mathrm{Id}_{\Omega^n_\mathcal{L}}$ for all $n \geq 2$. The equations then become

$$\phi^1_n \circ \rho^\vee_1 + \sum_{1<m<n} \phi^m_n \circ \rho^\vee_m = -\rho^\vee_n, \qquad n \geq 2.$$

In particular, for $n = 2$, this reduces to

$$\phi^1_2 \circ \rho^\vee_1 = -\rho^\vee_2.$$

Since $\mathcal{L}$ is transitive, the injective map $\rho^\vee_1 : \Omega^1_\mathcal{M} \to \Omega^1_\mathcal{L}$ admits a splitting $h_1 : \Omega^1_\mathcal{L} \to \Omega^1_\mathcal{M}$. Thus

$$\phi^1_2 = -\rho^\vee_2 \circ h_1$$

solves the equation. Proceeding inductively, one constructs ϕ^1_n for all n, and hence $\Phi^\sharp$. $\square$

Let $\mathcal{L}$ be a strict transitive L_∞ algebroid over $\mathcal{M}$, and let

$$\mathcal{L}_{red} := \ker \rho_1.$$

By Proposition 2.6, $\mathcal{L}_{red}$ is a graded vector bundle over $\mathcal{M}$.

Lemma 3.11. *For all $n \geq 1$, one has*

$$\rho_1(l_n(X_1, \dots, X_n)) = 0$$

whenever at least one of the elements $X_i \in \mathcal{L}$ belongs to $\mathcal{L}_{red}$.

Proof. Recall that for all $n \geq 3$, one has $\rho_1 \circ l_n = 0$. While for $n = 1, 2$, we have

$$\rho_1(l_1(X)) = [Q_{\mathcal{M}}, \rho_1(X)], \quad \rho_1\big(l_2(X_1, X_2)\big) = \big[\rho_1(X_1), \rho_1(X_2)\big].$$

These expressions vanish whenever X or at least one of X_1, X_2 belongs to $\mathcal{L}_{red}$. □

In particular, equipped with the induced multi-brackets from $\mathcal{L}$, $\mathcal{L}_{red}$ is an L_∞ algebra bundle over $\mathcal{M}$, called the *isotropy L_∞ algebra bundle*.

3.2. **Commutative algebraic perspective.** Let V be a non-positively graded vector space of finite total rank. Let $(V_0, \mathcal{O}_V)$ denote the associated ringed space as defined in the previous section. Let $\mathbb{V}$ be a graded $\mathcal{O}_V$-module free of finite total rank. We denote by $(V_0, \mathcal{O}_{\mathbb{V}})$ the ringed space whose structure sheaf is given by

$$\mathcal{O}_{\mathbb{V}} := \widehat{\mathrm{Sym}}_{\mathcal{O}_V}(\mathbb{V}^\vee),$$

where $\widehat{\mathrm{Sym}}_{\mathcal{O}_V}(\mathbb{V}^\vee)$ is the completion of $\mathrm{Sym}_{\mathcal{O}_V}(\mathbb{V}^\vee)$ with respect to its canonical descending filtration by symmetric degree.

Definition 3.12. *A* (projected) filtered graded manifold *is a ringed space $\mathcal{C} = (C, \mathcal{O}_{\mathcal{C}})$, where C is a smooth manifold and $\mathcal{O}_{\mathcal{C}}$ is a sheaf of graded commutative algebras on C, together with an ideal $\mathcal{I}_{\mathcal{C}}$ and a splitting of the short exact sequence*

$$0 \longrightarrow \mathcal{I}_{\mathcal{C}} \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow \mathcal{O}_{\mathcal{C}}/\mathcal{I}_{\mathcal{C}} \longrightarrow 0$$

of sheaves of graded commutative algebras.

These data are required to satisfy the following condition: for every $x \in C$, there exists an open neighborhood $U \ni x$ and a commutative diagram of ringed spaces

$$\begin{array}{ccc} (U, \mathcal{O}_{\mathcal{C}}|_U) & \longrightarrow & (V_0, \mathcal{O}_{\mathbb{V}}) \\ \downarrow & & \downarrow \\ (U, (\mathcal{O}_{\mathcal{C}}/\mathcal{I}_{\mathcal{C}})|_U) & \longrightarrow & (V_0, \mathcal{O}_V) \end{array}$$

where the horizontal arrows are isomorphisms.

By definition, $\mathcal{O}_{\mathcal{C}}$ is complete with respect to the $\mathcal{I}_{\mathcal{C}}$-adic filtration

$$F^p\mathcal{O}_{\mathcal{C}} = \mathcal{I}_{\mathcal{C}}^p, \qquad p \in \mathbb{N}.$$

Moreover, the ringed space

$$\mathrm{Gr}^0\mathcal{C} := (C, \mathrm{Gr}^0\mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}/\mathcal{I}_{\mathcal{C}})$$

is a graded manifold, called the *base* of $\mathcal{C}$.

Let $\mathcal{C}$ be a filtered graded manifold. For each open $U \subset M$, define

$$\mathcal{T}_{\mathcal{C}}(U) := \mathrm{Der}(\mathcal{O}_{\mathcal{C}}(U)).$$

The assignment $\mathcal{T}_{\mathcal{C}}$ defines a sheaf, called the *tangent sheaf* of $\mathcal{C}$. Global sections of $\mathcal{T}_{\mathcal{C}}$ are called *vector fields* on $\mathcal{C}$. Note that $\mathcal{T}_{\mathcal{C}}$ carries a natural filtration given by

$$F^p\mathcal{T}_{\mathcal{C}} = \{X \in \mathcal{T}_{\mathcal{C}} : X(F^\bullet\mathcal{O}_{\mathcal{C}}) \subset F^{\bullet+p}\mathcal{O}_{\mathcal{C}}\}, \qquad p \in \mathbb{Z}.$$

Definition 3.13. *A* quasi-dg manifold *is a filtered graded manifold $\mathcal{C}$ equipped with a degree 1 vector field $Q_{\mathcal{C}}$, which satisfies*

- $[Q_{\mathcal{C}}, Q_{\mathcal{C}}] = 0$*;*
- $Q_{\mathcal{C}}(\mathcal{I}_{\mathcal{C}}) \subset \mathcal{I}_{\mathcal{C}}$*, or equivalently,* $Q_{\mathcal{C}} \in F^0\mathcal{T}_{\mathcal{C}}(M)$*.*

Equipped with $Q_{\mathcal{C}}$ and the $\mathcal{I}_{\mathcal{C}}$-adic filtration, $\mathcal{O}_{\mathcal{C}}$ becomes a sheaf of filtered cdgas. It follows that $\mathrm{Gr}^0\mathcal{C}$, endowed with $\mathrm{Gr}^0Q_{\mathcal{C}}$, is a dg manifold. Note that the morphism of graded commutative algebras $\mathrm{Gr}^0\mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}$ is not, in general, a morphism of cdgas.

Example 3.14. *Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a dg manifold.*

- *$\mathcal{M}$ equipped with the trivial dg ideal defines a quasi-dg manifold over itself.*
- *The ringed space $\mathcal{M}_{dR} := (M, \widehat{\Omega}_{\mathcal{M}})$, equipped with the differential $D_{\mathcal{M}} = d_{dR} + L_{Q_{\mathcal{M}}}$ and the dg ideal $\widehat{\Omega}^{>0}_{\mathcal{M}}$ of completed differential forms whose projection to functions vanishes, defines a quasi-dg manifold over $\mathcal{M}$, called the* de Rham space *of $\mathcal{M}$.*

Definition 3.15. *A* morphism *between quasi-dg manifolds $\mathcal{C}_1$ and $\mathcal{C}_2$ consists of a pair $F = (f, F^\sharp)$, where $f : C_1 \to C_2$ is a smooth map and $F^\sharp : \mathcal{O}_{\mathcal{C}_2} \longrightarrow f_*\mathcal{O}_{\mathcal{C}_1}$ is a morphism of sheaves of filtered cdgas.*

Such F is called base-preserving *if the following diagram of sheaves of graded commutative algebras*

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{C}_2} & \xrightarrow{F^\sharp} & f_*\mathcal{O}_{\mathcal{C}_1} \\ \uparrow & & \uparrow \\ \mathrm{Gr}^0\mathcal{O}_{\mathcal{C}_2} & \xrightarrow{\mathrm{Gr}^0F^\sharp} & f_*\mathrm{Gr}^0\mathcal{O}_{\mathcal{C}_1} \end{array}$$

commutes. A base-preserving F is called base-fixing *if* $\mathrm{Gr}^0F^\sharp = \mathrm{id}$.

We denote by $\mathrm{Gr}^0F = (f, \mathrm{Gr}^0F^\sharp)$ the dg manifold morphism between $\mathrm{Gr}^0\mathcal{C}_1$ and $\mathrm{Gr}^0\mathcal{C}_2$.

Definition 3.16. *A quasi-dg manifold morphism $F : \mathcal{C}_1 \to \mathcal{C}_2$ is called a* weak equivalence *if $F^\sharp_{C_2} : \mathcal{O}_{\mathcal{C}_2}(C_2) \to \mathcal{O}_{\mathcal{C}_1}(C_1)$ is a filtered quasi-isomorphism; that is, if $\mathrm{Gr}\, F^\sharp_{C_2} : \mathrm{Gr}\,\mathcal{O}_{\mathcal{C}_2}(C_2) \to \mathrm{Gr}\,\mathcal{O}_{\mathcal{C}_1}(C_1)$ is a quasi-isomorphism.*

Lemma 3.17. *There exists a unique base-fixing morphism between quasi-dg manifolds*

$$H : \mathcal{C} \longrightarrow (\mathrm{Gr}^0\mathcal{C})_{dR}.$$

Moreover, for any base-preserving morphism $F : \mathcal{C}_1 \to \mathcal{C}_2$ between quasi-dg manifolds, the following diagram of quasi-dg manifolds

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} \\ (\mathrm{Gr}^0\mathcal{C}_1)_{dR} & \xrightarrow{dR(\mathrm{Gr}^0F)} & (\mathrm{Gr}^0\mathcal{C}_2)_{dR} \end{array}$$

commutes.

Proof. As a filtered graded commutative algebra, $\widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}}$ is generated by functions and exact 1-forms. We define $H^\sharp : \widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}} \to \mathcal{O}_\mathcal{C}$ by specifying its action on the generators:

$$H^\sharp(f) = f, \qquad H^\sharp(df) = (Q_\mathcal{C} - \mathrm{Gr}^0 Q_\mathcal{C})f,$$

for all $f \in \mathrm{Gr}^0\mathcal{O}_\mathcal{C}$. It is straightforward to verify that H satisfies the desired properties. □

Definition 3.18. *A quasi-dg manifold $\mathcal{C}$ is called* fibrant *if*

$$\mathrm{Gr}\, H^\sharp : \Omega_{\mathrm{Gr}^0\mathcal{C}} \to \mathrm{Gr}\,\mathcal{O}_\mathcal{C}$$

is injective.

Let $\mathbf{L}_\infty\mathbf{Algd}$ denote the category of L_∞ algebroids over dg manifolds. Let $\mathbf{QdgM}$ denote the category of quasi-dg manifolds, and $\mathbf{QdgM}^{\mathrm{bp}} \subset \mathbf{QdgM}$ denote the subcategory with the same objects and base-preserving morphisms.

Consider the assignment

$$\mathbf{CE}_{\mathbf{Algd}} : \mathbf{L}_\infty\mathbf{Algd} \longrightarrow \mathbf{QdgM}^{\mathrm{bp}}$$

defined on objects $\mathcal{L}/\mathcal{M}$ by

$$\mathbf{CE}_{\mathbf{Algd}}(\mathcal{L}/\mathcal{M}) = (M, \mathrm{CE}(\mathcal{L})),$$

and on morphisms $\Phi = ((f, f^\sharp), \Phi^\sharp) : \mathcal{L}_1/\mathcal{M}_1 \to \mathcal{L}_2/\mathcal{M}_2$ by

$$\mathbf{CE}_{\mathbf{Algd}}(\Phi) = (f, \Phi^\sharp) : (M_1, \mathrm{CE}(\mathcal{L}_1)) \longrightarrow (M_2, \mathrm{CE}(\mathcal{L}_2)).$$

$\mathbf{CE}_{\mathbf{Algd}}$ is obviously a well-defined functor, called the *Chevalley–Eilenberg functor*.

Theorem 3.19. *The Chevalley–Eilenberg functor $\mathbf{CE}_{\mathbf{Algd}}$ yields an equivalence of categories. Moreover, it detects weak equivalences and fibrant objects.*

Proof. By definition, $\mathbf{CE}_{\mathbf{Algd}}$ is fully faithful. We need to show that it is essentially surjective. Let $\mathcal{C}$ be a quasi-dg manifold. Since we are in the smooth setting, the short exact sequence of locally free graded $\mathrm{Gr}^0\mathcal{O}_\mathcal{C}$-modules

$$0 \longrightarrow F^2\mathcal{O}_\mathcal{C} \longrightarrow F^1\mathcal{O}_\mathcal{C} \longrightarrow \mathrm{Gr}^1\mathcal{O}_\mathcal{C} \longrightarrow 0.$$

admits a splitting $\mathrm{Gr}^1\mathcal{O}_\mathcal{C} \to F^1\mathcal{O}_\mathcal{C}$. Let $\mathcal{L} = (\mathrm{Gr}^1\mathcal{O}_\mathcal{C}[1])^\vee$. By the universal property of the symmetric algebra, the splitting induces a graded commutative $\mathrm{Gr}^0\mathcal{O}_\mathcal{C}$-algebra morphism

$$\Omega_\mathcal{L} = \mathrm{Sym}_{\mathrm{Gr}^0\mathcal{O}_\mathcal{C}}(\mathcal{L}[1]^\vee) \longrightarrow \mathcal{O}_\mathcal{C},$$

which extends uniquely to a morphism of filtered graded commutative algebras

$$\widehat{\Omega}_\mathcal{L} \longrightarrow \mathcal{O}_\mathcal{C}.$$

This morphism is locally an isomorphism and therefore an isomorphism globally. We can use it to transfer the differential $Q_\mathcal{C}$ on $\mathcal{O}_\mathcal{C}$ to a differential $D_\mathcal{L}$ on $\widehat{\Omega}_\mathcal{L}$. $\mathcal{L}$ is then an L_∞ algebroid over $\mathrm{Gr}^0\mathcal{C}$.

Let $\Phi = (f, \{\phi_n^\sharp\}_{n=1}^\infty) : \mathcal{L}_1/\mathcal{M}_1 \to \mathcal{L}_2/\mathcal{M}_2$ be a morphism of L_∞ algebroids. The functor detects weak equivalences: by Lemmas 2.22 and 2.23, the map

$$\mathrm{Gr}\,\Phi^\sharp_{M_2} : \mathrm{Sym}_{\mathcal{O}_{\mathcal{M}_2}(M_2)}\big(\mathcal{L}_2[1]^\vee(M_2)\big) \longrightarrow \mathrm{Sym}_{\mathcal{O}_{\mathcal{M}_1}(M_1)}\big(\mathcal{L}_1[1]^\vee(M_1)\big)$$

is a quasi-isomorphism if and only if both

$$f^\sharp_{M_2} : \mathcal{O}_{\mathcal{M}_2}(M_2) \to \mathcal{O}_{\mathcal{M}_1}(M_1) \quad \text{and} \quad \phi^\sharp_{M_1} : \mathcal{L}_1[1](M_1) \to f^*\mathcal{L}_2[1](M_1)$$

are quasi-isomorphisms.

The functor detects fibrant objects: $\mathrm{Gr}\, H^\sharp : \Omega_{\mathrm{Gr}^0\mathcal{C}} \to \mathrm{Gr}\, \mathcal{O}_\mathcal{C}$ is injective if and only if

$$\mathrm{Gr}^1 H^\sharp : \Omega^1_{\mathrm{Gr}^0\mathcal{C}} \cong (\mathcal{T}_{\mathrm{Gr}^0\mathcal{C}}[1])^\vee \to \mathrm{Gr}^1\mathcal{O}_\mathcal{C} \cong \mathcal{L}[1]^\vee$$

is injective, which holds precisely when the dual map $\mathcal{L} \to \mathcal{T}_{\mathrm{Gr}^0\mathcal{C}}$ is surjective. □

4. L_∞ spaces over dg manifolds

In this section, we adapt Costello's notion of L_∞ spaces to the dg manifold setting, and prove an equivalence of the categories of transitive L_∞ algebroids and L_∞ spaces over dg manifolds.

4.1. **Curved L_∞ algebras.** Let (R, D_R) be a cdga and $I \subset R$ a dg ideal. Assume that R is complete with respect to the I-adic filtration $F^pR = I^p$. Let $\mathfrak{g}$ be a graded R-module, endowed with the induced filtration

$$F^p\mathfrak{g} := I^p \cdot \mathfrak{g}.$$

Assume further that $\mathfrak{g}$ is complete with respect to this filtration.

Definition 4.1. *A* curved L_∞ algebra *structure on $\mathfrak{g}$ consists of the following data:*

- *an element of degree* 1
$$l_0 \in F^1\mathfrak{g}[1];$$
- *a filtration-preserving $\mathbb{R}$-linear map of degree* 1
$$l_1 : \mathfrak{g}[1] \to \mathfrak{g}[1];$$
- *for each $n \geq 2$, a filtration-preserving R-linear map of degree* 1
$$l_n : \mathrm{Sym}^n_R(\mathfrak{g}[1]) \longrightarrow \mathfrak{g}[1].$$

These operations satisfy the following identities:

- *(Leibniz rule) for every $r \in R$ and $x \in \mathfrak{g}[1]$;*
$$l_1(rx) = D_R(r)\, x + (-1)^{|r|} r\, l_1(x);$$
- *(Strong homotopy Jacobi identities) for every $n \geq 0$ and $x_1, \ldots, x_n \in \mathfrak{g}[1]$,*
$$\sum_{p+q=n} \sum_{\sigma\in\Sigma_n} \frac{\epsilon(\sigma)}{p!\, q!}\, l_{q+1}\big(l_p(x_{\sigma(1)}, \ldots, x_{\sigma(p)}), x_{\sigma(p+1)}, \ldots, x_{\sigma(n)}\big) = 0.$$

In particular, one has

$$l_1(l_0) = 0, \quad \textit{and} \quad l_2(l_0, x) + l_1(l_1(x)) = 0.$$

Accordingly, l_0 is called the curvature *of $\mathfrak{g}$.*

Lemma 4.2. *The associated $\mathbb{N}$-graded operator*

$$\mathrm{Gr}\, l_1 : \mathrm{Gr}\, \mathfrak{g}[1] \longrightarrow \mathrm{Gr}\, \mathfrak{g}[1]$$

is a differential.

Proof. Since $l_0 \in F^1\mathfrak{g}[1]$ and l_n preserves the filtration on $\mathfrak{g}[1]$, we have

$$l_1(l_1(x)) = -\,l_2(l_0, x) \in F^{p+1}\mathfrak{g}[1]$$

for any $x \in F^p\mathfrak{g}[1]$. Thus, $\operatorname{Gr} l_1 \circ \operatorname{Gr} l_1 = 0$. □

Let $\mathfrak{g}^\vee$ denote the R-dual of $\mathfrak{g}$. The filtration on $\mathfrak{g}$ induces a filtration on $\mathfrak{g}^\vee$:

$$F^p\mathfrak{g}^\vee = \{\alpha \in \mathfrak{g}^\vee \mid \alpha(F^q\mathfrak{g}) \subset F^{p+q}R \text{ for all } q\}.$$

Lemma 4.3. *If $\mathfrak{g}$ is projective and finitely generated over R, then the filtration on $\mathfrak{g}^\vee$ is also induced by the filtration on R, i.e.,*

$$F^p\mathfrak{g}^\vee = F^pR \cdot \mathfrak{g}^\vee.$$

Proof. We only need to show that $F^p\mathfrak{g}^\vee \subset F^pR \cdot \mathfrak{g}^\vee$. Without loss of generality, assume that $\mathfrak{g}$ is free of finite rank with basis $\{v_i\}$. If $\alpha \in F^p\mathfrak{g}^\vee$, then $\alpha(v_i) \in F^pR$ for all i, so

$$\alpha = \sum_i \alpha(v_i) \cdot v_i^\vee \in F^pR \cdot \mathfrak{g}^\vee,$$

where $\{v_i^\vee\}$ denotes the dual basis. □

For our purposes, we will henceforth assume that $\mathfrak{g}$ is finitely generated and projective. Consider the following two natural filtrations on $\operatorname{Sym}_R(\mathfrak{g}[1]^\vee)$:

- Internal filtration defined by

$$F_I^p\operatorname{Sym}_R(\mathfrak{g}[1]^\vee) := F^pR \cdot \operatorname{Sym}_R(\mathfrak{g}[1]^\vee)$$

- Symmetric filtration defined by

$$F_S^p\operatorname{Sym}_R(\mathfrak{g}[1]^\vee) := \bigoplus_{q \geq p} \operatorname{Sym}_R^q(\mathfrak{g}[1]^\vee).$$

These two filtrations can be combined into a single filtration by convolution:

$$F^p\operatorname{Sym}_R(\mathfrak{g}[1]^\vee) := \sum_{i+j=p} F_I^i\operatorname{Sym}_R(\mathfrak{g}[1]^\vee) \odot F_S^j\operatorname{Sym}_R(\mathfrak{g}[1]^\vee),$$

which is the smallest filtration containing both. We use $\widehat{\operatorname{Sym}}_R(\mathfrak{g}^\vee[-1])$ to denote the completion with respect to this filtration.

The multi-brackets $\{l_n\}_{n=0}^\infty$ on $\mathfrak{g}$ determines a filtration-preserving differential

$$D_\mathfrak{g} : \widehat{\operatorname{Sym}}_R(\mathfrak{g}[1]^\vee) \longrightarrow \widehat{\operatorname{Sym}}_R(\mathfrak{g}[1]^\vee).$$

More precisely, $D_\mathfrak{g}$ is determined by its action on the generators

$$D_\mathfrak{g}(r) = D_R(r), \qquad D_\mathfrak{g}(\alpha) = \sum_{n=0}^\infty l_n^\vee(\alpha)$$

for $r \in R$ and $\alpha \in \mathfrak{g}[1]^\vee$. Here $l_n^\vee$ denotes the R-linear dual of l_n for $n \neq 1$, while for $n = 1$ we define

$$(l_1^\vee(\alpha))(x) = D_R(\alpha(x)) - (-1)^{|\alpha|}\alpha(l_1(x)).$$

The condition $D_\mathfrak{g}^2 = 0$ is equivalent to the strong homotopy Jacobi identities of $\{l_n\}_{n=0}^\infty$.

Definition 4.4. *We call the filtered cdga*

$$\mathrm{CE}(\mathfrak{g}) := (\widehat{\mathrm{Sym}}_R(\mathfrak{g}^\vee[-1]), D_\mathfrak{g})$$

the Chevalley–Eilenberg algebra *of* $\mathfrak{g}$.

Definition 4.5. *A* (base-preserving) morphism *between curved* L_∞ *algebras* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *is a morphism between filtered cdgas*

$$\Phi : \mathrm{CE}(\mathfrak{g}_2) \longrightarrow \mathrm{CE}(\mathfrak{g}_1)$$

such that $\Phi(R_2) \subset R_1$.

Such Φ is uniquely determined by a filtered cdga morphism

$$\Phi|_{R_2} : (R_2, D_{R_2}) \to (R_1, D_{R_1}),$$

a degree 0 element

$$\phi_0 \in F^1\mathfrak{g}_2[1],$$

and a family of degree 0 R_2-linear maps

$$\phi_n : \mathrm{Sym}^n_{R_1}(\mathfrak{g}_1[1]) \to \mathfrak{g}_2[1], \quad n \geq 1.$$

$\{\phi_n\}_{n=0}^\infty$ intertwines the multi-brackets on $\mathfrak{g}_1$ and $\mathfrak{g}_2$:

$$\sum_{p+q=n} \sum_{\sigma\in\Sigma_n} \frac{\epsilon(\sigma)}{p!q!}\, \phi_{q+1}\big(l_p(x_{\sigma(1)}, \dots), \dots, x_{\sigma(n)}\big)$$

$$= \sum_{k_1+\cdots+k_j=n} \sum_{\sigma\in\Sigma_n} \frac{\epsilon(\sigma)}{j!k_1!\cdots k_j!}\, l_j\big(\phi_{k_1}(x_{\sigma(1)}, \dots), \dots, \phi_{k_j}(\dots, x_{\sigma(n)})\big).$$

In particular, for $n = 0$ and $n = 1$, one has

$$\phi_1(l_0) = l_0 + \sum_{j=1}^\infty \frac{1}{j!} l_j(\phi_0^{\otimes j}),$$

$$\phi_1(l_1(x)) + \phi_2(l_0, x) = l_1(\phi_1(x)) + \sum_{j=1}^\infty \frac{1}{j!}\, l_{j+1}(\phi_0^{\otimes j}, \phi_1(x)).$$

Lemma 4.6. *The associated* $\mathbb{N}$*-graded morphism*

$$\mathrm{Gr}\,\phi_1 : \mathrm{Gr}\,\mathfrak{g}_2[1] \longrightarrow \mathrm{Gr}\,\mathfrak{g}_1[1]$$

commutes with the differentials $\mathrm{Gr}\, l_1$.

Proof. Since $l_0 \in F^1\mathfrak{g}_1[1]$ and $\phi_0 \in F^1\mathfrak{g}_2[1]$, we have

$$\phi_1(l_1(x)) - l_1(\phi_1(x)) \in F^{p+1}\mathfrak{g}_2[1]$$

for any $x \in F^p\mathfrak{g}[1]$. Thus, $\mathrm{Gr}\,\phi_1 \circ \mathrm{Gr}\, l_1 = \mathrm{Gr}\, l_1 \circ \mathrm{Gr}\,\phi_1$. □

Definition 4.7. Φ *is called a* weak equivalence *if*

$$\Phi|_{R_2} : (R_2, D_{R_2}) \to (R_1, D_{R_1})$$

is a filtered quasi-isomorphism, and

$$\mathrm{Gr}\,\phi_1 : (\mathrm{Gr}\,\mathfrak{g}_1[1], \mathrm{Gr}\, l_1) \longrightarrow (\mathrm{Gr}\,\mathfrak{g}_2[1], \mathrm{Gr}\, l_1)$$

is a quasi-isomorphism.

Equivalently, a curved L_∞ algebra morphism $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ can be regarded as a degree 0 element

$$\alpha \in \mathfrak{g}_2 \mathbin{\widehat{\otimes}_{R_2}} F^1\mathrm{CE}(\mathfrak{g}_1),$$

where $\widehat{\otimes}_{R_2}$ denotes the completed tensor product of filtered graded R_2-modules. α satisfies the *Maurer–Cartan equation*:

$$\mathrm{MC}(\alpha) := l_0 \otimes 1 + (l_1 \otimes 1 + 1 \otimes D_{\mathfrak{g}_1})(\alpha) + \sum_{n\geq 2}^{\infty} \frac{1}{n!}(l_n \otimes 1)(\alpha^{\otimes n}) = 0.^1$$

The following definition (and Definitions 4.11) will not be used in the rest of the paper and may be skipped on a first reading.

Definition 4.8. *Let (S, D_S) be a cdga equipped with a dg ideal $J \subset S$, complete with respect to the J-adic filtration, together with a cdga morphism $(R, D_R) \to (S, D_S)$ sending the dg ideal I of R into J.*

The set of S-points of $\mathfrak{g}$ *is defined as*

$$\mathrm{MC}(\mathfrak{g}; S) := \left\{\alpha \in (\mathfrak{g} \mathbin{\widehat{\otimes}_R} J)_0 \middle| \mathrm{MC}(\alpha) = 0\right\},$$

where $\mathrm{MC}(\alpha) := l_0 \otimes 1 + (l_1 \otimes 1 + 1 \otimes D_S)(\alpha) + \sum_{n\geq 2}^{\infty} \frac{1}{n!}(l_n \otimes 1)(\alpha^{\otimes n})$.

4.2. L_∞ **spaces.** Let $\mathcal{M}$ be a dg manifold

Definition 4.9. *An* L_∞ space *over $\mathcal{M}$ is a pair $B\mathfrak{g} = (\mathcal{M}, \mathfrak{g})$, where $\mathfrak{g}$ is a sheaf of graded $\widehat{\Omega}_{\mathcal{M}}$-modules, locally free of finite total rank, equipped with a curved L_∞ algebra structure over $(\widehat{\Omega}_{\mathcal{M}}, D_{\mathcal{M}})$ with dg ideal $\widehat{\Omega}^{>0}_{\mathcal{M}}$.*

An L_∞ space over a point is simply a finite dimensional L_∞ algebra.

Definition 4.10. *A* morphism $B\mathfrak{g}_1 \to B\mathfrak{g}_2$ *of L_∞ spaces is a pair $\Phi = (f, \Phi^\sharp)$, where*

- *$f : \mathcal{M}_1 \to \mathcal{M}_2$ is a dg manifold morphism;*
- *$\Phi^\sharp : \mathrm{CE}(\mathfrak{g}_2) \to f_*\mathrm{CE}(\mathfrak{g}_1)$ is a morphism of sheaves of curved L_∞ algebra, such that*

$$\Phi^\sharp|_{\widehat{\Omega}_{\mathcal{M}_2}} : \widehat{\Omega}_{\mathcal{M}_2} \to f_*\widehat{\Omega}_{\mathcal{M}_1}$$

  is given by the pullback $f^\sharp_{dR}$ along f.

Such Φ is a weak equivalence *if*

$$\Phi^\sharp_{M_2} : \mathrm{CE}(\mathfrak{g}_2)(M_2) \to \mathrm{CE}(\mathfrak{g}_1)(M_1)$$

is a weak equivalence of curved L_∞ algebras.

Unlike L_∞ algebroids over dg manifolds, L_∞ spaces over dg manifolds admit pullbacks along morphisms of dg manifolds. Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of dg manifolds, and let $B\mathfrak{g}_2$ be an L_∞ space over $\mathcal{M}_2$. We define the pullback L_∞ space $Bf^*\mathfrak{g}_2 = (M_1, f^*\mathfrak{g}_2)$ over $\mathcal{M}_1$ by

$$f^*\mathfrak{g}_2 := \widehat{\Omega}_{\mathcal{M}_1} \mathbin{\widehat{\otimes}_{f^{-1}\widehat{\Omega}_{\mathcal{M}_2}}} f^{-1}\mathfrak{g}_2.$$

[1]The Maurer–Cartan equation is well-defined because for $n \geq 2$,

$$(l_n \otimes 1)(\alpha^{\otimes n}) \in F^n(\mathfrak{g}_1 \mathbin{\widehat{\otimes}_{R_2}} \mathrm{CE}(\mathfrak{g}_2)),$$

and the filtration on $\mathfrak{g}_1 \mathbin{\widehat{\otimes}_{R_2}} F^1\mathrm{CE}(\mathfrak{g}_2)$ is complete.

A morphism of L_∞ spaces $\Phi : B\mathfrak{g}_1 \to B\mathfrak{g}_2$ is equivalent to a degree 0 element

$$\alpha \in \mathfrak{g}_2 \otimes_{\widehat{\Omega}_{\mathcal{M}_2}} f_* F^1\mathrm{CE}(\mathfrak{g}_1) \cong f_* \left(f^*\mathfrak{g}_2 \otimes_{\widehat{\Omega}_{\mathcal{M}_1}} F^1\mathrm{CE}(\mathfrak{g}_1) \right)$$

satisfying the Maurer–Cartan equation.

Definition 4.11. *Let $\mathcal{C} = (C, \mathcal{O}_\mathcal{C})$ be a quasi-dg manifold. Let $B\mathfrak{g} = (\mathcal{M}, \mathfrak{g})$ be an L_∞ space. The set of* $\mathcal{C}$-points of $B\mathfrak{g}$ *is*

$$\mathrm{MC}(B\mathfrak{g}; \mathcal{C}) := \bigsqcup_{f : \mathrm{Gr}^0\mathcal{C} \to \mathcal{M}} \mathrm{MC}(f^*\mathfrak{g}; \mathcal{O}_\mathcal{C}),$$

where is the disjoint union is taken over all dg manifold morphisms $f : \mathrm{Gr}^0\mathcal{C} \to \mathcal{M}$.

Let $\mathbf{L}_\infty\mathbf{Sp}$ denote the category of L_∞ spaces over dg manifolds. Let $\mathbf{QdgM}^{\mathrm{bp}}_{\mathrm{fib}}$ denote the category of fibrant quasi-dg manifolds with base-preserving morphisms.

Consider the assignment

$$\mathbf{CE_{Sp}} : \mathbf{L}_\infty\mathbf{Sp} \longrightarrow \mathbf{QdgM}^{\mathrm{bp}}_{\mathrm{fib}}$$

defined on objects $B\mathfrak{g} = (\mathcal{M}, \mathfrak{g})$ by

$$\mathbf{CE_{Sp}}(B\mathfrak{g}) = (M, \mathrm{CE}(\mathfrak{g})),$$

and on morphisms $\Phi = ((f, f^\sharp), \Phi^\sharp) : B\mathfrak{g}_1 \to B\mathfrak{g}_2$ by

$$\mathbf{CE_{Sp}}(\Phi) = (f, \Phi^\sharp) : (M_1, \mathrm{CE}(\mathfrak{g}_1)) \longrightarrow (M_2, \mathrm{CE}(\mathfrak{g}_2)).$$

$\mathbf{CE_{Sp}}$ is obviously a well-defined functor, called the *Chevalley–Eilenberg functor*.

Theorem 4.12. $\mathbf{CE_{Sp}}$ *yields an equivalence of categories and detects weak equivalences.*

Proof. By definition, $\mathbf{CE_{Algd}}$ is fully faithful. We need to show that it is essentially surjective. Let $\mathcal{C}$ be a fibrant quasi-dg manifold. By Theorem 3.19 and Lemma 3.10, we know that there exists a strict transitive L_∞ algebroid $\mathcal{L}$ over $\mathrm{Gr}^0\mathcal{C}$ such that

$$(\mathcal{O}_\mathcal{C}, Q_\mathcal{C}) \cong (\mathrm{CE}(\mathcal{L}), D_\mathcal{L})$$

as filtered cdga over $(\widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}}, D_{\mathrm{Gr}^0\mathcal{C}})$. We now set

$$\mathfrak{g} := \widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}} \,\widehat{\otimes}_{\mathrm{Gr}^0\mathcal{O}_\mathcal{C}}\, \mathcal{L}_{red},$$

where $\mathcal{L}_{red} = \ker \rho_1$ and $\rho_1 : \mathcal{L} \to \mathcal{T}_{\mathrm{Gr}^0\mathcal{C}}$ is the unary anchor of $\mathcal{L}$. Fix a splitting

$$\sigma : \mathcal{T}_{\mathrm{Gr}^0\mathcal{C}} \longrightarrow \mathcal{L}$$

of the short exact sequence of graded vector bundles

$$0 \longrightarrow \mathcal{L}_{red} \longrightarrow \mathcal{L} \longrightarrow \mathcal{T}_{\mathrm{Gr}^0\mathcal{C}} \longrightarrow 0.$$

We can use σ to define a $\mathrm{Gr}^0\mathcal{O}_\mathcal{C}$-linear map

$$\mathcal{L}[1]^\vee \cong \Omega^1_{\mathrm{Gr}^0\mathcal{C}} \otimes_{\mathrm{Gr}^0\mathcal{O}_\mathcal{C}} \mathcal{L}_{red} \hookrightarrow \mathfrak{g}[1]^\vee.$$

By the universal property of the symmetric algebra, this map induces a graded commutative $\mathrm{Gr}^0\mathcal{O}_\mathcal{C}$-algebra morphism

$$\Omega_\mathcal{L} = \mathrm{Sym}_{\mathrm{Gr}^0\mathcal{O}_\mathcal{C}}(\mathcal{L}[1]^\vee) \longrightarrow \mathrm{Sym}_{\widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}}}(\mathfrak{g}[1]^\vee)$$

which extends uniquely to a morphism of filtered graded commutative algebras

$$\widehat{\Omega}_\mathcal{L} = \widehat{\mathrm{Sym}}_{\mathrm{Gr}^0\mathcal{O}_\mathcal{C}}(\mathcal{L}[1]^\vee) \longrightarrow \widehat{\mathrm{Sym}}_{\widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}}}(\mathfrak{g}[1]^\vee),$$

which is locally an isomorphism and therefore an isomorphism globally. We can use it to transfer the differential $D_{\mathcal{L}}$ on $\widehat{\Omega}_{\mathcal{L}}$ to a differential on $\widehat{\mathrm{Sym}}_{\widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}}}(\mathfrak{g}[1]^\vee)$. Since all higher anchor maps of $\mathcal{L}$ vanish, the induced differential $D_{\mathfrak{g}}$ satisfies the desired Leibniz rule

$$D_{\mathfrak{g}}(\omega\alpha) = D_{\mathrm{Gr}^0\mathcal{C}}(\omega)\,\alpha + (-1)^{|\omega|}\,\omega\,D_{\mathfrak{g}}(\alpha),$$

where $\omega \in \widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}}$ and $\alpha \in \widehat{\mathrm{Sym}}_{\widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}}}(\mathfrak{g}[1]^\vee)$. Thus, $\mathfrak{g}$ is a sheaf of curved L_∞ algebras over $(\widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}}, D_{\mathrm{Gr}^0\mathcal{C}})$, and $B\mathfrak{g} := (\mathrm{Gr}^0\mathcal{C}, \mathfrak{g})$ defines an L_∞ space over $\mathrm{Gr}^0\mathcal{C}$.

Let $\Phi : B\mathfrak{g}_1 \to B\mathfrak{g}_2$ be a morphism of L_∞ spaces, and let $\widetilde{\Phi} : \mathcal{L}_1/\mathcal{M}_1 \to \mathcal{L}_2/\mathcal{M}_2$ be the corresponding morphism of L_∞ algebroids. By Theorem 3.19, to show that the functor $\mathbf{CE_{Sp}}$ detects weak equivalences, it suffices to show that $\widetilde{\Phi}$ is a weak equivalence of L_∞ algebroids whenever Φ is a weak equivalence of L_∞ spaces, which follows from Lemmas 2.22–2.24.

More precisely, by Lemmas 2.22–2.24,

$$\Phi^\sharp|_{\widehat{\Omega}_{\mathcal{M}_2}} : \widehat{\Omega}_{\mathcal{M}_2} \longrightarrow \widehat{\Omega}_{\mathcal{M}_1}$$

is a filtered quasi-isomorphism if and only if $\mathcal{M}_1 \to \mathcal{M}_2$ is a weak equivalence of dg manifolds. On the other hand, let $\phi_1^\sharp$ be the linear component of $\Phi^\sharp$. Then

$$\mathrm{Gr}\,\phi_1^\sharp : \mathfrak{g}_1 \cong \Omega_{\mathcal{M}_1} \otimes_{\mathcal{O}_{\mathcal{M}_1}} (\mathcal{L}_1)_{red} \longrightarrow \mathfrak{g}_2 \cong \Omega_{\mathcal{M}_2} \otimes_{\mathcal{O}_{\mathcal{M}_2}} (\mathcal{L}_2)_{red}$$

is a quasi-isomorphism if and only if

$$(\mathcal{L}_1)_{red} \longrightarrow (\mathcal{L}_2)_{red}$$

is a weak equivalence of dg vector bundles. Note that Φ induces a morphism of short exact sequences of dg vector bundles

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathcal{L}_1)_{red} & \longrightarrow & \mathcal{L}_1 & \longrightarrow & \mathcal{T}_{\mathcal{M}_1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (\mathcal{L}_2)_{red} & \longrightarrow & \mathcal{L}_2 & \longrightarrow & \mathcal{T}_{\mathcal{M}_2} & \longrightarrow & 0
\end{array}.$$

Using Lemma 2.24, we conclude that $(\mathcal{L}_1)_{red} \longrightarrow (\mathcal{L}_2)_{red}$ is a weak equivalence if and only if $\mathcal{L}_1 \to \mathcal{L}_2$ is a weak equivalence. □

Corollary 4.13. *There is an equivalence between the category of L_∞ spaces over dg manifolds and that of transitive L_∞ algebroids over dg manifolds, which detects weak equivalences.*

At first glance, Corollary 4.13 may seem surprising, since L_∞ spaces and L_∞ algebroids appear to be of quite different nature. In particular, one may wonder how the curvature of an L_∞ space arises under this equivalence. The following example helps dispel the mystery.

Example 4.14. *Let M be a manifold. Let L be a transitive Lie algebroid over M; that is, a strict transitive L_∞ algebroid over M whose only non-vanish multi-bracket is the binary bracket*

$$[\cdot,\cdot]_L : L \times_M L \to L.$$

Let L_{red} be the isotropy Lie algebra bundle of L, i.e. the kernel of $\rho_L : L \to T_M$.

To construct the corresponding L_∞ space over M, we follow the proof of Theorem 4.12 to choose a vector bundle splitting

$$\sigma : T_M \longrightarrow L$$

of the short exact sequence of Lie algebroids

$$0 \longrightarrow L_{red} \longrightarrow L \xrightarrow{\rho_L} T_M \longrightarrow 0.$$

This splitting induces a connection on L_{red} *by*

$$\nabla^\sigma_X(s) := [\sigma(X), s]_L$$

The connection ∇^σ *is well-defined since*

$$[\nabla^\sigma_X, f](s) = \rho_L \circ \sigma(X)(f)s = X(f)s, \quad \rho_1(\nabla^\sigma_X(s)) = [X, \rho_L(s)] = 0.$$

Moreover, it is compatible with the bracket

$$\nabla^\sigma_X([s_1, s_2]_L) = [\nabla^\sigma_X(s_1), s_2]_L + [s_1, \nabla^\sigma_X(s_2)]_L.$$

And its curvature R_{∇^σ} *measures the failure of* σ *to be a Lie algebroid splitting. Explicitly,*

$$R_{\nabla^\sigma}(X, Y) = [\sigma(X), \sigma(Y)]_L - \sigma([X, Y]).$$

We are now ready to describe the corresponding L_∞ *space* $B\mathfrak{g}^\sigma = (M, \mathfrak{g}^\sigma)$ *over* M*. For an open* $U \subset M$*, we set*

$$\mathfrak{g}^\sigma(U) := \Omega_M(U) \otimes_{C^\infty(U)} \Gamma\big(L_{red}|_U\big).$$

The non-vanishing multi-brackets on $\mathfrak{g}^\sigma$ *are*

$$l_0 = R_{\nabla^\sigma} \in F^1\mathfrak{g}^\sigma[1], \qquad l_1 = d_{\nabla^\sigma} : \mathfrak{g}^\sigma[1] \longrightarrow \mathfrak{g}^\sigma[1],$$

and

$$l_2 : \mathrm{Sym}^2_{\Omega_M}(\mathfrak{g}^\sigma[1]) \longrightarrow \mathfrak{g}^\sigma[1],$$

defined by

$$l_2(\alpha \otimes s, \beta \otimes t) = (\alpha \wedge \beta) \otimes [s, t]_L.$$

One can readily verify that the strong homotopy Jacobi identities for $\{l_n\}_{n=0}^2$ *hold precisely by virtue of the second Bianchi identity for* R_{∇^σ}*, the compatibility between* ∇^σ *and* $[\cdot, \cdot]_L$*, and the Jacobi identity for* $[\cdot, \cdot]_L$*.*

More generally, let $\mathcal{L}/\mathcal{M}$ be a strict transitive L_∞ algebroid, and let $\sigma\colon \mathcal{T}_\mathcal{M} \to \mathcal{L}$ be a graded vector bundle splitting of the short exact sequence of strict L_∞ algebroids

$$0 \longrightarrow \mathcal{L}_{red} \longrightarrow \mathcal{L} \xrightarrow{\rho_1} \mathcal{T}_\mathcal{M} \longrightarrow 0.$$

Then the curvature of the associated L_∞ space is precisely the obstruction to σ being a strict morphism of strict L_∞ algebroids.

5. The fibrant replacement functor

In this section, we construct a faithful functor

$$\mathbf{Fib} : \mathbf{L_\infty Algd} \longrightarrow \mathbf{L_\infty Sp}$$

satisfying the following properties:

- it restricts to the identity on L_∞ algebras;
- it assigns to each dg manifold its Fedosov resolution;
- it detect weakly equivalences.

More precisely, **Fib** is obtained as the composition

$$\mathbf{L}_\infty\mathbf{Algd} \xrightarrow{\mathbf{CE}_{\mathbf{Algd}}} \mathbf{QdgM}^{\mathrm{bp}} \xrightarrow{\mathbf{J}^\infty} \mathbf{QdgM}^{\mathrm{bp}}_{\mathrm{fib}} \xrightarrow{\mathbf{CE}^{-1}_{\mathbf{Sp}}} \mathbf{L}_\infty\mathbf{Sp},$$

where $\mathbf{J}^\infty$ is the main functor we will construct in this section, called the *jet space functor*.

5.1. **Infinite jets and D-modules.** Let $\mathcal{M}$ be a dg manifold.

Definition 5.1. *A* filtered graded vector bundle *on $\mathcal{M}$ is a graded (left) $\mathcal{O}_\mathcal{M}$-module $\mathcal{E}$ equipped with a descending filtration*

$$\mathcal{E} \supset \cdots \supset F^{p-1}\mathcal{E} \supset F^p\mathcal{E} \supset F^{p+1}\mathcal{E} \supset \cdots \supset 0,$$

satisfying the following conditions:

- *the filtration is exhaustive, separated, and complete, i.e.*

$$\bigcup_{p\in\mathbb{Z}} F^p\mathcal{E} = \mathcal{E}, \qquad \bigcap_{p\in\mathbb{Z}} F^p\mathcal{E} = 0, \qquad \mathcal{E} \cong \varprojlim_p \mathcal{E}/F^p\mathcal{E};$$

- *for each $p \in \mathbb{Z}$, the graded $\mathcal{O}_\mathcal{M}$-module*

$$\mathrm{Gr}^p\mathcal{E} = F^p\mathcal{E}/F^{p+1}\mathcal{E}$$

is a graded vector bundle, i.e. locally free of total finite rank.

We recommend that our readers consult the appendices to familiarize themselves with our notions of differential operators, infinite jet bundles, and D-modules on graded manifolds before reading the rest of this section.

Definition 5.2. *A* quasi-dg vector bundle *on $\mathcal{M}$ is a filtered graded vector bundle $\mathcal{E}$ equipped with a differential operator*

$$D_\mathcal{E} : \mathcal{E} \to \mathcal{E},$$

of cohomological degree 1*, order* 1*, and filtration degree* 0*, such that*

- *$D_\mathcal{E}$ is a differential, i.e. $D_\mathcal{E} \circ D_\mathcal{E} = 0$;*
- *the principal symbol*

$$\sigma_1^0(D_\mathcal{E}) \in \mathcal{T}_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathrm{Gr}^0\mathrm{End}_\mathcal{E}$$

of $D_\mathcal{E}$ is given by $Q_\mathcal{M} \otimes \mathrm{id}_{\mathrm{Gr}\,\mathcal{E}}$; or equivalently,

$$[D_\mathcal{E}, f]s = Q_\mathcal{M}(f)s \mod F^{p+1}\mathcal{E}$$

for all $f \in \mathcal{O}_\mathcal{M}$ and $s \in F^p\mathcal{E}$.

Such $\mathcal{E}$ is called a filtered dg vector bundle *if*

$$[D_\mathcal{E}, f] = Q_\mathcal{M}(f)$$

for all $f \in \mathcal{O}_\mathcal{M}$.

By a slight abuse of notation, we denote a quasi-dg vector bundle by $\mathcal{E}/\mathcal{M}$, or simply by $\mathcal{E}$ when the base $\mathcal{M}$ is clear from context.

Lemma 5.3. *Let $\mathcal{E}$ be a quasi-dg vector bundle. Then*

$$\mathrm{Gr}^\bullet D_\mathcal{E} : \mathrm{Gr}^\bullet\mathcal{E} \to \mathrm{Gr}^\bullet\mathcal{E}$$

makes $\mathrm{Gr}^\bullet\mathcal{E}$ into a dg vector bundle.

Proof. By definition, one has $(\mathrm{Gr}^\bullet D_\mathcal{E})^2 = 0$, and

$$[\mathrm{Gr}^\bullet D_\mathcal{E}, f] = Q_\mathcal{M}(f)$$

for all $f \in \mathcal{O}_\mathcal{M}$. □

Example 5.4. *A dg vector bundle $\mathcal{E}$ can be regarded as a filtered dg vector bundle with the trivial filtration*

$$F^p\mathcal{E} = \begin{cases} \mathcal{E}, & p \leq 0, \\ 0, & p > 0. \end{cases}$$

Example 5.5. *Let $\mathcal{C}$ be a quasi-dg manifold. The structure sheaf $\mathcal{O}_\mathcal{C}$ of $\mathcal{C}$ defines a quasi-dg vector bundle over its base dg manifold $\mathrm{Gr}^0\mathcal{C}$.*

Definition 5.6. *A* morphism *between quasi-dg vector bundles $\mathcal{E}_1/\mathcal{M}_1 \to \mathcal{E}_2/\mathcal{M}_2$ consists of a pair $\Phi = (f, \Phi^\sharp)$, where $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a dg manifold morphism and*

$$\Phi^\sharp : f_*\mathcal{E}_1 \to \mathcal{E}_2$$

is an $\mathcal{O}_{\mathcal{M}_2}$-linear map of degree 0, compatible with both the filtrations and differentials.

Such Φ is called base-fixing *if $f = \mathrm{id}$, and a* weak equivalence *if $\mathrm{Gr}^\bullet\Phi$ is a weak equivalence of dg vector bundles.*

One can further introduce a category of quasi-dg vector bundles over quasi-dg manifolds. This level of generality becomes necessary when constructing fibrant replacements for representations of L_∞ algebroids. We will not pursue this direction in the present paper.

Example 5.7. *Every morphism of dg vector bundles is a morphism of quasi-dg vector bundles (with respect to the trivial filtration).*

Example 5.8. *Viewing the structure sheaf of a quasi-dg manifold as a quasi-dg vector bundle over its base dg manifold, every base-fixing morphism $\mathcal{C}_1 \to \mathcal{C}_2$ of quasi-dg manifolds yields a morphism of quasi-dg vector bundles $\mathcal{O}_{\mathcal{C}_2} \to \mathcal{O}_{\mathcal{C}_1}$ in the opposite direction.*

Let $\mathbf{QdgM}(\mathcal{M})$ and $\mathbf{QdgVB}(\mathcal{M})$ denote the categories of quasi-dg manifolds and quasi-dg vector bundles over $\mathcal{M}$ with base-fixing morphisms, respectively. By Examples 5.5 and 5.8, there is a faithful functor

$$\mathbf{QdgM}(\mathcal{M})^{op} \longrightarrow \mathbf{QdgVB}(\mathcal{M}).$$

Lemma 5.9. $\mathbf{QdgM}(\mathcal{M})^{op} \longrightarrow \mathbf{QdgVB}(\mathcal{M})$ *detects weak equivalences*

Proof. This is tautological from the definitions. □

Let $\mathcal{E}$ be a quasi-dg vector bundle over $\mathcal{M}$. Let $\mathcal{J}^\infty_\mathcal{E}$ be the infinite jet bundle of $\mathcal{E}$ (viewed as a filtered graded vector bundle). By Proposition B.18, the differential $D_\mathcal{E}$ lifts to a differential $j^\infty(D_\mathcal{E})$ on $\mathcal{J}^\infty_\mathcal{E}$. Moreover, since $D_\mathcal{E}$ is of order 1 and filtration degree 0, $j^\infty(D_\mathcal{E})$ decreases the filtration on $\mathcal{J}^\infty_\mathcal{E}$ by one level.

Let ∇^G denote the Grothendieck connection on $\mathcal{J}^\infty_\mathcal{E}$. Since ∇^G is flat, we have

$$[\nabla^G_{Q_\mathcal{M}}, \nabla^G_{Q_\mathcal{M}}] = \nabla^G_{[Q_\mathcal{M}, Q_\mathcal{M}]} = 0.$$

Thus, $\nabla^G_{Q_\mathcal{M}}$ is a differential on $\mathcal{J}^\infty_\mathcal{E}$, which also lowers the filtration by one level. Since $j^\infty(D_\mathcal{E})$ is compatible with the left $D_\mathcal{M}$-action on $\mathcal{J}^\infty_\mathcal{E}$, we have

$$[\nabla^G_{Q_\mathcal{M}}, j^\infty(D_\mathcal{E})] = 0.$$

Proposition 5.10. *Equipped with the total differential*

$$D_{\mathcal{J}^\infty_\mathcal{E}} := \nabla^G_{Q_\mathcal{M}} + j^\infty(D_\mathcal{E}),$$

$\mathcal{J}^\infty_\mathcal{E}$ becomes a filtered dg vector bundle, called the infinite jet bundle *of $\mathcal{E}$.*

Proof. $\nabla^G_{Q_\mathcal{M}}$ is a differential operator of order 1, while $j^\infty(D_\mathcal{E})$ is of order 0. Their sum $D_{\mathcal{J}^\infty_\mathcal{E}}$ is therefore a differential operator of order 1.

We now show that $D_{\mathcal{J}^\infty_\mathcal{E}}$ preserves the filtration on $\mathcal{J}^\infty_\mathcal{E}$. Since both $\nabla^G_{Q_\mathcal{M}}$ and $j^\infty(D_\mathcal{E})$ lower the filtration by one level, it suffices to show that the sum

$$\operatorname{Gr}\nabla^G_{Q_\mathcal{M}} + \operatorname{Gr} j^\infty(D_\mathcal{E}) : \operatorname{Gr}^\bullet \mathcal{J}^\infty_\mathcal{E} \longrightarrow \operatorname{Gr}^{\bullet-1}\mathcal{J}^\infty_\mathcal{E}$$

vanishes.

Recall that from Appendix A that

$$\operatorname{Gr}\mathcal{J}^\infty_\mathcal{E} \cong \operatorname{Sym}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}^\vee_\mathcal{M}) \otimes_{\mathcal{O}_\mathcal{M}} \operatorname{Gr}\mathcal{E},$$

and that the associated graded operator $\operatorname{Gr} d_{\nabla^G}$ of the exterior covariant derivative d_{∇^G} on

$$\operatorname{Sym}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}[1]^\vee) \otimes_{\mathcal{O}_\mathcal{M}} \operatorname{Sym}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}^\vee_\mathcal{M}) \otimes_{\mathcal{O}_\mathcal{M}} \operatorname{Gr}\mathcal{E},$$

is given by minus the universal Koszul differential

$$\operatorname{Gr} d_{\nabla^G} = -\delta_K \otimes \mathrm{id}_{\operatorname{Gr}\mathcal{E}}.$$

It follows that

$$\operatorname{Gr}\nabla^G_{Q_\mathcal{M}} = -\iota_{Q_\mathcal{M}} \otimes \mathrm{id}_{\operatorname{Gr}\mathcal{E}},$$

where $\iota_{Q_\mathcal{M}} : \operatorname{Sym}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}^\vee_\mathcal{M}) \to \operatorname{Sym}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}^\vee_\mathcal{M})$ denotes the contraction with $Q_\mathcal{M}$. On the other hand, by Proposition B.18, $\operatorname{Gr} j^\infty(D_\mathcal{E})$ is given by contraction with the principal symbol $\sigma^0_1(D_\mathcal{E})$ of $D_\mathcal{E}$, which precisely cancels $\operatorname{Gr}\nabla^G_{Q_\mathcal{M}}$.

To complete the proof, observe that

$$[D_{\mathcal{J}^\infty_\mathcal{E}}, f] = [\nabla^G_{Q_\mathcal{M}}, f] = Q_\mathcal{M}(f)$$

for all $f \in \mathcal{O}_\mathcal{M}$. □

Definition 5.11. *A* (left) D-module *on $\mathcal{M}$ is a filtered dg vector bundle $\mathcal{E}$ on $\mathcal{M}$, together with a D-module structure on $\mathcal{E}$ (viewed as a filtered graded vector bundle) satisfying*

$$D_\mathcal{E}(D \cdot s) = L_{Q_\mathcal{M}}(D) \cdot s + (-1)^{|D|} D \cdot D_\mathcal{E}(s),$$

for all $D \in \mathcal{D}_\mathcal{M}$ and $s \in \mathcal{E}$, where $L_{Q_\mathcal{M}}(D) = [Q_\mathcal{M}, D]$ is the Lie derivative of D along $Q_\mathcal{M}$.

Remark 5.12. *Equivalently, the compatibility between the left $\mathcal{D}_\mathcal{M}$-action and $D_\mathcal{E}$ reads*

$$[D_\mathcal{E}, f] = L_{Q_\mathcal{M}}(f)$$

for $f \in \mathcal{O}_\mathcal{M}$, which holds automatically by the definition of a filtered dg vector bundle, together with the condition

$$[D_\mathcal{E}, X\cdot] = [Q_\mathcal{M}, X]\cdot$$

for $X \in \mathcal{D}_\mathcal{M}$, since $\mathcal{D}_\mathcal{M}$ is generated by functions and vector fields.

Recall from Appendix B that the de Rham complex of a D-module $\mathcal{E}$ on $\mathcal{M}$ (viewed as a graded manifold) is given by

$$dR(\mathcal{E}) = \big(\widehat{\Omega}_{\mathcal{M}} \,\widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}}\, \mathcal{E},\ d_{\nabla}\big),$$

where ∇ is the flat connection on $\mathcal{E}$ induced by left $\mathcal{D}_{\mathcal{M}}$-action. Explicitly,

$$\nabla_X(s) = X \cdot s.$$

We now adapt this construction from the graded manifold setting to the dg manifld setting.

Definition 5.13. *The* de Rham complex *of a D-module $\mathcal{E}$ on $\mathcal{M}$ is the following quasi-dg vector bundle on $\mathcal{M}$:*

$$dR(\mathcal{E}) := \big(\widehat{\Omega}_{\mathcal{M}} \,\widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}}\, \mathcal{E},\ d_{\nabla} + L_{D_{\mathcal{E}}}\big),$$

where ∇ is the flat connection on $\mathcal{E}$ induced by left $\mathcal{D}_{\mathcal{M}}$-action, and $L_{D_{\mathcal{E}}}$ is given by

$$L_{D_{\mathcal{E}}}(\alpha \otimes s) = L_{Q_{\mathcal{M}}}(\alpha) \otimes s + (-1)^{|\alpha|}\alpha \otimes D_{\mathcal{E}}(s).$$

Proposition 5.14. *The de Rham complex $dR(\mathcal{E})$ is well-defined.*

Proof. By definition, $L_{D_{\mathcal{E}}}$ is well-defined and squares to zero. Moreover,

$$[L_{D_{\mathcal{E}}}, f]s = Q_{\mathcal{M}}(f)s, \quad d_{\nabla^G}(s) = 0 \mod F^{p+1}dR(\mathcal{E}),$$

for all $s \in F^p dR(\mathcal{E})$. It remains to verify that $[d_{\nabla}, L_{D_{\mathcal{E}}}] = 0$. Since $[d_{\nabla}, L^{\nabla}_{Q_{\mathcal{M}}}] = 0$, it suffices to show that

$$\nabla(D_{\mathcal{E}}(s)) + L_{D_{\mathcal{E}}}\nabla(s) = 0.$$

Let x^{μ} be local coordinates on $\mathcal{M}$. We compute

$$\begin{aligned} dx^{\mu} \otimes \nabla_{\frac{\partial}{\partial x^{\mu}}}(D_{\mathcal{E}}(s) + L_{D_{\mathcal{E}}}(dx^{\mu} \otimes \nabla_{\frac{\partial}{\partial x^{\mu}}}(s)) &= L_{Q_{\mathcal{M}}}(dx^{\mu}) \otimes \nabla_{\frac{\partial}{\partial x^{\mu}}}(s) - dx^{\mu} \otimes [D_{\mathcal{E}}, \nabla_{\frac{\partial}{\partial x^{\mu}}}](s) \\ &= L_{Q_{\mathcal{M}}}(dx^{\mu}) \otimes \nabla_{\frac{\partial}{\partial x^{\mu}}}(s) - dx^{\mu} \otimes \nabla_{[Q_{\mathcal{M}}, \frac{\partial}{\partial x^{\mu}}]}(s) \\ &= 0, \end{aligned}$$

which completes the proof. □

One readily sees that the infinite jet bundle $\mathcal{J}^{\infty}_{\mathcal{E}}$ of a quasi-dg vector bundle $\mathcal{E}$ carries a compatible D-module structure. More precisely, we have

$$[D_{\mathcal{J}^{\infty}_{\mathcal{E}}}, X\cdot] = [D_{\mathcal{J}^{\infty}_{\mathcal{E}}}, \nabla^G_X] = [\nabla^G_{Q_{\mathcal{M}}}, \nabla^G_X] = \nabla^G_{[Q_{\mathcal{M}}, X]} = [Q_{\mathcal{M}}, X]\cdot,$$

where we use Proposition B.18 and the flatness of ∇^G. For simplicity, we denote by

$$(\mathfrak{J}^{\infty}_{\mathcal{E}} := \widehat{\Omega}_{\mathcal{M}} \,\widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}}\, \mathcal{J}^{\infty}_{\mathcal{E}},\ D_{\mathfrak{J}^{\infty}_{\mathcal{E}}} := d_{\nabla^G} + L_{D_{\mathcal{J}^{\infty}_{\mathcal{E}}}})$$

the de Rham complex of $\mathcal{J}^{\infty}_{\mathcal{E}}$. Note that the differential $L_{D_{\mathcal{J}^{\infty}_{\mathcal{E}}}}$ can be written as

$$L_{D_{\mathcal{J}^{\infty}_{\mathcal{E}}}} = L^{\nabla^G}_{Q_{\mathcal{M}}} + \mathrm{id}_{\widehat{\Omega}_{\mathcal{M}}} \otimes j^{\infty}(D_{\mathcal{E}}).$$

Proposition 5.15. *The associated graded operator* $\mathrm{Gr}\, D_{\mathfrak{J}^{\infty}_{\mathcal{E}}}$ *on*

$$\mathrm{Gr}\,\mathfrak{J}^{\infty}_{\mathcal{E}} \cong \mathcal{K}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Gr}\,\mathcal{E}, \quad \textit{with} \quad \mathcal{K}_{\mathcal{M}} := \Omega_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Sym}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}^{\vee}_{\mathcal{M}}),$$

is given by

$$\mathrm{Gr}\, D_{\mathfrak{J}^{\infty}_{\mathcal{E}}} = -\delta_K \otimes \mathrm{id}_{\mathrm{Gr}\,\mathcal{E}} + L_{Q_{\mathcal{M}}} \otimes \mathrm{id}_{\mathrm{Gr}\,\mathcal{E}} + \mathrm{id}_{\mathcal{K}_{\mathcal{M}}} \otimes \mathrm{Gr}\, D_{\mathcal{E}},$$

where $L_{Q_{\mathcal{M}}}$ denotes the Lie derivative on $\mathcal{K}_{\mathcal{M}}$ along $Q_{\mathcal{M}}$.

Proof. It suffices to show that

$$\operatorname{Gr}\left(L^{\nabla^G}_{Q_{\mathcal{M}}} + \mathrm{id}_{\widehat{\Omega}_{\mathcal{M}}} \otimes j^\infty(D_{\mathcal{E}})\right) = L_{Q_{\mathcal{M}}} \otimes \mathrm{id}_{\operatorname{Gr}\mathcal{E}} + \mathrm{id}_{\mathcal{K}_{\mathcal{M}}} \otimes \operatorname{Gr} D_{\mathcal{E}}.$$

The most straightforward way to verify this is by working in local coordinates. Let x^μ be local coordinates on $\mathcal{M}$, dx^μ and y^μ be the corresponding fiber coordinates on $\mathcal{T}_{\mathcal{M}}[1]^\vee$ and $\mathcal{T}_{\mathcal{M}}$, and u^a be local fiber coordinates on $\mathcal{E}$. Locally identifying $\mathcal{J}^\infty_{\mathcal{E}}$ with $\widehat{\operatorname{Sym}}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}^\vee_{\mathcal{M}}) \mathbin{\widehat{\otimes}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{E}$ using the trivial connection, we can write

$$d_{\nabla^G} = dx^\mu \frac{\partial}{\partial x^\mu} - \left(dx^\mu \frac{\partial}{\partial y^\mu}\right) \otimes \mathrm{id}_{\mathcal{E}}.$$

Thus, $L^{\nabla^G}_{Q_{\mathcal{M}}}$ can be expressed as

$$L^{\nabla^G}_{Q_{\mathcal{M}}} = Q^\mu_{\mathcal{M}} \frac{\partial}{\partial x^\mu} - \left(dx^\nu \frac{\partial Q^\mu_{\mathcal{M}}}{\partial x^\nu} \frac{\partial}{\partial dx^\mu} + Q^\mu_{\mathcal{M}} \frac{\partial}{\partial y^\mu}\right) \otimes \mathrm{id}_{\mathcal{E}}.$$

On the other hand, $D_{\mathcal{E}}$ can be locally written as

$$D_{\mathcal{E}} = D^\mu_{\mathcal{E}}(x,u) \frac{\partial}{\partial x^\mu} + D^0_{\mathcal{E}}(x,u),$$

with $D^0_{\mathcal{E}}$ denoting the local zeroth-order part of $D_{\mathcal{E}}$. By definition, we know that

$$D^\mu_{\mathcal{E}}(x,u) = Q^\mu_{\mathcal{M}}(x) + \cdots,$$

where $\cdots$ denotes extra terms of filtration degrees ≥ 1. Then we can write

$$\operatorname{Gr} D_{\mathcal{E}} = Q^\mu_{\mathcal{M}} \frac{\partial}{\partial x^\mu} + \operatorname{Gr} D^0_{\mathcal{E}},$$

and

$$j^\infty(D_{\mathcal{E}}) = \sum_\alpha \frac{y^\alpha}{\alpha!} \left(\frac{\partial D^\mu_{\mathcal{E}}}{\partial x^\alpha} \frac{\partial}{\partial y^\mu} + \frac{\partial D^0_{\mathcal{E}}}{\partial x^\alpha}\right).^2$$

Since y^μ has filtration degree 1, we have

$$\mathrm{id}_{\widehat{\Omega}_{\mathcal{M}}} \otimes j^\infty(D_{\mathcal{E}}) = \mathrm{id}_{\widehat{\Omega}_{\mathcal{M}}} \otimes \left(Q^\mu_{\mathcal{M}} \frac{\partial}{\partial y^\mu} + y^\nu \frac{\partial Q^\mu_{\mathcal{M}}}{\partial x^\nu} \frac{\partial}{\partial y^\mu}\right) \otimes \mathrm{id}_{\mathcal{E}} + \mathrm{id}_{\widehat{\Omega}_{\mathcal{M}}} \otimes \operatorname{Gr} D^0_{\mathcal{E}} + \cdots,$$

where $\cdots$ denotes extra terms of filtration degrees ≥ 1.

To sum up, we have

$$\begin{aligned}
\operatorname{Gr}\left(L^{\nabla^G}_{Q_{\mathcal{M}}} + 1 \otimes j^\infty(D_{\mathcal{E}})\right) &= \left(Q^\mu_{\mathcal{M}} \frac{\partial}{\partial x^\mu} - dx^\nu \frac{\partial Q^\mu_{\mathcal{M}}}{\partial x^\nu} \frac{\partial}{\partial dx^\mu} + y^\nu \frac{\partial Q^\mu_{\mathcal{M}}}{\partial x^\nu} \frac{\partial}{\partial y^\mu}\right) \otimes \mathrm{id}_{\operatorname{Gr}\mathcal{E}} \\
&\quad + \mathrm{id}_{\mathcal{K}_{\mathcal{M}}} \otimes \left(Q^\mu_{\mathcal{M}} \frac{\partial}{\partial x^\mu} + \operatorname{Gr} D^0_{\mathcal{E}}\right) \\
&= L_{Q_{\mathcal{M}}} \otimes \mathrm{id}_{\operatorname{Gr}\mathcal{E}} + \mathrm{id}_{\mathcal{K}_{\mathcal{M}}} \otimes \operatorname{Gr} D_{\mathcal{E}},
\end{aligned}$$

which completes the proof. □

Let $j^\infty : \mathcal{E} \to \mathcal{J}^\infty_{\mathcal{E}}$ be the infinite jet prolongation of $\mathcal{E}$. By a slight abuse of notation, we write

$$j^\infty : \mathcal{E} \longrightarrow \mathfrak{J}^\infty_{\mathcal{E}}$$

for the composition $\mathcal{E} \xrightarrow{j^\infty} \mathcal{J}^\infty_{\mathcal{E}} \hookrightarrow \mathfrak{J}^\infty_{\mathcal{E}}$.

[2] See Appendix A for the multi-index notation.

Proposition 5.16. *The $\mathbb{R}$-linear map*

$$j_M^\infty : \mathcal{E}(M) \longrightarrow \mathfrak{J}_\mathcal{E}^\infty(M)$$

is a filtered quasi-isomorphism; that is,

$$\operatorname{Gr} j_M^\infty : \operatorname{Gr}\mathcal{E}(M) \longrightarrow \operatorname{Gr}\mathfrak{J}_\mathcal{E}^\infty(M) \cong \mathcal{K}_\mathcal{M}(M) \otimes_{\mathcal{O}_\mathcal{M}(M)} \operatorname{Gr}\mathcal{E}(M)$$

is a quasi-isomorphism.

Proof. $\operatorname{Gr} j^\infty$ is given by the canonical inclusion of $\operatorname{Gr}\mathcal{E}$. It suffices to prove that the complex

$$(\operatorname{Gr}\mathfrak{J}_\mathcal{E}^\infty(M) \cong \mathcal{K}_\mathcal{M}(M) \otimes_{\mathcal{O}_\mathcal{M}(M)} \operatorname{Gr}\mathcal{E}(M), \operatorname{Gr}\mathcal{D}_{\mathfrak{J}_\mathcal{E}^\infty})$$

is quasi-isomorphic to $(\operatorname{Gr}\mathcal{E}(M), \operatorname{Gr} D_\mathcal{E})$. This complex can be regarded as a double complex with horizontal and vertical differentials

$$\delta_K \otimes \mathrm{id}_{\operatorname{Gr}\mathcal{E}} \qquad L_{Q_\mathcal{M}} \otimes \mathrm{id}_{\operatorname{Gr}\mathcal{E}} + \mathrm{id}_{\mathcal{K}_\mathcal{M}} \otimes \operatorname{Gr} D_\mathcal{E}.$$

Since the cohomology of δ_K is $\mathcal{O}_\mathcal{M}(M)$ and $\operatorname{Gr}^\bullet\mathcal{E}(M)$ is finitely generated and projective over $\mathcal{O}_\mathcal{M}(M)$, we have

$$H_{\delta_K \otimes \mathrm{id}_{\operatorname{Gr}\mathcal{E}}} \cong \operatorname{Gr}\mathcal{E}(M).$$

Now observe that

$$(L_{Q_\mathcal{M}} \otimes \mathrm{id}_{\operatorname{Gr}\mathcal{E}} + \mathrm{id}_{\mathcal{K}_\mathcal{M}} \otimes \operatorname{Gr} D_\mathcal{E})|_{\operatorname{Gr}\mathcal{E}(M)} = \operatorname{Gr} D_\mathcal{E}.$$

It follows that the E_1 page of the spectral sequence associated to this double complex is isomorphic to

$$(\operatorname{Gr}\mathcal{E}(M), \operatorname{Gr} D_\mathcal{E}),$$

which completes the proof. □

5.2. **The jet space functor.** Let $\mathcal{M}$ be a dg manifold. Let $\mathcal{J}_\mathcal{M}^\infty$ be the sheaf of infinite jets on $\mathcal{M}$, i.e. the infinite jet bundle of the structure sheaf $\mathcal{O}_\mathcal{M}$. Applying the construction in the previous subsection to $\mathcal{O}_\mathcal{M}$, regarded as a quasi-dg vector bundle, we obtain the quasi-dg vector bundle

$$\mathfrak{J}_\mathcal{M}^\infty := \widehat{\Omega}_\mathcal{M} \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}\, \mathcal{J}_\mathcal{M}^\infty,$$

equipped with the total differential

$$D_{\mathfrak{J}_\mathcal{M}^\infty} = d_{\nabla^G} + L_{Q_\mathcal{M}},$$

where $L_{Q_\mathcal{M}}$ denotes the (total) Lie derivative on $\mathfrak{J}_\mathcal{M}^\infty$ along $Q_\mathcal{M}$

$$L_{Q_\mathcal{M}} = L_{Q_\mathcal{M}}^h + L_{Q_\mathcal{M}}^v, \tag{5.1}$$

with the horizontal and vertical Lie derivatives given by the formulas

$$L_{Q_\mathcal{M}}^h = L_{Q_\mathcal{M}}^{\nabla^G}, \qquad L_{Q_\mathcal{M}}^v = \mathrm{id}_{\widehat{\Omega}_\mathcal{M}} \otimes j^\infty(Q_\mathcal{M}),$$

respectively.

Recall from Appendix A that $\mathcal{J}_\mathcal{M}^\infty$ is a filtered graded commutative algebra. It follows that $\mathfrak{J}_\mathcal{M}^\infty$ is also a filtered graded commutative algebra. It is straightforward to verify that $D_{\mathfrak{J}_\mathcal{M}^\infty}$ is a derivation of $\mathfrak{J}_\mathcal{M}^\infty$. Therefore, $\mathfrak{J}_\mathcal{M}^\infty$ is a filtered cdga.

Proposition 5.17. *The pair $\mathcal{M}_{jet} := (M, \mathfrak{J}^\infty_{\mathcal{M}})$ defines a fibrant quasi-dg manifold over $\mathcal{M}$, called the* jet space *of $\mathcal{M}$. Moreover, the morphism*

$$\mathcal{M}_{jet} = (M, \mathfrak{J}^\infty_{\mathcal{M}}) \xrightarrow{(\mathrm{id}_M, j^\infty)} \mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$$

is a weak equivalence in **QdgM**.

Note that $(\mathrm{id}_M, j^\infty)$ is not base-preserving.

Proof. We need to show that $\mathcal{M}_{jet}$ is well-defined. We have

$$\mathrm{Gr}^0(\mathfrak{J}^\infty_{\mathcal{M}}) \cong \Omega^0_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Gr}^0(\mathcal{J}^\infty_{\mathcal{M}}) \cong \mathcal{O}_{\mathcal{M}}.$$

Both the splitting $\mathcal{O}_{\mathcal{M}} \to \mathfrak{J}^\infty_{\mathcal{M}}$ and the morphism $H^\sharp : \widehat{\Omega}_{\mathcal{M}} \to \mathfrak{J}^\infty_{\mathcal{M}}$ are given by the canonical inclusions. $\mathrm{Gr}\, H^\sharp$ is obviously injective.

The morphism $(\mathrm{id}_M, j^\infty)$ is well-defined by Lemma A.8. It is a weak equivalence by Proposition 5.16. □

Remark 5.18. *Using a formal exponential map*

$$\mathrm{fem} : \mathcal{J}^\infty_{\mathcal{M}} \longrightarrow \widehat{\mathrm{Sym}}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}^\vee_{\mathcal{M}})$$

of $\mathcal{M}$ to identify $\mathfrak{J}^\infty_{\mathcal{M}}$ with $\widehat{\mathrm{Sym}}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}[1]^\vee \oplus \mathcal{T}^\vee_{\mathcal{M}})$, the jet space $\mathcal{M}_{jet}$ is then identified with the Fedosov resolution of $\mathcal{M}$ as defined in [LSX25].

Now let $\mathcal{C}$ be a quasi-dg manifold. Applying again the construction in the previous subsection to $\mathcal{O}_{\mathcal{C}}$, regarded as a quasi-dg vector bundle, we obtain a quasi-dg vector bundle

$$\mathfrak{J}^\infty_{\mathcal{O}_{\mathcal{C}}} = \widehat{\Omega}_{\mathcal{M}} \,\widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}}\, \mathcal{J}^\infty_{\mathcal{O}_{\mathcal{C}}},$$

equipped with the total differential

$$D_{\mathfrak{J}^\infty_{\mathcal{O}_{\mathcal{C}}}} = d_{\nabla^G} + L^{\nabla^G}_{Q_{\mathcal{M}}} + \mathrm{id}_{\widehat{\Omega}_{\mathcal{M}}} \otimes j^\infty(Q_{\mathcal{C}}).$$

Similarly, one can show that

Proposition 5.19. *The pair $\mathcal{C}_{jet} := (M, \mathfrak{J}^\infty_{\mathcal{O}_{\mathcal{C}}})$ defines a fibrant quasi-dg manifold over $\mathcal{M}$, called the* jet space *of $\mathcal{C}$. Moreover, the morphism*

$$\mathcal{C}_{jet} = (M, \mathfrak{J}^\infty_{\mathcal{O}_{\mathcal{C}}}) \xrightarrow{(\mathrm{id}_M, j^\infty)} \mathcal{C} = (M, \mathcal{O}_{\mathcal{C}})$$

is a weak equivalence in **QdgM**.

Let $F = (f, F^\sharp) : \mathcal{C}_1 = (M_1, \mathcal{O}_{\mathcal{C}_1}) \to \mathcal{C}_2 = (M_2, \mathcal{O}_{\mathcal{C}_2})$ be a morphism of quasi-dg manifolds. Using Propositions A.16, B.17, and B.18 , one can straightforwardly verify that

Proposition 5.20. *The following diagram of filtered cdgas*

$$\begin{array}{ccc}
\mathfrak{J}^\infty_{\mathcal{O}_{\mathcal{C}_2}} = \widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}_2} \,\widehat{\otimes}_{\mathrm{Gr}^0\mathcal{O}_{\mathcal{C}_2}}\, \mathcal{J}^\infty_{\mathcal{O}_{\mathcal{C}_2}} & \xrightarrow{(\mathrm{Gr}^0 F)^\sharp_{dR} \otimes j^\infty(F^\sharp)} & f_*\mathfrak{J}^\infty_{\mathcal{O}_{\mathcal{C}_1}} = f_*\left(\widehat{\Omega}_{\mathrm{Gr}^0\mathcal{C}_1} \,\widehat{\otimes}_{\mathrm{Gr}^0\mathcal{O}_{\mathcal{C}_1}}\, \mathcal{J}^\infty_{\mathcal{O}_{\mathcal{C}_1}}\right) \\
{\scriptstyle j^\infty}\uparrow & & \uparrow{\scriptstyle f_* j^\infty} \\
\mathcal{O}_{\mathcal{C}_2} & \xrightarrow{\qquad F^\sharp \qquad} & f_*\mathcal{O}_{\mathcal{C}_1}.
\end{array}$$

commutes, where $j^\infty(F^\sharp)$ is defined in Proposition A.16.

Consider the assignment

$$\mathbf{J}^\infty : \mathbf{QdgM} \longrightarrow \mathbf{QdgM}_{\mathrm{fib}}$$

defined on objects $\mathcal{C}$ by

$$\mathbf{J}^\infty(\mathcal{C}) = \mathcal{C}_{jet},$$

and on morphisms $F = (f, F^\sharp) : \mathcal{C}_1 = (M_1, \mathcal{O}_{\mathcal{C}_1}) \to \mathcal{C}_2 = (M_2, \mathcal{O}_{\mathcal{C}_2})$ by

$$\mathbf{J}^\infty(F) := \big(f,\ (\mathrm{Gr}^0 F)^\sharp_{dR} \otimes j^\infty(F^\sharp)\big).$$

Our previous results fit together to yield the following theorem.

Theorem 5.21. $\mathbf{J}^\infty$ *is a well-defined functor, called the* jet space functor. *Moreover, there is a natural transformation*

$$j^\infty : \mathbf{J}^\infty \Rightarrow \mathrm{id}_{\mathbf{QdgM}},$$

which is a component-wise weak equivalence.[3]

It then follows from the 2-out-of-3 property of weak equivalences that the jet space functor $\mathbf{J}^\infty$ detects weak equivalences.

Corollary 5.22. *The following composition*

$$\mathbf{Fib} : \mathbf{L}_\infty\mathbf{Algd} \xrightarrow{\mathbf{CE_{Algd}}} \mathbf{QdgM}^{\mathrm{bp}} \xrightarrow{\mathbf{J}^\infty} \mathbf{QdgM}^{\mathrm{bp}}_{\mathrm{fib}} \xrightarrow{\mathbf{CE}^{-1}_{\mathbf{Sp}}} \mathbf{L}_\infty\mathbf{Sp}$$

detects weak equivalences.

Proof. This follows from Theorems 3.19, 4.12, and 5.21. □

Appendix A. Differential operators and jets

Let $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ be a graded manifold. In this appendix, we collect and prove some basic facts about differential operators and jets on $\mathcal{M}$.

Definition A.1. *The* sheaf of differential operators $\mathcal{D}_\mathcal{M}$ *on* $\mathcal{M}$ *is defined by assigning to each open set* $U \subset M$ *the subalgebra of* $\mathrm{End}_\mathbb{R}(\mathcal{O}_\mathcal{M}(U))$ *generated by left multiplications by* $\mathcal{O}_\mathcal{M}(U)$ *and by vector fields* $\mathcal{T}_\mathcal{M}(U)$.

$\mathcal{D}_\mathcal{M}$ is naturally a graded $(\mathcal{O}_\mathcal{M}, \mathcal{O}_\mathcal{M})$-algebra. It carries a compatible descending filtration

$$\mathcal{D}_\mathcal{M} \cdots \supset F^{-1}\mathcal{D}_\mathcal{M} \supset F^0\mathcal{D}_\mathcal{M} \supset F^1\mathcal{D}_\mathcal{M} = 0,$$

given by $F^0\mathcal{D}_\mathcal{M} = \mathcal{O}_\mathcal{M}$, and inductively,

$$F^{-k}\mathcal{D}_\mathcal{M} = \big\{D \in \mathcal{D}_\mathcal{M} \,\big|\, [D, f] \in F^{-k+1}\mathcal{D}_\mathcal{M} \text{ for all } f \in \mathcal{O}_\mathcal{M}\big\}$$

for $k > 0$, where $[D, f] = D \circ f - (-1)^{|f||D|} f \circ D$. In other words, $F^{-k}\mathcal{D}_\mathcal{M}$ consists of differential operators *of order at most* k. There is a canonical isomorphism of graded $\mathcal{O}_\mathcal{M}$-modules

$$\mathrm{Gr}^{-k}\mathcal{D}_\mathcal{M} \cong \mathrm{Sym}^k_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}).$$

This identification induces a map

$$\sigma_k : F^{-k}\mathcal{D}_\mathcal{M} \longrightarrow \mathrm{Sym}^k_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}),$$

called the *symbol map*. For $D \in F^{-k}\mathcal{D}_\mathcal{M}$, if $\sigma_k(D) \neq 0$, then D is said to be *of order* k. The element $\sigma_k(D) \in \mathrm{Sym}^k_{\mathcal{O}_\mathcal{M}}\mathcal{T}_\mathcal{M}$ is called the *principal symbol* of D.

[3]Here, we regard $\mathbf{J}^\infty$ as a functor from **QdgM** to **QdgM**.

$\mathcal{D}_\mathcal{M}$ is also a graded cocommutative $(\mathcal{O}_\mathcal{M}, \mathcal{O}_\mathcal{M})$-coalgebra. The counit map $\epsilon : \mathcal{D}_\mathcal{M} \to \mathcal{O}_\mathcal{M}$ is defined by

$$\epsilon(D) := D(1),$$

and the coproduct map $\Delta : \mathcal{D}_\mathcal{M} \to \mathcal{D}_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{D}_\mathcal{M}$ is defined by

$$\Delta(D)(f \otimes_\mathbb{R} g) := D(fg)$$

where we identify $\mathcal{D}_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{D}_\mathcal{M}$ with the set of bi-differential operators. Equivalently, Δ is characterized by the following formulas [Xu01]

$$\Delta(1) = 1 \otimes_{\mathcal{O}_\mathcal{M}} 1, \tag{A.1}$$

$$\Delta(X) = 1 \otimes_{\mathcal{O}_\mathcal{M}} X + X \otimes_{\mathcal{O}_\mathcal{M}} 1, \quad \text{for all } X \in \mathcal{T}_\mathcal{M}, \tag{A.2}$$

$$\Delta(D_1 \circ D_2) = \Delta(D_1) \circ \Delta(D_2), \quad \text{for all } D_1, D_2 \in \mathcal{D}_\mathcal{M}. \tag{A.3}$$

Moreover, ϵ and Δ are compatible with the filtration on $\mathcal{D}_\mathcal{M}$:

$$\epsilon(F^k\mathcal{D}_\mathcal{M}) \subset F^k\mathcal{O}_\mathcal{M}, \qquad \Delta(F^k\mathcal{D}_\mathcal{M}) \subset \sum_{i+j=k} F^i\mathcal{D}_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} F^j\mathcal{D}_\mathcal{M},$$

where $\mathcal{O}_\mathcal{M}$ is equipped with the trivial filtration. Indeed, $F^\bullet\mathcal{D}_\mathcal{M}$ is precisely the $\mathcal{I}$-adic filtration determined by the coideal $\mathcal{I} := F^0\mathcal{D}_\mathcal{M}$:

$$F^{-p}\mathcal{D}_\mathcal{M} = \left\{ D \in \mathcal{D}_\mathcal{M} \;\middle|\; \Delta^{(p)}(D) \in \sum_{k=0}^{p} \mathcal{D}_\mathcal{M}^{\otimes k} \otimes \mathcal{I} \otimes \mathcal{D}_\mathcal{M}^{\otimes(p-k)} \right\},$$

for all $p \in \mathbb{N}$, where $\Delta^{(p)}$ denotes the p-fold iteration of Δ.

Lemma A.2. *There is an one-to-one correspondence between splittings*

$$\mathrm{pbw}^1 : \mathrm{Sym}^2_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}) \longrightarrow F^{-2}\mathcal{D}_\mathcal{M}$$

of the short exact sequence of graded (left) $\mathcal{O}_\mathcal{M}$-modules

$$0 \longrightarrow F^{-1}\mathcal{D}_\mathcal{M} \longrightarrow F^{-2}\mathcal{D}_\mathcal{M} \longrightarrow \mathrm{Sym}^2_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}) \longrightarrow 0,$$

and torsion-free affine connections ∇ on $\mathcal{M}$.

Proof. The correspondence is given explicitly by the formula

$$\mathrm{pbw}^1(X \odot Y) = \frac{1}{2}\left((X \circ Y - \nabla_X Y) + (-1)^{|X||Y|}(Y \circ X - \nabla_Y X)\right), \tag{A.4}$$

where $X \odot Y$ denotes the symmetric product between X and Y. □

The formula (A.4) extends inductively to maps

$$\mathrm{pbw}^n : \mathrm{Sym}^{n+1}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}) \longrightarrow F^{-n-1}\mathcal{D}_\mathcal{M}.$$

The map $\mathrm{pbw}^0 : \mathcal{T}_\mathcal{M} \to F^{-1}\mathcal{D}_\mathcal{M} \cong \mathcal{O}_\mathcal{M} \oplus \mathcal{T}_\mathcal{M}$ is the canonical splitting. For $n \geq 1$, we define

$$\mathrm{pbw}^n(X_0 \odot X_1 \odot \cdots \odot X_n) = \frac{1}{n+1}\sum_{k=0}^{n} \epsilon_k \left(X_k \circ \mathrm{pbw}^{n-1}(X^{\{k\}}) - \mathrm{pbw}^{n-1}\big(\nabla_{X_k}(X^{\{k\}})\big)\right),$$

where ∇ is a torsion-free affine connection on $\mathcal{M}$, $X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$, and $\epsilon_k = (-1)^{|X_k|(|X_0|+\cdots+|X_{k-1}|)}$.

Proposition A.3 (Theorem 3.3 in [LS19]). *The map*

$$\mathrm{pbw} : \mathrm{Sym}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}) \to \mathcal{D}_\mathcal{M},$$

induced by a torsion-free affine connection ∇ on $\mathcal{M}$, is a filtered graded cocommutative $\mathcal{O}_\mathcal{M}$-coalgebra isomorphism.

The converse of the above statement does not hold.

Definition A.4. *The* sheaf of k-jets $\mathcal{J}^k_\mathcal{M}$ *on $\mathcal{M}$ is defined as*

$$\mathcal{J}^k_\mathcal{M} := F^{-k}\mathcal{D}^\vee_\mathcal{M} = \mathrm{Hom}_{\mathcal{O}_\mathcal{M}}(F^{-k}\mathcal{D}_\mathcal{M}, \mathcal{O}_\mathcal{M}),$$

where $\mathrm{Hom}_{\mathcal{O}_\mathcal{M}}$ *denotes the internal Hom in the category of graded left $\mathcal{O}_\mathcal{M}$-modules.*

$\{\mathcal{J}^k_\mathcal{M}\}_{k\geq 0}$ *together with the canonical epimorphisms $\{\mathcal{J}^j_\mathcal{M} \to \mathcal{J}^i_\mathcal{M}\}_{j\geq i\geq 0}$ forms an inverse system. The* sheaf of ∞-jets $\mathcal{J}^\infty_\mathcal{M}$ *on $\mathcal{M}$ is defined as the inverse limit*

$$\mathcal{J}^\infty_\mathcal{M} := \varprojlim \mathcal{J}^k_\mathcal{M} = \mathrm{Hom}_{\mathcal{O}_\mathcal{M}}(\varinjlim F^{-k}\mathcal{D}_\mathcal{M}, \mathcal{O}_\mathcal{M}) = \mathcal{D}^\vee_\mathcal{M}.$$

By definition, $\mathcal{J}^\infty_\mathcal{M}$ carries an $(\mathcal{O}_\mathcal{M}, \mathcal{O}_\mathcal{M})$-bimodule structure induced by the one on $\mathcal{D}_\mathcal{M}$. It also carries a descending filtration

$$\mathcal{J}^\infty_\mathcal{M} = F^0\mathcal{J}^\infty_\mathcal{M} \supset F^1\mathcal{J}^\infty_\mathcal{M} \supset \cdots,$$

induced by the filtration on $\mathcal{D}_\mathcal{M}$. More precisely,

$$F^k\mathcal{J}^\infty_\mathcal{M} = \{J \in \mathcal{J}^\infty_\mathcal{M} : J(F^l\mathcal{D}_\mathcal{M}) \subset F^{k+l}\mathcal{O}_\mathcal{M}\}.$$

There are canonical graded $\mathcal{O}_\mathcal{M}$-module isomorphisms:

$$\mathrm{Gr}^k\mathcal{J}^\infty_\mathcal{M} \cong \mathrm{Sym}^k_{\mathcal{O}_\mathcal{M}}(\mathcal{T}^\vee_\mathcal{M}).$$

Moreover, $\mathcal{J}^\infty_\mathcal{M}$ is a filtered graded commutative $(\mathcal{O}_\mathcal{M}, \mathcal{O}_\mathcal{M})$-algebra, whose product map $\mathcal{J}^\infty_\mathcal{M} \widehat{\otimes}_{\mathcal{O}_\mathcal{M}} \mathcal{J}^\infty_\mathcal{M} \to \mathcal{J}^\infty_\mathcal{M}$ is defined as the dual of the coproduct Δ on $\mathcal{D}_\mathcal{M}$

$$(J_1 \cdot J_2)(D) = (J_1 \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}\, J_2)(\Delta D),$$

where $\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}$ denotes the completed tensor product over $\mathcal{O}_\mathcal{M}$, and we use the identification

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{M}}(\mathcal{D}_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{D}_\mathcal{M}, \mathcal{O}_\mathcal{M}) = \mathcal{J}^\infty_\mathcal{M} \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}\, \mathcal{J}^\infty_\mathcal{M}.$$

Likewise, $F^\bullet\mathcal{J}_\mathcal{M}$ is precisely the $\mathcal{I}$-adic filtration determined by the ideal $\mathcal{I} := F^1\mathcal{J}_\mathcal{M}$:

$$F^k\mathcal{J}_\mathcal{M} = (\mathcal{I})^p,$$

for all $p \in \mathbb{N}$.

Definition A.5. *A* formal exponential map *on $\mathcal{M}$ is a filtered graded commutative $\mathcal{O}_\mathcal{M}$-algebra isomorphism*

$$\mathrm{fem} : \mathcal{J}^\infty_\mathcal{M} \xrightarrow{\cong} \widehat{\mathrm{Sym}}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}^\vee_\mathcal{M}).$$

Each formal exponential map on $\mathcal{M}$ corresponds to a filtered graded cocommutative $\mathcal{O}_\mathcal{M}$-coalgebra isomorphism

$$\mathrm{Sym}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}) \xrightarrow{\cong} D_\mathcal{M},$$

and vice versa. In particular, each torsion free affine connection on $\mathcal{M}$ determines a formal exponential map on $\mathcal{M}$.

Remark A.6. *A formal exponential map on an ordinary manifold M can be also defined as an equivalence class $[\phi]$ of generalized exponential maps* [BCM12, CMW19]. *Recall that a* generalized exponential map *on M is a smooth map*

$$\phi : U \to M, \quad (x, y \in U_x) \mapsto \phi_x(y),$$

where U is an open neighborhood of the zero section in TM, such that for all $x \in M$:

$$\phi_x(0) = x, \quad d_y\phi_x|_{y=0} = \mathrm{id}.$$

Two generalized exponential maps are said to be equivalent if, at each point of M, all their partial derivatives in the vertical directions agree at the zero section.

Given such an equivalence class $[\phi]$, one obtains a morphism

$$\mathrm{fem} : J^\infty_M \to \widehat{\mathrm{Sym}}_{C^\infty_M}(T^\vee_M)$$

by setting

$$\mathrm{fem}([f])(X_1, \dots, X_n)(x) = \frac{d}{dt_1}\bigg|_{t_1=0} \cdots \frac{d}{dt_n}\bigg|_{t_n=0} f_x(\phi_x(t_1X_1(x) + \cdots + t_nX_n(x))),$$

where $[f]$ is a section of J^∞_M, and f_x is a local smooth function representing $[f]$ at $x \in M$.[4] *It is straightforward to check that* fem *is a filtered graded commutative algebra isomorphism over C^∞_M.*

Definition A.7. *The k-*th (resp. infinite) jet prolongation *is the $\mathbb{R}$-linear morphism*

$$j^k : \mathcal{O}_\mathcal{M} \to \mathcal{J}^k_\mathcal{M} \quad \textit{(resp. } j^\infty : \mathcal{O}_\mathcal{M} \to \mathcal{J}^\infty_\mathcal{M}\textit{)},$$

defined by the evaluation map

$$f \mapsto (D \mapsto (-1)^{|D||f|}D(f)),$$

where D ranges over differential operators of order $\leq k$ (resp. arbitrary order).

Lemma A.8. *j^∞ is a filtered graded commutative algebra morphism. Moreover, it induces the canonical right $\mathcal{O}_\mathcal{M}$-module structure of $J^\infty_\mathcal{M}$, i.e.*

$$(Jj^\infty(f))(D) = (-1)^{|D||f|}J(D \circ f), \tag{A.5}$$

where J is an ∞-jet, D is a differential operator, and f is a function on $\mathcal{M}$.

Proof. Let g be a function and X a vector field over $\mathcal{M}$. Using (A.1), we obtain

$$(Jj^\infty(f))(g) = J \otimes j^\infty(f)(\Delta g) = J(1)j^\infty(f)(g) = J(1)fg = (-1)^{|g||f|}J(gf).$$

Now assume that D satisfies (A.5) for all ∞-jets and functions. Using (A.2) and (A.3), we compute

$$\begin{aligned}
(Jj^\infty(f))(D \circ X) &= J \otimes_\mathbb{R} j^\infty(f)(\Delta D \circ (1 \otimes X + X \otimes 1)) \\
&= (-1)^{|X|(|D|+|f|)}\left((Jj^\infty(X(f)))(D) + (J_Xj^\infty(f))(D)\right) \\
&= (-1)^{(|D|+|X|)|f|}(J(D \circ X(f) + (-1)^{|X||f|}D \circ f \circ X) \\
&= (-1)^{(|D|+|X|)|f|}J(D \circ X \circ f),
\end{aligned}$$

[4] Such an f exists by Borel's lemma.

where $J_X(\cdot) := J(\cdot \circ X) \in \mathcal{J}^\infty_{\mathcal{M}}$. By induction, (A.5) holds for all differential operators. It follows that

$$(j^\infty(g)j^\infty(f))(D) = (-1)^{|D||f|} j^\infty(g)(D \circ f) = (-1)^{|D||f|+|g|(|D|+|f|)} D(fg) = j^\infty(gf)(D).$$

This completes the proof. □

It is useful to give a local description of j^∞. Let $\{x^\mu\}$ be local coordinates on $\mathcal{M}$, and let $\{y^\nu\}$ denote the corresponding fiber coordinates on $\mathcal{T}^\vee_{\mathcal{M}}$. Then, for a local function $f(x)$,

$$j^\infty(f)(x,y) = \sum_\alpha \frac{1}{\alpha!} \frac{\partial^\alpha f(x)}{\partial x^\alpha}\, y^\alpha, \tag{A.6}$$

where $\alpha = (k_1, \dots, k_n; k_{n+1}, \dots, k_{n+m}) \in \mathbb{N}^n \times \{0,1\}^m$ is a multi-index, with $n|m$ the superdimension of $\mathcal{M}$. (We assume that the coordinates $x^1, \dots, x^n$ are even, while $x^{n+1}, \dots, x^{n+m}$ are odd.) We set $\alpha! = k_1! \cdots k_n!$,

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{k_{n+m}}}{(\partial x^{n+m})^{k_{n+m}}} \circ \cdots \circ \frac{\partial^{k_1}}{(\partial x^1)^{k_1}}, \qquad y^\alpha = (y^1)^{k_1} \cdots (y^{n+m})^{k_{n+m}}.$$

In other words, under the local identification of $J^\infty_{\mathcal{M}}$ with $\widehat{\mathrm{Sym}}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}^\vee_{\mathcal{M}})$ induced by a trivial local affine connection, $j^\infty(f)$ is the Taylor expansion of f.

Lemma A.9. *The graded $\mathcal{O}_{\mathcal{M}}$-module $J^\infty_{\mathcal{M}}$ is generated by $j^\infty(\mathcal{O}_{\mathcal{M}})$ over $\mathcal{O}_{\mathcal{M}}$.*

Proof. Since everything is smooth, it suffices to verify the statement locally. By (A.6), all monomials y^α lie in the local $\mathcal{O}_{\mathcal{M}}$-span of $j^\infty(\mathcal{O}_{\mathcal{M}})$, since

$$j^\infty(x^\alpha) = y^\alpha + \cdots .$$

This completes the proof. □

Let $f : \mathcal{M} \to \mathcal{N}$ be a morphism of graded manifold.

Proposition A.10. *There exists a unique filtered graded commutative algebra morphism*

$$j^\infty(f^\sharp) : \mathcal{J}^\infty_{\mathcal{N}} \longrightarrow f_* \mathcal{J}^\infty_{\mathcal{M}},$$

such that the following diagram

$$\begin{array}{ccc} \mathcal{J}^\infty_{\mathcal{N}} & \xrightarrow{\;j^\infty(f^\sharp)\;} & f_*\mathcal{J}^\infty_{\mathcal{M}} \\ {\scriptstyle j^\infty}\uparrow & & \uparrow{\scriptstyle f_* j^\infty} \\ \mathcal{O}_{\mathcal{N}} & \xrightarrow{\;f^\sharp\;} & f_*\mathcal{O}_{\mathcal{M}} \end{array}$$

commutes.

Proof. Since the image $j^\infty(\mathcal{O}_{\mathcal{N}})$ generates $J^\infty_{\mathcal{N}}$, we define $j^\infty(f^\sharp)$ simply by setting

$$j^\infty(f^\sharp)(g) = f^\sharp(g), \qquad j^\infty(f^\sharp)\big(j^\infty(g)\big) = j^\infty\big(f^\sharp(g)\big),$$

for all $g \in \mathcal{O}_{\mathcal{N}}$. To check well-definedness, it suffices to show that

$$\sum_{i\in I} f^\sharp(g_i)\, j^\infty\big(f^\sharp(h_i)\big) = 0 \quad \text{whenever} \quad \sum_{i\in I} g_i\, j^\infty(h_i) = 0.$$

This follows from the compatibility of $f^\sharp$ with derivations. Indeed, locally,

$$f^\sharp(g)\,\frac{\partial}{\partial x^\mu_{\mathcal{M}}} f^\sharp(h_i) = f^\sharp(g)\,\frac{\partial f^\sharp(x^\nu_{\mathcal{N}})}{\partial x^\mu_{\mathcal{M}}}\, f^\sharp\left(\frac{\partial h_i}{\partial x^\nu_{\mathcal{N}}}\right) = \pm\frac{\partial f^\sharp(x^\nu_{\mathcal{N}})}{\partial x^\mu_{\mathcal{M}}}\, f^\sharp\left(g\,\frac{\partial h_i}{\partial x^\nu_{\mathcal{N}}}\right),$$

which implies the claim. Here, $x^\mu_{\mathcal{M}}$ and $x^\nu_{\mathcal{N}}$ are coordinates on $\mathcal{M}$ and $\mathcal{N}$, respectively. □

Remark A.11. *Let $y^\mu_{\mathcal{M}}$ and $y^\nu_{\mathcal{N}}$ denote the fiber coordinates on $\mathcal{T}^\vee_{\mathcal{M}}$ and $\mathcal{T}^\vee_{\mathcal{N}}$, respectively. By Proposition A.10, one obtains the local expression*

$$j^\infty(f^\sharp)(y^\mu_{\mathcal{N}}) = j^\infty\big(f^\sharp(x^\mu_{\mathcal{N}})\big) - f^\sharp(x^\mu_{\mathcal{N}}) = \sum_{|\alpha|>0} \frac{y^\alpha_{\mathcal{M}}}{\alpha!}\frac{\partial^\alpha f^\sharp(x^\mu_{\mathcal{N}})}{\partial x^\alpha_{\mathcal{M}}}.$$

Let $\mathcal{E}$ be a filtered graded vector bundle on $\mathcal{M}$.

Definition A.12. *The* sheaf of differential operators *on $\mathcal{E}$ is defined as*

$$\mathcal{D}_{\mathcal{E}} := \mathcal{E}\,\widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}}\,\mathcal{D}_{\mathcal{M}}\,\widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}}\,\mathcal{E}^\vee.$$

Let $k \in \mathbb{N}$ and $p \in \mathbb{Z}$. Sections of

$$F^{-k}_O\mathcal{D}_{\mathcal{E}} := \mathcal{E}\,\widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}}\,F^{-k}\mathcal{D}_{\mathcal{M}}\,\widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}}\,\mathcal{E}^\vee$$

are differential operators *of order at most k* on $\mathcal{E}$, while sections of

$$F^p_I\mathcal{D}_{\mathcal{E}} := \{D_{\mathcal{E}} \in \mathcal{D}_{\mathcal{E}} : D_{\mathcal{E}}(F^\bullet\mathcal{E})^5 \subset F^{\bullet+p}\mathcal{E}\}$$

are differential operators *of filtration degree at most p* on $\mathcal{E}$. In particular, $F^0_I\mathcal{D}_{\mathcal{E}}$ consists of filtration-preserving differential operators.

$\mathcal{D}_{\mathcal{E}}$ is a filtered graded $(\mathcal{O}_{\mathcal{M}}, \mathcal{O}_{\mathcal{M}})$-algebra, whose product is given by composition of differential operators. Moreover, the default filtration $F^\bullet\mathcal{D}_{\mathcal{E}}$ on $\mathcal{D}_{\mathcal{E}}$ is the same as the convolution of the two filtrations $F^\bullet_O\mathcal{D}_{\mathcal{E}}$ and $F^\bullet_I\mathcal{D}_{\mathcal{E}}$:

$$F^p\mathcal{D}_{\mathcal{E}} := \sum_{i+j=p} F^i_I\mathcal{D}_{\mathcal{E}} \circ F^j_O\mathcal{D}_{\mathcal{E}}.$$

There is a canonical isomorphism of graded $\mathcal{O}_{\mathcal{M}}$-modules

$$\mathrm{Gr}^p\mathcal{D}_{\mathcal{E}} \cong \bigoplus_{q\in\mathbb{N}} \mathrm{Sym}^q_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}) \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Gr}^{p+q}\mathrm{End}_{\mathcal{E}},$$

where $\mathrm{End}_{\mathcal{E}} := \widehat{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{E},\mathcal{E})$ and $\widehat{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{M}}}$ denotes the completed internal Hom in the category of filtered graded left $\mathcal{O}_{\mathcal{M}}$-modules. This identification induces a map

$$\sigma^p_k : F^{-k}_O\mathcal{D}_{\mathcal{E}} \cap F^p_I\mathcal{D}_{\mathcal{E}} \subset F^{p-k}\mathcal{D}_{\mathcal{E}} \longrightarrow \mathrm{Sym}^k_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}) \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Gr}^p\mathrm{End}_{\mathcal{E}},$$

called the *symbol map*. For $D_{\mathcal{E}} \in F^{-k}_O\mathcal{D}_{\mathcal{E}} \cap F^p_I\mathcal{D}_{\mathcal{E}}$, if $\sigma^p_k(D_{\mathcal{E}}) \neq 0$, then $D_{\mathcal{E}}$ is said to be *of order k and filtration degree p*. The element

$$\sigma^p_k(D) \in \mathrm{Sym}^k_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}) \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Gr}^p\mathrm{End}_{\mathcal{E}}$$

is called the *principal symbol* of $D_{\mathcal{E}}$.

[5] Let $D_{\mathcal{E}} = e \otimes D \otimes e^\vee \in \mathcal{D}_{\mathcal{E}}$ and $s \in \mathcal{E}$. Then $D_{\mathcal{E}}$ acts on s by

$$D_{\mathcal{E}}(s) = eD(e^\vee(s)).$$

Definition A.13. *The* ∞-jet bundle *of* $\mathcal{E}$ *is the filtered graded vector bundle*

$$\mathcal{J}^\infty_\mathcal{E} := \mathcal{J}^\infty_\mathcal{M} \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}\, \mathcal{E},$$

where $\mathcal{J}^\infty_\mathcal{M}$ *is viewed as a right* $\mathcal{O}_\mathcal{M}$*-module.*

To see that $\mathcal{J}^\infty_\mathcal{E}$ is well-defined as a filtered graded vector bundle, choose a splitting

$$\sigma : \mathcal{E} \longrightarrow \mathcal{J}^\infty_\mathcal{E}$$

of the short exact sequence of graded left $\mathcal{O}_\mathcal{M}$-modules

$$0 \longrightarrow F^1\mathcal{J}^\infty_\mathcal{M} \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}\, \mathcal{E} \longrightarrow \mathcal{J}^\infty_\mathcal{M} \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}\, \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Consider the morphism of filtered graded left $\mathcal{O}_\mathcal{M}$-modules

$$\begin{aligned} \Phi^\sigma : \mathcal{E} \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}\, J^\infty_\mathcal{M} &\longrightarrow J^\infty_\mathcal{M} \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}\, \mathcal{E} \\ s \otimes j &\longmapsto (-1)^{|j||s|} j \cdot \sigma(s). \end{aligned}$$

Its associated graded morphism identifies with

$$\begin{aligned} \operatorname{Gr}\Phi^\sigma : \operatorname{Gr}\,\mathcal{E} \otimes_{\mathcal{O}_\mathcal{M}} \operatorname{Sym}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}^\vee_\mathcal{M}) &\longrightarrow \operatorname{Sym}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}^\vee_\mathcal{M}) \otimes_{\mathcal{O}_\mathcal{M}} \operatorname{Gr}\mathcal{E} \\ s \otimes \alpha &\longmapsto (-1)^{|\alpha||s|}\alpha \otimes s, \end{aligned}$$

which is obviously an isomorphism. Since the filtrations are complete and exhaustive, it follows that Φ^σ itself is an isomorphism. Therefore, $\mathcal{J}^\infty_\mathcal{E}$ is (non-canonically) isomorphic, as a filtered graded left $\mathcal{O}_\mathcal{M}$-module, to the filtered graded vector bundle $\mathcal{E} \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}\, J^\infty_\mathcal{M}$.

Lemma A.14. *Let* $\{(U, \varphi_U)\}$ *be a family of local trivializations of* $\mathcal{E}$*. The local* $\mathbb{R}$*-linear morphisms* $j^\infty|_U : \mathcal{E}(U) \to \mathcal{J}^\infty_\mathcal{E}(U)$*, defined via the commutative diagram*

$$\begin{array}{ccc} \mathcal{E}(U) & \xrightarrow{\quad j^\infty|_U \quad} & \mathcal{J}^\infty_\mathcal{M}(U) \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}(U)}\, \mathcal{E}(U) \\ {\scriptstyle\varphi_U}\big\downarrow & & \big\uparrow{\scriptstyle 1 \,\widehat{\otimes}\, \varphi_U^{-1}} \\ \mathcal{O}_\mathcal{M}(U) \,\widehat{\otimes}_\mathbb{R}\, F & \xrightarrow[j^\infty|_U \otimes 1]{} & \mathcal{J}^\infty_\mathcal{M}(U) \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}(U)}\, (\mathcal{O}_\mathcal{M}(U) \,\widehat{\otimes}_\mathbb{R}\, F), \end{array}$$

where F *denotes the fiber of* $\mathcal{E}$*. These local morphisms glue to well-defined, filtration-preserving* $\mathbb{R}$*-linear map of degree* 0

$$j^\infty \colon \mathcal{E} \to \mathcal{J}^\infty_\mathcal{E},$$

called the infinite jet prolongation *of* $\mathcal{E}$*.*

Proof. Without loss of generality, we assume that $\mathcal{E}$ is a graded vector bundle.

Let $\{e_i\}$ be a basis of F. The induced sections $s_i := \varphi_U^{-1}(1 \otimes e_i)$ form a basis for the free $\mathcal{O}_\mathcal{M}(U)$-module $\mathcal{E}(U)$. For any local section $s = f^i s_i \in \mathcal{E}(U)$, where $f^i \in \mathcal{O}_\mathcal{M}(U)$, the infinite jet prolongation acts by

$$j^\infty|_U(s) = j^\infty|_U(f^i)s_i.$$

Now consider another trivialization $(U', \varphi_{U'})$ with $U \cap U' \neq \emptyset$. Let $\{s'_j := \varphi_{U'}^{-1}(1 \otimes e_j)\}$ be the corresponding basis of $\mathcal{E}(U')$, and let $s_i|_{U\cap U'} = g^j_i s'_j|_{U\cap U'}$ for transition functions

$g^i_j \in \mathcal{O}_{\mathcal{M}}(U \cap U')$. On the intersection $U \cap U'$, we verify the compatibility:

$$\begin{aligned} j^\infty|_{U\cap U'}(s|_{U\cap U'}) &= j^\infty|_{U\cap U'}(f^i|_{U\cap U'})s_i|_{U\cap U'} \\ &= j^\infty|_{U\cap U'}(f^i|_{U\cap U'})g^i_j s'_j|_{U\cap U'} \\ &= j^\infty|_{U\cap U'}(f^i|_{U\cap U'}g^i_j)s'_j|_{U\cap U'} \\ &= j^\infty|_{U\cap U'}((f')^j|_{U\cap U'})s'_j|_{U\cap U'}, \end{aligned}$$

where $(f')^j = f^i g^j_i$. We use Lemma A.8 to pass to the third line. □

Lemma A.15. *The filtered graded $\mathcal{O}_{\mathcal{M}}$-module $J^\infty_{\mathcal{E}}$ is generated by $j^\infty(\mathcal{E})$ over $\mathcal{O}_{\mathcal{M}}$.*

Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of graded manifold. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be filtered graded vector bundles over $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Let

$$F^\sharp : \mathcal{E}_2 \to f_*\mathcal{E}_1$$

be a morphism of filtered graded $\mathcal{O}_{\mathcal{M}_2}$-modules. (Note that the pair $(f, F^\sharp)$ does not define a morphism of filtered graded vector bundles, since $F^\sharp$ goes in the opposite direction to the usual bundle map.)

Proposition A.16. *There exists a unique filtered graded $\mathcal{O}_{\mathcal{M}_2}$-module morphism*

$$j^\infty(F^\sharp) : J^\infty_{\mathcal{E}_2} \to f_* J^\infty_{\mathcal{E}_1},$$

such that the following diagram

$$\begin{array}{ccc} \mathcal{J}^\infty_{\mathcal{E}_2} & \xrightarrow{j^\infty(F^\sharp)} & f_*\mathcal{J}^\infty_{\mathcal{E}_1} \\ {\scriptstyle j^\infty}\uparrow & & \uparrow{\scriptstyle f_*j^\infty} \\ \mathcal{E}_2 & \xrightarrow{F^\sharp} & f_*\mathcal{E}_1 \end{array}$$

commutes.

The proofs of Lemma A.15 and Proposition A.16 follow the same arguments as in the proofs of Lemma A.9 and Proposition A.10, and are therefore omitted.

Appendix B. De Rham complexes of D-modules

Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a graded manifold. In this appendix, we consider (left) D-modules on $\mathcal{M}$ and their de Rham complexes.

Definition B.1. *A* (left) D-module *on $\mathcal{M}$ is a filtered graded vector bundle $\mathcal{E}$ on $\mathcal{M}$, together with a left action of the filtered graded $(\mathcal{O}_{\mathcal{M}}, \mathcal{O}_{\mathcal{M}})$-algebra $\mathcal{D}_{\mathcal{M}}$ on $\mathcal{E}$:*

$$F^p\mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} F^q\mathcal{E} \longrightarrow F^{p+q}\mathcal{E}.$$

Let $\mathcal{E}$ be a D-module. Then for every vector field $X \in \mathcal{T}_{\mathcal{M}}$, one has

$$X \cdot F^\bullet\mathcal{E} \subset F^{\bullet-1}\mathcal{E}.$$

Definition B.2. *Let $\mathcal{E}$ be a filtered graded vector bundle. A* connection *on $\mathcal{E}$ is an $\mathbb{R}$-linear map*

$$\nabla : F^\bullet\mathcal{E} \to \mathcal{T}^\vee_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} F^{\bullet-1}\mathcal{E}$$

such that

$$\nabla(fs) = df \otimes s + (-1)^{||f|} f \nabla s.$$

We use the standard notation ∇_X for the composition $\mathcal{E} \xrightarrow{\nabla} \mathcal{T}_{\mathcal{M}}^{\vee} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{E} \xrightarrow{\iota_X \otimes \mathrm{id}} \mathcal{E}$.

Remark B.3. *If $\mathcal{E}$ is a graded vector bundle, regarded as a filtered graded vector bundle with the trivial filtration, then all left $\mathcal{D}_{\mathcal{M}}$-actions and connections on $\mathcal{E}$ are automatically compatible with the trivial filtration.*

Example B.4. *A connection on the tangent bundle $\mathcal{T}_{\mathcal{M}}$ is called an* affine connection. *Such a connection ∇ is called* torsion-free *if*

$$\nabla_X Y - (-1)^{|X||Y|} \nabla_Y X = [X, Y].$$

Using partition of unity, one can show that there always exists a torsion-free affine connection.

Recall that the sheaf of completed forms on $\mathcal{M}$ is

$$\widehat{\Omega}_{\mathcal{M}} := \widehat{\mathrm{Sym}}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}[1]^{\vee}),$$

where $\widehat{\mathrm{Sym}}_{\mathcal{O}_{\mathcal{M}}}$ denotes the completed symmetric power over $\mathcal{O}_{\mathcal{M}}$. We define the *exterior covariant derivative* of ∇ as the $\mathbb{R}$-linear map

$$\begin{aligned} d_{\nabla} : \widehat{\Omega}_{\mathcal{M}} \,\widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}}\, \mathcal{E} &\to \widehat{\Omega}_{\mathcal{M}} \,\widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}}\, \mathcal{E} \\ \alpha \otimes e &\mapsto d_{dR}\alpha \otimes e + (-1)^{|\alpha|} \alpha \wedge \nabla e. \end{aligned}$$

By definition, d_{∇} preserves the filtration on $\widehat{\Omega}_{\mathcal{M}} \widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}} \mathcal{E}$.

Definition B.5. *A connection ∇ on $\mathcal{E}$ is called* flat *if $d_{\nabla} \circ d_{\nabla} = 0$.*

It is straightforward to show that

Lemma B.6. *There is a one-to-one correspondence between flat connections on $\mathcal{E}$ and D-module structures on $\mathcal{E}$.*

Let $\mathcal{E}$ be a D-module. Let ∇ be the corresponding flat connection on $\mathcal{E}$. Explicitly,

$$\nabla_X(s) = X \cdot s.$$

Definition B.7. *We call the filtered dg module*

$$dR(\mathcal{E}) := (\widehat{\Omega}_{\mathcal{M}} \,\widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}}\, \mathcal{E}, d_{\nabla})$$

over $(\widehat{\Omega}_{\mathcal{M}}, d_{dR})$ the de Rham complex *of $\mathcal{E}$.*

Definition B.8. *Let X be a vector field on $\mathcal{M}$. The* covariant Lie derivative *on $dR(\mathcal{E})$*

$$L_X^{\nabla} : F^{\bullet} dR(\mathcal{E}) \longrightarrow F^{\bullet - 1} dR(\mathcal{E})$$

along X is defined by

$$L_X^{\nabla}(\alpha \otimes s) = L_X(\alpha) \otimes s + (-1)^{|X||\alpha|} \alpha \otimes \nabla_X(s),$$

for $\alpha \in \widehat{\Omega}_{\mathcal{M}}$ and $s \in \mathcal{E}$, where $L_X(\alpha)$ is the Lie derivative of α.

Remark B.9. *It is straightforward to verify that*

$$L_X^{\nabla} = [\iota_X \otimes \mathrm{id}, d_{\nabla}],$$

where $\iota_X : \widehat{\Omega}_{\mathcal{M}} \to \widehat{\Omega}_{\mathcal{M}}$ is the contraction by X. Moreover, since ∇ is flat, one has

$$[d_{\nabla}, L_X^{\nabla}] = 0, \quad [L_X^{\nabla}, L_Y^{\nabla}] = L_{[X,Y]}^{\nabla}.$$

Let $\mathcal{J}^\infty_{\mathcal{M}}$ be the sheaf of infinite jets on $\mathcal{M}$.

Definition B.10. *The* Grothendieck connection *on $\mathcal{J}^\infty_{\mathcal{M}}$, denoted by ∇^G, is the flat connection defined by dualizing the canonical left $D_{\mathcal{M}}$-module structure of $D_{\mathcal{M}}$. Explicitly,*

$$(\nabla^G_X J)(D) = X(J(D)) - (-1)^{|X||J|} J(X \circ D).$$

For any function f on $\mathcal{M}$, we have

$$\begin{aligned}\nabla^G_X(j^\infty(f))(D) &= (-1)^{|X||f|}\big(X(D(f)) - (\nabla^G_X D)(f)\big)\\ &= (-1)^{|X||f|}\big(X(D(f)) - (X \circ D)(f)\big)\\ &= 0.\end{aligned}$$

We use this observation to determine a local formula for ∇^G. Indeed,

$$\nabla^G\big(j^\infty(x^\mu)\big) = \nabla^G(x^\mu + y^\mu) = dx^\mu + \nabla^G(y^\mu) = 0.$$

Since ∇^G is a derivation, it follows that it must take the local form

$$\nabla^G = dx^\mu \frac{\partial}{\partial x^\mu} - dx^\mu \frac{\partial}{\partial y^\mu}. \tag{B.1}$$

Lemma B.11. *The associated $\mathbb{N}$-graded operator of the differential d_{∇^G} on $dR(\mathcal{J}^\infty_{\mathcal{M}})$ is minus the* universal Koszul differential

$$\delta_{\mathrm{K}} : \Omega^k_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Sym}^l_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}^\vee_{\mathcal{M}}) \longrightarrow \Omega^{k+1}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Sym}^{l-1}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}^\vee_{\mathcal{M}}).$$

Proof. The associated $\mathbb{N}$-graded algebra of $dR(\mathcal{J}^\infty_{\mathcal{M}})$ is given by the universal Koszul algebra of the graded $\mathcal{O}_{\mathcal{M}}$-module $\mathcal{T}_{\mathcal{M}}$:

$$\mathrm{Gr}\, dR(\mathcal{J}^\infty_{\mathcal{M}}) \cong \Omega_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Sym}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}^\vee_{\mathcal{M}}).$$

Note that $dx^\mu\wedge$ increases the filtration degree by 1, while $\frac{\partial}{\partial y^\mu}$ decreases it by 1. Using (B.1), we obtain the local formula

$$-\mathrm{Gr}\, d_{\nabla^G} = dx^\mu \frac{\partial}{\partial y^\mu},$$

which is independent of local coordinates and is exactly the universal Koszul differential. □

Using (A.2) and (A.3), one can easily check that ∇^G_X is a derivation of $\mathcal{J}^\infty_{\mathcal{M}}$. It follows that d_{∇^G}, as well as any covariant Lie derivative on $dR(\mathcal{J}^\infty_{\mathcal{M}})$, are also derivations. Consequently, $dR(\mathcal{J}^\infty_{\mathcal{M}})$ is a filtered cdga over $(\widehat{\Omega}_{\mathcal{M}}, d_{dR})$.

Proposition B.12. *The infinite jet prolongation*

$$j^\infty : \mathcal{O}_{\mathcal{M}} \longrightarrow dR(\mathcal{J}^\infty_{\mathcal{M}})$$

is a filtered quasi-isomorphism of cdgas. More precisely, a stronger statement holds: on the E_0 pages of the associated spectral sequences,

$$\mathrm{Gr}\, j^\infty : \mathcal{O}_{\mathcal{M}} \longrightarrow \mathrm{Gr}\, dR(\mathcal{J}^\infty_{\mathcal{M}})$$

is already a quasi-isomorphism of cdgas.

Proof. It is straightforward to verify that $\operatorname{Gr} j^\infty$ is given by the canonical inclusion

$$\mathcal{O}_\mathcal{M} \hookrightarrow \Omega_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \operatorname{Sym}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}^\vee).$$

The statement then follows from the fact the cohomology of the universal Koszul algebra of a finitely generated projective $\mathcal{O}_\mathcal{M}$-module is the base $\mathcal{O}_\mathcal{M}$. □

Proposition B.13. *Every differential operator D of order k on $\mathcal{M}$ uniquely lifts to a morphism of D-modules*

$$j^\infty(D) : F^\bullet \mathcal{J}^\infty_\mathcal{M} \longrightarrow F^{\bullet-k} \mathcal{J}^\infty_\mathcal{M},$$

whose associated $\mathbb{N}$-graded operator

$$\operatorname{Gr} j^\infty(D) : \operatorname{Sym}^\bullet_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}^\vee) \longrightarrow \operatorname{Sym}^{\bullet-k}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}^\vee)$$

is given by contraction with the principal symbol $\sigma_k(D) \in \operatorname{Sym}^k_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M})$ of D.

Moreover, the following diagram

$$\begin{array}{ccc} \mathcal{J}^\infty_\mathcal{M} & \xrightarrow{j^\infty(D)} & \mathcal{J}^\infty_\mathcal{M} \\ {\scriptstyle j^\infty}\uparrow & & \uparrow{\scriptstyle j^\infty} \\ \mathcal{O}_\mathcal{M} & \xrightarrow{D} & \mathcal{O}_\mathcal{M} \end{array}$$

commutes.

Proof. Since the image $j^\infty(\mathcal{O}_\mathcal{M})$ generates $J^\infty_\mathcal{M}$, we define $j^\infty(D)$ simply by setting

$$j^\infty(D)\big(j^\infty(f)\big) := j^\infty\big(D(f)\big).$$

We need to verify that this is well-defined, i.e. that

$$\sum_{i\in I} g_i\, j^\infty\big(D(f_i)\big) = 0 \quad \text{whenever} \quad \sum_{i\in I} g_i\, j^\infty(f_i) = 0,$$

where I is some finite index set.

Recall that $j^\infty(f_i) = \sum_\alpha \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial x^\alpha}(f_i)\, y^\alpha$. It follows that

$$\sum_{i\in I} g_i \frac{\partial^\alpha}{\partial x^\alpha}(f_i) = 0$$

for all multi-indices α. Therefore,

$$\begin{aligned} \sum_{i\in I} g_i\, j^\infty\big(D(f_i)\big) &= \sum_\alpha \frac{y^\alpha}{\alpha!} \sum_{i\in I} g_i \frac{\partial^\alpha}{\partial x^\alpha}\big(D(f_i)\big) \\ &= \sum_{|\beta|\le k} \sum_\alpha \frac{y^\alpha}{\alpha!} \sum_{i\in I} g_i \frac{\partial^\alpha}{\partial x^\alpha}\big(D_\beta \partial_\beta(f_i)\big) \\ &= 0, \end{aligned}$$

where D_β is some local function. In local coordinates, $j^\infty(D)$ is given by

$$j^\infty(D) = \sum_{|\beta|\le k} \sum_\alpha \frac{\partial^\alpha D_\beta}{\partial x^\alpha} \frac{y^\alpha}{\alpha!} \frac{\partial^\beta}{\partial y^\beta}.$$

It follows that $j^\infty(D)$ lowers the filtration degree by k and

$$\operatorname{Gr} j^\infty(D) = \sum_{|\beta|=k} D_\beta \frac{\partial^\beta}{\partial y^\beta},$$

which is exactly contraction with $\sigma_k(D)$.

To see that $j^\infty(D)$ is a morphism of D-modules. We compute

$$j^\infty(D)(\nabla^G_X(gj^\infty(f))) = j^\infty(D)(X(g)j^\infty(f)) = \pm X(g)j^\infty(D(f)) = \pm\nabla^G_X(gj^\infty(D(f))),$$

which completes the proof. □

Let X be a vector field on $\mathcal{M}$, viewed as a differential operator.

Corollary B.14. *The operator*

$$L^v_X := 1 \otimes j^\infty(X)$$

is a derivation of $dR(\mathcal{J}^\infty_\mathcal{M})$, called the vertical Lie derivative *on $dR(\mathcal{J}^\infty_\mathcal{M})$.*

Respectively, we refer to the covariant Lie derivative $L^{\nabla^G}_X$ on $dR(\mathcal{J}^\infty_\mathcal{M})$ as the *horizontal Lie derivative* and denote it by L^h_X. The sum

$$L_X := L^h_X + L^v_X,$$

is called the *(total) Lie derivative* on $dR(\mathcal{J}^\infty_\mathcal{M})$. Note that both L^v_X and L^h_X, and hence also L_X, commute with d_{∇^G}.

Remark B.15. *The* Lie derivative *L_X on $\mathcal{D}_\mathcal{M}$ along X is defined by*

$$L_X D := [X, D].$$

L_X is a coderivation of $\mathcal{D}_\mathcal{M}$. It extends to $dR(\mathcal{D}_\mathcal{M}) = \Omega_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{D}_\mathcal{M}$ via the Leibniz rule:

$$L_X(\alpha \otimes D) = L_X\alpha \otimes D + (-1)^{|X||\alpha|}\alpha \otimes L_X D.$$

We leave it to the reader as an exercise to verify that the Lie derivative L_X on $dR(\mathcal{D}_\mathcal{M})$ is dual to the Lie derivative L_X on $dR(\mathcal{J}^\infty_\mathcal{M})$ under the identification

$$\operatorname{Hom}_{\mathcal{O}_\mathcal{M}}(\Omega_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{D}_\mathcal{M}, \mathcal{O}_\mathcal{M}) \cong \widehat{\Omega}_\mathcal{M} \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}\, \mathcal{J}^\infty_\mathcal{M}.$$

Let $\mathcal{E}$ be a filtered graded vector bundle on $\mathcal{M}$. Let $\mathcal{J}^\infty_\mathcal{E} = \mathcal{J}^\infty_\mathcal{M} \,\widehat{\otimes}_{\mathcal{O}_\mathcal{M}}\, \mathcal{E}$ be the infinite jet bundle of $\mathcal{E}$.

Definition B.16. *$\mathcal{J}^\infty_\mathcal{E}$ carries a canonical flat connection ∇^G, given by*

$$\nabla^G_X(J \otimes s) := \nabla^G_X(J) \otimes s. \tag{B.2}$$

∇^G is called the Grothendieck connection *of $J^\infty_\mathcal{E}$.*

We need to verify that (B.2) is well-defined. We have

$$\nabla^G_X(J \otimes fs) = \nabla^G_X(J) \otimes fs = (\nabla^G_X(J)j^\infty(f)) \otimes s.$$

for all functions f. Using Lemma A.8, we have

$$\nabla^G_X(J)j^\infty(f) = \nabla^G_X(Jf).$$

Thus, $\nabla^G_X(J \otimes fs) = \nabla^G_X(Jf \otimes s)$.

Proposition B.17. *The infinite jet prolongation*

$$j^\infty : \mathcal{E} \longrightarrow dR(\mathcal{J}^\infty_\mathcal{E})$$

is a filtered quasi-isomorphism. That is, on the E_0 pages of the associated spectral sequences,

$$\operatorname{Gr} j^\infty : \operatorname{Gr} \mathcal{E} \longrightarrow \operatorname{Gr} dR(\mathcal{J}^\infty_\mathcal{E})$$

is a quasi-isomorphism.

Proposition B.18. *Every differential operator $D_\mathcal{E} : \mathcal{E} \to \mathcal{E}$ of order k and filtration degree p uniquely lifts to a morphism of D-modules*

$$j^\infty(D_\mathcal{E}) : F^\bullet \mathcal{J}^\infty_\mathcal{E} \longrightarrow F^{\bullet+p-k} \mathcal{J}^\infty_\mathcal{E},$$

whose associated $\mathbb{N}$-graded operator

$$\operatorname{Gr} j^\infty(D_\mathcal{E}) : \operatorname{Sym}^\bullet_{\mathcal{O}_\mathcal{M}}(\mathcal{T}^\vee_\mathcal{M}) \otimes_{\mathcal{O}_\mathcal{M}} \operatorname{Gr}^\bullet \mathcal{E} \longrightarrow \operatorname{Sym}^{\bullet-k}_{\mathcal{O}_\mathcal{M}}(\mathcal{T}^\vee_\mathcal{M}) \otimes_{\mathcal{O}_\mathcal{M}} \operatorname{Gr}^{\bullet+p} \mathcal{E}$$

is given by contraction with the principal symbol $\sigma^p_k(D_\mathcal{E}) \in \operatorname{Sym}^k_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}) \otimes_{\mathcal{O}_\mathcal{M}} \operatorname{Gr}^p \operatorname{End}_\mathcal{E}$ of D.

Moreover, the following diagram

$$\begin{array}{ccc} \mathcal{J}^\infty_\mathcal{E} & \xrightarrow{j^\infty(D_\mathcal{E})} & \mathcal{J}^\infty_\mathcal{E} \\ {\scriptstyle j^\infty}\uparrow & & \uparrow{\scriptstyle j^\infty} \\ \mathcal{E} & \xrightarrow{D_\mathcal{E}} & \mathcal{E} \end{array}$$

commutes.

The proofs of Propositions B.17 and B.18 follow the same arguments as in the proofs of Propositions B.12 and B.13, and are therefore omitted.

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