BV PUSHFORWARD AS A QUASI-ISOMORPHISM

ALBERTO S. CATTANEO AND PAVEL MNEV

Abstract. Given a BV theory on a space of fields split into two subspaces ("infrared" and "ultraviolet"), one has the BV pushforward map P_*, sending observables to observables of the effective theory on the infrared space. This note proves that P_* is a quasi-isomorphism of BV complexes, by realizing it as a part of a strong deformation retraction constructed using the homological perturbation lemma. Two proofs are given: (i) comparing Feynman diagrams for P_* with "cable diagrams" arising from homological perturbation theory and (ii) using topological quantum mechanics. This construction gives a formula for the quasi-inverse i_{int} of P_*—the map lifting observables of the effective theory to the full theory. The topological quantum mechanics perspective—and its realization as an AKSZ theory—allows one to write i_{int} as a path integral (realizing cable diagrams for i_{int} as Feynman diagrams) and to study its classical limit.

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1. Introduction

Consider a gauge theory in the Batalin–Vilkovisky formalism with the space of fields $\mathcal{F}$ (a graded vector space) equipped with a symplectic structure ω of degree -1 and an action S – a function satisfying the quantum master equation (QME) $\frac{1}{2}\{S,S\}-\mathrm{i}\hbar\Delta S=0$.

One has the "BV complex" of the theory – the space of functions on $\mathcal{F}$ equipped with differential $Q_\hbar=\{S,-\}-\mathrm{i}\hbar\Delta$. Its cocycles are the gauge-invariant observables.

Assume that $\mathcal{F}$ is split as a sum of two symplectic subspaces

$$\mathcal{F}=\mathcal{F}'\oplus\mathcal{F}'' \tag{1}$$

– the "infrared" and "ultraviolet" subspaces. One has:

(i) The effective action S' on $\mathcal{F}'$ defined by the fiber BV integral

$$e^{\frac{\mathrm{i}}{\hbar}S'}=\int_{\mathcal{L}\subset\mathcal{F}''}e^{\frac{\mathrm{i}}{\hbar}S}. \tag{2}$$

Here $\mathcal{L}$ is some fixed Lagrangian subspace of $\mathcal{F}''$ and the integral in the r.h.s. is considered as a family over $\mathcal{F}'$. The effective action automatically satisfies the QME on $\mathcal{F}'$.

(ii) The BV pushforward map between BV complexes of the full and infrared (effective) theory[1]

$$\begin{array}{rccc} P_*\colon & (\mathrm{Fun}(\mathcal{F}), Q_\hbar) & \to & (\mathrm{Fun}(\mathcal{F}'), Q'_\hbar) \\ & O & \mapsto & O' = e^{-\frac{\mathrm{i}}{\hbar}S'} \int_{\mathcal{L}} e^{\frac{\mathrm{i}}{\hbar}S} O \end{array} \tag{3}$$

It is automatically a chain map and in particular sends gauge-invariant observables of the full theory to those of the effective theory.

Assume that $S = S_0 + S_{\mathrm{int}}$ is a perturbation of a free theory S_0, associated to a differential d on $\mathcal{F}$ compatible with ω and with the splitting (1), so that d is acyclic on $\mathcal{F}''$. Also, assume that $\mathcal{F}$ is finite-dimensional. Under these assumptions we prove the following.

Theorem A.[2] *The map P_* is a quasi-isomorphism.*

The proof is by realizing P_* as the map p_{int} in the strong deformation retraction (SDR) of BV complexes

$$K_{\mathrm{int}} \circlearrowleft (\mathcal{F}, Q_\hbar) \underset{p_{\mathrm{int}}}{\overset{i_{\mathrm{int}}}{\leftrightarrows}} (\mathcal{F}', Q'_\hbar). \tag{4}$$

This SDR is constructed by the homological perturbation lemma[3] (HPL), starting from the SDR for the free classical BV theory, and then deforming the differential $Q_0 = \{S_0, -\}$ of the classical free BV complex to $Q_\hbar = Q_0 + \{S_{\mathrm{int}}, -\} - \mathrm{i}\hbar\Delta$ – the differential of the quantum interacting theory. Then, Theorem A is an immediate corollary of (ii) of the following.

Theorem B.[4]

(i) The induced differential $Q'_\hbar$ in (4), defined by HPL formula, has the form $Q'_\hbar = \{S', -\} - \mathrm{i}\hbar\Delta'$, i.e., is generated by the effective action S' as defined by the fiber BV integral (2).

(ii) One has $p_{\mathrm{int}} = P_*$.

We give two proofs:

(1) By comparing "cable diagrams" of homological perturbation theory on one side with Feynman graphs for the BV integrals defining P_* and S' on the other side, see Section 5.

(2) By realizing the ingredients of the SDR (4) in terms of topological quantum mechanics (TQM) – an auxiliary 1d quantum field theory τ constructed out of the BV theory $(\mathcal{F}, \omega, S)$, see Section 6.1.[5]

The observable-lifting map i_{int}. Having $P_* = p_{\mathrm{int}}$ as a part of the SDR package (4) guarantees that it is a quasi-isomorphism and gives i_{int} as its quasi-inverse – a map lifting gauge-invariant observables of the infrared theory to gauge-invariant observables of the full theory.

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[1] Here $\mathrm{Fun}(\mathcal{F})$ stands for $\widehat{S}\mathcal{F}^*[[\hbar]]$ – the algebra of polynomials on $\mathcal{F}$ completed to formal power series, with coefficients in formal power series in $\hbar$.

[2] Corollary 5.2 in the main text.

[3] Lemma 3.2.

[4] Theorem 5.1.

[5] In fact, if $(\mathcal{F}, \omega, S)$ is itself an n-dimensional theory, then τ is an $(n+1)$-dimensional theory on a cylinder, cf. Section 8.3. One can think of τ as a "bulk theory" inducing $(\mathcal{F}, \omega, S)$ on the boundary, cf. Remark 8.5.

Having such an observable-lifting map is of particular interest in certain situations. For instance, in [6, Section 7.1], [5, Section 5.2], 4d Yang–Mills theory is realized as an effective theory for the topological $BF + B^2$ theory. Then, the computation of a correlation function of observables in Yang–Mills (e.g. Wilson loops) can be done by lifting those observables to the topological theory and calculating the correlator there. Cf. Section 5.2 for the abelian case.

Two "easy" and two "hard" maps. In the SDR (4), the deformed differential $Q_\hbar$ is given as input and four other objects $Q'_\hbar, i_{\mathrm{int}}, p_{\mathrm{int}}, K_{\mathrm{int}}$ are computed by HPL. By complexity of the result, they split into two pairs:

- Two "easy" objects, p_{int} and $Q'_\hbar$, which can be expressed in terms of BV integrals – cable diagrams for them simplify to Feynman graphs computing perturbative BV integrals.
- Two "hard" objects, i_{int} and K_{int}. The combinatorics of cable diagrams for them is more complicated and they do not simplify to Feynman diagrams (for BV integrals associated to theory $(\mathcal{F}, \omega, S)$). However – and this is one of the results of the paper – $i_{\mathrm{int}}, K_{\mathrm{int}}$ can be recognized as BV integrals (and cable diagrams can be recognized as Feynman graphs) for an auxiliary AKSZ theory – a Lagrangian description of the TQM τ.[6]

1.1. **Topological quantum mechanics τ.** Topological quantum mechanics τ is a 1d functorial QFT assigning to a point the space of states $\mathcal{H}_{\mathrm{pt}} = \mathrm{Fun}(\mathcal{F})$ equipped with the differential $Q_\hbar$.[7] Thus, the space of states of τ is the BV complex of theory $(\mathcal{F}, \omega, S)$. Additionally, $\mathcal{H}_{\mathrm{pt}}$ is equipped with a second differential of degree -1 – the gauge-fixing operator $\widehat{\kappa}$ – the extension of a chain contraction κ of $\mathcal{F}$ (defining the gauge-fixing Lagrangian $\mathcal{L} = \mathrm{im}(\kappa)$ in (2)) to a derivation of $\mathrm{Fun}(\mathcal{F})$. Then, the Hamiltonian of the TQM τ is $H = [Q_\hbar, \widehat{\kappa}]$ and the partition function for an interval of length T is

$$Z_{T,\mathrm{d}T} = e^{-TH + \mathrm{d}T\,\widehat{\kappa}} \in \Omega^\bullet(\mathbb{R}_+) \otimes \mathrm{End}(\mathcal{H}_{\mathrm{pt}}) \tag{5}$$

– a nonhomogeneous form on the space $\mathbb{R}_+$ of lengths of an interval valued in operators on $\mathcal{H}_{\mathrm{pt}}$; we will write Z_T for its 0-form component e^{-TH}. By construction this partition function satisfies the "topologicity" equation $(\mathrm{d}_T + [Q_\hbar, -])Z_{T,\mathrm{d}T} = 0$, which implies that $\int_0^\infty Z_{T,dT}$ is a chain homotopy between identity and $\lim_{T\to\infty} Z_T = P_{\ker H}$ – the projection onto $\ker H$ along $\mathrm{im}(H)$. Moreover, from the theorem below we have $\ker H \simeq \mathrm{Fun}(\mathcal{F}')$, where the isomorphism is given by $p_{\mathrm{int}}|_{\ker H}$ and its inverse is i_{int}).

One can recover the data of the SDR (4) from the TQM τ:

Theorem C.[8] *One has*

$$\begin{aligned} i_{\mathrm{int}} &= \lim_{T\to\infty} Z_T \circ i, \\ p_{\mathrm{int}} &= \lim_{T\to\infty} p \circ Z_T, \\ K_{\mathrm{int}} &= \int_0^T Z_{T,\mathrm{d}T}, \\ i_{\mathrm{int}} \circ p_{\mathrm{int}} &= \lim_{T\to\infty} Z_T. \end{aligned} \tag{6}$$

[6] See Theorem 8.7 and Section 8.2.1.

[7] This is an example of topological quantum mechanics in the sense of A. Losev [27]; τ is the name we use for our special example of topological quantum mechanics.

[8] Theorem 6.1.

Here i is the pullback by the projection $\pi\colon \mathcal{F} \to \mathcal{F}'$ and p is the pullback by the inclusion $\iota\colon \mathcal{F}' \to \mathcal{F}$ in the splitting (1).

The proof of Theorem C is based on the perturbation series for the exponential of a perturbed operator, where $H = \mathcal{L}_E + \cdots$ is seen as a perturbation of the Lie derivative along the Euler vector field E on $\mathcal{F}$ assigning degree 1 to ultraviolet fields and degree 0 to infrared ones.

TQM application 1: second proof of Theorem B. The TQM perspective and Theorem C have two immediate applications. First, they lead to the second proof of Theorem B, not relying on combinatorics of cable diagrams and Feynman graphs, see Section 6.1. To give the idea of the proof, let us specialize to (ii) of Theorem B in the case $\mathcal{F}' = 0$. Fixing an observable $O \in \mathrm{Fun}(\mathcal{F})$, we have

$$p_{\mathrm{int}}(O) = P_{\ker H}(O) = \ll O, 1 \gg, \tag{7}$$

with $\ll, \gg$ a pairing on $\mathrm{Fun}(\mathcal{F})$ such that H is self-adjoint and 1 has norm 1. It turns out that the pairing

$$\ll O_1, O_2 \gg = c \int_{\mathcal{L} \subset \mathcal{F}} e^{\frac{\mathrm{i}}{\hbar} S} O_1 O_2, \tag{8}$$

with c a normalization constant, does the job. This immediately implies that the r.h.s. of (7) is the BV pushforward $P_*(O)$.

TQM application 2: classical limit of maps $i_{\mathrm{int}}, p_{\mathrm{int}}, K_{\mathrm{int}}$. Let $H^{\mathrm{cl}} = H \bmod \hbar$ be the classical TQM Hamiltonian viewed as a vector field on $\mathcal{F}$. For X an object valued in formal power series in $\hbar$, we will denote X^{cl} its constant term in $\hbar$ (i.e., X^{cl} is the "classical limit" $\hbar \to 0$ of X).

The differential $Q^{\mathrm{cl}} = \{S^{\mathrm{cl}}, -\}$ is a cohomological vector field on $\mathcal{F}$ vanishing at zero. It equips $\mathcal{F}[-1]$ with the structure of an L_∞ algebra. Likewise, $Q'^{\mathrm{cl}} = \{S'^{\mathrm{cl}}, -\}'$ equips $\mathcal{F}'[-1]$ with the structure of an L_∞ algebra, obtained by homotopy transfer from $(\mathcal{F}[-1], Q^{\mathrm{cl}})$.[9]

Let $\Phi_T\colon \mathcal{F} \to \mathcal{F}$ be the flow of the vector field $-H^{\mathrm{cl}}$ in time T.

Theorem D.[10] *One has the following:*

- *The classical limit of i_{int} is $i_{\mathrm{int}}^{\mathrm{cl}} = \lim_{T\to\infty} \Phi_T^* \circ i\colon \mathrm{Fun}(\mathcal{F}') \to \mathrm{Fun}(\mathcal{F})$. It is the pullback by a nonlinear map $\pi_{\mathrm{int}}^{\mathrm{cl}} = \lim_{T\to\infty} \pi \circ \Phi_T\colon \mathcal{F} \to \mathcal{F}'$ which is an L_∞ morphism of L_∞ algebras.*
- *The classical limit of p_{int} is $p_{\mathrm{int}}^{\mathrm{cl}} = \lim_{T\to\infty} p \circ \Phi_T^*\colon \mathrm{Fun}(\mathcal{F}) \to \mathrm{Fun}(\mathcal{F}')$. It is the pullback by a nonlinear map $\iota_{\mathrm{int}}^{\mathrm{cl}} = \lim_{T\to\infty} \Phi_T \circ \iota\colon \mathcal{F}' \to \mathcal{F}$ which is an L_∞ morphism of L_∞ algebras. $\iota_{\mathrm{int}}^{\mathrm{cl}}$ maps a point $x' \in \mathcal{F}'$ to the critical point of S restricted to $x' + \mathcal{L}$.*
- *The classical limit of K_{int} is*

$$K_{\mathrm{int}}^{\mathrm{cl}}(O)(x) = \int_0^\infty O\big((\mathrm{id} + \mathrm{d}T\,\kappa) \circ \Phi_T(x)\big), \tag{9}$$

for $O \in \mathrm{Fun}(\mathcal{F})$.

Here the mutually quasi-inverse L_∞ morphisms $\iota_{\mathrm{int}}^{\mathrm{cl}}$, $\pi_{\mathrm{int}}^{\mathrm{cl}}$ are, respectively "hard" and "easy" maps (also, the homotopy transferred L_∞ structure Q'^{cl} is "easy" and the chain homotopy $K_{\mathrm{int}}^{\mathrm{cl}}$ is "hard").[11]

[9] This statement is the classical limit of (i) of Theorem B.

[10] Corollary 7.3 and Remark 7.4.

[11] This difference in complexity is related to the behavior of the vector field $-H^{\mathrm{cl}}$: it is tangent to submanifolds $x' + \mathcal{L}$ and its flow converges to the critical point of S on $x' + \mathcal{L}$. So, in the

AKSZ description of the theory τ. One can describe the TQM above by a path integral, starting from its presentation in the Lagrangian formalism, as a 1d AKSZ sigma model [1] on the interval $[0,T]$. Its space of fields is $\mathcal{F}^\tau = \Omega^\bullet([0,T],T^*\mathcal{F})$, where $T^*\mathcal{F}$ is the target (with $\mathcal{F}$ the space of fields of the BV theory we started with), equipped with canonical cotangent symplectic structure and the degree +1 "cohomological function" $\Theta = \omega^{-1}(p,\frac{\partial S}{\partial x})+\frac{1}{2}\omega^{-1}(p,p)$, satisfying $\{\Theta,\Theta\}=0$. Here x,p are the base and fiber coordinates on $T^*\mathcal{F}$. The action is:

$$S^\tau = \int_0^T \langle \widetilde{p}, \mathrm{d}_t \widetilde{x}\rangle + \Theta(\widetilde{x},\widetilde{p}), \tag{10}$$

where $\widetilde{x}\in\Omega^\bullet([0,T],\mathcal{F})$, $\widetilde{p}\in\Omega^\bullet([0,T],\mathcal{F}^*)$ are the "AKSZ superfields" (nonhomogeneous forms on $[0,T]$).

We find a gauge-fixing Lagrangian $\mathcal{L}^\tau\subset\mathcal{F}^\tau$ (268) for the AKSZ theory τ, such that one has the following path integral formulae for the SDR data (4):

Theorem E.[12] *One has:*

$$i_{\mathrm{int}}(O')(x_{\mathrm{out}}) = \int_{\widetilde{p}|_{t=0}=0,\ \widetilde{x}|_{t=T}=x_{\mathrm{out}}} O'(\pi(\widetilde{x}|_{t=0}))\ e^{\frac{\mathrm{i}}{\hbar}S^\tau}\Big|_{\mathcal{L}^\tau}, \tag{11}$$

$$p_{\mathrm{int}}(O)(x'_{\mathrm{out}}) = \int_{\widetilde{p}|_{t=0}=0,\ \widetilde{x}|_{t=T}=x'_{\mathrm{out}}} O(\widetilde{x}|_{t=0})\ e^{\frac{\mathrm{i}}{\hbar}S^\tau}\Big|_{\mathcal{L}^\tau}, \tag{12}$$

$$K_{\mathrm{int}}(O)(x_{\mathrm{out}}) = \int_0^\infty \int_{\widetilde{p}|_{t=0}=0,\ \widetilde{x}|_{t=T}=x_{\mathrm{out}}} O(\widetilde{x}|_{t=0,\,\mathrm{d}t=\mathrm{d}T})\ e^{\frac{\mathrm{i}}{\hbar}S^\tau}\Big|_{\mathcal{L}^\tau}, \tag{13}$$

for any $O\in\mathrm{Fun}(\mathcal{F})$, $O'\in\mathrm{Fun}(\mathcal{F}')$, $x_{\mathrm{out}}\in\mathcal{F}$, $x'_{\mathrm{out}}\in\mathcal{F}'$.

The Feynman diagram expansions of these path integrals recover the HPL cable diagrams for $i_{\mathrm{int}}, p_{\mathrm{int}}, K_{\mathrm{int}}$. In particular, the "hard" objects $i_{\mathrm{int}}, K_{\mathrm{int}}$ are now interpreted as BV integrals for the theory τ.

If $(\mathcal{F},\omega,S)$ is itself an AKSZ theory on an n-manifold M, then τ is an AKSZ theory on an $(n+1)$-dimensional cylinder $[0,T]\times M$, cf. Section 8.3.

1.2. **Plan of the paper.** We start with reminders on the BV pushforward (Section 2) and on strong deformation retractions and homological perturbation lemma (Section 3). We pay extra attention to the construction of SDR for algebras of polynomial functions (the symmetrized tensor power construction) and show different presentations of the chain homotopy, Section 3.1.

In Section 4 we construct the SDR for a free BV theory. Starting from a retraction of the complex of fields onto infrared fields, $\kappa\ \circlearrowleft\ (\mathcal{F},d)\underset{\pi}{\overset{\iota}{\leftrightarrows}}(\mathcal{F}',d')$, we apply the polynomial algebra construction to obtain SDR for the classical BV complex of the free BV theory $K\ \circlearrowleft\ (\mathrm{Fun}(\mathcal{F}),Q_0)\underset{p}{\overset{i}{\leftrightarrows}}(\mathrm{Fun}(\mathcal{F}'),Q'_0)$. Then we deform the differential on the BV complex to the quantum one, $Q_0\to Q_{0,\hbar}=Q_0-\mathrm{i}\hbar\Delta$, and construct the associated deformed SDR using homological perturbation lemma:

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formula for $\iota^{\mathrm{cl}}_{\mathrm{int}}$, the flow starts at x' and proceeds on the submanifold $x'+\mathcal{L}$, converging to the conditional critical point of S. In the formulae for $\pi^{\mathrm{cl}}_{\mathrm{int}}$ or $K^{\mathrm{cl}}_{\mathrm{int}}$, the initial condition is generic and the flow is more complicated.

[12] Theorem 8.7.

$K_\hbar \circlearrowleft (\mathrm{Fun}(\mathcal{F}), Q_{0,\hbar}) \underset{p_\hbar}{\overset{i}{\leftrightarrows}} (\mathrm{Fun}(\mathcal{F}'), Q'_{0,\hbar})$. We prove that $p_\hbar$ is the expectation value map (BV pushforward) of the free theory.

In Section 5 we consider an interacting BV theory with action $S = S_0 + S_{\mathrm{int}}$. Accordingly, we deform the differential in the BV complex to the interacting one $Q_{0,\hbar} \to Q_\hbar = Q_{0,\hbar} + Q_{\mathrm{int}}$ and consider the associated deformed SDR constructed by homological perturbation lemma, $K_{\mathrm{int}} \circlearrowleft (\mathrm{Fun}(\mathcal{F}), Q_\hbar) \underset{p_{\mathrm{int}}}{\overset{i_{\mathrm{int}}}{\leftrightarrows}} (\mathrm{Fun}(\mathcal{F}'), Q'_\hbar)$. We prove Theorem B by a diagrammatic argument. As a corollary we have Theorem A.

In Section 6 we introduce the topological quantum mechanics τ and prove Theorem C. Then we use it to provide the second, non-diagrammatic proof of Theorem B. In Section 7 we use the TQM perspective to study the classical limit of the SDR of interacting BV theory (4) and have Theorem D as an immediate corollary of Theorem C. In Section 8 we introduce the Lagrangian description of the TQM τ as an AKSZ theory and present the corresponding path integral formulae for the maps of the SDR package of the interacting theory (4) – Theorem E.

As a running toy example throughout the paper (alongside more involved examples) we use the "toy scalar field" – a model with $\mathcal{F} = \mathbb{R} \oplus \mathbb{R}[1]$ and action $S_0 = \frac{x^2}{2}$ or $S = \frac{x^2}{2} + P(x)$ with P a polynomial, with $\mathcal{F}' = 0$. This toy example is revisited several times as the machinery is being developed in the paper.

1.3. **Comparison to literature.** Connections between BV formalism, homological perturbation lemma, L_∞ algebras and homotopy transfer of those were known and discussed for a while, see e.g. [20], [26], [24], [29], [30], [27], [12], [28], [22].

The fact that the expectation value of an observable in a free BV theory is given by homological perturbation lemma (the map $p_\hbar$ in our notations) was pointed out in [18], [19]. The observation that the BV pushforward of a class of observables in BF theory can be understood as an L_∞ quasi-isomorphism was made in [29, Theorem 2]. BV pushforward as a variant of Hochschild–Kostant–Rosenberg map was discussed in [16], [17], [25].

The fact that the effective BV action can be described in terms of homological perturbation theory was known (a) classically, via comparison to homotopy transfer of L_∞ algebras – see references above, and (b) at a quantum level, see [31], [32], [12], [22].

In that sense, Theorem B is covered by existing literature. However, the approach via topological quantum mechanics is, to our knowledge, new. We should mention that formulae for the classical limits of maps of the SDR (4) arising from the classical limit of topological quantum mechanics appeared recently, independently of this work, in [14].

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2. Reminder: BV pushforward

In this section we give a brief reminder on the BV integral and BV pushforward, following [7, Sections 2.2.1, 2.2.2].

2.1. **BV integral.** Let $\mathcal{F}$ (the "space of fields") be a $\mathbb{Z}$-graded manifold equipped with a degree -1 symplectic form ω.

Throughout the paper we will assume by default that $\mathcal{F}$ is finite-dimensional.[13]

On the space of $\mathbb{C}$-valued half-densities on $\mathcal{F}$, $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$, one has a canonical differential operator Δ (the BV Laplacian) of degree 1, satisfying $\Delta^2 = 0$, see [23], [34]. In local Darboux coordinates (x^i, ξ_i) on $\mathcal{F}$, one has $\Delta = \frac{\partial^2}{\partial x^i \partial \xi_i}$.

Given a Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}$, one has a map

$$\int_{\mathcal{L}} : \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}) \to \mathbb{C}, \quad \rho \mapsto \int_{\mathcal{L}} \rho|_{\mathcal{L}} \tag{14}$$

– the BV integral. It takes a half-density on $\mathcal{F}$, restricts it to $\mathcal{L}$, resulting in a 1-density on $\mathcal{L}$, which is subsequently integrated over $\mathcal{L}$.

Writing integrals, we assume that the input is such that they converge. In the cases relevant to us, the integrals are perturbed Gaussian integrals, nondegenerate on $\mathcal{L}$, and computed perturbatively, via Feynman diagrams.

The two key properties of the BV integral are:

(i) The integral of a Δ-exact half-density vanishes:

$$\int_{\mathcal{L}} \Delta\sigma = 0. \tag{15}$$

(ii) Given a smooth family of Lagrangians $\mathcal{L}_t$ and a half-density satisfying $\Delta\rho = 0$, the BV integral of ρ is t-independent:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{L}_t} \rho = 0. \tag{16}$$

2.2. **BV pushforward.** Let $(\mathcal{F}, \omega)$ be the direct product of (-1)-symplectic graded manifolds $(\mathcal{F}', \omega')$ and $(\mathcal{F}'', \omega'')$:

$$\mathcal{F} = \mathcal{F}' \times \mathcal{F}'', \quad \omega = \omega' + \omega'' \tag{17}$$

– one thinks of this as splitting fields into "slow" and "fast" fields, or "infrared" and "ultraviolet," or "residual fields" and "fluctuations."

Fix a Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}''$. One has a map from half-densities on $\mathcal{F}$ to half-densities on $\mathcal{F}'$ given by the BV integral over $\mathcal{L} \subset \mathcal{F}''$

$$\int_{\mathcal{L}} : \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}) = \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}') \widehat{\otimes} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}'') \xrightarrow{\mathrm{id}\otimes\int_{\mathcal{L}}} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}'). \tag{18}$$

This map is the *fiber BV integral* or *BV pushforward* associated to the (-1)-symplectic fibration $P\colon \mathcal{F} \to \mathcal{F}'$. We will also denote the map (18) by $P^{\mathcal{L}}_*$.

The key properties of the BV pushforward are:

[13] Extension to field theory and infinite-dimensional spaces requires extra care (in particular, with regularization/renormalization of the BV Laplacian and with perturbative BV integrals) and generally goes beyond the scope of this paper. We will, however, have some field-theoretic examples with infinite-dimensional $\mathcal{F}$, see Sections 5.2, 7.1, 7.2, 8.3 and Examples 7.2, 7.5.

(i) The BV pushforward is a chain map with respect to BV Laplacians Δ, Δ' on half-densities on $\mathcal{F}$, $\mathcal{F}'$:

$$\int_{\mathcal{L}} \Delta\rho = \Delta' \int_{\mathcal{L}} \rho, \quad \rho \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}). \tag{19}$$

(ii) Let $\mathcal{L}_t$ be a family of Lagrangians in $\mathcal{F}''$ given as an image of $\mathcal{L}_0$ by the Hamiltonian flow generated by a time-dependent Hamiltonian $\Psi_t \in C^\infty(\mathcal{F})_{-1}$.[14] Let ρ be a half-density on $\mathcal{F}$ satisfying $\Delta\rho = 0$. Then one has[15]

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{L}_t} \rho = \Delta' \int_{\mathcal{L}_t} \Psi_t \rho. \tag{20}$$

I.e., the BV pushforward of a Δ-closed half-density changes with deformation of $\mathcal{L}$ by a Δ'-exact term.

2.3. **Exponential Δ-cocycles.**

2.3.1. *Quantum master equation.* Of particular interest are Δ-closed half-densities of the form[16]

$$\rho = \mu\, e^{\frac{\mathrm{i}}{\hbar} S} \tag{21}$$

with μ some fixed Δ-closed reference half-density on $\mathcal{F}$ and $S \in C^\infty(\mathcal{F})$ a *BV action*. The condition $\Delta\rho = 0$ is equivalent to the *quantum master equation* (QME) for S:

$$\frac{1}{2}\{S, S\} - \mathrm{i}\hbar\Delta_\mu S = 0. \tag{22}$$

Here $\{-,-\}$ is the degree 1 Poisson bracket on $C^\infty(\mathcal{F})$ associated with the symplectic form ω and $\Delta_\mu \colon f \mapsto \mu^{-1}\Delta(\mu f)$ is the BV Laplacian on functions on $\mathcal{F}$.

2.3.2. *Canonical transformations.* A family of Δ-closed exponential half-densities $\rho_t = \mu\, e^{\frac{\mathrm{i}}{\hbar} S_t}$ satisfies the property that $\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = \Delta(\rho_t R_t)$ for some $R_t \in C^\infty(\mathcal{F})_{-1}$ (in particular, the variation of ρ_t is Δ-exact) if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t} S_t = \{S_t, R_t\} - \mathrm{i}\hbar\Delta_\mu R_t \tag{23}$$

– then one says that S_t for different t are related by canonical transformations (or "quantum canonical BV transformations").

2.3.3. *Effective action.* Assume the splitting (17) and let μ, μ', μ'' be reference Δ-closed half-densities on $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ satisfying $\mu = \mu' \otimes \mu''$. Fix a solution S of the QME on $\mathcal{F}$. Then we have a Δ-closed element $\rho = \mu\, e^{\frac{\mathrm{i}}{\hbar} S}$. Its BV pushforward has the form

$$\mu' e^{\frac{\mathrm{i}}{\hbar} S'} = \int_{\mathcal{L}} \mu\, e^{\frac{\mathrm{i}}{\hbar} S} \tag{24}$$

with $S' \in C^\infty(\mathcal{F}')$ the *effective action*, induced on infrared fields.

The effective action satisfies the following properties due to (19), (20):

(i) The QME for S on $\mathcal{F}$ implies the QME for S' on $\mathcal{F}'$.

---

[14] Any smooth family of Lagrangians in an odd-symplectic manifold can be described in such a way, see [33].

[15] A more general formula, if we do not assume $\Delta\rho = 0$, is: $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{L}_t}\rho = \Delta'\int_{\mathcal{L}_t}\Psi_t\rho + \int_{\mathcal{L}_t}\Psi_t\Delta\rho$.

[16] In this section we are thinking of $\hbar$ as a finite positive number, with an eye toward considering $\hbar \to 0$ asymptotics.

(ii) A canonical transformation of S induces, via BV pushforward, a canonical transformation of S'.
(iii) A variation of the Lagrangian $\mathcal{L}$ induces a canonical transformation of S'.

2.3.4. *Observables, BV complex.* Given a solution S of QME on $\mathcal{F}$, one can study half-densities of the form

$$\rho_O = \mu\, e^{\frac{\mathrm{i}}{\hbar}S} O \tag{25}$$

with $O \in C^\infty(\mathcal{F})$. One has

$$-\mathrm{i}\hbar\Delta\rho_O = \rho_{Q_\hbar O}, \tag{26}$$

where

$$Q_\hbar = \{S, -\} - \mathrm{i}\hbar\Delta_\mu \tag{27}$$

is the BV differential on $C^\infty(\mathcal{F})$. We call $(C^\infty(\mathcal{F}), Q_\hbar)$ the *quantum BV complex*. Its cocycles – elements O satisfying $Q_\hbar O = 0$ – are the *quantum observables*.[17]

The full BV integral (14) maps ρ_O to the vacuum expectation value (or "correlator") $\langle O\rangle$ of O in the theory determined by the action S.

The BV pushforward maps ρ_O to a half-density on $\mathcal{F}'$ of the form $\rho_{O'} = \mu'\, e^{\frac{\mathrm{i}}{\hbar}S'} O'$ with O' the "effective" (or "induced") observable on the infrared fields. By construction, O' has the same expectation value as O.

The map

$$O \mapsto O' = \mu'^{-1} e^{-\frac{\mathrm{i}}{\hbar}S'} \int_{\mathcal{L}} \mu\, e^{\frac{\mathrm{i}}{\hbar}S} O \tag{28}$$

determined by the BV pushforward is a chain map of BV complexes

$$(C^\infty(\mathcal{F}), Q_\hbar) \to (C^\infty(\mathcal{F}'), Q'_\hbar) \tag{29}$$

with $Q'_\hbar = \{S', -\}' - \mathrm{i}\hbar\Delta'$ the BV differential on functions on infrared fields.

3. Reminder: homological perturbation lemma

In this section we review the definition of strong deformation retraction between cochain complexes and state the result on the effect of a deformation of the differential – the homological perturbation lemma. These are well known in the literature, see [15], [11] for details.

By default, we consider cochain complexes of finite-dimensional vector spaces over $\mathbb{R}$.

Definition 3.1. A *strong deformation retraction* (SDR) from a cochain complex $V^\bullet$ with differential d onto a cochain complex $V'^\bullet$ with differential d' is a triple of (i) a chain inclusion $i\colon V'^\bullet \to V^\bullet$, (ii) a chain projection $p\colon V^\bullet \to V'^\bullet$ and (iii) a chain homotopy $V^\bullet \to V^{\bullet-1}$ satisfying

$$dK + Kd = \mathrm{id} - ip, \tag{30}$$

$$pi = \mathrm{id} \tag{31}$$

and "side conditions"

$$K^2 = 0,\ Ki = 0,\ pK = 0. \tag{32}$$

[17] In the following we will often use the word "observable" more broadly – for any element of the BV complex. In that terminology, a cocycle is a "gauge-invariant observable."

We call the triple (i,p,K) "SDR data" and denote it by

$$(V,d) \overset{(i,p,K)}{\rightsquigarrow} (V',d') \tag{33}$$

or equivalently

$$K \circlearrowleft (V,d) \underset{p}{\overset{i}{\leftrightarrows}} (V',d'). \tag{34}$$

Note that, given an SDR, i and p are automatically *quasi-isomorphisms*, i.e., they induce mutually inverse isomorphisms i_*, p_* between the cohomology of (V,d) and cohomology of (V',d'). We say that i, p are mutually quasi-inverse.

Lemma 3.2 (Homological perturbation lemma)**.** *Fix SDR data (34) and consider a deformation of the differential on V from d to $\widetilde{d} = d + \delta$ for some $\delta\colon V \to V$ satisfying $(d+\delta)^2 = 0$. Then the deformed complex $(V,\widetilde{d})$ is quasi-isomorphic to V' equipped with deformed differential*

$$\widetilde{d'} = d' + \sum_{k\geq 0} p(-\delta K)^k \delta i \tag{35}$$

and one has the deformed SDR data

$$\widetilde{K} \circlearrowleft (V,\widetilde{d}) \underset{\widetilde{p}}{\overset{\widetilde{i}}{\leftrightarrows}} (V',\widetilde{d'}). \tag{36}$$

with

$$\widetilde{i} = \sum_{k\geq 0} (-K\delta)^k i, \;\; \widetilde{p} = \sum_{k\geq 0} p(-\delta K)^k, \;\; \widetilde{K} = \sum_{k\geq 0} K(-\delta K)^k. \tag{37}$$

Here we are assuming that δ is "sufficiently small," so that the sums over k above are convergent.[18]

3.1. **SDR on polynomial functions.** Given SDR (34), one has a dual SDR between the dual complexes,

$$K^\vee \circlearrowleft (V^*,d^\vee) \underset{p^\vee}{\overset{i^\vee}{\leftrightarrows}} (V'^*,d'^\vee). \tag{38}$$

Passing to the tensor algebras $TV^* = \bigoplus_{n\geq 0}(V^*)^{\otimes n}$ and TV'^*, with differentials given by extension of $d^\vee, d'^\vee$ by Leibniz identity, one has the SDR data

$$K^\otimes \circlearrowleft (TV^*,d^\vee) \underset{i^*}{\overset{p^*}{\leftrightarrows}} (TV'^*,d'^\vee), \tag{39}$$

where p^*, i^* are pullbacks by p, i (or, equivalently, extensions of $p^\vee, i^\vee$ to algebra maps). The chain homotopy on $(V^*)^{\otimes n}$ is given by

$$K^\otimes = \sum_{k=1}^{n} P'^{\otimes k-1} \otimes K^\vee \otimes (\mathrm{id})^{\otimes n-k}, \tag{40}$$

where we denoted $P' = p^\vee i^\vee$. This formula corresponds to the composition of SDRs

$$V\otimes V\otimes\cdots\otimes V \rightsquigarrow V'\otimes V\otimes\cdots\otimes V \rightsquigarrow \cdots \rightsquigarrow V'\otimes V'\cdots\otimes V', \tag{41}$$

[18]Typically, in examples, δ shifts some auxiliary grading on V (e.g. power in a formal parameter $\hbar$ or polynomial degree if V is the algebra of polynomials) – and K does not – which makes the convergence of sums above automatic. Also, it happens in some examples that sums terminate at finite k for a degree reason.

contracting the V factors one-by-one, going from left to right. One can choose a different order for contractions, which would result in permuting the factors in (40).

Passing to the symmetric algebras of V^*, V'^* (algebras of polynomials on V, V') by quotienting the tensor algebras by the symmetric group, one has the SDR

$$K^{\mathrm{sym}} \circlearrowleft (S^\bullet V^*, d^\vee) \underset{i^*}{\overset{p^*}{\leftrightarrows}} (S^\bullet V'^*, d'^\vee). \tag{42}$$

Here p^*, i^* the pullback of polynomials by p, i. The chain homotopy acting on $S^n V^*$ is the symmetrization of (40) – averaging over possible orderings of factors:[19]

$$K^{\mathrm{sym}} = \sum_{k=1}^{n} \sum_{X_i \in \{P', \mathrm{id}\}} c_{\#\{P'\},\#\{\mathrm{id}\}} X_1 \otimes \cdots \otimes X_{k-1} \otimes K^\vee \otimes X_{k+1} \otimes \cdots \otimes X_n. \tag{43}$$

In the sum each factor (except the one where $K^\vee$ is inserted) can be either P' or id and the coefficient is $c_{M,N} = \frac{M!N!}{(M+N+1)!}$ with M, N the numbers of P' and id factors.

3.2. **Equivalent formulae for K^{sym}.**

3.2.1. *Topological quantum mechanics formula.* Equivalently to (43), one can write K^{sym} as an integral[20]

$$K^{\mathrm{sym}}(f)(x) = \int_0^\infty f(e^{-(\mathrm{id}-ip)t + dt\, K} x) \tag{44}$$

in the variable t. Here f is a polynomial on V and $x \in V$ is a variable.

Remark 3.3. Topological quantum mechanics in the sense of A. Losev [27, Section 2.4] is a graded vector space V (the space of states) equipped with two differentials d and G of degree $+1$ and -1, respectively, such that the (d-exact) Hamiltonian $H = [d, G]$ is non-negative, with kernel $\ker H = V'$.[21] Additionally, we assume that G vanishes on $\ker H$. The partition function of the TQM on an interval is defined as

$$Z(t, \mathrm{d}t) = e^{-tH + \mathrm{d}t\, G} \tag{45}$$

– a nonhomogeneous form on $[0, +\infty)$ (the space of lengths of the interval) valued in operators on V.[22] Note that one has $Z|_{t=0} = \mathrm{id}$, $Z|_{t\to\infty}$ is the projection P' onto $\ker H = V'$, $\int_0^\infty Z(t, \mathrm{d}t) = G(H + cP')^{-1}$ is the chain homotopy contracting V onto V' (for any $c > 0$).

If we set $G = K$, $Z(t, \mathrm{d}t)$ becomes the operator $e^{-t[d,K] + \mathrm{d}t\, K}$ appearing in the integrand in (44).

---

[19] It might not be immediately obvious that the symmetrized chain homotopy satisfies $(K^{\mathrm{sym}})^2 = 0$. However, this follows from the fact that all terms in the sum (43) mutually anti-commute.

[20] This formula is related by a change of variable t to formulae (325,326) in [30] and (19) in [8]: $t_{\text{loc. cit.}} = 1 - e^{-t}$.

[21] This structure is also known as $\mathcal{N} = 2$ supersymmetric quantum mechanics with supercharges $Q_1 = d + G$, $Q_2 = i(d - G)$.

[22] Alternatively, one can think of $Z(t, \mathrm{d}t)$ as a heat flow for a d-exact heat operator (the 0-form component of Z), together with its infinitesimal homotopy (the 1-form component of Z).

Proof of (44). The r.h.s. of (44) can be written as

$$\int_0^\infty f((e^{-t}\mathrm{id} + (1-e^{-t})ip + \mathrm{d}t\, e^{-t}K)x). \tag{46}$$

Evaluated on a monomial f on V, it returns the r.h.s. of (43) (evaluated on $x \odot \cdots \odot x$), with correct coefficients, since

$$\begin{aligned}\int_0^\infty \mathrm{d}t\, e^{-t}(e^{-t})^N(1-e^{-t})^M &\underset{\tau=1-e^{-t}}{=} \int_0^1 \mathrm{d}\tau\, \tau^M(1-\tau)^N \\ &= B(M+1,N+1) = \frac{M!N!}{(M+N+1)!} = c_{M,N}\end{aligned} \tag{47}$$

is the Euler's beta function integral. □

Remark 3.4. The fact that the r.h.s. of (44) satisfies the chain homotopy property (30) can be shown directly as follows: the TQM partition function (45) satisfies $(\mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t} - [d,-])Z = 0$. Hence, the integrand Z^*f (the pullback by Z) in (44) satisfies

$$(\mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t} - [d^\vee,-])Z^* = 0. \tag{48}$$

Integrating over t and using Stokes', we obtain

$$Z^*\big|_{t=\infty} - Z^*\big|_{t=0} + \left[d^\vee, \int_0^\infty Z^*\right] = p^*i^* - \mathrm{id} + [d^\vee, K^{\mathrm{sym}}] = 0 \tag{49}$$

– the chain homotopy property.

3.2.2. *The P', P'' formula.* Let $P'' = \mathrm{id} - p^\vee i^\vee$ – the projector onto the second summand in $V = p^\vee(V'^*) \oplus V''^*$, with $P' = p^\vee i^\vee$ the projector onto the first summand. Sometimes it is convenient to rewrite (43) in terms of insertions of P', P'' instead of P', id as

$$K^{\mathrm{sym}} = \sum_{k=1}^n \sum_{Y_i \in \{P', P''\}} \frac{1}{\#\{P''\}+1} Y_1 \otimes \cdots \otimes Y_{k-1} \otimes K^\vee \otimes Y_{k+1} \otimes \cdots \otimes Y_n. \tag{50}$$

The fact that the coefficient of the term with M insertions of P' and N of P'' is $1/(N+1)$ is due to the fact that (46) can be written as

$$\int_0^\infty f((ip + e^{-t}(\mathrm{id} - ip) + \mathrm{d}t\, e^{-t}K)x) \tag{51}$$

and

$$\int_0^\infty \mathrm{d}t\, e^{-t}1^M(e^{-t})^N = \frac{1}{N+1}. \tag{52}$$

3.2.3. *Euler vector field formula for K^{sym}.* Let $\nu\colon SV^* \to SV^*$ be the linear operator mapping f to $\frac{1}{N}f$ for f homogeneous, of polynomial degree $N \geq 1$ in V'', and returning zero on SV'^*. Let $\widehat{K}$ be the extension of $K^\vee$ to a derivation of SV^*.

One can rewrite (50) as[23]

$$K^{\mathrm{sym}} = \nu\widehat{K}. \tag{53}$$

[23]This formula appears in the proof of Proposition 2.5.5 in [18].

Equivalently, introducing the "Lie derivative along the Euler vector field on V''" $\mathcal{L}_E$ – the extension of $\mathrm{id} - p^\vee i^\vee$ to a derivation of SV^* – one can write

$$K^{\mathrm{sym}} = (\mathcal{L}_E + c\, p^* i^*)^{-1} \widehat{K} \tag{54}$$

with $c > 0$ an arbitrary positive constant.

4. SDR data for free BV theory

Consider a linear version of the setting of Section 2.

Let the space of fields $(\mathcal{F}, d)$ be a cochain complex equipped with a degree -1 nondegenerate pairing ω (a constant (-1)-symplectic form) compatible with d (i.e., $\omega(x, dy) = \pm\omega(dx, y)$) and a constant "reference" half-density $\mu \in (\mathrm{Det}\, \mathcal{F}^*)^{\otimes \frac{1}{2}}$.

The algebra of polynomial functions on fields $S\mathcal{F}^*$ has the structure of a BV algebra, with degree 1 Poisson bracket and the BV Laplacian induced by ω. Using a basis in $\mathcal{F}$, one has

$$\{f, g\} = (\omega^{-1})^{ij} f \frac{\overleftarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial}}{\partial x^j} g, \tag{55}$$

$$\Delta f = \frac{1}{2} (\omega^{-1})^{ij} \frac{\partial^2}{\partial x^i \partial x^j} f. \tag{56}$$

Assume that $\mathcal{F}$ is split as a direct sum of complexes

$$\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}'' \tag{57}$$

with d', d'' the differentials on $\mathcal{F}', \mathcal{F}''$ and assume that $(\mathcal{F}'', d'')$ is acyclic. Furthermore, assume that $\mathcal{F}', \mathcal{F}''$ are equipped with compatible degree -1 pairings ω', ω'' and constant reference half-densities μ', μ'' satisfying $\omega = \omega' + \omega''$ and $\mu = \mu' \otimes \mu''$.

Consider the quadratic function on $\mathcal{F}$

$$S_0(x) = \frac{1}{2} \omega(x, dx) \tag{58}$$

– the "free BV action." It satisfies the QME (22) by virtue of $d^2 = 0$ and d being traceless (by degree reason).

Similarly, we have free BV actions for $\mathcal{F}'$ and $\mathcal{F}''$:

$$S_0'(x') = \frac{1}{2} \omega'(x', d'x'), \quad S_0''(x'') = \frac{1}{2} \omega''(x'', d''x'') \tag{59}$$

with $x' \in \mathcal{F}'$, $x'' \in \mathcal{F}''$.

4.1. **SDR for classical free BV theory.** Let ι, π be the inclusion of the first summand and the projection to the first summand in (57). Choose a chain homotopy κ between id and $\iota\pi$ satisfying the side conditions (32), so that one has a SDR

$$\kappa \circlearrowleft (\mathcal{F}, d) \underset{\pi}{\overset{\iota}{\leftrightarrows}} (\mathcal{F}', d') \tag{60}$$

Additionally, we assume compatibility of the SDR data with ω, ω':

$$\omega(x, \iota(y')) = \omega'(\pi(x), y')), \quad \omega(x, \kappa(y)) = \pm\omega(\kappa(x), y). \tag{61}$$

Equivalently, we require that the SDR (60) is self-dual with respect to ω, ω', cf. (38) for the definition of duality of SDR.

Next, we promote the SDR (60) to an SDR for polynomial functions using the construction of Section 3.1:

$$K = \kappa^{\mathrm{sym}} \circlearrowleft (S\mathcal{F}^*, Q_0) \underset{p=\iota^*}{\overset{i=\pi^*}{\leftrightarrows}} (S\mathcal{F}'^*, Q_0'). \tag{62}$$

Here $Q_0 = \{S_0, -\}$, $Q_0' = \{S_0', -\}'$ is the extension of dual differentials $d^\vee, d'^\vee$ to derivations of $S\mathcal{F}^*, S\mathcal{F}'^*$.

Example 4.1.[24] Given an SDR of cochain complexes

$$\kappa_W \circlearrowleft (W, d_W) \underset{\pi_W}{\overset{\iota_W}{\leftrightarrows}} (W', d_{W'}), \tag{63}$$

set the spaces of fields to be the cotangent complexes

$$\mathcal{F} = T^*[-1]W = W \oplus W^*[-1], \quad \mathcal{F}' = T^*[-1]W' = W' \oplus W'^*[-1] \tag{64}$$

equipped with differentials $d = d_W \oplus d_W^\vee$, $d' = d_{W'} \oplus d_{W'}^\vee$ and with symplectic forms ω, ω' induced by the canonical pairing $\langle , \rangle$ between W and W^* and between W' and W'^*.

The BV action on W is

$$S_0(A, B) = \langle B, d_W A \rangle \tag{65}$$

with fields $A \in W$, $B \in W^*[-1]$, and S_0' on W' is similar.

One obtains SDR (60) as a direct sum of (63) and its dual:

$$\kappa_W \oplus \kappa_W^\vee \circlearrowleft (\mathcal{F}, d) \underset{\pi_W \oplus \iota_W^\vee}{\overset{\iota_W \oplus \pi_W^\vee}{\leftrightarrows}} (\mathcal{F}', d'). \tag{66}$$

Note that there are no self-duality conditions (61) imposed on (63), whereas for (66) they are fulfilled automatically.

Example 4.2 (Toy free scalar theory)**.** Set $\mathcal{F} = \mathbb{R} \oplus \mathbb{R}[-1]$ with even degree 0 coordinate x and odd degree -1 coordinate ξ. We take $\omega = \delta x \wedge \delta\xi$ for the symplectic form and set $S_0 = \frac{1}{2}x^2$ for the action. This action corresponds to the differential $d\colon e \mapsto \epsilon$ on $\mathcal{F}$, with e, ϵ basis vectors of degree $0, 1$ in the two summands in $\mathcal{F}$. We set $\mathcal{F}' = 0$ (note that $(\mathcal{F}, d)$ is a contractible complex). The SDR (60) is unique

$$\kappa \circlearrowleft (\mathbb{R} \oplus \mathbb{R}[-1], d) \underset{0}{\overset{0}{\leftrightarrows}} 0 \tag{67}$$

with $\kappa\colon \epsilon \mapsto e$. The associated SDR (62) on polynomial functions on fields is

$$K \circlearrowleft (\mathbb{C}[x, \xi], Q_0) \underset{p}{\overset{i}{\leftrightarrows}} \mathbb{C}. \tag{68}$$

Here $Q_0 = x\frac{\partial}{\partial\xi}\colon x^N\xi \mapsto x^{N+1}$; i is the inclusion of constants into polynomials of x, ξ; p extracts the constant term of a polynomial and

$$K\colon \begin{cases} x^N \mapsto x^{N-1}\xi, & N \geq 1, \\ 1 \mapsto 0. \end{cases} \tag{69}$$

[24]This example corresponds to "abstract abelian BF theory" in the terminology of [29], [30], or "cotangent theory" in the terminology of [10].

Example 4.3 (Toy free scalar cont'd)**.** Generalizing the previous example, set $\mathcal{F} = \mathbb{R}^n \oplus \mathbb{R}^n[-1]$ with even degree 0 coordinates $x^1, \dots, x^n$ and odd degree -1 coordinates $\xi_1, \dots, \xi_n$. Set $\omega = \delta x^i \wedge \delta \xi_i$ and set $S_0 = \frac{1}{2} M_{ij} x^i x^j$, where M is a fixed *nondegenerate* symmetric $n \times n$ matrix. This S_0 corresponds to the differential $d\colon e_i \mapsto M_{ij}\epsilon^j$ on $\mathcal{F}$, with $\{e_i\}, \{\epsilon^i\}$ the standard basis on the first and second summand in $\mathcal{F}$. By nondegeneracy of M, $(\mathcal{F}, d)$ is again contractible, so one can set $\mathcal{F}' = 0$ and one has SDR

$$\kappa \circlearrowleft (\mathbb{R}^n \oplus \mathbb{R}^n[-1], d) \underset{0}{\overset{0}{\leftrightarrows}} 0 \tag{70}$$

with $\kappa\colon \epsilon^i \mapsto (M^{-1})^{ij} e_j$ as the unique possible chain contraction. Passing to polynomial functions of fields, one has

$$K \circlearrowleft (\mathbb{C}[x^1, \dots, x^n, \xi_1, \dots, \xi_n], Q_0) \underset{p}{\overset{i}{\leftrightarrows}} \mathbb{C} \tag{71}$$

with $Q_0 = M_{ij} x^i \frac{\partial}{\partial \xi_j}$; i, p are the inclusion of constants and extraction of constant term. The chain homotopy acts on a homogeneous polynomial as

$$K\colon \begin{cases} f(x,\xi) \mapsto \frac{1}{\deg_{x,\xi}(f)} (M^{-1})^{ij} \xi_i \frac{\partial}{\partial x^j} f(x,\xi) & \text{if} \quad \deg_{x,\xi}(f) \geq 1, \\ \text{const} \mapsto 0. \end{cases} \tag{72}$$

with $\deg_{x,\xi}(f)$ the polynomial degree of f (where x and ξ are counted with degree $+1$).

We remark that formula (72) is a special case of (53).

Example 4.4 (Free massive scalar on a graph)**.** Given a graph Γ (possibly with lengths assigned to edges) with V the set of vertices, one can consider the toy free massive scalar field theory on Γ – the special case of Example 4.3 with $\mathcal{F} = T^*[-1]\mathbb{R}^V$ and $M = \Delta_\Gamma + m^2$ with Δ_Γ the graph Laplacian and $m > 0$ a fixed mass parameter.

4.2. **Free quantum theory.**

Theorem 4.5. *Deformation of the differential $Q_0 \to Q_{0,\hbar} = Q_0 - \mathrm{i}\hbar\Delta$ on $S\mathcal{F}^*$ induces, via the homological perturbation lemma (Lemma 3.2), the following SDR of BV complexes (Section 2.3.4):*

$$K_\hbar \circlearrowleft (S\mathcal{F}^*, Q_{0,\hbar}) \underset{p_\hbar}{\overset{i_\hbar}{\leftrightarrows}} (S\mathcal{F}'^*, Q'_{0,\hbar}), \tag{73}$$

where

(i) $Q'_{0,\hbar} = Q'_0 - \mathrm{i}\hbar\Delta'$.
(ii) $i_\hbar = i = \pi^*$.
(iii) $p_\hbar$ *is the BV pushforward map (28) defined by free actions* S_0, S'_0,

$$p_\hbar\colon O \mapsto O' = \mu'^{-1} e^{-\frac{\mathrm{i}}{\hbar} S'_0} \int_{\mathcal{L}} \mu\, e^{\frac{\mathrm{i}}{\hbar} S_0} O, \tag{74}$$

with the Lagrangian subspace

$$\mathcal{L} = \mathrm{im}(\kappa) \subset \mathcal{F}''. \tag{75}$$

This theorem (in the case $\mathcal{F}' = 0$) is essentially contained in [18], [19]; the case with $\mathcal{F}' \neq 0$ is contained in [12, Proof of Theorem 2].

Proof. Item (i): by homological perturbation lemma (35), the deformed differential on the retract is

$$Q'_0 - \iota^*(\mathrm{i}\hbar\Delta)\pi^* - \sum_{k\geq 1} \iota^*(\mathrm{i}\hbar\Delta)(K(\mathrm{i}\hbar\Delta))^k\pi^*. \tag{76}$$

Note that $\Delta\pi^* = \pi^*\Delta'$. This implies that (a) the second term in (76) is $-\mathrm{i}\hbar\Delta'$ and (b) $K\Delta\pi^* = 0$ and hence the third term in (76) vanishes. Thus, the deformed differential is $Q'_{0,\hbar} = Q'_0 - \mathrm{i}\hbar\Delta'$.

Item (ii): by equation (37) of homological perturbation lemma, the deformed inclusion is

$$i_\hbar = \pi^* + \sum_{k\geq 1} (K(\mathrm{i}\hbar\Delta))^k\pi^*. \tag{77}$$

Using again that $K\Delta\pi^* = 0$, the second term above vanishes, yielding $i_\hbar = \pi^*$.

Item (iii): By homological perturbation lemma (37), we have

$$p_\hbar = \sum_{k\geq 0} \iota^*(\mathrm{i}\hbar\Delta K)^k. \tag{78}$$

Let x^i be a basis in $\mathcal{F}^*$ splitting into a basis $x^{i'}$ in $\mathcal{F}'^*$ and $x^{i''}$ in $\mathcal{F}''^*$. Consider the $k=1$ term in (78) applied to some $O \in S\mathcal{F}^*$:

(79)

$$\begin{aligned}\iota^*\mathrm{i}\hbar\Delta KO &\underset{(53)}{=} \mathrm{i}\hbar\,\iota^*\Delta\widehat{\kappa}\nu O \underset{(56)}{=} \left(\mathrm{i}\hbar\frac{1}{2}(\omega^{-1})^{ij}\frac{\partial^2}{\partial x^i\partial x^j}(\kappa^\vee)^{k''}{}_{l''}\,x^{l''}\frac{\partial}{\partial x^{k''}}\nu O\right)\Big|_{x''=0}\\ &= \left(\mathrm{i}\hbar(\omega''^{-1}\kappa^\vee)^{i''k''}\frac{\partial^2}{\partial x^{i''}\partial x^{k''}}\nu O\right)\Big|_{x''=0} = \left(\frac{\mathrm{i}\hbar}{2}h^{i''k''}\frac{\partial^2}{\partial x^{i''}\partial x^{k''}}O\right)\Big|_{x''=0}\\ &= \frac{\mathrm{i}\hbar}{2}\eta O|_{x''=0}.\end{aligned}$$

Here ν is as in Section 3.2.3 and we denoted

$$h = \omega''^{-1}\kappa^\vee \in \mathrm{Hom}(\mathcal{F}''^*, \mathcal{F}'')_{\mathrm{self-dual}} \cong S^2\mathcal{F}'' \tag{80}$$

– the *propagator*. We also denoted[25]

$$\eta = \omega''^{-1}\left(\frac{\partial}{\partial x''}, \kappa^\vee\frac{\partial}{\partial x''}\right) \tag{81}$$

the extension of h to a second-order derivation of $S\mathcal{F}^*$. Transitioning to the second line, we used that the derivative in x^i or x^j must hit $x^{l''}$ factor, otherwise the term would not survive when restricting to $x''=0$. In the last equality we used that for the restriction to $x''=0$ to be nonzero, O must be of order 2 in x'' and hence $\nu O = \frac{1}{2}O$.

The $k=2$ term in (78) is computed by replacing $O \to \mathrm{i}\hbar\Delta KO$ in (79):

$$\begin{aligned}&\iota^*(\mathrm{i}\hbar\Delta K)^2 O =\\ &\quad = (\mathrm{i}\hbar)^2\frac{1}{2}h^{i''_1k''_1}\frac{\partial^2}{\partial x^{i''_1}x^{k''_1}}\frac{1}{2}(\omega^{-1})^{i_2j_2}\frac{\partial^2}{\partial x^{i_2}\partial x^{j_2}}(\kappa^\vee)^{k''_2}{}_{l''_2}\,x^{l''_2}\frac{\partial}{\partial x^{k''_2}}\nu O\Big|_{x''=0}.\end{aligned} \tag{82}$$

Here we note that $x^{l''_2}$ must be "eaten" by the derivative in x^{i_2} or x^{j_2}, but not by derivative in $x^{i''_1}$ or $x^{k''_1}$, since in the latter cases the result would vanish due

[25]We remark that in similar notations the derivation $\widehat{\kappa}$ is $\langle x'', \kappa^\vee\frac{\partial}{\partial x''}\rangle$ with $\langle,\rangle$ the canonical pairing between $\mathcal{F}$ and $\mathcal{F}^*$.

$h\kappa^\vee = \omega''^{-1}(\kappa^\vee)^2 = 0$, which follows from $\kappa^2 = 0$. Also, for the expression to be nonzero O must be of order 4 in x'' and so $\nu O = \frac{1}{4}O$. Thus, one has

$$\iota^*(\mathrm{i}\hbar\Delta K)^2 O = \\ = \frac{(\mathrm{i}\hbar)^2}{2\cdot 4} h^{i_1''k_1''} \frac{\partial^2}{\partial x^{i_1''}\partial x^{k_1''}} h^{i_2''k_2''} \frac{\partial^2}{\partial x^{i_2''}\partial x^{k_2''}} O\Big|_{x''=0} = \frac{(\mathrm{i}\hbar)^2}{2\cdot 4}\eta^2 O\Big|_{x''=0}. \tag{83}$$

Iterating the construction, for the k-th term in (79) we obtain

$$\iota^*(\mathrm{i}\hbar\Delta K)^k O = \frac{(\mathrm{i}\hbar)^k}{2^k k!}\eta^k O\Big|_{x''=0}. \tag{84}$$

Thus, for $p_\hbar$ we obtain

$$p_\hbar O = \sum_{k\geq 0}\frac{(\mathrm{i}\hbar)^k}{2^k k!}\eta^k O\Big|_{x''=0} = e^{\frac{\mathrm{i}\hbar}{2}\eta} O\Big|_{x''=0} \\ = \left(c\int_{\mathcal{L}} \mu'' e^{\frac{\mathrm{i}}{\hbar}S_0''(x'')} O(x'+x'')\right)\Big|_{x''=0}, \tag{85}$$

with $c = \left(\int_{\mathcal{L}} \mu''\, e^{\frac{\mathrm{i}}{\hbar}S_0''}\right)^{-1}$ a normalization constant. Here the last transition uses a standard formula for the perturbative expansion of a Gaussian momentum of O in terms of Wick contractions. The operator η applies a single Wick contraction, and η^k applies k Wick contractions; at the end the remaining variables are set to be infrared. This finishes the proof of (74). □

Remark 4.6. We augment Theorem 4.5 with a formula for $K_\hbar$: for $O \in S\mathcal{F}^*$ of degree $n \geq 1$ in x'', one has

$$K_\hbar O = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \frac{(\mathrm{i}\hbar)^j}{n(n-2)\cdots(n-2j)}\widehat{\kappa}\eta^j O, \tag{86}$$

and for O independent of x'', $K_\hbar O = 0$.

The proof follows the similar logic to that of (iii) of Theorem 4.5. By (37),

$$K_\hbar = K + \mathrm{i}\hbar K\Delta K + (\mathrm{i}\hbar)^2 K\Delta K\Delta K + \cdots \tag{87}$$

Applying to O, by (53), the first term is $\frac{1}{n}\widehat{\kappa}O$, which is the first term in (86). The second term in (87) yields

$$\mathrm{i}\hbar\,\widehat{\kappa}\nu\Delta\widehat{\kappa}\nu O = \frac{\mathrm{i}\hbar}{n(n-2)}(\kappa^\vee)^{a''}{}_{b''}\, x^{a''}\frac{\partial}{\partial x^{b''}}\frac{1}{2}(\omega^{-1})^{ij}\frac{\partial^2}{\partial x^i\partial x^j}(\kappa^\vee)^{k''}{}_{l''}\, x^{l''}\frac{\partial}{\partial x^{k''}}O. \tag{88}$$

Here the factor $\frac{1}{n(n-2)}$ comes from the two ν factors. The factor $x^{l''}$ must be "eaten" by a derivative in x^i or x^j: otherwise it is either (a) eaten by the derivative in $x^{b''}$, in which case the expression vanishes by $\kappa^2 = 0$, or (b) not eaten at all, in which case one has a cancellation due to signs.[26] Thus, (88) evaluates to

$$\frac{\mathrm{i}\hbar}{n(n-2)}\widehat{\kappa}\eta O. \tag{89}$$

[26]The cancellation is $(\kappa^\vee\otimes\mathrm{id})\circ(\mathrm{id}\otimes\kappa^\vee) + (\mathrm{id}\otimes\kappa^\vee)\circ(\kappa^\vee\otimes\mathrm{id}) = \kappa^\vee\otimes\kappa^\vee - \kappa^\vee\otimes\kappa^\vee = 0$.

Subsequent terms in (87) are computed similarly, yielding the terms of (86). The range for j in (86) is such that $\eta^j O$ is of positive degree in x'' and so $\widehat{\kappa}\eta^j O$ can be nonzero.

An equivalent way to write $K_\hbar$, as follows immediately from (86), is

$$K_\hbar = \sum_{j\geq 0} \widehat{\kappa}\nu(\mathrm{i}\hbar\eta\nu)^j = \widehat{\kappa}\nu(1-\mathrm{i}\hbar\eta\nu)^{-1}. \tag{90}$$

Remark 4.7. Since the differential $Q_{0,\hbar}$ is the conjugation of the BV Laplacian, $Q_{0,\hbar} = e^{-\frac{\mathrm{i}}{\hbar}S_0}(-\mathrm{i}\hbar\Delta)e^{\frac{\mathrm{i}}{\hbar}S_0}$, we have an isomorphism of complexes

$$(S\mathcal{F}^*, Q_{0,\hbar}) \xrightarrow{\cdot e^{\frac{\mathrm{i}}{\hbar}S_0}} (\mathrm{Fun}(\mathcal{F}), -\mathrm{i}\hbar\Delta), \tag{91}$$

where $\mathrm{Fun}(\mathcal{F}) = \{e^{\frac{\mathrm{i}}{\hbar}S_0}O \mid O \text{ polynomial on } \mathcal{F}\}$. We also have a similar isomorphism for functions of infrared fields:

$$(S\mathcal{F}'^*, Q'_{0,\hbar}) \xrightarrow{\cdot e^{\frac{\mathrm{i}}{\hbar}S'_0}} (\mathrm{Fun}(\mathcal{F}'), -\mathrm{i}\hbar\Delta'), \tag{92}$$

with $\mathrm{Fun}(\mathcal{F}') = \{e^{\frac{\mathrm{i}}{\hbar}S'_0}O' \mid O' \text{ polynomial on } \mathcal{F}'\}$. One can then transport SDR data (73) along these isomorphisms to an SDR

$$K_\Delta \circlearrowleft (\mathrm{Fun}(\mathcal{F}), -\mathrm{i}\hbar\Delta) \underset{p_\Delta}{\overset{i_\Delta}{\leftrightarrows}} (\mathrm{Fun}(\mathcal{F}'), -\mathrm{i}\hbar\Delta') \tag{93}$$

where

$$i_\Delta \colon f' \mapsto e^{\frac{\mathrm{i}}{\hbar}S''_0}\pi^*(f'), \quad p_\Delta \colon f \mapsto c\int_{\mathcal{L}} \mu''\, f, \quad K_\Delta \colon f \mapsto e^{\frac{\mathrm{i}}{\hbar}S_0}K_\hbar(e^{-\frac{\mathrm{i}}{\hbar}S_0}f). \tag{94}$$

4.2.1. *Example: toy free scalar field – quantum version.* Returning to Example 4.2 and turning on the deformation of the differential $Q_0 = \xi\frac{\partial}{\partial x}$ in (68) by $\Delta = \frac{\partial^2}{\partial x \partial \xi}$, we obtain

$$K_\hbar \circlearrowleft (\mathbb{C}[x,\xi], Q_0 - \mathrm{i}\hbar\Delta) \underset{p_\hbar}{\overset{i_\hbar = i}{\leftrightarrows}} \mathbb{C}. \tag{95}$$

Here the deformed projection is

$$p_\hbar \colon O(x) \mapsto c\int_{-\infty}^{\infty} \mathrm{d}x\, e^{\frac{\mathrm{i}}{\hbar}\frac{x^2}{2}} O(x) \tag{96}$$

and maps functions linear in ξ to zero; the normalization constant is

$$c = (2\pi\hbar)^{-\frac{1}{2}} e^{-\frac{\pi\mathrm{i}}{4}}. \tag{97}$$

In particular, $p_\hbar$ acts on monomials by

$$p_\hbar(x^n) = \begin{cases} (\mathrm{i}\hbar)^m(2m-1)!!, & n = 2m \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \tag{98}$$

The deformed chain homotopy can be easily computed from (86), with $\widehat{\kappa} = \xi\frac{\partial}{\partial x}$, $\eta = \frac{\partial^2}{\partial x^2}$. One finds

$$\begin{aligned} K_\hbar \colon x^n \mapsto \xi(x^{n-1} + \mathrm{i}\hbar(n-1)x^{n-3} + (\mathrm{i}\hbar)^2(n-1)(n-3)x^{n-5} + \cdots) \\ = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} (\mathrm{i}\hbar)^j \frac{(n-1)!!}{(n-1-2j)!!}\xi x^{n-1-2j} \end{aligned} \tag{99}$$

for $n \geq 1$. In the first line, the sum stops when the exponent for x reaches 0 (for n odd) or 1 (for n even). One can also write the result as follows:

$$K_\hbar \colon O(x) \mapsto \left[\xi \left(1 - \frac{\mathrm{i}\hbar}{x} \frac{\partial}{\partial x} \right)^{-1} \frac{1}{x} O(x) \right]_{\mathrm{reg}}, \tag{100}$$

where $[\cdots]_{\mathrm{reg}}$ means that we subtract the singular part of the Laurent series at $x = 0$. Yet another way to write $K_\hbar$ is

$$K_\hbar \colon O(x) \mapsto \xi \frac{\mathrm{i}}{\hbar} e^{-\frac{\mathrm{i}}{\hbar}\frac{x^2}{2}} \int^x \mathrm{d}y \, e^{\frac{\mathrm{i}}{\hbar}\frac{y^2}{2}} (O(y) - p_\hbar(O)). \tag{101}$$

Here the integral is taken over the contour γ_x from $e^{\mathrm{i}\alpha} \cdot (+\infty)$ to x on the complex y plane, with some fixed $\alpha \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$. It is easy to see that the result does not depend on α.[27]

The conjugate SDR data (93) in this example is

$$K_\Delta \circlearrowleft (e^{\frac{\mathrm{i}}{\hbar}\frac{x^2}{2}} \mathbb{C}[x, \xi], -\mathrm{i}\hbar\Delta) \underset{p_\Delta \colon f \mapsto c \int_{-\infty}^{\infty} \mathrm{d}x f(x)}{\overset{e^{\frac{\mathrm{i}}{\hbar}\frac{x^2}{2}} \hookleftarrow 1 \colon i_\Delta}{\leftrightarrows}} \mathbb{C} \tag{102}$$

with

$$K_\Delta \colon f(x) \mapsto \xi \frac{\mathrm{i}}{\hbar} \int^x \mathrm{d}y \left(f(y) - e^{\frac{\mathrm{i}}{\hbar}\frac{y^2}{2}} c \int_{-\infty}^{\infty} \mathrm{d}z f(z) \right) \tag{103}$$

for $f(x) = e^{\frac{\mathrm{i}}{\hbar}\frac{x^2}{2}} O(x)$, $O(x)$ a polynomial. Written in this form, it is obvious that K_Δ inverts $-\mathrm{i}\hbar\Delta = -\mathrm{i}\hbar \frac{\partial^2}{\partial x \partial \xi}$.

4.3. **Cable diagrams.** It is convenient to visualize operators on the symmetric algebra $S\mathcal{F}^*$ arising in homological perturbation lemma by "cable diagrams." A cross-section of a cable with n threads represents an element of $S^n\mathcal{F}^*$; individual threads carry elements of $\mathcal{F}^*$. An operation can apply linear operators to individual threads, or it can have several inputs and/or outputs in $\mathcal{F}^*$, merging and/or splitting threads. A composition of operations is represented as a horizontal concatenation of cables.

We will show elements of $\mathcal{F}''^*$ as solid lines, elements of $\mathcal{F}'^*$ as dashed lines and elements of $\mathcal{F}^*$ that could be either in $\mathcal{F}'^*$ or in $\mathcal{F}''^*$ as thick solid lines.

Terms in formula (50) in the context of (62) apply $\kappa^\vee$ to one thread (denoted by a black dot) and distribute P', P'' over the other threads (shown as dashed or solid threads passing through). BV Laplacian Δ merges a pair of lines, replacing two linear forms $\alpha, \beta \in \mathcal{F}^*$ with the number $\omega^{-1}(\alpha, \beta)$. Moreover Δ splits into $\Delta' + \Delta''$, with Δ' merging a pair of dashed lines and Δ'' merging a pair of solid lines.

With these conventions, terms in the homological perturbation formula (78) for $p_\hbar$,

$$p_\hbar = p + \mathrm{i}\hbar p\Delta K + (\mathrm{i}\hbar)^2 p\Delta K\Delta K + \cdots \tag{104}$$

[27](101) is proven by evaluating on monomials x^n using iterated integration by parts and comparing to (99). Indeed: let $\Phi_n = \frac{\mathrm{i}}{\hbar} e^{-\frac{\mathrm{i}}{\hbar}\frac{x^2}{2}} \int^x \mathrm{d}y \, e^{\frac{\mathrm{i}}{\hbar}\frac{y^2}{2}} y^n$. Integrating by parts, we have $\Phi_n = x^{n-1} + \mathrm{i}\hbar(n-1)\Phi_{n-2}$. Iterating this relation, for $n = 2m+1$ odd, we obtain $\Phi_n = x^{n-1} + \mathrm{i}\hbar(n-1)x^{n-3} + (\mathrm{i}\hbar)^2(n-1)(n-3)x^{n-5} + \cdots + (\mathrm{i}\hbar)^m(n-1)!!$. Thus, taking into account $p_\hbar(x^{2m+1}) = 0$, (101) agrees with (99) on odd-degree monomials. For $n = 2m$ even, we get $\Phi_n = x^{n-1} + \mathrm{i}\hbar(n-1)x^{n-3} + (\mathrm{i}\hbar)^2(n-1)(n-3)x^{n-5} + \cdots + (\mathrm{i}\hbar)^{m-1}(n-1)!!x + (\mathrm{i}\hbar)^m(n-1)!!\Phi_0$. Thus, (99) gives $K_\hbar(x^n) = \xi(\Phi_n - (\mathrm{i}\hbar)^m(n-1)!!\Phi_0)$, which agrees with (101).

correspond to cable diagrams as in Figure 1. Note that a black dot on one of the two merging threads can be slid over to the other thread, by self-adjointness of κ (61).

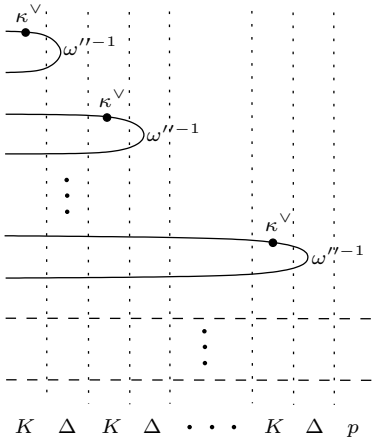

FIGURE 1. General cable diagram for a term in the homological perturbation series (104 for $p_\hbar$.

Such cable diagrams correspond – by pinching the left side of the cable, forgetting the horizontal tiered structure, and recalling that $\omega''^{-1}\kappa^\vee = h$ – to Feynman diagrams computing the BV pushforward of an observable (74) by Wick's contractions, Figure 2.

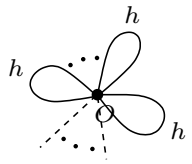

FIGURE 2. Feynman graph for the BV pushforward of an observable O in a free theory. (O corresponds to the input monomial on the left side of the cable in Figure 1.)

In Figure 3 we show the general nonvanishing cable diagrams computing the terms in the homological perturbation series (87) for $K_\hbar$.

5. SDR of the interacting theory and BV pushforward. Cable diagrams for $Q'_\hbar$, p_{int} as Feynman diagrams

In this section and onward, we will consider $\hbar$ to be a formal infinitesimal parameter and will switch the model for functions on $\mathcal{F}$ from polynomials $S\mathcal{F}^*$ to $\widehat{S}\mathcal{F}^*[[\hbar]]$, where $\widehat{S}\mathcal{F}^*$ is the completion of polynomials to formal power series. This change is needed for the convergence of homological perturbation series we encounter below.[28]

[28]Another possibility is to keep $\hbar$ finite but introduce a formal infinitesimal coupling constant g, require that $S_{\mathrm{int}} \in gS\mathcal{F}^*[[g]]$ is of degree ≥ 1 in g, and set the model for function on $\mathcal{F}$ to be $S\mathcal{F}^*[[g]]$.

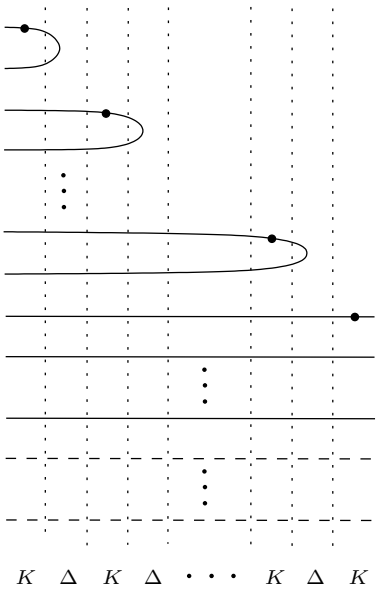

FIGURE 3. Cable diagrams for $K_\hbar$.

For functions on $\mathcal{F}'$, we similarly take $\widehat{S}\mathcal{F}'^*[[\hbar]]$. Integrals (BV pushforwards) we encounter in this section are understood perturbatively – as perturbed Gaussian integrals computed via Feynman diagram expansion.

In the setting of Section 4, consider a deformation of the BV action on $\mathcal{F}$, $S_0 \to S = S_0 + S_{\mathrm{int}}$, where $S_{\mathrm{int}} = \sum_{n\geq 3} S_n$, with S_n a polynomial of degree n on $\mathcal{F}$. We assume that S satisfies QME (22) or equivalently

$$Q_0 S_{\mathrm{int}} + \frac{1}{2}\{S_{\mathrm{int}}, S_{\mathrm{int}}\} - \mathrm{i}\hbar\Delta S_{\mathrm{int}} = 0. \tag{105}$$

We denote $Q_{\mathrm{int}} = \{S_{\mathrm{int}}, -\}$ – a derivation of $\widehat{S}\mathcal{F}^*[[\hbar]]$ (a formal vector field on $\mathcal{F}$). We have

$$Q_{\mathrm{int}} = \sum_{n\geq 3} Q_n \tag{106}$$

with $Q_n = \{S_n, -\}$ – a vector field on $\mathcal{F}$, with components of polynomial degree $n-1$.

Consider the deformation of differential $Q_{0,\hbar} \to Q_\hbar = Q_{0,\hbar} + Q_{\mathrm{int}} = \{S, -\} - \mathrm{i}\hbar\Delta$ on $\widehat{S}\mathcal{F}^*[[\hbar]]$. It induces a deformation of SDR data of free quantum theory (73) by homological perturbation lemma to the SDR

$$K_{\mathrm{int}} \circlearrowleft (\widehat{S}\mathcal{F}^*[[\hbar]], Q_\hbar) \underset{p_{\mathrm{int}}}{\overset{i_{\mathrm{int}}}{\leftrightarrows}} (\widehat{S}\mathcal{F}'^*[[\hbar]], Q'_\hbar). \tag{107}$$

Theorem 5.1.[29]

[29]Part (i) of the theorem is proven in [32], in the case of scalar theory perturbed by a polynomial potential. This reference gives a combinatorial proof of the fact that the coefficients of cable diagrams and Feynman graphs agree. Our proof of that fact is based on a different idea, see Remark 5.4. Paper [12] contains a different proof of part (ii) of the theorem (Theorem 3 in loc.

(i) *The induced differential $Q'_\hbar$ in (107) has the form*

$$Q'_\hbar = \{S', -\}' - \mathrm{i}\hbar\Delta' \tag{108}$$

with S' the effective action induced on $\mathcal{F}'$ via BV pushforward (24), with Lagrangian (75) determined by chain homotopy κ.

(ii) *The deformed projection in p_{int} is the BV pushforward map of observables (28),*

$$p_{\mathrm{int}} \colon O \mapsto O' = \mu'^{-1} e^{-\frac{\mathrm{i}}{\hbar}S'} \int_{\mathcal{L}} \mu\, e^{\frac{\mathrm{i}}{\hbar}S} O. \tag{109}$$

An immediate consequence of (ii) of the theorem is the statement in the title of the paper:

Corollary 5.2. *The BV pushforward map P_* (28) is a quasi-isomorphism of BV complexes.*

Proof. Indeed, $P_* = p_{\mathrm{int}}$, and p_{int}, being part of an SDR (107), is automatically a quasi-isomorphism. □

Remark 5.3 (Alternative SDR from conjugation). Note that one can also obtain "alternative" SDR data for the interacting theory from the free quantum case (73) by exploiting the chain isomorphisms of BV complexes

$$(\mathrm{Fun}(\mathcal{F}), Q_\hbar) \xrightarrow{\cdot e^{\frac{\mathrm{i}}{\hbar}S_{\mathrm{int}}}} (\mathrm{Fun}(\mathcal{F}), Q_{0,\hbar}), \tag{110}$$

$$(\mathrm{Fun}(\mathcal{F}'), Q'_\hbar) \xrightarrow{\cdot e^{\frac{\mathrm{i}}{\hbar}(S'-S'_0)}} (\mathrm{Fun}(\mathcal{F}'), Q'_{0,\hbar}). \tag{111}$$

(Here we are being intentionally vague with the classes of allowed functions on $\mathcal{F}, \mathcal{F}'$ and allowed dependence on $\hbar$.) Transported along these isomorphisms, the SDR dta (73) yields

$$K_{\mathrm{int}}^{\mathrm{conj}} \circlearrowleft (\mathrm{Fun}(\mathcal{F}, Q_\hbar)) \underset{p_{\mathrm{int}}^{\mathrm{conj}}}{\overset{i_{\mathrm{int}}^{\mathrm{conj}}}{\leftrightarrows}} (\mathrm{Fun}(\mathcal{F}'), Q'_\hbar) \tag{112}$$

with

$$i_{\mathrm{int}}^{\mathrm{conj}} \colon O' \mapsto e^{-\frac{\mathrm{i}}{\hbar}S_{\mathrm{int}}} i(e^{\frac{\mathrm{i}}{\hbar}(S'-S'_0)} O'), \tag{113}$$

$$p_{\mathrm{int}}^{\mathrm{conj}} \colon O \mapsto e^{-\frac{\mathrm{i}}{\hbar}(S'-S'_0)} p_\hbar(e^{-\frac{\mathrm{i}}{\hbar}S_{\mathrm{int}}} O) = \mu'^{-1} e^{-\frac{\mathrm{i}}{\hbar}S'} \int_{\mathcal{L}} \mu\, e^{\frac{\mathrm{i}}{\hbar}S} O, \tag{114}$$

$$K_{\mathrm{int}}^{\mathrm{conj}} \colon O \mapsto e^{-\frac{\mathrm{i}}{\hbar}S_{\mathrm{int}}} K_\hbar(e^{\frac{\mathrm{i}}{\hbar}S_{\mathrm{int}}} O). \tag{115}$$

Comparing this "conjugation" SDR data with the one obtained by homological perturbation (107), we have that:

- The induced differentials coincide – it is $Q'_\hbar$ in both cases, by (i) of Theorem 5.1.
- The projections coincide $p_{\mathrm{int}}^{\mathrm{conj}} = p_{\mathrm{int}}$, by (ii) of Theorem 5.1.
- Inclusions $i_{\mathrm{int}}^{\mathrm{conj}}$ and i_{int} generally do not coincide. Moreover, the first one does not map formal power series in $\hbar$ to formal power series in $\hbar$, and in particular does not have a "classical limit" – truncation mod $\hbar$. On the other hand i_{int} has a classical limit, which moreover is a morphism of

cit.). Part (i) of the theorem is present in [12] as well, obtained from part (ii) by essentially the same argument as the one we give in the beginning of Section 6.1.

commutative dg algebras, and thus is a pullback by an L_∞ morphism (a nonlinear map) $\pi_{\mathrm{int}}\colon \mathcal{F} \to \mathcal{F}'$, see Corollary 7.3 (b) and Remark 7.4 below.

- Chain homotopies $K_{\mathrm{int}}^{\mathrm{conj}}$ and K_{int} generally do not coincide. For instance, the former one does not vanish on constants while the latter one does. Also, as in the case of $i_{\mathrm{int}}^{\mathrm{conj}}$ above, $K_{\mathrm{int}}^{\mathrm{conj}}$ does not map formal power series in $\hbar$ to formal power series in $\hbar$, while K_{int} does, see Example 5.6.

Remark 5.4. Here below we will give a sketch of a diagrammatic proof of Theorem 5.1. The "sketchy" part is that we don't check that coefficients of the diagrams agree. Later, in Section 6.1, we will give an independent proof based on a presentation of homological perturbation theory via topological quantum mechanics. In particular, it implies that the coefficients of diagrams in the sketch below agree.

Sketch of proof of Theorem 5.1. Item (i): by homological perturbation lemma, we have

$$Q'_\hbar = Q'_{0,\hbar} + p_\hbar Q_{\mathrm{int}} i - p_\hbar Q_{\mathrm{int}} K_\hbar Q_{\mathrm{int}} i + \cdots \tag{116}$$

Contributions to terms in this series (excluding $Q'_{0,\hbar}$) are given by cable diagrams of the form shown in Figure 4. Graphically, Q_n splits a thread into $n-1$ threads

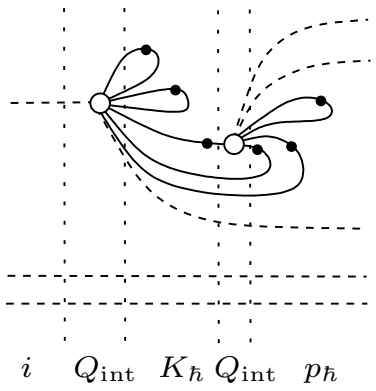

FIGURE 4. A typical cable diagram for a term in $Q'_\hbar$.

(each of the incoming and outgoing threads for Q_n can be infrared or ultraviolet).[30] Note that here a nontrivial transformation occurs only to one input thread,[31] thus the diagram determines a derivation of $\widehat{S}\mathcal{F}^*[[\hbar]]$ (a formal vector field on $\mathcal{F}'$). Such

[30] An important property of admissible cable diagrams for $Q'_\hbar$ (and for i_{int} below) is that the black dot $\kappa^\vee$ occurs only in two possible situations: (a) just before the merging of two threads on one of the merging threads, by the mechanism of Figures 1, 3 (or equivalently formulae (85), (90)) and (b) on a thread that is about to split by Q_{int}, just before Q_{int}. One can show the latter by looking at the transpose of terms in (116). (Note that passing to the transpose corresponds to reading the cable diagram right-to-left instead of left-to-right.) One has $S\mathcal{F}' \xrightarrow{p_\hbar^\vee} S\mathcal{F}' \otimes S\mathcal{L} \xrightarrow{Q_{\mathrm{int}}^\vee} S\mathcal{F}' \otimes S\mathcal{L} \otimes \mathcal{F} \xrightarrow{K_\hbar^\vee} S\mathcal{F}' \otimes S\mathcal{L} \xrightarrow{Q_{\mathrm{int}}^\vee} \cdots \xrightarrow{Q_{\mathrm{int}}^\vee} S\mathcal{F}' \otimes S\mathcal{L} \otimes \mathcal{F} \xrightarrow{K_\hbar^\vee} S\mathcal{F}' \otimes S\mathcal{L}$. Each time the separate $\mathcal{F}$ factor appears, it corresponds to the input of Q_{int}, and that is the only factor which κ (the rightmost dot in Figure 3) from the subsequent $K^\vee$ can hit.

[31] Proven by induction, going left-to-right in the cable diagram: The first Q_{int} splits one thread, acting as identity on the remaining dashed threads $T_1, \dots, T_n$. The subsequent $K_\hbar$ also acts trivially on T_i and puts the rightmost black dot $\kappa^\vee$ on one of the output threads of the first

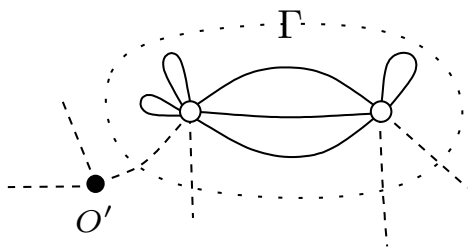

FIGURE 5. Hamiltonian vector field generated by the contribution of a Feynman graph Γ to the effective action S' acting on an infrared observable O'.

a diagram can be seen as representing $\{S'_\Gamma, O'\}'$ – the action of the Hamiltonian vector field generated by a contribution S'_Γ of a connected Feynman graph Γ to the effective action (24) on an element $O' \in \widehat{S}\mathcal{F}'^*[[\hbar]]$ (an infrared observable[32]), see Figure 5.

The translation of a cable diagram to a Feynman graph is by pinching the left side of the cable diagram and pairing to O', interpreting solid lines with a black dot as propagators h, and forgetting the left-to-right tiered structure of the cable diagram. The reverse translation of a connected Feynman graph Γ (with external dashed edges and internal solid edges) to a cable diagram is as follows: choose a "height function" (or "Morse function") χ on Γ (seen as a 1d cell complex), satisfying the following:

- χ is a continuous function on Γ valued in an interval $[a, b]$.
- $\chi = a$ at the tip of one dashed edge, $\chi = b$ at the tips of the other dashed edges.
- χ is monotonous on dashed edges. On solid edges it is either monotonous or has a unique local maximum in the interior of the edge.
- For every vertex v, χ is decreasing along exactly one of the incident edges (dashed or solid).

Such a Morse function converts Γ into a cable diagram, where cross-sections of the cable are given by level sets of χ. We understand that a black dot (corresponding to an insertion of $\kappa^\vee$ on the thread, or K on the cable) should be inserted just before the merging of two threads on one of them, and also on a thread that is about to split, see Figure 4. Local maxima of χ on edges correspond to merging a pair of threads with ω''^{-1} (or insertion of Δ on the cable), see Figure 1. Vertices of Γ correspond to Q_{int} factors in (116).[33]

Q_{int}, which then has to be followed by the second Q_{int} by footnote 30. Repeating the argument, threads T_i are left untransformed by all subsequent operations, and finally are left untransformed by $p_\hbar$ (cf. Figure 1).

[32]Here when we say "observable," we just mean an element of $\widehat{S}\mathcal{F}'^*[[\hbar]]$, not necessarily a $Q'_\hbar$-closed element (i.e., we don't insist on gauge-invariant observables).

[33]An equivalent construction is to consider a combinatorial Morse function $\bar{\chi}$ on Γ in the sense of R. Forman, assigning values in $[a, b]$ to vertices and edges, such that for each edge e, $\bar{\chi}(e)$ is smaller than at $\bar{\chi}$ on at most one of the incident vertices, and for each vertex v, $\bar{\chi}(v)$ is greater than $\bar{\chi}$ on at most one incident edge.

This process describes the creation of the nontrivial part of the cable diagram of $Q'_\hbar$, to which one can add an arbitrary number of dashed threads passing through, like the bottom two threads in Figure 4.

In summary, terms in (116) (cable diagrams) match the terms in the Feynman graph expansion of (108), $Q'_\hbar = Q'_{0,\hbar} + \sum_\Gamma \{S'_\Gamma, -\}'$, which proves item (i) of the theorem.

Item (ii): By homological perturbation lemma, we have

$$p_{\mathrm{int}} = p_\hbar - p_\hbar Q_{\mathrm{int}} K_\hbar + p_\hbar Q_{\mathrm{int}} K_\hbar Q_{\mathrm{int}} K_\hbar - \cdots \tag{117}$$

Terms here correspond to cable diagrams like the one in Figure 6. Note that footnote 30 applies here as well.

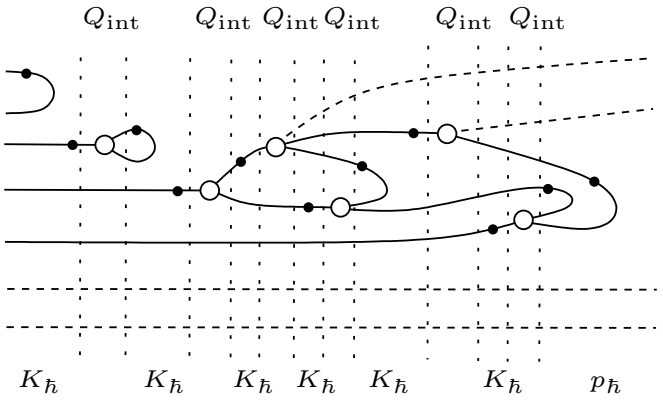

FIGURE 6. Typical cable diagram for a term in p_{int}.

Acting on an element $O \in \widehat{S}\mathcal{F}^*[[\hbar]]$ (an input for the left side of the cable), such a diagram reproduces a Feynman graph in the perturbative expansion of (109), see Figure 7. A Feynman graph here is a connected graph with a distinguished vertex decorated by O, other vertices decorated by S_n (with n the valence of the vertex), external (dashed) edges allowed (decorated by the input in $\mathcal{F}'$) and internal (solid) edges decorated by the propagator h.

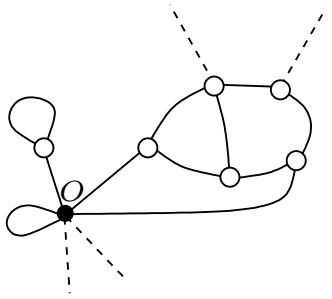

FIGURE 7. A Feynman graph for the integral (109) computing the BV pushforward of an observable O.

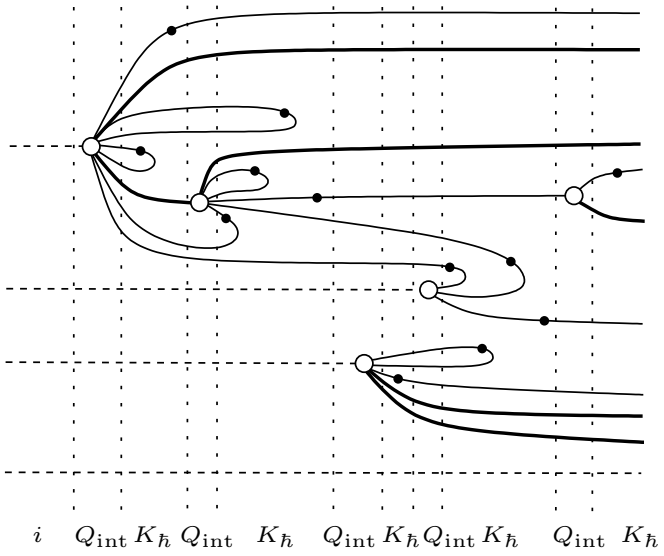

FIGURE 8. A typical cable diagram for $i_{\rm int}$. Thick threads stand for elements of $\mathcal{F}^*$ (i.e., thick thread = dashed thread $\mathcal{F}'^*$ plus thin thread $\mathcal{F}''^*$).

Translation of a cable diagram to a Feynman graph is similar to (i). The reverse process is also similar, with an amendment that χ assigns the (minimal) value a to the O-vertex and (maximal) value b to tips of dashed edges.

Again, terms in (117) match the Feynman graph contributions to (109), which proves (ii) of the theorem. □

Remark 5.5. The observable-lifting map $i_{\rm int}$ and the chain homotopy $K_{\rm int}$ in (107) are more complicated than $Q_\hbar$ and $p_{\rm int}$, and the corresponding cable diagrams cannot be immediately interpreted as Feynman graphs for a BV integral for the theory $(\mathcal{F}, \omega, S)$, see Figures 8, 9. Nevertheless, in Section 8.2 below we will interpret them as Feynman diagrams for the BV integral for a different theory build out of $(\mathcal{F}, \omega, S)$.

5.1. **Example: toy interacting quantum scalar field.** Consider the toy scalar theory (Example 4.2 and Section 4.2.1) and perturb the action $S_0(x) = \frac{x^2}{2}$ to

$$S(x) = \frac{x^2}{2} + P(x) \tag{118}$$

with $P(x) = S_{\rm int}(x) = \sum_{n\geq 3} \frac{P_n}{n!} x^n$ a polynomial potential. The corresponding perturbation of the BV differential is

$$Q_{0,\hbar} = x\frac{\partial}{\partial\xi} - \mathrm{i}\hbar\frac{\partial^2}{\partial x\partial\xi} \quad \to \quad Q_\hbar = x\frac{\partial}{\partial\xi} - \mathrm{i}\hbar\frac{\partial^2}{\partial x\partial\xi} + \underbrace{P'(x)\frac{\partial}{\partial\xi}}_{Q_{\rm int}} \tag{119}$$

In this case the SDR (107) is

$$K_{\rm int} \circlearrowleft (\mathbb{C}[[x,\xi]][[\hbar]], Q_\hbar) \underset{p_{\rm int}}{\overset{i_{\rm int}=i}{\leftrightarrows}} \mathbb{C}[[\hbar]]. \tag{120}$$

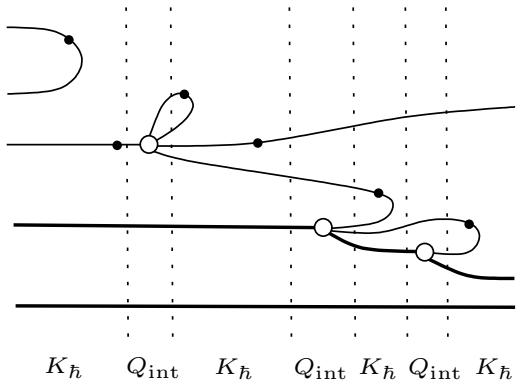

FIGURE 9. A typical cable diagram for K_{int}.

Here i_{int} is still the tautological inclusion of constants. The projection

$$p_{\mathrm{int}}\colon O(x) \mapsto c\int_{-\infty}^{\infty} \mathrm{d}x\, e^{\frac{\mathrm{i}}{\hbar}(\frac{x^2}{2}+P(x))}O(x) \tag{121}$$

– expectation value of an observable w.r.t. the perturbed action, with c as in (97).

Consider the chain homotopy K_{int}. It vanishes for degree reason on elements of the form $O(x,\xi) = \xi f(x)$, so it suffices to inspect $K_\hbar$ on observables of the form $O(x)$. Using (100), we find

$$\begin{aligned} O(x) &\xrightarrow{K_\hbar} \xi\mathcal{R}\left(1-\frac{\mathrm{i}\hbar}{x}\right)^{-1}\frac{O(x)}{x} \\ &\xrightarrow{Q_{\mathrm{int}}} P'(x)\mathcal{R}\left(1-\frac{\mathrm{i}\hbar}{x}\frac{\partial}{\partial x}\right)^{-1}\frac{O(x)}{x} \\ &\xrightarrow{K_\hbar} \xi\mathcal{R}\left(1-\frac{\mathrm{i}\hbar}{x}\frac{\partial}{\partial x}\right)^{-1}\frac{P'(x)}{x}\mathcal{R}\left(1-\frac{\mathrm{i}\hbar}{x}\frac{\partial}{\partial x}\right)^{-1}\frac{O(x)}{x} \to \cdots \end{aligned} \tag{122}$$

where the operator $\mathcal{R}$ subtracts the singular part of the Laurent series at $x=0$, as $[\cdots]_{\mathrm{reg}}$ in (100). Thus, we have

$$\begin{aligned} K_{\mathrm{int}}\colon O(x) &\mapsto (K_\hbar - K_\hbar Q_{\mathrm{int}} K_\hbar + K_\hbar Q_{\mathrm{int}} K_\hbar Q_{\mathrm{int}} K_\hbar - \cdots)(O) \\ &= \sum_{n\ge 0}\xi\mathcal{R}\left(-\left(1-\frac{\mathrm{i}\hbar}{x}\frac{\partial}{\partial x}\right)^{-1}\frac{P'(x)}{x}\mathcal{R}\right)^n\left(1-\frac{\mathrm{i}\hbar}{x}\frac{\partial}{\partial x}\right)^{-1}\frac{O(x)}{x} \\ &= \xi\mathcal{R}\left(1-\frac{\mathrm{i}\hbar}{x}\frac{\partial}{\partial x}+\frac{P'(x)}{x}\mathcal{R}\right)^{-1}\frac{O(x)}{x}. \end{aligned} \tag{123}$$

We make the following remarks:

(a) Formula (123) has a simple limit $\hbar \to 0$ (classical limit):[34]

$$K_{\mathrm{int}}^{\mathrm{cl}}(O) = \xi\frac{O(x)-O(0)}{x+P'(x)} = \xi\frac{O(x)-O(0)}{S'(x)}. \tag{124}$$

[34]Here and later when discussing toy interacting scalar field, the prime on S denotes the derivative in x, not the effective action.

(b) If $P(x)$ is an even function (contains only even powers of x) and $O(x)$ is an odd function, $\mathcal{R}$ acts tautologically in (123) and one has

$$K_{\mathrm{int}}(O) = \xi \left(1 - \frac{\mathrm{i}\hbar}{x}\frac{\partial}{\partial x} + \frac{P'(x)}{x}\right)^{-1} \frac{O(x)}{x} \tag{125}$$

Let us compare the "conjugation SDR" (112) of Remark 5.3 with the SDR (120) coming from homological perturbation theory. It is obvious that $i_{\mathrm{int}}^{\mathrm{conj}} = i_{\mathrm{int}}$, $p_{\mathrm{int}}^{\mathrm{conj}} = p_{\mathrm{int}}$. As for $K_{\mathrm{int}}^{\mathrm{conj}}$, we first observe that

$$e^{-\frac{\mathrm{i}}{\hbar}P(x)}\frac{\partial}{\partial x}e^{\frac{\mathrm{i}}{\hbar}P(x)} = \frac{\mathrm{i}}{\hbar}P'(x) + \frac{\partial}{\partial x}. \tag{126}$$

Thus, we have

$$\begin{aligned} K_{\mathrm{int}}^{\mathrm{conj}}(O) &= e^{-\frac{\mathrm{i}}{\hbar}P(x)}K_\hbar(e^{\frac{\mathrm{i}}{\hbar}P(x)}O(x)) \\ &= \xi e^{-\frac{\mathrm{i}}{\hbar}P(x)}\mathcal{R}\left(1 - \frac{\mathrm{i}\hbar}{x}\frac{\partial}{\partial x}\right)^{-1} e^{\frac{\mathrm{i}}{\hbar}P(x)}\frac{O(x)}{x} \\ &= \xi e^{-\frac{\mathrm{i}}{\hbar}P(x)}\mathcal{R}e^{\frac{\mathrm{i}}{\hbar}P(x)}\left(1 - \frac{\mathrm{i}\hbar}{x}\frac{\partial}{\partial x} + \frac{P'(x)}{x}\right)^{-1}\frac{O(x)}{x}. \end{aligned} \tag{127}$$

In the special case when $S(x)$ is an even function and $O(x)$ is an odd function, $\mathcal{R}$ in (127) acts tautologically and one has

$$K_{\mathrm{int}}^{\mathrm{conj}}(O) = K_{\mathrm{int}}(O), \tag{128}$$

cf. (125).

Without the assumption that S is even and O is odd, $K_{\mathrm{int}}^{\mathrm{conj}}(O)$ generally does not coincide with $K_{\mathrm{int}}(O)$. Moreover, $K_{\mathrm{int}}^{\mathrm{conj}}(O)$ is generally not a power series in $\hbar$ (and so does not have a classical limit).

Example 5.6. Let $S = \frac{x^2}{2} + gx^4$, $O = 1$. Then, expanding $K_{\mathrm{int}}^{\mathrm{conj}}(1)$ in powers of the coupling constant g, we have

$$\begin{aligned} K_{\mathrm{int}}^{\mathrm{conj}}(1) = \xi \Bigg(&\frac{\mathrm{i}}{\hbar}g(x^3 + \mathrm{i}\hbar\, 3x) \\ &+ \left(\frac{\mathrm{i}}{\hbar}\right)^2 \frac{g^2}{2}\left(-x^7 + \mathrm{i}\hbar x^5 + (\mathrm{i}\hbar)^2 7\cdot 5x^3 + (\mathrm{i}\hbar)^3 7\cdot 5\cdot 3x\right) + \mathcal{O}(g^3)\Bigg), \end{aligned} \tag{129}$$

which contains negative powers of $\hbar$.

5.1.1. *An alternative computation (and another formula) for K_{int}.* We are looking for an operator $K_{\mathrm{int}}\colon O(x) \mapsto \xi \underline{K} O$, $\xi O(x) \mapsto 0$ satisfying

$$[K_{\mathrm{int}}, Q_\hbar] = \mathrm{id} - i \circ p_{\mathrm{int}} \tag{130}$$

with $Q_\hbar = \left(S'(x) - \mathrm{i}\hbar\frac{\partial}{\partial x}\right)\frac{\partial}{\partial \xi}$. Equation (130), by specializing to functions $O(x)$ and $\xi O(x)$, is equivalent to the system

$$\left(S'(x) - \mathrm{i}\hbar\frac{\partial}{\partial x}\right)\underline{K}O = O(x) - c_{\mathrm{int}}\int_{\mathbb{R}} \mathrm{d}y\, e^{\frac{\mathrm{i}}{\hbar}S(y)}O(y), \tag{131}$$

$$\underline{K}\left(S'(x) - \mathrm{i}\hbar\frac{\partial}{\partial x}\right)O = O, \tag{132}$$

with $c_{\mathrm{int}} = \left(\int_{\mathbb{R}} \mathrm{d}x\, e^{\frac{\mathrm{i}}{\hbar}S(x)}\right)^{-1}$. Note that the differential operator

$$\left(S'(x) - \mathrm{i}\hbar\frac{\partial}{\partial x}\right) = e^{-\frac{\mathrm{i}}{\hbar}S}(-\mathrm{i}\hbar)\frac{\partial}{\partial x}e^{\frac{\mathrm{i}}{\hbar}S} \tag{133}$$

contained in $Q_\hbar$ is a conjugation of the derivative. Let $\underline{K} = e^{-\frac{\mathrm{i}}{\hbar}S}K^c e^{\frac{\mathrm{i}}{\hbar}S}$ for some operator K^c. The system (131), (132) reads

$$-\mathrm{i}\hbar\frac{\partial}{\partial x}K^c\widetilde{O} = \widetilde{O} - e^{\frac{\mathrm{i}}{\hbar}S}c_{\mathrm{int}}\int_{\mathbb{R}}\mathrm{d}y\,\widetilde{O}(y), \qquad K^c\left(-\mathrm{i}\hbar\frac{\partial}{\partial x}\right)\widetilde{O} = \widetilde{O}, \tag{134}$$

where $\widetilde{O} = e^{\frac{\mathrm{i}}{\hbar}S}O$. This system has a *unique* solution[35]

$$K^c(\widetilde{O})(x) = \frac{\mathrm{i}}{\hbar}\left(\int_{-\infty}^{x}\mathrm{d}y\,\widetilde{O}(y) - \int_{-\infty}^{x}\mathrm{d}y\,e^{\frac{\mathrm{i}}{\hbar}S(y)}c_{\mathrm{int}}\int_{\mathbb{R}}\mathrm{d}z\,\widetilde{O}(z)\right) \tag{135}$$

and thus K_{int} is

$$\begin{aligned} K_{\mathrm{int}}(O)(x) = \xi\frac{\mathrm{i}}{\hbar}e^{-\frac{\mathrm{i}}{\hbar}S(x)}\int_{-\infty}^{x}\mathrm{d}y\,e^{\frac{\mathrm{i}}{\hbar}S(y)}\left(O(y) - c_{\mathrm{int}}\int_{\mathbb{R}}\mathrm{d}z\,e^{\frac{\mathrm{i}}{\hbar}S(z)}O(z)\right) \\ = \xi\int_{\mathbb{R}}\mathrm{d}y\,G(x,y)O(y) \end{aligned} \tag{136}$$

with the "Green's function"

$$G(x,y) = \frac{\mathrm{i}}{\hbar}e^{\frac{\mathrm{i}}{\hbar}(-S(x)+S(y))}\left(\theta(x-y) - c_{\mathrm{int}}\int_{-\infty}^{x}\mathrm{d}z\,e^{\frac{\mathrm{i}}{\hbar}S(z)}\right). \tag{137}$$

Formula (136) generalizes the integral formula (101) to the interacting case. By uniqueness, this chain homotopy coincides with (123).

For comparison, as follows from (101), $K_{\mathrm{int}}^{\mathrm{conj}}$ is given by

$$\begin{aligned} K_{\mathrm{int}}^{\mathrm{conj}}(O)(x) = \xi\frac{\mathrm{i}}{\hbar}e^{-\frac{\mathrm{i}}{\hbar}S(x)}\int_{-\infty}^{x}\mathrm{d}y\left(e^{\frac{\mathrm{i}}{\hbar}S(y)}O(y) - e^{\frac{\mathrm{i}}{\hbar}S_0(y)}c\int_{\mathbb{R}}\mathrm{d}z\,e^{\frac{\mathrm{i}}{\hbar}S(z)}O(z)\right) \\ = \xi\int_{\mathbb{R}}\mathrm{d}y\,G^{\mathrm{conj}}(x,y)O(y) \end{aligned} \tag{138}$$

with

$$G^{\mathrm{conj}}(x,y) = \frac{\mathrm{i}}{\hbar}e^{\frac{\mathrm{i}}{\hbar}(-S(x)+S(y))}\left(\theta(x-y) - c\int_{-\infty}^{x}\mathrm{d}z\,e^{\frac{\mathrm{i}}{\hbar}S_0(z)}\right). \tag{139}$$

Comparing to (137), we note that in the second term the interacting action $S(x)$ is replaced with the free action $S_0(x)$ and the normalization constant c_{int} of the interacting theory is replaced with its free counterpart (97).

We also note that if $P(x)$ is an even function and $O(x)$ is an odd function, the second terms in (136) and in (138) vanish and we recover the result (128).

5.2. **Example: lifting Wilson loop observable from Yang–Mills to $BF+B^2$ theory, abelian case.**

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[35] Uniqueness of chain-homotopy is a feature of two-term complexes: Let $K \circlearrowleft (V,d) \underset{p}{\overset{i}{\leftrightarrows}} (V',d')$ be an SDR of chain complexes with V concentrated in two neighboring degrees n, $n+1$. Then if i and p are fixed, K is determined uniquely (is non-deformable). In the case at hand, (V,d) is the BV complex in the left side of (120), concentrated in degrees $-1, 0$.

5.2.1. *Abelian $BF + B^2$ theory.* Consider the abelian $BF + B^2$ theory on a closed oriented 4-manifold M defined by the BV action

$$S = \int_M \mathcal{B} \wedge \mathrm{d}\mathcal{A} + \frac{\lambda}{2} \mathcal{B} \wedge \mathcal{B} \tag{140}$$

with $\lambda \neq 0$ a coupling constant. The fields $(\mathcal{A}, \mathcal{B})$ (the AKSZ superfields) are two nonhomogeneous forms on M,

$$(\mathcal{A}, \mathcal{B}) \in \ \Omega^\bullet(M)[1] \oplus \Omega^\bullet(M)[2] =: \mathcal{F}. \tag{141}$$

We introduce the following notations for components of $\mathcal{A}, \mathcal{B}$:

$$\mathcal{A} = c + A + B^+ + \tau^+ + \phi^+, \qquad \mathcal{B} = \phi + \tau + B + A^+ + c^+, \tag{142}$$

where form degrees and ghost numbers are as follows:

	c	A	B^+	τ^+	ϕ^+	ϕ	τ	B	A^+	c^+
form degree	0	1	2	3	4	0	1	2	3	4
ghost number	1	0	-1	-2	-3	2	1	0	-1	-2

The (-1)-symplectic form on $\mathcal{F}$ is

$$\omega = \int_M \delta\mathcal{B} \wedge \delta\mathcal{A} = \int_M \sum_{\Phi \in \{A, B, c, \tau, \phi\}} \delta\Phi^+ \wedge \delta\Phi. \tag{143}$$

5.2.2. *Gauge-fixing.* Next, assume that M is equipped with a Riemannian metric g and let $*$ be the corresponding Hodge star operator. We split the 2-forms into self-dual forms $*\alpha = \alpha$ and anti-self-dual forms $*\beta = -\beta$: $\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M)$, with projectors to the two summands being $P_\pm = \frac{\mathrm{id} \pm *}{2}$.[36] Thus, we split the 2-forms fields into self-dual and anti-self-dual parts:

$$B = B_+ + B_-, \quad B^+ = B^+_+ + B^+_-. \tag{144}$$

The metric g on M also induces the Hodge decomposition of forms into harmonic, exact and coexact, $\Omega^\bullet(M) = \Omega^\bullet_{\mathrm{harm}}(M) \oplus \Omega^\bullet_{\mathrm{ex}}(M) \oplus \Omega^\bullet_{\mathrm{coex}}(M)$.

We introduce the infrared and ultraviolet complexes as follows. First, we introduce two complexes $\mathcal{F}_1, \mathcal{F}_2$:

$$\mathcal{F}_1 = \begin{array}{ccccccc} \Omega^0 & \xrightarrow{\mathrm{d}} & \Omega^1 & \xrightarrow{P_+ \mathrm{d}} & \Omega^2_+ & & \\ & & \oplus & & \oplus & & \\ & & \Omega^2_+ & \xrightarrow{\mathrm{d}} & \Omega^3 & \xrightarrow{\mathrm{d}} & \Omega^4 \end{array} \tag{145}$$

Here the first line corresponds to fields c, A, B^+_+ and the second line to fields B_+, A^+, c^+. We define $\mathcal{F}_2$ as the complex

$$\mathcal{F}_2 = \quad \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{P_- \mathrm{d}} \Omega^2_- \xrightarrow{0} \Omega^2_- \xrightarrow{\mathrm{d}} \Omega^3 \xrightarrow{\mathrm{d}} \Omega^4, \tag{146}$$

where, going left-to-right, the terms correspond to fields $\phi, \tau, B_-, B^+_-, \tau^+, \phi^+$.

We define the infrared complex as $\mathcal{F}_1$ plus the harmonic part of $\mathcal{F}_2$ (which is isomorphic to cohomology of $\mathcal{F}_2$):

$$\mathcal{F}' = \mathcal{F}_1 \oplus \underbrace{(\mathcal{F}_2 \cap \Omega^\bullet_{\mathrm{harm}})}_{\simeq H^\bullet(\mathcal{F}_2)} \tag{147}$$

[36] Note that for α_+ a self-dual 2-form and β_- an anti-self-dual 2-form, one has $\int_M \alpha_+ \wedge \beta_- = 0$.

and the ultraviolet complex as $\mathcal{F}_2$ with harmonic part removed (so as to make $\mathcal{F}''$ acyclic):

$$\mathcal{F}'' = \mathcal{F}_2 \cap (\Omega^\bullet_{\mathrm{ex}} \oplus \Omega^\bullet_{\mathrm{coex}}). \tag{148}$$

Viewed as a graded vector space, the space of fields (141) splits as $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$.

We can write the action (140) as

$$S = \underbrace{\int_M B_+ \mathrm{d}A + A^+ \mathrm{d}c}_{S'_0} + \underbrace{\int_M B^+_- \mathrm{d}\tau + \tau^+ \mathrm{d}\phi}_{S''_0} + \underbrace{\int_M B_- \mathrm{d}A + B^+_+ \mathrm{d}\tau + \lambda \left(\frac{1}{2} B_+ B_+ + \frac{1}{2} B_- B_- + A^+ \tau + c^+ \phi \right)}_{S_{\mathrm{int}}}. \tag{149}$$

Here S'_0, S''_0 are the free BV actions associated to the complexes $\mathcal{F}'$, $\mathcal{F}''$, and S_{int} is the perturbation.

We want to consider the BV pushforward defined by the gauge-fixing Lagrangian $\mathcal{L} = \mathcal{F}_2 \cap \Omega^\bullet_{\mathrm{coex}} \subset \mathcal{F}''$ – the coexact part of the complex $\mathcal{F}''$ (or equivalently $\mathcal{F}_2$). This Lagrangian can be written as $\mathcal{L} = \mathrm{im}(\kappa)$ with κ the chain contraction of $\mathcal{F}''$ induced by the following contraction of the complex $\mathcal{F}_2$:

$$\kappa\colon \quad \Omega^0 \xleftarrow{\mathrm{d}^* G} \Omega^1 \xleftarrow{2\mathrm{d}^* G} \Omega^2_- \xleftarrow{0} \Omega^2_- \xleftarrow{2P_- \mathrm{d}^* G} \Omega^3 \xleftarrow{\mathrm{d}^* G} \Omega^4. \tag{150}$$

Here $G = (\Delta_g + P_{\mathrm{harm}})^{-1}$ is the Green's function; $\Delta_g = [\mathrm{d}, \mathrm{d}^*]$ is the Laplace–de Rham operator. The dual of κ extended to a derivation is then

$$\widehat{\kappa} = \int_M G\mathrm{d}^*\tau \frac{\delta}{\delta \phi} + 2G\mathrm{d}^* B_- \frac{\delta}{\delta \tau} + 2P_- G\mathrm{d}^*\tau^+ \frac{\delta}{\delta B^+_-} + G\mathrm{d}^*\phi^+ \frac{\delta}{\delta \tau^+}. \tag{151}$$

Thus, we have an SDR (60) with the inclusion/projection ι, π obvious by construction and with κ as in (150).

The effective action on $\mathcal{F}'$, computed by BV pushforward, reads (up to a constant)

$$S' = \underbrace{\int_M B_+ \mathrm{d}A + A^+ \mathrm{d}c + \frac{\lambda}{2} B_+ B_+}_{S'_{\mathrm{YM}}} + \underbrace{\lambda \int_M \frac{1}{2} B_{-\mathrm{harm}} B_{-\mathrm{harm}} + A^+_{\mathrm{harm}} \tau_{\mathrm{harm}} + c^+_{\mathrm{harm}} \phi_{\mathrm{harm}}}_{S'_{\mathrm{zm}}}. \tag{152}$$

Here the first term in the abelian Yang–Mills action in the first-order formalism.[37] The second term is the contribution of zero-modes (cohomology) of $\mathcal{F}_2$.

[37]Note that integrating out the field B_+ in S'_{YM}, one obtains $\int_M -\frac{1}{4\lambda} \mathrm{d}A \wedge *\mathrm{d}A + A^+ \mathrm{d}c$ – the BV action of abelian Yang–Mills in the second-order formalism.

5.2.3. *Lifting abelian Wilson loop observable.* Assume that the first Betti number of M vanishes (so that $A^+_{\mathrm{harm}} = 0$).

Consider the abelian Wilson loop

$$W_\gamma(A) = \exp\left(\mathrm{i}\alpha \oint_\gamma A\right) \tag{153}$$

as an element of the infrared BV complex. Here $\gamma \subset M$ a closed curve and $\alpha \in \mathbb{R}$ a coupling constant. Note that W_γ is a BV cocycle: $\{S', W_\gamma\}' - \mathrm{i}\hbar\Delta' W_\gamma = 0$.[38]

Proposition 5.7. *The lifting of W_γ to abelian $BF + B^2$ theory is given by*

$$i_{\mathrm{int}}(W_\gamma) = \exp\left(\mathrm{i}\alpha \oint_\gamma (A + \lambda 2G\mathrm{d}^* B_-)\right). \tag{154}$$

Proof. By homological perturbation lemma, we have

$$i_{\mathrm{int}}(W_\gamma) = (i - K(Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta)i + K(Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta)K(Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta)i - \cdots)W_\gamma \tag{155}$$

with

$$\begin{aligned} Q_{\mathrm{int}} = \{S_{\mathrm{int}}, -\} = \int_M (P_+\mathrm{d}\tau)\frac{\delta}{\delta B_+} + (P_-\mathrm{d}A)\frac{\delta}{\delta B^+_-} + \mathrm{d}B_-\frac{\delta}{\delta A_+} + \mathrm{d}B^+_+\frac{\delta}{\delta \tau^+} \\ + \lambda\left(\phi\frac{\delta}{\delta c} + \tau\frac{\delta}{\delta A} + B_+\frac{\delta}{\delta B^+_+} + B_-\frac{\delta}{\delta B^+_-} + A^+\frac{\delta}{\delta\tau^+} + c^+\frac{\delta}{\delta\phi^+}\right). \end{aligned} \tag{156}$$

Next we compute the terms in the series (155):

$$\begin{aligned} &W_\gamma(A) \xrightarrow{Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta} \mathrm{i}\alpha\left(-\oint_\gamma \lambda\tau\right) W_\gamma(A) \xrightarrow{K} \mathrm{i}\alpha\left(-\oint_\gamma \lambda 2G\mathrm{d}^*B_-\right) W_\gamma(A) \\ &\xrightarrow{Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta} (\mathrm{i}\alpha)^2\left(-\oint_\gamma \lambda 2G\mathrm{d}^*B_-\right)\left(-\oint_\gamma \lambda\tau\right) W_\gamma(A) \xrightarrow{K} \frac{1}{2}(\mathrm{i}\alpha)^2\left(-\oint_\gamma \lambda 2G\mathrm{d}^*B_-\right)^2 W_\gamma(A) \\ &\cdots \to (K(Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta))^k(W_\gamma) = \frac{1}{k!}(\mathrm{i}\alpha)^k\left(-\oint_\gamma \lambda 2G\mathrm{d}^*B_-\right)^k W_\gamma(A). \end{aligned} \tag{157}$$

Here we use the formula (53) for K, $K = \nu\widehat{\kappa}$, and the operator ν is responsible for the appearance of the factor $\frac{1}{k!}$ above. Thus, we have

$$\begin{aligned} i_{\mathrm{int}}(W_\gamma) = \sum_{k\geq 0}\frac{(\mathrm{i}\alpha)^k}{k!}\left(\oint_\gamma \lambda 2G\mathrm{d}^*B_-\right)^k W_\gamma(A) \\ = \exp\left(\mathrm{i}\alpha\oint_\gamma (A + \lambda 2G\mathrm{d}^*B_-)\right). \end{aligned} \tag{158}$$

□

[38] Here we use the assumption of vanishing first Betti number. Otherwise, we would have $\{S', W_\gamma\}' - \mathrm{i}\hbar\Delta' W_\gamma = \mathrm{i}\alpha\lambda\left(\oint_\gamma \tau_{\mathrm{harm}}\right)\cdot W_\gamma(A) \neq 0$. Note that instead of assuming $B_1 = 0$ we could have assumed that γ is trivial in the first homology of M – that would also guarantee that W_γ is a BV cocycle.

As a consistency check, we can check that the r.h.s. of (154) – denote it $\widetilde{W}_\gamma$ – is indeed a BV cocycle in the abelian $BF + B^2$ theory:

$$(Q - \mathrm{i}\hbar\Delta)\widetilde{W}_\gamma = \widetilde{W}_\gamma \cdot \mathrm{i}\alpha\, Q \oint_\gamma (A + \lambda 2G\mathrm{d}^* B_-) \\ = \widetilde{W}_\gamma \cdot (-\mathrm{i}\alpha) \oint_\gamma (\mathrm{d}c + \lambda\tau - \lambda 2G\mathrm{d}^* P_- \mathrm{d}\tau)\,. \tag{159}$$

Note that we have

$$2G\mathrm{d}^* \underbrace{P_-}_{\frac{1}{2}-\frac{1}{2}*} \mathrm{d}\tau = Gd^*\mathrm{d}\tau = G(\Delta_g - \mathrm{dd}^*)\tau = \tau - \underbrace{P_{\mathrm{harm}}\tau}_{=0 \text{ by } B_1=0} - \mathrm{dd}^* G\tau. \tag{160}$$

Therefore, continuing the computation (159), we have

$$(Q - \mathrm{i}\hbar\Delta)\widetilde{W}_\gamma = \widetilde{W}_\gamma \cdot (-\mathrm{i}\alpha) \oint_\gamma \mathrm{d}(c + \lambda \mathrm{d}^* G\tau) = 0 \tag{161}$$

since γ is closed.

Remark 5.8. If we drop the assumption that the first Betti number of M vanishes, the computation of the lifting (154) still goes through. However, neither W_γ nor its lift $\widetilde{W}_\gamma$ will be BV cocycles.

6. A topological quantum mechanics perspective

Consider a topological quantum mechanics τ (cf. Remark 3.3) defined by the space of states for a point $V = \widehat{S}\mathcal{F}^*[[\hbar]]$ equipped with the BV differential $Q_\hbar$ and a second degree -1 differential $G = \widehat{\kappa}$. The Hamiltonian is

$$H = [Q_\hbar, \widehat{\kappa}] = [Q_0, \widehat{\kappa}] - \mathrm{i}\hbar[\Delta, \widehat{\kappa}] + [Q_{\mathrm{int}}, \widehat{\kappa}] \\ = \mathcal{L}_E - \mathrm{i}\hbar\eta + [Q_{\mathrm{int}}, \widehat{\kappa}] \tag{162}$$

with $\mathcal{L}_E$ the Lie derivative along the Euler vector field assigning degree 1 to coordinates on $\mathcal{F}''$ and degree 0 to coordinates on $\mathcal{F}'$, η as in (81), $\widehat{\kappa}$ the extension of $\kappa^\vee$ to a derivation of $S\mathcal{F}^*$.

Let

$$Z_{T,\mathrm{d}T} = e^{-TH + \mathrm{d}TG} = e^{-TH} + \mathrm{d}T\, G e^{-TH} \in \Omega^\bullet(\mathbb{R}_+) \otimes \mathrm{End}(\widehat{S}\mathcal{F}^*) \tag{163}$$

be the partition function of the topological quantum mechanics. We denote its zero-form part along T by $Z_T = e^{-TH}$.

Theorem 6.1. *SDR data (107) can be expressed as follows in terms of the TQM partition function (163):*[39]

$$i_{\mathrm{int}} = \lim_{T\to\infty} Z_T \circ i, \tag{164}$$

$$p_{\mathrm{int}} = \lim_{T\to\infty} p \circ Z_T, \tag{165}$$

$$K_{\mathrm{int}} = \int_0^\infty Z_{T,\mathrm{d}T}. \tag{166}$$

Also, one has

$$\lim_{T\to\infty} Z_T = i_{\mathrm{int}} \circ p_{\mathrm{int}} \tag{167}$$

[39]While working on the paper, we realized that independently the $\hbar \to 0$ limit of the formulae of this theorem were obtained by Ezra Getzler, [14].

Proof. Viewing the hamiltonian H as a perturbation of $\mathcal{L}_E$ by $[Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta, \widehat{\kappa}]$, we have

$$Z_T = e^{-T(\mathcal{L}_E + [Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta, \widehat{\kappa}])} \tag{168}$$
$$= \sum_{k\geq 0} (-1)^k \int_{t_0,\dots,t_k>0, t_0+\dots+t_k=T} \mathrm{d}t_1 \dots \mathrm{d}t_k e^{-t_0\mathcal{L}_E}[Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta, \widehat{\kappa}] e^{-t_1\mathcal{L}_E}[Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta, \widehat{\kappa}] \cdots \cdots [Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta, \widehat{\kappa}] e^{-t_k\mathcal{L}_E}$$

Note that

$$\lim_{t\to\infty} e^{-t\mathcal{L}_E} = i \circ p, \tag{169}$$
$$e^{-t\mathcal{L}_E} \circ i = i, \tag{170}$$
$$p \circ e^{-t\mathcal{L}_E} = p. \tag{171}$$

Precomposing (168) with i and taking $T \to \infty$ we have

$$\lim_{T\to\infty} Z_T \circ i = \tag{172}$$
$$= \sum_{k\geq 0} (-1)^k \int_0^\infty \cdots \int_0^\infty \mathrm{d}t_0 \cdots \mathrm{d}t_{k-1} e^{-t_0\mathcal{L}_E}[Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta, \widehat{\kappa}] \cdots e^{-t_{k-1}\mathcal{L}_E}[Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta, \widehat{\kappa}] i$$
$$= \sum_{k\geq 0} (-1)^k \int_0^\infty \cdots \int_0^\infty \mathrm{d}t_0 \cdots \mathrm{d}t_{k-1} e^{-t_0\mathcal{L}_E}\widehat{\kappa}(Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta) \cdots e^{-t_{k-1}\mathcal{L}_E}\widehat{\kappa}(Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta) i$$
$$= \sum_{k\geq 0} (-1)^k (\underbrace{\mathcal{L}_E^{-1}\widehat{\kappa}}_{K}(Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta))^k i = i_{\mathrm{int}}.$$

This recovers the homological perturbation formula for i_{int} and proves (164). Here we used the formula (54) for K; $\mathcal{L}_E^{-1}\widehat{\kappa}$ is a shorthand for $(\mathcal{L}_E + c\,ip)^{-1}\widehat{\kappa}$ with arbitrary $c > 0$.

Likewise, postcomposing (168) with p and taking $T \to \infty$, we get

$$\lim_{T\to\infty} p \circ Z_T = \tag{173}$$
$$= \sum_{k\geq 0} (-1)^k \int_0^\infty \cdots \int_0^\infty \mathrm{d}t_1 \cdots \mathrm{d}t_k p[Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta, \widehat{\kappa}] e^{-t_1\mathcal{L}_E} \cdots [Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta, \widehat{\kappa}] e^{-t_k\mathcal{L}_E}$$
$$= \sum_{k\geq 0} (-1)^k \int_0^\infty \cdots \int_0^\infty \mathrm{d}t_1 \cdots \mathrm{d}t_k p(Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta)\widehat{\kappa} e^{-t_1\mathcal{L}_E} \cdots (Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta)\widehat{\kappa} e^{-t_k\mathcal{L}_E}$$
$$= \sum_{k\geq 0} (-1)^k p((Q_{\mathrm{int}} - \mathrm{i}\hbar\Delta)\underbrace{\widehat{\kappa}\mathcal{L}_E^{-1}}_{K})^k = p_{\mathrm{int}}.$$

This recovers the homological perturbation formula for p_{int}, proving (165).

Note that in the limit $T \to \infty$, the integral in (168) precomposed with i is supported in the region $t_k \to \infty$, $t_0, \dots, t_{k-1}$ finite. If instead we postcompose with p, the integral is supported at $t_0 \to \infty$, $t_1, \dots, t_k$ finite.

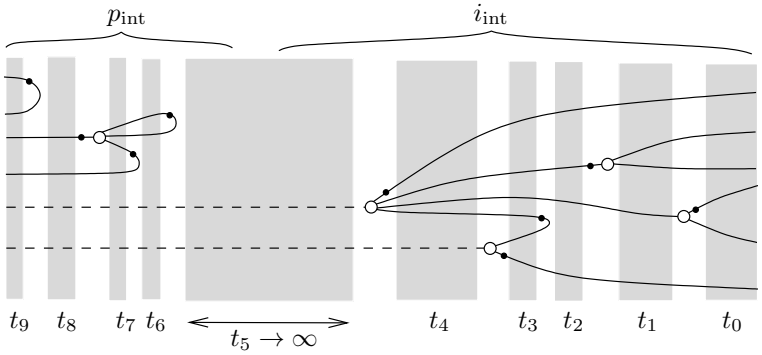

FIGURE 10. Graphical representation for a term in $Z_{T\to\infty}$. Grey boxes are $e^{-t_i\mathcal{L}_E}$, with t_i the width of the box. Between the boxes, terms of $[\widehat{\kappa}, Q_{\mathrm{int}}]$ or η are applied.

Next, composing (168) with $\widehat{\kappa}$ and integrating over T, we have

$$\int_0^\infty Z_{T,\mathrm{d}T} = \int_0^\infty \mathrm{d}T\,\widehat{\kappa} Z_T \tag{174}$$

$$= \sum_{k\geq 0}(-1)^k \int_{t_0,\dots,t_k>0} \mathrm{d}t_0\cdots\mathrm{d}t_k \widehat{\kappa} e^{-t_0\mathcal{L}_E}[Q_{\mathrm{int}}-\mathrm{i}\hbar\Delta,\widehat{\kappa}]e^{-t_1\mathcal{L}_E}\cdots[Q_{\mathrm{int}}-\mathrm{i}\hbar\Delta,\widehat{\kappa}]e^{-t_k\mathcal{L}_E}$$

$$= \sum_{k\geq 0}(-1)^k \int_{t_0,\dots,t_k>0} \mathrm{d}t_0\cdots\mathrm{d}t_k \widehat{\kappa} e^{-t_0\mathcal{L}_E}(Q_{\mathrm{int}}-\mathrm{i}\hbar\Delta)\widehat{\kappa}e^{-t_1\mathcal{L}_E}\cdots(Q_{\mathrm{int}}-\mathrm{i}\hbar\Delta)\widehat{\kappa}e^{-t_k\mathcal{L}_E}$$

$$= \sum_{k\geq 0}(-1)^k \underbrace{\widehat{\kappa}\mathcal{L}_E^{-1}}_{K}((Q_{\mathrm{int}}-\mathrm{i}\hbar\Delta)\underbrace{\widehat{\kappa}\mathcal{L}_E^{-1}}_{K})^k = K_{\mathrm{int}}.$$

This recovers the homological perturbation formula for K_{int}, proving (166).

To prove (167), note that in the limit $T\to\infty$, in the integral (168) one t_i must be very large. The corresponding factor in the integrand $e^{-t_i\mathcal{L}_E}\to i\circ p$ splits the the integrand into a composition of the integrands for i_{int} and for p_{int}, see Figure 10. □

Definition 6.2. Let us call a linear map $\bar{\kappa}\colon :\mathcal{F}^\bullet\to\mathcal{F}^{\bullet-1}$ a "non-normalized chain homotopy" (or "non-normalized gauge-fixing operator") compatible with the SDR (60) if

- $\bar{\kappa}$ induces an isomorphism $\mathrm{im}(d)\to\mathrm{im}(\kappa)$,
- $\bar{\kappa}$ vanishes on $\mathcal{F}'\oplus\mathrm{im}(\kappa)$,
- $[\bar{\kappa},d]$ is a non-negative operator (with kernel $\mathcal{F}'$, as follows from the points above).

Remark 6.3 (Generalization to non-normalized gauge-fixing operators). In the construction of this subsection, instead of choosing $G=\widehat{\kappa}$ for TQM, one can take $G=\widehat{\bar{\kappa}}$ – an extension to a derivation of $\widehat{S}\mathcal{F}^*$ of the dual of a non-normalized chain homotopy $\bar{\kappa}\colon\mathcal{F}^\bullet\to\mathcal{F}^{\bullet-1}$ compatible with the SDR (60). Then for the topological quantum mechanics (163) with $G=\widehat{\bar{\kappa}}$ and the Hamiltonian $H=[Q_\hbar,\widehat{\bar{\kappa}}]$, the statement of Theorem 6.1 still holds.

The reason is that one can arrange the formulae in the proof of Theorem 6.1 as involving $\widehat{\kappa}$ and only in the combination $[Q_0,\widehat{\kappa}]^{-1}\widehat{\kappa}$ and one has $[Q_0,\widehat{\kappa}]^{-1}\widehat{\kappa} = [Q_0,\widehat{\widehat{\kappa}}]^{-1}\widehat{\widehat{\kappa}}$.

Example 6.4. In the example of toy interacting scalar field (Section 5.1) with action $S(x) = \frac{x^2}{2} + P(x)$, we have

$$\widehat{H} = \Big[\underbrace{\xi\frac{\partial}{\partial x}}_{\widehat{\kappa}}, \underbrace{S'(x)\frac{\partial}{\partial\xi} - \mathrm{i}\hbar\frac{\partial^2}{\partial x\partial\xi}}_{Q_\hbar}\Big] = S'(x)\frac{\partial}{\partial x} + S''(x)\xi\frac{\partial}{\partial\xi} - \mathrm{i}\hbar\frac{\partial^2}{\partial x^2}, \tag{175}$$

and the TQM formula (166) for the chain homotopy becomes

$$K_{\mathrm{int}}(O) = \xi\int_0^\infty \mathrm{d}T\frac{\partial}{\partial x}e^{-T\left(S'(x)\frac{\partial}{\partial x} - \mathrm{i}\hbar\frac{\partial^2}{\partial x^2}\right)}O(x). \tag{176}$$

6.1. **Proof of Theorem 5.1 via TQM.** We give a second proof of Theorem 5.1, relying on the topological quantum mechanics picture above.

(i)⇐(ii). First, remark that if (ii) of Theorem 5.1 is known, i.e., p_{int} is the BV pushforward P_* of observables (109), then (i) of the Theorem follows automatically, by the following argument. We know by Remark 5.3 that $P_* = p_{\mathrm{int}}^{\mathrm{conj}}$ intertwines the BV differential $Q_\hbar$ on the full BV complex with the BV differential $Q'_\hbar$ on the infrared BV complex. On the other hand, by homological perturbation lemma, p_{int} intertwines $Q_\hbar$ with some induced differential $Q'^{\mathrm{HPL}}_\hbar$. But, by surjectivity of p_{int}, it can only intertwine $Q_\hbar$ with a unique differential on $\widehat{S}\mathcal{F}'^*[[\hbar]]$. Hence, $Q'_\hbar = Q'^{\mathrm{HPL}}_\hbar$.

Now we proceed to prove (ii) of Theorem 5.1. We will first consider the case $\mathcal{F}' = 0$ (we will generalize afterwards).

Proof of (ii) of Theorem 5.1 in the case $\mathcal{F}' = 0$. From Theorem 6.1 we know that, for $O \in \widehat{S}\mathcal{F}^*[[\hbar]]$,

$$p_{\mathrm{int}}O = \lim_{T\to\infty} pe^{-TH}O = P_{\ker H}O \tag{177}$$

– the projection onto the kernel of H, given by constants (for $\mathcal{F}' = 0$), along $\mathrm{im}H$.

To anticipate the next step: if we manage to find a pairing on $\widehat{S}\mathcal{F}^*[[\hbar]]$ with respect to which $\ker H$ and $\mathrm{im}H$ are orthogonal (and 1 has norm 1), we can construct the desired projection onto $\ker H$ as pairing with 1.

Consider the following pairing on observables

$$\ll,\gg\colon \widehat{S}\mathcal{F}^*\otimes\widehat{S}\mathcal{F}^* \to \mathbb{C}[[\hbar]], \quad \ll O_1, O_2 \gg = c\int_{\mathcal{L}} e^{\frac{\mathrm{i}}{\hbar}S}O_1O_2 \tag{178}$$

with $c = (\int_{\mathcal{L}} e^{\frac{\mathrm{i}}{\hbar}S})^{-1}$ the normalization factor, so that $\ll 1, 1 \gg = 1$.

The key property of pairing (178) is that H is self-adjoint with respect to it:

$$\ll HO_1, O_2 \gg = \ll O_1, HO_2 \gg . \tag{179}$$

To prove this, note that

$$H = [\widehat{\kappa}, Q_\hbar] = \left[\left\langle\kappa(x), \frac{\partial}{\partial x}\right\rangle, \omega^{-1}\left(\frac{\partial S}{\partial x}, \frac{\partial}{\partial x}\right) - \frac{\mathrm{i}\hbar}{2}\omega^{-1}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)\right], \tag{180}$$

hence the action of H restricted to $\mathcal{L}$ is

$$(HO)|_{\mathcal{L}} = h^{ij}\left(\frac{\partial S(x_\kappa)}{\partial x^i_\kappa}\frac{\partial}{\partial x^j_\kappa} - \mathrm{i}\hbar\frac{\partial}{\partial x^i_\kappa}\frac{\partial}{\partial x^j_\kappa}\right)O(x_\kappa). \tag{181}$$

Here $h = \omega^{-1}\kappa^\vee$ as in (80) (recall that h is symmetric); x^i_κ are coordinates on $\mathcal{L} = \mathrm{im}(\kappa) \subset \mathcal{F}$. Then, we have

$$
\begin{aligned}
(182)\quad \ll HO_1, O_2 \gg =& \\
&= c\int_{\mathcal{L}} Dx_\kappa e^{\frac{\mathrm{i}}{\hbar}S} h^{ij}\left(\frac{\partial S}{\partial x^i_\kappa}\frac{\partial}{\partial x^j_\kappa}O_1 - \mathrm{i}\hbar\frac{\partial}{\partial x^i_\kappa}\frac{\partial}{\partial x^j_\kappa}O_1\right)O_2 \\
&= c\int_{\mathcal{L}} Dx_\kappa e^{\frac{\mathrm{i}}{\hbar}S} O_1 h^{ij}\Big(-\big(\underbrace{\frac{\mathrm{i}}{\hbar}\frac{\partial S}{\partial x^i_\kappa}\frac{\partial S}{\partial x^j_\kappa}}_{a} + \underbrace{\frac{\partial^2 S}{\partial x^i_\kappa \partial x^j_\kappa}}_{b} + \underbrace{\frac{\partial S}{\partial x^i_\kappa}\frac{\partial}{\partial x^j_\kappa}}_{c}\big)O_2 - \\
&\quad - \mathrm{i}\hbar\Big(\underbrace{2\frac{\mathrm{i}}{\hbar}\frac{\partial S}{\partial x^i_\kappa}\frac{\partial}{\partial x^j_\kappa}}_{d} + \underbrace{\left(\frac{\mathrm{i}}{\hbar}\right)^2\frac{\partial S}{\partial x^i_\kappa}\frac{\partial S}{\partial x^j_\kappa}}_{e} + \underbrace{\frac{\mathrm{i}}{\hbar}\frac{\partial^2 S}{\partial x^i_\kappa \partial x^j_\kappa}}_{f} + \frac{\partial^2}{\partial x^i_\kappa \partial x^j_\kappa}\Big)O_2\Big).
\end{aligned}
$$

Here transition to the third line is integration by parts (once for the first term, twice for the second), transferring the derivatives from O_1 to O_2. Note that terms a and e cancel, b and f cancel, and c cancels half of d, leaving

(183)

$$
\ll HO_1, O_2 \gg = c\int_{\mathcal{L}} Dx_\kappa e^{\frac{\mathrm{i}}{\hbar}S} O_1 h^{ij}\left(\frac{\partial S}{\partial x^i_\kappa}\frac{\partial}{\partial x^j_\kappa} - \mathrm{i}\hbar\frac{\partial^2}{\partial x^i_\kappa \partial x^j_\kappa}\right)O_2 = \ll O_1, HO_2 \gg,
$$

which proves (179).

Next, given an observable $O \in \widehat{S}\mathcal{F}^*[[\hbar]]$, we split it as

$$
O = O_{\ker H} + O_{\mathrm{im}H}, \tag{184}
$$

according to the splitting $\widehat{S}\mathcal{F}^*[[\hbar]] = \ker H \oplus \mathrm{im}H$. Here $O_{\ker H} = p_{\mathrm{int}}O$ and is a constant, cf. (177). We have

$$
\begin{aligned}
(185)\quad \ll O, 1 \gg = \ll O_{\ker H}, 1 \gg + \ll H(\cdots), 1 \gg &= p_{\mathrm{int}}(O) + \ll H(\cdots), 1 \gg \\
&\underset{(179}{=} p_{\mathrm{int}}(O) + \ll \cdots, \underbrace{H(1)}_{0} \gg = p_{\mathrm{int}}(O).
\end{aligned}
$$

The left hand side is the BV integral $c\int_{\mathcal{L}} e^{\frac{\mathrm{i}}{\hbar}S}O$, which proves (ii) of Theorem 5.1 in the case $\mathcal{F}' = 0$.

Generalization of the argument to $\mathcal{F}'$ possibly nonzero. We consider the pairing

(186)

$$
\ll, \gg \colon \widehat{S}\mathcal{F}^*[[\hbar]] \otimes \widehat{S}\mathcal{F}^*[[\hbar]] \to \widehat{S}\mathcal{F}'^*[[\hbar]], \quad \ll O_1, O_2 \gg = e^{-\frac{\mathrm{i}}{\hbar}S'}\int_{\mathcal{L}} e^{\frac{\mathrm{i}}{\hbar}S} O_1 O_2.
$$

With respect to it, self-adjoint of H (179) still holds (and the proof (182), (183) goes through without change, except that now S, O_1, O_2 additionally depend on $x' \in \mathcal{F}'$ and the normalization is $c = e^{-\frac{\mathrm{i}}{\hbar}S'}$).

Consider an observable $O \in \widehat{S}\mathcal{F}^*[[\hbar]]$ and split it as in (184). We have

$$
\begin{aligned}
(187)\quad \ll O, 1 \gg = \ll O_{\ker H} + O_{\mathrm{im}H}, 1 \gg = \ll O_{\ker H}, 1 \gg + &\underbrace{\ll H(\cdots), 1 \gg}_{\underset{(179)}{=} \ll \cdots, H(1) \gg = 0} \\
&= \ll O_{\ker H}, 1 \gg
\end{aligned}
$$

Note that, by Theorem 6.1, we have

$$O_{\ker H} = \lim_{T\to\infty} Z_T O = i_{\mathrm{int}} p_{\mathrm{int}} O = \lim_{T\to\infty} Z_T i p_{\mathrm{int}} O. \tag{188}$$

Using this, we continue (187):

$$\ll O, 1 \gg = \lim_{T\to\infty} \ll e^{-TH} i p_{\mathrm{int}} O, 1 \gg \underset{(179)}{=} \lim_{T\to\infty} \ll i p_{\mathrm{int}} O, \underbrace{e^{-TH} 1}_{1} \gg \\ = \ll i p_{\mathrm{int}} O, 1 \gg = p_{\mathrm{int}} O. \tag{189}$$

Here in the last step we use the obvious property $\ll i(O'), 1 \gg = O'$. Since the l.h.s. of (189) is the BV pushforward of O, we have a proof of (ii) of Theorem 5.1.

Together with the argument in the beginning of this subsection ((ii) implies (i)), this concludes the second proof of Theorem 5.1.

7. Classical limit of $i_{\mathrm{int}}, p_{\mathrm{int}}, K_{\mathrm{int}}$

Let us describe the classical limit (reduction modulo $\hbar$) of the SDR (107) – let us denote this reduction by

$$K^{\mathrm{cl}}_{\mathrm{int}} \circlearrowleft (\widehat{S}\mathcal{F}^*, Q^{\mathrm{cl}}) \underset{p^{\mathrm{cl}}_{\mathrm{int}}}{\overset{i^{\mathrm{cl}}_{\mathrm{int}}}{\leftrightarrows}} (\widehat{S}\mathcal{F}'^*, Q'^{\mathrm{cl}}) \tag{190}$$

with $Q^{\mathrm{cl}} = \{S^{\mathrm{cl}}, -\} = Q_0 + Q^{\mathrm{cl}}_{\mathrm{int}}$, $Q'^{\mathrm{cl}} = \{S'^{\mathrm{cl}}, -\}'$. The superscript cl means "modulo $\hbar$," i.e., $S^{\mathrm{cl}} = S \bmod \hbar$ etc. The maps $i^{\mathrm{cl}}_{\mathrm{int}}, p^{\mathrm{cl}}_{\mathrm{int}}, K^{\mathrm{cl}}_{\mathrm{int}}$ are given by the homological perturbation formulae (37) with i, p, K of the free classical theory (62) and with $\delta = Q^{\mathrm{cl}}_{\mathrm{int}}$ the perturbation of the differential.

Consider the reduction modulo $\hbar$ of the TQM Hamiltonian (162),

$$H^{\mathrm{cl}} = [\widehat{\kappa}, Q^{\mathrm{cl}}] = \left[\left\langle \kappa(x), \frac{\partial}{\partial x}\right\rangle, \omega^{-1}\left(\frac{\partial S^{\mathrm{cl}}}{\partial x}, \frac{\partial}{\partial x}\right)\right] \\ = \underbrace{\omega^{-1}\left(\frac{\partial S^{\mathrm{cl}}}{\partial x}, \kappa^\vee \frac{\partial}{\partial x}\right)}_{H^{\mathrm{cl}}_1} + \underbrace{\omega^{-1}\left(\left\langle \kappa(x), \frac{\partial}{\partial x}\right\rangle \frac{\partial S^{\mathrm{cl}}}{\partial x}, \frac{\partial}{\partial x}\right)}_{H^{\mathrm{cl}}_2} \\ = h^{ij} \frac{\partial S^{\mathrm{cl}}}{\partial x^i_\kappa} \frac{\partial}{\partial x^j_\kappa} + \kappa^i{}_j x^j_d (\omega^{-1})^{IJ} \frac{\partial^2 S^{\mathrm{cl}}}{\partial x^i_\kappa \partial x^I} \frac{\partial}{\partial x^J}. \tag{191}$$

Here x^i_d are coordinates on $\mathrm{im}(d'')$ and x^I are coordinates on the entire $\mathcal{F} = \mathcal{F}' \oplus \mathrm{im}(\kappa) \oplus \mathrm{im}(d'')$.

Note that H^{cl} is a derivation of $\widehat{S}\mathcal{F}^*$ or, equivalently, a formal vector field on $\mathcal{F}$ (or: a vector field on a formal neighborhood of zero in $\mathcal{F}$; we denote the latter $\mathcal{F}^{\mathrm{formal}}$).[40]

Let $\mathrm{cr}\colon \mathcal{F}'^{\mathrm{formal}} \to \mathcal{L}^{\mathrm{formal}}$ be the formal nonlinear map sending x' to the solution $x_\kappa \in \mathcal{L}$ of the equation

$$\frac{\partial S^{\mathrm{cl}}(x' + x_\kappa)}{\partial x_\kappa} = 0. \tag{192}$$

[40] In fact, if S has positive convergence radius in $\mathcal{F}$, one can replace formal neighborhoods of zero in this subsection with finite open neighborhoods of zero.

Note that in the formal setting such a solution exists and is unique (i.e., for x' infinitesimal there exists a unique infinitesimal x_κ). Also note that (192) means that $x' + x_\kappa$ is a critical point of S^{cl} restricted to the affine space $x' + \mathcal{L}$. Denote

$$\mathcal{Z} \colon = \mathrm{graph}(\mathrm{cr}) = \{x' + x_\kappa(x') \mid (192) \text{ holds}\} \subset (\mathcal{F}' \oplus \mathcal{L})^{\mathrm{formal}}. \tag{193}$$

By the remark above, the projection π onto $\mathcal{F}'$ restricted to $\mathcal{Z}$ yields a (nonlinear) isomorphism

$$\pi|_{\mathcal{Z}} \colon \mathcal{Z} \xrightarrow{\sim} \mathcal{F}'^{\mathrm{formal}}. \tag{194}$$

Properties of the vector field H^{cl}:

(i) Locus of zeros (fixed points) of H^{cl} is $\mathcal{Z}$. This zero locus is attracting for the flow of $-H^{\mathrm{cl}}$.[41]

(ii) H^{cl} is tangent to $x' + \mathcal{L}$ for any $x' \in \mathcal{F}'^{\mathrm{formal}}$. In the restriction of H^{cl} to $x' + \mathcal{L}$, only the term H^{cl}_1 in (191) survives, whereas H^{cl}_2 vanishes.

Denote

$$\Phi_T = \mathrm{Flow}_T(-H^{\mathrm{cl}}) \colon \mathcal{F}^{\mathrm{formal}} \to \mathcal{F}^{\mathrm{formal}} \tag{195}$$

the flow of the vector field $-H^{\mathrm{cl}}$ in time $T \in \mathbb{R}$.

Example 7.1. In free theory, $S = S_0 = \frac{1}{2}\omega(x, dx)$, we have $H^{\mathrm{cl}} = \mathcal{L}_E = \langle x'', \frac{\partial}{x''} \rangle$ the Euler vector field. Its zero locus is $\mathcal{Z} = \mathcal{F}'$; it is repulsive for $\mathcal{L}_E$ and attractive for $-\mathcal{L}_E$. The flow Φ_T maps $x = x' + x''$ to $x' + e^{-T}x''$.

Example 7.2. Consider Chern–Simons theory on a 3-manifold M with structure quadratic Lie algebra $\mathfrak{g}$, with action

$$S = \int_M \frac{1}{2}\langle \mathcal{A}, \mathrm{d}\mathcal{A} \rangle + \frac{1}{6}\langle \mathcal{A}, [\mathcal{A}, \mathcal{A}] \rangle, \tag{196}$$

with $\mathcal{A} \in \Omega^\bullet(M, \mathfrak{g})[1] = \mathcal{F}$ the AKSZ superfield. Equipping M with a metric g we have a Hodge decomposition $\Omega^\bullet = \mathrm{Harm} \oplus \mathrm{im}(\mathrm{d}) \oplus \mathrm{im}(\mathrm{d}^*)$ into harmonic, exact and coexact forms. It induces a splitting $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ with $\mathcal{F}' = \mathrm{Harm} \otimes \mathfrak{g}[1] \simeq H^\bullet(M, \mathfrak{g})[1]$ and $\mathcal{F}''$ given by exact and coexact forms. For $\mathcal{L}$ we choose the coexact forms. In this setting, if we use non-normalized chain homotopy $\bar{\kappa} = \mathrm{d}^*$ (cf. Remark 6.3), the vector field (191) becomes

$$H^{\mathrm{cl}} = \langle \underbrace{\Delta\mathcal{A}}_{\text{free part}} + \underbrace{\mathrm{d}^*\frac{1}{2}[\mathcal{A}, \mathcal{A}] + [\mathcal{A}, \mathrm{d}^*\mathcal{A}]}_{\text{perturbation}}, \frac{\delta}{\delta\mathcal{A}} \rangle = \underbrace{\langle \mathrm{d}^* F_{\mathcal{A}}, \frac{\delta}{\delta\mathcal{A}} \rangle}_{H^{\mathrm{cl}}_1} + \underbrace{\langle \mathrm{d}_{\mathcal{A}}\mathrm{d}^*\mathcal{A}, \frac{\delta}{\delta\mathcal{A}} \rangle}_{H^{\mathrm{cl}}_2}, \tag{197}$$

with $F_{\mathcal{A}} = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]$ and $\mathrm{d}_{\mathcal{A}} = \mathrm{d} + [\mathcal{A}, -]$. The zero-locus of H^{cl} restricted to ghost number zero is

$$\mathcal{Z}|_{\mathrm{gh}=0} = \{A \in \Omega^1(M, \mathfrak{g}) \mid \mathrm{d}^* F_A = 0,\ \mathrm{d}^* A = 0\}. \tag{198}$$

Note that in the abelian case $\mathfrak{g} = \mathbb{R}$, this is the space of connections satisfying the abelian Yang–Mills equation $\mathrm{d}^*\mathrm{d}A = 0$ and the Lorenz gauge condition.

The flow Φ_T maps a connection 1-form A to a time-dependent connection A_T – a solution of a nonlinear heat equation

$$\frac{\partial}{\partial T} A_T = -\mathrm{d}^* F_{A_T} - \mathrm{d}_{A_T}\mathrm{d}^* A_T \tag{199}$$

[41] Thus, $-H^{\mathrm{cl}}$ behaves like the gradient vector field of a Morse-Bott function with critical locus $\mathcal{Z}$ of index zero.

with initial condition $A_{T=0} = A$. In the abelian case $\mathfrak{g} = \mathbb{R}$, equation (199) becomes the linear heat equation on a 1-form, $\partial_T A_T = -\Delta A_T$.

The following is an immediate consequence of Theorem 6.1 considered modulo $\hbar$.

Corollary 7.3.[42] *In the SDR (190):*

(a) $p^{\mathrm{cl}}_{\mathrm{int}} \colon (\widehat{S}\mathcal{F}^*, Q^{\mathrm{cl}}, \cdot) \to (\widehat{S}\mathcal{F}'^*, Q'^{\mathrm{cl}}, \cdot)$ *is a morphism of dg commutative algebras. It is the pullback by the nonlinear formal map*

$$\iota^{\mathrm{cl}}_{\mathrm{int}} \colon \mathcal{F}'^{\mathrm{formal}} \to \mathcal{F}^{\mathrm{formal}}, \quad x' \mapsto \lim_{T\to\infty} \Phi_T(x') = (\pi|_{\mathcal{Z}})^{-1} x' = x' + \mathrm{cr}(x'), \tag{200}$$

with $\pi|_{\mathcal{Z}}$ as in (194). Thus, $\iota^{\mathrm{cl}}_{\mathrm{int}}$ maps x' to the critical point of S^{cl} restricted to $x' + \mathcal{L}$.[43]

(b) $i^{\mathrm{cl}}_{\mathrm{int}} \colon (\widehat{S}\mathcal{F}'^*, Q'^{\mathrm{cl}}, \cdot) \to (\widehat{S}\mathcal{F}^*, Q^{\mathrm{cl}}, \cdot)$ *is a morphism of dg commutative algebras. It is the pullback of the nonlinear formal map*

$$\pi^{\mathrm{cl}}_{\mathrm{int}} \colon \mathcal{F}^{\mathrm{formal}} \to \mathcal{F}'^{\mathrm{formal}}, \quad x \to \lim_{T\to\infty} \pi \circ \Phi_T(x). \tag{201}$$

(c) The chain homotopy in (190) is

$$K^{\mathrm{cl}}_{\mathrm{int}}(O) = \int_0^\infty \mathrm{d}T\, \widehat{\kappa}\, \Phi_T^* O \tag{202}$$

for $O \in \widehat{S}\mathcal{F}^*$, or, equivalently,

$$K^{\mathrm{cl}}_{\mathrm{int}}(O)(x) = \int_0^\infty O\Big((\mathrm{id} + \mathrm{d}T\, \kappa) \circ \Phi_T(x)\Big) \tag{203}$$

for $x \in \mathcal{F}^{\mathrm{formal}}$.

Remark 7.4 (Language of L_∞ algebras and morphisms). The differential Q^{cl} equips $\mathcal{F}[-1]$ with the structure of an L_∞ algebra (and ω additionally makes it a cyclic L_∞ algebra with pairing of degree -3).[44] Likewise, Q'^{cl} equips $\mathcal{F}'[-1]$ with the structure of an L_∞ algebra (again, cyclic of degree -3 by virtue of ω') – the homotopy transfer of the L_∞ algebra $(\mathcal{F}[-1], Q^{\mathrm{cl}})$ onto $\mathcal{F}'[-1]$, using SDR (60). In this language, maps (200), (201) are L_∞ morphisms,

$$\iota^{\mathrm{cl}}_{\mathrm{int}} \colon (\mathcal{F}'[-1], Q'^{\mathrm{cl}}) \xrightarrow{L_\infty \text{ mor}} (\mathcal{F}[-1], Q^{\mathrm{cl}}), \tag{204}$$

$$\pi^{\mathrm{cl}}_{\mathrm{int}} \colon (\mathcal{F}[-1], Q^{\mathrm{cl}}) \xrightarrow{L_\infty \text{ mor}} (\mathcal{F}'[-1], Q'^{\mathrm{cl}}). \tag{205}$$

They provide nonlinear (L_∞) deformations of the original linear chain maps ι, π from (60).

---

[42] A version of this statement appeared, independently of our work, in [14]. Also, the statement that $i^{\mathrm{cl}}_{\mathrm{int}}$, defined by the homological perturbation series $\sum_{n\geq 0}(-KQ^{\mathrm{cl}}_{\mathrm{int}})^n i$, is a morphism of commutative dg algebras (and so defines an L_∞ morphism (205) was proven in [4].

[43] By properties of H^{cl}, Φ_T moves x' along $x' + \mathcal{L}$, converging as $T \to \infty$ to the fiberwise critical point of S, $x' + \mathrm{cr}(x')$.

[44] One reads off the multilinear operations $l_n \colon \wedge^n \mathcal{F}[-1] \to \mathcal{F}[-1]$ of degrees $2 - n$ from the Taylor expansion of Q^{cl}: $l_1 = d$ the differential and, for $n \geq 2$, l_n is the dual of Q^{cl}_{n+1} in the notations of (106).

Example 7.5. Let us revisit the example of Section 5.2 – BV pushforward from abelian $BF + B^2$ theory to abelian Yang–Mills. The vector field H^{cl} restricted to fields of ghost degree zero is:

$$H^{\mathrm{cl}}_{\mathrm{gh}=0} = \int_M (\mathrm{id} - P_{\mathrm{harm}})B_- \frac{\delta}{\delta B_-} + 2GP_+\mathrm{d}\mathrm{d}^*B_- \frac{\delta}{\delta B_+} + \lambda 2G\mathrm{d}^*B_- \frac{\delta}{\delta A}. \tag{206}$$

As it is a linear vector field on the space of triples $(A, B_+, B_-) \in \Omega^1 \oplus \Omega^2_+ \oplus \Omega^2_-$, its flow in time T is easy to compute:

$$\begin{aligned}\Phi_T(A, B_+, B_-) = (&A + (1 - e^{-T})\lambda 2G\mathrm{d}^*B_- \ , \\ &B_+ + (1 - e^{-T})2GP_+\mathrm{d}\mathrm{d}^*B_- \ , \ e^{-T}(\mathrm{id} - P_{\mathrm{harm}})B_- + P_{\mathrm{harm}}B_-).\end{aligned} \tag{207}$$

Therefore, we have

$$\begin{aligned}\pi_{\mathrm{int}} = \lim_{T\to\infty} \pi \circ \Phi_T \colon (A, B_+, B_-) \mapsto \\ \mapsto (A + \lambda 2G\mathrm{d}^*B_- \ , \ B_+ + 2GP_+\mathrm{d}\mathrm{d}^*B_- \ , \ P_{\mathrm{harm}}B_-).\end{aligned} \tag{208}$$

Hence, the observable-lifting map, in the classical limit and restricted to $\mathrm{gh} = 0$ fields, reads

$$\begin{aligned}i^{\mathrm{cl}}_{\mathrm{int}} \colon O'(A, B_+, B_{-\mathrm{harm}}) \mapsto \\ \mapsto O(A, B_+, B_-) = O'(A + \lambda 2G\mathrm{d}^*B_- \ , \ B_+ + 2GP_+\mathrm{d}\mathrm{d}^*B_- \ , \ P_{\mathrm{harm}}B_-).\end{aligned} \tag{209}$$

Applying this map to the Wilson loop, we recover the result of Proposition 5.7 (there are no quantum corrections to the lifting map $i^{\mathrm{cl}}_{\mathrm{int}}$ in this case).

We remark that instead of using the chain homotopy (150) one can use an equivalent non-normalized chain homotopy $\bar\kappa$:

$$\bar\kappa \colon \quad \Omega^0 \xleftarrow{\mathrm{d}^*} \Omega^1 \xleftarrow{\mathrm{d}^*} \Omega^2_- \xleftarrow{0} \Omega^2_- \xleftarrow{P_-\mathrm{d}^*} \Omega^3 \xleftarrow{\mathrm{d}^*} \Omega^4. \tag{210}$$

The corresponding vector field (restricted to $\mathrm{gh} = 0$ fields) is

$$H^{\mathrm{cl}}_{\mathrm{gh}=0} = [\widehat{\bar\kappa}, Q]\Big|_{\mathrm{gh}=0} = \int_M \frac{1}{2}\Delta B_- \frac{\delta}{\delta B_-} + P_+\mathrm{d}\mathrm{d}^*B_- \frac{\delta}{\delta B_+} + \lambda\mathrm{d}^*B_- \frac{\delta}{\delta A} \tag{211}$$

with Δ the Laplace-de Rham operator. Its flow is

$$\begin{aligned}\Phi_T(A, B_+, B_-) = \Bigg(&A + \lambda\mathrm{d}^* \frac{\mathrm{id} - e^{-\frac{1}{2}T\Delta}}{\frac{1}{2}\Delta} B_- \ , \\ &B_+ + P_+\mathrm{d}\mathrm{d}^* \frac{\mathrm{id} - e^{-\frac{1}{2}T\Delta}}{\frac{1}{2}\Delta} B_- \ , \ e^{-\frac{1}{2}T\Delta}B_-\Bigg).\end{aligned} \tag{212}$$

Here $e^{-\frac{1}{2}T\Delta}$ is the heat flow operator. Taking the limit $T \to \infty$ and projecting to $\mathcal{F}'$, we obtain the same formula for π_{int} as before (208), using the normaized homotopy κ.

7.1. **Example: 1d BF theory on an interval.**[45] Consider 1d BF theory on the interval $I = [0, 1]$ with structure Lie algebra $\mathfrak{g}$, which we assume to be unimodular. It is defined by the BV action

$$S = \int_I \langle \mathcal{B}, \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]\rangle \tag{213}$$

[45]The setup of this example is taken from [29], [30].

– a function on the space of fields

$$\mathcal{F} = T^*[-1]\Omega^\bullet(I,\mathfrak{g})[1] \cong \Omega^\bullet(I,\mathfrak{g})[1] \oplus \Omega^\bullet_{\mathrm{distr}}(I,\mathfrak{g}^*)[-1] \ni (\mathcal{A},\mathcal{B}) \tag{214}$$

with (-1)-symplectic form $\omega = \int_I \langle \delta\mathcal{B} \overset{\curvearrowleft}{,} \delta\mathcal{A}\rangle$. Subscript "distr" in the second summand means that we allow distributional forms, more specifically, delta-like 1-forms supported at the endpoints and discontinuity of 0-forms at the endpoints.[46]

We split the "superfields" $\mathcal{A},\mathcal{B}$ into components according to form degree on I:

$$\mathcal{A} = c + A, \tag{215}$$

$$\mathcal{B} = A^+ + c^+, \tag{216}$$

where the first field on the right is a 0-form on I and the second is a 1-form. The ghost numbers for c, A, A^+, c^+ are: $1, 0, -1, -2$.

Let

$$\mathcal{F}' = C^\bullet(I,\mathfrak{g})[1] \oplus C_{-\bullet}(I,\mathfrak{g}^*)[-2] \tag{217}$$

with $C_\bullet$ the cellular chains of the interval with standard CW decomposition into two 0-cells (endpoints) and one 1-cell (bulk), and $C^\bullet$ the cellular cochains. We denote the cellular basis in cochains $C^\bullet(I)$ by e_0, e_1, e_I and the dual basis in chains $C_\bullet(I)$ by e^0, e^1, e^I. Thus, an infrared field is

$$(\mathcal{A}' = e_0c_0 + e_1c_1 + e_IA_I, \mathcal{B}' = e^0c_0^+ + e^1c_1^+ + e^IA_I^+), \tag{218}$$

where

$$c_0, c_1 \in \mathfrak{g}[1],\ A_I \in \mathfrak{g},\ c_0^+, c_1^+ \in \mathfrak{g}^*[-2],\ A_I^+ \in \mathfrak{g}^*[-1]. \tag{219}$$

Consider the SDR

$$\kappa_{\mathcal{A}} \circlearrowleft (\Omega^\bullet(I,\mathfrak{g})[1], \mathrm{d}) \underset{\pi_{\mathcal{A}}}{\overset{\iota_{\mathcal{A}}}{\leftrightarrows}} (C^\bullet(I,\mathfrak{g})[1], \delta), \tag{220}$$

with δ the cellular coboundary operator, given by

$$\iota_{\mathcal{A}}\colon \quad e_0c_0 + e_1c_1 + e_IA_I \quad \mapsto \quad (1-t)c_0 + tc_1 + \mathrm{d}tA_I, \tag{221}$$

$$\pi_{\mathcal{A}}\colon \quad c + A \quad \mapsto \quad e_0c(0) + e_1c(1) + e_I\int_0^1 A, \tag{222}$$

$$\kappa_{\mathcal{A}}\colon \quad c + A \quad \mapsto \quad \int_{[0,1]\ni t'} (\theta(t-t') - t)A(t'). \tag{223}$$

One also has the dual SDR

$$\kappa_{\mathcal{B}} \circlearrowleft (\Omega^\bullet(I,\mathfrak{g}^*)[-1], \mathrm{d}) \underset{\pi_{\mathcal{B}}}{\overset{\iota_{\mathcal{B}}}{\leftrightarrows}} (C_{-\bullet}(I,\mathfrak{g}^*)[-2], \partial), \tag{224}$$

[46] BF theory with field $\mathcal{B}$ treated as a distribution (de Rham current) – one can call it "canonical" BF theory, cf. [30], [9, Section 6.3] – on a manifold with boundary is a pure BV theory rather than a "BV-BFV theory" [7] (a family of BV theories parametrized by boundary conditions). Here the effective action (and the action on $\mathcal{F}$, in a regularized sense) satisfy the usual quantum master equation, rather than one corrected by a boundary BFV operator. In this "canonical" setting, only the field $\mathcal{A}$ can be pulled back by the inclusion of the boundary, whereas $\mathcal{B}$ has the opposite functoriality (can be extended from the boundary into the bulk). Thus, a canonical BF theory does not behave as a functorial field theory in the sense of Segal's axioms: it does not assign symplectic phase spaces (classically) or spaces of states (at the quantum level) to boundaries – rather, it assigns to boundaries an odd-symplectic space with a boundary BV action. Nevertheless, there is a version of gluing formula for effective actions in canonical BF theory by an inclusion-exclusion formula, cf. [30, Section 5.4].

where ∂ is the cellular boundary operator. Here:

$$\iota_{\mathcal{B}}\colon e^0c_0^+ + e^1c_1^+ + e^IA_I^+ \mapsto \quad \delta(t)\mathrm{d}t\, c_0^+ + \delta(1-t)\mathrm{d}t\, c_1^+ + A_I^+, \tag{225}$$

$$\pi_{\mathcal{B}}\colon A^+ + c^+ \mapsto \quad e^0\int_0^1 (1-t)c^+ + e^1\int_0^1 tc^+ + e^I\int_0^1 \mathrm{d}t\, A^+, \tag{226}$$

$$\kappa_{\mathcal{B}}\colon A^+ + c^+ \mapsto \quad -\int_{[0,1]\ni t'} (\theta(t'-t)-t')c^+(t'). \tag{227}$$

Putting together SDRs (220) and (224) ($\mathcal{A}$- and $\mathcal{B}$-sectors), one has the full SDR of fields onto infrared fields,

$$\kappa = \kappa_{\mathcal{A}}\oplus\kappa_{\mathcal{B}} \circlearrowleft (\mathcal{F}, d) \underset{\pi=\pi_{\mathcal{A}}\oplus\pi_{\mathcal{B}}}{\overset{\iota=\iota_{\mathcal{A}}\oplus\iota_{\mathcal{B}}}{\leftrightarrows}} (\mathcal{F}', d'=\delta\oplus\partial) \tag{228}$$

We call this SDR the Whitney–Dupont gauge-fixing, since $\iota_{\mathcal{A}}$ realizes cell cochains by Whitney elementary forms on 1-simplex and $\kappa_{\mathcal{A}}$ is the Dupont chain contraction operator on forms on a simplex.[47]

The effective BV action on $\mathcal{F}'$ for this model was computed in [29], [30] to be

$$S' = \underbrace{\sum_{i=0}^1 \langle c_i^+, \frac{1}{2}[c_i,c_i]\rangle + \langle A_I^+, \mathbb{F}_+(\mathrm{ad}_{A_I})\circ c_1 - \mathbb{F}_-(\mathrm{ad}_{A_I})\circ c_0\rangle}_{S'^{\mathrm{cl}}} \underbrace{-\mathrm{i}\hbar\,\mathrm{tr}_{\mathfrak{g}}\mathbb{G}(\mathrm{ad}_{A_I})}_{S'^{1-\mathrm{loop}}}, \tag{229}$$

where

$$\mathbb{F}_+(x) = \frac{x}{1-e^{-x}} = \sum_{n\geq 0}\frac{B_n^+}{n!}x^n = 1+\frac{x}{2}+\frac{x^2}{12}-\frac{x^4}{720}+\cdots, \tag{230}$$

$$\mathbb{F}_-(x) = \frac{x}{e^x-1} = \sum_{n\geq 0}\frac{B_n^-}{n!}x^n = 1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^4}{720}+\cdots, \tag{231}$$

$$\begin{aligned}\mathbb{G}(x) &= \log\frac{\sinh(x/2)}{x/2}\\ &= \sum_{n\geq 0, n\neq 1}\frac{B_n}{n\cdot n!}x^n = 1+\frac{x^2}{2\cdot 12}-\frac{x^4}{4\cdot 720}+\cdots,\end{aligned} \tag{232}$$

with $B_n^\pm$ the Bernoulli numbers. The induced differential Q'^{cl} on $\mathcal{F}'$ is, by Theorem 5.1, the Hamiltonian vector field on $\mathcal{F}'$ generated by S'^{cl} or, equivalently, the cotangent lift of

$$Q'_{\mathcal{A}} = \sum_{i=0}^1 \langle\frac{1}{2}[c_i,c_i], \frac{\partial}{\partial c_i}\rangle + \langle\mathbb{F}_+(\mathrm{ad}_{A_I})\circ c_1 - \mathbb{F}_-(\mathrm{ad}_{A_I})\circ c_0, \frac{\partial}{\partial A_I}\rangle \tag{233}$$

– a formal vector field on $C^\bullet(I,\mathfrak{g})[1]$.

[47] This construction generalizes to higher-dimensional simplices, see [13], [29], [30]).

The vector field H^{cl} (191) in this example is

$$H^{\mathrm{cl}} = \int_0^1 \left\langle \kappa(F_{\mathcal{A}}) + d_{\mathcal{A}}\kappa(\mathcal{A}), \frac{\delta}{\delta \mathcal{A}} \right\rangle + \left\langle \kappa(d_{\mathcal{A}}\mathcal{B}) + d_{\mathcal{A}}\kappa(\mathcal{B})), \frac{\delta}{\delta \mathcal{B}} \right\rangle \tag{234}$$
$$= \int_0^1 \left\langle \kappa(d_A c) + [c, \kappa(A)], \frac{\delta}{\delta c} \right\rangle + \underbrace{\left\langle d_A\kappa(A), \frac{\delta}{\delta A} \right\rangle}_{(a)} +$$
$$+ \left\langle \kappa(d_A A^+ + [c, c^+]) + [c, \kappa(c^+)], \frac{\delta}{\delta A^+} \right\rangle + \left\langle d_A\kappa(c^+), \frac{\delta}{\delta c^+} \right\rangle.$$

7.1.1. *The L_∞ projection $\pi^{\mathrm{cl}}_{\mathrm{int}}$.* The flow Φ_T of the vector field $-H^{\mathrm{cl}}$ applies to a connection 1-form A a T-dependent gauge transformation relative to the endpoints of the interval I (cf. term (a) in (234)) which, as $T \to \infty$, "straightens" A to a gauge-equivalent constant connection[48] on I. Thus, we have the following.

Proposition 7.6. *$\pi^{\mathrm{cl}}_{\mathrm{int}}$ applied to A yields*

$$\pi^{\mathrm{cl}}_{\mathrm{int}}(A) = \lim_{T\to\infty} \pi(\Phi_T(A)) = -e_I \log P\exp\left(-\int_0^1 A\right) \in C^1(I, \mathfrak{g}) \cong \mathfrak{g} \tag{235}$$

– log of the parallel transport (path ordered exponential) of A along I.

Note that (235) is a nonlinear deformation of the integration map $\pi\colon A \mapsto \int_I A$.

More generally, applied to a general point in $\mathcal{F}^{\mathrm{formal}}$, $\pi^{\mathrm{cl}}_{\mathrm{int}}$ gives

(236)
$$\pi^{\mathrm{cl}}_{\mathrm{int}}(c + A, A^+ + c^+) = e_0 c(0) + e_1 c(1) - e_I \log P \exp\left(-\int_0^1 A\right) + \alpha(c, A, A^+, c^+),$$

with $\alpha \in C_{-\bullet}(I, \mathfrak{g}^*)[-2]$ a certain chain with complicated dependence on the input fields.[49]

Remark 7.7. Let $A_T = \Phi_T(A)$ be the flow of $-H^{\mathrm{cl}}$ in time T, applied to a connection A. The flow equation

$$\frac{\partial}{\partial T} A_T = -\mathrm{d}_{A_T}\kappa(A_T) \tag{237}$$

with initial condition $A_{T=0} = A$ can be rewritten in terms of Fourier modes $A_{T,n}$ (seen as functions of $T \in [0, \infty)$) of the connection $A_T = \sum_{n\in\mathbb{Z}} A_{T,n} e^{2\pi i n t} dt$, as an infinite system of quadratic ODEs

$$\frac{\partial}{\partial T} A_{T,n} = -A_{T,n}(1 - \delta_{n,0}) - \sum_{k\neq 0} \frac{1}{2\pi i k} A_{T,k}(A_{T,n-k} - A_{T,n}). \tag{238}$$

subject to the initial condition $A_{T=0,n} = A_n$.

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[48] Note that a connection is constant iff it is annihilated by κ.

[49] In special cases α simplifies: if $c^+ \in \mathrm{im}(\iota)$, we have $\alpha(c + A, A^+) = e^I \int_0^1 \mathrm{d}t\, A^+$. If $c = 0$, we have $\alpha(A, A^+ + c^+) = e^I \int_0^1 \mathrm{d}t\, A^+ + \beta(A, c^+)$, with β a complicated 0-chain.

7.1.2. *The L_∞ inclusion $\iota^{\mathrm{cl}}_{\mathrm{int}}$.* We find

$$\begin{aligned}(239)\quad \iota^{\mathrm{cl}}_{\mathrm{int}}(\mathcal{A}' = e_0c_0 + e_1c_1 + e_IA_I, \mathcal{B}' = e^0c_0^+ + e^1c_1^+ + e^IA_I^+) = \\ = \lim_{T\to\infty} \Phi_T\iota(\mathcal{A}', \mathcal{B}') \\ = \frac{e^{-t\mathrm{ad}_{A_I}} - e^{-\mathrm{ad}_{A_I}}}{1-e^{-\mathrm{ad}_{A_I}}}\circ c_0 + \frac{1-e^{-t\mathrm{ad}_{A_I}}}{1-e^{-\mathrm{ad}_{A_I}}}\circ c_1 + \mathrm{d}t\, A_I \\ + \delta(t)\mathrm{d}t\, c_0^+ + \delta(1-t)\mathrm{d}t\, c_1^+ + \frac{\mathrm{ad}^*_{A_I}}{1-e^{-\mathrm{ad}^*_{A_I}}}e^{-t\mathrm{ad}^*_{A_I}}\circ A_I^+\end{aligned}$$

Here the c component is the solution of $\mathrm{d}_{A_I}c = \mathrm{const}$, $c(0) = c_0, c(1) = c_1$; the A^+ component is the solution of $\mathrm{d}_{A_I}A^+ = 0$, $\int_0^1 \mathrm{d}t\, A^+ = A_I^+$ (this is the conditional critical point equation (192) supplemented by the condition that the field projects onto the respective infrared field).

7.1.3. *Magnus expansion.*[50] Comparing formula (235) with the homological perturbation formula

$$(240)\qquad i^{\mathrm{cl}}_{\mathrm{int}} = \sum_{n\geq 0}(-KQ_{\mathrm{int}})^n i,$$

one recovers the Magnus expansion of the log of parallel transport (the "continuous version" of the Baker–Campbell–Hausdorff formula).

As an illustration, let us compute the terms $n = 0, 1$ in the series (240) for a linear observable $O' = \langle \xi, A_I\rangle$, with $\xi \in \mathfrak{g}^*$ a parameter:

$$\begin{aligned}(241)\quad \langle \xi, A_I\rangle \xrightarrow{i} \left\langle \xi, \int_0^1 \mathrm{d}t\underline{A}(t)\right\rangle \xrightarrow{Q_{\mathrm{int}}} \left\langle \xi, -\int_0^1 \mathrm{d}t[\underline{A}(t), c(t)]\right\rangle \\ \xrightarrow{K} \left\langle \xi, \frac{1}{2}\int_0^1 \mathrm{d}t\left[\underline{A}(t) + \int_0^1 \mathrm{d}t''\underline{A}(t''), \int_0^1 \mathrm{d}t'(\theta(t-t') - t)\underline{A}(t')\right]\right\rangle \\ = \left\langle \xi, \frac{1}{2}\int_0^1 \mathrm{d}t\int_0^1 \mathrm{d}t'[\underline{A}(t), \underline{A}(t')]\underbrace{(\theta(t-t') - t - t' + \frac{1}{2})}_{f(t,t')}\right\rangle \\ = \left\langle \xi, \frac{1}{2}\int_0^1 \mathrm{d}t\int_0^t \mathrm{d}t'[\underline{A}(t), \underline{A}(t')]\right\rangle\end{aligned}$$

Here we were writing $A = \mathrm{d}t\underline{A}$ with $\underline{A}$ a $\mathfrak{g}$-valued function on the interval. In the transition to the last line we used that $[\underline{A}(t), \underline{A}(t')]$ is skew-symmetric under the swap of t and t' and that skew-symmetrizations of $f(t,t')$ and $\theta(t-t')$ agree. In the transition to the second line we used that $(\kappa^\vee)^{\mathrm{sym}}(a\otimes b) = \frac{\mathrm{id}+ip}{2}(a)\otimes\kappa^\vee(b) + \kappa^\vee(a)\otimes\frac{\mathrm{id}+ip}{2}(b)$, cf. (43). Thus, we have

$$\begin{aligned}(242)\quad i^{\mathrm{cl}}_{\mathrm{int}}\langle \xi, A_I\rangle = (i - KQ_{\mathrm{int}}i + \cdots)\langle \xi, A_I\rangle \\ = \left\langle \xi, \int_0^1 \mathrm{d}t\underline{A}(t) - \frac{1}{2}\int_0^1 \mathrm{d}t\int_0^t \mathrm{d}t'[\underline{A}(t), \underline{A}(t')] + \cdots\right\rangle\end{aligned}$$

where in the r.h.s. one recognizes ξ paired with the Magnus expansion of $-\log P\exp\left(-\int_0^1 A\right)$.

[50]The observation that one can recover Magnus expansion from homological perturbation lemma was made in [2] and in [3].

7.2. **Example: 1d BF theory on a circle: two gauge-fixings.**

7.2.1. *Whitney–Dupont gauge (circle with a distinguished point).* The example of Section 7.1 can be straightforwardly modified to 1d BF theory on a circle, with the action (213) and fields (214), with the interval I replaced by a circle S^1, which we parametrize by $t \in \mathbb{R}/\mathbb{Z}$. For the infrared fields we take

$$\mathcal{F}' = C^\bullet(S^1, \mathfrak{g})[1] \oplus C_{-\bullet}(S^1, \mathfrak{g}^*)[-2] \tag{243}$$

– the cellular cochains and chains of S^1 with CW decomposition into one 0-cell $t = 0$ and one 1-cell. We denote the cellular bases in cochains and chains e_0, e_{S^1} and e^0, e^{S^1}, respectively. Thus, an infrared field is

$$(\mathcal{A}' = e_0 c_{\mathrm{zm}} + e_{S^1} A_{\mathrm{zm}}, \mathcal{B}' = e^0 c^+_{\mathrm{zm}} + e^{S^1} A^+_{\mathrm{zm}}), \tag{244}$$

with $c_{\mathrm{zm}} \in \mathfrak{g}[1]$, $A_{\mathrm{zm}} \in \mathfrak{g}$, $c^+_{\mathrm{zm}} \in \mathfrak{g}^*[-2]$, $A^+_{\mathrm{zm}} \in \mathfrak{g}^*[-1]$. The subscript zm stands for "zero-mode."

One has SDR for the theory on S^1 obtained from (228) by gluing the endpoints of the interval,

$$\kappa_{\mathrm{WD}} \circlearrowleft (\mathcal{F}, d) \underset{\pi_{\mathrm{WD}}}{\overset{\iota_{\mathrm{WD}}}{\leftrightarrows}} (\mathcal{F}', 0) \tag{245}$$

with

(246)

$$\iota_{\mathrm{WD}} \colon (e_0 c_{\mathrm{zm}} + e_{S^1} A_{\mathrm{zm}}, e^{S^1} A^+_{\mathrm{zm}} + e^0 c^+_{\mathrm{zm}}) \mapsto (c_{\mathrm{zm}} + \mathrm{d}t\, A_{\mathrm{zm}},\ A^+_{\mathrm{zm}} + \delta(t)\mathrm{d}t\, c^+_{\mathrm{zm}}),$$

$$\pi_{\mathrm{WD}} \colon (c + A, A^+ + c^+) \mapsto \Big(e_0 c(0) + e_{S^1} \oint A,\ e^{S^1} \oint \mathrm{d}t A^+ + e^0 \oint c^+\Big),$$

$$\kappa_{\mathrm{WD}} \colon (c + A, A^+ + c^+) \mapsto \Big(\oint (\theta(t - t') - t) A(t'),\ \oint (-\theta(t' - t) + t') c^+(t')\Big).$$

Here $\oint$ is the integral over the entire S^1; subscript WD stands for "Whitney–Dupont."

The effective action of the model on the circle is

$$S' = \underbrace{\langle c^+_{\mathrm{zm}}, \frac{1}{2}[c_{\mathrm{zm}}, c_{\mathrm{zm}}]\rangle}_{S'^{\mathrm{cl}}} + \langle A^+_{\mathrm{zm}}, [A_{\mathrm{zm}}, c_{\mathrm{zm}}]\rangle - \mathrm{i}\hbar \underbrace{\mathrm{tr}_{\mathfrak{g}} \mathbb{G}(\mathrm{ad}_{A_{S^1}})}_{S'^{1-\mathrm{loop}}} \tag{247}$$

with $\mathbb{G}$ as in (232). The induced differential on $\mathcal{F}'$ is the Hamiltonian vector field generated by S'^{cl} or, equivalently, the cotangent lift of

$$Q'_{\mathcal{A}} = \langle \frac{1}{2}[c_{\mathrm{zm}}, c_{\mathrm{zm}}], \frac{\partial}{\partial c_{\mathrm{zm}}}\rangle + \langle [A_{\mathrm{zm}}, c_{\mathrm{zm}}], \frac{\partial}{\partial A_{\mathrm{zm}}}\rangle. \tag{248}$$

The vector field H^{cl} is again given by (234), and yields the following L_∞ maps $\iota^{\mathrm{cl}}_{\mathrm{WD,int}}, \pi^{\mathrm{cl}}_{\mathrm{WD,int}}$:

$$\begin{aligned} &\iota^{\mathrm{cl}}_{\mathrm{WD,int}}(e_0 c_{\mathrm{zm}} + e_{S^1} A_{\mathrm{zm}}, e^{S^1} A^+_{\mathrm{zm}} + e^0 c^+_{\mathrm{zm}}) \\ &\qquad = c_{\mathrm{zm}} + \mathrm{d}t\, A_{\mathrm{zm}} + \frac{\mathrm{ad}^*_{A_{\mathrm{zm}}}}{1 - e^{-\mathrm{ad}^*_{A_{\mathrm{zm}}}}} e^{-t\mathrm{ad}^*_{A_{\mathrm{zm}}}} \circ A^+_{\mathrm{zm}} + \delta(t)\mathrm{d}t\, c^+_{\mathrm{zm}}, \end{aligned} \tag{249}$$

$$\pi^{\mathrm{cl}}_{\mathrm{WD,int}}(c + A, A^+ + c^+) = e_0 c(0) - e_{S^1} \log P \exp\left(-\int_0^1 A\right) + \alpha, \tag{250}$$

with $\alpha \in C_{-\bullet}(S^1, \mathfrak{g}^*)[-2]$ a chain with complicated dependence on c, A, A^+, c^+.

7.2.2. *Lorenz gauge (circle without a distinguished point).* One can identify infrared fields (243) with doubled cohomology of the circle,

$$\mathcal{F}' = H^\bullet(S^1,\mathfrak{g})[1]\oplus H^\bullet(S^1,\mathfrak{g}^*)[-1]. \tag{251}$$

Thus, we identify e_0 and e^{S^1} as the generator 1 of $H^0(S^1)$ and we identify e_{S^1} and e^0 as the generator $[dt]$ of $H^1(S^1)$.

One can consider the following SDR

$$\kappa_{\mathrm{L}} \circlearrowleft (\mathcal{F},d) \underset{\pi_{\mathrm{L}}}{\overset{\iota_{\mathrm{L}}}{\leftrightarrows}} (\mathcal{F}',0) \tag{252}$$

with

$$\begin{aligned}
&\iota_{\mathrm{L}}\colon (e_0c_{\mathrm{zm}}+e_{S^1}A_{\mathrm{zm}},\ e^{S^1}A^+_{\mathrm{zm}}+e^0c^+_{\mathrm{zm}}) \mapsto (c_{\mathrm{zm}}+\mathrm{d}t\,A_{\mathrm{zm}}, A^+_{\mathrm{zm}}+\mathrm{d}t\,c^+_{\mathrm{zm}}),\\
&\pi_{\mathrm{L}}\colon (c+A,A^++c^+) \mapsto \Big(e_0\oint \mathrm{d}t\,c + e_{S^1}\oint A,\ e^{S^1}\oint \mathrm{d}t\,A^+ + e^0\oint c^+\Big),\\
&\kappa_{\mathrm{L}}\colon (c+A,A^++c^+) \mapsto \Big(\oint(\theta(t-t')-t+t'-\frac{1}{2})A(t'),\ \oint(\theta(t-t')-t+t'-\frac{1}{2})c^+(t')\Big).
\end{aligned} \tag{253}$$

Here L stands for "Lorenz." One has a non-normalized chain homotopy compatible with this SDR (cf. Remark 6.3), given by

$$\bar\kappa_{\mathrm{L}}\colon (\mathcal{A},\mathcal{B}) \mapsto (\mathrm{d}^*\mathcal{A},\mathrm{d}^*\mathcal{B}), \tag{254}$$

with d^* the Hodge codifferential associated with the metric $(\mathrm{d}t)^2$ on S^1, hence the name "Lorenz SDR": gauge-fixing $\bar\kappa(\mathcal{A},\mathcal{B})=0$ is the usual Lorenz gauge condition.

The vector field H^{cl} associated to $\bar\kappa$ is

$$\begin{aligned}
H^{\mathrm{cl}} = \oint \Big\langle \mathrm{d}^*\mathrm{d}_A c + [c,\mathrm{d}^*A], \frac{\delta}{\delta c}\Big\rangle + \Big\langle \mathrm{d}_A\mathrm{d}^*A, \frac{\delta}{\delta A}\Big\rangle + \\
+ \Big\langle \mathrm{d}^*(\mathrm{d}_A A^+ + [c,c^+]) + [c,\mathrm{d}^*c^+], \frac{\delta}{\delta A^+}\Big\rangle + \Big\langle \mathrm{d}_A\mathrm{d}^*c^+, \frac{\delta}{\delta c^+}\Big\rangle.
\end{aligned} \tag{255}$$

The corresponding L_∞ maps $\iota^{\mathrm{cl}}_{\mathrm{L,int}}, \pi^{\mathrm{cl}}_{\mathrm{L,int}}$ are:

$$\iota^{\mathrm{cl}}_{\mathrm{L,int}}(e_0c_{\mathrm{zm}}+e_{S^1}A_{\mathrm{zm}}, e^{S^1}A^+_{\mathrm{zm}}+e^0c^+_{\mathrm{zm}}) = (c_{\mathrm{zm}}+\mathrm{d}t\,A_{\mathrm{zm}},\ A^+_{\mathrm{zm}}+\mathrm{d}t\,c^+_{\mathrm{zm}}), \tag{256}$$

$$\begin{aligned}
&\pi^{\mathrm{cl}}_{\mathrm{L,int}}(c+A,A^++c^+)\\
&\quad = e_0(\cdots) - e_{S^1}U(A)\left(\log P\exp\left(-\int_0^1 A\right)\right)U(A)^{-1} + e^{S^1}(\cdots) + e^0(\cdots).
\end{aligned} \tag{257}$$

Here $U(A)$ is an element of the group G integrating $\mathfrak{g}$ with complicated dependence on A; coefficients $(\cdots)$ also have complicated dependence on the fields.

As a consequence of (257), if we have an observable $O'(A_{\mathrm{zm}})$ given by a $\mathfrak{g}$-invariant function of the infrared connection $A_{\mathrm{zm}}\in\mathfrak{g}$, then

$$\imath^{\mathrm{cl}}_{\mathrm{int}}(O')(A) = ((\pi^{\mathrm{cl}}_{\mathrm{L,int}})^*O')(A) = O'\left(-\log P\exp\left(-\int_0^1 A\right)\right) \tag{258}$$

– the natural lift of a gauge-invariant observable. E.g., if $O'=\mathrm{tr}_R e^{-A_{\mathrm{zm}}}$, with tr_R the trace in some representation R of G, then

$$\imath^{\mathrm{cl}}_{\mathrm{int}}(O')(A) = \mathrm{tr}_R P\exp\left(-\oint A\right). \tag{259}$$

We summarize the differences between Whitney–Dupont gauge and Lorenz gauge:

- in WD gauge, the point $t = 0$ plays a distinguished role, in Lorenz gauge it does not.
- ι_{WD} and ι_{L} differ in the c^+ term. π_{WD} and π_{L} differ in the c term.
- Lorenz SDR treats the $\mathcal{A}$ and $\mathcal{B}$ sectors symmetrically, whereas Whitney–Dupont SDR does not.
- ι_{L} lands in smooth forms and κ_{L} maps smooth forms to smooth forms. On the other hand, ι_{WD} lands in distributional forms and κ_{WD} maps smooth c^+ forms to 0-forms discontinuous at $t = 0$.
- In Lorenz gauge, ι does not get undeformed by the interaction, while in WD gauge the A^+ component attains a deformation.
- π gets deformed in both gauges, but in WD case the answer is more explicit (e.g. the A-component is known exactly, not up to an implicit conjugation).

8. TQM as a 1d AKSZ theory

Given an interacting BV theory $(\mathcal{F}, \omega, S = S_0 + S_{\mathrm{int}})$, consider a 1d AKSZ theory τ on the interval $[0, T]$ with target $\mathcal{N} = T^*\mathcal{F}$.[51]

We will assume for simplicity that S does not depend on $\hbar$. The quantum master equation for S splits into

$$\{S, S\} = 0, \tag{260}$$

$$\Delta S = 0. \tag{261}$$

We denote x^i the coordinates on $\mathcal{F}$ and p_i the dual coordinates on the cotangent fiber $\mathcal{F}^*$. The target $\mathcal{N}$ is equipped canonical with 0-symplectic structure $\omega_{\mathcal{N}} = \langle \delta p, \delta x \rangle$ and AKSZ Hamiltonian

$$\begin{aligned} \Theta_{\mathcal{N}}(x,p) &= \omega^{-1}\left(p, \frac{\partial S}{\partial x}\right) + \frac{1}{2}\omega^{-1}(p,p) \\ &= \langle p, d_{\mathcal{F}} x\rangle + \underbrace{\omega^{-1}\left(p, \frac{\partial S_{\mathrm{int}}}{\partial x}\right)}_{\Theta_{\mathrm{int}}} + \frac{1}{2}\omega^{-1}(p,p) \quad \in C^\infty(\mathcal{N})_1 \end{aligned} \tag{262}$$

of ghost number 1. In this subsection we denote the differential on $\mathcal{F}$ by $d_{\mathcal{F}}$ to avoid confusion with other differentials. The Hamiltonian vector field generated by $\Theta_{\mathcal{N}}$ is the cohomological vector field on $\mathcal{N}$

$$Q_{\mathcal{N}} = Q^{\text{cot. lift}} + \omega^{-1}\left(p, \frac{\partial}{\partial x}\right), \tag{263}$$

where the first term is the cotangent lift of $Q = \{S, -\}$ from $\mathcal{F}$ to $\mathcal{N}$.[52]

Thus, the space of fields of the 1d AKSZ theory τ is

$$\mathcal{F}^\tau = \mathrm{Map}(T[1][0,T], T^*\mathcal{F}) = \Omega^\bullet([0,T], \mathcal{F}) \oplus \Omega^\bullet([0,T], \mathcal{F}^*). \tag{264}$$

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[51] This theory is a classical Lagrangian description of the topological quantum mechanics τ of Section 6, see Remark 8.2, hence we use the same label τ for it.

[52] One has an isomorphism $T^*\mathcal{F} \simeq T[1]\mathcal{F}$ induced fiberwise by $(\omega^\#)^{-1}$. Thus, the target $\mathcal{N}$ can be identified with $T[1]\mathcal{F}$ with base coordinates x^i and shifted tangent fiber coordinates θ^i. Under the identification $C^\infty(\mathcal{N}) = C^\infty(T[1]\mathcal{F}) \cong \Omega^\bullet(\mathcal{F})$, the target AKSZ Hamiltonian (262) is $\Theta_{\mathcal{N}} = \mathsf{d}S + \omega$ – a form on $\mathcal{F}$ of mixed degree 1 and 2. The cohomological vector field (263), as a derivation of $\Omega^\bullet(\mathcal{F})$, is $Q_{\mathcal{N}} = \mathcal{L}_Q + \mathsf{d}$, with d the de Rham operator on forms on $\mathcal{F}$. The target 0-symplectic form is $\omega_{\mathcal{N}} = \omega(\delta\theta, \delta x)$.

It is parametrized by two AKSZ superfields

$$\widetilde{x} = x + p^+ \in \Omega^\bullet([0,T],\mathcal{F}), \quad \widetilde{p} = p + x^+ \in \Omega^\bullet([0,T],\mathcal{F}^*) \tag{265}$$

with x, p being 0-forms along $[0,T]$ and p^+, x^+ being 1-forms. The BV action of the 1d AKSZ theory τ is

$$\begin{aligned} S^\tau &= \int_0^T \langle p, \mathrm{d}_t x\rangle + \Theta_{\mathcal{N}}(\widetilde{x},\widetilde{p}) \\ &= \int_0^T \langle p, \mathrm{d}_t x\rangle + \langle \widetilde{p}, d_{\mathcal{F}}\widetilde{x}\rangle + \frac{1}{2}\omega^{-1}(\widetilde{p},\widetilde{p}) + \Theta_{\mathrm{int}}(\widetilde{x},\widetilde{p}). \end{aligned} \tag{266}$$

The (-1)-symplectic form on $\mathcal{F}^\tau$ is

$$\omega^\tau = \int_0^T \langle \delta\widetilde{p}, \delta\widetilde{x}\rangle = \int_0^T \langle \delta x, \delta x^+\rangle + \langle \delta p, \delta p^+\rangle. \tag{267}$$

We consider the following gauge-fixing Lagrangian $\mathcal{L}^\tau \subset \mathcal{F}^\tau$:

$$\mathcal{L}^\tau = \Big\{ \Big(\widetilde{x} = x + \mathrm{d}t\,\kappa(x), \widetilde{p} = p - \mathrm{d}t\,\kappa^\vee(p)\Big) \;\Big|\; (x,p) \in \Omega^0([0,T],\mathcal{F}\oplus\mathcal{F}^*)\Big\}, \tag{268}$$

i.e., on $\mathcal{L}^\tau$ the 0-form components of superfields are free and 1-form components are dependent on 0-form components.[53] Here t is the coordinate on the interval $[0,T]$.

The gauge-fixed action is:

$$\begin{aligned} S^\tau|_{\mathcal{L}^\tau} &= \int_0^T \mathrm{d}t\big(\langle p, (\partial_t - P'')x\rangle - \omega^{-1}(p,\kappa^\vee(p))\big) + \\ &\quad + \Theta_{\mathrm{int}}(x + \mathrm{d}t\,\kappa(x), p - \mathrm{d}t\,\kappa^\vee(p)) \\ &= \int_0^T \mathrm{d}t(\langle p, \partial_t x\rangle - \mathbb{H}(x,p)). \end{aligned} \tag{269}$$

Here $P'' = \mathrm{id} - \iota\circ\pi$ is the projector onto the second summand in $\mathcal{F} = \mathcal{F}'\oplus\mathcal{F}''$ and

$$\begin{aligned} \mathbb{H}(x,p) &= \{\Theta_{\mathcal{N}}, \big\langle p, \kappa(x)\big\rangle\}_{\mathcal{N}} \\ &= \omega^{-1}\left(p, \kappa^\vee\frac{\partial S}{\partial x}\right) + \omega^{-1}\left(p, \Big\langle \kappa(x), \frac{\partial}{\partial x}\Big\rangle \frac{\partial S}{\partial x}\right) + \omega^{-1}(p,\kappa^\vee(p)) \quad \in C^\infty(\mathcal{N})_0 \end{aligned} \tag{270}$$

is the classical Hamiltonian.

Remark 8.1. The phase space that the AKSZ theory τ assigns to a point is the target Hamiltonian dg manifold

$$(\mathcal{N} = T^*\mathcal{F}, \omega_{\mathcal{N}}, \Theta_{\mathcal{N}}, Q_{\mathcal{N}}). \tag{271}$$

Remark 8.2. The canonical quantization $x \mapsto x\cdot$, $p \mapsto -\mathrm{i}\hbar\frac{\partial}{\partial x}$ prescription makes the following assignments:

(i) The AKSZ target $\mathcal{N}$ (as a symplectic manifold) quantizes to $\widehat{S}\mathcal{F}^*[[\hbar]]$.

[53] One can express $\mathcal{L}^\tau$ by realizing $\mathcal{F}^\tau$ as the cotangent bundle of the "trivial Lagrangian" $\mathcal{L}^\tau_0 = \Omega^0([0,T],\mathcal{F}\oplus\mathcal{F}^*)$ (given by setting 1-forms to zero) and then deforming $\mathcal{L}^\tau_0$ to the graph Lagrangian $\mathcal{L}^\tau = \mathrm{graph}(\delta\Psi)$ for the gauge-fixing fermion $\Psi = \int_0^T \mathrm{d}t\,\langle p, \kappa(x)\rangle \in C^\infty(\mathcal{L}^\tau_0)_{-1}$.

(ii) The target AKSZ Hamiltonian $\Theta_{\mathcal{N}}$ quantizes to the BV differential $-\mathrm{i}\hbar Q_\hbar$.[54] Thus, the target as Hamiltonian dg manifold (271) quantizes to the BV complex $(\widehat{S}\mathcal{F}^*[[\hbar]], Q_\hbar)$.
(iii) The classical Hamiltonian $\mathbb{H}$ quantizes to $-\mathrm{i}\hbar H$, with $H = \mathcal{L}_E - \mathrm{i}\hbar\eta + [\widehat{\kappa}, Q_{\mathrm{int}}]$, as in (162).[55]

Remark 8.3. More appropriately, one should say that the target is a formal neighborhood of zero in $T^*\mathcal{F}$, $\Theta_{\mathcal{N}}$ and $\mathbb{H}$ are formal functions, $Q_{\mathcal{N}}$ is a formal vector field. To simplify the exposition we are suppressing the "formal" qualifier.

Remark 8.4. For gauge-fixing of the 1d AKSZ theory in (268), (269), (270), one can use a non-normalized chain homotopy $\bar{\kappa}$ instead of a normalized one κ, cf. Remark 6.3.

Remark 8.5. One can rewrite the action (266) – completing it to a full square in $\widetilde{p} + \cdots$ – as

$$S^\tau = \int_0^T \frac{1}{2}\omega^{-1}\left(\widetilde{p} + \omega^\# \mathrm{d}_t x + \frac{\partial S}{\partial x}(\widetilde{x}), \widetilde{p} + \omega^\# \mathrm{d}_t x + \frac{\partial S}{\partial x}(\widetilde{x})\right) - \frac{1}{2}\underbrace{\omega^{-1}\left(\frac{\partial S}{\partial x}(\widetilde{x}), \frac{\partial S}{\partial x}(\widetilde{x})\right)}_{0 \text{ by CME}} - \mathrm{d}_t S(\widetilde{x}) \tag{272}$$

Here the second term vanishes by classical master equation $\{S, S\} = 0$. Integrating out $\widetilde{p}$ we obtain a theory with the action

$$-\int_0^T \mathrm{d}_t S(\widetilde{x}) = S(x|_{t=0}) - S(x|_{t=T}). \tag{273}$$

Thus, fixing $x_{\mathrm{in}}, x_{\mathrm{out}} \in \mathcal{F}$ and fixing some extension $\widetilde{x}$ as a form on $[0,T]$ restricting to x_{in}, x_{out} at $t = 0, T$, we have

$$\int \mathcal{D}\widetilde{p}\, e^{\frac{\mathrm{i}}{\hbar}S^\tau} = e^{\frac{\mathrm{i}}{\hbar}S(x_{\mathrm{in}})} e^{-\frac{\mathrm{i}}{\hbar}S(x_{\mathrm{out}})}. \tag{274}$$

We stress that this is a different gauge-fixing than $\mathcal{L}^\tau$ (268) used in the rest of this section.

Example 8.6. Consider the toy interacting scalar field of Section 5.1. In this case, the 1d AKSZ theory τ has target

$$\mathcal{N} = \underset{\pi,p}{T^*}(T^*\underset{\xi}{[-1]}\mathbb{R}\underset{x}{)}, \tag{275}$$

with coordinates x, ξ on $T^*[-1]\mathbb{R}$ of ghost degrees $0, -1$ and coordinates p, π (of ghost degree $0, 1$) in the cotangent fiber over (x, ξ). The structure on $\mathcal{N}$ is:

$$\omega_{\mathcal{N}} = \delta p\, \delta x + \delta\pi\, \delta\xi, \quad \Theta_N = \pi S'(x) + \pi p. \tag{276}$$

The AKSZ superfields are

$$\widetilde{x} = \underset{0}{x} + \mathrm{d}t\, \underset{-1}{p^+},\ \widetilde{\xi} = \underset{-1}{\xi} + \mathrm{d}t\, \underset{-2}{\pi^+},\ \widetilde{p} = \underset{0}{p} + \mathrm{d}t\, \underset{-1}{x^+},\ \widetilde{\pi} = \underset{1}{\pi} + \mathrm{d}t\, \underset{0}{\xi^+} \tag{277}$$

[54]The two natural orderings – "p to right of x" vs. "p to the left of x" yield the same quantization due to $\Delta S = 0$ (261).

[55]More specifically, the three terms in the second line in (270) quantize to $-\mathrm{i}\hbar Q\widehat{\kappa}$, $-\mathrm{i}\hbar\widehat{\kappa}Q$ and $(-\mathrm{i}\hbar)^2\eta$.

where we indicated the ghost degrees of the components. The AKSZ action is

$$S^\tau = \int_0^T \mathrm{d}t(p\dot{x} - \pi\dot{\xi} + \Theta_{\mathcal{N}}(\widetilde{x},\widetilde{\xi},\widetilde{p},\widetilde{\pi})) \\ = \int_0^T \mathrm{d}t(p\dot{x} - \pi\dot{\xi} + \xi^+ S'(x) + p^+\pi S''(x) + x^+\pi + \xi^+ p) \tag{278}$$

The gauge-fixing is

$$\mathcal{L}^\tau \colon \widetilde{x} = x + \mathrm{d}t\,\xi,\ \widetilde{\xi} = \xi,\ \widetilde{p} = p,\ \widetilde{\pi} = \pi + \mathrm{d}t\,p. \tag{279}$$

The gauge-fixed AKSZ action is therefore

$$S^\tau|_{\mathcal{L}^\tau} = \int_0^T \mathrm{d}t(p\dot{x} - \pi\dot{\xi} + \mathbb{H}(x,\xi,p,\pi)) \tag{280}$$

with

$$\mathbb{H}(x,\xi,p,\pi) = \{\Theta_{\mathcal{N}}, p\xi\}_{\mathcal{N}} = pS'(x) + \xi\pi S''(x) + p^2. \tag{281}$$

8.1. **Path integral formulae for i_{int}, p_{int}, K_{int} in terms of 1d AKSZ theory.**

Theorem 8.7. *One has the following path integral formulae for $i_{\mathrm{int}}, p_{\mathrm{int}}, K_{\mathrm{int}}$ in terms of gauge-fixed 1d AKSZ theory (269):*

$$i_{\mathrm{int}}(O')(x_{\mathrm{out}}) = \lim_{T\to\infty} \int_{p(0)=0,\ x(T)=x_{\mathrm{out}}} \mathcal{D}[x(t)]\mathcal{D}[p(t)]\, O'(\pi(x(0)))\, e^{\frac{\mathrm{i}}{\hbar}S^\tau|_{\mathcal{L}^\tau}}, \tag{282}$$

$$p_{\mathrm{int}}(O)(x'_{\mathrm{out}}) = \lim_{T\to\infty} \int_{p(0)=0,\ x(T)=\iota(x'_{\mathrm{out}})} \mathcal{D}[x(t)]\mathcal{D}[p(t)]\, O(x(0))\, e^{\frac{\mathrm{i}}{\hbar}S^\tau|_{\mathcal{L}^\tau}}, \tag{283}$$

$$K_{\mathrm{int}}(O)(x_{\mathrm{out}}) \\ = \int_0^\infty \int_{p(0)=0,\ x(T)=x_{\mathrm{out}}} \mathcal{D}[x(t)]\mathcal{D}[p(t)]\ \underbrace{O(\widetilde{x}|_{t=0,\mathrm{d}t=\mathrm{d}T})}_{O\left(x(0)+\mathrm{d}T\kappa(x(0))\right)}\ e^{\frac{\mathrm{i}}{\hbar}S^\tau}\Big|_{\mathcal{L}^\tau}, \tag{284}$$

for any input observables $O \in \widehat{S}\mathcal{F}^*$, $O' \in \widehat{S}\mathcal{F}'^*$. The outer integral in (284) is in T (one extracts the $\mathrm{d}T$ component from the integrand, coming from the observable).

Here the path integrals are understood perturbatively, writing $S^\tau|_{\mathcal{L}^\tau}$ as the last line of (269) and understanding $\mathbb{H}$ as perturbation.

To prove the theorem, first we need the following lemma.

Lemma 8.8. *One has*

$$(e^{-TH}O)(x_{\mathrm{out}}) = \int_{p(0)=0,\ x(T)=x_{\mathrm{out}}} \mathcal{D}[x(t)]\mathcal{D}[p(t)]\, O(x(0))\, e^{\frac{\mathrm{i}}{\hbar}S^\tau|_{\mathcal{L}^\tau}}, \tag{285}$$

with any $T > 0$, $O \in \widehat{S}\mathcal{F}^*$ and with H as in (162) – the quantum Hamiltonian of the TQM.

Proof. First, write

$$x(t) = \overline{x}_{\mathrm{out}}(t) + \underline{x}(t), \tag{286}$$

where the first term is $\overline{x}_{\mathrm{out}}(t) = \begin{cases} x_{\mathrm{out}},\ t = T, \\ 0,\ t < T \end{cases}$ – the discontinuous extension of x_{out} by zero to $t \in [0, T)$.[56] The r.h.s. of (285) then reads

$$\int_{p(0)=0,\, \underline{x}(T)=0} \mathcal{D}[\underline{x}(t)]\mathcal{D}[p(t)]\, O(\underline{x}(0)) e^{\frac{\mathrm{i}}{\hbar}\left(\langle p(T), x_{\mathrm{out}}\rangle + \int_0^T \mathrm{d}t(\langle p, \partial_t \underline{x}\rangle - \mathbb{H}(\underline{x}, p))\right)} \tag{287}$$

Consider free expectation values

$$\langle f\rangle_0 = \int_{p(0)=0,\, \underline{x}(T)=0} \mathcal{D}[\underline{x}(t)]\mathcal{D}[p(t)]\, e^{\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t\langle p, \partial_t \underline{x}\rangle} \cdot f, \tag{288}$$

with f a function on the space of paths $(\underline{x}(t), p(t))$. One has propagators (two-point correlation functions)

$$\langle p(t) \otimes \underline{x}(t')\rangle_0 = -\mathrm{i}\hbar\theta(t-t') \otimes \mathrm{id}, \quad \langle \underline{x}(t) \otimes \underline{x}(t')\rangle_0 = 0, \quad \langle p(t) \otimes p(t')\rangle_0 = 0. \tag{289}$$

Computing (287) by Wick's lemma, we obtain

$$\left. e^{\langle x_{\mathrm{out}}, \frac{\partial}{\partial \underline{x}}\rangle} e^{-TH} O(\underline{x}) \right|_{\underline{x}=0} = (e^{-TH} O)(x_{\mathrm{out}}), \tag{290}$$

which proves the lemma.

Here we were understanding $\mathbb{H}$ as a normally-ordered expression, prohibiting self-contractions, which is tantamount to displacing p in $\mathbb{H}(\underline{x}, p)$ to a slightly earlier time t than $\widehat{x}$:

$$:\mathbb{H}(\underline{x}(t), p(t)) := \lim_{\epsilon \to +0} \mathbb{H}(\underline{x}(t), p(t-\epsilon)). \tag{291}$$

□

Proof of Theorem 8.7. Theorem 8.7 follows from Theorem 6.1 and Lemma 8.8. Indeed:

- Set $O = i(O')$ in (285) and take the limit $T \to \infty$. Then the l.h.s. becomes the r.h.s. of (164) applied to O' and evaluated at x_{out}, and the r.h.s. becomes the r.h.s. of (282), which proves (282).
- Set $x_{\mathrm{out}} = \iota(x'_{\mathrm{out}})$ in (285) and take the limit $T \to \infty$. Then the l.h.s. becomes the r.h.s. of (165) applied to O and evaluated at x'_{out}, whereas the r.h.s. becomes the r.h.s. of (283).
- Set $O = \mathrm{d}T\, \widehat{\kappa} O$ in (285) and integrate in T from $T = 0$ to $T = \infty$. Then the l.h.s. becomes the r.h.s. of (166) (applied to O and evaluated at x_{out}) while the r.h.s. becomes the r.h.s. of (284).

□

Example 8.9. For the toy interacting scalar field (Example 8.6), the path integral formula for the chain homotopy (284), evaluated on an observable $O(x)$, reads

$$K_{\mathrm{int}}(O)(x_{\mathrm{out}}, \xi_{\mathrm{out}}) = \tag{292}$$
$$\int_0^\infty \int_{p(0)=\pi(0)=0,\, x(T)=x_{\mathrm{out}},\, \xi(T)=\xi_{\mathrm{out}}} \mathcal{D}[x(t)]\mathcal{D}[\xi(t)]\mathcal{D}[p(t)]\mathcal{D}[\pi(t)]\, O(x(0) + \mathrm{d}T\, \xi(0))\, e^{\frac{\mathrm{i}}{\hbar} S^\tau|_{\mathcal{L}^\tau}},$$

with $S^\tau|_{\mathcal{L}^\tau}$ as in (280).

[56] Cf. discontinuous extension of boundary fields into the bulk in [7, Section 2.4].

8.2. **Cable diagrams for i_{int} as Feynman graphs for 1d AKSZ path integral.** Consider the path integral (285):

$$(293)\quad (e^{-TH}O)(x_{\mathrm{out}}) = \int_{p(0)=0,\ x(T)=x_{\mathrm{out}}} \mathcal{D}[x(t)]\mathcal{D}[p(t)]\, O(x(0))\, e^{\frac{\mathrm{i}}{\hbar}S^\tau|_{\mathcal{L}^\tau}}$$

$$= \int_{p(0)=0,\ \underline{x}(T)=0} \mathcal{D}[\underline{x}(t)]\mathcal{D}[p(t)]\, O(\underline{x}(0))\cdot \exp\frac{\mathrm{i}}{\hbar}\Big(\int_0^T \mathrm{d}t\,\Big(\langle p, (\partial_t - P'')\underline{x}\rangle -$$

$$-\underbrace{\omega^{-1}\left(\kappa^\vee(p), \frac{\partial S}{\partial x}\right)}_{\mathbb{H}_a} - \underbrace{\omega^{-1}\left(p, \left\langle \kappa(x), \frac{\partial}{\partial x}\right\rangle \frac{\partial S}{\partial x}\right)}_{\mathbb{H}_b} - \underbrace{\omega^{-1}(p, \kappa^\vee(p))}_{\mathbb{H}_\eta}\Big)\Big|_{x\to\underline{x}} + \underbrace{\langle p(T), x_{\mathrm{out}}\rangle}_{\mathbb{B}}\Big).$$

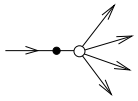
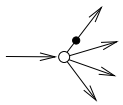
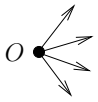
Here we used the splitting $x(t) = \bar{x}_{\mathrm{out}}(t) + \underline{x}(t)$ as in (286). Let us consider $\int_0^T \mathrm{d}t\langle p, (\partial_t - P'')\widehat{x}\rangle$ as the leading (kinetic) term and the rest of the exponential as a perturbation. We denote $\langle\cdots\rangle$ the Gaussian average with this kinetic term.

The propagator in this perturbation theory – the line in Feynman graphs – is

$$(294)\quad \alpha = \frac{\mathrm{i}}{\hbar}\langle p(t)\otimes \underline{x}(t')\rangle = \theta(t-t')e^{-(t-t')P''}\otimes \mathrm{id} \quad \in \Omega^0_{\mathrm{distr}}([0,T]^{\times 2}, \mathcal{F}^*\otimes\mathcal{F}).$$

We will draw it in Feynman graphs as an edge directed from $\underline{x}$ to p.

Perturbations in (293) correspond to vertices in Feynman graphs:

term in (293)	vertex in Feynman graphs
$\mathbb{H}_a \in \mathrm{Hom}(\mathcal{F}^*, S\mathcal{F}^*)$	
$\mathbb{H}_b \in \mathrm{Hom}(\mathcal{F}^*, S\mathcal{F}^*)$	
$\mathbb{H}_\eta \in \mathrm{Hom}(S^2\mathcal{F}^*, \mathbb{R})$	
$\mathbb{B} \in \mathrm{Hom}(\mathcal{F}^*, \mathbb{R})$	
$O \in S\mathcal{F}^*$	

Here incoming half-edges are decorated by p and outgoing ones by $\underline{x}$. Black dot decorating one incident half-edge is a part of the graphical notation for $\mathbb{H}_a$, $\mathbb{H}_b$ (and corresponds to $\kappa^\vee$ or κ hitting one p or x argument in the formulae for $\mathbb{H}_a$, $\mathbb{H}_b$); black dot in $\mathbb{H}_\eta$ vertex refers to $\kappa^\vee$ in the formula for $\mathbb{H}_\eta$. We also adopt the (optional) graphical arrangement convention that outgoing half-edges go to the right, and incoming ones come from the left.

The perturbative formula for the path integral (293) is then

$$(295)\qquad (293) = \sum_\Gamma \frac{(-\mathrm{i}\hbar)^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\Phi^\tau_\Gamma(O, x_{\mathrm{out}}, T),$$

where the sum is over oriented connected graphs Γ with a unique vertex O and arbitrary finite numbers of $\mathbb{H}_a$, $\mathbb{H}_b$, $\mathbb{H}_\eta$, $\mathbb{B}$ vertices (see Figure 11). We do not allow short loops (edges connecting a vertex to itself) in Γ. $\chi(\Gamma)$ is the Euler characteristic

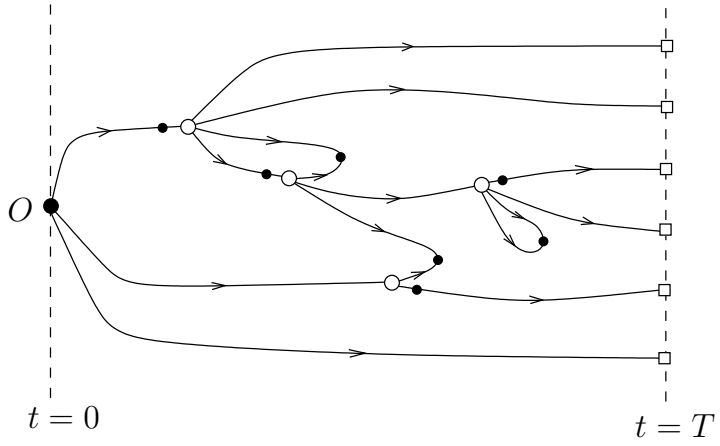


FIGURE 11. A typical Feynman graph contributing to the perturbative 1d AKSZ path integral (295). A particular configuration of times t_v in the integrand in (296) corresponds to a particular horizontal placement of vertices.

of the graph and $|\mathrm{Aut}(\Gamma)|$ is the order of the automorphism group. The weight of the graph is

$$\Phi^\tau_\Gamma(O, x_{\mathrm{out}}, T) \tag{296}$$
$$= \int_{[0,T]^{V_{\mathrm{bulk}}}} \left\langle \bigotimes_{v\in V_{\mathrm{bulk}}} \mathrm{d}t_v \mathbb{H}_{\lambda(v)} \otimes O \otimes \mathbb{B}^{\otimes V_{\mathbb{B}}}, \bigotimes_{\text{edges } e=(vu)} \pi_{uv}^* \alpha \right\rangle_\Gamma \Bigg|_{t_{v_O}=0, t_{v_{\mathbb{B}}}=T}$$

Here:

- The set of vertices of Γ is $V = V_{\mathrm{bulk}} \sqcup \{v_O\} \sqcup V_{\mathbb{B}}$ – "bulk" vertices (i.e. those integrated in t) of types $\mathbb{H}_a, \mathbb{H}_b, \mathbb{H}_\eta$, the unique vertex O and the vertices $\mathbb{B}$; $\lambda(v) \in \{a, b, \eta\}$ is the type of a bulk vertex.
- $\pi_{uv} \colon [0,T]^V \to [0,T]^{\times 2}$ is the map $(t_1, \dots, t_V) \mapsto (t_u, t_v)$.
- $\langle -, - \rangle_\Gamma$ is the canonical pairing between $\mathcal{F}$ and $\mathcal{F}^*$ extended to tensor powers, with factors matched according to the combinatorics of Γ.
- The integral is over the variables $t_v \in [0,T]$, $v \in V_{\mathrm{bulk}}$.

Note that, due to the form of the propagator (294), the integrand in (296) can be nonzero only if for every edge $\langle p(t)\underline{x}(t')\rangle$ in the graph one has $t > t'$ (which is consistent with edges oriented left-to-right and t increasing left-to-right in Figure 11).

8.2.1. *Specialization to i_{int}.* Evaluating (293), (295) on an infrared observable $O = i(O')$ and taking the limit $T \to \infty$, we obtain

$$i_{\mathrm{int}}(O')(x_{\mathrm{out}}) = \lim_{T\to\infty} \sum_\Gamma \frac{(-\mathrm{i}\hbar)^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \Phi^\tau_\Gamma(i(O), x_{\mathrm{out}}, T) \tag{297}$$

Here the contributing graphs – see Figure 12 – are the same as in (295), except O is now replaced with O', connected by edges $\frac{\mathrm{i}}{\hbar}\langle p(t)\pi(\underline{x}(0))\rangle = P'$ to other vertices; we draw these "infrared" edges as dashed. Note that O' cannot be connected by

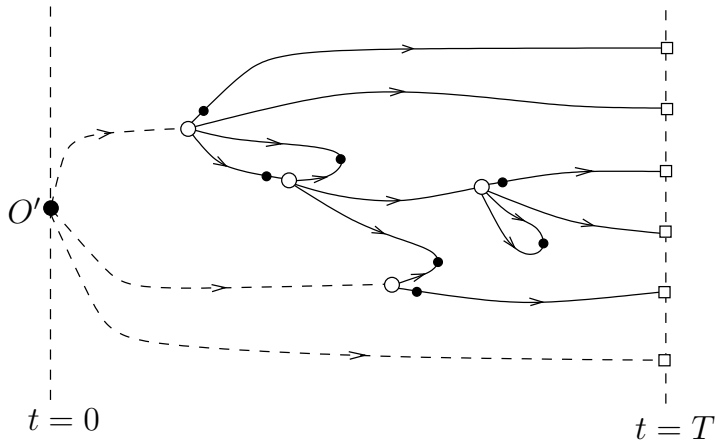


FIGURE 12. Typical Feynman graph contributing to the expansion of i_{int} (297.

edges to $\mathbb{H}_a$ vertices or $\mathbb{H}_\eta$ vertices, otherwise the graph vanishes trivially.

Also note that, due to the exponential decay of the propagator (294), the integral (296) in the limit $T \to \infty$ is supported on configurations $\{t_v\}$ where all t_v are within $\mathcal{O}(1)$ of T. Thus, a typical configuration is where the dashed edges are very long and the other edges are finite.

Finally, in the Feynman graphs appearing in (297) we immediately recognize the cable diagrams for i_{int}, Figure 8. Thick edges in the cable diagram correspond to edges without a black dot in Figure 12.

Remark 8.10. One can also recover cable diagrams for p_{int}, by specializing (293), (295) to $x_{\mathrm{out}} = \iota(x'_{\mathrm{out}})$ and taking $T \to \infty$:

$$(p_{\mathrm{int}}O)(x'_{\mathrm{out}}) = \lim_{T\to\infty} \sum_\Gamma \frac{(-\mathrm{i}\hbar)^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \Phi^\tau_\Gamma(O, \iota(x'_{\mathrm{out}}), T). \tag{298}$$

In this case, one has dashed "infrared" edges $\frac{\mathrm{i}}{\hbar}\langle \iota^\vee(p(T))\underline{x}(t)\rangle = P'$ connecting to the $\mathbb{B}$-vertices. In this case the graphs Γ cannot contain $\mathbb{H}_b$ vertices, otherwise the graph vanishes. See Figure 13 for a typical contributing graph. The integral (296) in the limit $T \to \infty$ is supported on configurations $t_v = \mathcal{O}(1)$, i.e., the dashed edges are long while the other edges are finite.

Likewise, one recovers cable diagrams for K_{int} by replacing $O \to dT\widehat{\kappa}(O)$ in (293), (295) and integrating in $T \in [0, \infty)$:

$$(K_{\mathrm{int}}O)(x_{\mathrm{out}}) = \int_0^T \sum_\Gamma \frac{(-\mathrm{i}\hbar)^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \Phi^\tau_\Gamma(\mathrm{d}T\,\widehat{\kappa}(O), x_{\mathrm{out}}, T). \tag{299}$$

See Figure 14 for a typical contributing graph.

Remark 8.11. In the case $\mathcal{F}' = 0$, we can consider the path integral (283) using the gauge-fixing of Remark 8.5 instead of $\mathcal{L}^\tau$:

$$\int \mathcal{D}x_{\mathrm{in}} \int \mathcal{D}\widetilde{p}\, O(x_{\mathrm{in}})\, e^{\frac{\mathrm{i}}{\hbar}S^\tau} \underset{(274)}{=} \int \mathcal{D}x_{\mathrm{in}}\, O(x_{\mathrm{in}})\, e^{\frac{\mathrm{i}}{\hbar}S(x_{\mathrm{in}})}. \tag{300}$$

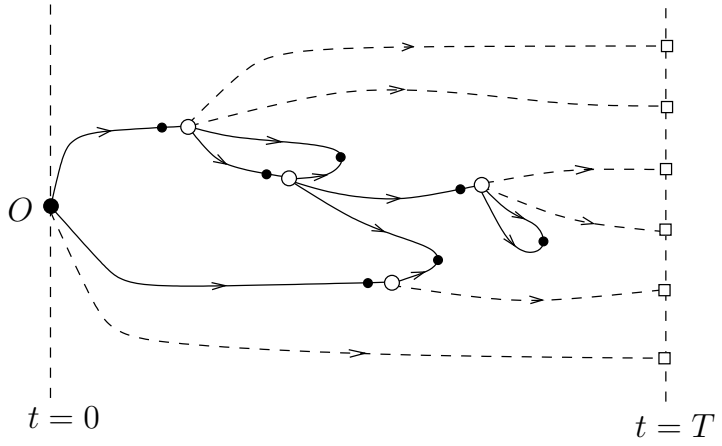


FIGURE 13. 1d AKSZ Feynman graph for p_{int}.

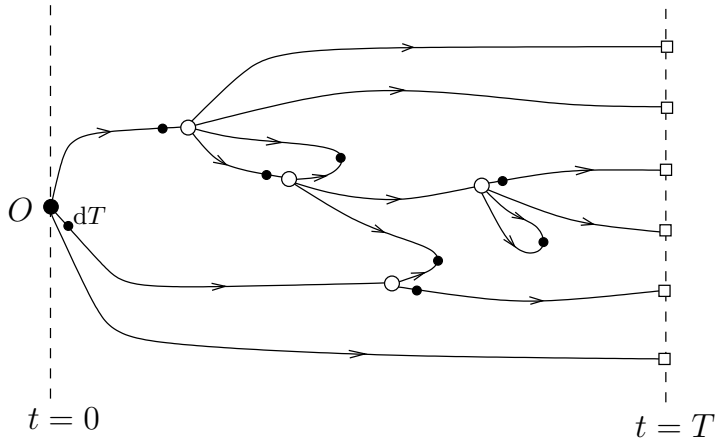


FIGURE 14. 1d AKSZ Feynman graph for K_{int}. A new element compared to Figure 11 is an extra black dot adjacent to O (understood in this case as $\mathrm{d}T\,\kappa$).

Here we understand that for each x_{in} we fix some extension $\widetilde{x}$ of it from $t = 0$ to the interval $[0, T]$, restricting to $x_{\mathrm{out}} = 0$ at $t = T$. The integral over $\widetilde{p}$ in the l.h.s. is with free boundary conditions. The integral in the r.h.s. still needs to be gauge-fixed, and then it yields the BV pushforward of O. This yields another (formal) proof of Theorem 5.1 (ii) in the case $\mathcal{F}' = 0$. Indeed, using the formal independence of the BV integral in theory τ on gauge-fixing and (283), we have that the r.h.s. (300) equals p_{int} as defined by homological perturbation series. Note that with this gauge-fixing the integral (300) does not depend on T and taking the limit $T \to \infty$ is a tautological operation.

8.3. **A class of examples: the case when $(\mathcal{F},\omega,S)$ is itself an AKSZ theory.** Let $(\mathcal{F},\omega,S)$ be an AKSZ theory on a closed n-manifold M with

$$\mathcal{F}=\mathrm{Map}(T[1]M,Y)\cong\Omega^\bullet(M,Y) \tag{301}$$

with target Y a graded vector space equipped with:

- A constant symplectic form ω_Y of ghost degree $n-1$.
- A target Hamiltonian $\Theta_Y\in C^\infty(Y)$ of ghost degree n satisfying the classical master equation $\{\Theta_Y,\Theta_Y\}_{\omega_Y}=0$.

We denote y a point in Y and $\bar y$ a point in $\mathcal{F}$ (an AKSZ superfield).

The AKSZ action is

$$S(\bar y)=\int_M\frac{1}{2}\omega_Y(\bar y,\mathrm{d}_M\bar y)+\Theta_Y(\bar y) \tag{302}$$

and the (-1)-symplectic form on the space of fields is

$$\omega=\int_M\frac{1}{2}\omega_Y(\delta\bar y,\delta\bar y). \tag{303}$$

We can identify the cotangent bundle $T^*\mathcal{F}$ with[57]

$$T^*\mathcal{F}\simeq\mathrm{Map}(T[1]M,T^*[n]Y)\cong\Omega^\bullet(M,Y)\oplus\Omega^\bullet(M,Y^*)[n]\quad\ni(\bar y,\bar\psi), \tag{304}$$

where the canonical symplectic form on $T^*\mathcal{F}$ is $\int_M\langle\delta\bar\psi,\delta\bar y\rangle$. In our notations, ψ is the cotangent fiber coordinate on $T^*[n]Y$. Note that, comparing with notations of (266), we have $x=\bar y$, $p=\bar\psi$.

The AKSZ theory τ then has the space of fields (264):

$$\begin{aligned}\mathcal{F}^\tau&=\mathrm{Map}(T[1][0,T],\underbrace{\mathrm{Map}(T[1]M,T^*[n]Y)}_{T^*\mathcal{F}})\\&\cong\mathrm{Map}(T[1]([0,T]\times M),T^*[n]Y).\end{aligned} \tag{305}$$

In particular, it can be seen as an $(n+1)$-dimensional theory on a cylinder $N=[0,T]\times M$. The target AKSZ structure on $\mathcal{Y}=T^*[n]Y$ is:

- The canonical degree n symplectic structure $\omega_{\mathcal{Y}}=\langle\delta\psi,\delta y\rangle$.
- The target AKSZ Hamiltonian

$$\Theta_{\mathcal{Y}}=\omega_Y^{-1}\left(\psi,\frac{\partial\Theta_Y}{\partial y}\right)+\frac{1}{2}\omega_Y^{-1}(\psi,\psi)\quad\in C^\infty(\mathcal{Y}) \tag{306}$$

  of ghost degree $n+1$.

Then the AKSZ action

$$S^\tau(\widetilde y,\widetilde\psi)=\int_N\langle\widetilde\psi,\mathrm{d}_N\widetilde y\rangle+\Theta_{\mathcal{Y}}(\widetilde y,\widetilde\psi) \tag{307}$$

is the action (266). We denoted $\widetilde y,\widetilde\psi$ the AKSZ superfields on $[0,T]\times M$ (as opposed to $\bar y,\bar\psi$ – forms on M). The (-1)-symplectic structure (267) is

$$\omega^\tau=\int_N\langle\delta\widetilde\psi,\delta\widetilde y\rangle. \tag{308}$$

[57]For $\mathcal{F}$ an infinite-dimensional graded vector space, we understand the dual space $\mathcal{F}^*$ in the weak sense, i.e., as a graded vector space W equipped with a degree zero bilinear form $b\colon\mathcal{F}\otimes W\to\mathbb{R}$ such that the induced map $b^\#\colon\mathcal{F}\to W^*$ is injective.

Example 8.12. Let $(\mathcal{F}, \omega, S)$ be Chern–Simons theory on a 3-manifold M with coefficients in a quadratic Lie algebra $\mathfrak{g}$ (cf. Example 7.2) – an AKSZ theory with target $Y = \mathfrak{g}[1] \ni y$, $\Theta_Y = \frac{1}{6}\langle y, [y, y]\rangle$, $\omega_Y = \frac{1}{2}\langle \delta y, \delta y\rangle$. Then τ is the 4d AKSZ theory on the cylinder $N = [0, T] \times M$ with target

$$\mathcal{Y} \cong \mathfrak{g}[1] \oplus \mathfrak{g}[2] \ni (y, \psi), \tag{309}$$

with symplectic structure $\omega_{\mathcal{Y}} = \langle \delta\psi, \delta y\rangle$ and the Hamiltonian

$$\Theta_{\mathcal{Y}} = \langle \psi, \frac{1}{2}[y, y]\rangle + \frac{1}{2}\langle \psi, \psi\rangle. \tag{310}$$

The AKSZ theory τ has the space of fields

$$\mathcal{F}^\tau = \Omega^\bullet(N, \mathfrak{g})[1] \oplus \Omega^\bullet(N, \mathfrak{g})[2] \quad \ni (\mathcal{A}, \mathcal{B}) \tag{311}$$

with (-1)-symplectic form $\int_N \langle \delta\mathcal{B}, \delta\mathcal{A}\rangle$ and the action

$$S^\tau = \int_N \langle \mathcal{B}, \mathrm{d}_N\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]\rangle + \frac{1}{2}\langle \mathcal{B}, \mathcal{B}\rangle. \tag{312}$$

One recognizes the 4d "$BF + B^2$" theory, or "BF theory with cosmological term."

Example 8.13. Let $(\mathcal{F}, \omega, S)$ be BF theory on an n-manifold M with coefficients in a unimodular Lie algebra $\mathfrak{g}$ – an AKSZ theory with target $Y = T^*[n-1]\mathfrak{g}[1] = \mathfrak{g}[1] \oplus \mathfrak{g}^*[n-2] \ni (y, p_y)$, with $\omega_Y = \langle \delta p_y, \delta y\rangle$ and $\Theta_Y = \langle p_y, \frac{1}{2}[y, y]\rangle$. Then τ is the AKSZ theory on the $(n+1)$-dimensional cylinder $N = [0, T] \times M$ with target

$$\begin{aligned} \mathcal{Y} &= T^*[n]T^*[n-1]\mathfrak{g}[1] \\ &\cong (\underset{y}{\mathfrak{g}[1]} \oplus \underset{P_\psi}{\mathfrak{g}^*[n-2]}) \oplus (\underset{P_y}{\mathfrak{g}^*[n-1]} \oplus \underset{\psi}{\mathfrak{g}[2]}) \cong T^*[n](\underset{y}{\mathfrak{g}[1]} \oplus \underset{\psi}{\mathfrak{g}[2]}), \end{aligned} \tag{313}$$

where we indicated the names y, ψ, P_y, P_ψ for the components of an element in $\mathcal{Y}$. The target $\mathcal{Y}$ is equipped with n-symplectic structure

$$\omega_{\mathcal{Y}} = \langle \delta P_y, \delta y\rangle + \langle \delta P_\psi, \delta\psi\rangle \tag{314}$$

and the Hamiltonian

$$\Theta_{\mathcal{Y}} = \langle P_y, \frac{1}{2}[y, y]\rangle + \langle P_\psi, [y, \psi]\rangle + \langle P_y, \psi\rangle \tag{315}$$

of degree $n+1$. AKSZ theory τ has the space of fields

$$\mathcal{F}^\tau = \Omega^\bullet(N, \mathfrak{g})[1] \oplus \Omega^\bullet(N, \mathfrak{g})[2] \oplus \Omega^\bullet(N, \mathfrak{g}^*)[n-1] \oplus \Omega^\bullet(N, \mathfrak{g}^*)[n-2] \quad \ni (\mathcal{A}, \alpha, \mathcal{B}, \beta) \tag{316}$$

equipped with (-1)-symplectic form

$$\omega^\tau = \int_N \langle \delta\mathcal{B}, \delta\mathcal{A}\rangle + \langle \delta\beta, \delta\alpha\rangle \tag{317}$$

and the action

$$S^\tau = \int_N \langle \mathcal{B}, \mathrm{d}_N\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]\rangle + \langle \beta, \mathrm{d}_N\alpha + [\mathcal{A}, \alpha]\rangle + \langle \mathcal{B}, \alpha\rangle. \tag{318}$$

This is BF theory on N with coefficients in $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g}[1]$ – the Lie algebra $\mathfrak{g}$ extended by the (shifted) adjoint module, seen as a *differential graded* Lie algebra with differential $\mathfrak{g} \xleftarrow{\mathrm{id}} \mathfrak{g}[1]$.

Example 8.14. Consider Example 8.13 with $M = S^1$, i.e., $(\mathcal{F}, \omega, S)$ is BF theory on a circle. Consider the Lorenz gauge of Section 7.2.2 and an infrared observable $O'(A_{\mathrm{zm}})$ given by a G-invariant function on $\mathfrak{g}$. In this case, τ is the 2d BF theory on a cylinder $N = S^1 \times [0, T]$ with coefficients in the dg Lie algebra $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g}[1]$. Formula (282) becomes

$$i_{\mathrm{int}}(O')(A_{\mathrm{out}}) = \lim_{T\to\infty} \int O' \left(\oint_{S^1} A\big|_{t=0} \right) e^{\frac{\mathrm{i}}{\hbar} S^\tau|_{\mathcal{L}^\tau}}. \tag{319}$$

Here the outer integral is over fields $\mathcal{A}, \mathcal{B}, \alpha, \beta$ on the cylinder N, subject to boundary conditions $\mathcal{A}|_{t=T} = A_{\mathrm{out}}, \beta|_{t=T} = 0$, $\alpha|_{t=0} = \mathcal{B}|_{t=0} = 0$, and restricted to the gauge-fixing Lagrangian $\mathcal{L}^\tau$ (which fixes the components of fields which are 1-forms along $[0, T]$).[58]

The equations of motion generated by the gauge-fixed action $S^\tau|_{\mathcal{L}^\tau}$ extend A_{out} to a flat connection in the cylinder[59] (so the conjugacy class of its holonomy is constant for each cross-section of the cylinder). In the asymptotics $T \to \infty$, this extension restricts to a constant connection on the in-circle $S^1 \times \{t = 0\}$, which proves that semi-classically one has the result (258):

$$i^{\mathrm{cl}}_{\mathrm{int}}(O')(A_{\mathrm{out}}) = O'(-\log P \exp\left(- \oint_{S^1} A_{\mathrm{out}} \right)). \tag{320}$$

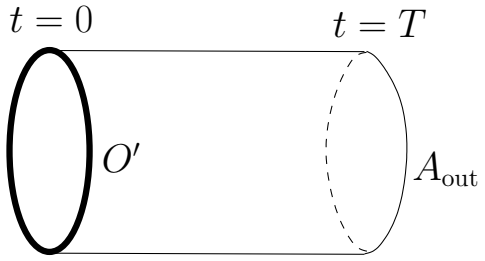


FIGURE 15. Illustration for Example 8.14: BF theory on a cylinder with boundary condition for A on one side ($t = T$) and observable $O'\left(A_{\mathrm{zm}} = \oint_{S^1\times\{0\}} A\right)$ on the other side yields in the limit $T \to \infty$ the lift of the observable $i_{\mathrm{int}}(O')(A_{\mathrm{out}})$, cf. (319).

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[58] Due to a conflict between notations for fields convenient for 2d theory vs. 1d theory, the fields of 1d BF theory on S^1 in this example are $\mathcal{A}, \beta$ instead of $\mathcal{A}, \mathcal{B}$ as in Section 7.2.2.

[59] More explicitly: let s be the coordinate on S^1 and t be the coordinate on $[0, T]$. Decomposing the $\mathfrak{g}$-valued 1-form on the cylinder as $A = A_s ds + A_t dt$, flatness reads $\partial_t A_s = \partial_s A_t + [A_s, A_t]$. The equation of $\mathcal{L}^\tau$ (268), using the non-normalized chain homotopy $\bar{\kappa} = \mathrm{d}^*_s$ associated to Lorenz gauge (254), yields $A_t = -\partial_s A_s$. Thus, we have a non-linear heat equation for A_s: $\partial_t A_s = -\partial_s^2 A_s - [A_s, \partial_s A_s]$.

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Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

Email address: cattaneo@math.uzh.ch

University of Notre Dame, Notre Dame, IN 46556, USA

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

Email address: pmnev@nd.edu