BV Pushforward of Palatini–Cartan gravity

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Abstract

The goal of this note is to show that the standard BV formulation of gravity in the Palatini–Cartan formalism is equivalent, using a BV pushforward, to an AKSZ-like version compatible with BFV data on the boundary.

1 Introduction

Gravity in the Palatini–Cartan (PC) formalism[1] has the advantage of using differential forms, which makes the boundary (and lower-dimensional corners) analysis more transparent. On the flip side, in dimension 4 and higher, its BV formulation [BV77] is not directly compatible with the boundary BFV formulation [BF83]. As such, PC gravity cannot be directly presented as an extended field theory, which would be the first, semiclassical, step towards a quantization satisfying Segal's axioms [Seg88].

A possible way out, on a "globally hyperbolic" space–time $M = \Sigma \times I$, Σ being a Cauchy surface, was studied in [CCS21a] (see also [AV98; Ale00; Bar01; DKS09; RP09; BTT13; CS19b; MRC19; Mon+20; CCS21b]), where the AKSZ construction [Ale+97] was applied to the BFV description of the boundary's reduced phase space. The resulting theory automatically has a compatible BV-BFV structure [CMR14] and is BV embedded in the BV description of PC gravity. The goal of this paper is to show that this embedding is actually a BV equivalence, via BV pushforward.

More explicitly, we consider PC theory in N space–time dimensions, $N \geq 4$. We pick once and for all a vector bundle $\mathcal{V}$ over space–time M isomorphic to its tangent bundle and carrying a Minkowski (or Euclidean) inner product. The action reads

$$S[\mathbf{e}, \boldsymbol{\omega}] = \int_M \frac{1}{(N-2)!} \mathbf{e}^{N-2} F_{\boldsymbol{\omega}},$$

where $\mathbf{e}\colon TM \xrightarrow{\sim} \mathcal{V}$ is a coframe and $\boldsymbol{\omega}$ is an orthogonal connection for $\mathcal{V}$. The BV action for this theory was studied in [Pig00; CS19a] and is recalled in Section 3.1.

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[1] that is, working with Cartan's coframes instead of metric tensors and letting the connection be an independent field, à la Palatini, to be determined by the equations of motion [Pal19; Car22]

If M has a boundary Σ, we require $\mathbf{e}$ to induce a nondegenerate metric on it (we might also more strictly require the metric induced by $\mathbf{e}$ to be positive-definite, which would make Σ space-like). Under this assumption, it was shown in [CS19b] that the reduced phase space can be obtained as the reduction of a coisotropic submanifold (i.e., defined by first-class constraints) of a symplecting space of boundary fields consisting $\mathcal{V}$-coframes and connections on Σ satisfying an appropriate condition (called the structural constraint). As such, the reduced phase space can be cohomologically resolved in terms of the BFV formalism. This is explicitly done in [CCS21b].

The problem is that the bulk BV theory does not communicate well with the boundary BV theory, due to some irregularity in a quotient appearing in the course of the construction [CS19a].

In this paper, also extending the classical construction of [CC25], we show how to remedy this when working on a space–time M of the form $\Sigma \times I$, where I is an interval or a circle, and $\mathbf{e}$ is required to induce a nondegenerate metric on each slice $\Sigma \times \{t\}$, $t \in I$. (Under the stricter condition that each of these metrics is positive-definite, this makes M into a globally hyperbolic manifold for each allowed coframe field $\mathbf{e}$.) We can now split the BV fields of the theory into their Σ and I components. It turns out that part of the Σ component of the connection is redundant, as already observed at the classical level in [CC25]. Here we show that upon gauge fixing it can be integrated away. This is a partial BV integration, known as a BV pushforward, which in the present case leads to an equivalence between the original theory and the one resulting after pushforward. The latter is then shown to be that described in [CCS21a] as a result of the AKSZ construction for the BFV structure associated to Σ. In conclusion, we prove that PC gravity, on $\Sigma \times I$, is equivalent to a theory with a well-defined BV-BFV structure, in the sense specified in Section 2.2. In particular, this means that the expectation values of corresponding observables in the two theories are the same, hence carrying the same physical information.

Summary of the paper

In Section 2, we recall the BV pushforward and its properties. In Section 3, we recall the BV formalism for PC theory and for the AKSZ construction starting from BFV on a boundary. In Sections 4 and 5, we compute the BV pushforward showing the equivalence between the two theories, ignoring boundary effects. This is legitimate if I is a circle or if I is an open interval and we let the fields decay to zero towards its boundary. If I is a compact interval, which is the interesting case to relate bulk to boundary, we need a more refined analysis, taking care of boundary conditions. This is explained in Section 6.

2 Generalities on the BV pushforward

A short introduction to the BV-pushforward can be found in [CR05; Mne08; CM09], in [Mne17][Section 4.7] and in [CMR18][Section 2.2.2].

Let $(\mathcal{F}, \varpi)$ be a graded symplectic manifold with symplectic form of degree -1. Let us suppose that there exist two graded symplectic manifolds with symplectic forms of degree -1, $(\mathcal{F}_1, \varpi_1)$ and $(\mathcal{F}_2, \varpi_2)$ such that

$$\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \qquad\qquad \varpi = \varpi_1 + \varpi_2.$$

Definition 1. Given a Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}_2$, the BV-pushforward along $\mathcal{L}$ is a partial[2] map between half-densities,

$$\mathcal{P}_{\mathcal{L}} : \mathrm{Dens}^{\frac{1}{2}}\mathcal{F} \to \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}_1$$

[2]Its definition domain consists of half-densities integrable on $\mathcal{L}$.

defined as an $\mathbb{R}$-linear map sending $\phi = \phi_1 \otimes \phi_2 \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}_1 \otimes \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}_2 \subset \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}$ to

$$\mathcal{P}_{\mathcal{L}}\phi := \phi_1 \cdot \int_{\mathcal{L}} \phi_2$$

where the integral is the BV-integral over $\mathcal{L}$.

The importance of this map is related to the fact that it relates solutions of the Quantum Master Equation (QME) on $\mathcal{F}$ to solutions of the QME on $\mathcal{F}_1$. Let us now briefly recap the results that lead to this statement. The first is an adaptation of the standard BV theorem.

Theorem 2 ([Mne08])**.** *Let Δ, Δ_1 and Δ_2 be the canonical BV laplacians on respectively $\mathcal{F}, \mathcal{F}_1$ and $\mathcal{F}_2$.*

1. *Given any Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}_2$, $\mathcal{P}_{\mathcal{L}}$ is a chain map, i.e.*

$$\Delta_1 \mathcal{P}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}} \Delta.$$

2. *If $\mathcal{L} \sim \mathcal{L}'$ are two homologous Lagrangians in $\mathcal{F}_2$ and $\phi \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}$ is such that $\Delta\phi = 0$, then the difference*

$$\mathcal{P}_{\mathcal{L}}\phi - \mathcal{P}_{\mathcal{L}'}\phi$$

is Δ_1-exact.

Let us now assume that the symplectic manifolds $(\mathcal{F}, \varpi), (\mathcal{F}_1, \varpi_1)$ and $(\mathcal{F}_2, \varpi_2)$ are equipped with compatible Berezinians μ, μ_1 and μ_2 and such that $\mu = \mu_1\mu_2$. Then we have the following corollary.

Corollary 3. *Let $S \in C^\infty(\mathcal{F})[[\hbar]]$. Then there exists $S_1 \in C^\infty(\mathcal{F}_1)[[\hbar]]$ such that*

$$\mu_1^{\frac{1}{2}} e^{\frac{i}{\hbar}S_1} = \mathcal{P}_{\mathcal{L}} \left(\mu^{\frac{1}{2}} e^{\frac{i}{\hbar}S} \right).$$

If S satisfies the Quantum Master Equation on $\mathcal{F}$

$$\Delta_\mu \left(e^{\frac{i}{\hbar}S} \right) = 0,$$

then so does S_1 on $\mathcal{F}_1$:

$$\Delta_{\mu_1} \left(e^{\frac{i}{\hbar}S_1} \right) = 0.$$

Furthermore, if we have S_1 and S_1' for two homologous Lagrangians $\mathcal{L} \sim \mathcal{L}' \subset \mathcal{F}_2$, then

$$e^{\frac{i}{\hbar}S_1} - e^{\frac{i}{\hbar}S_1'}$$

is Δ_1-exact.

2.1 BV bundles

The setting explained at the beginning of this section can (and must) be extended, with some care, to the case of a fiber bundle $\pi\colon \mathcal{F} \to \mathcal{F}_1$ that is locally the product of odd symplectic manifolds. By choosing a lagrangian submanifold $\mathcal{L}$ of the fibers (we can in principle allow a

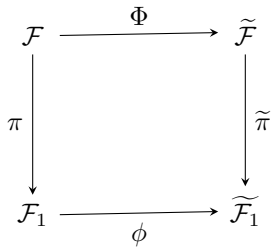
smoothly varying family of Lagrangian submanifolds $\mathcal{L}_x \subset \pi^{-1}(x)$, $\forall x \in \mathcal{F}_1$), we can get, via integration along $\mathcal{L}$, a map

$$\mathcal{P}_{\mathcal{L}} \colon \mathrm{Dens}^{\frac{1}{2}} \mathcal{F} \to \mathrm{Dens}^{\frac{1}{2}} \mathcal{F}_1.$$

What is not guaranteed in the general setting are the results of Theorem 2.

Conditions on the fiber bundle $\pi \colon \mathcal{F} \to \mathcal{F}_1$ to obtain Theorem 2 are discussed in [CMR18, Remark 2.13] under the name of a "BV hedgehog."

For the present note, we only need a simpler version, which we explain now. Namely, assume that we have a bundle isomorphism



where $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}_1} \times \widetilde{\mathcal{F}_2}$, $\widetilde{\pi}$ is the projection to the first factor, and both Φ and ϕ are symplectomorphisms. We then choose a Lagrangian submanifold $\widetilde{\mathcal{L}}$ of $\widetilde{\mathcal{F}_2}$ and perform the BV pushforward along it. What we are interested in is the composition

$$\widehat{\mathcal{P}}_{\widetilde{\mathcal{L}}} \coloneqq (\phi^{-1})_* \circ \mathcal{P}_{\widetilde{\mathcal{L}}} \circ \Phi_*.$$

Since the pushforwards of half-densities by symplectomorphisms are chain maps, the first statement of Theorem 2 now holds for $\widehat{\mathcal{P}}_{\widetilde{\mathcal{L}}}$. The second statement holds as well for homologous Lagrangians $\widetilde{\mathcal{L}} \sim \widetilde{\mathcal{L}}'$ in $\widetilde{\mathcal{F}_2}$. Similarly, we get the results of Corollary 3.

2.2 BV observables and quasiisomorphisms

Two theories connected through a BV pushforward can be considered equivalent in the sense specified in this section. Suppose S satisfies the QME on $\mathcal{F}$ with respect to a reference, Δ-closed, nowhere vanishing half-density $\mu^{\frac{1}{2}}$ (i.e., $\Delta_\mu \left(e^{\frac{i}{\hbar} S} \right) = 0$). Thanks to the QME, the degree-1 operator

$$\Omega_{(S,\mu)} \coloneqq (S, \ ) - i\hbar \Delta_\mu$$

acts on a functional $\mathcal{O}$ as

$$\Omega_{(S,\mu)} \mathcal{O} = -i\hbar \, e^{-\frac{i}{\hbar} S} \Delta_\mu \left(e^{\frac{i}{\hbar} S} \mathcal{O} \right).$$

Therefore, it squares to zero and defines a cohomology $H_{\Omega_{(S,\mu)}}(\mathcal{F})$. Note that

1. $\Omega_{(S,\mu)} \mathcal{O} = 0$ if and only if $\Delta \left(\mu^{\frac{1}{2}} e^{\frac{i}{\hbar} S} \mathcal{O} \right) = 0$.
2. $\mathcal{O} = \Omega_{(S,\mu)} \widetilde{\mathcal{O}}$ if and only if $\mu^{\frac{1}{2}} e^{\frac{i}{\hbar} S} \mathcal{O} = \Delta \left(-i\hbar \, \mu^{\frac{1}{2}} e^{\frac{i}{\hbar} S} \widetilde{\mathcal{O}} \right)$.

This implies that the expectation value

$$\langle \mathcal{O} \rangle_{\mathcal{L}} \coloneqq \frac{\int_{\mathcal{L}} \mu^{\frac{1}{2}} e^{\frac{i}{\hbar} S} \mathcal{O}}{\int_{\mathcal{L}} \mu^{\frac{1}{2}} e^{\frac{i}{\hbar} S}}$$

is invariant under deformations of $\mathcal{L}$ if $\mathcal{O}$ is $\Omega_{(S,\mu)}$-closed and vanishes if $\mathcal{O}$ is $\Omega_{(S,\mu)}$-exact. Therefore, the expectation value is a well-defined partial linear functional on the cohomology $H_{\Omega_{(S,\mu)}}(\mathcal{F})$, whose elements are accordingly called BV observables.

A BV symplectomorphism that maps (S,μ) to $(\widetilde{S},\widetilde{\mu})$ establishes an isomorphic chain map between the corresponding complexes and therefore an isomorphism in cohomology.

Another way to produce a chain map is via the BV pushforward. Namely, with the notations of Corollary 3, we have the partial chain map[3]

$$\begin{array}{rccc} p\colon & (\mathcal{F},\Omega_{(S,\mu)}) & \to & (\mathcal{F}_1,\Omega_{(S_1,\mu_1)}) \\ & \mathcal{O} & \mapsto & \mu_1^{-\frac{1}{2}} e^{-\frac{i}{\hbar}S_1}\mathcal{P}_{\mathcal{L}}\left(\mu^{\frac{1}{2}} e^{\frac{i}{\hbar}S}\mathcal{O}\right) \end{array}$$

which in turn induces a linear map $H_{\Omega_{(S,\mu)}}(\mathcal{F}) \to H_{\Omega_{(S_1,\mu_1)}}(\mathcal{F}_1)$. Note that we have

$$\int_{\mathcal{L}_1\times\mathcal{L}} \mu^{\frac{1}{2}} e^{\frac{i}{\hbar}S}\mathcal{O} = \int_{\mathcal{L}_1} \mu_1^{\frac{1}{2}} e^{\frac{i}{\hbar}S_1} p\mathcal{O},$$

so[4]

$$\langle \mathcal{O}\rangle_{\mathcal{L}_1\times\mathcal{L}} = \langle p\mathcal{O}\rangle_{\mathcal{L}_1}.$$

When the map induced by p in cohomology is an isomorphism, the chain map p is called a quasiisomorphism. This is equivalent to having a partial chain map $q\colon (\mathcal{F}_1,\Omega_{(S_1,\mu_1)}) \to (\mathcal{F},\Omega_{(S,\mu)})$ such that $q\circ p$ and $p\circ q$ are homotopic to the identity maps. It follows that, for a functional $\mathcal{O}_1$ on $\mathcal{F}_1$, we have

$$\langle \mathcal{O}_1\rangle_{\mathcal{L}_1} = \langle q\mathcal{O}_1\rangle_{\mathcal{L}_1\times\mathcal{L}}.$$

Therefore, when a BV pushforward defines a quasiisomorphism, the two theories are completely equivalent in the sense that there is a one-to-one correspondence between their BV observables and between the respective expectation values.

We will show that the two formulations of gravity considered in this paper are equivalent in this sense.

3 Two BV theories of PC gravity on cylinders

3.1 *Standard* BV theory

In this section we introduce the BV version of Palatini–Cartan theory. This description firstly appeared in [CS19a]. Using the notation of Section 2, this theory will play the role of $\mathcal{F}$ in the BV pushforward.

Let M be an N-dimensional manifold with $N > 2$.

Definition 4. The *standard* BV theory for Palatini–Cartan gravity is the quadruple

$$\mathfrak{F}_s(M) = (\mathcal{F}_s(M), S_s(M), \varpi_s(M), Q_s(M)),$$

where the quantities appearing above are defined as follows:

[3]This map depends on the choice of μ, S, and $\mathcal{L}$, which we do not write down explicitly in order to keep the notation light.

[4]Note that $\int_{\mathcal{L}_1\times\mathcal{L}} \mu^{\frac{1}{2}} e^{\frac{i}{\hbar}S} = \int_{\mathcal{L}_1} \mu_1^{\frac{1}{2}} e^{\frac{i}{\hbar}S_1}$, so the normalization factors are the same on both sides.

1. $\mathcal{F}_s(M)$ is the space of fields and it is defined as

$$\mathcal{F}_s(M) \coloneqq T^*[-1]\left(\Omega^1_{nd}(M,\mathcal{V}) \oplus \mathcal{A}(M) \oplus \mathfrak{X}[1](M) \oplus \Omega^0[1](M, \mathrm{ad}P)\right),$$

where the fields in the base are denoted by $(\mathbf{e}, \boldsymbol{\omega}, \boldsymbol{\xi}, \mathbf{c})$, while the corresponding variables in the cotangent fibre are denoted by $(\mathbf{e}^\dagger, \boldsymbol{\omega}^\dagger, \boldsymbol{\xi}^\dagger, \mathbf{c}^\dagger)$.

2. $\varpi_s(M)$ is a symplectic form on $\mathcal{F}_s(M)$ and its explicit expression is

$$\varpi_s(M) = \int_M \delta\mathbf{e}\delta\mathbf{e}^\dagger + \delta\boldsymbol{\omega}\delta\boldsymbol{\omega}^\dagger + \delta\mathbf{c}\delta\mathbf{c}^\dagger + \iota_{\delta\boldsymbol{\xi}}\delta\boldsymbol{\xi}^\dagger.$$

3. $S_s(M) \in C^\infty(\mathcal{F}_s(M))$ is the BV action functional and reads

$$\begin{aligned} S_s(M) = &\int_M \frac{1}{(N-2)!}\mathbf{e}^{N-2}F_{\boldsymbol{\omega}} + (\iota_{\boldsymbol{\xi}}F_{\boldsymbol{\omega}} - d_{\boldsymbol{\omega}}\mathbf{c})\,\boldsymbol{\omega}^\dagger - \left(L^{\boldsymbol{\omega}}_{\boldsymbol{\xi}}\mathbf{e} - [\mathbf{c},\mathbf{e}]\right)\mathbf{e}^\dagger \\ &+ \int_M \frac{1}{2}\left(\iota_{\boldsymbol{\xi}}\iota_{\boldsymbol{\xi}}F_{\boldsymbol{\omega}} - [\mathbf{c},\mathbf{c}]\right)\mathbf{c}^\dagger + \frac{1}{2}\iota_{[\boldsymbol{\xi},\boldsymbol{\xi}]}\boldsymbol{\xi}^\dagger, \end{aligned} \tag{1}$$

where $L^{\boldsymbol{\omega}}_{\boldsymbol{\xi}}\mathbf{e} = \iota_{\boldsymbol{\xi}}d_{\boldsymbol{\omega}}\mathbf{e} - d_{\boldsymbol{\omega}}\iota_{\boldsymbol{\xi}}\mathbf{e}$.

4. The cohomological vector field Q_s is defined by the equation $\iota_{Q_s}\varpi_s = \delta S_s$ and its action on components is

$$\begin{aligned} Q_s\mathbf{e} &= L^{\boldsymbol{\omega}}_{\boldsymbol{\xi}}\mathbf{e} - [\mathbf{c},\mathbf{e}] & Q_s\mathbf{e}^\dagger &= \mathbf{e}^{N-3}F_{\boldsymbol{\omega}} + L^{\boldsymbol{\omega}}_{\boldsymbol{\xi}}\mathbf{e}^\dagger - [\mathbf{c},\mathbf{e}^\dagger] \\ Q_s\boldsymbol{\omega} &= \iota_{\boldsymbol{\xi}}F_{\boldsymbol{\omega}} - d_{\boldsymbol{\omega}}\mathbf{c} & Q_s\boldsymbol{\omega}^\dagger &= \mathbf{e}^{N-3}d_{\boldsymbol{\omega}}\mathbf{e} - d_{\boldsymbol{\omega}}\iota_{\boldsymbol{\xi}}\boldsymbol{\omega}^\dagger - [\mathbf{c},\boldsymbol{\omega}^\dagger] + \iota_{\boldsymbol{\xi}}[\mathbf{e},\mathbf{e}^\dagger] - \frac{1}{2}d_{\boldsymbol{\omega}}\iota_{\boldsymbol{\xi}}\iota_{\boldsymbol{\xi}}\mathbf{c}^\dagger \\ Q_s\mathbf{c} &= \frac{1}{2}\iota_{\boldsymbol{\xi}}\iota_{\boldsymbol{\xi}}F_{\boldsymbol{\omega}} - \frac{1}{2}[\mathbf{c},\mathbf{c}] & Q_s\mathbf{c}^\dagger &= -d_{\boldsymbol{\omega}}\boldsymbol{\omega}^\dagger - [\mathbf{e},\mathbf{e}^\dagger] - [\mathbf{c},\mathbf{c}^\dagger] \\ Q_s\boldsymbol{\xi} &= \frac{1}{2}[\boldsymbol{\xi},\boldsymbol{\xi}] & Q_s\boldsymbol{\xi}^\dagger_\bullet &= F_{\boldsymbol{\omega}\bullet}\boldsymbol{\omega}^\dagger - (d_{\boldsymbol{\omega}\bullet}\mathbf{e})\mathbf{e}^\dagger + \iota_{\boldsymbol{\xi}}F_{\boldsymbol{\omega}\bullet}\mathbf{c}^\dagger + L^{\boldsymbol{\omega}}_{\boldsymbol{\xi}}\boldsymbol{\xi}^\dagger_\bullet + (d_{\boldsymbol{\omega}}\iota_{\boldsymbol{\xi}}\boldsymbol{\xi}^\dagger)_\bullet. \end{aligned}$$

Here we used the symbol $\bullet$ to remind the reader that $\boldsymbol{\xi}^\dagger$ is a one-form with values in densities, and on the right hand side we highlight the one-form part of the expression.

Suppose now that $M = \Sigma \times I$ where Σ is an $(N-1)$-dimensional closed manifold and I is a 1-dimensional manifold. The product structure of M induces a splitting on forms and on vector fields on M: let $\boldsymbol{\phi} \in \Omega^k(\Sigma \times I)$, then we get

$$\boldsymbol{\phi} = \phi + \underline{\phi_n}$$

where $\phi \in \Omega^k(\Sigma)\widehat{\otimes}C^\infty(I)$ and $\underline{\phi_n} \in \Omega^{k-1}(\Sigma)\widehat{\otimes}\Omega^1(I)$. Note that if $k = n$ the first addend does not exists, while for $k = 0$ only the first survives. On the other hand, for a vector field $v \in \mathfrak{X}(M)$ we obtain $\mathbf{v} = v + \overline{v_n}$ where $v \in \mathfrak{X}(\Sigma)\widehat{\otimes}C^\infty(I)$ and $\overline{v_n} \in \mathfrak{X}(I)\widehat{\otimes}C^\infty(\Sigma)$

Remark 5. Note that we underline forms that are one-forms along I. In particular, $\underline{\phi_n} = \phi_n dx^n$ where x^n is the coordinate along I. The parity of $\underline{\phi_n}$ is the same as $\boldsymbol{\phi}$, while ϕ_n has opposite parity.

The fields of the BV formulation of Palatini–Cartan theory then get split as

$$\begin{aligned} \mathbf{e} &= e + \underline{e_n} & \mathbf{e}^\dagger &= e^\dagger + \underline{e^\dagger_n} & \boldsymbol{\omega} &= \omega + \underline{\omega_n} & \boldsymbol{\omega}^\dagger &= \omega^\dagger + \underline{\omega^\dagger_n} \\ \mathbf{c} &= c & \mathbf{c}^\dagger &= \underline{c^\dagger_n} & \boldsymbol{\xi} &= \xi + \overline{\xi^n} & \boldsymbol{\xi}^\dagger &= \underline{\xi^\dagger} + \underline{\xi^\dagger{}_n} \end{aligned}$$

Hence, on a cylindrical manifold $\Sigma\times I$, we can write the symplectic form ϖ_s and S_s as follows:

$$\varpi_s = \int_{\Sigma\times I} \delta e\delta\underline{e_n^\dagger} + \delta\underline{e_n}\delta e^\dagger + \delta\omega\delta\underline{\omega_n}^\dagger + \delta\underline{\omega_n}\delta\omega^\dagger + \delta c\delta\underline{c_n^\dagger} + \iota_{\delta\xi}\delta\underline{\xi^\dagger} + \delta\xi^n\delta\underline{\xi_n^\dagger}. \tag{2}$$

$$\begin{aligned} S_s = \int_{\Sigma\times I} & \frac{1}{(N-3)!}e^{N-3}\underline{e}_n F_\omega + \frac{1}{(N-2)!}e^{N-2}\underline{F_{\omega_n}} - \left(\iota_\xi F_\omega + F_{\omega_n}\xi^n - d_\omega c\right)\underline{\omega_n^\dagger} \\ & - \left(\iota_\xi\underline{F_{\omega_n}} - \underline{d_{\omega_n}c}\right)\omega^\dagger + \left(L^\omega_\xi e + d_{\omega_n}e\xi^n + e_n d\xi^n - [c,e]\right)\underline{e}_n^\dagger \\ & + \left(\iota_\xi d_\omega\underline{e}_n + \iota_{\underline{\partial}_n\xi}e - \underline{d_{\omega_n}}(e_n\xi^n) - [c,\underline{e}_n]\right)e^\dagger \\ & - \left(\frac{1}{2}\iota_\xi\iota_\xi F_\omega + \iota_\xi F_{\omega_n}\xi^n - \frac{1}{2}[c,c]\right)\underline{c}_n^\dagger + \frac{1}{2}\iota_{[\xi,\xi]}\underline{\xi^\dagger} + \xi^n\iota_{\partial_n\xi}\underline{\xi^\dagger} + \iota_\xi d\xi^n\underline{\xi^\dagger}_n + \xi^n\partial_n\xi^n\underline{\xi^\dagger}_n, \end{aligned} \tag{3}$$

where we defined $\underline{F_{\omega_n}} = d_\omega\underline{\omega_n} + \underline{\partial_n}\omega$.

3.2 *Restricted* BV theory

Using the field e we can then define the following set of maps

$$\begin{aligned} W^{i,j}_{e^{N-3}}\colon \Omega^i\left(M,\textstyle\bigwedge^j\mathcal{V}\right) &\to \Omega^{i+N-3}\left(M,\textstyle\bigwedge^{j+N-3}\mathcal{V}\right) \\ Y &\mapsto e^{N-3}\wedge Y. \end{aligned}$$

The properties of these maps are collected in [Can24].

We now fix a nowhere vanishing section $\epsilon_n \in \Gamma(M,\mathcal{V})$ with the additional property that $d_I\epsilon_n = 0$. Once ϵ_n is fixed, we restrict the space of fields by the open condition that e and ϵ_n form a basis of V at every point. Since $\mathbf{e}$ is nondegenerate, e_n becomes a linear combination of e and ϵ_n, with nonzero ϵ_n-component. More generally we denote by $Y^{[\mu]}$ the components of a field Y with respect to the basis given by e and ϵ_n (i.e. $Y = Y^{[b]}e_b + Y^{[n]}\epsilon_n$). Let now α be a one-form on Σ, $\alpha = \sum_{a=1}^{N-1}\alpha_a dx^a$. For shortness of notation we define

$$\underline{z}^a := \underline{e}_n^{[a]} \qquad\qquad \underline{\lambda} := \underline{e}_n^{[n]}$$

so that

$$\underline{e}_n = \iota_{\underline{z}}e + \underline{\lambda}\epsilon_n$$

and extend this notation using the Leibnitz rule to any form on Σ. Note that if α is an odd quantity, then $\iota_{\underline{z}}\alpha$ is even and $\iota_z\alpha$ is odd and vice versa.

Using this notation we consider $\underline{\mathfrak{W}^\dagger} \in \Omega^{N-2}(\Sigma,\wedge^{N-2}\mathcal{V})\otimes\Omega^1(I)$ and $X\in\Omega^0(\Sigma\times I,\mathcal{V})$ defined as

$$\underline{\mathfrak{W}^\dagger} := \underline{\omega}_n^\dagger - \iota_{\underline{z}}\omega^\dagger - \iota_\xi\underline{c}_n^\dagger + \iota_{\underline{z}}c_n^\dagger\xi^n \tag{4}$$

$$X := [c,\epsilon_n] + L^\omega_\xi(\epsilon_n) - d_{\omega_n}\epsilon_n\xi^n \tag{5}$$

and define the following BV theory.

Definition 6. The *restricted* BV theory for Palatini–Cartan gravity is the quadruple

$$\mathfrak{F}_r = (\mathcal{F}_r, S_r, \varpi_r, Q_r)$$

where the quantities appearing above are defined as follows:

1. $\mathcal{F}_r$ is the space of fields and is defined as the subspace of $\mathcal{F}_s$ of fields satisfying the following equations (called *structural constraints*):

$$\mathfrak{W}^\dagger \in \mathrm{Im}(W^{1,1}_{e^{N-3}}) \tag{6}$$

$$\epsilon_n e^{(N-4)} d_\omega e - \epsilon_n e^{(N-4)} W^{-1}_{e^{N-3}}(\mathfrak{W}^\dagger) d\xi^n + X^{[a]}(\omega^\dagger_a - c^\dagger_{an}\xi^n) \in \mathrm{Im}(W^{1,1}_{e^{N-3}}), \tag{7}$$

   (ϵ_n, e) form a basis and the induced metric $g = \eta(e,e)$ is non-degenerate.

2. $\varpi_r = \varpi_s|_{\mathcal{F}_r}$;[5]

3. $S_r = S_s|_{\mathcal{F}_r}$;

4. $Q_r = Q_s$.

The importance of this theory was shown in [CCS21a] by proving that it is BV-isomorphic to the AKSZ theory of gravity.

In this note we want to show that the *restricted* BV theory described in Definition 6 can be obtained as a BV pushforward of the *standard* BV theory described in Definition 4. In particular we divide this in two steps. First we prove in Section 4 that the *standard* BV theory is symplectomorphic to a BV theory isomorphic to a product of BV spaces. We then show that its BV pushforward does indeed coincide with the *restricted* BV theory (Section 5).

4 From the standard BV PC theory to a BV product

The goal of this section is to show that, on cylindrical manifolds, the standard BV theory for Palatini–Cartan gravity $\mathfrak{F}_s$ introduced in Definition 4 is strongly BV equivalent (i.e. there exists a symplectomorphic compatible with the actions) to another BV theory $\mathfrak{F}_H$, in which the symplectic form can be written as the sum of the symplectic form of the restricted BV theory $\mathfrak{F}_r$ (see Definition 6) and a complement. Then, in Section 5.1 we will show that such theory $\mathfrak{F}_H$ is actually strongly BV equivalent to a BV product with basis given by the restricted BV theory $\mathfrak{F}_r$.

In order to construct the aforementioned symplectomorphism, we consider the following results:

Theorem 7. *[CS19b][Theorem 33] Suppose that the boundary metric g^∂ is nondegenerate. Given any $\omega \in \Omega(\Sigma, \wedge^2\mathcal{V})$, there is a unique decomposition*

$$\omega = \widehat{\omega} + v \tag{8}$$

with $\widehat{\omega}$ satisfying (7) and v such that $e^{N-3}v = 0$.

Lemma 8. *[CCS21b][Lemma 16] Let g^∂ be nondegenerate and let $X \in \Omega^{N-2,N-2}_\partial$. Then there exist a unique $Z \in \mathrm{Ker}W^{\partial,(1,2)}_{N-3}$ and a unique $Y \in \Omega^{1,1}_\partial$ such that*

$$\beta = e^{N-3}Y + \epsilon_n e^{N-4}[Z, e].$$

The decomposition (8) allows us to write ω in terms of a field $\widehat{\omega}$ satisfying the constraint (7), and hence being a field in $\mathcal{F}_r$, together with some residual field v.

[5] Note that the structural constraints above restrict the space, hence the symplectic form is only *written* in the same way.

Similarly Lemma 8 allows us to decompose $\underline{\mathfrak{W}^\dagger}$ as

$$\underline{\mathfrak{W}^\dagger} = e^{N-3}\underline{\tau}^\dagger + \epsilon_n e^{N-4}[\underline{\mu}^\dagger, e]$$

for some $\underline{\tau}^\dagger$ and $\underline{\mu}^\dagger$. Using the explicit expression of $\underline{\mathfrak{W}^\dagger}$ in (4), we can then define

$$\begin{aligned}\widehat{\underline{\omega}}_n^\dagger &:= e^{N-3}\underline{\tau}^\dagger + \iota_{\underline{z}}\omega^\dagger + \iota_\xi \underline{c}_n^\dagger - \iota_{\underline{z}} c_n^\dagger \xi^n \\ \underline{v}^\dagger &:= \epsilon_n e^{N-4}[\underline{\mu}^\dagger, e],\end{aligned}$$

and rewrite $\underline{\omega}_n^\dagger$ as

$$\underline{\omega}_n^\dagger = \widehat{\underline{\omega}}_n^\dagger + \underline{v}^\dagger. \tag{9}$$

Theorem 9. *On a cylindrical manifold $\Sigma \times I$, there exists a BV-symplectomorpshim*

$$\psi\colon (\mathcal{F}_s, \varpi_s, S_s) \to (\mathcal{F}_H, \varpi_H, S_H),$$

where $\mathcal{F}_s = \mathcal{F}_H$

$$\varpi_H = \varpi_r + \int_{\Sigma\times I} \delta v \delta \underline{v}^\dagger, \tag{10}$$

and

$$S_H = S_r + \int_{\Sigma\times I} \frac{1}{2(N-3)!}\underline{e_n} e^{N-3}[v,v] + \left(L_\xi^\omega v + [c,v] + \xi^n d_{\omega_n} v + d\xi^n \iota_z v\right)\underline{v}^\dagger.$$

In other words, the *standard* BV theory is BV symplectomorphic to the *restricted* one plus some additional parts proportional to v. We can then apply a BV pushforward to $(\mathcal{F}_H, \varpi_H, S_H)$, integrating v in order to obtain the *restricted* theory. This will be made explicit in Section 5.

Proof. We prove the claim by explicitely showing a symplectomorphism ψ.

Let $\underline{v}^\dagger$ as in (9), using Lemma 8 we can write $e^{N-4}\mu^\dagger d\xi^n = e^{N-3}\alpha + \epsilon_n e^{N-4}[\beta, e]$. Since $\epsilon_n^2 = 0$, $\epsilon_n e^{N-4}\mu^\dagger d\xi^n = \epsilon_n e^{N-3}\alpha = e^{N-3}\nu$ where we defined $\nu = \epsilon_n \alpha$.

Then the explicit expression of ψ reads:

$$\begin{aligned}
\widetilde{e} = e \qquad & \widetilde{\underline{e}}_n^\dagger = \underline{e}_n^\dagger + \frac{1}{N-3} e^{N-4}\left(d_{\widehat{\omega}}(\epsilon_n \underline{\mu}^\dagger) + \sigma \underline{\mu}^\dagger + \underline{\tau}^\dagger(\nu + v)\right) \\
& \qquad + X^{(a)}(Z_b \underline{\mu}^\dagger)_a^{(b)} + \iota_{\underline{z}}((\nu_a + v_a)Z)^{(a)} + \theta_N(5) e^{N-5}\epsilon_n \mu^\dagger d\xi^n \underline{\tau}^\dagger \\
\widetilde{\underline{e}}_n = \underline{e}_n \qquad & \widetilde{e}^\dagger = e^\dagger + ((\nu_a + v_a)Z)^{(a)} \\
\widehat{\widetilde{\omega}} = \widehat{\omega} + \nu \qquad & \widetilde{\underline{\omega}}_n^\dagger = \underline{\omega}_n^\dagger + [\epsilon_n, \iota_\xi (Z_a \underline{\mu}^\dagger)^{(a)}] \\
\widetilde{v} = v \qquad & \widetilde{\omega}^\dagger = \omega^\dagger + [\epsilon_n, (Z_a \mu^\dagger)^{(a)} \xi^n] \\
\widetilde{\underline{v}}^\dagger = \underline{v}^\dagger \qquad & \widetilde{\underline{\omega}}_n = \underline{\omega}_n + \iota_{\underline{z}}(\nu + v) + X^{(a)} \underline{\mu}_a^\dagger \\
\widetilde{\xi}^n = \xi^n \qquad & \widetilde{c} = c + \iota_\xi(\nu + v) + \iota_z(\nu + v)\xi^n + X^{(a)} \mu_a^\dagger \xi^n \\
\widetilde{\xi}^a = \xi^a \qquad & \widetilde{\underline{c}}_n^\dagger = \underline{c}_n^\dagger + [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}] \\
& \widetilde{\underline{\xi}}^\dagger = \underline{\xi}^\dagger + \underline{c}_n^\dagger(\nu + v) + d_\omega \epsilon_n (Z_a \underline{\mu}^\dagger)^{(a)} + \nu[\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}] \\
& \widetilde{\underline{\xi}}_n^\dagger = \underline{\xi}_n^\dagger + X^{(a)} \underline{c}_{an}^\dagger \mu^\dagger + d(e^{N-4}\epsilon \underline{\tau}^\dagger \mu^\dagger) + \iota_z \underline{c}_n^\dagger(\nu + v) + \iota_z(\nu + v)[\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}] \\
& \qquad + d_{\omega_n} \epsilon_n (Z_a \underline{\mu}^\dagger)^{(a)} + X^{(b)} \mu_b^\dagger [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}]
\end{aligned}$$

where $\widetilde{Z} = Z = \omega^\dagger + c_n^\dagger \xi^n$ and $X = [c + \iota_\xi v, \epsilon_n] + L^\omega_\xi(\epsilon_n) - d_{\omega_n}\epsilon_n \xi^n$ and $\theta_N(5) = 0$ for $N = 4$, $\theta_N(5) = \frac{1}{(N-4)!}$ for $N > 4$. The proof that $\psi^*(\varpi_H) = \varpi_s$ and $\psi^*(S_H) = S_s$ are long computations and are postponed to Appendix A. Let us now just recall the key points (for $N = 4$).

Using the decompositions (8), $\omega = \widehat{\omega} + v$ and (9), $\underline{\omega}_n^\dagger = \underline{\widehat{\omega}}_n^\dagger + \underline{v}^\dagger$ in the standard BV gravity symplectic form (2), we get

$$\varpi_s = \varpi_r + \int_{\Sigma\times I} \delta\widehat{\omega}\delta\underline{v}^\dagger + \delta v\delta\underline{\widehat{\omega}}_n^\dagger + \delta v\delta\underline{v}^\dagger. \tag{11}$$

The difference with ϖ_H (10) is then given by the terms $\delta\widehat{\omega}\delta\underline{v}^\dagger + \delta v\delta\underline{\widehat{\omega}}_n^\dagger$. We can write the first as

$$\delta\widehat{\omega}\delta(\epsilon_n[e, \underline{\mu}^\dagger]) = \delta(\epsilon_n[\widehat{\omega}, e])\delta\underline{\mu}^\dagger + \cdots$$

and then use (6) getting

$$\delta(\epsilon_n[\widehat{\omega}, e])\delta\underline{\mu}^\dagger = \delta(e\sigma)\delta\underline{\mu}^\dagger + \cdots$$

Similarly, for $\delta v\delta\underline{\widehat{\omega}}_n^\dagger$ we can use (7) and get

$$\delta v\delta\underline{\widehat{\omega}}_n^\dagger = \delta v\delta(e\underline{\tau}^\dagger) + \cdots = \delta e\delta(v\tau^\dagger) + \cdots .$$

Then these two terms are compensated by the corresponding term arising from the transformation of $\underline{e}_n^\dagger$. The same constraints are used in the same way in the computation for the actions. □

5 BV pushforward

5.1 The product structure

Proposition 10. *Let e_0 be a reference tetrad and define $\mathcal{F}_f = T^*[1](V)$ where $V = KerW_{e_0}$. Then $\mathcal{F}_H \to \mathcal{F}_r$ is a BV bundle as in Section 2.1 with typical fiber $\mathcal{F}_f$.*

Proof. We define π as the natural projection $\mathcal{F}_H \to \mathcal{F}_r$. We have to find a symplectomorphism $\pi^{-1}(\mathcal{F}_r) \to \mathcal{F}_r \times \mathcal{F}_f$. For each $x \in M$, there exists a unique transformation Λ_x such that $\Lambda_x e = e_0$. Correspondingly, we have $KerW^{(2,1)}_{e_0} = \Lambda_x^2 KerW^{(2,1)}_e$. We then apply Λ_x^2 to v_x and extend this to $v^\dagger$ as a cotangent lift. □

5.2 Nondegeneracy of the Gaussian integrand

We have now proven that the standard formulation of BV gravity is symplectomorphic to a BV product. We can then perform a Gaussian integration on the fiber in order to obtain the restricted theory. This can be done thanks to the following lemma:

Lemma 11. *The quadratic form in v*

$$\int_{\Sigma\times I} \frac{1}{2(N-3)!}\underline{e_n} e^{N-3}[v, v]$$

is nondegenerate.

In order to prove this lemma, we fix a specific set of coordinates in which $e^a_\mu = \delta^a_\mu$ and $(e_n)^a = \delta^a_N$. Using this basis the components of v have been characterized as follows:

Lemma 12. *[CCS21b] Let $v \in \Omega(\Sigma, \wedge^2\mathcal{V})$ such that $ev = 0$. Then in the basis for which $e^a_\mu = \delta^a_\mu$ and $(e_n)^a = \delta^a_N$, the components of v satisfy the following set of equations:*

$$\begin{aligned} v_i^{Nj} &= 0 \qquad 1 \le i, j \le N-1 \; i \neq j \\ \sum_{i\neq N, i\neq j} v_i^{iN} &= 0 \qquad 1 \le j \le N-1 \\ \sum_{i\neq N, i\neq j} v_i^{ij} &= 0 \qquad 1 \le j \le N-1. \end{aligned}$$

Proof of Lemma 11. We start from $N = 4$ (see also [CC25]). In components, we have the following expression of the quadratic form ($1 \le a, b, c, d, e, f \le 4$, $1 \le \mu, \nu, \rho \le 3$):

$$\begin{aligned} (e_n)^a e^b_\mu [v, v]^{cd}_{\nu\rho} \epsilon^{\mu\nu\rho} \epsilon_{abcd} &= (e_n)^a e^b_\mu v^{ce}_\nu v^{df}_\rho \eta_{ef} \epsilon^{\mu\nu\rho} \epsilon_{abcd} \\ &= \delta^a_4 \delta^b_\mu v^{ce}_\nu v^{df}_\rho \eta_{ef} \epsilon^{\mu\nu\rho} \epsilon_{abcd} \\ &= v^{ce}_\nu v^{df}_\rho \eta_{ef} \epsilon^{b\nu\rho} \epsilon_{bcd}, \end{aligned}$$

where now we also have $1 \le b, c, d \le 3$. Using now the identity

$$\epsilon^{b\nu\rho} \epsilon_{bcd} = \delta^\nu_c \delta^\rho_d - \delta^\nu_d \delta^\rho_c,$$

we get

$$\begin{aligned} (e_n)^a e^b_\mu [v, v]^{cd}_{\nu\rho} \epsilon^{\mu\nu\rho} \epsilon_{abcd} &= v^{ce}_\nu v^{df}_\rho \eta_{ef} \left(\delta^\nu_c \delta^\rho_d - \delta^\nu_d \delta^\rho_c\right) \\ &= v^{ce}_c v^{df}_d \eta_{ef} - v^{ce}_d v^{df}_c \eta_{ef}. \end{aligned}$$

From Lemma 12 we deduce that the free components of v are parametrized by

$$v_1^{23} \qquad v_2^{13} \qquad v_3^{12} \qquad v_1^{12} \qquad v_1^{13} \qquad v_2^{21} \tag{12}$$

and we have $v_3^{32} = -v_1^{12}$, $v_2^{23} = -v_1^{13}$ and $v_3^{31} = -v_2^{21}$, while all the other components are zero.

Then we have (note that now also $1 \le e, f \le 3$)

$$\begin{aligned} (e_n)^a e^b_\mu [v, v]^{cd}_{\nu\rho} \epsilon^{\mu\nu\rho} \epsilon_{abcd} &= v^{ce}_c v^{df}_d \eta_{ef} - v^{ce}_d v^{df}_c \eta_{ef} \\ &= v^{c1}_c v^{d1}_d + v^{c2}_c v^{d2}_d + v^{c3}_c v^{d3}_d - v^{c1}_d v^{d1}_c - v^{c2}_d v^{d2}_c - v^{c3}_d v^{d3}_c \\ &= v^{21}_2 v^{21}_2 + 2 v^{21}_2 v^{31}_3 + v^{31}_3 v^{31}_3 + v^{12}_1 v^{12}_1 + 2 v^{12}_1 v^{32}_3 + v^{32}_3 v^{32}_3 \\ &\quad + v^{13}_1 v^{13}_1 + 2 v^{23}_2 v^{13}_1 + v^{23}_2 v^{23}_2 - v^{21}_2 v^{21}_2 - 2 v^{31}_2 v^{21}_3 - v^{31}_3 v^{31}_3 \\ &\quad - v^{12}_1 v^{12}_1 - 2 v^{32}_1 v^{12}_3 - v^{32}_3 v^{32}_3 - v^{13}_1 v^{13}_1 - 2 v^{23}_1 v^{13}_2 - v^{23}_2 v^{23}_2 \\ &= 2 \left(v^{21}_2 v^{31}_3 + v^{12}_1 v^{32}_3 + v^{23}_2 v^{13}_1 - v^{31}_2 v^{21}_3 - v^{32}_1 v^{12}_3 - v^{23}_1 v^{13}_2 \right) \\ &= 2 \left(-(v^{21}_2)^2 - (v^{12}_1)^2 - (v^{13}_1)^2 - v^{13}_2 v^{12}_3 + v^{23}_1 v^{12}_3 - v^{23}_1 v^{13}_2 \right). \end{aligned}$$

Then, expressing the quadratic form as a matrix with respect to the basis (12), we get

$$A = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Then we can compute the determinant:

$$\det A = -\det \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = -\det \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} - \det \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} = -2 \neq 0.$$

Let now $N > 4$. In components, we have the following expression of the quadratic form ($1 \leq a_1, \dots a_N, b, c \leq N$, $1 \leq \mu_1, \dots \mu_{N-1} \leq N-1$):

$$\begin{aligned}(e_n)^{a_1} e^{a_2}_{\mu_1} \dots e^{a_{N-2}}_{\mu_{N-3}} [v,v]^{a_{N-1}a_N}_{\mu_{N-2}\mu_{N-1}} \epsilon^{\mu_1 \dots \mu_{N-1}} \epsilon_{a_1 \dots a_N} \\ &= (e_n)^{a_1} e^{a_2}_{\mu_1} \dots e^{a_{N-2}}_{\mu_{N-3}} v^{a_{N-1}b}_{\mu_{N-2}} v^{a_N c}_{\mu_{N-1}} \eta_{bc} \epsilon^{\mu_1 \dots \mu_{N-1}} \epsilon_{a_1 \dots a_N} \\ &= \delta^{a_1}_N \delta^{a_2}_{\mu_1} \dots \delta^{a_{N-2}}_{\mu_{N-3}} v^{a_{N-1}b}_{\mu_{N-2}} v^{a_N c}_{\mu_{N-1}} \eta_{bc} \epsilon^{\mu_1 \dots \mu_{N-1}} \epsilon_{a_1 \dots a_N} \\ &= v^{a_{N-1}b}_{\mu_{N-2}} v^{a_N c}_{\mu_{N-1}} \eta_{bc} \epsilon^{a_2 \dots a_{N-2} \mu_{N-2} \mu_{N-1}} \epsilon_{a_2 \dots a_{N-2} a_{N-1} a_N}\end{aligned}$$

where now we also have $1 \leq a_2 \dots a_N \leq N-1$. Using the identity

$$\epsilon^{a_2 \dots a_{N-2} \mu_{N-2} \mu_{N-1}} \epsilon_{a_2 \dots a_{N-2} a_{N-1} a_N} = \frac{1}{N-2} \left(\delta^{\mu_{N-2}}_{a_{N-1}} \delta^{\mu_{N-1}}_{a_N} - \delta^{\mu_{N-2}}_{a_N} \delta^{\mu_{N-1}}_{a_{N-1}} \right),$$

and denoting by $\nu = \mu_{N-2}$, $\rho = \mu_{N-1}$, $c = a_{N-1}$, $d = a_N$, modulo a factor, we get

$$\begin{aligned}(e_n)^{a_1} e^{a_2}_{\mu_1} \dots e^{a_{N-2}}_{\mu_{N-3}} [v,v]^{a_{N-1}a_N}_{\mu_{N-2}\mu_{N-1}} \epsilon^{\mu_1 \dots \mu_{N-1}} \epsilon_{a_1 \dots a_N} \\ &= v^{ce}_{\nu} v^{df}_{\rho} \eta_{ef} \left(\delta^{\nu}_c \delta^{\rho}_d - \delta^{\nu}_d \delta^{\rho}_c \right) \\ &= v^{ce}_c v^{df}_d \eta_{ef} - v^{ce}_d v^{df}_c \eta_{ef} \\ &= v^{ce}_c v^{de}_d - v^{ce}_d v^{de}_c\end{aligned}$$

where in the last line $1 \leq c, d, e \leq N-1$, they are all different and summation over c, d, e is understood. From Lemma 12 we deduce that the free components of v are parametrized by

$$v^{de}_c \qquad\qquad v^{ce}_c \tag{13}$$

where $\sum_{c \neq N, c \neq e} v^{ce}_c = 0$. Let us begin with the second term of the sum $-v^{ce}_d v^{de}_c$. If we select three different indexes c, d, e, the corresponding components v^{de}_c, v^{ce}_d, and v^{cd}_e appear in the terms

$$-v^{de}_c v^{ce}_d - v^{ed}_c v^{cd}_e - v^{ec}_d v^{dc}_e = -v^{de}_c v^{ce}_d + v^{de}_c v^{cd}_e - v^{ce}_d v^{cd}_e.$$

The matrix associated to this part of the quadratic form reads

$$A = \frac{1}{2} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix},$$

and its determinant is 1. Counting all possible triples c, d, e we get $\binom{3}{N-1}$ blocks of this type.

Let us now consider the terms $v^{ce}_c v^{de}_d$ and let us fix e. Let now $m = \max(1, \dots, N-1)$ with $m \neq e$. Then from the equation $\sum_{c \neq N, c \neq e} v^{ce}_c = 0$ we get

$$v^{me}_m = - \sum_{c \neq N, e, m} v^{ce}_c,$$

where all the other v_c^{ce} are now independent. Hence we get

$$\sum_{\substack{d,c\neq e\\ d\neq c}} v_c^{ce}v_d^{de} = \sum_{\substack{d,c\neq e,m\\ d\neq c}} v_c^{ce}v_d^{de} + 2\sum_{c\neq e,m} v_c^{ce}v_m^{me}$$
$$= \sum_{\substack{d,c\neq e,m\\ d\neq c}} v_c^{ce}v_d^{de} - 2\sum_{c,d\neq e,m} v_c^{ce}v_d^{de}.$$

Therefore the corresponding $(N-3)\times(N-3)$-matrix is

$$B = -\frac{1}{2}\begin{pmatrix} 4 & 1 & \dots & 1\\ 1 & 4 & \dots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \dots & 4 \end{pmatrix}.$$

We can compute its determinant as the sum of an invertible matrix (3Id) and the outer product of a vector $w = (1, \cdots, 1)$ with its transposed:

$$B = -\frac{1}{2}\left(3\mathrm{Id} + ww^T\right).$$

Hence

$$\det B = \left(-\frac{1}{2}\right)^{N-3}\left(1+\frac{1}{3}w^T\mathrm{Id}w\right)3^{N-3}\det\mathrm{Id} = \left(-\frac{3}{2}\right)^{N-3}\left(1+\frac{N-3}{3}\right) = \left(-\frac{3}{2}\right)^{N-3}\frac{N}{3}.$$

Note that in the matrix representing the quadratic form there are $(N-1)$ such blocks. Hence, since the determinant is always different from zero, we get the desired claim. □

Collecting all the information, we get the following theorem.

Theorem 13. *Let*

$$\widetilde{\mathcal{P}_V} : Dens^{\frac{1}{2}}\mathcal{F}_s \to Dens^{\frac{1}{2}}\mathcal{F}_r$$

be defined by

$$Dens^{\frac{1}{2}}\mathcal{F}_s \xrightarrow{\psi^*} Dens^{\frac{1}{2}}\mathcal{F}_H \xrightarrow{\mathcal{P}_V} Dens^{\frac{1}{2}}\mathcal{F}_r$$

where ψ is defined respectively in Section 4 and $\mathcal{P}_V$ is the BV pushforward with respect to the Lagrangian submanifold $V \coloneqq \{\underline{v}^\dagger = 0\}$ inside $T^*[1]V$. Then we have that

$$\mu_r^{\frac{1}{2}} e^{\frac{i}{\hbar}S_r} = \widetilde{\mathcal{P}_V}\left(\mu_s^{\frac{1}{2}} e^{\frac{i}{\hbar}S_s}\right)$$

where $\mu_r^{\frac{1}{2}}$ and $\mu_s^{\frac{1}{2}}$ are suitable half-densities on $\mathcal{F}_r$ and $\mathcal{F}_s$ respectively.

This theorem shows that the restricted BV theory can be obtained as the BV pushforward of the standard BV theory.

Remark 14. In this paper we have asssumed the cosmological constant $\Lambda = 0$. The same construction holds for a non vanishing Λ, just by adding a term

$$\int_{\Sigma\times I}\frac{1}{(N-1)!}\underline{e}_n e^{N-1}\Lambda$$

to S_r.

6 Dealing with the boundary

In the previous sections, we have ignored the fact that the interval I has boundary points. Two simple ways out are either to work on $\Sigma \times S^1$ or to take I as an open interval (we can even take $I = \mathbb{R}$) and to assume fast decay of the fields towards the boundary of I. In any of these two cases, the results of the paper hold exactly as stated so far.

It is however interesting to consider also the case when I is a compact interval (we also assume that Σ is a closed manifold). In this case we have to deal with the boundary structure and to impose appropriate boundary conditions when performing the BV pushforward.

6.1 The pre–BV-BFV formalism

If a BV theory, for a closed space–time manifold, is put on a compact manifold with boundary, the master equation may be spoiled. Actually, what typically does not hold anymore is the condition that the cohomological vector field Q is the hamiltonian vector field of the BV action S: in fact, the original equation $\iota_Q \omega = \delta S$, true for a closed space–time manifold, may now be spoiled by boundary terms arising by integration by parts in the computation of the functional derivative δS. We recollect here the details of the construction to circumvent this problem (this is an adapted reformulation of the construction of [CMR11; CMR14; CMR18]).

Definition 15 (Relaxed BV manifold)**.** A relaxed BV manifold is a quadruple $(\mathcal{F}, S, \varpi, Q)$, where $\mathcal{F}$ is a graded manifold and S, ϖ, Q are respectively a function, a closed 2-form, and a vector field on $\mathcal{F}$ satisfying the following:

- Q is cohomological, i.e., it is odd of degree 1 and $[Q, Q] = 0$;
- ϖ is odd of degree -1 and S is even of degree 0.[6]

Given this, we define

$$\underline{\alpha} \coloneqq \iota_Q \varpi - \delta S,$$
$$\underline{\varpi} \coloneqq \delta \underline{\alpha} = -L_Q \varpi.$$

Immediate consequences of Cartan's calculus, degree considerations, and the condition $[Q, Q] = 0$ are the following:

1. $\underline{\alpha}$ and $\underline{\varpi}$ are even of degree 0.
2. There exists a unique odd function $\underline{S}$ of degree 1 satisfying $\iota_Q \underline{\varpi} = \delta \underline{S}$.

This shows that $(\mathcal{F}, \underline{S}, \underline{\alpha}, Q)$ is an exact preBFV manifold according to the following

Definition 16 (BFV)**.** A preBFV manifold is a quadruple $(\widetilde{\mathcal{F}}, \widetilde{S}, \widetilde{\varpi}, \widetilde{Q})$, where $\widetilde{\mathcal{F}}$ is a graded manifold and $\widetilde{S}, \widetilde{\varpi}, \widetilde{Q}$ are respectively a function, a closed 2-form, and a vector field on $\mathcal{F}$ satisfying the following:

- $\widetilde{Q}$ is cohomological, i.e., it is odd of degree 1 and $[\widetilde{Q}, \widetilde{Q}] = 0$;
- $\widetilde{\varpi}$ is even of degree 0 and $\widetilde{S}$ is odd of degree 1;
- $\widetilde{Q}$ is a hamiltonian vector field of $\widetilde{S}$, i.e., $\iota_{\widetilde{Q}} \widetilde{\varpi} = \delta \widetilde{S}$.

[6] A generalization of this, called relaxed BF^{k}V manifold, replaces this condition by saying that S has degree k, ω has degree $k-1$, and they both have parity equal to their degree modulo 2.

The preBFV manifold is called a BFV manifold if $\widetilde{\varpi}$ is nondegenerate. If $\widetilde{\varpi}$ is exact with a specified potential $\widetilde{\alpha}$, i.e., $\widetilde{\varpi} = \delta\widetilde{\alpha}$, then we say that $(\widetilde{\mathcal{F}}, \widetilde{S}, \widetilde{\alpha}, \widetilde{Q})$ is an exact (pre)BFV manifold.

We also have a relation between the original relaxed BV manifold $(\mathcal{F}, S, \varpi, Q)$ and the resulting exact preBFV manifold $(\mathcal{F}, \underline{S}, \underline{\alpha}, Q)$ (namely, the defining equation $\underline{\alpha} = \iota_Q\varpi - \delta S$), which can be viewed as a special case of the following

Definition 17 (BV-BFV)**.** A preBV-BFV structure is a triple $((\mathcal{F}, S, \varpi, Q), (\widetilde{\mathcal{F}}, \widetilde{S}, \widetilde{\alpha}, \widetilde{Q}), \pi)$, where $(\mathcal{F}, S, \varpi, Q)$ is a relaxed BV manifold, $(\widetilde{\mathcal{F}}, \widetilde{S}, \widetilde{\alpha}, \widetilde{Q})$ is an exact preBFV manifold, and $\pi\colon \mathcal{F} \to \widetilde{\mathcal{F}}$ is a surjective submersion, such that

- Q and $\widetilde{Q}$ are π-related, i.e., for every function f on $\widetilde{\mathcal{F}}$, we have $Q\pi^*f = \pi^*\widetilde{Q}f$;
- $\iota_Q\varpi = \delta S + \pi^*\widetilde{\alpha}$.

It is then easy to compute how the "master equation" is violated in a general preBV-BFV structure:

Theorem 18. *Let $((\mathcal{F}, S, \varpi, Q), (\widetilde{\mathcal{F}}, \widetilde{S}, \widetilde{\alpha}, \widetilde{Q}), \pi)$ be a preBV-BFV structure. Then*

$$\frac{1}{2}\iota_Q\iota_Q\varpi = \pi^*\widetilde{S} \qquad \text{and} \qquad L_Q S = \pi^*(2\widetilde{S} - \iota_{\widetilde{Q}}\widetilde{\alpha}).$$

We can summarize what we have so far by saying that $((\mathcal{F}, S, \varpi, Q), (\mathcal{F}, \underline{S}, \underline{\alpha}, Q), \mathrm{id})$ is a preBV-BFV structure.

The reason for considering other (pre)BV-BFV structures is that the 2-form $\underline{\varpi} = \delta\underline{\alpha}$ is typically degenerate and we might want to remove part or all of the degeneracy by performing a quotient.

Let $\mathcal{Y}$ be a simple, integrable distribution on $\mathcal{F}$ such that for every Y in $\mathcal{Y}$ we have $\iota_Y\underline{\varpi} = 0$. Note that Q is automatically projectable, i.e., $[Q, Y] \in \mathcal{Y}$ for every Y in $\mathcal{Y}$. Let $\widetilde{\mathcal{F}}$ denote the leaf space of the distribution $\mathcal{Y}$ and $\pi\colon \mathcal{F} \to \widetilde{\mathcal{F}}$ the canonical projection. There are then uniquely determined $\widetilde{\varpi}$ and $\widetilde{Q}$ such that $\underline{\varpi} = \pi^*\widetilde{\varpi}$ and $Q\pi^*f = \pi^*\widetilde{Q}f$ for every function f on $\widetilde{\mathcal{F}}$. There is also a uniquely determined function $\widetilde{S}$ satisfying $\iota_{\widetilde{Q}}\widetilde{\varpi} = \delta\widetilde{S}$ and $\underline{S} = \pi^*\widetilde{S}$. If there is a (necessarily unique) 1-form $\widetilde{\alpha}$ on $\widetilde{\mathcal{F}}$ satisfying $\underline{\alpha} = \pi^*\widetilde{\alpha}$, then $((\mathcal{F}, S, \varpi, Q), (\widetilde{\mathcal{F}}, \widetilde{S}, \widetilde{\alpha}, \widetilde{Q}), \pi)$ is a preBV-BFV structure.[7]

In particular, let $\mathcal{Y}$ be the kernel of $\underline{\varpi}$, i.e., the module of all vector fields Y such that $\iota_Y\underline{\varpi} = 0$. This distribution is automatically involutive but not necessarily regular or simple. If it is, and $\underline{\alpha}$ is basic, we can perform the reduction as above. In this case, the resulting 2-form $\widetilde{\varpi}$ is nondegenerate, so symplectic. The notation $((\mathcal{F}, S, \varpi, Q), (\mathcal{F}^\partial, S^\partial, \alpha^\partial, Q^\partial), \pi^\partial)$ is reserved for the resulting BV-BFV structure.

6.2 Relaxed BV pushforward

Let $(\mathcal{F}, S, \varpi, Q)$ be a relaxed BV manifold. As a warm up, we are interested in the BV integral $Z_{\mathcal{L}} = \int_{\mathcal{L}} \left(\mu^{\frac{1}{2}} e^{\frac{i}{\hbar}S}\right)$, where μ is a compatible Berezinian for the graded symplectic manifold $(\mathcal{F}, \varpi)$ and $\mathcal{L}$ is a Lagrangian submanifold. At this point, $Z_{\mathcal{L}}$ might not be invariant under deformations of $\mathcal{L}$ because S possibly does not satisfy the quantum master equation. We assume for simplicity

[7] There is a more general version of a preBV-BFV structure, which we are not going to use in this paper, where $\widetilde{\varpi}$ is only locally exact, i.e., it is the curvature of a connection 1-form $\widetilde{\alpha}$ for some line bundle over $\widetilde{\mathcal{F}}$. We can view $\underline{\alpha}$ as a connection 1-form for the trivial line bundle over $\mathcal{F}$. Requiring that $\underline{\alpha}$ descend to the quotient as a connection is a weaker condition than requiring that it descend as a globally defined 1-form.

that $\Delta_\mu S = 0$ (however Δ_μ is regularized in the infinite-dimensional case). Still, the classical master equation is in general violated, in one of the forms appearing in Theorem 18. To avoid the problem, we then have to make a choice so that $\underline{S}$ and $\underline{\alpha}$ disappear (note that we consider the preBV-BFV structure in which $\widetilde{S} = \underline{S}$ and $\widetilde{\alpha} = \underline{\alpha}$). This choice also has to be compatible with all the structure. We make the following

Definition 19. A good b-condition[8] is a submanifold $\mathcal{B} \overset{\iota}{\hookrightarrow} \mathcal{F}$ such that

1. $\mathcal{B}$ is a symplectic submanifold of $(\mathcal{F}, \varpi)$;
2. $\mathcal{B}$ is Q-invariant;
3. $\iota^* \underline{S} = 0$;
4. $\iota^* \underline{\alpha} = 0$.

These conditions imply that $(\mathcal{B}, \iota^* S, \iota^* \varpi, Q)$ is a BV manifold.[9] The last condition moreover implies that $\mathcal{B}$ is an isotropic submanifold of $(\mathcal{F}, \underline{\varpi})$. It is convenient to assume that it is maximally isotropic.[10] Now it makes sense to consider $Z_{\mathcal{L}} = \int_{\mathcal{L}} \left(\mu^{\frac{1}{2}} e^{\frac{i}{\hbar} S} \right)$ for μ a compatible Berezinian and $\mathcal{L}$ the intersection with $\mathcal{B}$ of the given Lagrangian submanifold of $\mathcal{F}$. The usual BV theorem now implies that $Z_{\mathcal{L}}$ is invariant under deformations of $\mathcal{L}$.

We now move to the BV pushforward. Assume, as in Section 2, that $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and $\varpi = \varpi_1 + \varpi_2$. We also assume, as in Corollary 3, that we have a compatible Berezinian for the graded symplectic manifold $(\mathcal{F}, \varpi)$ and a lagrangian submanifold $\mathcal{L}$ of $\mathcal{F}_2$. We are interested in the BV pushforward

$$\mu_1^{\frac{1}{2}} e^{\frac{i}{\hbar} S_1} = \mathcal{P}_{\mathcal{L}} \left(\mu^{\frac{1}{2}} e^{\frac{i}{\hbar} S} \right).$$

As before, the problem is that S possibly does not satisfy the quantum master equation and we assume that $\Delta_\mu S = 0$. Still, the classical master equation is in general violated. To avoid the problem, we then have to make a suitable choice, adapting the notion of good b-condition.

Definition 20. A good b-condition is a submanifold $\mathcal{B} \overset{\iota}{\hookrightarrow} \mathcal{F}$ such that

- $\mathcal{B}$ is a symplectic submanifold of $(\mathcal{F}, \varpi)$;
- $\mathcal{B}$ is Q-invariant;
- the restriction $\pi_{\mathcal{B}} \colon \mathcal{B} \to \mathcal{F}_2$ of $\pi \colon \mathcal{F} \to \mathcal{F}_2$ is such that $\iota_Y \varpi = 0$ for all $\pi_{\mathcal{B}}$-vertical vector fields Y;
- $\iota^* \underline{S} = \pi^* \underline{S}_1$ for a (necessarily uniquely determined) function $\underline{S}_1$ on $\mathcal{F}_1$;
- $\iota^* \underline{\alpha} = \pi^* \underline{\alpha}_1$ for a (necessarily uniquely determined) 1-form $\underline{\alpha}_1$ on $\mathcal{F}_1$.

We now work on $\mathcal{B}$. Note the we have

$$\Delta_\mu e^{\frac{i}{\hbar} S} = \frac{1}{2} \left(\frac{i}{\hbar} \right)^2 L_Q S e^{\frac{i}{\hbar} S} = (2\pi^* \underline{S}_1 - \iota_Q \pi^* \underline{\alpha}_1) \frac{1}{2} \left(\frac{i}{\hbar} \right)^2 e^{\frac{i}{\hbar} S}.$$

[8] b stands for boundary, for in the application to field theory this is really a boundary condition

[9] The first condition implies that $\iota^* \varpi$ is an odd symplectic form on $\mathcal{B}$. The second condition implies that Q is an odd vector field on $\mathcal{B}$. The third condition implies that Q on $\mathcal{B}$ is Hamiltonian with Hamiltonian function $\iota^* S$. The last condition now implies that the master equation is satisfied, so Q is a cohomological vector field on $\mathcal{B}$.

[10] For applications in field theory, one usually wants to perturb around a critical point of S. Maximality of $\mathcal{B}$ is usually required to ensure that the space of critical points is not empty.

The BV pushforward Ω_1 of $2\pi^*\underline{S}_1 - \iota_Q\pi^*\underline{\alpha}_1$ along $\mathcal{L}$ determines the violation of the master equation for S_1.

In field theory, $\underline{S}$ and $\underline{\alpha}$, and therefore $\underline{S}_1$, $\underline{\alpha}_1$, and Ω_1, only depend on boundary fields (and their transversal jets). As a consequence, the violation of the master equation for S_1 also has this property. Moreover, a deformation of $\mathcal{L}$ leads to a change of $e^{\frac{i}{\hbar}S_1}$ by a Δ_{μ_1}-exact term and by a term that only depends on boundary fields (and their transversal jets).

For simplicity, we focus on the case, occurring in the current paper, that the $\mathcal{F}_1$-component Q_1 of Q depends only on $\mathcal{F}_2$. In this case, $\Omega_1 = 2\underline{S}_1 - \iota_{Q_1}\underline{\alpha}_1$. We also assume that S_1 is the evaluation of S at the critical point. As a result, $((\mathcal{F}_1, S_1, \varpi_1, Q_1), (\mathcal{F}_1, \underline{S}_1, \underline{\alpha}_1, Q_1), \mathrm{id})$ is the preBV-BFV structure for the effective theory. Moreover, a deformation of $\mathcal{L}$ changes $\mu_1^{\frac{1}{2}} e^{\frac{i}{\hbar}S_1}$ by an $(\Omega_1 - i\hbar\Delta_{\mu_1})$-exact term.

The BV pushforward now has all the properties as in Section 2.

6.3 Back to PC gravity

In presence of a boundary, we get a preBV-BFV structure to which we can apply a relaxed BV pushforward.

Proposition 21. *Denoting by $(\mathcal{F}^\partial, \varpi^\partial, S^\partial)$ the BFV theory described in [CCS21b, Theorem 30], the PC gravity preBV-BFV structure is given by the following data (listed as in Definition 17):*

- *The bulk space of fields is $\mathcal{F} = \mathcal{F}_H \times \mathcal{F}_2$.*
- *The bulk symplectic form is*

$$\varpi = \varpi_r + \int_{\Sigma\times I} \delta v \delta \underline{v}^\dagger.$$

- *The bulk action is*

$$S = S_r + \int_{\Sigma\times I} \frac{1}{2(N-3)!}\underline{e_n} e^{N-3}[v,v] + \left(L_\xi^\omega v + [c,v] + \xi^n d_{\omega_n} v + d\xi^n \iota_z v\right)\underline{v}^\dagger.$$

- $\widetilde{\mathcal{F}} = \mathcal{F}^\partial \times \mathcal{F}_{add}$ *where*

$$\mathcal{F}_{add} = \Omega^1(\Sigma, \wedge^2\mathcal{V}) \oplus \Omega^{N-2}(\Sigma, \wedge^{N-2}\mathcal{V}).$$

- $\widetilde{\varpi} = \varpi^\partial + \varpi_{add}$ *where*

$$\varpi_{add} = \int_\Sigma \delta v \delta(v^\dagger \xi^n).$$

- $\widetilde{S} = S^\partial + S_{add}$ *where*

$$S^\partial_{add} = \int_\Sigma \frac{1}{2(N-3)!} e_n \xi^n e^{N-3}[v,v] + \left(L_\xi^\omega v + [c,v] + d\xi^n \iota_z v\right) v^\dagger \xi^n.$$

- *The projection π is given by $(\pi^\partial, \mathrm{id}) : \mathcal{F}_H \times \mathcal{F}_2 \to \mathcal{F}^\partial \times \mathcal{F}_{add}$ where π^∂ is the composition of the inverse of the symplectomorphism described in [CCS21a, equation (78)] and the projection to the boundary.*

Proof. The bulk data is that of Theorem 9. Using the results in [CCS21a] it is a long but easy computation to show that these data form a preBV-BFV structure. □

In order now to perform a relaxed BV pushforward, we then have to take a good b-condition. We define

$$\mathcal{B} \coloneqq \{\iota_\partial^* v = 0 \text{ and } \iota_\partial^* (d_{\omega_n})^k v = 0 \, \forall k \in \mathbb{N}^*\}.$$

Proposition 22. *$\mathcal{B}$ is a good b-condition.*

Proof. Noting that $\iota_\partial^* \underline{S}_1 = 0$ and $\iota_\partial^* \underline{\alpha}_1 = 0$, all the conditions in Definition 20 are easily verified, only the Q-invariance requires some additional care. From the expression of S_H, we deduce that

$$\begin{aligned} Qv &= L_\xi^\omega v + [c, v] + \xi^n d_{\omega_n} v + d\xi^n \iota_z v \\ Q\omega_n &= \iota_\xi F_{\omega_n} + d_{\omega_n} c \end{aligned}$$

Hence $Qv|_{\mathcal{B}} = 0$ and

$$Q\left(d_{\omega_n} v\right) = [\iota_\xi F_{\omega_n} + d_{\omega_n} c, v] + d_{\omega_n}\left(L_\xi^\omega v + [c, v] + \xi^n d_{\omega_n} v + d\xi^n \iota_z v\right).$$

Hence, using $\iota_\partial^* (d_{\omega_n})^k v = 0$, it is easy to see, that $\mathcal{B}$ is Q-invariant. □

With this choice of a good b-condition, we can proceed and perform a relaxed BV pushforward exactly as in the case without boundary, with the same results.

A Detailed proof of Theorem 9

A.1 Symplectic forms

We prove here that

$$\psi^*(\varpi_H) = \varpi_s. \tag{14}$$

We have:

$$\varpi_H = \int_{\Sigma \times I} \delta\widetilde{e}\delta\underline{\widetilde{e}}_n^\dagger + \delta\underline{\widetilde{e}}_n\delta\widetilde{e}^\dagger + \delta\widetilde{\widehat{\omega}}\delta\underline{\widetilde{\widehat{\omega}}}_n^\dagger + \delta\underline{\widetilde{\omega}}_n\delta\widetilde{\omega}^\dagger + \delta\widetilde{c}\delta\underline{\widetilde{c}}_n^\dagger + \iota_{\delta\widetilde{\xi}}\delta\underline{\widetilde{\xi}}^\dagger + \delta\widetilde{\xi}^n\delta\underline{\widetilde{\xi}}_n^\dagger + \delta\widetilde{v}\delta\underline{\widetilde{v}}^\dagger$$

$$\begin{aligned} \varpi_s = \int_{\Sigma \times I} & \underline{\delta e \delta \underline{e}_n^\dagger}_1 + \underline{\delta \underline{e}_n \delta e^\dagger}_2 + \underline{\delta\widehat{\omega}\delta\underline{\widehat{\omega}}_n^\dagger}_3 + \underline{\delta\underline{\omega}_n\delta\omega^\dagger}_4 + \underline{\delta c \delta \underline{c}_n^\dagger}_5 + \underline{\iota_{\delta\xi}\delta\underline{\xi}^\dagger}_6 + \underline{\delta\xi^n\delta\underline{\xi}_n^\dagger}_7 \\ & + \underline{\delta v \delta \underline{v}^\dagger}_8 + \underline{\delta\widehat{\omega}\delta\underline{v}^\dagger}_9 + \underline{\delta v \delta \underline{\widehat{\omega}}_n^\dagger}_{10} \end{aligned} \tag{15}$$

Hence, using the explicit expression of ψ we get

$$\psi^*(\varpi_H) = \int_{\Sigma\times I} \underline{\delta e \delta \underline{e}^\dagger_n}_1 + \frac{1}{N-3}\underline{\delta e \delta \left(e^{N-4} d_{\widehat{\omega}}(\epsilon_n \underline{\mu}^\dagger) + e^{N-4}\sigma \underline{\mu}^\dagger\right)}_2 + \frac{1}{N-3}\underline{\delta e \delta \left(e^{N-4}\right)\underline{\tau}^\dagger \nu\big)}_3 \tag{16}$$

$$+ \frac{1}{N-3}\underline{\delta e \delta \delta \left(e^{N-4}\right)\underline{\tau}^\dagger v\big)}_4 + \underline{\delta e \delta \left(X^{(a)}(Z_b \underline{\mu}^\dagger)^{(b)}_a\right)}_5 + \underline{\delta e \delta \left(\iota_{\underline{z}}(v_a Z)^{(a)}\right)}_6$$
$$+ \underline{\delta e \delta \left(\iota_{\underline{z}}(\nu_a Z)^{(a)}\right)}_7 + \theta_N(5)\underline{\delta e \delta \left(e^{N-5}\epsilon_n \mu^\dagger d\xi^n \underline{\tau}^\dagger\right)}_8$$
$$+ \underline{\delta \underline{e}_n \delta e^\dagger}_9 + \underline{\delta \underline{e}_n \delta (v_a Z)^{(a)}}_{10} + \underline{\delta \underline{e}_n \delta (\nu_a Z)^{(a)}}_{11}$$
$$+ \underline{\delta \widehat{\omega} \delta \underline{\widehat{\omega}}^\dagger_n}_{12} + \underline{\delta \nu \delta \underline{\widehat{\omega}}^\dagger_n}_{13} + \underline{\delta \widehat{\omega} \delta [\epsilon_n, \iota_\xi (Z_a \underline{\mu}^\dagger)^{(a)}]}_{14} + \underline{\delta \nu \delta [\epsilon_n, \iota_\xi (Z_a \underline{\mu}^\dagger)^{(a)}]}_{15}$$
$$+ \underline{\delta \underline{\omega}_n \delta \omega^\dagger}_{16} + \underline{\delta \iota_{\underline{z}} \nu \delta \omega^\dagger}_{17} + \underline{\delta \iota_{\underline{z}} v \delta \omega^\dagger}_{18} + \underline{\delta X^{(a)} \underline{\mu}^\dagger_a \delta \omega^\dagger}_{19}$$
$$+ \underline{\delta \underline{\omega}_n \delta [\epsilon_n, (Z_a \mu^\dagger)^{(a)} \xi^n]}_{20} + \underline{\delta \iota_{\underline{z}} \nu \delta [\epsilon_n, (Z_a \mu^\dagger)^{(a)} \xi^n]}_{21} + \underline{\delta \iota_{\underline{z}} v \delta [\epsilon_n, (Z_a \mu^\dagger)^{(a)} \xi^n]}_{22}$$
$$+ \underline{\delta X^{(a)} \underline{\mu}^\dagger_a [\epsilon_n, (Z_a \mu^\dagger)^{(a)} \xi^n]}_{23} + \underline{\delta c \delta \underline{c}^\dagger_n}_{24} + \underline{\delta (\iota_\xi \nu) \delta \underline{c}^\dagger_n}_{25} + \underline{\delta (\iota_\xi v) \delta \underline{c}^\dagger_n}_{26} + \underline{\delta (\iota_z \nu \xi^n) \delta \underline{c}^\dagger_n}_{27}$$
$$+ \underline{\delta (\iota_z v \xi^n) \delta \underline{c}^\dagger_n}_{28} + \underline{\delta \left(X^{(a)} \mu^\dagger_a \xi^n\right) \delta \underline{c}^\dagger_n}_{29} + \underline{\delta c \delta [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}]}_{30} + \underline{\delta (\iota_\xi \nu) \delta [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}]}_{31}$$
$$+ \underline{\delta (\iota_\xi v) \delta [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}]}_{32} + \underline{\delta (\iota_z \nu \xi^n) \delta [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}]}_{33} + \underline{\delta (\iota_z v \xi^n) \delta [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}]}_{34}$$
$$+ \underline{\delta \left(X^{(a)} \mu^\dagger_a \xi^n\right) \delta [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}]}_{35} + \underline{\iota_{\delta\xi} \delta \underline{\xi}^\dagger}_{36} + \underline{\iota_{\delta\xi} \delta (\underline{c}^\dagger_n \nu)}_{37} + \underline{\iota_{\delta\xi} \delta (\underline{c}^\dagger_n v)}_{38}$$
$$+ \underline{\iota_{\delta\xi} \delta (d_\omega \epsilon_n (Z_a \underline{\mu}^\dagger)^{(a)})}_{39} + \underline{\iota_{\delta\xi} \delta (\nu [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}])}_{40}$$
$$+ \underline{\delta \xi^n \delta \underline{\xi}^\dagger_n}_{41} + \underline{\delta \xi^n \delta (X^{(a)} \underline{c}^\dagger_{an} \mu^\dagger)}_{42} + \underline{\delta \xi^n \delta (d(e^{N-4} \epsilon \underline{\tau}^\dagger \mu^\dagger))}_{43} + \underline{\delta \xi^n \delta (\iota_z \underline{c}^\dagger_n \nu)}_{44}$$
$$+ \underline{\delta \xi^n \delta (\iota_z \underline{c}^\dagger_n v)}_{45} + \underline{\delta \xi^n \delta (\iota_z \nu [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}])}_{46} + \underline{\delta \xi^n \delta (d_{\omega_n} \epsilon_n (Z_a \underline{\mu}^\dagger)^{(a)})}_{47}$$
$$+ \underline{\delta \xi^n \delta (X^{(b)} \mu^\dagger_b [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}])}_{48} + \underline{\delta \xi^n \delta (\iota_z v [\epsilon_n, (Z_a \underline{\mu}^\dagger)^{(a)}])}_{49} + \underline{\delta v \delta \underline{v}^\dagger}_{50}.$$

Let us now compare the two expressions (15) and (16) and show that each term appears in both expressions. Let us divide the computation into some steps.

A The following terms appear verbatim in the two expressions:

$$(15.1) = (16.1); \qquad (15.2) = (16.9); \qquad (15.3) = (16.12);$$
$$(15.4) = (16.16); \qquad (15.5) = (16.24); \qquad (15.6) = (16.36);$$
$$(15.7) = (16.41); \qquad (15.8) = (16.50).$$

B The following terms in (16) are equal and simplify:

$$(16.15) = (16.31) + (16.40); \qquad (16.21) = (16.33) + (16.46);$$
$$(16.22) = (16.34) + (16.49); \qquad (16.23) = (16.35) + (16.48).$$

C Consider (16.4):

$$\frac{1}{N-3}\delta e \delta \left(e^{N-4} v \tau^\dagger\right) = \delta e^{N-3} \delta (v \tau^\dagger) = \delta \left(e^{N-3} \tau^\dagger\right) \delta v$$
$$= \delta \underline{\widehat{\omega}}^\dagger_n \delta v - \delta (\iota_{\underline{z}} Z) \delta v + \delta (\iota_\xi \underline{c}^\dagger_n) \delta v,$$

The first and last terms are respectively (15.10) and (16.26) + (16.38), while the second is compensated by (16.6), (16.10), (16.18), (16.28) and (16.45):

$$\delta(\iota_{\underline{z}}v)\delta\omega^\dagger - \delta(\iota_z v\xi^n)\delta\underline{c}^\dagger_n + \delta\xi^n\delta(\iota_z v\underline{c}^\dagger_n) = \delta(\iota_{\underline{z}}v)\delta(Z)$$

and

$$\delta e\delta\left(\iota_{\underline{z}}(v_aZ)^{(a)}\right) + \delta\underline{e}_n\delta\left((v_aZ)^{(a)}\right) = \iota_{\delta\underline{z}}\delta(vZ)$$

D Consider (16.2):

$$\begin{aligned}
\frac{1}{N-3}\delta e\delta\left(e^{N-4}d_{\widehat{\omega}}(\epsilon_n\underline{\mu}^\dagger) + e^{N-4}\sigma\underline{\mu}^\dagger\right) &= \delta e^{N-3}\delta\left(d_{\widehat{\omega}}(\epsilon_n\underline{\mu}^\dagger) + \sigma\underline{\mu}^\dagger\right)\\
&= \delta e^{N-3}[\delta\widehat{\omega},\epsilon_n\underline{\mu}^\dagger] + d_{\widehat{\omega}}(\delta e^{N-3})\epsilon_n\delta\underline{\mu}^\dagger + \delta(e^{N-3}\sigma)\delta\underline{\mu}^\dagger\\
&\overset{(7)}{=} \delta\widehat{\omega}\delta\left(\epsilon_n[e^{N-3},\underline{\mu}^\dagger]\right) + \epsilon_n\delta\underline{\mu}^\dagger[\delta\widehat{\omega},e^{N-3}] + d_{\widehat{\omega}}\delta e^{N-3}\epsilon_n\delta\underline{\mu}^\dagger\\
&\quad + \delta\left(\epsilon_n d_{\widehat{\omega}}e^{(N-3)} - \epsilon_n e^{(N-4)}\tau^\dagger d\xi^n + X^{[a]}Z_a\right)\delta\underline{\mu}^\dagger\\
&= \delta\widehat{\omega}\delta v^\dagger + \delta\left(-\epsilon_n e^{(N-4)}\tau^\dagger d\xi^n + X^{[a]}Z_a\right)\delta\underline{\mu}^\dagger.
\end{aligned}$$

The first term is (15.9). We consider the last terms in the following points.

E We consider $\delta\left(X^{[a]}Z_a\right)\delta\underline{\mu}^\dagger$ together with (16.19), (16.29), (16.42):

$$\delta X^{[a]}\delta\left(Z_a\underline{\mu}^\dagger\right) = X^{[b]}(\delta e_b)^{[a]}\delta\left(Z_a\underline{\mu}^\dagger\right) + (\delta X)^{[a]}\delta\left(Z_a\underline{\mu}^\dagger\right).$$

The first term is (16.5), while, recalling that

$$\delta X = [\delta(c+\iota_\xi v),\epsilon_n] + [\iota_\xi\delta\widehat{\omega},\epsilon_n] + \iota_{\delta\xi}d_{\widehat{\omega}}(\epsilon_n) - [\delta\omega_n,\epsilon_n]\xi^n + d_{\omega_n}\epsilon_n\delta\xi^n$$

the second is given by (16.30), (16.14), (16.39), (16.20), (16.32) and (16.47).

F Lastly we consider $\delta\left(-\epsilon_n e^{(N-4)}\tau^\dagger d\xi^n\right)\delta\underline{\mu}^\dagger$ together with (16.43) and (16.8):

$$\begin{aligned}
&\delta\left(-\epsilon_n e^{(N-4)}\tau^\dagger d\xi^n\right)\delta\underline{\mu}^\dagger + \delta\xi^n\delta(d(e^{N-4}\epsilon\underline{\tau}^\dagger\mu^\dagger)) + \theta_N(5)\delta e\delta\left(e^{N-5}\epsilon_n\mu^\dagger d\xi^n\underline{\tau}^\dagger\right)\\
&= \delta\tau^\dagger\delta\left(\epsilon_n e^{(N-4)}d\xi^n\underline{\mu}^\dagger\right) = \delta\tau^\dagger\delta\left(e^{(N-3)}\nu\right)\\
&= \delta\left(\tau^\dagger e^{(N-3)}\right)\delta\nu + \frac{1}{N-3}\delta e\delta\left(e^{N-4}\nu\tau^\dagger\right).
\end{aligned}$$

The last term is (16.3), while for the first we proceed exactly as in C and get (16.13), (16.25), (16.37), (16.7), (16.11), (16.17), (16.27) and (16.44).

A.2 Actions

We use the same strategy for the action. We rewrite S_s as the sum of S_r and an additional part:

$$\begin{aligned}
S_s(\Sigma\times I) = S_r(\Sigma\times I) + \int_{\Sigma\times I} &\underbrace{\frac{1}{(N-3)!}e^{N-3}\underline{e}_n d_{\widehat{\omega}}v}_{1} + \underbrace{\frac{1}{2(n-3)!}e^{N-3}\underline{e}_n[v,v]}_{2}\\
&+ \underbrace{\frac{1}{(N-2)!}e^{N-2}d_{\underline{\omega}_n}v}_{3} - \underline{\iota_\xi d_{\widehat{\omega}}v\widehat{\underline{\omega}}^\dagger_n}_4 + \underbrace{\frac{1}{2}\iota_\xi[v,v])\widehat{\underline{\omega}}^\dagger_n}_5 - \underline{[v,c]\widehat{\underline{\omega}}^\dagger_n}_6 + \underline{X(\underline{v}^\dagger)}_7\\
&- \underbrace{\iota_\xi d_{\underline{\omega}_n}v\omega^\dagger}_8 + \underline{[\iota_\xi v,e]\underline{e}^\dagger_n}_9 + \underline{[\iota_\xi v,\underline{e}_n]e^\dagger}_{10} - \underbrace{\frac{1}{2}\iota_\xi\iota_\xi d_{\widehat{\omega}}v\underline{c}^\dagger_n}_{11} + \underbrace{\frac{1}{4}\iota_\xi\iota_\xi[v,v]\underline{c}^\dagger_n}_{12}\\
&- \underline{d_{\omega_n}v\xi^n\widehat{\underline{\omega}}^\dagger_n}_{13} - \underline{\iota_\xi d_{\omega_n}v\xi^n\underline{c}^\dagger_n}_{14}
\end{aligned} \tag{17}$$

where (17.7) reads

$$X(\underline{v}^\dagger) = \Big(\underline{\iota_\xi F_{\widehat{\omega}}}_{15} + \underline{\iota_\xi d_{\widehat{\omega}} v}_{16} + \underline{[\iota_\xi v, v]}_{17} + \underline{d_{\widehat{\omega}}\omega_n\xi^n}_{18} + \underline{\partial_n\widehat{\omega}\xi^n}_{19} + \underline{d_{\omega_n} v\xi^n}_{20} - \underline{d_{\widehat{\omega}} c}_{21} - \underline{[v,c]}_{22}\Big)\,\underline{v^\dagger}$$

Before computing the pullback of the action S_H note the following:

$$\begin{aligned}
\psi^*\left(\iota_{\widetilde{\xi}}F_{\widehat{\omega}} + F_{\widetilde{\omega}_n}\widetilde{\xi}^n + d_{\widehat{\omega}}\widetilde{c}\right) &= \iota_\xi F_{\widehat{\omega}} + F_{\omega_n}\xi^n + d_{\widehat{\omega}} c + L^{\omega}_{\xi}\nu + \iota_z(\nu+v)d\xi^n \\
&\quad + X^{(a)}\mu_a^\dagger d\xi^n + d_{\omega_n}\nu\xi^n + [\nu, c] \\
\psi^*\left(\iota_{\widetilde{\xi}}\underline{F_{\widetilde{\omega}_n}} + \underline{d_{\widetilde{\omega}_n}\widetilde{c}}\right) &= \iota_\xi\underline{F_{\omega_n}} + \underline{d_{\omega_n}} c + \underline{\iota_{\partial_n\xi}\nu} + \underline{d_{\omega_n}\iota_\xi v} + \underline{d_{\omega_n}}\left(X^{(a)}\mu_a^\dagger\xi^n\right) \\
&\quad + \underline{d_{\omega_n}}\left(\iota_{\underline{z}}\nu + \iota_{\underline{z}}v\right) + [X^{(a)}\underline{\mu}_a^\dagger, c] + \iota_\xi d_{\widehat{\omega}}\left(\iota_{\underline{z}}\nu + \iota_{\underline{z}}v\right) \\
&\quad + \iota_\xi d_{\widehat{\omega}}\left(X^{(a)}\underline{\mu}_a^\dagger\right) + [\iota_{\underline{z}}\nu + \iota_{\underline{z}}v, c] \\
\psi^*\left(\frac{1}{2}\iota_{\widetilde{\xi}}\iota_{\widetilde{\xi}}F_{\widehat{\omega}} + \iota_{\widetilde{\xi}}F_{\widetilde{\omega}_n}\widetilde{\xi}^n + \frac{1}{2}[\widetilde{c},\widetilde{c}]\right) &= \frac{1}{2}\iota_\xi\iota_\xi F_{\widehat{\omega}} + \iota_\xi F_{\omega_n}\xi^n + \frac{1}{2}[c,c] + \frac{1}{2}\iota_\xi\iota_\xi d_{\widehat{\omega}}\nu \\
&\quad + \iota_\xi d_{\widehat{\omega}}\left(\iota_z v + \iota_z\nu\right)\xi^n + \iota_\xi d_{\widehat{\omega}}\left(X^{(a)}\mu_a^\dagger\right)\xi^n + \iota_\xi d_{\omega_n}\nu\xi^n \\
&\quad + [X^{(a)}\mu_a^\dagger, c] + [\iota_\xi(\nu+v), c] + [\iota_z(\nu+v)\xi^n, c]
\end{aligned}$$

We divide the computation in several parts. Let us start from the classical action:

$$\begin{aligned}
\psi^*\left(\frac{1}{(N-2)!}\widetilde{e}^{N-2}F_{\widetilde{\omega}_n} + \frac{1}{(N-3)!}\widetilde{e}^{N-3}\widetilde{\underline{e}}_n F_{\widetilde{\widehat{\omega}}}\right) &= \frac{1}{(N-2)!}e^{N-2}F_{\omega_n} + \frac{1}{(N-3)!}e^{N-3}\underline{e}_n F_{\widehat{\omega}} \\
&+ \frac{1}{(N-2)!}e^{N-2}\left(\underline{d_{\widehat{\omega}}\left(X^{(a)}\underline{\mu}_a^\dagger\right)}_1 + \underline{d_{\widehat{\omega}}\left(\iota_{\underline{z}}(\nu+v)\right)}_2 + \underline{\underline{d_{\omega_n}\nu}}_3 + \underline{[\nu, X^{(a)}\underline{\mu}_a^\dagger]}_4 + \underline{[\nu, \iota_{\underline{z}}(\nu+v)]}_5\right) \\
&+ \frac{1}{(N-3)!}\underline{e^{N-3}\underline{e}_n d_{\widehat{\omega}}\nu}_6 + \frac{1}{2(N-3)!}\underline{e^{N-3}\underline{e}_n[\nu,\nu]}_7
\end{aligned} \tag{18}$$

We then consider the terms with antifields $\widehat{\omega}_n^\dagger$ and $\omega^\dagger$

$$
\begin{aligned}
\psi^*\left((\iota_{\widetilde{\xi}}F_{\widetilde{\omega}}+F_{\widetilde{\omega}_n}\widetilde{\xi}^n+d_{\widehat{\omega}}\widetilde{c})\widetilde{\widehat{\omega}}_n^\dagger\right) &= (\iota_\xi F_{\widehat{\omega}}+F_{\omega_n}\xi^n+d_{\widehat{\omega}}c)\,\widehat{\omega}_n^\dagger \qquad (19)\\
&+\Big(\underline{L_\xi^\omega\nu}_1+\underline{\iota_z(\nu+v)d\xi^n}_2+\underline{X^{(a)}\mu_a^\dagger d\xi^n}_3+\underline{d_{\omega_n}\nu\xi^n}_4+\underline{[\nu,c]}_5+\underline{[\nu,\iota_\xi v]}_6+\underline{d_{\widehat{\omega}}\iota_\xi v}_7\Big)\,\widehat{\underline{\omega}}_n^\dagger\\
&+\Big(\underline{\iota_\xi F_{\widehat{\omega}}}_8+\underline{F_{\omega_n}\xi^n}_9+\underline{d_{\widehat{\omega}}c}_{10}+\underline{L_\xi^\omega\nu}_{11}+\underline{\iota_z(\nu+v)d\xi^n}_{12}+\underline{X^{(a)}\mu_a^\dagger d\xi^n}_{13}\Big)\,[\epsilon_n,\iota_\xi(Z_a\underline{\mu}^\dagger)^{(a)}]\\
&+\Big(\underline{d_{\omega_n}\nu\xi^n}_{14}+\underline{[\nu,c]}_{15}+\underline{[\nu,\iota_\xi v]}_{16}+\underline{d_{\widehat{\omega}}\iota_\xi v}_{17}\Big)\,[\epsilon_n,\iota_\xi(Z_a\underline{\mu}^\dagger)^{(a)}]\\
\psi^*\left((\iota_{\widetilde{\xi}}\underline{F_{\widetilde{\omega}_n}}+\underline{d_{\widetilde{\omega}_n}}\widetilde{c})\widetilde{\omega}^\dagger\right) &= \Big(\underline{\iota_\xi F_{\omega_n}}+\underline{d_{\omega_n}c}\Big)\,\omega^\dagger+\Big(\underline{\underline{\iota_{\partial_n\xi}\nu}}_{18}+\underline{\underline{d_{\omega_n}\iota_\xi v}}_{19}\Big)\,\omega^\dagger\\
&+\Big(\underline{d_{\omega_n}\Big(X^{(a)}\mu_a^\dagger\xi^n\Big)}_{20}+\underline{\underline{d_{\omega_n}}\left(\iota_{\underline{z}}\nu\xi^n+\iota_{\underline{z}}v\xi^n\right)}_{21}+\underline{[X^{(a)}\underline{\mu}_a^\dagger,c+\iota_\xi v]}_{22}\Big)\,\omega^\dagger\\
&+\Big(\underline{\iota_\xi d_{\widehat{\omega}}\left(\iota_{\underline{z}}\nu+\iota_{\underline{z}}v\right)}_{23}+\underline{\iota_\xi d_{\widehat{\omega}}\Big(X^{(a)}\underline{\mu}_a^\dagger\Big)}_{24}+\underline{[\iota_{\underline{z}}\nu+\iota_{\underline{z}}v,c+\iota_\xi v]}_{25}\Big)\,\omega^\dagger\\
&+\Big(\underline{\underline{\iota_\xi F_{\omega_n}}}_{26}+\underline{\underline{d_{\omega_n}c}}_{27}+\underline{\underline{\iota_{\partial_n\xi}\nu}}_{28}+\underline{\underline{d_{\omega_n}\iota_\xi v}}_{29}+\underline{\underline{d_{\omega_n}}\Big(X^{(a)}\mu_a^\dagger\xi^n\Big)}_{30}\Big)\,[\epsilon_n,(Z_a\mu^\dagger)^{(a)}\xi^n]\\
&+\Big(\underline{\underline{d_{\omega_n}}\left(\iota_{\underline{z}}\nu\xi^n+\iota_{\underline{z}}v\xi^n\right)}_{31}+\underline{[X^{(a)}\underline{\mu}_a^\dagger,c+\iota_\xi v]}_{32}+\underline{\iota_\xi d_{\widehat{\omega}}\left(\iota_{\underline{z}}\nu+\iota_{\underline{z}}v\right)}_{33}\Big)\,[\epsilon_n,(Z_a\mu^\dagger)^{(a)}\xi^n]\\
&+\Big(\underline{\iota_\xi d_{\widehat{\omega}}\Big(X^{(a)}\underline{\mu}_a^\dagger\Big)}_{34}+\underline{[\iota_{\underline{z}}\nu+\iota_{\underline{z}}v,c+\iota_\xi v]}_{35}\Big)\,[\epsilon_n,(Z_a\mu^\dagger)^{(a)}\xi^n]
\end{aligned}
$$

We now consider the terms with antifields $e_n^\dagger$ and $e^\dagger$:

$$
\begin{aligned}
\psi^*\left(\left(L_{\widetilde{\xi}}^{\widetilde{\widehat{\omega}}}\widetilde{e}+d_{\widetilde{\omega}_n}\widetilde{e}\widetilde{\xi}^n+\widetilde{e}_n d\widetilde{\xi}^n-[\widetilde{c},\widetilde{e}]\right)\widetilde{\underline{e}}_n^\dagger\right) &= \left(L_\xi^\omega e+d_{\omega_n}e\xi^n+e_n d\xi^n-[c,e]\right)\underline{e}_n^\dagger \qquad (20)\\
&+\underline{[\iota_\xi v,e]\underline{e}_n^\dagger}_1+\frac{1}{N-3}e^{N-4}L_\xi^\omega e\,\Big(\underline{d_{\widehat{\omega}}(\epsilon_n\underline{\mu}^\dagger)+\sigma\underline{\mu}^\dagger}_2+\underline{\underline{\tau}^\dagger(\nu+v)}_3\Big)\\
&+L_\xi^\omega e\,\left(\underline{X^{(a)}(Z_b\underline{\mu}^\dagger)_a^{(b)}}_4+\underline{\iota_{\underline{z}}((\nu_a+v_a)Z)^{(a)}}_5+\underline{\frac{\theta_N(5)e^{N-5}}{N-4}\epsilon_n\mu^\dagger d\xi^n\underline{\tau}^\dagger}_6\right)\\
&+\frac{1}{N-3}e^{N-4}d_{\omega_n}e\xi^n\,\Big(\underline{d_{\widehat{\omega}}(\epsilon_n\underline{\mu}^\dagger)+\sigma\underline{\mu}^\dagger}_7+\underline{\underline{\tau}^\dagger(\nu+v)}_8\Big)\\
&+d_{\omega_n}e\xi^n\,\left(\underline{X^{(a)}(Z_b\underline{\mu}^\dagger)_a^{(b)}}_9+\underline{\iota_{\underline{z}}((\nu_a+v_a)Z)^{(a)}}_{10}+\underline{\frac{\theta_N(5)e^{N-5}}{N-4}\epsilon_n\mu^\dagger d\xi^n\underline{\tau}^\dagger}_{11}\right)\\
&+\frac{1}{N-3}e^{N-4}e_n d\xi^n\,\Big(\underline{d_{\widehat{\omega}}(\epsilon_n\underline{\mu}^\dagger)+\sigma\underline{\mu}^\dagger}_{12}+\underline{\underline{\tau}^\dagger(\nu+v)}_{13}\Big)\\
&+e_n d\xi^n\,\left(\underline{X^{(a)}(Z_b\underline{\mu}^\dagger)_a^{(b)}}_{14}+\underline{\iota_{\underline{z}}((\nu_a+v_a)Z)^{(a)}}_{15}+\underline{\frac{\theta_N(5)e^{N-5}}{N-4}\epsilon_n\mu^\dagger d\xi^n\underline{\tau}^\dagger}_{16}\right)\\
&+\frac{1}{N-3}e^{N-4}[c+\iota_\xi v,e]\,\Big(\underline{d_{\widehat{\omega}}(\epsilon_n\underline{\mu}^\dagger)+\sigma\underline{\mu}^\dagger}_{17}+\underline{\underline{\tau}^\dagger(\nu+v)}_{18}\Big)\\
&+[c+\iota_\xi v,e]\,\left(\underline{X^{(a)}(Z_b\underline{\mu}^\dagger)_a^{(b)}}_{19}+\underline{\iota_{\underline{z}}((\nu_a+v_a)Z)^{(a)}}_{20}+\underline{\frac{\theta_N(5)e^{N-5}}{N-4}\epsilon_n\mu^\dagger d\xi^n\underline{\tau}^\dagger}_{21}\right)\\
\psi^*\left(\left(L_{\widetilde{\xi}}^{\widetilde{\widehat{\omega}}}\widetilde{\underline{e}}_n+\underline{d_{\widetilde{\omega}_n}}(\widetilde{e}_n\widetilde{\xi}^n)+\underline{\iota_{\partial_n\widetilde{\xi}}}\widetilde{e}-[\widetilde{c},\widetilde{e}_n]\right)\widetilde{e}^\dagger\right) &= \Big(\iota_\xi d_\omega\underline{e}_n+\underline{\iota_{\partial_n\xi}e}-\underline{d_{\omega_n}}(e_n\xi^n)-[c,\underline{e}_n]\Big)\,e^\dagger\\
&+\underline{[\iota_\xi v,\underline{e}_n]e^\dagger}_{22}+\Big(\underline{\iota_\xi d_\omega\underline{e}_n}_{23}+\underline{\iota_{\partial_n\xi}e}_{24}-\underline{\underline{d_{\omega_n}}(e_n\xi^n)}_{25}-\underline{[c+\iota_\xi v,\underline{e}_n]}_{26}\Big)\,((\nu_a+v_a)Z)^{(a)}
\end{aligned}
$$

Next we consider the terms with $c_n^\dagger$

$$\psi^*\left(\left(\frac{1}{2}\iota_{\widetilde{\xi}}\iota_{\widetilde{\xi}}F_{\widehat{\omega}}+\iota_{\widetilde{\xi}}F_{\widetilde{\omega}_n}\widetilde{\xi}^n+\frac{1}{2}[\widetilde{c},\widetilde{c}]\right)\widetilde{\underline{c}}_n^\dagger\right)=\left(\frac{1}{2}\iota_\xi\iota_\xi F_{\widehat{\omega}}+\iota_\xi F_{\omega_n}\xi^n+\frac{1}{2}[c,c]\right)\underline{c}_n^\dagger \tag{21}$$
$$+\left(\frac{1}{2}\underline{\iota_\xi\iota_\xi d_{\widehat{\omega}}\nu}_1+\underline{\iota_\xi d_{\widehat{\omega}}\left(\iota_z v+\iota_z\nu\right)\xi^n}_2+\underline{\iota_\xi d_{\widehat{\omega}}\left(X^{(a)}\mu_a^\dagger\right)\xi^n}_3+\underline{\iota_\xi d_{\omega_n}\nu\xi^n}_4\right)\underline{c}_n^\dagger$$
$$+\left(\underline{[X^{(a)}\mu_a^\dagger\xi^n,c]}_5+\underline{[\iota_\xi(\nu+v),c]}_6+\underline{[\iota_z(\nu+v)\xi^n,c]}_7\right)\underline{c}_n^\dagger$$
$$+\left(\underline{\frac{1}{4}\iota_\xi\iota_\xi[v,v]}_8+\underline{\frac{1}{2}\iota_\xi\iota_\xi[\nu,v]}_9+\underline{[\iota_z(\nu+v)\xi^n,\iota_\xi v]}_{10}+\underline{[X^{(a)}\mu_a^\dagger\xi^n,\iota_\xi v]}_{11}\right)\underline{c}_n^\dagger$$
$$+\left(\underline{\frac{1}{2}\iota_\xi\iota_\xi F_{\widehat{\omega}}}_{12}+\underline{\iota_\xi F_{\omega_n}\xi^n}_{13}+\underline{\frac{1}{2}[c,c]}_{14}\right)[\epsilon_n,(Z_a\underline{\mu}^\dagger)^{(a)}]$$
$$+\left(\underline{\frac{1}{2}\iota_\xi\iota_\xi d_{\widehat{\omega}}\nu}_{15}+\underline{\iota_\xi d_{\widehat{\omega}}\left(\iota_z v+\iota_z\nu\right)\xi^n}_{16}+\underline{\iota_\xi d_{\widehat{\omega}}\left(X^{(a)}\mu_a^\dagger\right)\xi^n}_{17}+\underline{\iota_\xi d_{\omega_n}\nu\xi^n}_{18}\right)[\epsilon_n,(Z_a\underline{\mu}^\dagger)^{(a)}]$$
$$+\left(\underline{[X^{(a)}\mu_a^\dagger\xi^n,c]}_{19}+\underline{[\iota_\xi(\nu+v),c]}_{20}+\underline{[\iota_z(\nu+v)\xi^n,c]}_{21}\right)[\epsilon_n,(Z_a\underline{\mu}^\dagger)^{(a)}]$$
$$+\left(\underline{\frac{1}{4}\iota_\xi\iota_\xi[v,v]}_{22}+\underline{\frac{1}{2}\iota_\xi\iota_\xi[\nu,v]}_{23}+\underline{[\iota_z(\nu+v)\xi^n,\iota_\xi v]}_{24}+\underline{[X^{(a)}\mu_a^\dagger\xi^n,\iota_\xi v]}_{25}\right)[\epsilon_n,(Z_a\underline{\mu}^\dagger)^{(a)}]$$

We now consider the terms with $\xi_n^\dagger$ and $\xi^\dagger$:

$$\psi^*\left(\iota_{\widetilde{\xi}}d\widetilde{\xi}^n\widetilde{\underline{\xi}}_{-n}^\dagger+\widetilde{\xi}^n\partial_n\widetilde{\xi}^n\widetilde{\underline{\xi}}_{-n}^\dagger\right)=\iota_\xi d\xi^n\underline{\xi}_n^\dagger+\xi^n\partial_n\xi^n\underline{\xi}_n^\dagger \tag{22}$$
$$+\iota_\xi d\xi^n\left(\underline{X^{(a)}\underline{c}_{an}^\dagger\mu^\dagger}_1+\underline{d(e^{N-4}\epsilon\underline{\tau}^\dagger\mu^\dagger)}_2+\underline{\iota_z\underline{c}_n^\dagger(\nu+v)}_3+\underline{\iota_z(\nu+v)[\epsilon_n,(Z_a\underline{\mu}^\dagger)^{(a)}]}_4\right)$$
$$+\iota_\xi d\xi^n\left(\underline{d_{\omega_n}\epsilon_n(Z_a\underline{\mu}^\dagger)^{(a)}}_5+\underline{X^{(b)}\mu_b^\dagger[\epsilon_n,(Z_a\underline{\mu}^\dagger)^{(a)}]}_6\right)$$
$$+\xi^n\partial_n\xi^n\left(\underline{X^{(a)}\underline{c}_{an}^\dagger\mu^\dagger}_7+\underline{d(e^{N-4}\epsilon\underline{\tau}^\dagger\mu^\dagger)}_8+\underline{\iota_z\underline{c}_n^\dagger(\nu+v)}_9+\underline{\iota_z(\nu+v)[\epsilon_n,(Z_a\underline{\mu}^\dagger)^{(a)}]}_{10}\right)$$
$$+\xi^n\partial_n\xi^n\left(\underline{d_{\omega_n}\epsilon_n(Z_a\underline{\mu}^\dagger)^{(a)}}_{11}+\underline{X^{(b)}\mu_b^\dagger[\epsilon_n,(Z_a\underline{\mu}^\dagger)^{(a)}]}_{12}\right)$$
$$\psi^*\left(\widetilde{\xi}^n\iota_{\partial_n\widetilde{\xi}}\widetilde{\underline{\xi}}^\dagger+\frac{1}{2}\iota_{[\widetilde{\xi},\widetilde{\xi}]}\widetilde{\underline{\xi}}^\dagger\right)=\xi^n\iota_{\partial_n\xi}\underline{\xi}^\dagger+\frac{1}{2}\iota_{[\xi,\xi]}\underline{\xi}^\dagger$$
$$+\xi^n\iota_{\partial_n\xi}\left(\underline{\underline{c}_n^\dagger(\nu+v)}_{13}+\underline{d_\omega\epsilon_n(Z_a\underline{\mu}^\dagger)^{(a)}}_{14}+\underline{\nu[\epsilon_n,(Z_a\underline{\mu}^\dagger)^{(a)}]}_{15}\right)$$
$$+\frac{1}{2}\iota_{[\xi,\xi]}\left(\underline{\underline{c}_n^\dagger(\nu+v)}_{16}+\underline{d_\omega\epsilon_n(Z_a\underline{\mu}^\dagger)^{(a)}}_{17}+\underline{\nu[\epsilon_n,(Z_a\underline{\mu}^\dagger)^{(a)}]}_{18}\right)$$

Lastly we consider all the remaining terms:

$$\psi^*\left(\frac{1}{2(N-3)!}\widetilde{e}^{N-3}\widetilde{\underline{e}}_n[\widetilde{v},\widetilde{v}]+\left(L_{\widetilde{\xi}}^{\widetilde{\widehat{\omega}}}\widetilde{v}+[\widetilde{c},\widetilde{v}]+\widetilde{\xi}^n d_{\widetilde{\omega}_n}v+d\widetilde{\xi}^n\iota_z\widetilde{v}\right)\widetilde{\underline{v}}^\dagger\right) \tag{23}$$
$$=\underline{\frac{1}{2(N-3)!}e^{N-3}\underline{e}_n[v,v]}_1\left(\underline{L_\xi^\omega v}_2+\underline{[c,v]}_3+\underline{\xi^n d_{\omega_n}v}_4+\underline{d\xi^n\iota_z v}_5+\underline{[\iota_\xi v,v]}_6\right)\underline{v}^\dagger$$

The terms that are not underlined correspond to the ones in S_r. Everything else should be found in (17) or vanish.

A The following terms simplify:

$$
\begin{array}{llll}
(17.2) = (23.1); & (17.3) = 0; & (18.5) = 0; & (18.4) = 0; \\
(19.26) = (21.13); & (19.12) = (22.4); & (19.13) = (22.6); & (19.14) = (21.18); \\
(17.22) = (23.3); & (17.20) = (23.4); & (17.17) = (23.6); & (17.9) = (20.1); \\
(17.10) = (20.22); & (19.16) = (21.23); & (19.30) = (22.12); & (19.31) = (22.10); \\
(19.28) = (22.15); & (19.33) = (21.16); & (19.34) = (21.17); &
\end{array}
$$

$$
\begin{array}{ll}
(19.32) = (21.19) + (21.25); & (19.35) = (21.21) + (21.24); \\
(21.15) + (19.11) = (22.18); & (17.8) + (19.18) + (19.19) + (22.13) = (20.24); \\
(18.5) + (18.7) = (23.5). &
\end{array}
$$

B We consider the terms containing the ghost c. Using (6) and (7) we get:

$$
\begin{aligned}
&(20.17) + (20.18) + (20.21) \qquad (24) \\
&= d_{\widehat{\omega}}(c + \iota_\xi v)\underline{v}^\dagger + [c + \iota_\xi v, X^{(a)} Z_a]\underline{\mu}^\dagger + \frac{1}{N-3}[c + \iota_\xi v, \epsilon_n]^{(a)} \underline{\mu}_a^\dagger e^{N-3} \left(d_{\widehat{\omega}} e + \tau^\dagger d\xi^n\right) \\
&\quad + [c + \iota_\xi v, \epsilon_n]^{(n)} X^{(a)} Z_a \underline{\mu}^\dagger + \frac{1}{N-3}[c + \iota_\xi v, v + \nu] e^{N-3} \underline{\tau}^\dagger. \qquad (25)
\end{aligned}
$$

We also have that

$$
(19.22) + (21.5) + (21.11) = [c, X^{(a)} \underline{\mu}_a^\dagger] Z + [\iota_\xi v, X^{(a)} \underline{\mu}_a^\dagger] Z.
$$

compensating the second term in (25) and that

$$
\begin{aligned}
&(19.5) + (19.6) + (17.5) + (17.6) + (19.25) + (21.6) + (21.7) + (21.8) + (21.9) + (21.10) \\
&\quad = \frac{1}{N-3}[c + \iota_\xi v, v + \nu] e^{N-3} \underline{\tau}^\dagger
\end{aligned}
$$

Compensating the last term of (25). Furthermore the following identity holds:

$$
\begin{aligned}
&(21.14) + (19.29) + (19.17) + (19.15) + (21.20) + (21.22) + (19.27) + (19.10) + (20.19) \\
&\quad = [c + \iota_\xi v, \epsilon_n]^{(n)} X^{(a)} Z_a \underline{\mu}^\dagger + [c + \iota_\xi v, \epsilon_n]^{(a)} \left(L_\xi^{\widehat{\omega}} \epsilon_n + d_{\omega_n} \epsilon^n \xi^n\right)^{(n)} Z_a \underline{\mu}^\dagger \\
&\qquad + L_\xi^{\widehat{\omega}} \left([c + \iota_\xi v, \epsilon_n]^{(a)}\right) Z_a \underline{\mu}^\dagger + d_{\omega_n} \left([c + \iota_\xi v, \epsilon_n]^{(a)}\right) \xi^n Z_a \underline{\mu}^\dagger \\
&\qquad + \left(L_\xi^{\widehat{\omega}} e_a + d_{\omega_n} e_a \xi^n\right) [c + \iota_\xi v, \epsilon_n]^{(a)} (Z_b \underline{\mu}^\dagger)^{(b)}.
\end{aligned}
$$

Lastly we also have

$$
(20.20) + (20.26) = \lambda [c + \iota_\xi v, \epsilon_n]^{(a)} (\nu_a + v_a) Z.
$$

C Similarly, for the terms containing ξ we get

$$
\begin{aligned}
&(20.2) + (20.3) + (20.6) + (22.2) \\
&\quad = \iota_\xi F_{\widehat{\omega}} \underline{v}^\dagger + L_\xi^{\widehat{\omega}} (X^{(a)} Z_a) \underline{\mu}^\dagger + \frac{1}{N-3} \left(L_\xi^{\widehat{\omega}} \epsilon_n\right)^{(a)} \underline{\mu}_a^\dagger e^{N-3} \left(d_{\widehat{\omega}} e + \tau^\dagger d\xi^n\right) \\
&\qquad + \left(L_\xi^{\widehat{\omega}} \epsilon_n\right)^{(n)} X^{(a)} Z_a \underline{\mu}^\dagger + \frac{1}{N-3} L_\xi^{\widehat{\omega}} (v + \nu) e^{N-3} \underline{\tau}^\dagger
\end{aligned}
$$

Furthermore

$$\begin{aligned}(19.1)+(19.7)+(17.4)+(19.23)+(21.1)+(17.11)+(17.12)+(21.2)+(22.16)\\ = L_\xi^{\widehat{\omega}}(\nu+v)e\underline{\tau}^\dagger+\iota_{[\xi,\underline{z}]}(\nu+v)Z\end{aligned}$$

and

$$\begin{aligned}(19.24)+(21.3)+(22.17)+(22.5)+(22.14)+(19.8)+(19.9)+(21.12)+(20.4)\\ = [c+\iota_\xi v,\epsilon_n]^{(a)}\left(L_\xi^{\widehat{\omega}}\epsilon_n\right)^{(n)}Z_a\underline{\mu}^\dagger+L_\xi^{\widehat{\omega}}\left([c+\iota_\xi v,\epsilon_n]^{(a)}\right)Z_a\underline{\mu}^\dagger\\ +L_\xi^{\widehat{\omega}}e_a[c+\iota_\xi v,\epsilon_n]^{(a)}(Z_b\underline{\mu}^\dagger)^{(b)}+\left(L_\xi^{\widehat{\omega}}\epsilon_n\right)^{(a)}(d_{\omega_n}\epsilon^n\xi^n)^{(n)}Z_a\underline{\mu}^\dagger\\ +d_{\omega_n}\left(\left(L_\xi^{\widehat{\omega}}\epsilon_n\right)^{(a)}\right)\xi^nZ_a\underline{\mu}^\dagger+d_{\omega_n}e_a\xi^n\left(L_\xi^{\widehat{\omega}}\epsilon_n\right)^{(a)}(Z_b\underline{\mu}^\dagger)^{(b)}.\end{aligned}$$

and the terms in the first line cancel with equal ones in B. Lastly

$$(20.5)+(20.23)=\iota_{[\xi,\underline{z}]}(\nu+v)Z+\lambda\left(L_\xi^{\widehat{\omega}}\epsilon_n\right)^{(a)}(\nu_a+v_a)Z.$$

D For some of the terms containing ξ^n we get

$$\begin{aligned}(20.7)+(20.8)+(20.11)+(18.3)+(22.8)\\ = F_{\omega_n}\xi^n\underline{v}^\dagger+d_{\omega_n}(X^{(a)}Z_a)\underline{\mu}^\dagger\xi^n+\frac{1}{N-3}\left(d_{\omega_n}\epsilon_n\xi^n\right)^{(a)}\underline{\mu}_a^\dagger e^{N-3}\left(d_{\widehat{\omega}}e+\tau^\dagger d\xi^n\right)\\ +\left(d_{\omega_n}\epsilon_n\xi^n\right)^{(n)}X^{(a)}Z_a\underline{\mu}^\dagger+\frac{1}{N-3}d_{\omega_n}(v+\nu)\xi^ne^{N-3}\underline{\tau}^\dagger.\end{aligned}$$

The last term is compensated by

$$\begin{aligned}(19.4)+(17.13)+(17.14)+(21.4)+(19.21)\\ =\frac{1}{N-3}d_{\omega_n}(v+\nu)\xi^ne^{N-3}\underline{\tau}^\dagger+\iota_{\underline{z}}(\nu+v)\partial_n\xi^n\omega^\dagger+\iota_{\partial_n\underline{z}}(\nu+v)\xi^n\omega^\dagger.\end{aligned}$$

We also have

$$\begin{aligned}(19.20)+(22.7)+(22.11)+(20.9)\\ =\left(d_{\omega_n}\epsilon_n\xi^n\right)^{(n)}X^{(a)}Z_a\underline{\mu}^\dagger+d_{\omega_n}(X^{(a)}Z_a)\underline{\mu}^\dagger\xi^n+[c+\iota_\xi v,\epsilon_n]^{(a)}\left(d_{\omega_n}\epsilon^n\xi^n\right)^{(n)}Z_a\underline{\mu}^\dagger\\ +d_{\omega_n}\left([c+\iota_\xi v,\epsilon_n]^{(a)}\right)\xi^nZ_a\underline{\mu}^\dagger+\left(d_{\omega_n}e_a\xi^n\right)[c+\iota_\xi v,\epsilon_n]^{(a)}(Z_b\underline{\mu}^\dagger)^{(b)}\\ +\left(L_\xi^{\widehat{\omega}}\epsilon_n\right)^{(a)}\left(d_{\omega_n}\epsilon^n\xi^n\right)^{(n)}Z_a\underline{\mu}^\dagger+d_{\omega_n}\left(\left(L_\xi^{\widehat{\omega}}\epsilon_n\right)^{(a)}\right)\xi^nZ_a\underline{\mu}^\dagger\\ +d_{\omega_n}e_a\xi^n\left(L_\xi^{\widehat{\omega}}\epsilon_n\right)^{(a)}(Z_b\underline{\mu}^\dagger)^{(b)}\end{aligned}$$

where the second and third line in the result are canceled by the same terms appearing in items B and C respectively. Furthermore

$$\begin{aligned}(20.10)+(20.25)+(22.9)=\iota_{\underline{z}}(\nu+v)\partial_n\xi^n\omega^\dagger+\iota_{\partial_n\underline{z}}(\nu+v)\xi^n\omega^\dagger\\ +\lambda\left(d_{\omega_n}\epsilon_n\xi^n\right)^{(a)}(\nu_a+v_a)Z.\end{aligned}$$

E Summing up items B, C and D we get

$$d_{\widehat{\omega}}(c+\iota_\xi v)+\iota_\xi F_{\widehat{\omega}}\underline{v}^\dagger+F_{\omega_n}\xi^n\underline{v}^\dagger+\frac{1}{N-3}X^{(a)}\underline{\mu}_a^\dagger e^{N-3}\left(d_{\widehat{\omega}}e+\tau^\dagger d\xi^n\right)+\lambda X^{(a)}(\nu_a+v_a)Z.$$

The first three terms are given by (17.21), (17.16), (23.2), (17.15), (18.18) and (17.19). We also have that

$$(19.3)+(20.14)+(22.1)=X^{(a)}\mu_a^\dagger d\xi^n e^{N-3}\underline{\tau}^\dagger+\iota_z\left(X^{(a)}Z_a\right)d\xi^n\underline{\mu}^\dagger$$

and

$$(18.1)=\frac{1}{N-3}X^{(a)}\underline{\mu}_a^\dagger e^{N-3}d_{\widehat{\omega}}e$$

Furthermore

$$\begin{aligned}(20.12)+(20.16)+(18.2)+(18.6)=&\frac{1}{N-3}e^{N-4}\lambda\nu\epsilon_n d\xi^n\underline{\tau}^\dagger+\lambda\nu X^{(a)}Z_a\\&+\iota_z\left(X^{(a)}Z_a\right)d\xi^n\underline{\mu}^\dagger+\frac{1}{N-2}e^{N-2}d_{\widehat{\omega}}\iota_{\underline{z}}v,\end{aligned}$$

$$(20.13)+(20.15)+(22.3)+(19.2)=\frac{1}{N-3}\lambda e^{N-4}\epsilon_n d\xi^n(\nu+v)\underline{\tau}^\dagger$$

and

$$(17.1)=\frac{1}{N-3}e^{N-4}\lambda\epsilon_n d\xi^n v\underline{\tau}^\dagger+\lambda X^{(a)}v_aZ+\frac{1}{N-2}e^{N-2}d_{\widehat{\omega}}\iota_{\underline{z}}v.$$

Summing up all the terms we get the desired claim.

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