GRAVITY COUPLED WITH SCALAR, SU(N), AND SPINOR FIELDS ON MANIFOLDS WITH NULL-BOUNDARY

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Abstract. In this paper, we present a theory for gravity coupled with scalar, SU(n) and spinor fields on manifolds with null-boundary. We perform the symplectic reduction of the space of boundary fields and give the constraints of the theory in terms of local functionals of boundary vielbein and connection. For the three different couplings, the analysis of the constraint algebra shows that the set of constraints does not form a first class system.

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1. Introduction

The concept of gauge theories is a central aspect of modern mathematical physics, serving as the basis for the formulation of many fundamental physical theories. In a gauge theory, the physical fields are described by means of a principal G-bundle over a base manifold M (possibly with boundary) and an action functional S that embodies the symmetry of the theory and from which the field equations are derived. The conserved quantities of the theory come from the invariance of the action functional under the symmetry group, while interactions are introduced by gauging these symmetries, making them local. The mathematical representation of

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a gauge theory is achieved through a principal bundle P and the gauge group can consequently be defined.

Two of the most widely accepted theories in fundamental physics are the Standard Model (SM) of particle physics and General Relativity (GR). The SM, with its symmetry group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, is a gauge theory that explains three of the four fundamental interactions. Meanwhile, GR is a theory that accounts for gravity, with physical fields representing the geometry of the base manifold.

GR is originally formulated using metric and Christoffel symbols, and is not formally equivalent to a Yang–Mills theory like the SM. In the SM, symmetries are encoded in physical fields through a principal connection. In order to study GR within a framework that would make the gauge formulation similar to the one of the SM, the coframe formalism, a.k.a. Palatini–Cartan (PC) theory, can be used. Within this formalism, the physical fields of GR are represented as coframes, a.k.a. vielbein, and local connections (see e.g. [Tec19b], [Thi07] and [Cat23]) and gravity becomes an explicit $\mathrm{SO}(3,1)$ gauge theory.

In this paper, we examine the boundary structure of GR in the coframe formulation on manifolds with boundary, focusing specifically on the scenario where the boundary is null, resulting in a degenerate boundary metric. Our study extends the findings of the previous work [CCF22], in which the authors investigated the geometric structures of gravity coupled to scalar, $\mathrm{SU}(n)$, and spinor fields for a non-degenerate boundary metric. Our aim is to extend these findings to boundaries with the most general structure, namely including a possible degenerate metric. From a different perspective, this paper extends the study conducted in [CCT21], where the authors analyze the degenerate boundary structure of the Palatini–Cartan theory, in order to incorporate gravity coupled with matter and gauge fields. Therefore, the generalization of these results will lay the foundation for formulating the SM on manifolds with boundary (see [Can+23]).

The boundary structure is recovered performing the method established by Kijowski and Tulczijew (KT) in [KT79] (for an introduction see also [Cat23] and references therein). This method involves characterizing the reduced phase space as a reduction, i.e., a quotient space, of the space of free boundary fields, rather than using the approach proposed by Dirac in [Dir58]. The KT method has several mathematical advantages, including a more straightforward procedure for formulating constraints and compatibility with the BV-BFV formalism as described in [CMR14] (in the case of PC gravity this is done in [CS19a], [CCS21a] and [CCS21b]). Additionally, the quantization procedure within the BV-BFV formalism as outlined in [CMR18] can be more readily applied to the theory when using the KT approach.

Gravity in the coframe formalism is expressed though the so called Palatini–Cartan action. The structure of the symplectic form of the boundary fields poses a major challenge in the constraint analysis of the theory. This form is defined on a quotient space of the restrictions of the bulk fields to the boundary, determined by an equivalence relation given by the kernel of the map $e\wedge$, where e denotes the coframe. To simplify the analysis, we describe this phase space using a fixed representative instead of working with equivalence classes, by introducing a suitable structural constraint. In prior works, such as [CS19b], [CCS21a] and [CCF22], this method has been successfully applied to space-like and time-like boundaries. However, for a null-boundary, the structural constraint must be adapted, as it only fixes the representative uniquely when the induced metric on the boundary is non-degenerate. We extend the solution proposed for space- and time-like boundaries to a null-boundary by adapting the structural constraint for all three different couplings (scalar, $\mathrm{SU}(n)$, and spinor). The solution is slightly more involved and

gives rise to second class constraints, compared to the non-degenerate case, where all constraints are first class.

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2. Geometrical background of gravity

2.1. **Coframe formalism.** In the following section we will examine the geometrical background of the theory, i.e. the coframe formalism and the Palatini–Cartan action (see for example [Tec19b; Thi07] and references therein).

The general set-up consists of

- An N-dimensional differentiable oriented[1] pseudo-riemannian manifold M with boundary Σ;
- A principal $\mathrm{GL}(N,\mathbb{R})$-bundle LM called the *frame bundle*, which can be reduced to a principal $\mathrm{SO}(N-1,1)$-bundle P;
- An associated vector bundle $\mathcal{V} \coloneqq P \times_\rho V$ called the *Minkowski bundle*, where V is an N-dimensional real pseudo-riemannian vector space with reference metric $\eta = \mathrm{diag}(1,...,-1)$ and $\rho\colon \mathrm{SO}(N-1,1) \to \mathrm{Aut}(V)$ is the fundamental representation of $\mathrm{SO}(N-1,1)$.

Then, we define the vielbein via a reduction of the frame bundle.

Definition 1. We define the *vielbein* $\tilde{e}\colon P \to LM$ as the principal bundle isomorphism such that the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\tilde{e}} & LM \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathcal{V} & \underset{e}{\rightleftarrows} & TM \end{array}$$

where $e\colon TM \to \mathcal{V}$ is the vector bundle isomorphism induced by $\tilde{e}\colon P \to LM$ and p', π' the corresponding associated bundle maps. This means that the vielbein consists of the elements in $\Omega^1(M,\mathcal{V})$ possessing smooth inverse. We can call this space $\tilde{\Omega}^1(M,\mathcal{V})$.

Remark 2.

- Given $i\colon \mathrm{SO}(N-1,1) \to \mathrm{GL}(N,\mathbb{R})$ as the canonical embedding, we recall that, in order for $\tilde{e}$ to be a principal bundle isomorphism, it must be an isomorphism of fiber bundles and also satisfy the equivariance condition

$$\mathcal{R}_{i(g)} \circ \tilde{e} = \tilde{e} \circ \mathcal{R}_g \quad \text{for all} \quad g \in G. \tag{1}$$

  This is equivalent to asking that the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\tilde{e}} & LM \\ {\scriptstyle \mathcal{R}_g}\downarrow & & \downarrow{\scriptstyle \mathcal{R}_{i(g)}} \\ P & \xrightarrow[\tilde{e}]{} & LM \end{array}$$

- The existence and uniqueness of the map can be guaranteed through the use of the universal property of the quotient for the bundle isomorphism $\pi' \circ \tilde{e}\colon P \to TM$. This is possible thanks to the equivariance condition of $\tilde{e}$. The isomorphism property of the map $e\colon TM \to \mathcal{V}$ is simply inherited from $\tilde{e}$ by passing to the quotient.

[1] Orientability is not necessary (see, e.g., [CCS21a, Section 2.1]), but we assume it here for simplicity of notations.

– Since the map $e\colon TM \to \mathcal{V}$ is an isomorphism of vector bundles, it acts like a linear isomorphism on the fibers. It means it can be written in the following way:

$$\begin{aligned} {}_x e\colon T_x M &\to V \\ v &\mapsto v^a = v^\mu e^a_\mu, \end{aligned} \tag{2}$$

where $v = v^\mu \partial_\mu \in T_x M$. Consider now the dual basis $\{dx^\mu\}$. We can collect the components of the isomorphism into the covector $e^a_\nu dx^\nu(\partial_\mu) = e^a_\mu$, since a covector is a linear map over the tangent space. Given that a basis of the cotangent space can be seen as a family of N covectors $e^a_\mu dx^\mu$ and also that an isomorphism sends a basis to another basis, on a chart over $U \in M$, we can identify the map $e\colon TM \to \mathcal{V}$ with a family of N covector fields or directly with a V-valued covector field in $\Omega^1(U, V)$. Therefore, if M is parallelizable, we can identify the whole map e with a V-valued covector field $e \in \Omega^1(M, V)$.

– The name coframe formalism comes from the fact that e not only defines an isomorphism, but, thanks to the fact that it is obtained from the reduction of the structure group of the frame bundle to the pseudo-orthogonal group $\mathrm{SO}(N-1,1)$, it is also a linear isometry on the fibers. In fact, the reduction to $\mathrm{SO}(N-1,1)$ means by definition that the frames of the frame bundle are orthonormal, namely we have on the fibers $g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab}$. On the other hand, in terms of their dual basis (coframes) $\{e^a\}$, we have $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, which can be written as

$$g = e^* \eta. \tag{3}$$

This means that e is a linear isometry.

Proposition 3. *The inner product on V allows the identification $\mathfrak{so}(N-1,1) \cong \bigwedge^2 V$.*

Because of this proposition, we can identify $\mathfrak{so}(N-1,1)$-valued forms[2] with $\bigwedge^2 \mathcal{V}$-valued forms and we will use the following shortened notation to indicate the spaces of i-forms on M with values in the jth wedge product of $\mathcal{V}$

$$\Omega^{i,j} := \Omega^i\left(M, \bigwedge^j \mathcal{V}\right), \tag{4}$$

which is generalized to all possible $i, j \in \mathbb{N}$.

Remark 4. These spaces form indeed a graded algebra with graded product

$$\begin{aligned} \wedge\colon \Omega^{i,j} \times \Omega^{k,l} &\to \Omega^{i+k, j+l} \qquad \text{for } i+k \le N,\ j+l \le N \\ (\alpha, \beta) &\mapsto \alpha \wedge \beta = (-1)^{(i+j)(k+l)} \beta \wedge \alpha. \end{aligned}$$

We will refer to an alement in $\Omega^{i,j}$ also as an (i,j)-form.

Definition 5. A connection form ω on a principal G-bundle P is a $\mathfrak{g}$-valued 1-form on P such that:

– It is adjoint-equivariant;
– For each $\xi \in \mathfrak{g}$ and fundamental vector field X_ξ, it holds $\omega(X_\xi) = \xi$.

We will refer to the space of principal connections on P as $\mathcal{A}(P)$.

[2]In the sense of a vector bundle with fibers $\mathfrak{so}(N-1,1)$

Remark 6. If we consider a principal connection form on the principal $\mathrm{SO}(N-1,1)$-bundle P, namely an element $\omega \in \Omega^1(P, \bigwedge^2 V)$ (thanks to Proposition 3), we can pull it back using local sections. We will obtain a family of local connections $\omega_\alpha \in \Omega^1(U_\alpha, \bigwedge^2 \mathcal{V})$. These forms define a covariant derivative on M (see Definition 10).

The action of the Lie algebra on $\bigwedge^j \mathcal{V}$-differential forms will be denoted in the following way:

Definition 7. Let $\alpha \in \Omega^{i,j}$ and $\beta \in \Omega^{k,l}$. Then, we define the generalized Lie bracket

$$[\ ,\ ]\colon \Omega^{i,j} \times \Omega^{k,l} \to \Omega^{i+k,j+l-2}$$

through

$$\begin{aligned}&[\alpha,\beta]^{a_1\dots a_{j+l-2}}_{\mu_1\dots\mu_{i+k}} = \\ &= \sum_{\sigma_{i+k}} \sum_{\sigma_{j+l-2}} \mathrm{sign}(\sigma_{i+k})\mathrm{sign}(\sigma_{j+l-2}) \alpha^{a_{\sigma(1)}\dots a_{\sigma(j-1)}a}_{\mu_{\sigma(1)}\dots\mu_{\sigma(i)}} \beta^{a_{\sigma(j)}\dots a_{\sigma(j+l-2)}b}_{\mu_{\sigma(i+1)}\dots\mu_{\sigma(i+k)}} \iota(\rho)_{ab},\end{aligned} \tag{5}$$

where $\iota(\rho)$ is the contraction map that $\bigwedge^m \mathcal{V}$ inherits[3] from the representation ρ of $\mathrm{SO}(N-1,1)$. For the fundamental representation, this map is just the contraction with the η.

Remark 8. Shortly speaking, the brackets act as a wedge product on both space-time and internal indices not contracted with the contraction map.

Remark 9. The contraction map $\iota(\rho)$ is obtained from the representation map of the algebra $\mathrm{d}\rho\colon \mathfrak{so}(N-1,1) \to \mathrm{End}(V)$ composed with the isomorphism of Proposition 3.

Definition 10. Local connections define an exterior covariant derivative for $\bigwedge^j \mathcal{V}$-valued i-forms on M. We denote such a map with

$$d_\omega\colon \Omega^{i,j} \to \Omega^{i+1,j}. \tag{6}$$

Explicitly, it reads

$$d_\omega \alpha = d\alpha + [\omega, \alpha], \tag{7}$$

where $\alpha \in \Omega^{i,j}$.

Remark 11. Note that the representation of the brackets is the fundamental one. This is due to the fact that $\mathcal{V}$ is the associated bundle to P through the fundamental representation. In the case of a different associated bundle, through a different representation, the brackets will be replaced by the given representation.

Definition 12. Let $\omega \in \mathcal{A}(P)$ be a principal connection. Then, the associated local connections on M define a global 2-form $F_\omega \in \Omega^{2,2}$, which satisfies, in any arbitrary trivialization chart (U_α, s_α),

$$F_\omega|_{U_\alpha} = d\omega_\alpha + \frac{1}{2}[\omega_\alpha, \omega_\alpha], \tag{8}$$

with $\omega_\alpha = s_\alpha^* \omega$.

A more detailed derivation of this definition can be found in [Tec19b].

[3]The representation ρ induces an algebra representation $\mathrm{d}\rho$ and we can translate that to $\bigwedge^2 \mathcal{V}$ thanks to Proposition 3. Then, we can easily generalize this action to $\bigwedge^m \mathcal{V}$.

Definition 13. The classical Palatini–Cartan theory is the assignment of the pair $(\mathcal{F}_{PC}, \mathcal{S}_{PC})_M$ to every pseudo riemannian N-dimensional manifold and vector space V with reference metric[4] η with space of fields

$$\mathcal{F}_{PC} = \tilde{\Omega}^{1,1} \times \mathcal{A}(P) \ni (e, \omega) \tag{9}$$

and action functional

$$\mathcal{S}_{PC} = \int_M \frac{1}{(N-2)!} e^{N-2} F_\omega + \frac{1}{N!} \Lambda e^N, \tag{10}$$

where $\Lambda \in \mathbb{R}$ and the powers in e are in terms of the wedge product.

Remark 14. In Eq. (10), we have omitted both the wedge product and the trace. The trace operator is the map $\mathrm{Tr}\colon \bigwedge^N V \to \mathbb{R}$ such that, given a basis $\{u_i\}_{i=1,\dots,N}$ of V, it holds that $\mathrm{Tr}[u_{i_1} \wedge \cdots \wedge u_{i_n}] = \varepsilon_{i_1 \dots i_n}$, thus the trace works as a choice of the orientation on M, which must be compatible with the $\mathrm{SO}(N-1,1)$ reduction.

Remark 15. In the subsequent sections, we will avoid reiterating a similar definition for each distinct case. Rather, we will provide the space of fields on M, and it is important to bear in mind that the definitions of the upcoming theories will be straightforward generalizations of Definition 13.

The Euler–Lagrange equations coming from the action principle $\delta \mathcal{S}_{PC} = 0$ are, respectively for the variations in e and ω, Euler–Lagrange equations of Palatini–Cartan theory read:

$$\frac{1}{(N-3)!} e^{N-3} F_\omega - \frac{1}{(N-1)!} \Lambda e^{N-1} = 0 \tag{11}$$

$$e^{N-3} d_\omega e = 0, \tag{12}$$

which, in $N = 4$, reduce to

$$e F_\omega - \frac{1}{3!} \Lambda e^3 = 0 \tag{13}$$

$$e d_\omega e = 0. \tag{14}$$

By injectivity of the map $e \wedge \cdot$ on $(2,1)$-forms, Eq. (12) is equivalent to

$$d_\omega e = 0, \tag{15}$$

which is the torsion-free condition. Therefore, this is the equation that identifies the Levi-Civita connection for the metric (3).

2.2. **Symplectic reduction on the boundary.** The geometrical method implemented to the study of the boundary structure of the theory is the KT construction described in [KT79].
The construction starts from a space of bulk fields denoted with $\mathcal{F}$ and an action functional of such fields denoted with $\mathcal{S}$. In the case of the Palatini–Cartan theory, these are precisely $\mathcal{F}_{PC}$ and $\mathcal{S}_{PC}$. We notice that the integration by parts in the variation of the action $\mathcal{S}$ gives rise to a boundary term

$$\alpha = \int_\Sigma \frac{1}{(N-2)!} e^{N-2} \delta\omega, \tag{16}$$

which we call the Noether 1-form.

By considering the pull-back of the fields in $\mathcal{F}$ to the boundary Σ via the natural inclusion $i\colon \Sigma \to M$, we obtain the space of pulled-back fields denoted by $\tilde{\mathcal{F}}_\Sigma$. In this setting, the boundary term α defined in (16) can be interpreted as a 1-form on

[4]Note that any particular choice of the Lorentzian structure on V is immaterial, since a change in V would just isomorphically reflect to the space of fields without changing $\mathcal{S}_{PC}$.

the space of pulled-back fields. Furthermore, the variational operator δ is regarded as a de Rham differential of the complex of differential forms on $\tilde{\mathcal{F}}_\Sigma$.

Note that the 2-form on $\tilde{\mathcal{F}}_\Sigma$ defined via

$$\tilde{\varpi} = \delta\alpha = \int_\Sigma \frac{1}{(N-3)!} e^{N-3}\delta e \delta\omega \tag{17}$$

is closed.

Remark 16. It is important to note that a closed 2-form does not necessarily have to be non-degenerate. The form $\tilde{\varpi}$ defined in Eq. (17) may have a non-trivial kernel. This is the case with both the free theory and the all three different couplings analyzed in this paper. A closed 2-form with possibly degenerate kernel is called a pre-symplectic form and a space endowed with such a form is called a pre-symplectic space.

Definition 17. We define the *space of pre-boundary fields* for the Palatini–Cartan theory as the pre-symplectic space $(\tilde{\mathcal{F}}_\Sigma, \tilde{\varpi})$, where $\tilde{\mathcal{F}}_\Sigma$ is the space of pulled-back fields on the boundary Σ and $\tilde{\varpi}$ is the pre-symplectic form.

Remark 18. We will use the following definition for bundle valued differential forms on the boundary

$$\Omega^{i,j}_\Sigma \coloneqq \Omega^i(\Sigma, \textstyle\bigwedge^j i^*\mathcal{V}). \tag{18}$$

As we pointed out in Remark 16, the pre-symplectic form might be indeed not symplectic (it might be degenerate). In order to obtain a symplectic space, we could just quotient by the distribution given by the kernel of the pre-symplectic form.

Definition 19. We define the *geometric phase space* of the theory as the symplectic space $(\mathcal{F}_\Sigma, \varpi)$ obtained as the quotient of the space of pre-boundary fields by the kernel of its pre-symplectic form[5]

$$\mathcal{F}_\Sigma \coloneqq \frac{\tilde{\mathcal{F}}_\Sigma}{\ker(\tilde{\varpi})} \tag{19}$$

and with symplectic form ϖ, the unique 2-form on $\mathcal{F}_\Sigma$ such that $p^*\varpi = \tilde{\varpi}$, where $p\colon \tilde{\mathcal{F}}_\Sigma \to \mathcal{F}_\Sigma$ is the canonical projection.

In field theory, it is commonly understood that not all field equations are dynamical, and on a manifold with boundary, this is equivalent to having some field equations that are non-transverse with respect to the boundary. The resulting non-dynamical equations can be interpreted as constraints that must be satisfied by the boundary fields.
We can give these constraints the form of local functionals on $\mathcal{F}_\Sigma$ (or $\tilde{\mathcal{F}}_\Sigma$), just by integrating the pulled-back equations on Σ. We denote this set of constraints as $\mathcal{C}$ (or $\tilde{\mathcal{C}}$).

The first understanding of the nature of a set of constraints on a symplectic space is due to Dirac [Dir50]. He pointed out correctly that the nature of the constraints, which he divided in first- and second-class, had important implications on the local degrees of freedom of the theory[6]. More precisely, the hamiltonian vector fields of the first-class constraints generate the algebra of the symmetry group of the theory and the ones of the second-class constraints do not. In Section 4, a more detailed discussion of first- and second-class constraints is presented.

[5]This quotient is to be intended in the sense of distributions on the tangent bundle. Note that $\ker(\tilde{\varpi})$ is involutive, since $\tilde{\varpi}$ is closed.

[6]The local degrees of freedom are defined as the dimension of the reduced phase space and the dimension of a space is define as the rank of the fiber or its dimension as a C^∞-module.

The vanishing locus of these integral constraints, quotiented by the action of the algebra generated by their hamiltonian vector fields, is called the *reduced phase space*. Roughly speaking, this is the space of the non-gauge equivalent (thanks to the quotient) initial conditions (the fields are on the boundary) for the dynamical field equations of the theory (since we have considered the vanishing locus of the constraints).

In Table 1, we summarize all the steps to the reduced phase space[7].

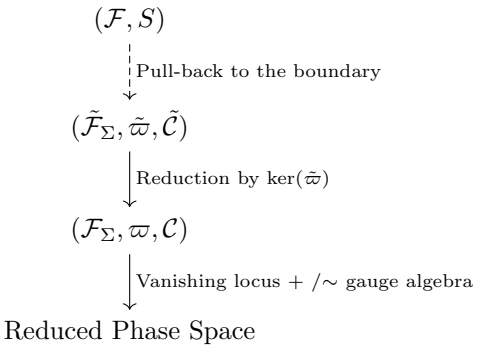

TABLE 1.

2.3. **The structural and degeneracy constraints.** From now on, we will work in $N = 4$.

On the boundary Σ, the injectivity property of the map $e \wedge \cdot$ acting on boundary $(2,1)$-forms is lost.[8] This property guaranteed the equivalence of $d_\omega e = 0$ and $e d_\omega e = 0$ in the bulk. This situation is indeed problematic. In fact, in the bulk we have two perfectly equivalent conditions, namely two equivalent ways of writing one of the field equations. When we pull these back to the boundary, we want this equivalence to hold in order to make sense of the field equations on the boundary themselves. In other words, since in the bulk $e d_\omega e = 0$ must give rise to the same solution space of $d_\omega e = 0$, if the solutions space of these two equations on the boundary were to differ, then the Cauchy problem would be ill-defined. I.e., the pull-back to the boundary of the solutions obtained from the field equations in the bulk would be different from the boundary fields obtained from the solutions of the fields equations on the boundary. It means that one has to impose some additional conditions in order to maintain this equivalence on the boundary. We call part of the family of these extra conditions the *structural constraint*.

This problem is present in both the non-degenerate and degenerate cases; however, the form of the structural constraint strictly depends on the nature of the boundary (null or non-null). In fact, in the non-degenerate case, the structural constraint alone is sufficient to ensure the aforementioned equivalence on the boundary. On the other hand, on a null boundary, the extra conditions split into a structural and a degeneracy constraint. We will see that, from a different perspective, the structural constraint of the non-degenerate case is just a specific characterization of the structural and the degeneracy constraint where the latter is trivial.

Note that the core of this section, as we will mention again later in Remark 24, is maintained rather general, namely independent of the field equations. The application of these results to the Palatini–Cartan theory is, on the one hand, a fundamental building block for the subsequent sections and, on the other hand, a useful way to get a solid grasp on the ideas behind the main results of the section itself.

[7] This table is taken from [CCT21]

[8] See [CS19b].

First, we start by giving some definitions.

Definition 20. Let $e \in \tilde{\Omega}^{1,1}_{\Sigma}$ and $e^k \in \Omega^{k,k}_{\Sigma}$ be the wedge product of k elements e. Then, we define the following maps:

$$\begin{aligned} W_k^{\Sigma,(i,j)} \colon \Omega^{i,j}_{\Sigma} &\longrightarrow \Omega^{i+k,j+k}_{\Sigma} \\ \alpha &\longmapsto e^k \wedge \alpha \end{aligned} \tag{20}$$

$$\begin{aligned} \varrho^{(i,j)} \colon \Omega^{i,j}_{\Sigma} &\longrightarrow \Omega^{i+1,j-1}_{\Sigma} \\ \alpha &\longmapsto [e, \alpha] \end{aligned} \tag{21}$$

$$\begin{aligned} \tilde{\varrho}^{(i,j)} \colon \Omega^{i,j}_{\Sigma} &\longrightarrow \Omega^{i+1,j-1}_{\Sigma} \\ \alpha &\longmapsto [\tilde{e}, \alpha], \end{aligned} \tag{22}$$

with $\tilde{e} \in \tilde{\Omega}^{1,1}_{\Sigma}$ being a degenerate vielbein, namely $\tilde{e}^* \eta = 0$.

We also give the definitions of three geometrical objects that we will require in the following theorems.

Definition 21. Let J be a complement[9] in $\Omega^{2,1}_{\Sigma}$ of the space $\operatorname{Im} \varrho^{(1,2)}|_{\operatorname{Ker} W_1^{\Sigma,(1,2)}}$. Then, we define the following subspaces:

$$\mathcal{T} \coloneqq \operatorname{Ker} W_1^{\Sigma(2,1)} \cap J \subset \Omega^{2,1}_{\Sigma} \tag{23}$$

$$\mathcal{S} \coloneqq \operatorname{Ker} W_1^{\Sigma,(1,3)} \cap \operatorname{Ker} \tilde{\varrho}^{(1,3)} \subset \Omega^{1,3}_{\Sigma} \tag{24}$$

$$\mathcal{K} \coloneqq \operatorname{Ker} W_1^{\Sigma,(1,2)} \cap \operatorname{Ker} \varrho^{(1,2)} \subset \Omega^{1,2}_{\Sigma}. \tag{25}$$

We present the initial key result for the degenerate theory, which will ensure the equivalence between $d_\omega e = 0$ and $e d_\omega e = 0$ at the boundary. While it may appear initially quite redundant with respect to Lemma 66, it will have profound implications for the geometry of the theory, as highlighted in Remark 25.

Lemma 22 (Corollary of Lemma 66)**.** *Let $e_n \in \Omega^{0,1}_{\Sigma}$ be fixed such that, for a chosen vielbein $e \in \tilde{\Omega}^{1,1}_{\Sigma}$, $\{e(v_1), e(v_2), e(v_3), e_n\}$[10] is a basis of $i^*\mathcal{V}$, where $\{v_1, v_2, v_3\}$ is a basis of $T\Sigma$. Moreover, let $\alpha \in \Omega^{2,1}_{\Sigma}$. Then, we have that*

$$\alpha = 0$$

if and only if

$$\begin{cases} \alpha \in \operatorname{Ker} W_1^{\Sigma,(2,1)} \\ e_n(\alpha - p_{\mathcal{T}}\alpha) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_{\mathcal{T}}\alpha = 0, \end{cases} \tag{26}$$

where $p_{\mathcal{T}}$ is the projector onto $\mathcal{T}$. We call the second and third conditions in (26) *respectively the structural and the degeneracy constraints.*

The next lemma provides a formulation of the degeneracy constraint in terms of an integral functional.

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[9]To obtain an explicit expression for the complement, one can follow these steps. Start by selecting an arbitrary Riemannian metric on the boundary manifold Σ and extend it to the space $\Omega^{2,1}$. Then, the orthogonal complement of the image of the map $\varrho^{(1,2)}|\operatorname{Ker} W_1^{\Sigma,(1,2)}$ in $\Omega^{2,1}_{\Sigma}$ can be identified as the space J, with respect to the chosen Riemannian metric.

[10]Notice in particular that, in any neighborhood of e of the space of boundary fields, we are allowed to pick e_n independently of the dynamics of the vielbein e. In other words, we can state that e_n is constant in the field e. This trivially implies that e_n has no variation along e.

Lemma 23. *Let $\alpha \in \Omega^{2,1}_\Sigma$. Then, we have the following equivalence*

$$p_{\mathcal{T}}\alpha = 0 \quad \iff \quad \int_\Sigma \tau\alpha = 0 \quad \forall \tau \in \mathcal{S}. \tag{27}$$

Proof. See [CCT21]. □

Remark 24. As long as we do not specify any α, these two lemmas remain purely geometrical and do not depend on the properties of the field equations. We will then be able to use these results for the interactive theories where the equivalence condition on the boundary will differ from $d_\omega e = 0$ and $e d_\omega e = 0$ (since the field equations will be different themselves). Therefore, in general, we need to specify α for each different theory. In particular, for the Palatini–Cartan theory, $\alpha = d_\omega e$ and the structural and the degeneracy constraints read

$$\begin{cases} e_n(d_\omega e - p_{\mathcal{T}} d_\omega e) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_{\mathcal{T}} d_\omega e = 0. \end{cases} \tag{28}$$

Remark 25. It is important to emphasize that Eq.s (26) are trivially equivalent to the structural constraint

$$e_n \alpha \in \operatorname{Im} W_1^{\Sigma,(1,1)} \tag{29}$$

in the non-degenerate case. Nonetheless, the introduction of this split plays a crucial role in the analysis of the degenerate theory. More specifically, apart from $p_{\mathcal{T}}$ not being trivial, Eq. (29) alone will not be sufficient to uniquely fix a representative of the equivalence class defining the symplectic space (see Theorem 26). In other words, since in the non-degenerate case $p_{\mathcal{T}}\alpha = 0$ holds trivially, we can infer that the second equation in (26) is the most general form of the structural constraint of the theory, whose geometrical implications are only visible in the degenerate case. In fact, the peculiar integral condition of the degenerate case, introduced in Lemma 23, carries significant consequences. It can be interpreted as a modification of the set of constraints of the theory by incorporating a new functional constraint. For $\alpha = d_\omega e$ (the case of the Palatini–Cartan theory), this is denoted as

$$R_\tau = \int_\Sigma \tau d_\omega e. \tag{30}$$

Further discussions of this matter will be presented in the next section.

2.4. **Fixing the representative.** The reduction by the kernel of the presymplectic form, as shown in [CS19b], is equivalent to a quotient space with an equivalence relation on the connection form, as stated in the following theorem.

Theorem 26. *The geometric phase space for the Palatini–Cartan theory is the symplectic manifold $(\mathcal{F}_\Sigma, \varpi)$ given by the following equivalence relation on the space of pre-boundary fields $\tilde{\mathcal{F}}_\Sigma$*

$$\omega' \sim \omega \quad \iff \quad \omega' - \omega \in \operatorname{Ker} W_1^{\Sigma,(1,2)} \tag{31}$$

and the symplectic form

$$\varpi = \int_\Sigma e\delta e\delta[\omega]. \tag{32}$$

We refer to this equivalence class as $\mathcal{A}(i^*P)_{red}$.

Proof. See [CS19b]. □

Remark 27. To study the reduced phase space of the theory, we make use of representatives for the equivalence classes defined in (31). In the non-degenerate case, these representatives are uniquely determined by the structural constraint itself. In other words, ensuring the equivalence of $d_\omega e = 0$ and $ed_\omega e = 0$ on the boundary, is enough to determine uniquely the representatives of the equivalence classes defined in (31). However, in the degenerate case, the structural constraint and the degenerate constraint (or its integral form R_τ), despite the fact that they indeed ensure on the boundary the equivalence mentioned above, are not sufficient to uniquely assign a representative to each equivalence class. Therefore, it is necessary to seek an alternative way to guarantee the unambiguous determination of these representatives.

We can accomplish this through the following lemma.

Lemma 28. *Let i^*g be degenerate. Then, given $\omega \in \Omega^{1,2}_\Sigma$ and $e_n \in \Omega^{0,1}_\Sigma$ as in Lemma 22, the conditions*

$$\begin{cases} e_n(d_\omega e - p_{\mathcal{T}}(d_\omega e)) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_{\mathcal{K}}\omega = 0 \end{cases} \tag{33}$$

uniquely define a representative of the equivalence class $[\omega] \in \mathcal{A}(i^*P)_{red}$.

Proof. See [CCT21]. □

Remark 29. In [CCT21], it has been proved that the analysis is independent of the choice of the representative of the equivalence class (31). In more rigorous terms, for each choice of the representatives there is a canonical symplectomorphism between the symplectic space defined by representatives and the geometric phase space of the theory.

Remark 30. It is important to highlight that, in the non-degenerate case, the subspaces $\mathcal{T}$, $\mathcal{S}$, and $\mathcal{K}$ of Definition 21 are trivial. It follows that the projectors $p_{\mathcal{K}}$ and $p_{\mathcal{T}}$ are also trivial. Once again, this means that, in the non-degenerate theory, the structural constraint alone serves the purpose of establishing the equivalence between $d_\omega e = 0$ and $ed_\omega e = 0$ on the boundary, as well as uniquely determining the representatives of the equivalence classes defined in Eq. (31).

We have seen that, on a null-boundary, we need both the structural and the degeneracy constraints together with the additional equation $p_{\mathcal{K}}\omega = 0$ in order to both guarantee the equivalence between $d_\omega e = 0$ and $ed_\omega e = 0$ on the boundary and uniquely fix the representative of the equivalence class $[\omega] \in \mathcal{A}(i^*P)_{red}$.
More specifically, the role of the structural constraint together with the integral constraint R_τ is the one of ensuring the aforementioned equivalence condition, whereas, the structural constraint together with $p_{\mathcal{K}}\omega = 0$ will uniquely fix the representatives.

We display now the constraints of the theory.

Definition 31. Let[11] $c \in \Omega^{0,2}_\Sigma[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$ and $\tau \in \mathcal{S}[1]$. Then, we define the following functionals

$$L_c = \int_\Sigma ced_\omega e \tag{34}$$

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \iota_\xi(\omega - \omega_0)ed_\omega e \tag{35}$$

$$H_\lambda = \int_\Sigma \lambda e_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3\Big) \tag{36}$$

$$R_\tau = \int_\Sigma \tau d_\omega e. \tag{37}$$

We refer to these as the constraints of the Palatini–Cartan (degenerate) theory.

We are now able to determine the algebra of the constraints of the theory. This differs from the one of the non-degenerate theory, since the new constraint R_τ changes the nature of the Poisson brackets, which become second class.

Theorem 32. *Let i^*g be degenerate. Then the structure of the Poisson brackets of the constraints L_c, P_ξ, H_λ and R_τ is given by the following expressions*

$$\begin{aligned}
&\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]} && \{P_\xi, P_\xi\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}\\
&\{L_c, P_\xi\} = L_{\mathcal{L}^{\omega_0}_\xi c} && \{H_\lambda, H_\lambda\} \approx 0\\
&\{L_c, R_\tau\} = -R_{p_{\mathcal{S}}[c,\tau]} && \{P_\xi, R_\tau\} = R_{p_{\mathcal{S}}\mathcal{L}^{\omega_0}_\xi \tau}.\\
&\{R_\tau, H_\lambda\} \approx G_{\lambda\tau} && \{R_\tau, R_\tau\} \approx F_{\tau\tau}\\
&\{L_c, H_\lambda\} = -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}}\\
&\{P_\xi, H_\lambda\} = P_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(n)}}
\end{aligned}$$

with $X = [c, \lambda e_n]$ and $Y = \mathcal{L}^{\omega_0}_\xi(\lambda e_n)$ and where the superscripts (a) and (n) describe their components with respect to e_a, e_n. Furthermore $F_{\tau\tau}$ and $G_{\lambda\tau}$ are functionals of e, ω, τ and λ that are not proportional to any other constraint.

Remark 33. The symbol $\approx$ indicates the identity on the zero locus of the constraints. In particular, this means that those brackets written with this symbol are not a linear combination of the constraints themselves. On the other hand, all the brackets written with a = vanish on the zero locus, for example $\{L_c, L_c\} \approx 0$.

For the details of the proof and the definition of first and second class constraints, we refer to [CCT21].

Remark 34. The former results still hold in the case where the boundary metric has some extra degeneracy, apart from the one coming from the nature of the null-boundary. In the upcoming sections, we will, for the sake of simplicity, assume that the restriction of the metric, excluding the degenerate direction of the boundary, remains non-degenerate.

Remark 35. The distinctive feature of the degenerate theory, highlighted in [CCT21] and summarized in Section 4, is that the additional constraint R_τ turn out to be second-class (see Definition 62). As discussed in Section 4, this implies that the requirement to uniquely determine a representative for $[\omega]$ results in the reduced phase space of the theory having a dimension of two, compared to the non-degenerate theory, which had a dimension of four.

[11] The notation [1] indicates a shift in parity.

The first step in each of the following sections will be the one of finding the correct set of equations as a choice for the structural constraint. Then, we will find the relations for uniquely fixing the representatives. Once established the correct geometrical set-up, we will proceed by computing the algebra of the constraints of the theory at hand. This will be done in the cases of scalar, SU(n) and spinor couplings.

3. Coupling terms: degenerate structure

3.1. **Scalar field.** In the following section, we will dive into the case of the scalar coupling. We stress that we refer to [CCF22] for what concerns the non-degenerate case.
In the canonical formalism, the Palatini–Cartan theory coupled with a scalar field maintains the very same geometrical background of the previous sections with the addition of two new fields, the scalar field ϕ and its conjugate momentum (upon equation of motion) Π.

We must define the building blocks of our scalar Palatini–Cartan theory, starting with the space of fields on M, which reads

$$\mathcal{F}^{\phi} = \tilde{\Omega}^{1,1} \times \mathcal{A}(P) \times C^{\infty}(M) \times \Omega^{0,1} \ni (e, \omega, \phi, \Pi), \tag{38}$$

and the action functional

$$\mathcal{S}^{\phi} = \mathcal{S}_{PC} + \int_M \frac{1}{6} e^3 \Pi d\phi + \frac{1}{48} e^4(\Pi, \Pi), \tag{39}$$

where the brackets indicates the inner product of the Minkowski bundle. It follows that the Euler-Lagrange equations of the theory are given by

$$e d_{\omega} e = 0 \tag{40}$$

$$e F_{\omega} + \frac{\Lambda}{6} e^3 + \frac{1}{2} e^2 \Pi d\phi + \frac{1}{12} e^3(\Pi, \Pi) = 0 \tag{41}$$

$$d(e^3 \Pi) = 0 \tag{42}$$

$$e^3(d\phi - (e, \Pi)) = 0. \tag{43}$$

The variation of the action leads to the following Noether 1-form on the space of pre-boundary fields[12]

$$\tilde{\alpha} = \int_{\Sigma} \frac{1}{2} e^2 \delta\omega + \frac{1}{6} e^3 \Pi \delta\phi, \tag{44}$$

which gives rise to the following pre-symplectic form

$$\tilde{\varpi} = \delta\tilde{\alpha} = \int_{\Sigma} e \delta e \delta\omega + \frac{1}{6} \delta(e^3 \Pi) \delta\phi. \tag{45}$$

Similarly to the previous section, we can define the space of pre-boundary fields $\tilde{\mathcal{F}}^{\phi}_{\Sigma}$, as in Definition 17 for the Palatini–Cartan theory, by pulling back the fields to the boundary Σ. Also in this case, we will write the fields on the boundary with the same letters as for those in the bulk.

As shown in [CCF22], we are now able to define the geometric phase space of the theory via a reduction through the kernel of the pre-symplectic form.
Here is a generalization of Theorem 26.

[12]Note that we keep the same notation for the Noether and the pre-symplectic forms of the previous section even though these are different.

Theorem 36. *The geometric phase space for the scalar Palatini–Cartan theory is the symplectic manifold $(\mathcal{F}^{\phi}_{\Sigma}, \varpi)$ given by the following equivalence relations on the space of pre-boundary fields $\tilde{\mathcal{F}}^{\phi}_{\Sigma}$*

$$\omega' \sim \omega \quad \iff \quad \omega' - \omega \in \mathrm{Ker} W_1^{\Sigma,(1,2)} \tag{46}$$

$$\Pi' \sim \Pi \quad \iff \quad \Pi' - \Pi \in \mathrm{Ker} W_3^{\Sigma,(0,1)} \tag{47}$$

and the symplectic form

$$\varpi = \int_{\Sigma} e\delta e\delta[\omega] + \frac{1}{6}\delta(e^3[\Pi])\delta\phi. \tag{48}$$

We refer to these equivalence classes as $\mathcal{A}(i^*P)_{red}$ and $\Omega^{0,1}_{\Sigma,red}$.

Proof. See [CCF22]. □

We notice that the first field equation (41) does not couple with the scalar field. Therefore, since this purely geometrical term is equivalent to the one of the Palatini–Cartan theory (namely $\alpha = d_\omega e$), the structural and degeneracy constraints possess the same form of the free theory. In fact, as we said, they serve the purpose of maintaining the equivalence between $ed_\omega e = 0$ and $d_\omega e = 0$ on the boundary. We recall here the aforementioned constraints, which thus read

$$\begin{cases} e_n(d_\omega e - p_{\mathcal{T}} d_\omega e) \in \mathrm{Im}\, W_1^{\Sigma,(1,1)} \\ p_{\mathcal{T}} d_\omega e = 0. \end{cases} \tag{49}$$

Similarly to the Palatini–Cartan theory, we focus on fixing the representative of the equivalence classes defined in Theorem 36. The purely gravitational part remains the same, since it follows uniquely from the kernel of the piece of (45) equal to the pre-symplectic form of the free Palatini–Cartan theory. In other words, since in the present case the equivalence class $[\omega]$ is defined in the same way of the Palatini–Cartan theory, as well as the structural constraint, it follows that Lemma 28 applies verbatim to the scalar field theory.
Although, we are left to fix the representative of the equivalence class for Π. For this purpose, we give the following lemma.

Lemma 37. *Let i^*g be degenerate. Then, given $\phi \in C^\infty(\Sigma)$, $\Pi \in \Omega^{0,1}_\Sigma$ and $e_n \in \Omega^{0,1}_\Sigma$ as in Lemma 22, the conditions*

$$\begin{cases} d\phi - (e, \Pi) = 0 \\ p_W \Pi = 0, \end{cases} \tag{50}$$

with[13] *$W = e(\mathrm{Ker}(i^*g))$, uniquely define a representative of the equivalence class $[\Pi] \in \Omega^{0,1}_{\Sigma,red}$.*

Proof. We first notice that, if we consider a vector field along the degenerate direction, namely $X \in \mathrm{Ker}(i^*g)$, and we take the contraction of the field equations with it, we obtain the condition

$$\iota_X d\phi = 0. \tag{51}$$

[13]Here, we regard the boundary metric as a map $i^*g \colon T\Sigma \to T^*\Sigma$ and therefore we have that $\mathrm{Ker}(i^*g) = \{\xi \in \mathfrak{X}(\Sigma) \mid \iota_\xi(i^*g) = 0\} \subset \mathfrak{X}(\Sigma)$.

What happens is that the degeneracy in the boundary metric decouples ϕ and Π along the degenerate direction[14] and this is precisely why we need, compared to the non-degenerate case, an extra condition in order to fix the representative of the equivalence class of Π.

We can decompose the field $\Pi \in \Omega^{0,1}_{\Sigma}$ in the following way[15]

$$\Pi = \pi^n e_n + \pi^a e_a, \tag{52}$$

with $a = 1, 2, 3$. Then, we notice that, thanks to definition of the wedge product, $e^3 e_a = 0$ for every a and therefore $\pi = \pi^a e_a \in \mathrm{Ker} W_3^{\Sigma,(0,1)}$. This means that fixing a certain π^n uniquely defines an equivalence class $[\Pi] \in \Omega^{0,1}_{\Sigma,red}$ and vice versa. We are thus left to show that the conditions (50) fix also uniquely $\pi = \pi^a e_a$, as a function of π^n. Now, we recall that $\dim(\mathrm{Ker}(i^* g)) = 1$ and e is injective and, therefore, we have that $W \subset e(T\Sigma)$ is a 1-dimensional subspace. Furthermore, for any open neighbourhood of $e(T\Sigma)$, without loss of generality, we can assume that the basis given by the vielbein $\{e_1, e_2, e_3\}$ is such that, say, e_3 spans W. From Eq. (52), it follows that the condition

$$p_W \Pi = 0 \tag{53}$$

implies

$$\pi^n e^3_n + \pi^3 = 0. \tag{54}$$

Moreover, with such a choice of basis, we can write the exterior derivative of the scalar field as

$$d\phi = \partial_i \phi dx^i = e^a_1 \partial_a \phi dx^1 + e^a_2 \partial_a \phi dx^2, \tag{55}$$

where we implemented the condition $\iota_X d\phi = 0$, which reads $e^a_3 \partial_a \phi = 0$. Lastly, we can write the field equations implementing Eq. (55), obtaining

$$\begin{aligned} d\phi - (e, \Pi) &= \partial_i \phi dx^i - (e^a_i dx^i e_a, \pi^b e_b + \pi^n e_n) && (56)\\ &= e^a_i \partial_a \phi dx^i - e^a_i \pi^b g_{ab} dx^i - e^a_i \pi^n g_{an} dx^i && (57)\\ &= e^a_i (\partial_a \phi - \pi^b g_{ab} - \pi^n g_{an}) dx^i = 0, && (58) \end{aligned}$$

where g is the metric and $b = 1, 2$. Since the restricted inner product is non-degenerate, we have

$$\partial_a \phi - \pi^b g_{ab} - \pi^n g_{an} = 0 \tag{59}$$

and thus we deduce the following equation for π^b (with $b = 1, 2$)

$$\pi^b = g^{ab} (\partial_a \phi - \pi^n g_{an}). \tag{60}$$

It follows that Eqs. (54) and (60) completely fix the components of π in terms of π^n. Hence, since fixing π^n is equivalent to fixing a representative for $[\Pi]$ and vice versa, we have that, given an equivalence class (or equivalently a π^n), the conditions (50) fix uniquely the representative of $[\Pi]$. On the other hand, given a representative, the conditions (50) fix unambiguously a π^n and therefore an equivalence class $[\Pi]$. □

[14] This gives a condition on the derivative of ϕ. More specifically, the degeneracy of the boundary metric complicates the selection of the components of the fields in the orthogonal direction to the boundary. This implies that we could potentially have some spurious components of the field ϕ generating the diffeomorphisms along the orthogonal direction. Therefore, we can interpret the condition of Eq. (51) as a geometrical constraint that selects the only component of the symmetry transformations orthogonal to the boundary which are actually generated by the Hamiltonian vector field h^{ϕ}_{λ} of Eq. (69). In other words, one could say that Eq. (51) selects the "physically meaningful" components of the derivative of the field ϕ.

[15] We take the basis of $i^*\mathcal{V}$ given by the vielbein and the completion e_n. Notice that, as a section of $i^*\mathcal{V}$, e_n will have components along the vielbein in general.

We have uniquely determined the representatives for the equivalence classes that define the symplectic space of boundary fields. As a result, we can now write the set of constraints of the theory as functionals of the representatives themselves.

Definition 38. Let $c \in \Omega^{0,2}_\Sigma[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$ and $\tau \in \mathcal{S}[1]$. Then, we define the following functionals

$$L_c = \int_\Sigma ced_\omega e \tag{61}$$

$$P^\phi_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \frac{1}{3!}\iota_\xi(e^3\Pi)d\phi + \iota_\xi(\omega-\omega_0)ed_\omega e \tag{62}$$

$$H^\phi_\lambda = \int_\Sigma \lambda e_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3 + \frac{1}{2}e^2\Pi d\phi + \frac{1}{2\cdot 3!}e^3(\Pi,\Pi)\Big) \tag{63}$$

$$R_\tau = \int_\Sigma \tau d_\omega e. \tag{64}$$

We refer to these as the constraints of the scalar Palatini–Cartan theory.

In the following theorem, we give the form of the Poisson brackets determining the constraint algebra of the theory.

Theorem 39. *Let i^*g be degenerate. Then the Poisson brackets of the constraints of Definition 38 read*

$$\begin{aligned}
&\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]} && \{P^\phi_\xi, P^\phi_\xi\} = \frac{1}{2}P^\phi_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}\\
&\{L_c, P^\phi_\xi\} = L_{\mathcal{L}^{\omega_0}_\xi c} && \{H^\phi_\lambda, H^\phi_\lambda\} \approx 0\\
&\{L_c, R_\tau\} = -R_{p_{\mathcal{S}}[c,\tau]} && \{P^\phi_\xi, R_\tau\} = R_{p_{\mathcal{S}}\mathcal{L}^{\omega_0}_\xi\tau}\\
&\{R_\tau, H^\phi_\lambda\} \approx G_{\lambda\tau} && \{R_\tau, R_\tau\} \approx F_{\tau\tau}\\
&\{L_c, H^\phi_\lambda\} = -P^\phi_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H^\phi_{X^{(n)}}\\
&\{P^\phi_\xi, H^\phi_\lambda\} = P^\phi_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H^\phi_{Y^{(n)}},
\end{aligned}$$

with $X = [c, \lambda e_n]$ and $Y = \mathcal{L}^{\omega_0}_\xi(\lambda e_n)$ and where the superscripts (a) and (n) describe their components with respect to e_a, e_n. Furthermore, $F_{\tau\tau}$ and $G_{\lambda\tau}$ are functionals of e, ω, τ and λ which are not proportional to any other constraint.[16]

Proof. First, we introduce the following notation[17]

$$P^\phi_\xi = P_\xi + p^\phi_\xi \qquad\qquad H^\phi_\lambda = H_\lambda + h^\phi_\lambda, \tag{65}$$

in order to simplify the computations.

[16] They are properly defined in [CCT21] (proof of Theorem 30).
[17] With P_ξ and H_λ of Definition 31.

In accordance with the results from [CCT21] and [CCF22], we possess knowledge of the some of the brackets as follows

$$\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]} \qquad \{L_c, P^\phi_\xi\} = L_{\mathcal{L}^{\omega_0}_\xi c}$$
$$\{P^\phi_\xi, P^\phi_\xi\} = \frac{1}{2}P^\phi_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi \iota_\xi F_{\omega_0}} \qquad \{L_c, R_\tau\} = -R_{p_{\mathcal{S}}[c,\tau]}$$
$$\{P_\xi, R_\tau\} = R_{p_{\mathcal{S}}\mathcal{L}^{\omega_0}_\xi \tau} \qquad \{R_\tau, H_\lambda\} \approx G_{\lambda\tau}$$
$$\{R_\tau, R_\tau\} \approx F_{\tau\tau} \qquad \{H^\phi_\lambda, H^\phi_\lambda\} \approx 0$$
$$\{L_c, H^\phi_\lambda\} = -P^\phi_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H^\phi_{X^{(n)}}$$
$$\{P^\phi_\xi, H^\phi_\lambda\} = P^\phi_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H^\phi_{Y^{(n)}}$$

with F and G non-identically-vanishing functional of τ and λ defined in [CCT21] (Theorem 30) and $X = [c, \lambda e_n] \in \Omega^{0,1}_\Sigma$ divided into a tangential component $X^{(a)} = [c, \lambda e_n]^{(a)}$ and a normal component $X^{(n)} = [c, \lambda e_n]^{(n)}$. We are therefore left to compute the brackets $\{R_\tau, h^\phi_\lambda\}$ and $\{R_\tau, p^\phi_\xi\}$. Also, we can recall the known results from [CCT21] and [CCT21] for what concerns all Hamiltonian vector fields. In particular, for p^ϕ_ξ, we have

$$\mathbb{p}^\phi_e = 0 \qquad \mathbb{p}^\phi_\omega = 0 \tag{66}$$
$$\mathbb{p}^\phi_\rho = -\mathcal{L}^{\omega_0}_\xi \rho \qquad \mathbb{p}^\phi_\phi = -\xi(\phi), \tag{67}$$

whereas, for h^ϕ_λ, we have

$$\mathbb{h}^\phi_e = 0 \qquad \mathbb{h}^\phi_\omega = \lambda e_n\big(\Pi d\phi + \frac{e}{4}(\Pi,\Pi)\big) - \frac{\lambda}{2}e\Pi(\Pi, e_n) \tag{68}$$
$$\mathbb{h}^\phi_\rho = \frac{1}{2}d_\omega(\lambda e_n e^2\Pi) \qquad \mathbb{h}^\phi_\phi = -\lambda(e_n, \Pi), \tag{69}$$

where we have defined a new field $\rho \coloneqq \frac{1}{3!}e^3\Pi \in \Omega^{3,4}_\Sigma$.

Next, it is helpful to write explicitly the variation[18] of R_τ, which reads[19]

$$\delta R_\tau = \int_\Sigma \delta\tau d_\omega e - \tau[\delta\omega, e] + \tau d_\omega \delta e \tag{70}$$
$$= \int_\Sigma (g(\tau,\omega,e) + d_\omega\tau)\delta e + [\tau, e]\delta\omega, \tag{71}$$

where we have introduced the formal expression $g = g(\tau, e, \omega)$ which encodes the dependence of τ on e (see [CCT21] Theorem 30 for further details). It follows that the Hamiltonian vector fields are

$$e\mathbb{R}_e = [\tau, e] \qquad e\mathbb{R}_\omega = g(\tau,\omega,e) + d_\omega\tau \tag{72}$$
$$\mathbb{R}_\rho = 0 \qquad \mathbb{R}_\phi = 0. \tag{73}$$

Now that we possess all Hamiltonian vector fields, we are ready to compute the Poisson brackets of the remaining constraints. First, we notice that

$$\{R_\tau, p^\phi_\xi\} = 0 \tag{74}$$

[18]We compute the Hamiltonian vector fields in the following manner. Let $\mathbb{X}$ be the Hamiltonian vector field of the functional F for the symplectic form ϖ, then it holds $\iota_{\mathbb{X}}\varpi - \delta F = 0$, where δF is the functional derivative of F.

[19]Since τ is defined on $\mathcal{S}$ and the latter is defined making use of e, it follows that τ has a non-trivial variation along e.

since p^{ϕ}_{ξ} has trivial Hamiltonian vector fields along e or ω. Then, we compute

$$\{R_\tau, h^{\phi}_{\lambda}\} = \int_\Sigma -\lambda e_n \Pi d\phi[\tau, e] - \frac{e}{4}\lambda e_n(\Pi,\Pi)[\tau, e] + \frac{\lambda}{2} e\Pi(\Pi, e_n)[\tau, e]. \tag{75}$$

Here, the last two terms are zero thanks to $e[\tau, e] = 0$ (following from $e\tau = 0$ in the definition of $\mathcal{S}$). For the term bracket, we have

$$\begin{aligned}
\int_\Sigma -\lambda e_n \Pi d\phi[\tau, e] &= \int_\Sigma -\lambda \Pi d\phi \tau[e_n, e] && (76)\\
&= \int_\Sigma -\lambda \Pi d\phi \tau(e_n, e) && (77)\\
&\overset{(50)}{=} \int_\Sigma -\lambda \Pi(e, \Pi)\tau(e_n, e) && (78)\\
&= \int_\Sigma -\lambda \Pi(e, \Pi) e_n[\tau, e] && (79)\\
&= 0, && (80)
\end{aligned}$$

where we implemented the Leibniz identity for the squared brackets, the definition of $\tau \in \mathcal{S}$ and Proposition 71, thanks to the fact that there are no derivatives in the integral[20].

Finally, in order to complete the proof, we can simply exploit the linearity of the Poisson brackets and recall the definition of the split introduced in Eq. (65) together with the known results mentioned above. □

3.2. **Yang–Mills field.** In this section, we will examine the case of an SU(n)-gauge-field[21], namely a principal connection A of a principal SU(n)-bundle over M denoted with P (see [Tec19b] Section 5). It follows that the space of gauge fields is locally modelled on $\Omega^1(M, \mathfrak{su}(n))$, via the pull-backs along the sections of G. In the Standard Model of particle physics, this kind of field is responsible for the mediation of a variety of interactions, in particular, the Electroweak and the Strong interaction. Moreover, similarly to what we did in the previous section, we associate to the gauge field[22] A an independent field $B \in \Gamma\big(\bigwedge^2 \mathcal{V} \otimes \mathfrak{su}(n)\big)$.

Hence, the Yang–Mills–Palatini–Cartan theory is defined by the following space of fields

$$\mathcal{F}^A = \tilde{\Omega}^{1,1} \times \mathcal{A}(P) \times \mathcal{A}(G) \times \Gamma\big(\textstyle\bigwedge^2 \mathcal{V} \otimes \mathfrak{su}(n)\big) \ni (e, \omega, A, B), \tag{81}$$

and the action functional

$$\mathcal{S}^A = \mathcal{S}_{PC} + \int_M \frac{1}{4} e^2 \mathrm{Tr}(BF_A) + \frac{1}{46!} e^4 \mathrm{Tr}(B, B), \tag{82}$$

where $\Omega^2(M, \mathfrak{su}(n)) \ni F_A = dA + \frac{1}{2}[A, A]$ is the field strength, $(\ ,\ )$ is the canonical pairing in $\bigwedge^2 \mathcal{V}$ and $\mathrm{Tr}\colon \mathfrak{su}(n) \to \mathbb{R}$ is the trace over the algebra.

[20] Roughly speaking, we can "diagonalize" the vielbein.

[21] All the considerations below work with a general Lie algebra $\mathfrak{g}$.

[22] Note that we refer to both A and its pull-back as the gauge field.

The Euler-Lagrange equations are as follows

$$d_\omega e = 0 \tag{83}$$

$$e(F_\omega + \mathrm{Tr}(BF_A)) + \frac{e^3}{6}(\Lambda + \frac{1}{2}\mathrm{Tr}(B,B)) = 0 \tag{84}$$

$$e^2\big(F_A + \frac{1}{2}(e^2, B)\big) = 0 \tag{85}$$

$$d_A(e^2 B) = 0, \tag{86}$$

whereas the Noether 1-form becomes

$$\tilde{\alpha} = \int_\Sigma \frac{e^2}{2}\delta\omega + \frac{e^2}{2}\mathrm{Tr}(B\delta A). \tag{87}$$

It follows that the pre-symplectic form of the theory is

$$\tilde{\varpi} = \delta\tilde{\alpha} = \int_\Sigma e\delta e\delta\omega + \mathrm{Tr}(eB\delta e\delta A) + \frac{1}{2}\mathrm{Tr}(e^2\delta B\delta A). \tag{88}$$

This is a 2-form over the space of pre-boundary fields obtained as the pull-back of bulk fields along $i\colon \Sigma \to M$ and denoted in this case as $\tilde{\mathcal{F}}^A_\Sigma$. Notice that, also in this case, we refer to boundary fields with the same notation of bulk fields.

Theorem 40. *The geometric phase space for the Yang–Mills–Palatini–Cartan theory is the symplectic manifold $(\mathcal{F}^A_\Sigma, \varpi)$ given by the following equivalence relations on the space of pre-boundary fields $\tilde{\mathcal{F}}^A_\Sigma$*

$$\omega' \sim \omega \quad \Longleftrightarrow \quad \omega' - \omega \in \mathrm{Ker} W_1^{\Sigma,(1,2)} \tag{89}$$

$$B' \sim B \quad \Longleftrightarrow \quad B' - B \in \mathrm{Ker} W_2^{\Sigma,(0,2*)}, \tag{90}$$

where $2*$ indicates that the $\bigwedge^2 \mathcal{V}$-algebra is tensored with $\mathfrak{su}(n)$, and the symplectic form

$$\varpi = \int_\Sigma e\delta e\delta[\omega] + \frac{1}{2}\mathrm{Tr}(\delta(e^2[B])\delta A). \tag{91}$$

We refer to these equivalence classes as $\mathcal{A}(i^*P)_{red}$ and $\Gamma\big(\bigwedge^2 i^*\mathcal{V} \otimes \mathfrak{su}(n)\big)_{red}$.

Proof. See [CCF22]. □

Remark 41. In the context of the Yang–Mills–Palatini–Cartan theory, we can indeed establish unique representatives for these equivalence classes. Subsequently, we can proceed to formulate the constraints in a manner analogous to the approach we previously employed in the preceding section. The representative for $[\omega] \in \mathcal{A}(i^*P)_{red}$ is already uniquely fixed thanks to equivalent considerations to the ones articulated in the previous sections. Therefore, the structural and the degeneracy constraints for the Yang–Mills–Palatini–Cartan theory read

$$\begin{cases} e_n(d_\omega e - p_{\mathcal{T}} d_\omega e) \in \mathrm{Im}\, W_1^{\Sigma,(1,1)} \\ p_{\mathcal{T}} d_\omega e = 0. \end{cases} \tag{92}$$

We are therefore left with the problem of the representative for $[B] \in \Gamma\big(\bigwedge^2 i^*\mathcal{V} \otimes \mathfrak{su}(n)\big)_{red}$, which is determined by the following lemma.

Lemma 42. *Let i^*g be degenerate. Then, given $A \in \mathcal{A}(i^*G)$, $B \in \Gamma\big(\bigwedge^2 i^*\mathcal{V} \otimes \mathfrak{su}(n)\big)$ and $e_n \in \Omega^{0,1}_\Sigma$ as in Lemma 22, the conditions*

$$\begin{cases} F_A + \frac{1}{2}(e^2, B) = 0 \\ p_{\Omega^{0,1*}_e \wedge W} B = 0, \end{cases} \tag{93}$$

with $\Omega_e^{i,j*} \coloneqq \Omega^i\big(\Sigma, \bigwedge^j e(T\Sigma) \otimes \mathfrak{su}(n)\big)$ *where* $W = e(\mathrm{Ker}(i^*g))$*, uniquely define a representative of the equivalence class* $[B] \in \Gamma\big(\bigwedge^2 i^*\mathcal{V} \otimes \mathfrak{su}(n)\big)_{red}$.

Proof. We can decompose an element[23] $B \in \Gamma\big(\bigwedge^2 i^*\mathcal{V} \otimes \mathfrak{su}(n)\big)$ as[24]

$$B = b^{an} e_a e_n + \frac{1}{2} b^{ab} e_a e_b, \tag{94}$$

with $b^{an}, b^{ab} \in \Gamma(\mathfrak{su}(n))$ and $a, b = 1, 2, 3$. We notice that $b = b^{ab} e_a e_b \in \mathrm{Ker}(W_2^{\Sigma,(0,2*)})$, since $e^2 e_a e_b = 0$ for all $a, b = 1, 2, 3$. This directly implies that the components b^{an} are already uniquely determined by the equivalence class $[B]$ and vice versa. Now, as we did in the proof of Lemma 37, we observe that $\dim(\mathrm{Ker}(i^*g)) = 1$ and e is injective and, therefore, we have that $W \subset e(T\Sigma)$ is a 1-dimensional subspace. Hence, for any open neighbourhood of $e(T\Sigma)$, without loss of generality, we can take as a basis of $e(T\Sigma)$ the one given by the $\{e_1, e_2, e_3\}$ such that e_3 spans W. Then, since a basis of $\Omega_e^{0,1*} \wedge W$ is given by $\{e_1 e_3, e_2 e_3\} \otimes \mathfrak{su}(n)$, we have that, similarly to the scalar case, we first notice that the field equations imply the condition $\iota_X F_A = 0$, with $X \in \mathrm{Ker}(i^*g)$. Moreover, the condition $p_{\Omega_e^{0,1*} \wedge W} B = 0$ implies that

$$2b^{[1n} e_n^{3]} + b^{13} = 2b^{[2n} e_n^{3]} + b^{23} = 0, \tag{95}$$

where the square brackets in the indices denote the anti-symmetrization.

Next, consider the condition $\iota_X F_A = 0$. Then, we can write

$$F_A = \frac{1}{2} F_{ab} e_i^a e_j^b dx^i dx^j = \frac{1}{2} F_{12} dx^1 dx^2. \tag{96}$$

Furthermore, similarly to the preceding case, we can write

$$2F_A + (e^2, B) = \tag{97}$$

$$= F_{ij} dx^i dx^j + (\frac{1}{2} e_i^a e_j^b dx^i dx^j e_a e_b, b^{cd} e_c e_d + b^{cn} e_c e_n) \tag{98}$$

$$= F_{ab} e_i^a e_j^b dx^i dx^j + b^{cd} e_i^a e_j^b g_{ac} g_{bd} dx^i dx^j + b^{cn} e_i^a e_j^b g_{ac} g_{bn} dx^i dx^j \tag{99}$$

$$= e_i^a e_j^b (F_{ab} + b^{cd} g_{ac} g_{bd} + b^{cn} g_{ac} g_{bn}) dx^i dx^j = 0. \tag{100}$$

We observe that, since the restricted inner product is non-degenerate, we have

$$F_{ab} + b^{cd} g_{ac} g_{bd} + b^{cn} g_{ac} g_{bn} = 0 \tag{101}$$

and, given $a, b, c, d \neq 3$, we can use the inverse metric to write

$$b^{cd} = -(g^{ac} g^{bd} F_{ab} + g^{bd} g_{bn} b^{cn}). \tag{102}$$

This result together with Eq. (95) fixes uniquely the elements b^{ab} in terms of b^{an} (with $a, b = 1, 2, 3$). The completion of the proof follows from analogous considerations to the ones of the scalar case in the previous section. □

We are now able to give the definition of the constraints of the theory.

[23] We can consider the basis for $e(T\Sigma)$ given by the vielbein. See the proof of Lemma 37 for more details.

[24] Apart from the wedge product, in order to lighten the notation, we also omit the tensor product.

Definition 43. Let $c \in \Omega^{0,2}_\Sigma[1]$, $\mu \in C^\infty(\Sigma, \mathfrak{g})[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$ and $\tau \in \mathcal{S}[1]$. Moreover, let $\rho = e^2 B \in \Omega^{2,4*}_\Sigma$. Then, we define the following functionals

$$L_c = \int_\Sigma c e d_\omega e \tag{103}$$

$$M_\mu = \int_\Sigma \mathrm{Tr}(\mu d_A \rho) \tag{104}$$

$$P^A_\xi = \int_\Sigma \frac{1}{2}\iota_\xi e^2 F_\omega + \iota_\xi(\omega - \omega_0) e d_\omega e + \frac{1}{2}\mathrm{Tr}(\iota_\xi \rho F_A) \tag{105}$$

$$+ \mathrm{Tr}\big(\iota_\xi (A - A_0) d_A \rho\big) \tag{106}$$

$$H^A_\lambda = \int_\Sigma \lambda e_n \Big(e F_\omega + \frac{\Lambda}{3!} e^3 + e\mathrm{Tr}(BF_A) + \frac{1}{2\cdot 3!} e^3 \mathrm{Tr}(B,B)\Big) \tag{107}$$

$$R_\tau = \int_\Sigma \tau d_\omega e. \tag{108}$$

We refer to these as the constraints of the Yang–Mills-Palatini–Cartan theory.

Theorem 44. *Let i^*g be degenerate. Then the Poisson brackets of the constraints of Definition 43 read*

$$\begin{aligned}
&\{L_c, L_c\} = -\frac{1}{2} L_{[c,c]} && \{M_\mu, M_\mu\} = -\frac{1}{2} M_{[\mu,\mu]} \\
&\{L_c, P^A_\xi\} = L_{\mathcal{L}^{\omega_0}_\xi c} && \{H^A_\lambda, H^A_\lambda\} \approx 0 \\
&\{L_c, M_\mu\} = 0 && \{P_\xi, M_\mu\} = M_{\mathcal{L}^{A_0}_\xi \mu} \\
&\{H^A_\lambda, M_\mu\} = 0 && \{R_\tau, M_\mu\} = 0 \\
&\{L_c, R_\tau\} = -R_{p_{\mathcal{S}}[c,\tau]} && \{P^A_\xi, R_\tau\} = R_{p_{\mathcal{S}}\mathcal{L}^{\omega_0}_\xi \tau} \\
&\{R_\tau, H^A_\lambda\} \approx G_{\lambda\tau} + K^A_{\lambda\tau} && \{R_\tau, R_\tau\} \approx F_{\tau\tau} \\
&\{P^A_\xi, P^A_\xi\} = \frac{1}{2} P^A_{[\xi,\xi]} - \frac{1}{2} L_{\iota_\xi \iota_\xi F_{\omega_0}} - \frac{1}{2} M_{\iota_\xi \iota_\xi F_{\omega_0}} \\
&\{L_c, H^A_\lambda\} = -P^A_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H^A_{X^{(n)}} \\
&\{P^A_\xi, H^A_\lambda\} = P^A_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H^A_{Y^{(n)}},
\end{aligned}$$

with $X = [c, \lambda e_n]$ and $Y = \mathcal{L}^{\omega_0}_\xi(\lambda e_n)$ and where the superscripts (a) and (n) describe their components with respect to e_a, e_n. Furthermore, $F_{\tau\tau}$, $G_{\lambda\tau}$ and $K^A_{\lambda\tau}$ are functional of e, ω, A, B, τ and λ defined in the proof[25] which are not proportional to any other constraint.

Proof. Similarly to the proof of Theorem 39, we will introduce now a split in some of the constraints. In this case, we have

$$P^A_\xi = P_\xi + p^A_\xi \qquad\qquad H^A_\lambda = H_\lambda + h^A_\lambda. \tag{109}$$

[25] F and G are properly defined in [CCT21] (proof of Theorem 30).

Moreover, from [CCT21] and [CCF22], we have knowledge of the following brackets

$$\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]} \qquad \{L_c, P^A_\xi\} = L_{\mathcal{L}^{\omega_0}_\xi c}$$
$$\{P^A_\xi, P^A_\xi\} = \frac{1}{2}P^A_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi \iota_\xi F_{\omega_0}} \qquad \{L_c, R_\tau\} = -R_{p_{\mathcal{S}}[c,\tau]}$$
$$\{P_\xi, R_\tau\} = R_{p_{\mathcal{S}}\mathcal{L}^{\omega_0}_\xi \tau} \qquad \{R_\tau, H_\lambda\} \approx G_{\lambda\tau}$$
$$\{R_\tau, R_\tau\} \approx F_{\tau\tau} \qquad \{H^A_\lambda, H^A_\lambda\} \approx 0$$
$$\{M_\mu, M_\mu\} = -\frac{1}{2}M_{[\mu,\mu]} \qquad \{L_c, M_\mu\} = 0$$
$$\{P^A_\xi, M_\mu\} = M_{\mathcal{L}^{A_0}_\xi \mu} \qquad \{M_\mu, H^A_\lambda\} = 0$$
$$\{L_c, H^A_\lambda\} = -P^A_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H^A_{X^{(n)}}$$
$$\{P_\xi, H^A_\lambda\} = P^A_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H^A_{Y^{(n)}},$$

with F and G non-identically-vanishing functional of τ and λ defined in [CCT21] (Theorem 30), $X = [c, \lambda e_n]$ and $Y = \mathcal{L}^{\omega_0}_\xi(\lambda e_n)$. We are thus left with computing the remaining brackets.

Equivalently to the scalar case, the Hamiltonian vector fields for R_τ are given by

$$e\mathbb{R}_e = [\tau, e] \qquad e\mathbb{R}_\omega = g(\tau, \omega, e) + d_\omega \tau \tag{110}$$
$$\mathbb{R}_A = 0 \qquad \mathbb{R}_\rho = 0, \tag{111}$$

since in does not possess any variation along the gauge fields. We consider now the variation

$$\delta M_\mu = \int_\Sigma \mathrm{Tr}(\mu\delta(d_A\rho)) = \int_\Sigma \mathrm{Tr}(-\mu([\delta A, \rho] + d_A(\delta\rho)) \tag{112}$$
$$= \int_\Sigma \mathrm{Tr}\big([\mu, \rho]\delta A + d_A\mu\,\delta\rho\big), \tag{113}$$

and therefore we obtain the following Hamiltonian vector fields

$$\mathbb{M}_e = 0 \qquad \mathbb{M}_\omega = 0 \tag{114}$$
$$\mathbb{M}_\rho = [\mu, \rho] \qquad \mathbb{M}_A = d_A\mu. \tag{115}$$

From [CCF22], for p^A_ξ, we have

$$\mathbb{p}^A_e = 0 \qquad \mathbb{p}^A_\omega = 0 \tag{116}$$
$$\mathbb{p}^A_\rho = -\mathcal{L}^{A_0}_\xi \rho \qquad \mathbb{p}^A_A = -\mathcal{L}^{A_0}_\xi(A - A_0) - \iota_\xi F_{A_0}, \tag{117}$$

whereas, for h^A_λ, the Hamiltonian vector fields read

$$\mathbb{h}^A_e = 0 \qquad \mathbb{h}^A_\rho = d_A(\lambda e_n e B) \tag{118}$$
$$\mathbb{h}^A_A = \lambda(B, e_n e) \qquad e\mathbb{h}^A_\omega = \mathrm{Tr}\big(\lambda e_n B F_A + \lambda e_n \frac{e^2}{4}(B, B) - \lambda e B(B, e_n e)\big). \tag{119}$$

Now, we are left with computing the Poisson brackets of the constraints for $\{R_\tau, h^A_\lambda\}$, $\{R_\tau, p^A_\xi\}$ and $\{R_\tau, M_\mu\}$. We start with noticing that

$$\{R_\tau, p^A_\xi\} = \{R_\tau, M_\mu\} = 0 \tag{120}$$

since both p_ξ^A and M_μ have vanishing Hamiltonian vector fields along e and ω. Then, we are left with computing

$$\{R_\tau, h_\lambda^A\} = \int_\Sigma \mathrm{Tr}\Big(\lambda e_n B F_A W_1^{-1}[\tau, e] + \lambda e_n \frac{e}{4}(B,B)[\tau,e] \tag{121}$$

$$- \lambda B(B, e_n e)[\tau, e]\Big), \tag{122}$$

where the second term is zero because of $e[\tau, e] = 0$ and the first and third terms in general do not vanish. In fact, we have

$$\{R_\tau, h_\lambda^A\} = \tag{123}$$

$$= \int_\Sigma \mathrm{Tr}\Big(\lambda e_n B F_A W_1^{-1}[\tau, e] - \lambda B(B, e_n e)[\tau, e]\Big) \tag{124}$$

$$= \int_\Sigma \mathrm{Tr}\Big(\frac{\lambda e_n}{2} B(B, e^2) - \lambda B(B, e_n e)e\Big) W_1^{-1}[\tau, e] \tag{125}$$

$$= \int_\Sigma \mathrm{Tr}\Big(\lambda B\Big(\frac{e_n}{2}(B, e^2) - (B, e_n e)e\Big)\Big) W_1^{-1}[\tau, e] \tag{126}$$

$$\approx\colon K^A_{\lambda\tau}, \tag{127}$$

where $W_1^{-1}\colon \Omega^{2,2}_\Sigma \to \Omega^{1,1}_\Sigma$ indicates the inverse of the map $W_1^{\Sigma,(1,1)}$ and the symbol $\approx\colon$ means that we are defining the quantity $K^A_{\lambda\tau}$ on the constraint submanifold. Then, thanks to Corollary 12 of [CCT21], we can write the explicit form of $K^A_{\lambda\tau}$ by means of the independent components $\mathcal{X}$ and $\mathcal{Y}$ of τ, defined in Proposition 8 of[CCT21]. Hence, we define $K^A_{\lambda\tau}$ as

$$K^A_{\lambda\tau} = \int_\Sigma \mathrm{Tr}\Big(\lambda\Big(\frac{1}{2}\Big(\sum_{\mu=1}^{2} \mathcal{Y}_\mu \mathcal{C}^\mu_\mu - \sum_{\mu_1\neq\mu_2=1}^{2} \mathcal{X}^{\mu_2}_{\mu_1}\mathcal{C}^{\mu_1}_{\mu_2}\Big) \tag{128}$$

$$- \Big(\sum_{\mu=1}^{2} \mathcal{Y}_\mu \mathcal{D}^\mu_\mu - \sum_{\mu_1\neq\mu_2=1}^{2} \mathcal{X}^{\mu_2}_{\mu_1}\mathcal{D}^{\mu_1}_{\mu_2}\Big)\Big)\Big), \tag{129}$$

where $\mathcal{C}^\rho_\sigma \coloneqq (B^{\rho 3} - B^{\rho 4})(B, e^2)_{3\sigma}$ and $\mathcal{D}^\rho_\sigma \coloneqq (B^{\rho 3} - B^{\rho 4})(B, e_n e)_\sigma$.

Therefore, thanks to the linearity of the Poisson brackets together with the known results, this completes the proof. □

3.3. **Spinor field.** The concept of a spinor field is central in mathematical physics. The idea of a spinor field is funded on the definition a particular subalgebra of the tensor algebra over a vector space, called the *Clifford algebra*. In the following, we will recall the basic and fundamental results about the structure of these algebras in order to be able to write the Palatini–Cartan theory coupled with a Dirac spinor.

Definition 45. Let, V be a vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $g\colon V \times V \to \mathbb{K}$ be a symmetric bilinear form.[26] Moreover, let I_g be the two sided ideal in the tensor graded algebra $T(V)$ of V generated by

$$\{v \otimes v + g(v, v)1, v \in V\}, \tag{130}$$

where $1 \in T(V)$ is the unit element. Then, we define the Clifford algebra $\mathrm{Cl}(V, g)$ as the filtered algebra given by the quotient

$$\mathrm{Cl}(V, g) \coloneqq \frac{T(V)}{I_g}. \tag{131}$$

[26]We call this space a *quadratic vector space*.

Remark 46. The general definition of a Clifford algebra is given by means of a universal property in the category of unital associative algebras. One can recover Definition 45 by building a functor between the category of vector spaces endowed with a symmetric bilinear form and the category of unital associative algebras. Then, the universal property guarantees that morphisms extend uniquely to Clifford algebras homomorphisms.

In the following, we will state some results which are well-known facts in the literature. They will serve as a basis in order to build the theory of spin coframes, which can be regarded as a sort of generalization of the vielbein and the coframe formalism. We refer to [Wer19], [Fat18] and references therein for the proofs of these results as well as more details.

Definition 47. Let V be a quadratic vector space on $\mathbb{R}$ and let (p, q) be the signature of g. Moreover, let $\mathrm{Cl}^+(V, g) \coloneqq \mathrm{Cl}^0 \oplus \mathrm{Cl}^2 \oplus \mathrm{Cl}^4 \oplus \ldots$ be the subalgebra defined by the even grading. We define the group $\mathrm{Pin}_{p,q} \subset \mathrm{Cl}(V, g)$ as the subgroup of the group of units in $\mathrm{Cl}(V, g)$ generated by $v \in V$ such that $|g(v, v)| = 1$.
Then, we defined the group $\mathrm{Spin}_{p,q}$ as the subgroup of $\mathrm{Pin}_{p,q}$ given by

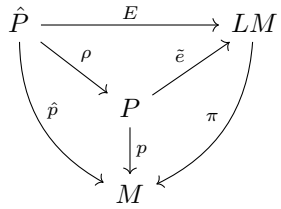
$$\mathrm{Spin}_{p,q} \coloneqq \mathrm{Pin}_{p,q} \cap \mathrm{Cl}^+(V, g). \tag{132}$$

Proposition 48. *Let V be a quadratic vector space on $\mathbb{R}$ and let (p, q) be the signature of g. Moreover, let $\rho\colon \mathrm{Spin}_{p,q} \to \mathrm{GL}\big(\mathfrak{spin}_{p,q}\big)$ be the adjoint representation. Then, we have the following:*

- $\mathfrak{spin}_{p,q} \subset \mathrm{Cl}(V, g)$*;*
- *The map ρ acts as $\mathrm{SO}(p, q)$ on V*[27] *(or, for its complexification, as $\mathrm{SO}(n) \times \mathrm{U}(1)$ with $n = p + q$);*
- *The map ρ defines a covering map*[28] *$\rho\colon \mathrm{Spin}_{p,q} \to \mathrm{SO}(p, q)$.*

Furthermore, the group $\mathrm{Spin}_{p,q}$ is simply connected and it is the universal cover of $\mathrm{SO}(p, q)$. Therefore, in particular, $\mathrm{Spin}_{3,1} \cong \mathrm{SL}(2, \mathbb{C})$.

Definition 49. Let $\hat{P}$ be a principal $\mathrm{Spin}_{p,q}$-bundle on M and LM the frame bundle. Then, we define the *spin map* $E\colon \hat{P} \to LM$ as the principal bundle morphism such that the following diagram commutes



where $\rho\colon \hat{P} \to P$ denotes the bundle morphism induced by the covering map of Proposition 48 and $\tilde{e}$ the vielbein of Definition 1.

The following result will be a particular example of the broader spectrum of the classification of Clifford algebras. In a nutshell, they exhibit a 2-periodicity in the complex case and a 8-periodicity in the real case.

Theorem 50. *Let V be a 4-dimensional quadratic vector space on $\mathbb{K}$ and, in particular, if $\mathbb{K} = \mathbb{R}$, let $(p, q) = (3, 1)$. Furthermore, let $\mathrm{M}_{4\times4}(\mathbb{K})_{Cl}$ denote the algebra*

[27] Here, we regard V as a first grade subspace of the Clifford algebra.

[28] By abuse of notation, we denote the covering map and the adjoint representation in the same manner.

of 4×4 *matrices on* $\mathbb{K}$ *endowed with the Clifford structure. Then, we have the following isomorphism*

$$\lambda\colon \mathrm{Cl}(V,g) \to \mathrm{M}_{4\times 4}(\mathbb{K})_{Cl}. \tag{133}$$

Remark 51. If we consider the complexification of the algebra $\mathfrak{spin}^{\mathbb{C}}_{3,1}$, as a consequence of Theorem 50, the adjoint representation $\rho\colon \mathrm{Spin}_{3,1} \to \mathrm{GL}\big(\mathfrak{spin}^{\mathbb{C}}_{3,1}\big)$ can be regarded as acting on $\mathrm{M}_{4\times 4}(\mathbb{C})_{Cl}$, since $\mathfrak{spin}_{p,q} \subset \mathrm{Cl}(V,g)$. Moreover, we know by Proposition 48 that ρ acts as $\mathrm{SO}(3,1)$ on V. Hence, this statement takes the form

$$\rho_S(\gamma^a) = S\gamma^a S^{-1} = \Lambda^a_b \gamma^b, \tag{134}$$

where $S \in \lambda\big(\mathrm{Spin}_{3,1}\big)$ and $\Lambda \in \mathrm{SO}(3,1)$ is the matrix associated to S under the covering map with $a, b = 1, 2, 3, 4$. In other words, the complexified algebra of the spin group, where the adjoint representation acts, can be expressed in terms of γ-matrices, which can be also labeled according to a basis of V, i.e. $\gamma = \gamma^a v_a \in V \otimes \mathrm{M}_{4\times 4}(\mathbb{C})_{Cl}$, such that the Clifford relation reads

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab} 1_{4\times 4}, \tag{135}$$

where the brackets denote the anti-commutators.
Furthermore, if we denote with $f\colon \mathrm{SO}(3,1) \to \mathrm{Aut}(V)$ the fundamental representation of $\mathrm{SO}(3,1)$, by composition with the adjoint representation of $\mathrm{Spin}_{3,1}$, we can construct the Minkowski bundle as the associated vector bundle to $\hat{P}$ under the composition, i.e.

$$\mathcal{V} \coloneqq \hat{P} \times_{f\circ\rho} V. \tag{136}$$

Note that the isomorphism of Theorem 50 defines a representation of the complexified group $\mathrm{Spin}^{\mathbb{C}}_{3,1}$ on $\mathbb{C}^4$. This representation is called the γ-representation and it corresponds to the representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of $\mathrm{SL}(2,\mathbb{C})$ (thanks to the group isomorphism $\mathrm{Spin}_{3,1} \cong \mathrm{SL}(2,\mathbb{C})$). This fact allows to have the following definition.

Definition 52. Let $\gamma\colon \mathrm{Spin}^{\mathbb{C}}_{3,1} \to \mathrm{Aut}(\mathbb{C}^4)$ be the γ-representation of the spin group. Then, we define the *spinor bundle* as the associated vector bundle to $\hat{P}$ under γ, namely

$$S \coloneqq \hat{P} \times_{\gamma} \mathbb{C}^4. \tag{137}$$

We define a *spinor field*[29] as a section of the odd-bundle ΠS, where Π indicates the parity reversal operation[30].

Remark 53. Notice that, in this context, we can regard the γ-matrices as elements $\gamma \in \Gamma\big(\mathcal{V} \otimes \mathrm{End}(\Pi S)\big)$. Note also that, by construction, the parity of a spinor field $\psi \in \Gamma(\Pi S)$ is given by $|\psi| = 1$.

Proposition 54. *Given a real vector space* V *and the isomorphism* $\mathfrak{so}(3,1) \cong \bigwedge^2 V$*, we have the following algebra isomorphism*

$$\mathrm{d}\rho\colon \mathfrak{spin}_{3,1} \to \bigwedge^2 V, \tag{138}$$

which is given by

$$\mathrm{d}\rho^{-1}(v \wedge w) = -\frac{1}{4}[\tilde{v}, \tilde{w}], \tag{139}$$

[29]In our case, we will only deal with Dirac spinors. Therefore, the term "spinor" refers uniquely to a Dirac one. In a more general setting, we must slightly generalize our definition in order to include other spin structures.

[30]Parity inversion is fundamental since we want spinors to be Grassmannian/odd quantities.

where $v, w \in V$, $\tilde{v}, \tilde{w} \in \mathfrak{spin}_{3,1}$ and $\rho\colon \mathrm{Spin}_{3,1} \to \mathrm{GL}\big(\mathfrak{spin}_{3,1}\big)$ is the adjoint representation.

If we consider the complexified Lie algebra $\mathfrak{spin}^{\mathbb{C}}_{3,1}$ and the isomorphism of Proposition 54, we can build a covariant derivative for spinor fields in terms of local connections in $\Omega^{1,2}$. Explicitly, it reads

$$d_\omega \psi = d\psi + [\omega, \psi] = d\psi - \frac{1}{4}\omega^{ab}\gamma_a\gamma_b\psi. \tag{140}$$

We define the covariant derivative for the conjugate of ψ such that $d_\omega\overline{\psi} = \overline{d_\omega\psi}$. Therefore, we have

$$d_\omega \overline{\psi} = d\overline{\psi} + [\omega, \overline{\psi}] = d\overline{\psi} - \frac{1}{4}\omega^{ab}\overline{\psi}\gamma_a\gamma_b. \tag{141}$$

By Remark 53, we can extend the definition of the covariant derivative also to the γ-matrices. It follows the upcoming lemma.

Lemma 55. *Let $\gamma \in \Gamma\big(\mathcal{V} \otimes \mathrm{End}(\Pi S)\big)$. Then, it holds*

$$d_\omega \gamma = 0. \tag{142}$$

Proof. See [CCF22]. □

The space of fields of the Spinor–Palatini–Cartan theory is given by[31]

$$\mathcal{F}^\psi = \tilde{\Omega}^{1,1} \times \mathcal{A}(P) \times \Gamma(\Pi S) \times \Gamma(\Pi \bar{S}) \ni (e, \omega, \psi, \overline{\psi}), \tag{143}$$

whereas the action functional reads

$$\mathcal{S}^\psi = \mathcal{S}_{PC} + \int_M \frac{i}{12} e^3 (\overline{\psi}\gamma d_\omega \psi - d_\omega \overline{\psi}\gamma\psi). \tag{144}$$

It follows that the field equations are the following Euler-Lagrange equations for the action $\mathcal{S}^\psi$

$$eF_\omega + \frac{i}{4}e^2(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi) = 0 \tag{145}$$

$$ed_\omega e + \frac{i}{6}(\overline{\psi}\gamma[e^3, \psi] - [e^3, \overline{\psi}]\gamma\psi) = 0 \tag{146}$$

$$\frac{e^3}{6}\gamma d_\omega\psi - \frac{1}{12}d_\omega e^3\gamma\psi = 0 \tag{147}$$

$$\frac{e^3}{6}d_\omega\overline{\psi}\gamma + \frac{1}{12}d_\omega e^3\overline{\psi}\gamma = 0, \tag{148}$$

where we define, for $X \in \Gamma(\mathcal{V})$ and $\alpha \in \Omega^{r,k}_\Sigma$, the contraction

$$\iota_X \alpha \coloneqq \frac{\eta_{ab}}{(k-1)!} X^a \alpha^{bi_2\cdots i_k} v_{i_2} \wedge \cdots \wedge v_{i_k} \tag{149}$$

and consequently, for $\chi \in \Omega^{i,j}_\Sigma$, the brackets

$$\begin{cases} [\chi, \psi] & \coloneqq \frac{1}{4(j-1)}\iota_\gamma\iota_\gamma\chi\psi \\ [\chi, \overline{\psi}] & \coloneqq -\frac{(-1)^{|\chi||\psi|}}{4(j-1)}\overline{\psi}\iota_\gamma\iota_\gamma\chi, \end{cases} \tag{150}$$

where $|\chi|$ is the parity of χ and $|\psi|$ the parity of ψ.

Similar to the preceding sections, the space of pre-boundary fields $\tilde{\mathcal{F}}^\psi_\Sigma$, as defined in Definition 17 for the Palatini–Cartan theory, can be established by pulling back the fields to the boundary Σ. Furthermore, we will keep denoting the fields on the boundary in the same way as those in the bulk.

As outlined in [CCF22], we can now define the geometric phase space of the theory through a reduction using the kernel of the pre-symplectic form.

[31] Where $\bar{S}$ is simply given by the conjugate representation $\bar{\gamma}$.

Theorem 56. *The geometric phase space for the Spinor–Palatini–Cartan theory is the symplectic manifold* $(\mathcal{F}^{\psi}_{\Sigma}, \varpi)$ *given by the following equivalence relations on the space of pre-boundary fields* $\tilde{\mathcal{F}}^{\psi}_{\Sigma}$

$$\omega' \sim \omega \quad \Longleftrightarrow \quad \omega' - \omega \in \mathrm{Ker} W_1^{\Sigma,(1,2)} \tag{151}$$

and the symplectic form

$$\varpi = \int_{\Sigma} e\delta e\delta\omega + i\frac{e^2}{4}\left(\overline{\psi}\gamma\delta\psi - \delta\overline{\psi}\gamma\psi\right)\delta e + i\frac{e^3}{3!}\delta\overline{\psi}\gamma\delta\psi. \tag{152}$$

We denote this equivalence class as $\mathcal{A}(i^*P)_{red}$.

Proof. See [CCF22]. □

Remark 57. Likewise the preceding cases, we notice that the equivalence class of ω, defining the geometric phase space, remains equal to the Palatini–Cartan theory. In fact, similarly to the previous couplings, this can be seen as a consequence of the fact that the symplectic form does not have any other piece along ω, but the one equal to the Palatini–Cartan case.

Remark 58. In the case at hand, the field equations see a substantial difference. Namely, the Levi-Civita (or torsion-free) condition $ed_\omega e = 0$ no longer holds. Indeed, the Lagrangian of the theory couples the connection with the spinor. Therefore, the structural and the degeneracy constraints take the form

$$\begin{cases} e_n(\alpha_\psi - p_{\mathcal{T}}\alpha_\psi) \in \mathrm{Im}\, W_1^{\Sigma,(1,1)} \\ p_{\mathcal{T}}\alpha_\psi = 0. \end{cases} \tag{153}$$

with

$$\alpha_\psi \coloneqq d_\omega e + \frac{i}{4}(\overline{\psi}\gamma[e^2,\psi] - [e^2,\overline{\psi}]\gamma\psi). \tag{154}$$

The following proposition will ensure that, although the form of α_ψ is sensibly different from the preceding cases, the form of the functional R^{ψ}_{τ} will coincide with the one of the Palatini–Cartan theory.

Proposition 59. *Let* $\tau \in \mathcal{S}$. *Then, we have the following identity*

$$\tau(\overline{\psi}\gamma[e^2,\psi] - [e^2,\overline{\psi}]\gamma\psi) = 0. \tag{155}$$

Proof. The proof comes by applying twice Lemma 74. Therefore, by means of Proposition 70, we have

$$\begin{aligned} \tau(\overline{\psi}\gamma[e^2,\psi] - [e^2,\overline{\psi}]\gamma\psi) &= e_n\beta(\overline{\psi}\gamma[e^2,\psi] - [e^2,\overline{\psi}]\gamma\psi) && (156)\\ &= e_n e^2(\overline{\psi}\gamma[\beta,\psi] - [\beta,\overline{\psi}]\gamma\psi) && (157)\\ &= e\beta(\overline{\psi}\gamma[e_n e,\psi] - [e_n e,\overline{\psi}]\gamma\psi) && (158)\\ &= 0, && (159) \end{aligned}$$

since $\beta \in \mathrm{Ker} W_1^{\Sigma,(1,2)}$. □

We are now able to properly give the constraints of the theory.

Definition 60. Let $c \in \Omega^{0,2}_\Sigma[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$ and $\tau \in \mathcal{S}[1]$. Then, we define the following functionals

$$L^\psi_c = \int_\Sigma ced_\omega e - i\frac{e^3}{2\cdot 3!}\left([c,\overline{\psi}]\gamma\psi - \overline{\psi}\gamma[c,\psi]\right) \tag{160}$$

$$P^\psi_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \iota_\xi(\omega-\omega_0)ed_\omega e - i\frac{e^3}{2\cdot 3!}\left(\overline{\psi}\gamma \mathrm{L}^{\omega_0}_\xi(\psi) - \mathrm{L}^{\omega_0}_\xi(\overline{\psi})\gamma\psi\right) \tag{161}$$

$$H^\psi_\lambda = \int_\Sigma \lambda e_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3 + i\frac{e^2}{4}\left(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi\right)\Big) \tag{162}$$

$$R^\psi_\tau = \int_\Sigma \tau d_\omega e. \tag{163}$$

We refer to these as the constraints of the Spinor–Palatini–Cartan (degenerate) theory.

Theorem 61. *Let i^*g be degenerate. Then, the Poisson brackets of the constraints of Definition 60 read*

$$\begin{aligned}
&\{L^\psi_c, L^\psi_c\} = -\frac{1}{2}L_{[c,c]} && \{P^\psi_\xi, P^\psi_\xi\} = \frac{1}{2}P^\psi_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}\\
&\{L^\psi_c, P^\psi_\xi\} = L^\psi_{\mathcal{L}^{\omega_0}_\xi c} && \{H^\psi_\lambda, H^\psi_\lambda\} \approx 0\\
&\{L^\psi_c, R^\psi_\tau\} = -R^\psi_{p_\mathcal{S}[c,\tau]} && \{R^\psi_\tau, P^\psi_\xi\} = R^\psi_{p_\mathcal{S}\mathcal{L}^{\omega_0}_\xi\tau}\\
&\{R^\psi_\tau, H^\psi_\lambda\} \approx G_{\lambda\tau} + K^\psi_{\lambda\tau} && \{R^\psi_\tau, R^\psi_\tau\} \approx F_{\tau\tau}\\
&\{L^\psi_c, H^\psi_\lambda\} = -P^\psi_{X^{(a)}} + L^\psi_{X^{(a)}(\omega-\omega_0)_a} - H^\psi_{X^{(n)}}\\
&\{P^\psi_\xi, H^\psi_\lambda\} = P^\psi_{Y^{(a)}} - L^\psi_{Y^{(a)}(\omega-\omega_0)_a} + H^\psi_{Y^{(n)}},
\end{aligned}$$

with $X = [c, \lambda e_n]$, $Y = \mathcal{L}^{\omega_0}_\xi(\lambda e_n)$ and where the superscripts (a) and (n) describe their components with respect to e_a, e_n. Furthermore, $F_{\tau\tau}$, $G_{\lambda\tau}$ and $K^\psi_{\lambda\tau}$ are functionals of $e, \omega, \psi, \overline{\psi}, \tau$ and λ defined in the proof which are not proportional to any other constraint.

Proof. First, we notice that the contraction of the symplectic form with a vector field $\mathbb{X} \in \mathfrak{X}(\mathcal{F}^\psi_\Sigma)$ is given by

$$\iota_\mathbb{X}\varpi = \int_\Sigma e\mathbb{X}_e\delta\omega + \left[e\mathbb{X}_\omega + \frac{i}{4}e^2(\overline{\psi}\gamma\mathbb{X}_\psi - \mathbb{X}_{\overline{\psi}}\gamma\psi)\right]\delta e \tag{164}$$

$$+ i\delta\overline{\psi}\left(-\frac{e^2}{4}\gamma\psi\mathbb{X}_e + \frac{e^3}{3!}\gamma\mathbb{X}_\psi\right) + i\left(\frac{e^2}{4}\overline{\psi}\gamma\mathbb{X}_e + \frac{e^3}{3!}\mathbb{X}_{\overline{\psi}}\gamma\right)\delta\psi. \tag{165}$$

Then, we start giving the Hamiltonian vector fields of the constraints. For L^ψ_c and P^ψ_ξ, from [CCF22], we have

$$\mathbb{L}^\psi_e = [c, e] \qquad \mathbb{L}^\psi_\psi = [c, \psi] \tag{166}$$

$$\mathbb{L}^\psi_\omega = d_\omega c \qquad \mathbb{L}^\psi_{\overline{\psi}} = [c, \overline{\psi}] \tag{167}$$

$$\mathbb{P}^\psi_e = -\mathcal{L}^{\omega_0}_\xi e \qquad \mathbb{P}^\psi_\psi = -\mathcal{L}^{\omega_0}_\xi(\psi) \tag{168}$$

$$\mathbb{P}^\psi_\omega = -\mathcal{L}^{\omega_0}_\xi(\omega-\omega_0) - \iota_\xi F_{\omega_0} \qquad \mathbb{P}^\psi_{\overline{\psi}} = -\mathcal{L}^{\omega_0}_\xi(\overline{\psi}). \tag{169}$$

Whereas, for H^{ψ}_{λ}, we have

$$\mathbb{H}^{\psi}_{e} = d_{\omega}(\lambda e_n) + \lambda\sigma + \frac{i}{4}\lambda\overline{\psi}\left(\iota_{\gamma}\iota_{\gamma}e_n e\gamma - \gamma\iota_{\gamma}\iota_{\gamma}e_n e\right)\psi \tag{170}$$

$$e\mathbb{H}^{\psi}_{\omega} = \lambda e_n\left(F_{\omega} + \frac{\Lambda}{2}e^2\right) - i\frac{\lambda e_n}{4}e(\overline{\psi}\gamma d_{\omega}\psi - d_{\omega}\overline{\psi}\gamma\psi) \tag{171}$$

$$\frac{e^3}{3!}\gamma\mathbb{H}^{\psi}_{\psi} = \frac{\lambda e_n}{2}e^2\gamma d_{\omega}\psi - \frac{\lambda e_n}{4}e d_{\omega}e\gamma\psi \tag{172}$$

$$+ \frac{i}{64}\lambda e\left[\overline{\psi}\left(\iota_{\gamma}\iota_{\gamma}(e_n e^2)\gamma - \gamma\iota_{\gamma}\iota_{\gamma}(e_n e^2)\right)\psi\right]\gamma\psi \tag{173}$$

$$\frac{e^3}{3!}\mathbb{H}^{\psi}_{\overline{\psi}}\gamma = \frac{\lambda e_n}{2}e^2 d_{\omega}\overline{\psi}\gamma + \frac{\lambda e_n}{4}e d_{\omega}e\overline{\psi}\gamma \tag{174}$$

$$- \frac{i}{64}\lambda e\overline{\psi}\gamma\left[\overline{\psi}\left(\iota_{\gamma}\iota_{\gamma}(e_n e^2)\gamma - \gamma\iota_{\gamma}\iota_{\gamma}(e_n e^2)\right)\psi\right], \tag{175}$$

where $\sigma \in \Omega^{1,1}_{\Sigma}$. Lastly, the Hamiltonian vector fields of R^{ψ}_{τ}, are given by

$$e\mathbb{R}^{\psi}_{e} = [\tau, e] \tag{176}$$

$$e\mathbb{R}^{\psi}_{\omega} = \frac{\delta\tau}{\delta e}d_{\omega}e + d_{\omega}\tau \tag{177}$$

$$\mathbb{R}^{\psi}_{\psi} = \mathbb{R}^{\psi}_{\overline{\psi}} = 0, \tag{178}$$

since they coincide with the ones of the Palatini–Cartan theory of Definition 31. Notice that, instead of using the function $g = g(\tau, e, \omega)$, we preferred expressing the variation of τ with respect to e by means of the functional derivative $\dfrac{\delta\tau}{\delta e}$. However, we have the relation

$$g(\tau, e, \omega) = \frac{\delta\tau}{\delta e}d_{\omega}e. \tag{179}$$

Now, we are ready to compute the Poisson brackets of the constraints. From [CCF22], we have already knowledge of the following Poisson brackets

$$\{P^{\psi}_{\xi}, P^{\psi}_{\xi}\} = \frac{1}{2}P^{\psi}_{[\xi,\xi]} - \frac{1}{2}L^{\psi}_{\iota_{\xi}\iota_{\xi}F_{\omega_0}} \qquad \{H^{\psi}_{\lambda}, H^{\psi}_{\lambda}\} = 0 \tag{180}$$

$$\{L^{\psi}_{c}, P^{\psi}_{\xi}\} = L^{\psi}_{\mathcal{L}^{\omega_0}_{\xi}c} \qquad \{L^{\psi}_{c}, L^{\psi}_{c}\} = -\frac{1}{2}L^{\psi}_{[c,c]} \tag{181}$$

$$\{L^{\psi}_{c}, H^{\psi}_{\lambda}\} = -P^{\psi}_{X^{(a)}} + L^{\psi}_{X^{(a)}(\omega-\omega_0)_a} - H^{\psi}_{X^{(n)}} \tag{182}$$

$$\{P^{\psi}_{\xi}, H^{\psi}_{\lambda}\} = P^{\psi}_{Y^{(a)}} - L^{\psi}_{Y^{(a)}(\omega-\omega_0)_a} + H^{\psi}_{Y^{(n)}}, \tag{183}$$

with $X = [c, \lambda e_n]$ and $Y = \mathcal{L}^{\omega_0}_{\xi}(\lambda e_n)$ as above. Therefore, we are left with computing the remaining constraints. First, we notice that

$$\{R^{\psi}_{\tau}, L^{\psi}_{c}\} = \{R_{\tau}, L_{c}\} = -R_{p_{\mathcal{S}}[c,\tau]} = -R^{\psi}_{p_{\mathcal{S}}[c,\tau]}. \tag{184}$$

Similarly, we can also compute the bracket

$$\{R^{\psi}_{\tau}, P^{\psi}_{\xi}\} = \{R_{\tau}, P_{\xi}\} = R_{p_{\mathcal{S}}\mathcal{L}^{\omega_0}_{\xi}\tau} = R^{\psi}_{p_{\mathcal{S}}\mathcal{L}^{\omega_0}_{\xi}\tau}. \tag{185}$$

Now, we move on to compute the brackets $\{R^{\psi}_{\tau}, R^{\psi}_{\tau}\}$ and $\{R^{\psi}_{\tau}, H^{\psi}_{\lambda}\}$. The first bracket is simply given by

$$\{R^{\psi}_{\tau}, R^{\psi}_{\tau}\} = \{R_{\tau}, R_{\tau}\} \approx F_{\tau\tau} \tag{186}$$

with $F_{\tau\tau}$ defined in Theorem 30 of [CCT21] and which is in general non-vanishing on the constraint submanifold. Whereas, for the second one, we obtain

$$\{R^\psi_\tau, H^\psi_\lambda\} = \int_\Sigma \Big(e_n \frac{\delta\beta}{\delta e} d_\omega e + d_\omega(e_n\beta)\Big)\Big(d_\omega(\lambda e_n) + \lambda\sigma \tag{187}$$

$$- i\lambda(\overline{\psi}\gamma[e_n e, \psi] - [e_n e, \overline{\psi}]\gamma\psi)\Big) \tag{188}$$

$$+ W_1^{-1}[e_n\beta, e]\big(\lambda e_n(F_\omega + \frac{\Lambda}{2}e^2) \tag{189}$$

$$- \frac{i}{4}\lambda e_n e(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi)\big) \tag{190}$$

$$\approx \int_\Sigma -i\lambda\beta d_\omega e_n(\overline{\psi}\gamma[e_n e, \psi] - [e_n e, \overline{\psi}]\gamma\psi) \tag{191}$$

$$- \frac{i}{4}[e_n\beta, e]\lambda e_n(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi) \tag{192}$$

$$+ G_{\lambda\tau}, \tag{193}$$

where, in the last passage, we used Lemma 74 and the fact that $e_n^2 = 0$. Moreover, the quantity $G_{\lambda\tau}$ and the map W_1^{-1} are defined respectively in Theorem 30 of [CCT21] and in the proof of Theorem 44. Now, we can notice that, thanks to Lemma 74, we can write

$$\lambda\beta d_\omega e_n(\overline{\psi}\gamma[e_n e, \psi] - [e_n e, \overline{\psi}]\gamma\psi) = \tag{194}$$

$$= \lambda e_n e d_\omega e_n(\overline{\psi}\gamma[\beta, \psi] - [\beta, \overline{\psi}]\gamma\psi) \tag{195}$$

$$= \lambda e\beta(\overline{\psi}\gamma[e_n d_\omega e_n, \psi] - [e_n d_\omega e_n, \overline{\psi}]\gamma\psi) \tag{196}$$

$$= 0, \tag{197}$$

obtaining

$$\{R^\psi_\tau, H^\psi_\lambda\} \approx G_{\lambda\tau} - \int_\Sigma \frac{i}{4}[e_n\beta, e]\lambda e_n(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi). \tag{198}$$

Finally, we can write the integral as

$$\{R^\psi_\tau, H^\psi_\lambda\} \approx G_{\lambda\tau} - \int_\Sigma \frac{i}{4}\lambda\tau[e_n, \hat{e}](\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi), \tag{199}$$

where we implemented again Proposition 70 and also the relation[32]

$$e_n[\tau, e] = \tau[e_n, \hat{e}] \tag{200}$$

with $\hat{e}$ defined as $\hat{e} \coloneqq e - \tilde{e}$ (see Eq. (22)). More specifically, using the definition of the independent components of τ, as we did in the proof of Theorem 44, we have

$$\{R^\psi_\tau, H^\psi_\lambda\} \approx G_{\lambda\tau} + K^\psi_{\lambda\tau}, \tag{201}$$

with

$$K^\psi_{\lambda\tau} \coloneqq -\int_\Sigma i\lambda\Big(\sum_{\mu=1}^{2} \mathcal{Y}_\mu\big(\hat{g}_n d_\omega J_\psi\big)^{3\mu}_{\mu} + \sum_{\mu_1\neq\mu_2=1}^{2} \mathcal{X}^{\mu_2}_{\mu_1}\big(\hat{g}_n d_\omega J_\psi\big)^{\mu_1}_{3\mu_2}\Big), \tag{202}$$

where $\hat{g}_n \coloneqq [e_n, \hat{e}] \in \Omega^{1,0}_\Sigma$ and $d_\omega J_\psi \coloneqq d_\omega(\overline{\psi}\gamma\psi) \in \Omega^{1,1}_\Sigma$.

This final result completes the proof. □

[32] It simply comes from the definition of $\mathcal{S}$.

4. First and second class constraints

In [CCT21], a study of first- and second-class constrains has been presented. In the following section, we will recall the main results and adapt them to the present analysis.

Definition 62. Consider a symplectic manifold $\mathcal{F}$ and a set of smooth maps $\phi_i \in C^\infty(\mathcal{F})$ defined on it. Let $C_{ij} = \{\phi_i, \phi_j\}$ represent the matrix of Poisson brackets associated with these maps. The count of second-class maps in the set corresponds to the rank of the matrix C_{ij} evaluated at the zero locus defined by the ϕ_is[33]. In particular, if $C_{ij} \approx 0$, we categorize all the maps as first-class.

Proposition 63. *Let $\mathcal{F}$ be a symplectic manifold and let $\psi_i, \phi_j \in C^\infty(\mathcal{F})$, where $i = 1 \dots n$ and $j = 1 \dots m$. Moreover, denote with $C_{jj'}, B_{ij}$ and $D_{ii'}$ the matrices representing, respectively, the Poisson brackets $\{\phi_j, \phi_{j'}\}$, $\{\psi_i, \phi_j\}$ and $\{\psi_i, \psi_{i'}\}$, with $i, i' = 1 \dots n$ and $j, j' = 1 \dots m$. Then, if D is invertible and $C = -B^T D^{-1} B$, the number of second-class constraints is n, i.e. the rank of the matrix D.*

Proof. See [CCT21]. □

Theorem 64. *Let the symbol $\bullet$ be such that $\bullet = \phi, A, \psi$. Then, the constraints $L_c^\bullet$, $P_\xi^\bullet$, $H_\lambda^\bullet$ and $R_\tau^\bullet$ do not form a first class system. In particular, R_τ is a second class constraint.*

Proof. The proof follows verbatim the one of [CCT21]. □

We can now determine the degrees of freedom of the reduced phase space. Let r denote the number of degrees of freedom in the reduced phase space, p the number of degrees of freedom in the geometric phase space, f the number of first-class constraints, and s the number of second-class constraints. The relationship among them is given by[34]

$$r = p - 2f - s. \tag{203}$$

For all the possible couplings, it follows that we obtain the same result of the Palatini–Cartan theory, i.e.,

$$r = 2. \tag{204}$$

Remark 65. We notice that in the non-degenerate case we would obtain $r = 4$. This reflects the existence of the constraint $R_\tau^\bullet$, which has been proven giving rise to a second-class system. We recall that such a constraint was implied by the geometry of the theory. In particular, together with some additional condition, it ensured the possibility of uniquely fixing a representative of the equivalence class of ω. In fact, on a null-boundary, the space $\mathcal{T}$ defined in Definition 21 is non-trivial.
In physics, it is well-known that GR carries four local degrees of freedom.[35] However, the constraint analysis of the degenerate theory sheds light of the fact that these local degrees of freedom, in the case of manifolds with a null-boundary, are reduced to only two. This fact has important implications regarding the study of black-holes, since the event horizon is a null-hypersurface.

[33] We assume the rank to be constant on the zero locus.

[34] The proof of this formula is contained in [HT92].

[35] Notice that sometimes the literature reports only two degrees of freedom. This is simply a consequence of considering the dimension of the phase space or just the one of the base manifold.

Appendix

Lemma 66. *Let $e_n \in \Omega^{0,1}_\Sigma$ be as in Lemma 22 and $\alpha \in \Omega^{2,1}_\Sigma$. Then, we have*

$$\alpha = 0$$

if and only if

$$\begin{cases} \alpha \in \mathrm{Ker} W_1^{\Sigma,(2,1)} \\ e_n \alpha \in \mathrm{Im}\, W_1^{\Sigma,(1,1)}. \end{cases} \tag{205}$$

Proof. See [CCS21a]. □

Corollary 67. *Let $e_n \in \Omega^{0,1}_\Sigma$ be as in Lemma 22 and $\gamma \in \Omega^{2,2}_\Sigma$. Then, we have the unique decomposition*

$$\gamma = e\sigma + e_n \alpha, \tag{206}$$

with $\sigma \in \Omega^{1,1}_\Sigma$ and $\alpha \in \mathrm{Ker} W_1^{\Sigma,(2,1)}$.

Proof. We define the map

$$W_1^{n,\Sigma,(i,j)} \colon \Omega^{i,j}_\Sigma \to \Omega^{i,j+1}_\Sigma \tag{207}$$

$$\kappa \mapsto e_n \kappa. \tag{208}$$

From Lemma 66, we know that the map $W_1^{n,\Sigma,(2,1)}|_{\mathrm{Ker} W_1^{\Sigma,(2,1)}}$ is injective[36], whereas, the proof of the injectivity of $W_1^{\Sigma,(1,1)}$ is given in [Can21]. Moreover, Lemma 66 basically states that the intersection $\mathrm{Im} W_1^{\Sigma,(1,1)} \cap \mathrm{Im} W_1^{n,\Sigma,(2,1)}|_{\mathrm{Ker} W_1^{\Sigma,(2,1)}}$ is trivial. We then have

$$\dim(\mathrm{Im} W_1^{\Sigma,(1,1)}) = \dim(\Omega^{1,1}_\Sigma) = 12 \tag{209}$$

and

$$\dim(\mathrm{Im} W_1^{n,\Sigma,(2,1)}|_{\mathrm{Ker} W_1^{\Sigma,(2,1)}}) = \dim(\mathrm{Ker} W_1^{\Sigma,(2,1)}) = 6, \tag{210}$$

since we know from [Can21] that $W_1^{\Sigma,(2,1)}$ is surjective. Given that

$$\dim(\Omega^{2,2}_\Sigma) = 18, \tag{211}$$

it follows the statement. □

Lemma 68. *Let $e_n \in \Omega^{0,1}_\Sigma$ be as in Lemma 22 and $v \in \Omega^{1,2}_\Sigma$. Then, we have*

$$v = 0 \tag{212}$$

if and only if

$$\begin{cases} v & \in \mathrm{Ker} W_1^{\Sigma,(1,2)} \\ e_n v & \in \mathrm{Im} W_1^{\Sigma,(0,2)}. \end{cases} \tag{213}$$

Proof. This statement is the precise analogous of Lemma 66 and the proof follows verbatim upon the substitution $W_1^{\Sigma,(1,1)} \to W_1^{\Sigma,(0,2)}$. □

Corollary 69. *Let $e_n \in \Omega^{0,1}_\Sigma$ be as in Lemma 22 and $\theta \in \Omega^{1,3}_\Sigma$. Then, we have the unique decomposition*

$$\theta = ec + e_n \beta, \tag{214}$$

with $c \in \Omega^{0,2}_\Sigma$ and $\beta \in \mathrm{Ker} W_1^{\Sigma,(1,2)}$.

[36] It is easy to see by setting $e_n \alpha = 0$.

Proof. Given the map $W_1^{n,\Sigma,(1,2)}$ defined in Corollary 67, from Lemma 68, we know that the map $W_1^{n,\Sigma,(1,2)}|_{\mathrm{Ker}W_1^{\Sigma,(1,2)}}$ is injective, whereas, the proof of the injectivity of $W_1^{\Sigma,(0,2)}$ is given in [Can21]. Moreover, Lemma 68 basically states that the intersection $\mathrm{Im}W_1^{\Sigma,(0,2)} \cap \mathrm{Im}W_1^{n,\Sigma,(1,2)}|_{\mathrm{Ker}W_1^{\Sigma,(1,2)}}$ is trivial. We then have

$$\dim(\mathrm{Im}W_1^{\Sigma,(0,2)}) = \dim(\Omega_\Sigma^{0,2}) = 6 \tag{215}$$

and

$$\dim(\mathrm{Im}W_1^{n,\Sigma,(1,2)}|_{\mathrm{Ker}W_1^{\Sigma,(1,2)}}) = \dim(\mathrm{Ker}W_1^{\Sigma,(1,2)}) = 6, \tag{216}$$

since we know from [Can21] that $W_1^{\Sigma,(1,2)}$ is surjective. Given that

$$\dim(\Omega_\Sigma^{1,3}) = 12, \tag{217}$$

it follows the statement. □

Proposition 70. *Let $\tau \in \mathcal{S}$. Then, $\tau = e_n\beta$ with $\beta \in \Omega_\Sigma^{1,2}[1]$ such that $e_n\beta \in \mathrm{Ker}\tilde{\varrho}^{1,3}$ and e_n defined as above.*

Proof. From Lemma 23, in particular, we have that

$$p_{\mathcal{T}}\alpha = 0 \quad \Longrightarrow \quad \int_\Sigma \tau\alpha = 0 \quad \forall \tau \in \mathcal{S}, \tag{218}$$

for $\alpha \in \Omega_\Sigma^{2,1}$. Now, consider an $\alpha \in \Omega_\Sigma^{2,1}$ such that $p_{\mathcal{T}}\alpha = 0$ holds together with the structural constraint $e_n(\alpha - p_{\mathcal{T}}\alpha) = e\sigma$ (notice that this subset of $\Omega_\Sigma^{2,1}$ is in general non-trivial because we do not require the condition $\alpha \in \mathrm{Ker}W_1^{\Sigma,(2,1)}$ as in Lemma 22), then it follows that

$$\int_\Sigma \tau\alpha = \int_\Sigma ec\alpha + e_n\beta\alpha = \int_\Sigma ecp_{\mathcal{T}^C}\alpha + \beta e\sigma = \int_\Sigma ecp_{\mathcal{T}^C}\alpha, \tag{219}$$

where $p_{\mathcal{T}^C}$ is the projection onto a complement of $\mathcal{T}$. Since the right hand side of (218) must hold for all $\tau \in \mathcal{S}$, if the intersection $\mathcal{S} \cap \mathrm{Im}W_1^{\Sigma,(0,2)}$ were not trivial, we would have an absurdum. This implies $c \in \mathrm{Ker}W_1^{\Sigma,(0,2)}$ for all $\tau \in \mathcal{S}$, which, thanks to the injectivity of $W_1^{\Sigma,(0,2)}$, is equivalent to $c = 0$.

Lastly, the fact that $e_n\beta \in \mathrm{Ker}\tilde{\varrho}^{1,3}$ follows immediately from the definition of $\mathcal{S}$. □

Proposition 71. *Let $\tau \in \mathcal{S}$ and e be a diagonal degenerate boundary vielbein, i.e. $e^*\eta = i^*\tilde{g}$ with $\eta = \mathrm{diag}(1,1,1-1)$ and $i^*\tilde{g} = \mathrm{diag}(1,1,0)$. Then, we have*

$$e_n[\tau, e] = 0. \tag{220}$$

Proof. Given $a = 1,2,3,4$ and let $\mu = 1,2,+$ be the coordinates on the boundary Σ such that we can write the diagonal degenerate boundary vielbein e as

$$\hat{e}^a = \begin{cases} e_1^a & = \delta_1^a \\ e_2^a & = \delta_2^a \end{cases} \tag{221}$$

$$e_+^a = \delta_3^a - \delta_4^a \tag{222}$$

$$e_n^a = \delta_3^a + \delta_4^a. \tag{223}$$

Then, the definition of $\tau \in \mathcal{S}$ implies the following relations

$$\tau_+^{abc} = 0 \quad \forall a, b, c \tag{224}$$

$$\tau_\mu^{123} = 0 \quad \mu = 1, 2 \tag{225}$$

$$\tau_\mu^{124} = 0 \quad \mu = 1, 2 \tag{226}$$

$$\tau_1^{234} = \tau_2^{134} \tag{227}$$

$$\tau_1^{134} = -\tau_2^{234}. \tag{228}$$

The proof follows simply by computing $e_n[\tau, e]$ in components implementing the explicit form of the diagonal vielbein above[37]. □

Lemma 72. *Let*[38] $A \in \Omega_\Sigma^{k,i}$ *with* $2 \leq i \leq 4$. *Then, it holds*

$$\gamma\iota_\gamma\iota_\gamma A = (-1)^{|A|}(\iota_\gamma\iota_\gamma A\gamma + 4(i-1)[\gamma, A]). \tag{229}$$

Proof.

$$\gamma\iota_\gamma\iota_\gamma A = (i-2)!\gamma^a\gamma^b\gamma^c v_a\iota_{v_b}\iota_{v_c}A \tag{230}$$

$$= -(i-2)!(\gamma^b\gamma^a\gamma^c + 2\eta^{ab}\gamma^c)v_a\iota_{v_b}\iota_{v_c}A \tag{231}$$

$$= -(i-2)!(-\gamma^b\gamma^c\gamma^a + 4\eta^{ab}\gamma^c)v_a\iota_{v_b}\iota_{v_c}A \tag{232}$$

$$= (-1)^{|A|}(\iota_\gamma\iota_\gamma A\gamma + 4(i-1)[\gamma, A]). \tag{233}$$

□

Remark 73. This lemma introduces a relation between the action of the brackets over the Clifford algebra and $\mathcal{V}$-algebra. In particular, it is consistent a triviality condition on the bracket in the Clifford algebra, i.e.

$$[A, \overline{\psi}\gamma\psi] = (-1)^{|A|}\overline{\psi}[A, \gamma]_{\mathcal{V}}\psi = \overline{\psi}\gamma[A, \psi]_{Cl} + [\overline{\psi}, A]_{Cl}\gamma\psi, \tag{234}$$

where we occasionally added some redundancy with the labels of the specific algebras, even if we will not use them in general.

Lemma 74. *Given* $A \in \Omega_\Sigma^{k,i}$ *and* $B \in \Omega_\Sigma^{l,j}$ *with* $i, j = 2, 3$ *such that* $i + j < 6$, *then we have*

$$B(\overline{\psi}\gamma[A, \psi] - [A, \overline{\psi}]\gamma\psi) = (-1)^{|A||B|}A(\overline{\psi}\gamma[B, \psi] - [B, \overline{\psi}]\gamma\psi). \tag{235}$$

Proof. The proof goes by direct computation of

$$B\gamma\iota_\gamma\iota_\gamma A = B\gamma\gamma^a\gamma^b[v_a, [v_b, A]] \tag{236}$$

$$= (-1)^{|B|}([v_a, B]\gamma + (-1)^{|B|}B\gamma_a)\gamma^a\gamma^b[v_b, A] \tag{237}$$

$$= (-1)^{|B|}([v_a, B]\gamma\gamma^a - 4(-1)^{|B|}B)\gamma^b[v_b, A] \tag{238}$$

$$= -\big([v_b, [v_a, A]]\gamma\gamma^a\gamma^b - (-1)^{|B|}([v_a, B]\gamma_b\gamma^a\gamma^b + 4[\gamma, B])\big)A \tag{239}$$

$$= -(-(-1)^{|B|}\gamma\iota_\gamma\iota_\gamma B - 6(-1)^{|B|}\iota_\gamma B)A \tag{240}$$

$$= (-1)^{|B|}(-1)^{|A|(|B|+1)}A(\gamma\iota_\gamma\iota_\gamma B + 6\iota_\gamma B) \tag{241}$$

[37] We refer to [Tec19a] for further details about this kind of computations.

[38] Notice that this may be also a *shifted* variable, like τ for example.

and

$$B\iota_\gamma\iota_\gamma A\gamma = (-1)^{|A||B|}\gamma^a\gamma^b[v_a,[v_b,A]]B\gamma \tag{242}$$

$$= (-1)^{|A||B|}(-1)^{|A}\gamma^a\gamma^b[v_b,A]([v_a,B]\gamma + (-1)^{|B|}\gamma_a B) \tag{243}$$

$$= -(-1)^{|A||B|}A\gamma^a\gamma^b\big([v_b,[v_a,B]]\gamma - (-1)^{|B|}([v_a,B]\gamma_b - \gamma_a[v_b,B])\big) \tag{244}$$

$$= -(-1)^{|A||B|}A\big(-\iota_\gamma\iota_\gamma B\gamma + (-1)^{|B|}(4[\gamma,B] + \gamma^a\gamma^b\gamma_a[v_b,B])\big) \tag{245}$$

$$= (-1)^{|A||B|}A(\iota_\gamma\iota_\gamma B\gamma - (-1)^{|B|}6\iota_\gamma B). \tag{246}$$

Then, we can conclude the proof by considering the four possible parities of A and B. □

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