WEIL COHOMOLOGY THEORIES AND THEIR MOTIVIC HOPF ALGEBROIDS

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In memory of Jacob Murre

Abstract. In this paper we discuss a general notion of Weil cohomology theories, both in algebraic geometry and in rigid analytic geometry. We allow our Weil cohomology theories to have coefficients in arbitrary commutative ring spectra. Using the theory of motives, we give three equivalent viewpoints on Weil cohomology theories: as a cohomology theory on smooth varieties, as a motivic spectrum and as a realization functor. We also associate to every Weil cohomology theory a motivic Hopf algebroid generalizing the construction we gave in [Ayo14b] for the Betti cohomology. Exploiting results and constructions from [Ayo20], we are able to prove that the motivic Hopf algebroids of all the classical Weil cohomology theories are connective. In particular, they give rise to motivic Galois groupoids which are spectral affine groupoid schemes.

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Introduction

Let k be a ground field. Given a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$, we introduced in [Ayo14b, Ayo14c] a motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k,\sigma)$ coacting on the Betti realization of motives over k in a universal way. The motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k,\sigma)$, which is derived by construction, is known to be connective by [Ayo14b, Corollaire 2.105] and hence defines a spectral affine group scheme $\mathcal{G}_{\mathrm{mot}}(k,\sigma)$ called the motivic Galois group. (In fact, it is conjectured that $\mathcal{H}_{\mathrm{mot}}(k,\sigma)$ is classical, i.e., concentrated in degree zero, but we will not discuss this conjecture in this paper.)

Key words and phrases. Motives, Weil cohomology theories, motivic Hopf algebras, motivic Galois groups, spectral algebraic geometry.

A motivic Hopf algebra can be associated to any Weil cohomology theory. This follows from considerations in [Ayo17, §3], but we will revisit the construction in Section 4 taking advantage of the modern language of higher category theory and higher algebra. Using the usual comparison isomorphisms relating Betti cohomology with ℓ-adic and de Rham cohomologies, it is easy to deduce from [Ayo14b, Corollaire 2.105] that the motivic Hopf algebras associated to the classical Weil cohomology theories are connective when k has characteristic zero.

Our initial motivation for writing this paper was the desire to extend the connectivity of the motivic Hopf algebras to the case where k has positive characteristic. Our proof of [Ayo14b, Corollaire 2.105] is relying crucially on an explicit formula for the Betti realization given in [Ayo14b, Corollaire 2.63], and no such formula is available for the other classical realizations. When k has positive characteristic, the lack a Betti realization for motives over k suggests, at first sight, that a new approach is necessary. This turned out not to be the case: we explain in this paper how to prove the connectivity of the motivic Hopf algebras in positive characteristic by reducing to the zero characteristic case! Our strategy is to use the new Weil cohomology theories introduced in [Ayo20]. More precisely, let K be a valued field of height 1, of unequal characteristic $(0, p)$ and with residue field k. Fix a complex embedding $K \hookrightarrow \mathbb{C}$. Then, there is a Weil cohomology theory $\Gamma_{\mathrm{new,B}}$ for smooth k-varieties which is constructed from the Betti realization for motives over K and the motivic rigid analytification functor associated to the valuation of K. The Weil cohomology theory $\Gamma_{\mathrm{new,B}}$ compares to all the classical Weil cohomology theories for k-varieties: if A is the coefficient ring of $\Gamma_{\mathrm{new,B}}$, there is a morphism $A \to \mathbb{Q}_\ell$, for every prime $\ell \neq p$, such that $\Gamma_{\mathrm{new,B}} \otimes_A \mathbb{Q}_\ell$ is canonically identified with the ℓ-adic cohomology theory. A similar identification exists also for Berthelot's rigid cohomology [Ber86]. Therefore, it is enough to show that the motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B}})$ associated to $\Gamma_{\mathrm{new,B}}$ is connective. Using that $\Gamma_{\mathrm{new,B}}$ is constructed from the Betti realization and the rigid analytification functor, it is possible to express the underlying algebra of $\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B}})$ very explicitly so that connectivity can be seen to hold directly. (See Section 7 for more details).

To implement the strategy described above, it is natural to adopt a generalized notion of a Weil cohomology theory. In particular, we allow our Weil cohomology theories to have coefficients in an arbitrary commutative ring spectrum (whereas in [Ayo20] we insisted on having ordinary rings). A large portion of the paper is devoted to recasting the theory of Weil cohomology theories in its natural generality, taking advantage of the modern techniques of higher category theory. In particular, we establish equivalences of ∞-categories that enable us to move freely between the different incarnations of Weil cohomology theories, namely: as a cohomology theory on smooth varieties, as a motivic spectrum, and as a realization functor. See Theorems 1.16 and 1.21. We also clarify in Section 3 the various relations between Weil cohomology theories in algebraic geometry and in rigid analytic geometry. A notable result is Theorem 3.4 which is an improved version of the key result used in [Ayo20] for constructing the new Weil cohomology theories. Another notable result is Theorem 3.21 which, roughly speaking, asserts that all Weil cohomology theories in rigid analytic geometry comes from Weil cohomology theories in algebraic geometry. In Section 4, we recall the definition of the motivic Hopf algebra associated to a Weil cohomology theory. In Section 6, we gather many examples of Weil cohomology theories. Finally, we prove our connectivity theorem for the motivic Hopf algebras in Section 7.

Notations and conventions.

Higher categories. We use the language of higher category theory following Lurie's books [Lur09], [Lur17] and [Lur18], and we assume that the reader is familiar with this language. Our notations pertaining to higher category theory are very close to that of loc. cit. Nevertheless, we list below some of the notations that we use frequently.

Fixing Grothendieck universes, we denote by Cat_∞ the ∞-category of small ∞-categories and CAT_∞ the ∞-category of locally small but possibly large ∞-categories. We denote by Pr^{L} (resp. Pr^{R}) the ∞-category of presentable ∞-categories and left (resp. right) adjoint functors. 1-Categories are typically referred to as just 'categories' and viewed as ∞-categories via the nerve construction. We denote by $\mathcal{S}$ the ∞-category of spaces, by $\mathcal{S}p$ the ∞-category of spectra and by $\mathcal{S}p_{\geq 0} \subset \mathcal{S}p$ the full subcategory of connective spectra.

If $C^\otimes$ is a symmetric monoidal ∞-category, we denote by $\mathrm{CAlg}(C)$ the ∞-category of commutative algebras in $C^\otimes$. (In particular, $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is the ∞-category of presentable symmetric monoidal ∞-categories and left adjoint symmetric monoidal functors.) If A is a commutative algebra in C, we denote by $\mathrm{Mod}_A(C)$ the ∞-category of A-modules. When $C = \mathcal{S}p$, we often write CAlg instead of $\mathrm{CAlg}(\mathcal{S}p)$ and Mod_A instead of $\mathrm{Mod}_A(\mathcal{S}p)$.

Given an ∞-category C, we denote by $\mathrm{Map}_C(x, y)$ the mapping space between two objects x and y in C. Given another ∞-category $\mathcal{D}$, we denote by $\mathrm{Fun}(C, \mathcal{D})$ the ∞-category of functors from C to $\mathcal{D}$. If C is small, we denote by $\mathcal{P}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$ the ∞-category of presheaves on C. If Λ is a commutative ring spectrum, we denote by $\mathrm{Psh}(C; \Lambda) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$ the ∞-category of presheaves of Λ-modules on C. Given a topology τ on C, we denote by $\mathrm{Shv}_\tau(C) \subset \mathcal{P}(C)$ and $\mathrm{Shv}_\tau(C; \Lambda) \subset \mathrm{Psh}(C; \Lambda)$ the full sub-∞-categories of τ-hypersheaves. For an object $X \in C$, we denote by $\Lambda_\tau(X) \in \mathrm{Shv}_\tau(C; \Lambda)$ the τ-hypersheaf associated to the presheaf of Λ-modules freely generated on X.

Varieties (algebraic, rigid analytic). We always denote by k the ground field for algebraic varieties and by K the ground field for rigid analytic varieties. By 'algebraic k-variety' we mean a finite type k-scheme and by 'rigid analytic K-variety' we mean an adic space over K, in the sense of Huber, which is locally of finite type. (We do not assume that rigid analytic K-varieties are quasi-compact since the analytification of an algebraic K-variety is rarely quasi-compact.) We denote by Sm_k the category of smooth k-varieties and by RigSm_K the category of smooth rigid analytic K-varieties. We also denote by $\mathrm{RigSm}_K^{\mathrm{qcqs}} \subset \mathrm{RigSm}_K$ the full subcategory of quasi-compact and quasi-separated smooth rigid analytic K-varieties. These three categories will be endowed with the étale topology which we abbreviate by 'ét'.

It will be convenient for us to allow the ground valued field K to be non necessary complete, and we denote by $\widehat{K}$ its completion. In particular, the expression 'rigid analytic K-variety' really means 'rigid analytic $\widehat{K}$-variety'. We usually assume that the residue field of K is the ground field k for algebraic varieties, although we will also consider algebraic varieties over K. In general, we write 'pt' for $\mathrm{Spec}(k)$ or $\mathrm{Spa}(\widehat{K})$. As usual, we write $\mathbb{A}^1$ for the affine line, $\mathbb{P}^1$ for the projective line and $\mathbb{B}^1$ for the Tate ball.

Motives (algebraic, rigid analytic). We fix a connective commutative ring spectrum Λ, and we always work Λ-linearly. (In particular, the tensor product of Λ-modules will be denoted by $- \otimes -$ instead of $- \otimes_\Lambda -$.) We will always assume that the exponent characteristic of k in invertible in Λ. This applies to the ground field for our algebraic varieties and to the residue field of the ground valued field for our rigid analytic varieties.

Given a scheme S, we denote by $\mathbf{SH}_{\text{ét}}(S;\Lambda)$ the Morel–Voevodsky ∞-category of étale motives over S with coefficients in Λ. Similarly, given a rigid analytic space S, we denote by $\mathbf{RigSH}_{\text{ét}}(S;\Lambda)$ the ∞-category of étale rigid analytic motives over S with coefficients in Λ. We are mainly interested in the case where S is the spectrum (resp. adic spectrum) of the ground field, and in this case we write $\mathbf{SH}_{\text{ét}}(k;\Lambda)$ (resp. $\mathbf{RigSH}_{\text{ét}}(K;\Lambda)$). Given a smooth algebraic k-variety (resp. smooth rigid analytic K-variety) X, we denote by $\mathrm{M}(X) = \Sigma^\infty_T \mathrm{L}_{\text{mot}}(\Lambda_{\text{ét}}(X))$ its associated motive.

1. Weil cohomology theories in algebraic geometry

Let k be a ground field and let $\Lambda \in \mathrm{CAlg}$ be a connective commutative ring spectrum. We will always assume that the exponent characteristic of k is invertible in $\pi_0(\Lambda)$. We denote by Sm_k the category of smooth algebraic k-varieties. In this article, we adopt the following general notion of a Weil cohomology theory. (Compare with [CD12, Definition 2.1.4] and [Ayo20, Définition 1.1].)

Definition 1.1. A Weil cohomology theory Γ_W for algebraic k-varieties is a presheaf of commutative Λ-algebras on Sm_k satisfying the following properties.

(1) ($\mathbb{A}^1$-invariance) The obvious morphism $\Gamma_W(\mathrm{pt}) \to \Gamma_W(\mathbb{A}^1)$ is an equivalence.
(2) The $\Gamma_W(\mathrm{pt})$-module $\Gamma_W(\mathbb{P}^1,\infty) = \mathrm{cofib}\{\Gamma_W(\mathrm{pt}) \to \Gamma_W(\mathbb{P}^1)\}$ is invertible.
(3) (Künneth formula) For every $X, Y \in \mathrm{Sm}_k$, the obvious morphism
$$\Gamma_W(X) \otimes_{\Gamma_W(\mathrm{pt})} \Gamma_W(Y) \to \Gamma_W(X \times Y)$$
is an equivalence.
(4) The presheaf Γ_W admits étale hyperdescent.

The commutative Λ-algebra $\Gamma_W(\mathrm{pt})$ is called the coefficient ring of Γ_W.

Definition 1.2. We denote by $\mathrm{WCT}(k;\Lambda)$ the ∞-category of Weil cohomology theories for algebraic k-varieties. This is the nonfull sub-∞-category of $\mathrm{Fun}((\mathrm{Sm}_k)^{\mathrm{op}}, \mathrm{CAlg}_{\Lambda\backslash})$ spanned by morphisms between Weil cohomology theories $\Gamma_W \to \Gamma_{W'}$ such that the induced morphism
$$\Gamma_W(\mathbb{P}^1,\infty) \otimes_{\Gamma_W(\mathrm{pt})} \Gamma_{W'}(\mathrm{pt}) \to \Gamma_{W'}(\mathbb{P}^1,\infty)$$
is an equivalence.

Remark 1.3. Let $\Gamma_W \in \mathrm{WCT}(k;\Lambda)$ be a Weil cohomology theory. For $n \in \mathbb{Z}$, we set:
$$\Gamma_W(n) = \Gamma_W \otimes_{\Gamma_W(\mathrm{pt})} \big(\Gamma_W(\mathbb{P}^1,\infty)[-2]\big)^{\otimes -n}.$$
This is an invertible Γ_W-module. It follows from the Künneth formula that
$$\Gamma_W(- \times (\mathbb{P}^1,\infty))(n)[2n] \simeq \Gamma_W(-)(n-1)[2n-2]. \tag{1.1}$$
Observe that, given a morphism of Weil cohomology theories $\Gamma_W \to \Gamma_{W'}$, we have natural morphisms of Γ_W-modules $\Gamma_W(n) \to \Gamma_{W'}(n)$ for all $n \in \mathbb{Z}$. This is clear for $n \geq 0$ and, for $n < 0$, we use the condition imposed in Definition 1.2. By the way, note that this condition depends only on the class of a morphism in the homotopy category of $\mathrm{Fun}((\mathrm{Sm}_k)^{\mathrm{op}}, \mathrm{CAlg}_{\Lambda\backslash})$, so that $\mathrm{WCT}(k;\Lambda)$ is a well defined nonfull sub-∞-category (see [Lur09, §1.2.11]).

Weil cohomology theories are representable in the Morel–Voevodsky stable homotopy category, and even in its étale localization. To explain this, we start by recalling a few basic definitions.

Definition 1.4. We denote by $\mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Lambda) \subset \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}_k;\Lambda)$ the full sub-∞-category of $\mathbb{A}^1$-local étale hypersheaves of Λ-modules on Sm_k. We denote by

$$\mathrm{L}_{\mathrm{mot}} : \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}_k;\Lambda) \to \mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Lambda)$$

the motivic localization functor. Note that $\mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Lambda)$ underlies a symmetric monoidal structure. We denote by $\mathbf{SH}_{\text{ét}}(k;\Lambda)^{\otimes}$ the symmetric monoidal ∞-category obtained from $\mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Lambda)^{\otimes}$ by inverting the object $T = \mathrm{L}_{\mathrm{mot}}(\Lambda_{\text{ét}}(\mathbb{P}^1,\infty))$ for the tensor product. Given $X \in \mathrm{Sm}_k$, we denote by $\mathrm{M}^{\mathrm{eff}}(X)$ and $\mathrm{M}(X)$ the objects $\mathrm{L}_{\mathrm{mot}}(\Lambda_{\text{ét}}(X))$ and $\Sigma^{\infty}_T\mathrm{L}_{\mathrm{mot}}(\Lambda_{\text{ét}}(X))$ in $\mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Lambda)$ and $\mathbf{SH}_{\text{ét}}(k;\Lambda)$.

Remark 1.5. By [Rob15, Corollary 2.22], the ∞-category $\mathbf{SH}_{\text{ét}}(k;\Lambda)$ is the limit of the tower

$$\cdots \xrightarrow{\Omega^1_T} \mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Lambda) \xrightarrow{\Omega^1_T} \mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Lambda).$$

Thus, an object of $\mathbf{SH}_{\text{ét}}(k;\Lambda)$ is a T-spectrum, i.e., a sequence $E = (E_n)_{n\in\mathbb{N}}$ of $\mathbb{A}^1$-local étale hypersheaves of Λ-modules together with equivalences $E_n \simeq \Omega^1_T(E_{n+1}) = \underline{\mathrm{Hom}}(T, E_{n+1})$. The functor $E \mapsto E_n$ will be denoted by Ev^n_T and its left adjoint will be denoted by Sus^n_T. For $n = 0$, these functors are more commonly denoted by Ω^{∞}_T and Σ^{∞}_T respectively.

Remark 1.6. The ∞-category $\mathbf{SH}_{\text{ét}}(k;\Lambda)$ is part of a six-functor formalism: for every scheme S, we have an ∞-category of motivic sheaves $\mathbf{SH}_{\text{ét}}(S;\Lambda)$ underlying a closed symmetric monoidal structure, and for every finite type morphism of schemes f we have functors f^*, f_*, $f_!$ and $f^!$. (See [Ayo07a, Ayo07b], and [AGV22, Theorem 4.6.1] for a modern and concise discussion.) We will occasionally make use of this formalism, for example in Lemma 1.12 below.

We will need the following simple fact concerning the notion of idempotent algebras in the sense of [Lur18, Definition 2.6.0.1].

Lemma 1.7. *Let $C^{\otimes}$ be a symmetric monoidal ∞-category, and denote by $\mathbf{1}$ its unit object. Let A be an E_0-algebra in C, i.e., an object of C endowed with a morphism $u : \mathbf{1} \to A$. Then the following conditions are equivalent.*

(1) The morphism $u \otimes \mathrm{id}_A : A \to A \otimes A$ is an equivalence. (We express this by saying that the E_0-algebra A is idempotent.)
(2) The endofunctor $\rho : C \to C$, $M \mapsto M \otimes A$, together with the natural transformation $\mathrm{id} \to \rho$ induced by u, defines a localization functor.

Moreover, if these conditions are satisfied, there is a unique commutative algebra structure on A extending the given E_0-algebra structure, and the commutative algebra A is idempotent in the sense of [Lur18, Definition 2.6.0.1].

Proof. The implication (1) $\Rightarrow$ (2) follows from [Lur09, Proposition 5.2.7.4] and the converse is obvious. To prove the second assertion, we note that the localization functor ρ is compatible with the symmetric monoidal structure on C in the sense of [Lur17, Definition 2.2.1.6 & Example 2.2.1.7]. It follows from [Lur17, Proposition 2.2.1.9] that ρ is right-lax monoidal. In particular, A is naturally a commutative algebra. For the unicity, we note that a commutative algebra structure on A extending the E_0-algebra structure determines (and is determined by) a symmetric monoidal functor $\widetilde{\rho} : C^{\otimes} \to \mathrm{Mod}_A(C)^{\otimes}$ whose underlying functor is $\rho : C \to \rho(C)$. Thus, we may invoke again [Lur17, Proposition 2.2.1.9] to conclude. □

Notation 1.8. If R be a commutative algebra in $\mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Lambda)$, we write $\mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;R)$ and $\mathbf{SH}_{\text{ét}}(k;R)$ for the ∞-categories $\mathrm{Mod}_R(\mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Lambda))$ and $\mathrm{Mod}_R(\mathbf{SH}_{\text{ét}}(k;\Lambda))$. Similarly, if R' is a commutative algebra in $\mathbf{SH}_{\text{ét}}(k;\Lambda)$, we write $\mathbf{SH}_{\text{ét}}(k;R')$ for the ∞-category $\mathrm{Mod}_{R'}(\mathbf{SH}_{\text{ét}}(k;\Lambda))$.

Proposition 1.9. *Let $\Gamma_W \in \mathrm{WCT}(k;\Lambda)$ be a Weil cohomology theory. Then Γ_W is a commutative algebra in $\mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Lambda)$ and there is a unique commutative algebra $\mathbf{\Gamma}_W$ in $\mathbf{SH}_{\text{ét}}(k;\Gamma_W)$ satisfying the following conditions:*

(1) the morphism $\Gamma_W \to \Omega^\infty_T(\mathbf{\Gamma}_W)$ is an equivalence;
(2) the underlying spectrum of $\mathbf{\Gamma}_W$ is given by $\Gamma_W(n)[2n]$ in level n and has assembly maps induced from the equivalence in (1.1)*.*

Moreover, $\mathbf{\Gamma}_W$ is an idempotent algebra in $\mathbf{SH}_{\text{ét}}(k;\Gamma_W)$.

Proof. The first statement is clear. By Remark 1.5, there is a T-spectrum $\mathbf{\Gamma}_W \in \mathbf{SH}_{\text{ét}}(k;\Gamma_W)$ satisfying the description in (2). We have a morphism $u : \Sigma^\infty_T \Gamma_W \to \mathbf{\Gamma}_W$ from the unit object of $\mathbf{SH}_{\text{ét}}(k;\Gamma_W)$ and, by Lemma 1.7, it is enough to show that $\mathbf{\Gamma}_W$ is idempotent as an E_0-algebra.

We first reduce to the case where k has finite virtual Λ-cohomological dimension in the sense of [AGV22, Definition 2.4.8]. To do so, consider the family $(k_\alpha)_\alpha$ of subfields of k that are finitely generated over their prime field. We claim that the obvious symmetric monoidal functor

$$\operatorname*{colim}_\alpha \mathbf{SH}^{(\mathrm{eff})}_{\text{ét}}(k_\alpha;\Lambda) \to \mathbf{SH}^{(\mathrm{eff})}_{\text{ét}}(k;\Lambda), \tag{1.2}$$

where the colimit is computed in Pr^{L}, is a localization. Indeed, we have a commutative square in Pr^{L}:

$$\begin{array}{ccc} \operatorname*{colim}_\alpha \mathbf{SH}^{(\mathrm{eff})}(k_\alpha;\Lambda) & \longrightarrow & \mathbf{SH}^{(\mathrm{eff})}(k;\Lambda) \\ \downarrow{\scriptstyle \mathrm{L}_{\text{ét}}} & & \downarrow{\scriptstyle \mathrm{L}_{\text{ét}}} \\ \operatorname*{colim}_\alpha \mathbf{SH}^{(\mathrm{eff})}_{\text{ét}}(k_\alpha;\Lambda) & \longrightarrow & \mathbf{SH}^{(\mathrm{eff})}_{\text{ét}}(k;\Lambda), \end{array}$$

where $\mathbf{SH}^{(\mathrm{eff})}(-;\Lambda)$ is the Nisnevich local version of $\mathbf{SH}^{(\mathrm{eff})}_{\text{ét}}(-;\Lambda)$. The vertical arrows in this square are localizations. (Indeed, colimits in Pr^{L} preserve localizations as it follows easily from their universal properties.) Moreover, the top horizontal arrow is an equivalence by [AGV22, Proposition 2.5.11]. This proves our claim.

Precomposing Γ_W with the base change functor $\mathrm{Sm}_{k_\alpha} \to \mathrm{Sm}_k$ yields a Weil cohomology theory $\Gamma_{W,\alpha} \in \mathrm{WCT}(k_\alpha;\Lambda)$. The associated motivic spectrum $\mathbf{\Gamma}_{W,\alpha}$ is nothing but the image of $\mathbf{\Gamma}_W$ by the functor $(k/k_\alpha)_* : \mathbf{SH}_{\text{ét}}(k;\Lambda) \to \mathbf{SH}_{\text{ét}}(k_\alpha;\Lambda)$. In fact, the families $(\Gamma_{W,\alpha})_\alpha$ and $(\mathbf{\Gamma}_{W,\alpha})_\alpha$ define objects of the ∞-categories

$$\operatorname*{colim}_\alpha \mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k_\alpha;\Lambda) \qquad \text{and} \qquad \operatorname*{colim}_\alpha \mathbf{SH}_{\text{ét}}(k_\alpha;\Lambda).$$

(See [Lur09, Corollary 5.5.3.4 & Theorem 5.5.3.18]). These objects are by construction the images of Γ_W and $\mathbf{\Gamma}_W$ by the right adjoint of the functor in (1.2). Since this functor is a localization, we deduce that Γ_W and $\mathbf{\Gamma}_W$ are also the images of $(\Gamma_{W,\alpha})_\alpha$ and $(\mathbf{\Gamma}_{W,\alpha})_\alpha$ by (1.2). It is therefore sufficient to prove that the E_0-algebras $\mathbf{\Gamma}_{W,\alpha} \in \mathbf{SH}_{\text{ét}}(k_\alpha,\Gamma_{W,\alpha})$ are idempotent. In this way, we are reduced to the case where k is finitely generated over its prime field, and hence of finite virtual Λ-cohomological dimension in the sense of [AGV22, Definition 2.4.8]. Now, the result follows from Lemma 1.10 below since $\Gamma_W(r) \otimes_{\Gamma_W} \Gamma_W(s) = \Gamma_W(r+s)$, for all $r, s \in \mathbb{Z}$. □

The following lemma is needed for the proof of Proposition 1.9.

Lemma 1.10. *Assume that k has finite virtual Λ-cohomological dimension in the sense of* [AGV22, Definition 2.4.8]. *Let R be a commutative algebra in $\mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Lambda)$, and let $E = (E_n)_{n\in\mathbb{N}}$ and $F =$*

$(F_n)_{n\in\mathbb{N}}$ be two T-spectra in $\mathbf{SH}_{\text{ét}}(k;R)$. Then the T-spectrum $E\otimes_R F$ is given in level n by

$$\mathrm{Ev}_T^n(E\otimes_R F) = \operatorname*{colim}_{r+s\geq n} \underline{\mathrm{Hom}}(T^{\otimes r+s-n}, E_r\otimes_R F_s). \tag{1.3}$$

Proof. We have $E\simeq \operatorname{colim}_r \mathrm{Sus}_T^r(E_r)$ and similarly for F. It follows that

$$E\otimes_R F = \operatorname*{colim}_{r,s\geq 0} \mathrm{Sus}_T^r(E_r)\otimes_R \mathrm{Sus}_T^s(F_s) \simeq \operatorname*{colim}_{r,s\geq 0} \mathrm{Sus}_T^{r+s}(E_r\otimes_R F_s). \tag{1.4}$$

(See [Ayo07b, Corollaire 4.3.72].) Using [Ayo07b, Théorème 4.3.61], we deduce that

$$\mathrm{Ev}_T^n(E\otimes_R F) = \operatorname*{colim}_{e+r+s\geq n} \underline{\mathrm{Hom}}(T^{\otimes e+r+s-n}, T^{\otimes e}\otimes E_r\otimes_R F_s). \tag{1.5}$$

Note that [Ayo07b, Hypothèse 4.3.56], which is implicitly assumed in the statement of [Ayo07b, Théorème 4.3.61], is satisfied when k has finite virtual Λ-cohomological dimension by [AGV22, Proposition 3.2.3]. We now notice that, in the colimit in (1.5), the map from the (e,r,s)-th term to the $(e,r+e,s)$-th term factors through the $(0,r+e,s)$-th term using the assembly morphism $T^{\otimes e}\otimes E_r\to E_{r+e}$. The same is true with s in place of r. This implies that the colimit in (1.5) is equivalent to the one in the statement. □

The commutative algebra $\mathbf{\Gamma}_W$ has the following remarkable property that was uncovered in [CD12, Theorem 2.6.2] in the case where the coefficient ring of Γ_W is a field.

Proposition 1.11. *Let $\Gamma_W\in \mathrm{WCT}(k;\Lambda)$ be a Weil cohomology theory, and let $\mathbf{\Gamma}_W$ be the associated motivic commutative ring spectrum. Then, the obvious functor*

$$\mathrm{Mod}_{\Gamma_W(\mathrm{pt})}\to \mathbf{SH}_{\text{ét}}(k;\mathbf{\Gamma}_W),\quad M\mapsto \mathbf{\Gamma}_W\otimes_{\Gamma_W(\mathrm{pt})} M \tag{1.6}$$

is an equivalence.

Proof. The functor (1.6) is fully faithful. Indeed, let P and Q be two $\Gamma_W(\mathrm{pt})$-modules. The motivic spectrum $\mathbf{\Gamma}_W\otimes_{\Gamma_W(\mathrm{pt})} P$ is given in level r by $\Gamma_W\otimes_{\Gamma_W(\mathrm{pt})} P(r)[2r]$, and similarly for Q in place of P. It follows that we have a chain of equivalences:

$$\begin{aligned}
&\mathrm{Map}_{\mathbf{SH}_{\text{ét}}(k;\mathbf{\Gamma}_W)}(\mathbf{\Gamma}_W\otimes_{\Gamma_W(\mathrm{pt})} P, \mathbf{\Gamma}_W\otimes_{\Gamma_W(\mathrm{pt})} Q)\\
\overset{(1)}{\simeq}\ &\mathrm{Map}_{\mathbf{SH}_{\text{ét}}(k;\Gamma_W)}(\mathbf{\Gamma}_W\otimes_{\Gamma_W(\mathrm{pt})} P, \mathbf{\Gamma}_W\otimes_{\Gamma_W(\mathrm{pt})} Q)\\
\overset{(2)}{\simeq}\ &\lim_r \mathrm{Map}_{\mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Gamma_W)}(\Gamma_W\otimes_{\Gamma_W(\mathrm{pt})} P(r)[2r], \Gamma_W\otimes_{\Gamma_W(\mathrm{pt})} Q(r)[2r])\\
\overset{(3)}{\simeq}\ &\mathrm{Map}_{\mathbf{SH}^{\mathrm{eff}}_{\text{ét}}(k;\Gamma_W)}(\Gamma_W\otimes_{\Gamma_W(\mathrm{pt})} P, \Gamma_W\otimes_{\Gamma_W(\mathrm{pt})} Q)\\
\overset{(4)}{\simeq}\ &\mathrm{Map}_{\mathrm{Mod}_{\Gamma_W(\mathrm{pt})}}(P, \Gamma(\mathrm{pt};\Gamma_W\otimes_{\Gamma_W(\mathrm{pt})} Q))\\
\overset{(5)}{\simeq}\ &\mathrm{Map}_{\mathrm{Mod}_{\Gamma_W(\mathrm{pt})}}(P, Q)
\end{aligned}$$

where:

(1) follows from the fact that $\mathbf{\Gamma}_W$ is an idempotent commutative algebra in $\mathbf{SH}_{\text{ét}}(k;\Gamma_W)$;
(2) follows from Remark 1.5;
(3) follows by noticing that the pro-system on the previous line is constant;
(4) is by adjunction;
(5) is obvious.

Since the functor (1.6) is colimit-preserving, it remains to see that its essential image contains a set of objects generating $\mathbf{SH}_{\text{ét}}(k;\mathbf{\Gamma}_W)$ under colimits. By Lemma 1.12 below, we are reduced to showing that $(\pi_{X,*}\Lambda)\otimes\mathbf{\Gamma}_W$ belongs to the image of functor (1.6). But, since $\pi_{X,*}\Lambda$ is dualizable with dual $\pi_{X,\sharp}\Lambda$, we have equivalences

$$(\pi_{X,*}\Lambda)\otimes\mathbf{\Gamma}_W \simeq \underline{\mathrm{Hom}}(\pi_{X,\sharp}\Lambda,\mathbf{\Gamma}_W)\simeq \pi_{X,*}\pi_X^*\mathbf{\Gamma}_W.$$

The motivic spectrum $\pi_{X,*}\pi_X^*\mathbf{\Gamma}_W$ is given in level r by $\Gamma_W(-\times X)(r)[2r]$ which, by the Künneth formula, can be identified with $\Gamma_W(-)\otimes_{\Gamma_W(\mathrm{pt})}\Gamma_W(X)(r)[2r]$. This shows that $\pi_{X,*}\pi_X^*\mathbf{\Gamma}_W$ is equivalent to $\mathbf{\Gamma}_W\otimes_{\Gamma_W(\mathrm{pt})}\Gamma_W(X)$ as needed. □

Given an algebraic k-variety X, we denote by $\pi_X : X\to\mathrm{pt}$ its structural morphism. The following lemma was used in the proof of Proposition 1.11.

Lemma 1.12. *For every $X\in\mathrm{Sm}_k$, the motive $\pi_{X,\sharp}\Lambda\in\mathbf{SH}_{\text{ét}}(k;\Lambda)$ is dualizable with dual $\pi_{X,*}\Lambda$. Moreover, the following holds.*

(1) The ∞-category $\mathbf{SH}_{\text{ét}}(k;\Lambda)$ is generated under colimits, desuspension and Tate twists by the objects $\pi_{X,\sharp}\Lambda$ for $X\in\mathrm{Sm}_k$.
(2) The ∞-category $\mathbf{SH}_{\text{ét}}(k;\Lambda)$ is generated under colimits, desuspension and Tate twists by the objects $\pi_{X,*}\Lambda$ for $X\in\mathrm{Sm}_k$.

Proof. It suffices to prove this for $\mathbf{SH}(k;\Lambda)$, the Nisnevich local version of $\mathbf{SH}_{\text{ét}}(k;\Lambda)$. The lemma then follows from [EK20, Theorem 3.2.1] which is based on [LYZR19, Corollary B.2]. Indeed, by loc. cit., the sub-∞-category of dualizable objects coincides with the stable thick sub-∞-category generated by $\pi_{X,\sharp}\Lambda(m)[n]$, for $X\in\mathrm{Sm}_k$ and $m,n\in\mathbb{Z}$. But since the duality functor $M\mapsto M^\vee$ is an anti-equivalence of the sub-∞-category of dualizable objects, we deduce that the latter is also generated by $\pi_{X,*}\Lambda(m)[n]$, for $X\in\mathrm{Sm}_k$ and $m,n\in\mathbb{Z}$. □

Proposition 1.13. *Let $R=(R_n)_{n\in\mathbb{N}}$ be a commutative algebra in $\mathbf{SH}_{\text{ét}}(k;\Lambda)$. Then the following conditions are equivalent.*

(1) There is a Weil cohomology theory $\Gamma_W\in\mathrm{WCT}(k;\Lambda)$ such that R is equivalent to $\mathbf{\Gamma}_W$.
(2) The presheaf of commutative algebras $R_0=\Omega^\infty_T(R)$ is a Weil cohomology theory and the R_0-modules R_n are invertible.
(3) For every R-module $M=(M_n)_{n\in\mathbb{N}}$, the morphism $R\otimes_{R_0(\mathrm{pt})}M_0(\mathrm{pt})\to M$ is an equivalence.

Proof. The implication (1) ⇒ (2) is clear. To prove the converse, we note that the condition in (2) implies that R, viewed as an E_0-algebra in $\mathbf{SH}_{\text{ét}}(k;R_0)$, is equivalent to the E_0-algebra associated to the Weil cohomology theory R_0 as in the proof of Proposition 1.9. The result then follows from Lemma 1.7. The implication (1) ⇒ (3) follows from Proposition 1.11. To finish the proof, it remains to see that (3) implies (2). Assume that $R=(R_n)_{n\in\mathbb{N}}$ satisfies (3). Since the T-spectrum $(R_{n+m})_{n\in\mathbb{N}}$ is naturally an R-module, we get that $R_m\simeq R_0\otimes_{R_0(\mathrm{pt})}R_m(\mathrm{pt})$. This gives the equivalences

$$R_0(\mathrm{pt})\simeq R_m((\mathbb{P}^1,\infty)^{\wedge m})\simeq R_0((\mathbb{P}^1,\infty)^{\wedge m})\otimes_{R_0(\mathrm{pt})}R_m(\mathrm{pt})$$

showing that $R_m(\mathrm{pt})$ is invertible. To conclude, it remains to see that R_0 is a Weil cohomology theory. Clearly, the presheaf R_0 is $\mathbb{A}^1$-invariant and admits étale hyperdescent. Also, we have just proven that $R_0(\mathbb{P}^1,\infty)$ is invertible with inverse $R_1(\mathrm{pt})$. We now check the Künneth formula for two smooth algebraic k-varieties X and Y. In $\mathbf{SH}_{\text{ét}}(k;\Lambda)$, we have an equivalence $\pi_{X,*}\Lambda\otimes\pi_{Y,*}\Lambda\simeq\pi_{X\times Y,*}\Lambda$ where π_X, π_Y and $\pi_{X\times Y}$ are the structural morphisms of X, Y and $X\times Y$. This follows for

instance from the fact that $\pi_{X,*}\Lambda$ is dualizable with dual $\pi_{X,\sharp}\Lambda$ (see Lemma 1.12). Tensoring with R, we obtain an equivalence

$$(R \otimes \pi_{X,*}\Lambda) \otimes_R (R \otimes \pi_{Y,*}\Lambda) \simeq R \otimes \pi_{X\times Y,*}\Lambda. \tag{1.7}$$

Using again that $\pi_{X,*}\Lambda$ is dualizable with dual $\pi_{X,\sharp}\Lambda$, we have $R \otimes \pi_{X,*}\Lambda \simeq \pi_{X,*}R$ and similarly for Y and $X \times Y$ in place of X. Thus, we can rewrite the equivalence (1.7) as

$$\pi_{X,*}R \otimes_R \pi_{Y,*}R \simeq \pi_{X\times Y,*}R. \tag{1.8}$$

Since $\pi_{X,*}R$ is an R-module, it is equivalent to $R \otimes_{R_0(\mathrm{pt})} R_0(X)$, and similarly for Y and $X \times Y$ in place of X. Thus, passing to the 0-level and taking global sections in the equivalence (1.8) yield the equivalence $R_0(X) \otimes_{R_0(\mathrm{pt})} R_0(Y) \simeq R_0(X \times Y)$. This finishes the proof. □

Definition 1.14. A Weil spectrum for algebraic k-varieties is a commutative algebra in the ∞-category $\mathbf{SH}_{\text{ét}}(k;\Lambda)$ satisfying the equivalent conditions in Proposition 1.13. We denote by $\mathrm{WSp}(k;\Lambda)$ the full sub-∞-category of $\mathrm{CAlg}(\mathbf{SH}_{\text{ét}}(k;\Lambda))$ spanned by Weil spectra.

Corollary 1.15. *Let $\Gamma_W \in \mathrm{WCT}(k;\Lambda)$ be a Weil cohomology theory, and let R be a commutative $\mathbf{\Gamma}_W$-algebra in $\mathbf{SH}_{\text{ét}}(k;\Lambda)$. Then R is a Weil spectrum and we have an equivalence*

$$R \simeq \mathbf{\Gamma}_W \otimes_{\Gamma_W(\mathrm{pt})} \mathrm{R}\Gamma(\mathrm{pt}; \Omega^\infty_T(R)). \tag{1.9}$$

Proof. The equivalence (1.9) follows from Proposition 1.11. Letting $A = \mathrm{R}\Gamma(\mathrm{pt}; \Omega^\infty_T(R))$, we deduce that R is given by $\Gamma_W(n)[2n] \otimes_{\Gamma_W(\mathrm{pt})} A$ in level n. In particular, R_0 is a Weil cohomology theory and the R_0-module R_n is invertible for every $n \geq 0$. We conclude using Proposition 1.13. □

Theorem 1.16. *The functor $\Omega^\infty_T : \mathrm{WSp}(k;\Lambda) \to \mathrm{WCT}(k;\Lambda)$ is an equivalence of ∞-categories.*

Proof. The functor in the statement is well defined. Indeed, Ω^∞_T takes a morphism of Weil spectra to a morphism of Weil cohomology theories, i.e., to a morphism of presheaves of commutative Λ-algebras satisfying the condition in Definition 1.2. This follows immediately from the property (3) in Proposition 1.13 by evaluating at $(\mathbb{P}^1, \infty)$.

We now construct a functor $\beta : \mathrm{WCT}(k;\Lambda) \to \mathrm{WSp}(k;\Lambda)$ sending a Weil cohomology theory Γ_W to its Weil spectrum $\mathbf{\Gamma}_W$. Consider the cocartesian fibrations

$$p^{(\mathrm{eff})} : \int_{\Gamma_W \in \mathrm{WCT}(k;\Lambda)} \mathbf{SH}^{(\mathrm{eff})}_{\text{ét}}(k;\Gamma_W) \to \mathrm{WCT}(k;\Lambda).$$

By Remark 1.5, the domain of p is the limit of a tower having the domain of p^{eff} on every stage and where the functors between successive stages are given by Ω^1_T fiberwise. In particular, we have

$$\mathrm{Sect}(p) = \lim\left(\cdots \xrightarrow{\Omega^1_T} \mathrm{Sect}(p^{\mathrm{eff}}) \xrightarrow{\Omega^1_T} \mathrm{Sect}(p^{\mathrm{eff}})\right).$$

For $r \in \mathbb{N}$, we have a section $\sigma_r \in \mathrm{Sect}(p)$ given by $\Gamma_W \mapsto \mathrm{Sus}^r_T(\Gamma_W(r)[2r])$. Noting that the section of p^{eff} sending Γ_W to the presheaf $\Gamma_W((\mathbb{P}^1,\infty)\times -)(r)[2r] \simeq \Gamma_W(r-1)[2r-2]$ maps naturally to the section $\mathrm{Ev}^{r-1}_T(\sigma_r)$, we obtain natural morphisms $\sigma_{r-1} \to \sigma_r$ for $r \geq 1$. We let $\sigma = \mathrm{colim}_r\, \sigma_r$. By Proposition 1.9, the E_0-algebra σ in $\mathrm{Sect}(p)$ (with unit $\sigma_0 \to \sigma$) is idempotent. Thus, σ is naturally a commutative algebra in $\mathrm{Sect}(p)$. Said differently, we can view σ as a section of the cocartesian fibration

$$\widetilde{p} : \int_{\Gamma_W \in \mathrm{WCT}(k;\Lambda)} \mathrm{CAlg}(\mathbf{SH}_{\text{ét}}(k;\Gamma_W)) \to \mathrm{WCT}(k;\Lambda).$$

Composing σ with the obvious projection to $\mathrm{CAlg}(\mathbf{SH}_{\text{ét}}(k;\Lambda))$, we obtain a functor

$$\beta : \mathrm{WCT}(k;\Lambda) \to \mathrm{CAlg}(\mathbf{SH}_{\text{ét}}(k;\Lambda))$$

whose image lies in the sub-∞-category $\mathrm{WSp}(k;\Lambda)$. By construction, the composition of

$$\mathrm{WCT}(k;\Lambda) \xrightarrow{\beta} \mathrm{WSp}(k;\Lambda) \xrightarrow{\Omega^\infty_T} \mathrm{WCT}(k;\Lambda)$$

is the identity functor. To see that the composition $\beta \circ \Omega^\infty_T$ is also the identity functor, we are reduced to showing that the section $\sigma' = \sigma \circ \Omega^\infty_T$ of the cocartesian fibration

$$\widetilde{p}' : \int_{\mathbf{\Gamma}_W \in \mathrm{WSp}(k;\Lambda)} \mathrm{CAlg}(\mathbf{SH}_{\text{ét}}(k;\Omega^\infty_T \mathbf{\Gamma}_W)) \to \mathrm{WSp}(k;\Lambda)$$

is equivalent to the obvious diagonal section δ. By Lemma 1.7, it is enough to do so for the underlying E_0-algebras, which is clear. □

Definition 1.17. Let $\Gamma_W \in \mathrm{WCT}(k;\Lambda)$ be a Weil cohomology theory. The realization functor associated to Γ_W is the functor $\mathrm{R}^*_W : \mathbf{SH}_{\text{ét}}(k;\Lambda) \to \mathrm{Mod}_{\Gamma_W(\mathrm{pt})}$ given by the composition of

$$\mathbf{SH}_{\text{ét}}(k;\Lambda) \xrightarrow{\mathbf{\Gamma}_W \otimes -} \mathbf{SH}_{\text{ét}}(k;\mathbf{\Gamma}_W) \simeq \mathrm{Mod}_{\Gamma_W(\mathrm{pt})},$$

where the equivalence is provided by Proposition 1.11. The functor R^*_W underlies a symmetric monoidal functor and it admits a right adjoint $\mathrm{R}_{W,*}$ sending a $\Gamma_W(\mathrm{pt})$-module M to $\mathbf{\Gamma}_W \otimes_{\Gamma_W(\mathrm{pt})} M$.

Remark 1.18. A Weil cohomology theory Γ_W can be recovered from the associated realization functor R^*_W since $\mathbf{\Gamma}_W \simeq \mathrm{R}_{W,*}\mathrm{R}^*_W\Lambda$.

Definition 1.19. A plain realization functor for algebraic k-varieties is a morphism

$$\mathrm{R}^* : \mathbf{SH}_{\text{ét}}(k;\Lambda)^\otimes \to \mathrm{Mod}^\otimes_A$$

in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, where $A \in \mathrm{CAlg}$ is a commutative ring spectrum. The ∞-category $\mathrm{Real}(k;\Lambda)$ of plain realization functors is the full sub-∞-category of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathbf{SH}_{\text{ét}}(k;\Lambda)^\otimes\backslash}$ spanned by functors with codomain of the form $\mathrm{Mod}^\otimes_A$ for some $A \in \mathrm{CAlg}$.

Proposition 1.20. *For* $\mathrm{R}^* \in \mathrm{Real}(k;\Lambda)$*, the commutative algebra* $\mathrm{R}_*\mathrm{R}^*\Lambda$ *is a Weil spectrum.*

Proof. Let $\mathrm{Mod}^\otimes_A$ be the codomain of R^*. The functor R^* factors through the functor

$$\widetilde{\mathrm{R}}^* : \mathbf{SH}_{\text{ét}}(k;\mathrm{R}_*A) \to \mathrm{Mod}_A, \quad M \mapsto \mathrm{R}^*(M) \otimes_{\mathrm{R}^*\mathrm{R}_*A} A.$$

By Corollary 1.13, it is enough to show that $\widetilde{\mathrm{R}}^*$ is an equivalence. To do so, we first reduce to the case where k has finite virtual Λ-cohomological dimension in the sense of [AGV22, Definition 2.4.8]. Consider the family $(k_\alpha)_\alpha$ of subfields of k that are finitely generated over their prime field. Precomposing with the functors $(k/k_\alpha)^*$ yields plain realizations functors $\mathrm{R}^*_\alpha \in \mathrm{Real}(k_\alpha;\Lambda)$ and we have the induced functor

$$\widetilde{\mathrm{R}}^*_\alpha : \mathbf{SH}_{\text{ét}}(k_\alpha;\mathrm{R}_{\alpha,*}A) \to \mathrm{Mod}_A.$$

It is easy to see that we have a commutative triangle

$$\begin{array}{ccc} \operatorname*{colim}_\alpha \mathbf{SH}_{\text{ét}}(k_\alpha;\mathrm{R}_{\alpha,*}A) & \xrightarrow{\phi} & \mathbf{SH}_{\text{ét}}(k;\mathrm{R}_*A) \\ & \searrow^{\widetilde{\mathrm{R}}^*_\infty} & \downarrow{\scriptstyle \widetilde{\mathrm{R}}^*} \\ & & \mathrm{Mod}_A \end{array}$$

where the colimit is taken in Pr^{L}. Moreover, it follows from [AGV22, Proposition 2.5.11] that the functor ϕ is a localization. (See the proof of Proposition 1.9.) If we knew that the R^*_α's were equivalences, we would deduce that the localization functor ϕ admits a retraction, and hence must be an equivalence, and we would be done. Thus, it suffices to treat the case where k is finitely generated over its prime field, and hence of finite virtual Λ-cohomological dimension in the sense of [AGV22, Definition 2.4.8].

We are now in the favourable situation were the ∞-category $\mathbf{SH}_{\text{ét}}(k;\Lambda)$ is compactly generated by its dualizable objects. (Use [AGV22, Proposition 3.2.3] and Lemma 1.12.) The same is true for $\mathbf{SH}_{\text{ét}}(k;\mathrm{R}_*A)$. Since dualizable objects in Mod_A are compact (by [Lur17, Proposition 7.2.4.4]), the functor $\widetilde{\mathrm{R}}^*$ preserves compact objects and its right adjoint $\widetilde{\mathrm{R}}_*$ is colimit-preserving. Since $\widetilde{\mathrm{R}}^*$ is clearly essentially surjective, we only need to show that the unit morphism $\mathrm{id} \to \widetilde{\mathrm{R}}_*\widetilde{\mathrm{R}}^*$ is an equivalence. Since the domain and codomain of this morphism are colimit-preserving, it is enough to check this after evaluation on compact generators, and hence on objects of the form $M \otimes \mathrm{R}_*A$ for $M \in \mathbf{SH}_{\text{ét}}(k;\Lambda)$ dualizable. In this case, the morphism we are considering can be written as $M \otimes \mathrm{R}_*A \to \mathrm{R}_*\mathrm{R}^*M$. We now conclude using [Ayo14b, Lemme 2.8]. □

Theorem 1.21. *The functor* $\mathrm{Real}(k;\Lambda) \to \mathrm{WSp}(k;\Lambda)$*, given by* $\mathrm{R}^* \mapsto \mathrm{R}_*\mathrm{R}^*\Lambda$*, is an equivalence of* ∞*-categories.*

Proof. Let us call α the functor of the statement. As in Definition 1.17, we also have a functor $\beta : \mathrm{WSp}(k;\Lambda) \to \mathrm{Real}(k;\Lambda)$ sending a Weil spectrum $\mathbf{\Gamma}_W$ to the associated realization functor R^*_W. By Remark 1.18, we have $\alpha \circ \beta = \mathrm{id}$. To prove that $\beta \circ \alpha = \mathrm{id}$, we observe that there is a commutative diagram

$$\begin{array}{ccc} \mathbf{SH}_{\text{ét}}(k;\Lambda) & & \\ {\scriptstyle \mathrm{R}_*(A)\otimes -}\downarrow & \searrow^{\mathrm{R}^*} & \\ \mathbf{SH}_{\text{ét}}(k;\mathrm{R}_*A) & \xrightarrow[\widetilde{\mathrm{R}}^*]{} & \mathrm{Mod}_A, \end{array}$$

which is natural in $\mathrm{R}^* \in \mathrm{Real}(k;\Lambda)$. But we have $\beta \circ \alpha(\mathrm{R}^*) = \widetilde{\mathrm{R}}^*(\mathrm{R}_*(A) \otimes -)$ since $\widetilde{\mathrm{R}}^*$ is a quasi-inverse to the obvious functor $\mathrm{Mod}_A \to \mathbf{SH}_{\text{ét}}(k;\mathrm{R}_*A)$. This finishes the proof. □

We end this section with the following result.

Proposition 1.22. *The functor* $\mathrm{WCT}(k;\Lambda) \to \mathrm{CAlg}_{\Lambda\backslash}$*, given by* $\Gamma_W \mapsto \Gamma_W(\mathrm{pt})$*, is a left fibration. The analogous statement is also true for* $\mathrm{WSp}(k;\Lambda)$ *and* $\mathrm{Real}(k;\Lambda)$*.*

Proof. Since the ∞-categories $\mathrm{WCT}(k;\Lambda)$, $\mathrm{WSp}(k;\Lambda)$ and $\mathrm{Real}(k;\Lambda)$ are all equivalent, it is enough to prove the first statement. Recall that a left fibration is a cocartesian fibration whose fibres are groupoids (see [Lur09, Proposition 2.4.2.4]). Clearly, $\mathrm{WCT}(k;\Lambda) \to \mathrm{CAlg}_{\Lambda\backslash}$ is a cocartesian fibration classified by the functor sending a commutative Λ-algebra A to the ∞-category $\mathrm{WCT}(k)_A$ of Weil cohomology theories Γ_W such that $\Gamma_W(\mathrm{pt}) = A$. Thus, to conclude, it is enough to show that a morphism of Weil cohomology theories $\Gamma_W \to \Gamma_{W'}$ is an equivalence provided that the induced morphism $\Gamma_W(\mathrm{pt}) \to \Gamma_{W'}(\mathrm{pt})$ is an equivalence. To prove this, we note that since $\mathbf{\Gamma}_{W'}$ is a $\mathbf{\Gamma}_W$-module, we have $\mathbf{\Gamma}_{W'} = \mathbf{\Gamma}_W \otimes_{\Gamma_W(\mathrm{pt})} \Gamma_{W'}(\mathrm{pt})$ by Proposition 1.11. □

2. Weil cohomology theories in rigid analytic geometry

Let K be a field endowed with a nontrivial valuation of height 1, and let k be the residue field of K. We denote by $\widehat{K}$ the completion of K, and we denote by K° and $\widehat{K}^\circ$ the rings of integers of

K and $\widehat{K}$. By 'rigid analytic K-variety' we mean a locally finite type adic $\widehat{K}$-space in the sense of Huber. We denote by RigSm_K the category of smooth rigid analytic K-varieties, and by $\mathrm{RigSm}_K^{\mathrm{qcqs}}$ its full subcategory spanned by the quasi-compact and quasi-separated ones. We fix a connective commutative ring spectrum $\Lambda \in \mathrm{CAlg}$ and assume that the exponent characteristic of k is invertible in $\pi_0(\Lambda)$. In this section, we extend the theory developed in Section 1 to the setting of rigid analytic K-varieties. This turns out to be straightforward most of the time.

Definition 2.1. A Weil cohomology theory Γ_W for rigid analytic K-varieties is a presheaf of commutative Λ-algebras on RigSm_K satisfying the following properties.

(1) ($\mathbb{B}^1$-invariance) The obvious morphism $\Gamma_W(\mathrm{pt}) \to \Gamma_W(\mathbb{B}^1)$ is an equivalence.
(2) The $\Gamma_W(\mathrm{pt})$-module $\Gamma_W(\mathbb{P}^1, \infty) = \mathrm{cofib}\{\Gamma_W(\mathrm{pt}) \to \Gamma_W(\mathbb{P}^1)\}$ is invertible.
(3) (Künneth formula) For every $X, Y \in \mathrm{RigSm}_K^{\mathrm{qcqs}}$, the obvious morphism
$$\Gamma_W(X) \otimes_{\Gamma_W(\mathrm{pt})} \Gamma_W(Y) \to \Gamma_W(X \times Y)$$
is an equivalence.
(4) The presheaf Γ_W admits étale hyperdescent.

The commutative Λ-algebra $\Gamma_W(\mathrm{pt})$ is called the coefficient ring of Γ_W.

Remark 2.2. Definition 2.1 corrects [Ayo20, Définition 2.15] where the Künneth formula was requested for all $X, Y \in \mathrm{RigSm}_K$. This is clearly an unreasonable demand since the tensor product of Λ-modules does not commute with infinite direct products in both variables, unless Λ is zero. However, using Proposition 2.6 below, one can prove that the Künneth formula holds if only X or Y is assumed to be quasi-compact and quasi-separated. Indeed, if X is quasi-compact and quasi-separated, the $\Gamma_W(\mathrm{pt})$-module $\Gamma_W(X)$ is dualizable, and hence the functor $\Gamma_W(X) \otimes_{\Gamma_W(\mathrm{pt})} (-)$ is limit-preserving. Without using Proposition 2.6, we still can conclude that the Künneth formula is true when $X = \mathbb{P}^1$ and Y is general using that $\Gamma_W(\mathbb{P}^1, \infty)$ is invertible. In particular, defining the twisted presheaves $\Gamma_W(n)$, for $n \in \mathbb{Z}$, as in Remark 1.3, we still have equivalences
$$\Gamma_W(- \times (\mathbb{P}^1, \infty))(n)[2n] \simeq \Gamma_W(-)(n-1)[2n-2] \tag{2.1}$$
of presheaves on RigSm_K.

Definition 2.3. We denote by $\mathrm{RigWCT}(K; \Lambda)$ the ∞-category of Weil cohomology theories for rigid analytic K-varieties. This is the nonfull sub-∞-category of $\mathrm{Fun}((\mathrm{RigSm}_K)^{\mathrm{op}}; \mathrm{CAlg}_{\Lambda\backslash})$ spanned by morphisms between Weil cohomology theories $\Gamma_W \to \Gamma_{W'}$ such that the induced morphism
$$\Gamma_W(\mathbb{P}^1, \infty) \otimes_{\Gamma_W(\mathrm{pt})} \Gamma_{W'}(\mathrm{pt}) \to \Gamma_{W'}(\mathbb{P}^1, \infty)$$
is an equivalence.

We need to recall the rigid analytic version of the Morel–Voevodsky stable homotopy category which is the natural home for rigid analytic Weil spectra.

Definition 2.4. We denote by $\mathbf{RigSH}^{\mathrm{eff}}_{\text{ét}}(K; \Lambda) \subset \mathrm{Shv}_{\text{ét}}(\mathrm{RigSm}_K; \Lambda)$ the full sub-∞-category of $\mathbb{B}^1$-local étale hypersheaves of Λ-modules on RigSm_K. We denote by
$$\mathrm{L}_{\mathrm{mot}} : \mathrm{Shv}_{\text{ét}}(\mathrm{RigSm}_K; \Lambda) \to \mathbf{RigSH}^{\mathrm{eff}}_{\text{ét}}(K; \Lambda)$$
the motivic localization functor. We denote by $\mathbf{RigSH}_{\text{ét}}(K; \Lambda)^{\otimes}$ the symmetric monoidal ∞-category obtained from $\mathbf{RigSH}^{\mathrm{eff}}_{\text{ét}}(K; \Lambda)^{\otimes}$ by inverting the object $T = \mathrm{L}_{\mathrm{mot}}(\Lambda_{\text{ét}}(\mathbb{P}^1, \infty))$ for the tensor product. Given $X \in \mathrm{RigSm}_K$, we denote by $\mathrm{M}^{\mathrm{eff}}(X)$ and $\mathrm{M}(X)$ the objects $\mathrm{L}_{\mathrm{mot}}\Lambda_{\text{ét}}(X)$ and $\Sigma^{\infty}_T \mathrm{L}_{\mathrm{mot}}\Lambda_{\text{ét}}(X)$ in $\mathbf{RigSH}^{\mathrm{eff}}_{\text{ét}}(K; \Lambda)$ and $\mathbf{RigSH}_{\text{ét}}(K; \Lambda)$.

Remarks 1.5 and 1.6 apply in the rigid analytic setting. Below, we will use the obvious analog of Notation 1.8.

Proposition 2.5. *Let $\Gamma_W \in \mathrm{RigWCT}(K;\Lambda)$ be a Weil cohomology theory. Then Γ_W is a commutative algebra in $\mathbf{RigSH}^{\mathrm{eff}}_{\text{ét}}(K;\Lambda)$ and there is a unique commutative algebra $\mathbf{\Gamma}_W$ in $\mathbf{RigSH}_{\text{ét}}(K;\Gamma_W)$ satisfying the following conditions:*

(1) the morphism $\Gamma_W \to \Omega^\infty_T(\mathbf{\Gamma}_W)$ is an equivalence;
(2) the underlying spectrum of $\mathbf{\Gamma}_W$ is given by $\Gamma_W(n)[2n]$ in level n and has assembly maps induced from the equivalence in (2.1)*.*

Moreover, $\mathbf{\Gamma}_W$ is an idempotent algebra in $\mathbf{RigSH}_{\text{ét}}(K;\Gamma_W)$.

Proof. The proof of Proposition 1.9 extends easily to the rigid analytic setting. We need to reduce to the case where $\widehat{K}$ has finite virtual Λ-cohomological dimension. To do so, we write K as the union of its subfields K_α which are non discrete and finitely generated over their prime field. Then, using [AGV22, Theorem 2.5.1] and arguing as in the proof of Proposition 1.9, we see that

$$\operatorname*{colim}_\alpha \mathbf{RigSH}_{\text{ét}}(K_\alpha;\Lambda) \to \mathbf{RigSH}_{\text{ét}}(K;\Lambda)$$

is a localization functor, as needed for the reduction. Finally, we note that the rigid analytic version of Lemma 1.10 holds true and can be proven similarly. □

Proposition 2.6. *Let $\Gamma_W \in \mathrm{RigWCT}(K;\Lambda)$ be a Weil cohomology theory, and let $\mathbf{\Gamma}_W$ be the associated motivic commutative ring spectrum. Then, the obvious functor*

$$\mathrm{Mod}_{\Gamma_W(\mathrm{pt})} \to \mathbf{RigSH}_{\text{ét}}(K;\mathbf{\Gamma}_W), \quad M \mapsto \mathbf{\Gamma}_W \otimes_{\Gamma_W(\mathrm{pt})} M \tag{2.2}$$

is an equivalence.

Proof. The proof of Proposition 1.11 extends literally to the rigid analytic setting provided that we have a rigid analytic version of Lemma 1.12. This is the subject of Lemma 2.8 below. □

Lemma 2.7. *Let $f : T \to S$ be a smooth and proper morphism of rigid analytic spaces. Let $M \in \mathbf{RigSH}_{\text{ét}}(T)$ be a dualizable object. Then f_*M is dualizable with dual $f_\sharp M^\vee$.*

Proof. For $N \in \mathbf{RigSH}_{\text{ét}}(S)$, we have the following chain of equivalences

$$\underline{\mathrm{Hom}}(f_*M,N) \overset{(1)}{\simeq} \underline{\mathrm{Hom}}(f_!M,N) \overset{(2)}{\simeq} f_*\underline{\mathrm{Hom}}(M,f^!N) \overset{(3)}{\simeq} f_*\underline{\mathrm{Hom}}(M,f^!\Lambda\otimes f^*N)$$

$$\overset{(4)}{\simeq} f_*\left(\underline{\mathrm{Hom}}(M,f^!\Lambda)\otimes f^*N\right) \overset{(5)}{\simeq} f_*\underline{\mathrm{Hom}}(M,f^!\Lambda)\otimes N \overset{(6)}{\simeq} f_!(\mathrm{Th}(\Omega_f)\otimes M^\vee)\otimes N$$

where:

(1) follows from the identification $f_! \simeq f_*$ (see [AGV22, Proposition 4.4.27]);
(2) follows from [AGV22, Corollary 4.5.4];
(3) follows from [AGV22, Theorem 4.4.29];
(4) follows from the assumption that M is dualizable;
(5) follows from [AGV22, Proposition 4.1.7];
(6) follows from [AGV22, Theorem 4.4.29].

This proves that f_*M is dualizable with dual $f_!(\mathrm{Th}(\Omega_f)\otimes M^\vee) = f_\sharp M^\vee$ as needed. □

Given a rigid analytic K-variety X, we denote by $\pi_X : X \to \mathrm{pt} = \mathrm{Spa}(\widehat{K})$ its structural morphism. The following lemma is used in the proof of Proposition 2.6.

Lemma 2.8. *For every $X \in \mathrm{RigSm}_K^{\mathrm{qcqs}}$, the motive $\pi_{X,\sharp}\Lambda \in \mathbf{RigSH}_{\text{ét}}(K;\Lambda)$ is dualizable with dual $\pi_{X,*}\Lambda$. Moreover, the following holds.*

(1) The ∞-category $\mathbf{RigSH}_{\text{ét}}(K;\Lambda)$ is generated under colimits, desuspension and Tate twists by the objects $\pi_{X,\sharp}\Lambda$ for $X \in \mathrm{RigSm}_K^{\mathrm{qcqs}}$.

(2) The ∞-category $\mathbf{RigSH}_{\text{ét}}(K;\Lambda)$ is generated under colimits, desuspension and Tate twists by the objects $\pi_{X,*}\Lambda$ for $X \in \mathrm{RigSm}_K^{\mathrm{qcqs}}$.

In fact, in (1) and (2), we may restrict to those X's with potential good reduction.

Proof. Since $\underline{\mathrm{Hom}}(\pi_{X,\sharp}\Lambda,\Lambda) = \pi_{X,*}\Lambda$, the dual of $\pi_{X,\sharp}\Lambda$ is necessarily $\pi_{X,*}\Lambda$ provided that $\pi_{X,\sharp}\Lambda$ is dualizable. Therefore, it suffices to show point (1) and that the thick stable sub-∞-category generated by $\pi_{X,\sharp}\Lambda(m)[n]$, for $m,n \in \mathbb{N}$ and $X \in \mathrm{RigSm}_K^{\mathrm{qcqs}}$, is closed under taking the dual. To prove this, we may employ the device used in the proof of Proposition 2.5 to reduce to the case where $\widehat{K}$ is has finite virtual Λ-cohomological dimension. In this case, we know that $\mathbf{RigSH}_{\text{ét}}(K;\Lambda)$ is compactly generated by the $\pi_{X,\sharp}\Lambda(m)[n]$, for $m,n \in \mathbb{N}$ and $X \in \mathrm{RigSm}_K^{\mathrm{qcqs}}$, and it remains to see that these generators are dualizable. (Indeed, the dual is then necessary compact, and the second point to check is automatic.)

By [AGV22, Proposition 3.7.17], the ∞-category $\mathbf{RigSH}_{\text{ét}}(K;\Lambda)$ is generated under colimits by objects of the form $\pi_{X,\sharp}\Lambda(-m)[-n]$ where $m,n \in \mathbb{Z}$ are integers and X has potentially good reduction. More precisely, we can assume that there are a finite étale extension $L/\widehat{K}$ and a smooth formal L°-scheme $\mathcal{X}$ such that $X = \mathcal{X}_\eta$. By Lemma 2.7 applied to $f : \mathrm{Spa}(L) \to \mathrm{Spa}(\widehat{K}) = \mathrm{pt}$, it is enough to show that $\pi_{X/L,\sharp}\Lambda$ is dualizable. Said differently, we may assume that X has good reduction, i.e., that $X = \mathcal{X}_\eta$ with $\mathcal{X}$ smooth over $\widehat{K}^\circ$. In this case, $\pi_{X,\sharp}\Lambda$ is the image of $\pi_{\mathcal{X}_\sigma,\sharp}\Lambda$ by the symmetric monoidal functor

$$\xi : \mathbf{SH}_{\text{ét}}(k;\Lambda) \simeq \mathbf{FSH}_{\text{ét}}(\widehat{K}^\circ;\Lambda) \to \mathbf{RigSH}_{\text{ét}}(K;\Lambda)$$

described in [AGV22, Notation 3.1.12]. (See also Remark 3.1 below.) By Lemma 1.12, $\pi_{\mathcal{X}_\sigma,\sharp}\Lambda$ is dualizable, and this enables us to conclude. □

Proposition 2.9. *Let $R = (R_n)_{n\in\mathbb{N}}$ be a commutative algebra in $\mathbf{RigSH}_{\text{ét}}(K;\Lambda)$. Then the following conditions are equivalent.*

(1) There is a Weil cohomology theory $\Gamma_W \in \mathrm{RigWCT}(K;\Lambda)$ such that R is equivalent to $\mathbf{\Gamma}_W$.

(2) The presheaf of commutative algebras $R_0 = \Omega_T^\infty(R)$ is a Weil cohomology theory and the R_0-modules R_n are invertible.

(3) For every R-module $M = (M_n)_{n\in\mathbb{N}}$, the morphism $R \otimes_{R_0(\mathrm{pt})} M_0(\mathrm{pt}) \to M$ is an equivalence.

Proof. The proof of Proposition 1.13 extends literally to the rigid analytic setting. Of course, one needs to use Lemma 2.8 in place of Lemma 1.12. □

Definition 2.10. A Weil spectrum for rigid analytic K-varieties is a commutative algebra in the ∞-category $\mathbf{RigSH}_{\text{ét}}(K;\Lambda)$ satisfying the equivalent conditions in Proposition 2.9. We denote by $\mathrm{RigWSp}(K;\Lambda)$ the full sub-∞-category of $\mathrm{CAlg}(\mathbf{RigSH}_{\text{ét}}(K;\Lambda))$ spanned by Weil spectra.

Corollary 2.11. *Let $\Gamma_W \in \mathrm{RigWCT}(K;\Lambda)$ be a Weil cohomology theory, and let R be a commutative $\mathbf{\Gamma}_W$-algebra in $\mathbf{RigSH}_{\text{ét}}(K;\Lambda)$. Then R is a Weil spectrum and we have an equivalence*

$$R \simeq \mathbf{\Gamma}_W \otimes_{\Gamma_W(\mathrm{pt})} \mathrm{R}\Gamma(\mathrm{pt};\Omega_T^\infty(R)). \tag{2.3}$$

Proof. The proof of Corollary 1.15 extends literally. □

Theorem 2.12. *The functor* $\Omega_T^\infty : \mathrm{RigWSp}(K;\Lambda) \to \mathrm{RigWCT}(K;\Lambda)$ *is an equivalence of* ∞*-categories.*

Proof. The proof of Theorem 1.16 extends literally. □

Definition 2.13. Let $\Gamma_W \in \mathrm{RigWCT}(K;\Lambda)$ be a Weil cohomology theory. The realization functor associated to Γ_W is the functor $\mathrm{R}_W^* : \mathbf{RigSH}_{\text{ét}}(K;\Lambda) \to \mathrm{Mod}_{\Gamma_W(\mathrm{pt})}$ given by the composition of

$$\mathbf{RigSH}_{\text{ét}}(K;\Lambda) \xrightarrow{\mathbf{\Gamma}_W \otimes -} \mathbf{RigSH}_{\text{ét}}(K;\mathbf{\Gamma}_W) \simeq \mathrm{Mod}_{\Gamma_W(\mathrm{pt})},$$

where the equivalence is provided by Proposition 2.6. The functor R_W^* underlies a symmetric monoidal functor and it admits a right adjoint $\mathrm{R}_{W,*}$ sending a $\Gamma_W(\mathrm{pt})$-module M to $\mathbf{\Gamma}_W \otimes_{\Gamma_W(\mathrm{pt})} M$.

Definition 2.14. A plain realization functor for rigid analytic K-varieties is a morphism

$$\mathrm{R}^* : \mathbf{RigSH}_{\text{ét}}(K;\Lambda)^\otimes \to \mathrm{Mod}_A^\otimes$$

in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, where A is a commutative ring spectrum. The ∞-category $\mathrm{RigReal}(K;\Lambda)$ of plain realization functors is the full sub-∞-category of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathbf{RigSH}_{\text{ét}}(K;\Lambda)^\otimes\backslash}$ spanned by functors with codomain of the form $\mathrm{Mod}_A^\otimes$ for some $A \in \mathrm{CAlg}$.

Proposition 2.15. *For* $\mathrm{R}^* \in \mathrm{RigReal}(K;\Lambda)$*, the commutative algebra* $\mathrm{R}_*\mathrm{R}^*\Lambda$ *is a Weil spectrum.*

Proof. The proof of Proposition 1.20 can be easily adapted to the rigid analytic setting. The reduction to the case where K has finite virtual Λ-cohomological dimension is obtained as in the proof of Proposition 2.5. In this case $\mathbf{RigSH}_{\text{ét}}(K;\Lambda)$ is compactly generated by its dualizable objects as it follows from Lemma 2.8 and [AGV22, Proposition 2.4.22]. □

Theorem 2.16. *The functor* $\mathrm{RigReal}(K;\Lambda) \to \mathrm{RigWSp}(K;\Lambda)$*, given by* $\mathrm{R}^* \mapsto \mathrm{R}_*\mathrm{R}^*\Lambda$*, is an equivalence of* ∞*-categories.*

Proof. The proof of Theorem 1.21 extends literally. □

Proposition 2.17. *The functor* $\mathrm{RigWCT}(K;\Lambda) \to \mathrm{CAlg}_{\Lambda\backslash}$*, given by* $\Gamma_W \mapsto \Gamma_W(\mathrm{pt})$*, is a left fibration. The analogous statement is also true for* $\mathrm{RigWSp}(K;\Lambda)$ *and* $\mathrm{RigReal}(K;\Lambda)$*.*

Proof. The proof of Proposition 1.22 extends literally. □

3. From algebraic geometry to rigid analytic geometry and back

We keep the running notations and assumptions from Sections 1 and 2. We will discuss the relations between Weil cohomology theories for algebraic varieties and for rigid analytic varieties. For this, we need to recall the two functors relating algebraic motives and rigid analytic motives.

Remark 3.1. There are two symmetric monoidal functors

$$\mathrm{Rig}^* : \mathbf{SH}_{\text{ét}}(K;\Lambda) \to \mathbf{RigSH}_{\text{ét}}(K;\Lambda) \qquad \text{and} \qquad \xi^* : \mathbf{SH}_{\text{ét}}(k;\Lambda) \to \mathbf{RigSH}_{\text{ét}}(K;\Lambda).$$

The functor Rig^* is induced by the analytification functor $X \mapsto X^{\mathrm{an}}$ sending a smooth K-variety X to its rigid analytification. (The effective version of this functor is constructed in [Ayo15, Proposition 1.3.6] and its stabilisation is introduced on page 101 of [Ayo15]; this functor is also recalled on page 34 of [AGV22] under the name 'An^*'.) Modulo the equivalence $\mathbf{FSH}_{\text{ét}}(\widehat{K}^\circ;\Lambda) \simeq \mathbf{SH}_{\text{ét}}(k;\Lambda)$ of [Ayo15, Corollaire 1.4.24], the functor ξ^* is induced by the functor $\mathcal{X} \mapsto \mathcal{X}_\eta$ taking a formal $\widehat{K}^\circ$-scheme to its Raynaud generic fibre. (This functor is discussed in [AGV22, §3.1].) As usual, we denote by Rig_* and ξ_* the right adjoints of Rig^* and ξ^*.

Proposition 3.2. *Let* $\mathbf{\Gamma}_W \in \mathrm{RigWSp}(K;\Lambda)$ *be a Weil spectrum for rigid analytic* K*-varieties. Then* $\mathrm{Rig}_*\mathbf{\Gamma}_W$ *and* $\xi_*\mathbf{\Gamma}_W$ *are Weil spectra for algebraic varieties. Moreover, if* R^*_W *is the realization functor associated to* $\mathbf{\Gamma}_W$*, then* $\mathrm{R}^*_W \circ \mathrm{Rig}^*$ *and* $\mathrm{R}^*_W \circ \xi^*$ *are the realization functors associated to* $\mathrm{Rig}_*\mathbf{\Gamma}_W$ *and* $\xi_*\mathbf{\Gamma}_W$ *respectively.*

Proof. It is clear that $\mathrm{R}^*_W \circ \mathrm{Rig}^*$ and $\mathrm{R}^*_W \circ \xi^*$ belong $\mathrm{Real}(K;\Lambda)$ and $\mathrm{Real}(k;\Lambda)$ respectively, and that the Weil spectra associated to these plain realization functors are $\mathrm{Rig}_*\mathbf{\Gamma}_W$ and $\xi_*\mathbf{\Gamma}_W$. □

Remark 3.3. We have a commutative diagram of ∞-categories

$$\begin{array}{ccccc}
\mathrm{Real}(k;\Lambda) & \xrightarrow{\sim} & \mathrm{WSp}(k;\Lambda) & \xrightarrow[\sim]{\Omega^\infty_T} & \mathrm{WCT}(k;\Lambda) \\
\uparrow{\scriptstyle -\circ\xi^*} & & \uparrow{\scriptstyle \xi_*} & & \uparrow{\scriptstyle \xi_*} \\
\mathrm{RigReal}(K;\Lambda) & \xrightarrow{\sim} & \mathrm{RigWSp}(K;\Lambda) & \xrightarrow[\sim]{\Omega^\infty_T} & \mathrm{RigWCT}(K;\Lambda) \\
\downarrow{\scriptstyle -\circ\mathrm{Rig}^*} & & \downarrow{\scriptstyle \mathrm{Rig}_*} & & \downarrow{\scriptstyle \mathrm{Rig}_*} \\
\mathrm{Real}(K;\Lambda) & \xrightarrow{\sim} & \mathrm{WSp}(K;\Lambda) & \xrightarrow[\sim]{\Omega^\infty_T} & \mathrm{WCT}(K;\Lambda)
\end{array}$$

where the horizontal arrows are equivalences. Moreover, the functor

$$\mathrm{Rig}_* : \mathrm{RigWCT}(K;\Lambda) \to \mathrm{WCT}(K;\Lambda)$$

is just the naive one given by composing a Weil cohomology theory on rigid analytic K-varieties with the rigid analytification functor $X \mapsto X^{\mathrm{an}}$.

The following result will play an important role in the sequel.

Theorem 3.4. *Let* $\mathbf{\Gamma}_W \in \mathrm{WSp}(K;\Lambda)$ *be a Weil spectrum for algebraic* K*-varieties. Then* $\mathrm{Rig}^*\mathbf{\Gamma}_W$ *is a Weil spectrum for rigid analytic* K*-varieties. Thus, we have a functor*

$$\mathrm{Rig}^* : \mathrm{WSp}(K;\Lambda) \to \mathrm{RigWSp}(K;\Lambda)$$

which is left adjoint to the functor Rig_* *from Remark 3.3.*

To prove Theorem 3.4, we need the following result.

Proposition 3.5. *The image of the functor* $\mathrm{Rig}^* : \mathbf{SH}_{\text{ét}}(K;\Lambda) \to \mathbf{RigSH}_{\text{ét}}(K;\Lambda)$ *generates the* ∞*-category* $\mathbf{RigSH}_{\text{ét}}(K;\Lambda)$ *under colimits.*

Proof. The case where Λ is a commutative $\mathbb{Q}$-algebra was discussed in [Ayo20, Proposition 2.31] and it is based on [Ayo15, Théorème 2.5.35]. (In loc. cit., Λ is supposed to be classical, but this is not a real restriction since the statement for $\Lambda = \mathbb{Q}$ implies the statement for any commutative $\mathbb{Q}$-algebra.) To prove the proposition, we reduce the general case to the case where Λ is a $\mathbb{Q}$-algebra using rigidity. Given $M \in \mathbf{RigSH}_{\text{ét}}(K;\Lambda)$, we may consider the cofibre sequence

$$M \to M \otimes \mathbb{Q} \to M \otimes \mathbb{Q}/\mathbb{Z}.$$

Since the forgetful functor $\mathrm{Mod}_{\Lambda\otimes\mathbb{Q}} \to \mathrm{Mod}_\Lambda$ is colimit-preserving, there is a commutative square

$$\begin{array}{ccc}
\mathbf{SH}_{\text{ét}}(K;\Lambda\otimes\mathbb{Q}) & \xrightarrow{\mathrm{Rig}^*\otimes\mathbb{Q}} & \mathbf{RigSH}_{\text{ét}}(K;\Lambda\otimes\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathbf{SH}_{\text{ét}}(K;\Lambda) & \xrightarrow{\mathrm{Rig}^*} & \mathbf{RigSH}_{\text{ét}}(K;\Lambda).
\end{array}$$

The object $M \otimes \mathbb{Q}$ belongs to the image of the right vertical functor. Thus, by [Ayo20, Proposition 2.31], $M \otimes \mathbb{Q}$ belongs to the localizing subcategory generated by the image of Rig^*. To conclude, it remains to prove the same for $M \otimes \mathbb{Q}/\mathbb{Z}$. Using [AGV22, Theorems 2.10.3 & 2.10.4], we are reduced to showing that the image of the functor

$$\mathrm{Shv}_{\text{ét}}(\text{Ét}_K; \Lambda) \to \mathrm{Shv}_{\text{ét}}(\text{Ét}_{\widehat{K}}; \Lambda)$$

generates the ∞-category $\mathrm{Shv}_{\text{ét}}(\text{Ét}_{\widehat{K}}; \Lambda)$ under colimits. This follows from the fact that every finite separable extension of $\widehat{K}$ is a base change of a separable extension of K (by Krasner's lemma). □

Notation 3.6. The functor $\mathrm{Rig}_* : \mathbf{RigSH}_{\text{ét}}(K; \Lambda) \to \mathbf{SH}_{\text{ét}}(K; \Lambda)$ is right-lax monoidal. In particular, $\mathrm{Rig}_*\Lambda$ is a commutative algebra in $\mathbf{SH}_{\text{ét}}(K; \Lambda)$. We have an adjunction

$$\widetilde{\mathrm{Rig}^*} : \mathbf{SH}_{\text{ét}}(K; \mathrm{Rig}_*\Lambda) \rightleftarrows \mathbf{RigSH}_{\text{ét}}(K; \Lambda) : \widetilde{\mathrm{Rig}_*},$$

where $\widetilde{\mathrm{Rig}^*}$ is given by the formula $\widetilde{\mathrm{Rig}^*}(M) = \mathrm{Rig}^*(M) \otimes_{\mathrm{Rig}^*\mathrm{Rig}_*\Lambda} \Lambda$.

Theorem 3.7. *The functor* $\widetilde{\mathrm{Rig}^*}$ *is an equivalence of* ∞*-categories.*

Proof. The functor $\widetilde{\mathrm{Rig}^*}$ commutes with colimits. Using Proposition 3.5, it remains to see that it is fully faithful, i.e., that the unit of the adjunction $\mathrm{id} \to \widetilde{\mathrm{Rig}_*}\widetilde{\mathrm{Rig}^*}$ is an equivalence. By Lemma 3.8 below, it is enough to check this on a set of objects generating $\mathbf{SH}_{\text{ét}}(K; \mathrm{Rig}_*\Lambda)$ under colimits. By Lemma 1.12 and Proposition 3.5, such a set is given by objects of the form $M \otimes \mathrm{Rig}_*\Lambda$, with $M \in \mathbf{SH}_{\text{ét}}(K; \Lambda)$ dualizable. The unit morphism evaluated at such an object coincides with the obvious morphism $M \otimes \mathrm{Rig}_*\Lambda \to \mathrm{Rig}_*\mathrm{Rig}^*M$ which is an equivalence by [Ayo14b, Lemme 2.8]. □

Lemma 3.8. *The functor* $\mathrm{Rig}_* : \mathbf{RigSH}_{\text{ét}}(K; \Lambda) \to \mathbf{SH}_{\text{ét}}(K; \Lambda)$ *is colimit-preserving.*

Proof. The result is easy when $\widehat{K}$ has finite Λ-cohomological dimension. Indeed, in this case, the ∞-categories $\mathbf{SH}_{\text{ét}}(K; \Lambda)$ and $\mathbf{RigSH}_{\text{ét}}(K; \Lambda)$ are compactly generated by their dualizable objects, as it follows by combining [AGV22, Propositions 2.4.22 & 3.2.3] with Lemmas 1.12 and 2.8. Thus, it is enough to show that the left adjoint functor

$$\mathrm{Rig}^* : \mathbf{SH}_{\text{ét}}(K; \Lambda) \to \mathbf{RigSH}_{\text{ét}}(K; \Lambda)$$

preserves compact objects. But this is clear since this functor is monoidal and thus preserves dualizable objects. In the sequel, we will only use this observation when K is general and Λ is replaced with $\Lambda \otimes \mathbb{Q}$, in which case the condition on $\widehat{K}$ is void.

Let $(M_i)_{i \in I}$ be an inductive system in $\mathbf{RigSH}_{\text{ét}}(K; \Lambda)$. We want to show that the morphism

$$\operatorname*{colim}_i \mathrm{Rig}_*(M_i) \to \mathrm{Rig}_*\left(\operatorname*{colim}_i M_i\right)$$

is an equivalence. It is enough to do so for the systems $(M_i \otimes \mathbb{Q})_i$ and $(M_i \otimes \mathbb{Q}/\mathbb{Z})_i$. For the system $(M_i \otimes \mathbb{Q})_i$, we use the commutative square

$$\begin{array}{ccc} \mathbf{RigSH}_{\text{ét}}(K; \Lambda \otimes \mathbb{Q}) & \xrightarrow{\mathrm{Rig}_*} & \mathbf{SH}_{\text{ét}}(K; \Lambda \otimes \mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathbf{RigSH}_{\text{ét}}(K; \Lambda) & \xrightarrow{\mathrm{Rig}_*} & \mathbf{SH}_{\text{ét}}(K; \Lambda) \end{array}$$

where all the functors are colimit-preserving, except possibly the horizontal bottom one. (Here we are using the above discussion.)

For the system $(M_i \otimes \mathbb{Q}/\mathbb{Z})_i$, we need to work a bit more. We first prove that the functor

$$\iota_* : \mathrm{Shv}_{\text{ét}}(\text{Ét}_{\widehat{K}}; \Lambda) \to \mathrm{Shv}_{\text{ét}}(\text{Ét}_K; \Lambda) \tag{3.1}$$

is colimit-preserving. To do so, it is enough to prove that the functor

$$\iota_* : \mathrm{Shv}_{t_{\amalg}}(\text{Ét}_{\widehat{K}}) \to \mathrm{Shv}_{t_{\amalg}}(\text{Ét}_K),$$

which is obviously colimit-preserving, preserves the ét-local equivalences. (Here, we denote by $t_{\amalg}$ the Grothendieck topology generated by covers of the form $(U_j \to \coprod_{i\in I} U_i)_{j\in I}$.) Given a finite separable extension L/K, any faithfully flat étale $\widehat{K} \otimes_K L$-algebra can be refined by an algebra of the form $\widehat{K} \otimes_K L'$ for a finite separable extension L'/L. This can be used to show that if a morphism $u : A \to B$ of pointed $t_{\amalg}$-sheaves of spaces on $\text{Ét}_{\widehat{K}}$ induces isomorphisms $\mathrm{L}_{\text{ét}}\pi_n(A) \simeq \mathrm{L}_{\text{ét}}\pi_n(B)$ for $n \in \mathbb{N}$, then $\iota_*(u)$ also induces isomorphisms $\mathrm{L}_{\text{ét}}\pi_n(\iota_*(A)) \simeq \mathrm{L}_{\text{ét}}\pi_n(\iota_*(B))$.

Since the functor (3.1) is colimit-preserving, it also preserves torsion objects. Consider the following square

$$\begin{array}{ccc} \mathrm{Shv}_{\text{ét}}(\text{Ét}_{\widehat{K}}; \Lambda)_{\mathrm{tor}} & \xrightarrow{\iota_*} & \mathrm{Shv}_{\text{ét}}(\text{Ét}_K; \Lambda)_{\mathrm{tor}} \\ \downarrow & & \downarrow \\ \mathbf{RigSH}_{\text{ét}}(K; \Lambda) & \xrightarrow{\mathrm{Rig}_*} & \mathbf{SH}_{\text{ét}}(K; \Lambda) \end{array} \tag{3.2}$$

where the vertical arrows are colimit-preserving, fully faithful and induce equivalences with the sub-∞-categories of torsion objects by [AGV22, Theorems 2.10.3 & 2.10.4]. This square is in fact commutative. Indeed, the vertical arrows send a torsion étale hypersheaf F on $\text{Ét}_{\widehat{K}}$ (resp. Ét_K) to the T-spectrum given in level n by $F(n)[2n]$ left Kan extended to RigSm_K (resp. Sm_K) and hypersheafified. It now clear how to conclude: the system $(M_i \otimes \mathbb{Q}/\mathbb{Z})_i$ belongs to the essential image of the left vertical arrow in the square (3.2). Since all the functors in this square are colimit-preserving, except possibly the horizontal bottom one, the result follows. □

We can now give the proof of Theorem 3.4.

Proof of Theorem 3.4. Set $A = \Gamma(\mathrm{pt}; \Omega^\infty_T \mathrm{Rig}^* \mathbf{\Gamma}_W)$. By Proposition 2.9, it is enough to show that the functor

$$\mathrm{Mod}_A \to \mathbf{RigSH}_{\text{ét}}(K; \mathrm{Rig}^* \mathbf{\Gamma}_W)$$

is an equivalence. By Theorem 3.7, we have an equivalence of ∞-categories

$$\mathbf{RigSH}_{\text{ét}}(K; \mathrm{Rig}^* \mathbf{\Gamma}_W) \simeq \mathbf{SH}_{\text{ét}}(K; \mathbf{\Gamma}_W \otimes \mathrm{Rig}_* \Lambda).$$

Thus, setting $B = \Gamma(\mathrm{pt}; \Omega^\infty_T(\mathbf{\Gamma}_W \otimes \mathrm{Rig}_* \Lambda))$, it is enough to show that the functor

$$\mathrm{Mod}_B \to \mathbf{SH}_{\text{ét}}(K; \mathbf{\Gamma}_W \otimes \mathrm{Rig}_* \Lambda)$$

is an equivalence. By Proposition 1.13, we need to show that $\mathbf{\Gamma}_W \otimes \mathrm{Rig}_* \Lambda$ is a Weil spectrum. Since $\mathbf{\Gamma}_W$ is a Weil spectrum, Proposition 1.11 furnishes an equivalence of commutative algebras $\mathbf{\Gamma}_W \otimes_{\Gamma_W(\mathrm{pt})} B \simeq \mathbf{\Gamma}_W \otimes \mathrm{Rig}_* \Lambda$, showing that $\mathbf{\Gamma}_W \otimes \mathrm{Rig}_* \Lambda$ is indeed a Weil spectrum. □

The following is a corollary of the proof of Theorem 3.4.

Corollary 3.9. *Let $\mathbf{\Gamma}_W \in \mathrm{WSp}(K; \Lambda)$ be a Weil spectrum for algebraic K-varieties, and let $\mathrm{R}^*_W : \mathbf{SH}_{\text{ét}}(K; \Lambda) \to \mathrm{Mod}_{\Gamma_W(\mathrm{pt})}$ be the associated realization.*

(1) The coefficient ring of the Weil spectrum $\mathrm{Rig}^* \mathbf{\Gamma}_W$ is equivalent to $\mathrm{R}^*_W(\mathrm{Rig}_* \Lambda)$.

(2) The realization associated to $\mathrm{Rig}^*\mathbf{\Gamma}_W$ *is given by the composition of*

$$\mathbf{RigSH}_{\text{ét}}(K;\Lambda) \xrightarrow{\widetilde{\mathrm{Rig}_*}} \mathbf{SH}_{\text{ét}}(K;\mathrm{Rig}_*\Lambda) \xrightarrow{\mathrm{R}^*_W} \mathrm{Mod}_{\mathrm{R}^*_W(\mathrm{Rig}_*\Lambda)}. \tag{3.3}$$

Proof. The coefficient ring of $\mathrm{Rig}^*\mathbf{\Gamma}_W$ is the commutative algebra $A = \Gamma(\mathrm{pt};\Omega^\infty_T\mathrm{Rig}^*\mathbf{\Gamma}_W)$ which is equivalent to the commutative algebra $B = \Gamma(\mathrm{pt};\Omega^\infty_T(\mathbf{\Gamma}_W \otimes \mathrm{Rig}_*\Lambda))$. This proves the first claim since the realization functor R^*_W is given by $\Gamma(\mathrm{pt};\Omega^\infty_T(\mathbf{\Gamma}_W \otimes -))$. To prove the second statement, we need to show that the right adjoint to the functor (3.3) sends $\mathrm{R}^*_W(\mathrm{Rig}_*\Lambda)$ to $\mathrm{Rig}^*\mathbf{\Gamma}_W$. This is clear since the equivalence $\widetilde{\mathrm{Rig}_*}$ sends $\mathrm{Rig}^*\mathbf{\Gamma}_W$ to $\mathbf{\Gamma}_W \otimes \mathrm{Rig}_*\Lambda$ which is also the image of $\mathrm{R}^*_W(\mathrm{Rig}_*\Lambda)$ by the right adjoint functor $\mathrm{R}_{W,*} : \mathrm{Mod}_{\mathrm{R}^*_W(\mathrm{Rig}_*\Lambda)} \to \mathbf{SH}_{\text{ét}}(K;\mathrm{Rig}_*\Lambda)$. □

Remark 3.10. Corollary 3.9 shows that the Weil cohomology theory represented by the Weil spectrum $\mathrm{Rig}^*\mathbf{\Gamma}_W$ is precisely the 'new' Weil cohomology theory associated to Γ_W in the sense of [Ayo20, §2E]. Here, contrary to loc. cit., we allow Weil cohomology theories with non necessarily classical coefficients rings.

In a similar vein, we have the following result.

Theorem 3.11. *Assume that K is algebraically closed. If $\mathbf{\Gamma}_W \in \mathrm{WSp}(k;\Lambda)$ is a Weil spectrum on algebraic k-varieties, then $\xi^*\mathbf{\Gamma}_W$ is a Weil spectrum on rigid analytic K-varieties whose ring of coefficients is equivalent to $\mathrm{R}^*_W(\xi_*\Lambda)$, where $\mathrm{R}^*_W : \mathbf{SH}_{\text{ét}}(k;\Lambda) \to \mathrm{Mod}_{\Gamma_W(\mathrm{pt})}$ is the realization functor associated to Γ_W. Thus, we have a functor*

$$\xi^* : \mathrm{WSp}(k;\Lambda) \to \mathrm{RigWSp}(K;\Lambda)$$

which is left adjoint to the functor ξ_* in Remark 3.3.

Proof. The proof of Theorem 3.11 is very similar to the proof of Theorem 3.4. Recall from [AGV22, Theorem 3.7.21] that the functor

$$\widetilde{\xi}_* : \mathbf{RigSH}_{\text{ét}}(K;\Lambda) \to \mathbf{SH}_{\text{ét}}(k;\xi_*\Lambda)$$

is an equivalence of ∞-categories. (In loc. cit., the adjunction (ξ^*,ξ_*) is denoted by (ξ,χ).) Thus, the functor ξ^* is equivalent to $-\otimes\xi_*\Lambda : \mathbf{SH}_{\text{ét}}(k;\Lambda) \to \mathbf{SH}_{\text{ét}}(k;\xi_*\Lambda)$. To conclude, it remains to see that the functor $\mathrm{Mod}_{\mathrm{R}^*_W(\xi_*\Lambda)} \to \mathbf{SH}_{\text{ét}}(k;\mathbf{\Gamma}_W \otimes \xi_*\Lambda)$ is an equivalence, which follows from the fact that $\mathbf{\Gamma}_W \otimes \xi_*\Lambda$ is a Weil spectrum on algebraic k-varieties with coefficient ring $\mathrm{R}^*_W(\xi_*\Lambda)$. □

To go further, we consider the following situation.

Situation 3.12. Assume that K is algebraically closed, and that the value group of K is finite dimensional over $\mathbb{Q}$. Let P be a set of primes and let $P^\times$ be the submonoid of $\mathbb{N}^\times$ generated by P. We assume that the characteristic of k is not in P and that P contains all the primes that are not invertible in $\pi_0(\Lambda)$. (This is possible since the exponent characteristic of k is invertible in $\pi_0(\Lambda)$; note that if Λ is a $\mathbb{Q}$-algebra, we may take $P = \emptyset$ so to have $P^\times = \{1\}$.) Let R be a strictly henselian regular ring, let $a_1,\dots,a_n \in R$ be a regular sequence in R, and let $\rho : R \to K^\circ$ be a local morphism such that $|\rho(a_1)|,\dots,|\rho(a_n)|$ form a basis of the value group of K over $\mathbb{Q}$. (The existence of such a morphism follows from de Jong's theorem on resolution of singularities by alterations [dJ96].) We consider the unique factorization

$$R \to \overline{R} \xrightarrow{\bar{\rho}} K^\circ$$

where $\overline{R}$ is the profinite R-algebra obtained by extracting all the r-th roots of the a_i's for the integers $r \in P^\times$. We set $S = \mathrm{Spec}(R)$ and, for $\emptyset \neq I \subset \{1,\dots,n\}$, we set $D_I = \mathrm{Spec}(R/(a_s, s \in I))$. We also

write C for $D_{\{1,\dots,n\}}$, and set $D = \bigcup_{s=1}^{n} D_s$ and $U = S \smallsetminus D$. We form the commutative diagram with cartesian squares (up to nil-immersions)

$$\begin{array}{ccccc}
\mathrm{Spec}(K) & \xrightarrow{\widehat{j}} & \mathrm{Spec}(K^\circ) & \xleftarrow{\widehat{i}} & \mathrm{Spec}(k) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{U} & \xrightarrow{\bar{j}} & \overline{S} & \xleftarrow{\bar{i}} & C \\
\downarrow & & \downarrow & & \| \\
U & \xrightarrow{j} & S & \xleftarrow{i} & C
\end{array}$$

where $\overline{S} = \mathrm{Spec}(\overline{R})$.

Given a presentable symmetric monoidal ∞-category $C^\otimes$, we denote by $\mathrm{S}(-) : C \to \mathrm{CAlg}(C)$ the functor sending an object to the associated free commutative algebra. (See [Lur17, Example 3.1.3.14].)

Theorem 3.13. *In Situation 3.12, we have equivalences*

$$\xi_*\Lambda \simeq (\bar{i}^*\bar{j}_*\Lambda)|_{\mathrm{Spec}(k)} \simeq \mathrm{S}(\Lambda_{\mathbb{Q}}^{\oplus n}(-1)[-1]) \tag{3.4}$$

of commutative algebras in $\mathbf{SH}_{\text{ét}}(k;\Lambda)$. *(Here and below we write* $\Lambda_{\mathbb{Q}}$ *for* $\Lambda \otimes \mathbb{Q}$*.)*

Proof. We will deduce this from [AGV22, Theorem 3.8.1]. Indeed, by loc. cit., we have

$$\xi_*\Lambda = \widehat{i}^*\widehat{j}_*\Lambda,$$

and the task is to compute the commutative algebra $\widehat{i}^*\widehat{j}_*\Lambda$. Using [dJ96], we can write K° as a filtered colimit of strictly henselian local R-algebras R_α with local homomorphisms $\rho_\alpha : R_\alpha \to K^\circ$ such that $S_\alpha = \mathrm{Spec}(R_\alpha)$ is regular and the inverse image of D in S_α is a normal crossing divisor. Thus, we can find a regular sequence $a_{\alpha,1},\dots,a_{\alpha,n}$ in R_α such that the $|\rho_\alpha(a_{\alpha,s})|$, for $1 \le s \le n$, form a basis of the value group of K over $\mathbb{Q}$ and the inverse image of D is $D_\alpha = \bigcup_{s=1}^{n} D_{\alpha,s}$ with $D_{\alpha,s} = \mathrm{Spec}(R_\alpha/(a_{\alpha,s}))$. Then we have relations in R_α

$$a_s = u_s \cdot a_{\alpha,1}^{e_{s,1}} \cdots a_{\alpha,n}^{e_{s,n}}$$

with u_s invertible in R_α and $e_{s,j} \in \mathbb{N}$. In particular, letting $U_\alpha = S_\alpha \smallsetminus D_\alpha$ and $C_\alpha = D_{\alpha,1} \cap \cdots \cap D_{\alpha,n}$, we can form a commutative diagram with cartesian squares (up to nil-immersions)

$$\begin{array}{ccccc}
\mathrm{Spec}(K) & \xrightarrow{\widehat{j}} & \mathrm{Spec}(K^\circ) & \xleftarrow{\widehat{i}} & \mathrm{Spec}(k) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{U}_\alpha & \xrightarrow{\bar{j}_\alpha} & \overline{S}_\alpha & \xleftarrow{\bar{i}_\alpha} & C_\alpha \\
\downarrow & & \downarrow & & \downarrow \\
U_\alpha & \xrightarrow{j_\alpha} & S_\alpha & \xleftarrow{i_\alpha} & C_\alpha
\end{array}$$

with $\overline{S}_\alpha = \mathrm{Spec}(\overline{R}_\alpha)$ as in Situation 3.12. Note that $\overline{S}_\alpha$ depends on the regular sequence $a_1,\dots,a_n$ in R, fixed in Situation 3.12, but not on the choice of the regular sequence $a_{\alpha,1},\dots,a_{\alpha,n}$ in R_α. Indeed, the latter choice is constrained by the condition that $a_{\alpha,s}$ is a generator of the ideal defining the divisor $D_{\alpha,s}$ (up to a permutation). Thus, another choice of a regular sequence $a'_{\alpha,1},\dots,a'_{\alpha,n} \in R_\alpha$ as above needs to satisfy $a'_{\alpha,s} = v_s \cdot a_{\alpha,s}$, with v_s invertible in R_α. Since R_α is strictly henselian,

it contains all the r-th roots of v_s's for all $r \in P^\times$. This shows that the two $\overline{R}_\alpha$'s obtained using $a_{\alpha,1}, \dots, a_{\alpha,n}$ and $a'_{\alpha,1}, \dots, a'_{\alpha,n}$ are isomorphic, and even canonically once a morphism to K° is fixed. Thus, we have a well defined inductive system $(\overline{S}_\alpha)_\alpha$, and we can use [AGV22, Proposition 3.2.4] to obtain the equivalence:

$$\widehat{i}^*\widehat{j}_*\Lambda \simeq \operatorname*{colim}_\alpha (\bar{i}_\alpha^*\bar{j}_{\alpha,*}\Lambda)|_{\mathrm{Spec}(k)}.$$

Therefore, to establish the first equivalence in (3.4), it suffices to show that the morphisms

$$(\bar{i}^*\bar{j}_*\Lambda)|_{C_\alpha} \to \bar{i}_\alpha^*\bar{j}_{\alpha,*}\Lambda$$

are equivalences. Consider the ring $A = \mathbb{Z}[\sqrt{-1}, P^{-1}]$. Using purity in the form of [Ayo14a, Théorème 7.4] and continuity [AGV22, Proposition 3.2.4], we are reduced to showing the analogous property for the morphism of A-algebras

$$A\left[x_1^{1/r}, \dots, x_n^{1/r} \mid r \in P^\times\right] \to A\left[u_1^{\pm 1/r}, \dots, u_n^{\pm 1/r}, y_1^{1/r}, \dots, y_n^{1/r} \mid r \in P^\times\right],$$

sending $x_s^{1/r}$ to $u_s^{1/r} \cdot y_1^{e_{s,1}/r} \cdots y_n^{e_{s,n}/r}$, in place of the morphism $\overline{R} \to \overline{R}_\alpha$. (Here, we are using that R has the structure of an A-algebra, which follows from the assumptions that R is strictly henselian and that the characteristic of k is not in P.) Consider the schemes:

- $T = \mathrm{Spec}(A[x_1^{\pm 1/r}, \dots, x_n^{\pm 1/r} \mid r \in P^\times])$,
- $B = \mathrm{Spec}(A[u_1^{\pm 1/r}, \dots, u_n^{\pm 1/r} \mid r \in P^\times])$,
- $T' = \mathrm{Spec}(A[u_1^{\pm 1/r}, \dots, u_n^{\pm 1/r}, y_1^{\pm 1/r}, \dots, y_n^{\pm 1/r} \mid r \in P^\times])$,

and the commutative square

$$\begin{array}{ccc} T' & \xrightarrow{h} & T \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & \mathrm{Spec}(\mathbb{Z}[P^{-1}]) \end{array}$$

where h is induced by the morphism sending $x_s^{1/r}$ to $u_s^{1/r} \cdot y_1^{e_{s,1}/r} \cdots y_n^{e_{s,n}/r}$. By Lemma 3.14 below, it suffices to show that the morphism

$$(p_*\Lambda)|_B \to q_*\Lambda$$

is an equivalence. This follows from Lemma 3.15 below by noticing that $(h, q) : T' \to T \times B$ can be viewed as a morphism of pro-tori over B. (This requires a change of coordinates on T' of the form $y'_s = u_1^{a_{s,1}} \cdots u_n^{a_{s,n}} \cdot y_s$, with the $a_{s,j}$'s in $\mathbb{Q}$.)

It remains to establish the second equivalence in (3.13). Using purity in the form of [Ayo14a, Théorème 7.4] and continuity [AGV22, Proposition 3.2.4], we are again reduced to showing the analogous property for the A-algebra

$$A\left[x_1^{1/r}, \dots, x_n^{1/r} \mid r \in P^\times\right].$$

The result follows from Lemma 3.14 below and Proposition A.1. □

Lemma 3.14. *Let S be a scheme, and consider the commutative diagram of schemes*

$$\begin{array}{ccccc} \mathbb{A}^n_S \smallsetminus H & \xrightarrow{j} & \mathbb{A}^n_S & \xleftarrow{i} & S \\ & {\scriptstyle q}\searrow & \downarrow{\scriptstyle p} & \swarrow & \\ & & S, & & \end{array}$$

where $H \subset \mathbb{A}^n_S$ is the union of the standard hyperplanes, j is the obvious inclusion and i is the zero section. Then there is a natural equivalence

$$q_*q^* \xrightarrow{\sim} i^*j_*q^*$$

between endofunctors of $\mathbf{SH}_{\text{ét}}(S;\Lambda)$.

Proof. The natural morphism $q_*q^* \to i^*j_*q^*$ is obtained by applying p_* to $j_*q^* \to i_*i^*j_*q^*$. We show that this morphism is an equivalence by induction on n. When $n = 1$, we need to show that $p_*j_!q^* \simeq 0$ which follows from the fact that the morphism $p_*p^* \to p_*i_*i^*p^*$ is an equivalence by homotopy invariance. The general case follows by induction using [Ayo07b, Théorème 3.3.10] for the canonical specialisation system to conclude that j_*q^*M restricted to the intersection of $n-1$ standard hyperplanes is the direct image along the inclusion $\mathbb{A}^1_S \smallsetminus 0_S \to \mathbb{A}^1_S$ of a motive pulled back from S. We leave the details to the reader. □

Lemma 3.15. *Consider a morphism of pro-tori*

$$\begin{array}{ccc} T' & \xrightarrow{\ e\ } & T \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle p} & \\ & S & \end{array}$$

over a scheme S. Let L and L' be the dual ind-lattices of T and T'. Assume that the induced morphism $e^* : L \otimes \mathbb{Q} \to L' \otimes \mathbb{Q}$ is an isomorphism and that L'/L is p-torsion free for every prime p which is not invertible in $\pi_0(\Lambda)$. Then the induced morphism $p_*\Lambda \to p'_*\Lambda$ is an equivalence in $\mathbf{SH}_{\text{ét}}(S;\Lambda)$.

Proof. Using localization [Ayo07b, Corollaire 4.5.47] and étale descent [AGV22, Proposition 3.2.1], we may replace S by $S[\sqrt{-1}]$ and assume that -1 is a square in $\mathcal{O}(S)$. If m is invertible in $\pi_0(\Lambda)$, multiplication by m on any pro-torus $q : E \to S$ induces an autoequivalence of $q_*\Lambda$. Indeed, it suffices to prove this when E is of finite type. Using étale descent [AGV22, Proposition 3.2.1], we further reduce to the case $E = (\mathbb{G}_{\mathrm{m},S})^{\times n}$. We then conclude using Proposition A.1.

Let $A \subset \mathbb{Q}$ be the localization of $\mathbb{Z}$ at all primes which are invertible in $\pi_0(\Lambda)$, and consider A as an ind-lattice on S. Given a pro-torus $q : E \to S$, we set $\widetilde{E} = \underline{\mathrm{Hom}}(A, E)$ and form the commutative triangle

$$\begin{array}{ccc} \widetilde{E} & \longrightarrow & E \\ & {\scriptstyle \widetilde{q}}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

defining a morphism of pro-tori over S. By the previous discussion, the induced morphism $q_*\Lambda \to \widetilde{q}_*\Lambda$ is an equivalence in $\mathbf{SH}_{\text{ét}}(S;\Lambda)$. Going back to the statement of the lemma, we see that it suffices to prove that the morphism $\widetilde{p}_*\Lambda \to \widetilde{p}'_*\Lambda$ is an equivalence. But, the morphism $\widetilde{e} : \widetilde{T'} \to \widetilde{T}$ induces an isomorphism on the dual ind-lattices, and hence is an isomorphism. □

Remark 3.16. The second equivalence in (3.4) depends on the choice of a compatible sequence of roots of the a_i's. Thus, the equivalence

$$\xi_*\Lambda \simeq \mathrm{S}(\Lambda_{\mathbb{Q}}^{\oplus n}(-1)[-1])$$

depends on the morphism $\rho : R \to K^\circ$, on the regular sequence $a_1, \ldots, a_n$ and the compatible families of r-roots of the a_i's for $r \in P^\times$. The same type of dependency applies to the functor ψ^* constructed below.

Construction 3.17. In Situation 3.12, we define the functor

$$\psi^* : \mathbf{RigSH}_{\text{ét}}(K;\Lambda) \to \mathbf{SH}_{\text{ét}}(k;\Lambda)$$

to be the composition of

$$\mathbf{RigSH}_{\text{ét}}(K;\Lambda) \xrightarrow{\widetilde{\xi}_*} \mathbf{SH}_{\text{ét}}(k;\xi_*\Lambda) \xrightarrow{-\otimes_{\xi_*\Lambda}\Lambda} \mathbf{SH}_{\text{ét}}(k;\Lambda),$$

where the second arrow is the base change functor along the morphism of commutative algebras $\mathrm{S}(\Lambda_{\mathbb{Q}}^{\oplus n}(-1)[-1]) \to \Lambda$ corresponding to the zero morphism $\Lambda_{\mathbb{Q}}^{\oplus n}(-1)[-1] \to \Lambda$ in $\mathbf{SH}_{\text{ét}}(k;\Lambda)$. Clearly, the functor ψ^* underlies a symmetric monoidal functor and admits a right adjoint ψ_*.

Remark 3.18. The functor ψ^* is a version of the motivic nearby functor. Compare with [Ayo15, Scholie 1.3.26(2)].

Lemma 3.19. *In Situation 3.12, let $\mathbf{\Gamma}_W \in \mathrm{WSp}(k;\Lambda)$ be a Weil spectrum on algebraic k-varieties. Then $\psi_*\mathbf{\Gamma}_W$ is a Weil spectrum on rigid analytic K-varieties. This defines a functor*

$$\psi_* : \mathrm{WSp}(k;\Lambda) \to \mathrm{RigWSp}(K;\Lambda).$$

Proof. Indeed, we have a functor $- \circ \psi^* : \mathrm{Real}(k;\Lambda) \to \mathrm{RigReal}(K;\Lambda)$. □

Proposition 3.20. *In Situation 3.12, let $\mathbf{\Gamma}_W \in \mathrm{WSp}(k;\Lambda)$ be a Weil spectrum on algebraic k-varieties. Then, there is an equivalence*

$$\psi_*\mathbf{\Gamma}_W \simeq (\xi^*\mathbf{\Gamma}_W) \otimes_{\mathrm{R}_W(\xi_*\Lambda)} \mathrm{R}_W^*(\Lambda)$$

in $\mathrm{RigWSp}(K;\Lambda)$.

Proof. Recall that the realization functor associated to the Weil spectrum $\xi^*\mathbf{\Gamma}_W$ is given by the composition of

$$\mathbf{RigSH}_{\text{ét}}(K;\Lambda) \xrightarrow{\widetilde{\xi}_*} \mathbf{SH}_{\text{ét}}(k;\xi_*\Lambda) \xrightarrow{\mathrm{R}_W^*} \mathrm{Mod}_{\mathrm{R}_W^*(\xi_*\Lambda)}.$$

The result follows from the commutative diagram

$$\begin{array}{ccccc}
\mathbf{RigSH}_{\text{ét}}(K;\Lambda) & \xrightarrow[\sim]{\widetilde{\xi}_*} & \mathbf{SH}_{\text{ét}}(k;\xi_*\Lambda) & \xrightarrow{\mathrm{R}_W^*} & \mathrm{Mod}_{\mathrm{R}_W^*(\xi_*\Lambda)} \\
& \searrow^{\psi^*} & \downarrow{\scriptstyle -\otimes_{\xi_*\Lambda}\Lambda} & & \downarrow{\scriptstyle -\otimes_{\mathrm{R}_W(\xi_*\Lambda)}\mathrm{R}_W^*(\Lambda)} \\
& & \mathbf{SH}_{\text{ét}}(k;\Lambda) & \xrightarrow{\mathrm{R}_W^*} & \mathrm{Mod}_{\Gamma_W(\mathrm{pt})}
\end{array}$$

showing that the realization functors associated to the Weil spectra under consideration are naturally equivalent. □

Theorem 3.21. *We work in Situation 3.12. Assume furthermore that Λ is a $\mathbb{Q}$-algebra. Let $\mathbf{\Gamma}_{W'} \in \mathrm{RigWSp}(K;\Lambda)$ be a Weil spectrum on rigid analytic K-varieties such that $\Gamma_{W'}(\mathrm{pt})(1)$ is connective. Then, there exists a Weil spectrum $\mathbf{\Gamma}_W \in \mathrm{WSp}(k;\Lambda)$ such that $\psi_*\mathbf{\Gamma}_W$ is equivalent to $\mathbf{\Gamma}_{W'}$. In fact, we may take $\mathbf{\Gamma}_W = \xi_*\mathbf{\Gamma}_{W'}$.*

Proof. Set $\mathbf{\Gamma}_W = \xi_*\mathbf{\Gamma}_{W'}$ and consider the counit morphism

$$\xi^*\mathbf{\Gamma}_W = \xi^*\xi_*\mathbf{\Gamma}_{W'} \to \mathbf{\Gamma}_{W'}.$$

Denote by $\theta : \mathrm{R}_W^*(\xi_*\Lambda) \to \Gamma_{W'}(\mathrm{pt}) = \Gamma_W(\mathrm{pt})$ the induced morphism on the ring of coefficients. By Proposition 2.17, we have an equivalence

$$\xi^*\mathbf{\Gamma}_W \otimes_{\mathrm{R}_W^*(\xi_*\Lambda),\theta} \Gamma_W(\mathrm{pt}) \simeq \mathbf{\Gamma}_{W'}.$$

By Theorem 3.13 and because Λ is a $\mathbb{Q}$-algebra, $\mathrm{R}^*_W(\xi_*\Lambda) \simeq \mathrm{S}_{\Gamma_W(\mathrm{pt})}(\Gamma_W(\mathrm{pt})^{\oplus n}(-1)[-1])$ is the free commutative algebra on the $\Gamma_W(\mathrm{pt})$-module $\Gamma_W(\mathrm{pt})^{\oplus n}(-1)[-1]$. Thus, θ is uniquely determined by its restriction

$$\theta_0 : \Gamma_W(\mathrm{pt})^{\oplus n}(-1)[-1] \to \Gamma_W(\mathrm{pt}).$$

To give a morphism θ_0 is equivalent to give n elements in $\pi_{-1}(\Gamma_W(\mathrm{pt})(1))$ which is zero since $\Gamma_W(\mathrm{pt})(1)$ is connective. This proves that θ is homotopic to the realization of the obvious morphism $\xi_*\Lambda \to \Lambda$, i.e., the one used in Construction 3.17. By Proposition 3.20, $\xi^*\mathbf{\Gamma}_W \otimes_{\mathrm{R}^*_W(\xi_*\Lambda),\theta} \Gamma_W(\mathrm{pt})$ is thus equivalent to $\psi_*\mathbf{\Gamma}_W$ as needed. □

Remark 3.22. A statement, similar to Theorem 3.21, was obtained recently by Binda–Gallauer–Vezzani in [BGV23, Corollary 4.34]. In loc. cit., the authors consider realization functors valued in abstract stable symmetric monoidal ∞-categories (and not only ∞-categories of modules over commutative ring spectra) satisfying a certain compatibility with the weight structure on rigid analytic motives. Contrary to Theorem 3.21 where the equivalence $\psi_*\xi_*\Gamma_{W'} \simeq \Gamma_{W'}$ depends on a path between θ and the obvious morphism, the equivalence in [BGV23, Corollary 4.34] is canonical.

4. The motivic Hopf algebroid of a Weil cohomology theory

In this section, we associate to every Weil cohomology theory Γ_W a motivic Hopf algebroid $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)$. When Γ_W is the Betti cohomology theory associated to a complex embedding of the ground field, we recover the motivic Hopf algebra introduced and studied in [Ayo14b, Ayo14c]. We start by recalling the notions of group and groupoid objects in a general ∞-category following [Lur09, Definition 6.1.2.7].

Definition 4.1. Let C be an ∞-category. A groupoid in C is a cosimplicial object $G : \mathbf{\Delta}^{\mathrm{op}} \to C$ such that, for every integer $n \geq 0$ and every covering $\{0, \dots, n\} = I \cup J$ with $I \cap J = \{m\}$ a singleton, the square

$$\begin{array}{ccc} G(\Delta^{\{0,\dots,n\}}) & \longrightarrow & G(\Delta^J) \\ \downarrow & & \downarrow \\ G(\Delta^I) & \longrightarrow & G(\Delta^{\{m\}}) \end{array} \tag{4.1}$$

is Cartesian. We say that G is a group if moreover $G(\Delta^0)$ is a final object.

Definition 4.2. Let $C^\otimes$ be a symmetric monoidal ∞-category. A Hopf algebroid (resp. algebra) in C is a cosimplicial object $H : \mathbf{\Delta} \to \mathrm{CAlg}(C)$ such that the corresponding simplicial object in $\mathrm{CAlg}(C)^{\mathrm{op}}$ is a groupoid (resp. group). We denote by $\mathrm{Hopf}(C)$ the full sub-∞-category of $\mathrm{CAlg}(C)^{\mathbf{\Delta}}$ spanned by Hopf algebroids. When $C^\otimes = \mathrm{Mod}^\otimes_\Lambda$, for a commutative ring spectrum Λ, we write $\mathrm{Hopf}(\Lambda)$ instead of $\mathrm{Hopf}(\mathrm{Mod}_\Lambda)$.

Lemma 4.3. *Let Λ be a commutative ring spectrum, and let $H \in \mathrm{Hopf}(\Lambda)$ be a Hopf algebroid. The following conditions are equivalent.*

(1) For every integer $n \geq 0$, the algebra $H(\Delta^n)$ is connective.
(2) The algebras $H(\Delta^0)$ and $H(\Delta^1)$ are connective.

When these conditions are satisfied, we say that the Hopf algebroid H is connective.

Proof. Indeed, by definition, there is an equivalence

$$H(\Delta^n) \simeq \overbrace{H(\Delta^1) \otimes_{H(\Delta^0)} \cdots \otimes_{H(\Delta^0)} H(\Delta^1)}^{n \text{ times}}.$$

This proves the implication (2)⇒(1). The other implication is obvious. □

Definition 4.4. Let $C^{\otimes}$ be a symmetric monoidal ∞-category and let H be a Hopf algebroid in C. An H-comodule is a module $M \in \mathrm{Mod}_H(C^{\mathbf{\Delta}})$ such that, for all integers $0 \leq m \leq n$, the natural morphism

$$M(\Delta^{\{m\}}) \otimes_{H(\Delta^{\{m\}})} H(\Delta^{\{0,\dots,n\}}) \to M(\Delta^{\{0,\dots,n\}})$$

is an equivalence. We denote by $\mathrm{coMod}_H(C)$ the full sub-∞-category of $\mathrm{Mod}_H(C^{\mathbf{\Delta}})$ spanned by H-comodules. Clearly, $\mathrm{coMod}_H(C)$ is closed under the tensor product on $\mathrm{Mod}_H(C^{\mathbf{\Delta}})$, and hence inherits the structure of a symmetric monoidal ∞-category. Evaluating at Δ^0 yields the forgetful functor $\mathrm{ff} : \mathrm{coMod}_H(C) \to \mathrm{Mod}_{H(\Delta^0)}(C)$, which is conservative and symmetric monoidal. (Conservativity follows immediately from the defining property of H-comodules.) When $C^{\otimes} = \mathrm{Mod}_{\Lambda}^{\otimes}$, for a commutative ring spectrum Λ, we write coMod_H instead of $\mathrm{coMod}_H(\mathrm{Mod}_{\Lambda})$.

Remark 4.5. Let G be a groupoid object in an ∞-category C. If C admits finite limits, the simplicial object $G \times_{\check{\mathrm{C}}(G_0)} G_0$, where $\check{\mathrm{C}}_{\bullet}(G_0)$ is the Čech nerve of G_0, is a group object in $C_{/G_0}$. Similarly, let $C^{\otimes}$ be a presentable symmetric monoidal ∞-category and let H be a Hopf algebroid in a C. Then the cosimplicial algebra $H \otimes_{\check{\mathrm{C}}(H^0)} H^0$, where $\check{\mathrm{C}}^{\bullet}(H^0)$ is the Čech conerve of H^0, is a Hopf algebra in $\mathrm{Mod}_{H^0}(C)$. (The Čech conerve of a commutative algebra is explicitated in Notation 4.6 below.) Given a commutative ring spectrum Λ and a Hopf algebroid $H \in \mathrm{Hopf}(\Lambda)$, if Λ and H are connective, then the associated Hopf H^0-algebra $H \otimes_{\check{\mathrm{C}}(H^0)} H^0$ is also connective.

Notation 4.6. Let $C^{\otimes}$ be a symmetric monoidal ∞-category and let $A \in \mathrm{CAlg}(C)$ be a commutative algebra in C. The Čech conerve of A is the cosimplicial commutative algebra

$$\check{\mathrm{C}}^{\bullet}(A) : \mathbf{\Delta} \to \mathrm{CAlg}(C)$$

which is the left Kan extension along the inclusion $\mathbf{\Delta}^{\leq 0} \subset \mathbf{\Delta}$ of the functor sending the unique object $[0]$ of $\mathbf{\Delta}^{\leq 0}$ to $A \in \mathrm{CAlg}(C)$. Informally, $\check{\mathrm{C}}^{\bullet}(A)$ is given as follows.

(1) For $n \in \mathbb{N}$, we have $\check{\mathrm{C}}^n(A) = A^{\otimes n+1}$.
(2) For $0 \leq i \leq n+1$, the i-th face morphism $\check{\mathrm{C}}^n(A) \to \check{\mathrm{C}}^{n+1}(A)$ is given by

$$A^{\otimes i} \otimes \mathbf{1} \otimes A^{\otimes n+1-i} \xrightarrow{\mathrm{u}} A^{\otimes i} \otimes A \otimes A^{\otimes n+1-i}$$

where u is the unit of A.
(3) For $0 \leq i \leq n-1$, the i-th codegeneracy morphism $\check{\mathrm{C}}^n(A) \to \check{\mathrm{C}}^{n-1}(A)$ is given by

$$A^{\otimes i} \otimes (A \otimes A) \otimes A^{\otimes n-i-1} \xrightarrow{\mathrm{m}} A^{\otimes i} \otimes A \otimes A^{\otimes n-i-1}$$

where m is the multiplication of A.

The Čech conerve of A has a natural augmentation given by $\check{\mathrm{C}}^{-1}(A) = \mathbf{1}$.

Situation 4.7. Let $e : C^{\otimes} \to \mathcal{M}^{\otimes}$ be a colimit-preserving symmetric monoidal functor between presentable symmetric monoidal ∞-categories, and denote by $d : \mathcal{M} \to C$ its right adjoint. Let A be a commutative algebra in $\mathcal{M}^{\otimes}$, and assume that the induced functor

$$\mathrm{Mod}_{d(A)}(C) \to \mathrm{Mod}_A(\mathcal{M})$$

is an equivalence of ∞-categories. More concretely, the following two conditions are satisfied.

(1) For every A-module M, the obvious morphism $A \otimes_{ed(A)} ed(M) \to M$ is an equivalence.
(2) For every $d(A)$-module L, the obvious morphism $L \to d(A \otimes_{ed(A)} e(L))$ is an equivalence.

Remark 4.8. With k, K and Λ as in the previous sections, we are mainly interested in the following two instances of Situation 4.7:

(1) $C^\otimes = \mathrm{Mod}_\Lambda^\otimes$, $\mathcal{M}^\otimes = \mathbf{SH}_{\text{ét}}(k;\Lambda)^\otimes$, e the obvious functor and $A = \mathbf{\Gamma}_W$ a Weil spectrum on algebraic k-varieties;
(2) $C^\otimes = \mathrm{Mod}_\Lambda^\otimes$, $\mathcal{M}^\otimes = \mathbf{RigSH}_{\text{ét}}(K;\Lambda)^\otimes$, e the obvious functor and $A = \mathbf{\Gamma}_W$ a Weil spectrum on rigid analytic K-varieties.

The assumption on A is satisfied in both cases by Propositions 1.11 and 2.6. Also, in these cases, the functor d is given by $\Gamma(\mathrm{pt};\Omega_T^\infty(-))$.

Theorem 4.9. *We work in Situation 4.7.*

(1) The cosimplicial commutative algebra $d(\check{\mathrm{C}}(A))$ is a Hopf algebroid in C.
(2) For every object $M \in \mathcal{M}$, the $d(\check{\mathrm{C}}(A))$-module $d(\check{\mathrm{C}}(A)\otimes M)$ is a comodule over the Hopf algebroid $d(\check{\mathrm{C}}(A))$.

Moreover, we have a symmetric monoidal functor

$$d(\check{\mathrm{C}}(A)\otimes -) : \mathcal{M}^\otimes \to \mathrm{coMod}_{d(\check{\mathrm{C}}(A))}(C)^\otimes. \tag{4.2}$$

Proof. To prove (1), we need to show that for every partition $\{0,\dots,n\} = I \cup J$ with $I \cap J = \{m\}$ a singleton, the natural map

$$d(\check{\mathrm{C}}^I(A)) \otimes_{d(\check{\mathrm{C}}^{\{m\}}(A))} d(\check{\mathrm{C}}^J(A)) \to d(\check{\mathrm{C}}^{\{0,\dots,n\}}(A)) \tag{4.3}$$

is an equivalence in C. To do so, we start by noting that we have an equivalence

$$\check{\mathrm{C}}^I(A) \otimes_{\check{\mathrm{C}}^{\{m\}}(A)} \check{\mathrm{C}}^J(A) \xrightarrow{\sim} \check{\mathrm{C}}^{\{0,\dots,n\}}(A) \tag{4.4}$$

in $\mathcal{M}$. By assumption, we also have equivalences

$$\check{\mathrm{C}}^{\{m\}}(A) \otimes_{ed(\check{\mathrm{C}}^{\{m\}}(A))} ed(\check{\mathrm{C}}^L(A)) \xrightarrow{\sim} \check{\mathrm{C}}^L(A)$$

for all subsets $L \subset \{0,\dots,n\}$ containing m. This shows that the domain of the equivalence in (4.4) is equivalent to

$$\check{\mathrm{C}}^{\{m\}}(A) \otimes_{ed(\check{\mathrm{C}}^{\{m\}}(A))} e(d(\check{\mathrm{C}}^I(A)) \otimes_{d(\check{\mathrm{C}}^{\{m\}}(A))} d(\check{\mathrm{C}}^J(A)))$$

whereas its codomain is equivalent to

$$\check{\mathrm{C}}^{\{m\}}(A) \otimes_{ed(\check{\mathrm{C}}^{\{m\}}(A))} ed(\check{\mathrm{C}}^{\{0,\dots,n\}}(A)).$$

Thus, up to natural identifications, we see that the equivalence (4.4) can be obtained from the morphism (4.3) by applying $A \otimes_{ed(A)} e(-)$. Since this functor is an equivalence by assumption, the result follows.

The proof of (2) is very similar to that of (1), but we include it for the reader's convenience. Here, for every integers $0 \le m \le n$, we need to check that the obvious morphism

$$d(\check{\mathrm{C}}^{\{0,\dots,n\}}(A)) \otimes_{d(\check{\mathrm{C}}^{\{m\}}(A))} d(\check{\mathrm{C}}^{\{m\}}(A)\otimes M) \to d(\check{\mathrm{C}}^{\{0,\dots,n\}}(A)\otimes M) \tag{4.5}$$

is an equivalence in C. As above, we start by noting that we have an equivalence

$$\check{\mathrm{C}}^{\{0,\dots,n\}}(A) \otimes_{\check{\mathrm{C}}^{\{m\}}(A)} (\check{\mathrm{C}}^{\{m\}}(A)\otimes M) \xrightarrow{\sim} \check{\mathrm{C}}^{\{0,\dots,n\}}(A)\otimes M \tag{4.6}$$

in $\mathcal{M}$. By assumption, we have equivalences

$$\check{\mathrm{C}}^{\{m\}}(A) \otimes_{ed(\check{\mathrm{C}}^{\{m\}}(A))} ed(\check{\mathrm{C}}^L(A)\otimes N) \xrightarrow{\sim} \check{\mathrm{C}}^L(A)\otimes N$$

for all subsets $L \subset \{0,\dots,n\}$ containing m and all objects $N \in \mathcal{M}$. Using this for $L = \{m\}$ or $L = \{0,\dots,n\}$ and $N = \mathbf{1}_\mathcal{M}$ or $N = M$, we see that, up to natural identifications, the equivalence in

(4.6) can be obtained from the morphism in (4.5) by applying the functor $A \otimes_{ed(A)} e(-)$. We then conclude using that this functor is an equivalence.

Finally, we explain how to obtain the symmetric monoidal functor in (4.2). Note that we have a right-lax symmetric monoidal functor

$$d(\check{\mathrm{C}}(A) \otimes -) : \mathcal{M} \to \mathrm{Mod}_{d(\check{\mathrm{C}}(A))}(C^{\Delta}).$$

By (2), this functor factors through $\mathrm{coMod}_{d(\check{\mathrm{C}}(A))}(C)$ and it remains to see that the resulting right-lax symmetric monoidal functor is actually monoidal. Since the forgetful functor

$$\mathrm{ff} : \mathrm{coMod}_{d(\check{\mathrm{C}}(A))}(C) \to \mathrm{Mod}_{d(A)}(C)$$

is monoidal and conservative, it suffices to show that $\mathrm{ff} \circ d(\check{\mathrm{C}}(A) \otimes -) = d(A) \otimes -$ is monoidal, which is clear. □

Definition 4.10. Applying Theorem 4.9 to the situations described in Remark 4.8 we obtain Hopf algebroids

$$\mathcal{H}_{\mathrm{mot}}(\Gamma_W) = \Gamma(\mathrm{pt}; \Omega^{\infty}_T(\check{\mathrm{C}}(\mathbf{\Gamma}_W))).$$

These are the motivic Hopf algebroid associated to the Weil cohomology theory $\Gamma_W = \Omega^{\infty}_T(\mathbf{\Gamma}_W)$. The functor corresponding to (4.2) is called the motivic realization associated to Γ_W and will be denoted by

$$\mathrm{R}^*_{W,\mathrm{mot}} : \mathbf{SH}_{\text{ét}}(k; \Lambda) \to \mathrm{coMod}_{\mathcal{H}_{\mathrm{mot}}(\Gamma_W)} \tag{4.7}$$

in the algebraic setting, and similarly in the rigid analytic setting.

Remark 4.11. Clearly, the motivic realization refines the plain realization associated to Γ_W: we have a commutative triangle

$$\begin{array}{ccc} \mathbf{SH}_{\text{ét}}(k;\Lambda) & \xrightarrow{\mathrm{R}^*_{W,\mathrm{mot}}} & \mathrm{coMod}_{\mathcal{H}_{\mathrm{mot}}(\Gamma_W)} \\ & \searrow^{\mathrm{R}^*_W} & \downarrow \mathrm{ff} \\ & & \mathrm{Mod}_{\Gamma_W(\mathrm{pt})} \end{array}$$

of symmetric monoidal functors. The same applies in the rigid analytic setting.

Conjecture 4.12. *Let $\Gamma_W \in \mathrm{WCT}(k; \Lambda)$ be a Weil cohomology theory for algebraic k-varieties. Assume that the commutative Λ-algebra $\Gamma_W(k)$ is connective and faithfully flat. Then, the motivic realization functor* (4.7) *associated to Γ_W becomes fully faithful when restricted to the thick stable sub-∞-category $\mathbf{SH}_{\text{ét, ct}}(k; \Lambda) \subset \mathbf{SH}_{\text{ét}}(k; \Lambda)$ generated by $\mathrm{M}(X)(n)$, for $X \in \mathrm{Sm}_k$ and $n \in \mathbb{Z}$.*

Remark 4.13. The full faithfulness of the functor $\mathbf{SH}_{\text{ét, ct}}(k; \Lambda) \to \mathrm{Mod}_{\mathcal{H}_{\mathrm{mot}}(\Gamma_W)}$, when Γ_W is Betti cohomology with rational coefficients, is equivalent to [Ayo14b, §2.4, Conjecture B]. A similar conjecture is also expected in the rigid analytic setting.

Lemma 4.14. *Let $\Gamma_W \to \Gamma'_W$ be a morphism of Weil cohomology theories for algebraic k-varieties (or rigid analytic K-varieties). Then, we have an equivalence of Hopf algebroids*

$$\mathcal{H}_{\mathrm{mot}}(\Gamma_W) \otimes_{\check{\mathrm{C}}(\Gamma_W(\mathrm{pt}))} \check{\mathrm{C}}(\Gamma_{W'}(\mathrm{pt})) \xrightarrow{\sim} \mathcal{H}_{\mathrm{mot}}(\Gamma_{W'}).$$

Proof. Indeed, by Proposition 1.22, we have an equivalence $\Gamma_W \otimes_{\Gamma_W(\mathrm{pt})} \Gamma_{W'}(\mathrm{pt}) \simeq \Gamma_{W'}$. This induces an equivalence of cosimplicial algebras in $\mathbf{SH}_{\text{ét}}(k; \Lambda)$:

$$\check{\mathrm{C}}(\mathbf{\Gamma}_W) \otimes_{\check{\mathrm{C}}(\Gamma_W(\mathrm{pt}))} \check{\mathrm{C}}(\Gamma_{W'}(\mathrm{pt})) \simeq \check{\mathrm{C}}(\mathbf{\Gamma}_{W'}).$$

Applying $\Gamma(\mathrm{pt}; \Omega^{\infty}_T(-))$ to this equivalence yields the desired result. □

We note the following simple but useful fact, where our running assumption that Λ is connective is actually important.

Corollary 4.15. *Let $\Gamma_W \to \Gamma'_W$ be a morphism of Weil cohomology theories for algebraic k-varieties (or rigid analytic K-varieties). Assume that $\Gamma_{W'}(\mathrm{pt})$ and $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)$ are connective. Then, $\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'})$ is also connective.*

Proof. Since Λ is connective, the cosimplicial commutative algebras $\check{\mathrm{C}}(\Gamma_W(\mathrm{pt}))$ and $\check{\mathrm{C}}(\Gamma_{W'}(\mathrm{pt}))$ are degreewise connective. Thus, the result follows from Lemma 4.14. □

To ease comparison with the construction in [Ayo14b], we end this section with an alternative construction of the motivic Hopf algebroid of a Weil cohomology theory in terms of the associated realization functor.

Construction 4.16. In Situation 4.7, we let $f : \mathcal{M}^{\otimes} \to \mathrm{Mod}_{d(A)}(C)^{\otimes}$ be the composite functor

$$\mathcal{M}^{\otimes} \to \mathrm{Mod}_A(\mathcal{M})^{\otimes} \simeq \mathrm{Mod}_{d(A)}(C)^{\otimes},$$

and $g : \mathrm{Mod}_{d(A)}(C) \to \mathcal{M}$ its right adjoint. (Note that in the situations described in Remark 4.8, the functor f is the associated realization functor R^*_W.) By [Lur17, Proposition 4.7.3.3] applied to the functor $(f)^{\mathrm{op}}$, the composite functor $g \circ f$ underlies a coalgebra structure in the ∞-category $\mathrm{EndFun}(\mathcal{M}^{\otimes})$ of lax symmetric monoidal endofunctors of $\mathcal{M}^{\otimes}$. Said differently, there is a cosimplicial object $\Phi_f^{\bullet} : \mathbf{\Delta} \to \mathrm{EndFun}(\mathcal{M}^{\otimes})$ which is informally given as follows.

(1) For $n \in \mathbb{N}$, we have $\Phi_f^n = (g \circ f)^{\circ n+1}$.
(2) For $0 \leq i \leq n+1$, the i-th face morphism $\Phi_f^n \to \Phi_f^{n+1}$ is given by

$$(g \circ f)^{\circ i} \circ \mathrm{id} \circ (g \circ f)^{\circ n+1-i} \xrightarrow{\eta} (g \circ f)^{\circ i} \circ (g \circ f) \circ (g \circ f)^{\circ n+1-i}$$

where η is the unit of the adjunction (f, g).
(3) For $0 \leq i \leq n-1$, the i-th codegeneracy morphism $\Phi_f^n \to \Phi_f^{n-1}$ is given by

$$g \circ (f \circ g)^{\circ i} \circ (f \circ g) \circ (f \circ g)^{\circ n-i-1} \circ f \xrightarrow{\delta} g \circ (f \circ g)^{\circ i} \circ \mathrm{id} \circ (f \circ g)^{\circ n-i-1} \circ f$$

where δ is the counit of the adjunction (f, g).

We may think of $\Phi_f^{\bullet}$ as a right-lax symmetric monoidal functor $\Phi_f : \mathcal{M}^{\otimes} \to (\mathcal{M}^{\mathbf{\Delta}})^{\otimes}$. In particular, $\Phi_f(\mathbf{1})$ is a cosimplicial commutative algebra in $\mathcal{M}^{\otimes}$.

Lemma 4.17. *We work in Situation 4.7 and we keep the notation from Construction 4.16. There is an equivalence of cosimplicial commutative algebras $\check{\mathrm{C}}(A) \xrightarrow{\sim} \Phi_f(\mathbf{1})$. Moreover, for $M \in \mathcal{M}$, the $\check{\mathrm{C}}(A)$-module $\check{\mathrm{C}}(A) \otimes M$ is equivalent to the $\Phi_f(\mathbf{1})$-module $\Phi_f(M)$.*

Proof. By construction, we have $\Phi_f(\mathbf{1})(\Delta^0) = g \circ f(\mathbf{1}) = A$. By the universal property of $\check{\mathrm{C}}(A)$, we deduce a morphism of cosimplicial commutative algebras $\check{\mathrm{C}}(A) \to \Phi_f(\mathbf{1})$. More generally, for $M \in \mathcal{M}$, we have a morphism $\check{\mathrm{C}}(A) \otimes M \to \Phi_f(M)$ given by the composition of

$$\check{\mathrm{C}}(A) \otimes M \to \Phi_f(\mathbf{1}) \otimes M \to \Phi_f(\mathbf{1}) \otimes \Phi_f(M) \to \Phi_f(M).$$

(Recall that the cosimplicial functor Φ_f is right-lax monoidal.) Using the natural equivalence $g \circ f \simeq A \otimes -$, it is easy to see that $\check{\mathrm{C}}(A) \otimes M \to \Phi_f(M)$ is an equivalence. □

Theorem 4.18. *We work in Situation 4.7 and we keep the notation from Construction 4.16.*

(1) *The cosimplicial commutative algebra* $d(\Phi_f(\mathbf{1}))$ *is a Hopf algebroid in* $C^{\otimes}$ *which is equivalent to* $d(\check{\mathrm{C}}(A))$.
(2) *For* $M \in \mathcal{M}$, *the* $d(\Phi_f(\mathbf{1}))$*-module* $d(\Phi_f(M))$ *is a comodule over the Hopf algebroid* $d(\Phi_f(\mathbf{1}))$ *which is equivalent to the comodule* $d(\check{\mathrm{C}}(A) \otimes M)$ *over* $d(\check{\mathrm{C}}(A))$.

Moreover, we have a symmetric monoidal functor

$$d(\Phi_f(-)) : \mathcal{M}^{\otimes} \to \mathrm{coMod}_{d(\Phi_f(\mathbf{1}))}(C)^{\otimes} \tag{4.8}$$

which is equivalent to the functor (4.2).

Proof. This follows immediately from Theorem 4.9 and Lemma 4.17. □

Remark 4.19. Let Γ_W be a Weil cohomology theory for algebraic k-varieties (or rigid analytic K-varieties), and let R^*_W be the associated realization functor. Then the motivic Hopf algebroid $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)$ is equivalent to $\Gamma(\mathrm{pt}; \Omega^\infty_T(\Phi_{\mathrm{R}^*_W}(\Lambda)))$ and the associated motivic realization functor is equivalent to $\Gamma(\mathrm{pt}; \Omega^\infty_T(\Phi_{\mathrm{R}^*_W}(-)))$. In particular, note that $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^1) = \mathrm{R}^*_W \mathrm{R}_{W,*}\Gamma_W(\mathrm{pt})$.

5. Complements

In this section, we gather a few more facts about Weil cohomology theories and their motivic Hopf algebroids. As in the previous sections, we fix a connective commutative ring spectrum Λ, and let k be a ground field whose exponent characteristic is invertible in $\pi_0(\Lambda)$. We also let K be a field endowed with a height 1 valuation, and whose residue field is k.

Theorem 5.1. *Let* k'/k *be a field extension and let* $e : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ *be the associated morphism. The right-lax monoidal functor* $e_* : \mathbf{SH}_{\text{ét}}(k'; \Lambda) \to \mathbf{SH}_{\text{ét}}(k; \Lambda)$ *takes Weil spectra to Weil spectra inducing a functor*

$$e_* : \mathrm{WSp}(k'; \Lambda) \to \mathrm{WSp}(k; \Lambda). \tag{5.1}$$

Assume now that the extension k'/k *is algebraic, then the same is true for the symmetric monoidal functor* $e^* : \mathbf{SH}_{\text{ét}}(k; \Lambda) \to \mathbf{SH}_{\text{ét}}(k'; \Lambda)$ *and the induced functor*

$$e^* : \mathrm{WSp}(k; \Lambda) \to \mathrm{WSp}(k'; \Lambda) \tag{5.2}$$

is left adjoint to the functor (5.1). *Moreover, given a Weil spectrum* $\mathbf{\Gamma}_W \in \mathrm{WSp}(k; \Lambda)$, *the coefficient ring of the Weil spectrum* $e^*\mathbf{\Gamma}_W$ *is equivalent to* $\mathrm{R}^*_W(e_*\Lambda)$ *where* R^*_W *is the realization functor associated to* Γ_W.

The first assertion is obvious. The proof of the part concerning e^* is similar to the proof of Theorem 3.4. We will need the following analog of Theorem 3.7.

Proposition 5.2. *Let* k'/k *be an algebraic extension and let* $e : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ *be the associated morphism. Then the functor*

$$\widetilde{e}_* : \mathbf{SH}_{\text{ét}}(k'; \Lambda) \to \mathbf{SH}_{\text{ét}}(k; e_*\Lambda)$$

is an equivalence of ∞*-categories.*

Proof. The functor $\widetilde{e}_*$ admits a left adjoint $\widetilde{e}^*$ sending an $e_*\Lambda$-module M to $e^*(M) \otimes_{e^*e_*\Lambda} \Lambda$. It is enough to prove the following two properties.

(1) The image of the functor $\widetilde{e}^*$ generates $\mathbf{SH}_{\text{ét}}(k'; \Lambda)$ under colimits.
(2) The unit of the adjunction $\mathrm{id} \to \widetilde{e}_*\widetilde{e}^*$ is an equivalence.

The first property follows from Lemma 5.3 below. To prove the second property, we use Lemmas 1.12 and 5.4 to reduce to showing that the unit of the adjunction is an equivalence when evaluated at objects of the form $M \otimes e_*\Lambda$, with $M \in \mathbf{SH}_{\text{ét}}(k;\Lambda)$ dualizable. The resulting morphism coincides with $M \otimes e_*\Lambda \to e_*e^*(M)$ which is indeed an equivalence by [Ayo14b, Lemme 2.8]. □

Lemma 5.3. *Let k'/k be an algebraic extension and let $e : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ be the associated morphism. The image of the functor $e^* : \mathbf{SH}_{\text{ét}}(k;\Lambda) \to \mathbf{SH}_{\text{ét}}(k';\Lambda)$ generates $\mathbf{SH}_{\text{ét}}(k';\Lambda)$ under colimits.*

Proof. Using [AGV22, Theorem 2.9.7], we may replace k and k' with their perfections and assume that k'/k is a separable extension. If $X \in \mathrm{Sm}_{k'}$, then X can be defined over a finite separable subextension l/k of k'/k. But, if $X_0 \in \mathrm{Sm}_l$ is such that $X_0 \otimes_l k' \simeq X$, then X is a clopen subscheme of $X_0 \otimes_k k'$. It follows that the motive of X is a direct summand of the base change along e of the motive of X_0. This enables us to conclude. □

Lemma 5.4. *Let k'/k be an algebraic extension and let $e : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ be the associated morphism. The functor $e_* : \mathbf{SH}_{\text{ét}}(k';\Lambda) \to \mathbf{SH}_{\text{ét}}(k;\Lambda)$ is colimit-preserving.*

Proof. The functor $e_* : \mathrm{Shv}_{t_{\amalg}}(\mathrm{Sm}_{k'}) \to \mathrm{Shv}_{t_{\amalg}}(\mathrm{Sm}_k)$ preserves étale local equivalences. (Indeed, if A is a strictly henselian ring over k, then $A \otimes_k k'$ can be written as a filtered colimit of finite products of strictly henselian rings.) It follows that

$$e_* : \mathrm{Shv}^{\wedge}_{\text{ét}}(\mathrm{Sm}_{k'};\Lambda) \to \mathrm{Shv}^{\wedge}_{\text{ét}}(\mathrm{Sm}_k;\Lambda) \tag{5.3}$$

is also colimit-preserving, and the same applies for the ∞-categories of T-prespectra. (Recall that a T-prespectrum $E = (E_n)_{n\in\mathbb{N}}$ is just a sequence of étale hypersheaves E_n with bounding maps $T \otimes E_n \to E_{n+1}$. It is called a T-spectrum if the induced maps $E_n \to \Omega^1_T(E_{n+1})$ are equivalences. Compare with Remark 1.5.) Since $\mathbf{SH}_{\text{ét}}(k;\Lambda)$ and $\mathbf{SH}_{\text{ét}}(k';\Lambda)$ are obtained from these ∞-categories of T-prespectra by localizations that are compatible with the functor (5.3), the result follows. □

Proof of Theorem 5.1. We need to show that $e^*\mathbf{\Gamma}_W$ is a Weil spectrum. By Proposition 5.2, the functor e^* is equivalent to $- \otimes e_*\Lambda : \mathbf{SH}_{\text{ét}}(k;\Lambda) \to \mathbf{SH}_{\text{ét}}(k; e_*\Lambda)$. Thus, it suffices to prove that the functor $\mathrm{Mod}_{\mathrm{R}^*_W(e_*\Lambda)} \to \mathbf{SH}_{\text{ét}}(k;\mathbf{\Gamma}_W \otimes e_*\Lambda)$ is an equivalence, which follows from the fact that the commutative algebra $\mathbf{\Gamma}_W \otimes e_*\Lambda \simeq \mathbf{\Gamma}_W \otimes_{\Gamma_W(\mathrm{pt})} \mathrm{R}^*_W(e_*\Lambda)$ is a Weil spectrum over k. □

We also have the analogs of Theorem 5.1 and Proposition 5.2 in the rigid analytic setting. For the ease of reference, we state these results leaving the details of their proofs to the reader.

Theorem 5.5. *Let K'/K be an extension of height 1 valued fields and let $e : \mathrm{Spa}(\widehat{K'}) \to \mathrm{Spa}(\widehat{K})$ be the induced morphism. The right-lax monoidal functor $e_* : \mathbf{RigSH}_{\text{ét}}(K';\Lambda) \to \mathbf{RigSH}_{\text{ét}}(K;\Lambda)$ takes Weil spectra to Weil spectra inducing a functor*

$$e_* : \mathrm{RigWSp}(K';\Lambda) \to \mathrm{RigWSp}(K;\Lambda). \tag{5.4}$$

Assume now that the extension K'/K is algebraic, then the same is true for the symmetric monoidal functor $e^* : \mathbf{RigSH}_{\text{ét}}(K;\Lambda) \to \mathbf{RigSH}_{\text{ét}}(K';\Lambda)$ and the induced functor

$$e^* : \mathrm{RigWSp}(K;\Lambda) \to \mathrm{RigWSp}(K';\Lambda) \tag{5.5}$$

is left adjoint to the functor (5.4)*. Moreover, given a Weil spectrum $\mathbf{\Gamma}_W \in \mathrm{RigWSp}(K;\Lambda)$, the coefficient ring of the Weil spectrum $e^*\mathbf{\Gamma}_W$ is equivalent to $\mathrm{R}^*_W(e_*\Lambda)$ where R^*_W is the realization functor associated to Γ_W.*

Proposition 5.6. *Let K'/K be an algebraic extension of height* 1 *valued fields and let* $e : \mathrm{Spa}(\widehat{K'}) \to \mathrm{Spa}(\widehat{K})$ *be the induced morphism. Then the functor*

$$\widetilde{e}_* : \mathbf{RigSH}_{\text{ét}}(K'; \Lambda) \to \mathbf{RigSH}_{\text{ét}}(K; e_*\Lambda)$$

is an equivalence of ∞*-categories.*

The next results describe the relation between motivic Hopf algebroids and the classical Galois groups of algebraic extensions. Given a profinite set P, we set

$$\Lambda^P = \operatorname*{colim}_{P \twoheadrightarrow E} \Lambda^E,$$

where the colimit is indexed by the set of finite quotients of P.

Proposition 5.7. *Let k'/k be a quasi-Galois algebraic extension with Galois group G. Let $\Gamma_{W'}$ be a Weil cohomology theory for algebraic k'-varieties and let $\Gamma_W = (k'/k)_*\Gamma_{W'}$. Then, there is a cocartesian diagram of Hopf algebroids*

$$\begin{array}{ccc} \Lambda^{\mathrm{B}_\bullet(G)} & \longrightarrow & \mathcal{H}_{\mathrm{mot}}(\Gamma_W) \\ \downarrow & & \downarrow \\ \Lambda & \longrightarrow & \mathcal{H}_{\mathrm{mot}}(\Gamma_{W'}). \end{array}$$

In particular, if $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)$ is connective, then so is $\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'})$.

Proof. Modulo the equivalence in Proposition 5.2, the Weil spectrum $\mathbf{\Gamma}_{W'} \in \mathrm{WSp}(k'; \Lambda)$ corresponds to an $e_*\Lambda$-algebra structure on the Weil spectrum $\mathbf{\Gamma}_W \in \mathbf{SH}_{\text{ét}}(k; \Lambda)$. Thus, we see that

$$\mathcal{H}_{\mathrm{mot}}(\Gamma_W) = \Gamma(\mathrm{pt}; \Omega^\infty_T \check{\mathrm{C}}(\mathbf{\Gamma}_W)) \quad \text{and} \quad \mathcal{H}_{\mathrm{mot}}(\Gamma_{W'}) = \Gamma(\mathrm{pt}; \Omega^\infty_T \check{\mathrm{C}}(\mathbf{\Gamma}_W/e_*\Lambda)).$$

Now, there is a commutative diagram with cocartesian squares

$$\begin{array}{ccccc} \Lambda^{\mathrm{B}(G)} & \longrightarrow & \check{\mathrm{C}}(e_*\Lambda) & \longrightarrow & \check{\mathrm{C}}(\mathbf{\Gamma}_W) \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda & \longrightarrow & e_*\Lambda & \longrightarrow & \check{\mathrm{C}}(\mathbf{\Gamma}_W/e_*\Lambda) \end{array}$$

in $\mathrm{CAlg}(\mathbf{SH}_{\text{ét}}(k; \Lambda))^{\mathbf{\Lambda}}$, where $\mathrm{B}_\bullet(G)$ is the classifying space of the profinite group G which can be identified with the simplicial profinite set $\pi_0(\check{\mathrm{C}}^\bullet(k'/k))$. It follows that $\check{\mathrm{C}}(\mathbf{\Gamma}_W/e_*\Lambda) \simeq \check{\mathrm{C}}(\mathbf{\Gamma}_W) \otimes_{\Lambda^{\mathrm{B}(G)}} \Lambda$. To conclude, it remains to see that the obvious morphism

$$\mathrm{R}\Gamma(\mathrm{pt}; \Omega^\infty_T(\check{\mathrm{C}}(\mathbf{\Gamma}_W))) \otimes_{\Lambda^{\mathrm{B}(G)}} \Lambda \to \mathrm{R}\Gamma(\mathrm{pt}; \Omega^\infty_T(\check{\mathrm{C}}(\mathbf{\Gamma}_W) \otimes_{\Lambda^{\mathrm{B}(G)}} \Lambda))$$

is an equivalence, and we may prove this degreewise. For $n \geq 0$, $\check{\mathrm{C}}^n(\mathbf{\Gamma}_W)$ is a Weil spectrum with ring of coefficients $A = \mathrm{R}\Gamma(\mathrm{pt}; \Omega^\infty_T(\check{\mathrm{C}}^n(\mathbf{\Gamma}_W)))$. Using Corollary 1.15, we deduce that $\check{\mathrm{C}}^n(\mathbf{\Gamma}_W) \otimes_{\Lambda^{\mathrm{B}_n(G)}} \Lambda$ is a Weil spectrum with ring of coefficients $A \otimes_{\Lambda^{\mathrm{B}_n(G)}} \Lambda$. This gives the desired equivalence. □

Proposition 5.8. *Let K'/K be an algebraic extension of height* 1 *valued fields, such that $\widehat{K'}/\widehat{K}$ is quasi-Galois with Galois group G. Let $\Gamma_{W'}$ be a Weil cohomology theory for rigid analytic K'-varieties and let $\Gamma_W = (K'/K)_*\Gamma_{W'}$. Then, there is a cocartesian diagram of Hopf algebroids*

$$\begin{array}{ccc} \Lambda^{\mathrm{B}_\bullet(G)} & \longrightarrow & \mathcal{H}_{\mathrm{mot}}(\Gamma_W) \\ \downarrow & & \downarrow \\ \Lambda & \longrightarrow & \mathcal{H}_{\mathrm{mot}}(\Gamma_{W'}). \end{array}$$

In particular, if $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)$ is connective, then so is $\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'})$.

Proof. The proof is identical to the proof of Proposition 5.7. □

Theorem 5.9. *Let k^{alg}/k be an algebraic closure and let G be the Galois group of k^{alg}/k. Let $\Gamma_{W'}$ be a Weil cohomology theory for algebraic k^{alg}-varieties and let $\Gamma_W = (k^{\mathrm{alg}}/k)_*\Gamma_{W'}$. Then, the natural morphism $\Lambda^{\mathrm{B}(G)} \to \mathcal{H}_{\mathrm{mot}}(\Gamma_W)$ from Proposition 5.7 induces an equivalence*

$$\left(\check{\mathrm{C}}(\Gamma_W(\mathrm{pt})) \otimes \Lambda^{\mathrm{B}(G)}\right)^\wedge_\ell \xrightarrow{\sim} \mathcal{H}_{\mathrm{mot}}(\Gamma_W)^\wedge_\ell \tag{5.6}$$

after ℓ-completion.

Proof. The morphism (5.6) is clearly an equivalence in cosimplicial degree zero. We need to see that it is also an equivalence in cosimplicial degree 1, i.e., that the morphism

$$\left(\Gamma_W(\mathrm{pt})^{\otimes 2} \otimes \Lambda^G\right)^\wedge_\ell \to \left(\mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^1)\right)^\wedge_\ell$$

is an equivalence. Recall from Remark 4.19 that $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^1)$ is equivalent to $\mathrm{R}^*_W\mathrm{R}_{W,*}\Gamma_W(\mathrm{pt})$ where $\mathrm{R}^*_W : \mathbf{SH}_{\text{ét}}(k;\Lambda) \to \mathrm{Mod}_{\Gamma_W(\mathrm{pt})}$ is the realization functor associated to Γ_W. The right adjoint $\mathrm{R}_{W,*} : \mathrm{Mod}_{\Gamma_W(\mathrm{pt})} \to \mathbf{SH}_{\text{ét}}(k;\Lambda)$ is colimit-preserving since it is equivalent to the forgetful functor $\mathbf{SH}_{\text{ét}}(k;\mathbf{\Gamma}_W) \to \mathbf{SH}_{\text{ét}}(k;\Lambda)$. In particular, it belongs to Pr^{L} and we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Mod}_{\Gamma_W(\mathrm{pt})} & \xrightarrow{\mathrm{R}_{W,*}} & \mathbf{SH}_{\text{ét}}(k;\Lambda) & \xrightarrow{\mathrm{R}^*_W} & \mathrm{Mod}_{\Gamma_W(\mathrm{pt})} \\
\downarrow{\scriptstyle (-)^\wedge_\ell} & & \downarrow{\scriptstyle (-)^\wedge_\ell} & & \downarrow{\scriptstyle (-)^\wedge_\ell} \\
(\mathrm{Mod}_{\Gamma_W(\mathrm{pt})})_{\ell\text{-cpl}} & \xrightarrow{\widehat{\mathrm{R}}_{W,*}} & \mathbf{SH}_{\text{ét}}(k;\Lambda)_{\ell\text{-cpl}} & \xrightarrow{\widehat{\mathrm{R}}^*_W} & (\mathrm{Mod}_{\Gamma_W(\mathrm{pt})})_{\ell\text{-cpl}}
\end{array}$$

where $\widehat{\mathrm{R}}_{W,*}$ is the right adjoint of $\widehat{\mathrm{R}}^*_W$. This gives an equivalence

$$\left(\mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^1)\right)^\wedge_\ell \simeq \widehat{\mathrm{R}}^*_W\widehat{\mathrm{R}}_{W,*}\Gamma_W(\mathrm{pt})^\wedge_\ell.$$

By rigidity [Bac21] (see also [AGV22, Theorem 2.10.4]), we have a commutative diagram where the vertical arrows are equivalences:

$$\begin{array}{ccccc}
(\mathrm{Mod}_{\Gamma_W(\mathrm{pt})})_{\ell\text{-cpl}} & \xrightarrow{\widehat{f_*}} & \mathrm{Shv}_{\text{ét}}(\acute{\mathrm{E}}\mathrm{t}_k;\Lambda)_{\ell\text{-cpl}} & \xrightarrow{\widehat{f^*}} & (\mathrm{Mod}_{\Gamma_W(\mathrm{pt})})_{\ell\text{-cpl}} \\
\| & & \downarrow{\scriptstyle \sim} & & \| \\
(\mathrm{Mod}_{\Gamma_W(\mathrm{pt})})_{\ell\text{-cpl}} & \xrightarrow{\widehat{\mathrm{R}}_{W,*}} & \mathbf{SH}_{\text{ét}}(k;\Lambda)_{\ell\text{-cpl}} & \xrightarrow{\widehat{\mathrm{R}}^*_W} & (\mathrm{Mod}_{\Gamma_W(\mathrm{pt})})_{\ell\text{-cpl}}.
\end{array}$$

Recall that $\mathrm{Shv}_{\text{ét}}(\acute{\mathrm{E}}\mathrm{t}_k;\Lambda)$ is the ∞-category of étale hypersheaves of Λ-modules on the small étale site $\acute{\mathrm{E}}\mathrm{t}_k$ of k. This gives an equivalence

$$\left(\mathcal{H}_{\mathrm{mot}}(\Gamma_{\ell,k})(\Delta^1)\right)^\wedge_\ell \simeq \widehat{f}^*\widehat{f}_*\Gamma_W(\mathrm{pt})^\wedge_\ell.$$

We claim that $\widehat{f}^*$ is the ℓ-completion of the composite functor

$$f^* : \mathrm{Shv}_{\text{ét}}(\acute{\mathrm{E}}\mathrm{t}_k;\Lambda) \xrightarrow{(k^{\mathrm{alg}}/k)^*} \mathrm{Mod}_\Lambda \to \mathrm{Mod}_{\Gamma_W(\mathrm{pt})}.$$

Indeed, since $\Gamma_W = (k^{\mathrm{alg}}/k)_*\Gamma_{W'}$, we have a natural factorization $\mathrm{R}^*_W = \mathrm{R}^*_{W'} \circ (k^{\mathrm{alg}}/k)^*$ where $\mathrm{R}^*_{W'} : \mathbf{SH}_{\text{ét}}(k^{\mathrm{alg}};\Lambda) \to \mathrm{Mod}_{\Gamma_W(\mathrm{pt})}$ is the realization functor associated to $\Gamma_{W'}$. It follows that $\widehat{f}^*$ is

equal to the composition of

$$\mathrm{Shv}_{\text{ét}}(\acute{\mathrm{E}}\mathrm{t}_k;\Lambda)_{\ell\text{-cpl}} \xrightarrow{(k^{\mathrm{alg}}/k)^*} \mathrm{Shv}_{\text{ét}}(\acute{\mathrm{E}}\mathrm{t}_{k^{\mathrm{alg}}};\Lambda)_{\ell\text{-cpl}} \xrightarrow{\sim} \mathbf{SH}_{\text{ét}}(k^{\mathrm{alg}};\Lambda)_{\ell\text{-cpl}} \xrightarrow{\widehat{\mathrm{R}}^*_{W'}} (\mathrm{Mod}_{\Gamma_W(\mathrm{pt})})_{\ell\text{-cpl}}.$$

But the composition of the last two functors has to be the obvious one since it is a morphism of $\mathrm{Mod}_\Lambda^\otimes$-modules in Pr^{L}. This proves our claim. To conclude, we use the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Mod}_{\Gamma_W(\mathrm{pt})} & \xrightarrow{f_*} & \mathrm{Shv}_{\text{ét}}(\acute{\mathrm{E}}\mathrm{t}_k;\Lambda) & \xrightarrow{f^*} & \mathrm{Mod}_{\Gamma_W(\mathrm{pt})} \\
\downarrow{\scriptstyle (-)^\wedge_\ell} & & \downarrow{\scriptstyle (-)^\wedge_\ell} & & \downarrow{\scriptstyle (-)^\wedge_\ell} \\
(\mathrm{Mod}_{\Gamma_W(\mathrm{pt})})_{\ell\text{-cpl}} & \xrightarrow{\widehat{f_*}} & \mathrm{Shv}_{\text{ét}}(\acute{\mathrm{E}}\mathrm{t}_k;\Lambda)_{\ell\text{-cpl}} & \xrightarrow{\widehat{f^*}} & (\mathrm{Mod}_{\Gamma_W(\mathrm{pt})})_{\ell\text{-cpl}}
\end{array}$$

showing that

$$\widehat{f^*}\,\widehat{f_*}\Gamma_W(\mathrm{pt})^\wedge_\ell \simeq (f^*f_*\Gamma_W(\mathrm{pt}))^\wedge_\ell \simeq (\Gamma_W(\mathrm{pt})^\otimes \otimes \Lambda^G)^\wedge_\ell.$$

This finishes the proof. □

Theorem 5.10. *Let K^{alg}/K be an algebraic closure of K. Choose an extension of the valuation of K to K^{alg}, and let G be the Galois group of $\widehat{K}^{\mathrm{alg}}/\widehat{K}$. Let $\Gamma_{W'}$ be a Weil cohomology theory for rigid analytic K^{alg}-varieties and let $\Gamma_W = (K^{\mathrm{alg}}/K)_*\Gamma_{W'}$. Then, the natural morphism $\Lambda^{\mathrm{B}(G)} \to \mathcal{H}_{\mathrm{mot}}(\Gamma_{\ell,K})$ from Proposition 5.8 induces an equivalence*

$$\left(\check{\mathrm{C}}(\Gamma_W(\mathrm{pt})) \otimes \Lambda^{\mathrm{B}(G)}\right)^\wedge_\ell \xrightarrow{\sim} \mathcal{H}_{\mathrm{mot}}(\Gamma_{\ell,K})^\wedge_\ell \tag{5.7}$$

after ℓ-completion.

Proof. The proof is identical to the proof of Theorem 5.9. □

We have the following continuity property for the motivic Hopf algebroids.

Proposition 5.11. *Let $(k_\alpha)_\alpha$ be a filtered family of subfields of k such that $k = \bigcup_\alpha k_\alpha$. Let $\Gamma_W \in \mathrm{WCT}(k;\Lambda)$ be a Weil cohomology theory for algebraic k-varieties. Denote by $\Gamma_{W,\alpha} \in \mathrm{WCT}(k_\alpha;\Lambda)$ the Weil cohomology theory for algebraic k_α-varieties obtained by precomposing with the base change functor $\mathrm{Sm}_{k_\alpha} \to \mathrm{Sm}_k$. Then, we have a canonical equivalence of Hopf algebroids*

$$\operatorname*{colim}_\alpha \mathcal{H}_{\mathrm{mot}}(\Gamma_{W,\alpha}) \simeq \mathcal{H}_{\mathrm{mot}}(\Gamma_W). \tag{5.8}$$

Proof. Recall from the proof of Proposition 1.9 that the obvious functor

$$\operatorname*{colim}_\alpha \mathbf{SH}_{\text{ét}}(k_\alpha;\Lambda) \to \mathbf{SH}_{\text{ét}}(k;\Lambda)$$

is a localization functor sending the object $(\mathbf{\Gamma}_{W,\alpha})_\alpha$ to $\mathbf{\Gamma}_W$. (As usual, the above colimit is computed in Pr^{L}.) This yields an equivalence in $\mathbf{SH}_{\text{ét}}(k;\Lambda)$

$$\operatorname*{colim}_\alpha (k/k_\alpha)^*\mathbf{\Gamma}_{W,\alpha} \simeq \mathbf{\Gamma}_W. \tag{5.9}$$

Taking tensor powers, we obtain the equivalences

$$\operatorname*{colim}_\alpha (k/k_\alpha)^*(\mathbf{\Gamma}_{W,\alpha})^{\otimes n+1} \simeq (\mathbf{\Gamma}_W)^{\otimes n+1}$$

for all $n \geq 0$. Unravelling Definition 4.10 and using Proposition 1.11, we can rewrite these equivalences as follows:

$$\operatorname*{colim}_\alpha (k/k_\alpha)^*(\mathbf{\Gamma}_{W,\alpha} \otimes_{\Gamma_W(\mathrm{pt})} \mathcal{H}_{\mathrm{mot}}(\Gamma_{W,\alpha})(\Delta^n)) \simeq \mathbf{\Gamma}_W \otimes_{\Gamma_W(\mathrm{pt})} \mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^n).$$

Tensoring over (5.9), we obtain the equivalence

$$\mathbf{\Gamma}_W \otimes_{\Gamma_W(\mathrm{pt})} \operatorname*{colim}_\alpha \mathcal{H}_{\mathrm{mot}}(\Gamma_{W,\alpha})(\Delta^n) \simeq \mathbf{\Gamma}_W \otimes_{\Gamma_W(\mathrm{pt})} \mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^n).$$

We can now apply Proposition 1.11 to conclude. □

Proposition 5.12. *Let $(K_\alpha)_\alpha$ be a filtered family of non discrete subfields of K such that $\widehat{K}$ is the completion of $\bigcup_\alpha K_\alpha$. Let $\Gamma_W \in \mathrm{RigWCT}(K;\Lambda)$ be a Weil cohomology theory for rigid analytic K-varieties. Denote by $\Gamma_{W,\alpha} \in \mathrm{RigWCT}(K_\alpha;\Lambda)$ the Weil cohomology theory for rigid analytic K_α-varieties obtained by precomposing with the base change functor $\mathrm{RigSm}_{K_\alpha} \to \mathrm{RigSm}_K$. Then, we have a canonical equivalence of Hopf algebroids*

$$\operatorname*{colim}_\alpha \mathcal{H}_{\mathrm{mot}}(\Gamma_{W,\alpha}) \simeq \mathcal{H}_{\mathrm{mot}}(\Gamma_W). \tag{5.10}$$

Proof. The proof is identical to the proof of Proposition 5.11. □

In the remainder of this section, we will establish some criteria for proving the connectivity of motivic Hopf algebras. We start with a converse to the last assertion in Proposition 5.7.

Proposition 5.13. *Let k'/k be a quasi-Galois algebraic extension with Galois group G. Let $\Gamma_{W'}$ be a Weil cohomology theory for algebraic k'-varieties and let $\Gamma_W = (k'/k)_*\Gamma_{W'}$. Assume that the following properties are satisfied. (Below, we write pt' for $\mathrm{Spec}(k')$.)*

(1) The Hopf algebroid $\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'})$ is connective.
(2) For every $g \in G$, there is a faithfully flat $\Gamma_{W'}(\mathrm{pt}')$-algebra A such that the Weil cohomology theories $\Gamma_{W'} \otimes_{\Gamma_{W'}(\mathrm{pt}')} A$ and $g_*(\Gamma_{W'}) \otimes_{\Gamma_{W'}(\mathrm{pt}')} A$ are equivalent.

Then, the Hopf algebroid $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)$ is connective.

Proof. Using [AGV22, Theorem 2.9.7], we may replace k and k' with their perfections and assume that k'/k is a separable extension. By Lemma 4.3, it is enough to show that

$$\mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^1) = \mathrm{R}\Gamma(\mathrm{pt}; \Omega^\infty_T(\mathbf{\Gamma}_W \otimes \mathbf{\Gamma}_W))$$

is connective. By Proposition 5.7, there is a morphism $\Lambda^G \to \mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^1)$, and it is enough to show that, for every prime ideal $\mathfrak{p}$ of $\pi_0(\Lambda)^G$, the localized algebra $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^1)_\mathfrak{p}$ is connective. Since every prime ideal of $\pi_0(\Lambda)^G$ is in the image of the morphism $\mathrm{Spec}(\Lambda^{\{g\}}) \to \mathrm{Spec}(\Lambda^G)$ for a unique $g \in G$, it is enough to show that $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^1) \otimes_{\Lambda^G} \Lambda^{\{g\}}$ is connective for every $g \in G$. Arguing as we did at the end of the proof of Proposition 5.7, we have:

$$\begin{aligned} \mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^1) \otimes_{\Lambda^G} \Lambda^{\{g\}} &= \mathrm{R}\Gamma(\mathrm{pt}; \Omega^\infty_T(\mathbf{\Gamma}_W \otimes \mathbf{\Gamma}_W)) \otimes_{\Lambda^G} \Lambda^{\{g\}} \\ &\simeq \mathrm{R}\Gamma(\mathrm{pt}; \Omega^\infty_T((\mathbf{\Gamma}_W \otimes \mathbf{\Gamma}_W) \otimes_{\Lambda^G} \Lambda^{\{g\}})). \end{aligned}$$

On the other hand, we have the following equivalences of commutative algebras in $\mathbf{SH}_{\text{ét}}(k; e_*\Lambda)$:

$$\begin{aligned} (\mathbf{\Gamma}_W \otimes \mathbf{\Gamma}_W) \otimes_{\Lambda^G} \Lambda^{\{g\}} &\simeq (\mathbf{\Gamma}_W \otimes \mathbf{\Gamma}_W) \otimes_{e_*\Lambda \otimes e_*\Lambda, \mathrm{id}\otimes g^*} e_*\Lambda \\ &\simeq \mathbf{\Gamma}_W \otimes_{e_*\Lambda} (\mathbf{\Gamma}_W \otimes_{e_*\Lambda, g^*} e_*\Lambda), \end{aligned}$$

where $g^* = \mathrm{Spec}(g)^* : e_*\Lambda \to e_*\Lambda$ is the action of $g \in G$ on the commutative algebra $e_*\Lambda$ which is induced from the morphism $\mathrm{Spec}(g) : \mathrm{Spec}(k') \to \mathrm{Spec}(k')$. Modulo the equivalence of ∞-categories $\mathbf{SH}_{\text{ét}}(k; e_*\Lambda) \simeq \mathbf{SH}_{\text{ét}}(k'; \Lambda)$, the $e_*\Lambda$-algebra $\mathbf{\Gamma}_W$ corresponds to $\mathbf{\Gamma}_{W'}$ and the $e_*\Lambda$-algebra $\mathbf{\Gamma}_W \otimes_{e_*\Lambda, g^*} e_*\Lambda$ corresponds to $g^*\mathbf{\Gamma}_{W'} \simeq g_*^{-1}\mathbf{\Gamma}_{W'}$. From this, we deduce the equivalence

$$\mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^1) \otimes_{\Lambda^G} \Lambda^{\{g\}} \simeq \mathrm{R}\Gamma(\mathrm{pt}'; \Omega^\infty_T(\mathbf{\Gamma}_{W'} \otimes g^*\mathbf{\Gamma}_{W'})).$$

By the second assumption in the statement, we can find a faithfully flat $\Gamma_{W'}(\mathrm{pt}')$-algebra A such that $g^*(\mathbf{\Gamma}_{W'}) \otimes_{\Gamma_{W'}(\mathrm{pt}')} A$ is equivalent to $\mathbf{\Gamma}_{W'} \otimes_{\Gamma_{W'}(\mathrm{pt}')} A$. It follows that

$$\begin{aligned}\left(\mathcal{H}_{\mathrm{mot}}(\Gamma_W)(\Delta^1) \otimes_{\Lambda^G} \Lambda^{\{g\}}\right) \otimes_{\Gamma_{W'}(\mathrm{pt}')^{\otimes 2}} A^{\otimes 2} &\simeq \mathrm{R}\Gamma(\mathrm{pt}'; \Omega^\infty_T(\mathbf{\Gamma}_{W'} \otimes \mathbf{\Gamma}_{W'})) \otimes_{\Gamma_{W'}(\mathrm{pt}')^{\otimes 2}} A^{\otimes 2} \\ &\simeq \left(\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'})(\Delta^1)\right) \otimes_{\Gamma_{W'}(\mathrm{pt}')^{\otimes 2}} A^{\otimes 2}.\end{aligned}$$

Since $\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'})(\Delta^1)$ is connective by assumption, the result follows. □

Proposition 5.14. *Let K'/K be an algebraic extension of height 1 valued fields, such that $\widehat{K'}/\widehat{K}$ is quasi-Galois with Galois group G. Let $\Gamma_{W'}$ be a Weil cohomology theory for rigid analytic K'-varieties and let $\Gamma_W = (K'/K)_*\Gamma_{W'}$. Assume that the following properties are satisfied. (Below, we write pt' for $\mathrm{Spa}(\widehat{K'})$.)*

(1) The Hopf algebroid $\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'})$ is connective.
(2) For every $g \in G$, there is a faithfully flat $\Gamma_{W'}(\mathrm{pt}')$-algebra A such that the Weil cohomology theories $\Gamma_{W'} \otimes_{\Gamma_{W'}(\mathrm{pt}')} A$ and $g_*(\Gamma_{W'}) \otimes_{\Gamma_{W'}(\mathrm{pt}')} A$ are equivalent.

Then, the Hopf algebroid $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)$ is connective.

Proof. The proof is identical to the proof of Proposition 5.13. Notice that the functor σ_* between $\mathrm{RigWCT}(K'; \Lambda)$ and $\mathrm{RigWCT}(K; \Lambda)$ is well defined since these categories depend only on the completed fields $\widehat{K}$ and $\widehat{K'}$. □

We will need also another criterion for the connectivity of motivic Hopf algebras.

Theorem 5.15. *We work in Situation 3.12. Let $\mathbf{\Gamma}_{W'} \in \mathrm{RigWSp}(K; \Lambda)$ be a Weil spectrum and set $\mathbf{\Gamma}_W = \xi_*\mathbf{\Gamma}_{W'}$. Assume that Λ is a $\mathbb{Q}$-algebra and that $\Gamma_{W'}(\mathrm{pt})(1)$ is connective. Then $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)$ is connective if and only if $\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'})$ is connective.*

Theorem 5.15 follows from Theorem 3.21, asserting that $\mathbf{\Gamma}_{W'}$ is equivalent to $\psi_*\mathbf{\Gamma}_W$, and Theorem 5.17 below describing the motivic Hopf algebroid of $\psi_*\mathbf{\Gamma}_W$ in terms of the motivic Hopf algebroid of $\mathbf{\Gamma}_W$. To state Theorem 5.17, we need the notion of a semi-direct tensor product of Hopf algebroids.

Construction 5.16. Let $C^\otimes$ be a presentable symmetric monoidal ∞-category. Let H be a Hopf algebroid in C and let L be a Hopf algebroid in $\mathrm{coMod}_H(C)$. Then H is given by a cosimplicial commutative algebra $H : \mathbf{\Delta} \to \mathrm{CAlg}(C)$ and L is given by a cosimplicial commutative algebra

$$L : \mathbf{\Delta} \to \mathrm{CAlg}(\mathrm{Mod}_H(C^{\mathbf{\Delta}})) = \mathrm{CAlg}(C^{\mathbf{\Delta}})_{H\backslash}.$$

Equivalently, we can view L as a bi-cosimplicial commutative algebra $L : \mathbf{\Delta}^2 \to \mathrm{CAlg}(C)$ together with an augmentation $H \to L$. It is easy to see that

$$L(\Delta^m, \Delta^n) \simeq H(\Delta^m) \otimes L(\Delta^0, \Delta^n) \simeq H(\Delta^1)^{\otimes m} \otimes L(\Delta^0, \Delta^1)^{\otimes n}.$$

This implies easily that the diagonal cosimplicial algebra $\mathrm{diag}(L)$ is a Hopf algebra in C. We denote it by $H \otimes^{\triangleright} L$ and call it the semi-direct tensor product of H with L.

Theorem 5.17. *We work in Situation 3.12. Let $\mathbf{\Gamma}_W \in \mathrm{WSp}(k; \Lambda)$ be a Weil spectrum and set $\mathbf{\Gamma}_{W'} = \psi_*\mathbf{\Gamma}_W$. There is an equivalence of Hopf algebroids*

$$\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'}) \simeq \mathcal{H}_{\mathrm{mot}}(\Gamma_W) \otimes^{\triangleright} \mathrm{S}(\Gamma_W(\mathrm{pt})^{\oplus n}_{\mathbb{Q}}(-1)). \tag{5.11}$$

In particular, $\mathcal{H}_{\mathrm{mot}}(\Gamma_W)$ is connective if and only if $\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'})$ is connective.

Proof. Recall that we have an equivalence $\mathbf{RigSH}_{\text{ét}}(K;\Lambda) \simeq \mathbf{SH}_{\text{ét}}(k;\xi_*\Lambda)$, modulo which ψ^* is given by base change along the obvious co-augmentation $\xi_*\Lambda \simeq \mathrm{S}(\Lambda_{\mathbb{Q}}^{\oplus n}(-1)[-1]) \to \Lambda$. (See Construction 3.17.) Unravelling the definitions, it follows that $\mathbf{\Gamma}_{W'} = \psi_*\mathbf{\Gamma}_W$ is equivalent to $\mathbf{\Gamma}_W$ viewed as a commutative algebra in $\mathbf{SH}_{\text{ét}}(k;\xi_*\Lambda)$ using the composite morphism

$$\xi_*\Lambda \to \Lambda \to \mathbf{\Gamma}_W.$$

Thus, the motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'})$ is obtained by applying $\Gamma(\mathrm{pt};\Omega_T^\infty(-))$ to the cosimplicial algebra $\check{\mathrm{C}}^\bullet(\mathbf{\Gamma}_W/\xi_*\Lambda)$ in $\mathbf{SH}_{\text{ét}}(k;\xi_*\Lambda)$. Consider the bi-cosimplicial algebra

$$A^{\bullet,\bullet} = \check{\mathrm{C}}^\bullet(\mathbf{\Gamma}_W) \otimes \check{\mathrm{C}}^\bullet(\Lambda/\xi_*\Lambda).$$

Since $\check{\mathrm{C}}^\bullet(\Lambda/\xi_*\Lambda)$ is naturally a cosimplicial $\xi_*\Lambda$-algebra, we may view $A^{\bullet,\bullet}$ as a bi-cosimplicial algebra in $\mathbf{SH}_{\text{ét}}(k;\xi_*\Lambda)$. Moreover, the $\xi_*\Lambda$-algebra $A^{0,0}$ is equivalent to $\mathbf{\Gamma}_W$. Thus, by the universal property of left Kan extensions, there is a unique morphism of cosimplicial $\xi_*\Lambda$-algebras

$$\check{\mathrm{C}}(\mathbf{\Gamma}_W/\xi_*\Lambda) \to \mathrm{diag}(A).$$

We claim that this morphism is an equivalence. Indeed, in degree n, we have

$$\begin{aligned} A^{n,n} &= \overbrace{\mathbf{\Gamma}_W \otimes \cdots \otimes \mathbf{\Gamma}_W}^{n+1 \text{ times}} \otimes \overbrace{\Lambda \otimes_{\xi_*\Lambda} \cdots \otimes_{\xi_*\Lambda} \Lambda}^{n+1 \text{ times}} \\ &\simeq \overbrace{(\mathbf{\Gamma}_W \otimes \Lambda) \otimes_{\xi_*\Lambda} \cdots \otimes_{\xi_*\Lambda} (\mathbf{\Gamma}_W \otimes \Lambda)}^{n+1 \text{ times}}. \end{aligned}$$

Modulo this identification, the morphism $A^{0,0} \to A^{n,n}$ corresponding to $\Delta^{\{i\}} \to \Delta^{\{0,\dots,n\}}$ is given by

$$\mathbf{\Gamma}_W \otimes \Lambda \simeq (\xi_*\Lambda)^{\otimes_{\xi_*\Lambda} i} \otimes_{\xi_*\Lambda} (\mathbf{\Gamma}_W \otimes \Lambda) \otimes_{\xi_*\Lambda} (\xi_*\Lambda)^{\otimes_{\xi_*\Lambda} n-i} \xrightarrow{u\otimes \mathrm{id} \otimes u} (\mathbf{\Gamma}_W \otimes \Lambda)^{\otimes_{\xi_*\Lambda} n+1}.$$

This proves our claim. Thus, at this point, we have proven that

$$\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'}) \simeq \mathrm{diag}(\mathrm{R}\Gamma(\mathrm{pt};\Omega_T^\infty(A))).$$

Recall that $\xi_*\Lambda = \mathrm{S}(\Lambda_{\mathbb{Q}}^{\oplus n}(-1)[-1])$. Since $\mathrm{S}(-)$ is a left adjoint functor, it follows that $\check{\mathrm{C}}^\bullet(\Lambda/\xi_*\Lambda)$ is equivalent to the free commutative algebra on the cosimplicial object of $\mathbf{SH}_{\text{ét}}(k;\xi_*\Lambda)$:

$$\check{\mathrm{C}}^\bullet(0/\Lambda_{\mathbb{Q}}^{\oplus n}(-1)[-1]) \simeq \underline{\mathrm{Hom}}(\mathrm{B}_\bullet(\mathbb{Z}), \Lambda_{\mathbb{Q}}^{\oplus n}(-1)).$$

Said differently, $\check{\mathrm{C}}(\Lambda/\xi_*\Lambda)$ is a Hopf algebra whose underlying commutative algebra is $\mathrm{S}(\Lambda_{\mathbb{Q}}^{\oplus n}(-1))$. By abuse of notation, we will write $\mathrm{S}(\Lambda_{\mathbb{Q}}^{\oplus n}(-1))$ instead of $\check{\mathrm{C}}(\Lambda/\xi_*\Lambda)$. Now, notice that the bi-cosimplicial algebra $\mathrm{R}\Gamma(\mathrm{pt};\Omega_T^\infty(A))$ encodes the Hopf algebra

$$\mathrm{R}^*_{W,\mathrm{mot}}(\mathrm{S}(\Lambda_{\mathbb{Q}}^{\oplus n}(-1))) = \mathrm{S}(\Gamma_W(\mathrm{pt})_{\mathbb{Q}}^{\oplus n}(-1))$$

in $\mathrm{coMod}_{\mathcal{H}_{\mathrm{mot}}(\Gamma_W)}$ obtained by applying the motivic realization functor (of Definition 4.10) to the Hopf algebra $\mathrm{S}(\Lambda_{\mathbb{Q}}^{\oplus n}(-1))$. Since $\mathcal{H}_{\mathrm{mot}}(\Gamma_{W'})$ is the diagonal of $\mathrm{R}\Gamma(\mathrm{pt};\Omega_T^\infty(A))$, this gives the equivalence (5.11) in the statement. □

6. Examples of Weil cohomology theories

In this section, we recall the constructions of the classical Weil cohomology theories, and we revisit the new Weil cohomology theories introduced in [Ayo20]. As in the previous sections, we fix a connective commutative ring spectrum Λ, and let k be a ground field whose exponent characteristic is invertible in $\pi_0(\Lambda)$. We also let K be a field endowed with a height 1 valuation, and whose residue field is k. We start with the ℓ-adic cohomology theories.

Notation 6.1. Let ℓ be a prime number. Given a presentable stable ∞-category C, we denote by $C_\ell \subset C$ its full sub-∞-category of ℓ-complete objects and by $(-)^\wedge_\ell : C \to C_\ell$ the ℓ-completion functor. (See [Lur18, Chapter 7] for a detailed discussion and [AGV22, Notation 2.10.1 & Remark 2.10.2] for a very quick review.) Below, we will assume that ℓ is invertible in k and we denote by Λ_ℓ the ℓ-completion of Λ.

Construction 6.2. Assume that k is algebraically closed. By rigidity [Bac21] (see also [AGV22, Theorem 2.10.4]), we have an equivalence of ∞-categories

$$(\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}} \xrightarrow{\sim} \mathbf{SH}_{\text{ét}}(k;\Lambda)_{\ell\text{-cpl}}.$$

Recall that $\mathbf{SH}_{\text{ét, ct}}(k;\Lambda) \subset \mathbf{SH}_{\text{ét}}(k;\Lambda)$ is the thick stable sub-∞-category generated by the motives $\mathrm{M}(X)(n)$, for $X \in \mathrm{Sm}_k$ and $n \in \mathbb{Z}$. In fact, $\mathbf{SH}_{\text{ét}}(k;\Lambda)$ is compactly generated, and $\mathbf{SH}_{\text{ét, ct}}(k;\Lambda)$ is its sub-∞-category of compact objects by [AGV22, Proposition 3.2.3]. By Lemma 1.12, the composite functor

$$\mathbf{SH}_{\text{ét, ct}}(k;\Lambda) \to \mathbf{SH}_{\text{ét}}(k;\Lambda) \xrightarrow{(-)^\wedge_\ell} \mathbf{SH}_{\text{ét}}(k;\Lambda)_{\ell\text{-cpl}} \simeq (\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}} \tag{6.1}$$

lands in the full sub-∞-category of $(\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}}$ spanned by dualizable objects. It also lands in the sub-∞-category spanned by eventually connective objects. Indeed, by Lemma 1.12, it is enough to show that the image of $\pi_{X,*}\Lambda$, for $X \in \mathrm{Sm}_k$, is eventually connective. But this image is given by $\lim_n \mathrm{R}\Gamma_{\text{ét}}(X;\Lambda/\ell^n\Lambda)$ which is $-2\dim(X)$-connective since the étale cohomological dimension of X is bounded by twice its dimension. On the other hand, by [Lur18, Corollary 8.3.5.9], the functor

$$(-)^\wedge_\ell : \mathrm{Mod}_{\Lambda_\ell} \to (\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}}$$

induces an equivalence between the full sub-∞-category of dualizable Λ_ℓ-modules and the full sub-∞-category of eventually connective and dualizable objects in $(\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}}$. Thus, by the previous discussion, we see that the functor (6.1) yields a functor

$$\mathrm{R}_\ell : \mathbf{SH}_{\text{ét, ct}}(k;\Lambda) \to \mathrm{Mod}_{\Lambda_\ell}. \tag{6.2}$$

Since $\mathbf{SH}_{\text{ét}}(k;\Lambda)$ is equivalent to the indization of $\mathbf{SH}_{\text{ét, ct}}(k;\Lambda)$, the functor (6.2) extends uniquely to a colimit-preserving functor

$$\mathrm{R}^*_\ell : \mathbf{SH}_{\text{ét}}(k;\Lambda) \to \mathrm{Mod}_{\Lambda_\ell},$$

which we call the plain ℓ-adic realization functor. Clearly, R^*_ℓ underlies a symmetric monoidal functor. We denote by $\mathbf{\Gamma}_\ell = \mathrm{R}_{\ell,*}\Lambda_\ell$ and $\Gamma_\ell = \Omega^\infty_T(\mathbf{\Gamma}_\ell)$ the Weil spectrum and the Weil cohomology theory associated to R^*_ℓ. To stress the dependence on k, we would write '$\mathrm{R}^*_{\ell,k}$', '$\mathbf{\Gamma}_{\ell,k}$' and '$\Gamma_{\ell,k}$'.

Remark 6.3. Construction 6.2 extends to the rigid analytic setting. Indeed, if K is algebraically closed, there is an equivalence of ∞-categories

$$(\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}} \xrightarrow{\sim} \mathbf{RigSH}_{\text{ét}}(K;\Lambda)_{\ell\text{-cpl}}$$

by [AGV22, Theorem 2.10.3]. Moreover, Lemma 2.8 implies that the composite functor analogous to (6.1) lands in the full sub-∞-category of eventually connective and dualizable objects. Using [Lur18, Corollary 8.3.5.9], we obtain the plain ℓ-adic realization functor

$$\mathrm{R}^*_\ell : \mathbf{RigSH}_{\text{ét}}(K;\Lambda) \to \mathrm{Mod}_{\Lambda_\ell}.$$

We also denote by $\mathbf{\Gamma}_\ell = \mathrm{R}_{\ell,*}\Lambda_\ell$ and $\Gamma_\ell = \Omega^\infty_T(\mathbf{\Gamma}_\ell)$ the Weil spectrum and the Weil cohomology theory associated to R^*_ℓ. To stress the dependence on K, we would write '$\mathrm{R}^*_{\ell,K}$', '$\mathbf{\Gamma}_{\ell,K}$' and '$\Gamma_{\ell,K}$'.

Lemma 6.4. *Let $\sigma : k \to k'$ be a morphism between algebraically closed fields. Then, there is an equivalence $\sigma_*\Gamma_{\ell,k'} \simeq \Gamma_{\ell,k}$ of Weil cohomology theories on algebraic k-varieties. A similar statement is also true in the rigid analytic setting.*

Proof. We prove that the associated plain realization functors agree, and it is enough to do so after restrictions to compact objects since these functors are colimit-preserving. Thus, we need to show that the following triangle commutes

$$\begin{array}{ccc} \mathbf{SH}_{\text{ét, ct}}(k;\Lambda) & \xrightarrow{\sigma^*} & \mathbf{SH}_{\text{ét, ct}}(k';\Lambda) \\ & {\scriptstyle \mathrm{R}^*_{\ell,k}} \searrow & \downarrow {\scriptstyle \mathrm{R}^*_{\ell,k'}} \\ & & \mathrm{Mod}_{\Lambda_\ell}. \end{array}$$

Since the functor σ^* commutes with ℓ-completion, it is enough to show that the following triangle commutes

$$\begin{array}{ccc} \mathbf{SH}_{\text{ét}}(k;\Lambda)_{\ell\text{-cpl}} & \xrightarrow{\sigma^*} & \mathbf{SH}_{\text{ét}}(k';\Lambda)_{\ell\text{-cpl}} \\ & \nwarrow^{\sim} & \uparrow {\scriptstyle \sim} \\ & & (\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}}, \end{array}$$

which is obvious. □

Definition 6.5. We extend the ℓ-adic cohomology theories to non necessary algebraically closed fields in the usual way. Given an algebraic closure k^{alg}/k of k, we set $\Gamma_{\ell,k} = (k^{\mathrm{alg}}/k)_*\Gamma_{\ell,k^{\mathrm{alg}}}$. Similarly, given an algebraic closure K^{alg}/K and an extension of the valuation of K to K^{alg}, we set $\Gamma_{\ell,K} = (K^{\mathrm{alg}}/K)_*\Gamma_{\ell,K^{\mathrm{alg}}}$.

Remark 6.6. Lemma 6.4 implies that the equivalence class of the ℓ-adic cohomology theory $\Gamma_{\ell,k}$ is independent of the choice of the algebraic closure k^{alg}/k. For this reason, when discussing ℓ-adic cohomology, we often keep the choice of the algebraic closure implicit. The same applies in the rigid analytic setting.

Proposition 6.7. *The three ℓ-adic cohomology theories $\Gamma_{\ell,k} \in \mathrm{WCT}(k;\Lambda)$, $\Gamma_{\ell,K} \in \mathrm{WCT}(K;\Lambda)$ and $\Gamma_{\ell,K} \in \mathrm{RigWCT}(K;\Lambda)$ are related as follows.*

(1) There is an equivalence of Weil spectra on algebraic K-varieties $\mathbf{\Gamma}_{\ell,K} \simeq \mathrm{Rig}_*(\mathbf{\Gamma}_{\ell,K})$.
(2) There is an equivalence of Weil spectra on algebraic k-varieties $\mathbf{\Gamma}_{\ell,k} \simeq \xi_*(\mathbf{\Gamma}_{\ell,K})$.
(3) In Situation 3.12 (and with the notation of Construction 3.17), there is an equivalence of Weil spectra on rigid analytic K-varieties $\mathbf{\Gamma}_{\ell,K} \simeq \psi_*(\mathbf{\Gamma}_{\ell,k})$.

Proof. We can assume that K is algebraically closed: for (3) this is the only case we need to consider and, for (1) and (2), we can reduce easily to this case. For each equivalence, we will show that the two associated plain realizations agree, and it is enough to show this after restriction to compact objects since these realizations are colimit-preserving. Thus, we need to show that the following triangles commute

$$\begin{array}{ccc} \mathbf{SH}_{\text{ét, ct}}(K;\Lambda) & \xrightarrow{\mathrm{Rig}^*} & \mathbf{RigSH}_{\text{ét, ct}}(K;\Lambda) \\ & {\scriptstyle \mathrm{R}^*_{\ell}} \searrow & \downarrow {\scriptstyle \mathrm{R}^*_{\ell}} \\ & & \mathrm{Mod}_{\Lambda_\ell}, \end{array} \qquad \begin{array}{ccc} \mathbf{SH}_{\text{ét, ct}}(k;\Lambda) & \xrightarrow{\xi^*} & \mathbf{RigSH}_{\text{ét, ct}}(K;\Lambda) \\ & {\scriptstyle \mathrm{R}^*_{\ell}} \searrow & \downarrow {\scriptstyle \mathrm{R}^*_{\ell}} \\ & & \mathrm{Mod}_{\Lambda_\ell}, \end{array}$$

$$\begin{array}{ccc} \mathbf{RigSH}_{\text{ét, ct}}(K;\Lambda) & \xrightarrow{\psi^*} & \mathbf{SH}_{\text{ét, ct}}(k;\Lambda) \\ & \searrow^{\mathrm{R}^*_\ell} & \downarrow{\scriptstyle \mathrm{R}^*_\ell} \\ & & \mathrm{Mod}_{\Lambda_\ell}. \end{array}$$

Since the functors Rig^*, ξ^* and ψ^* commute with ℓ-completion, it is enough to show that the following triangles commute

$$\begin{array}{ccc} \mathbf{SH}_{\text{ét}}(K;\Lambda)_{\ell\text{-cpl}} & \xrightarrow{\mathrm{Rig}^*} & \mathbf{RigSH}_{\text{ét}}(K;\Lambda)_{\ell\text{-cpl}} \\ & \overset{\sim}{\nwarrow} & \uparrow{\scriptstyle \sim} \\ & & (\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}}, \end{array} \qquad \begin{array}{ccc} \mathbf{SH}_{\text{ét}}(k;\Lambda)_{\ell\text{-cpl}} & \xrightarrow{\xi^*} & \mathbf{RigSH}_{\text{ét}}(K;\Lambda)_{\ell\text{-cpl}} \\ & \overset{\sim}{\nwarrow} & \uparrow{\scriptstyle \sim} \\ & & (\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}}, \end{array}$$

$$\begin{array}{ccc} \mathbf{RigSH}_{\text{ét}}(K;\Lambda)_{\ell\text{-cpl}} & \xrightarrow{\psi^*} & \mathbf{SH}_{\text{ét}}(k;\Lambda)_{\ell\text{-cpl}} \\ & \overset{\sim}{\nwarrow} & \uparrow{\scriptstyle \sim} \\ & & (\mathrm{Mod}_\Lambda)_{\ell\text{-cpl}}, \end{array}$$

which is obvious. □

We now discuss the Betti cohomology theory.

Construction 6.8. Let CpSm be the category of complex smooth varieties, and denote by cl the classical topology on this category. Also, denote by $\mathbb{D}^1 \subset \mathbb{C}$ the open unit disc. The ∞-category $\mathbf{AnSH}(\Lambda)$, obtained from the ∞-category of hypersheaves of Λ-modules $\mathrm{Shv}_{\mathrm{cl}}(\mathrm{CpSm};\Lambda)$ by $\mathbb{D}^1$-localization and $\otimes$-inversion of the Tate object $T = \mathrm{L}_{\mathrm{mot}}(\Lambda_{\mathrm{cl}}(\mathbb{P}^1,\infty))$, is equivalent to the ∞-category Mod_Λ via the obvious tensor functor $\mathrm{Mod}_\Lambda \to \mathbf{AnSH}(\Lambda)$. (See for example [Ayo10, Théorème 1.8].) On the other hand, the analytification functor $\mathrm{Sm}_{\mathbb{C}} \to \mathrm{CpSm}$ gives rise to a functor $\mathrm{An}^* : \mathbf{SH}_{\text{ét}}(\mathbb{C};\Lambda) \to \mathbf{AnSH}(\Lambda)$. We define the Betti realization functor

$$\mathrm{B}^* : \mathbf{SH}_{\text{ét}}(\mathbb{C};\Lambda) \to \mathrm{Mod}_\Lambda$$

to be the composite of

$$\mathbf{SH}_{\text{ét}}(\mathbb{C};\Lambda) \xrightarrow{\mathrm{An}^*} \mathbf{AnSH}(\Lambda) \simeq \mathrm{Mod}_\Lambda.$$

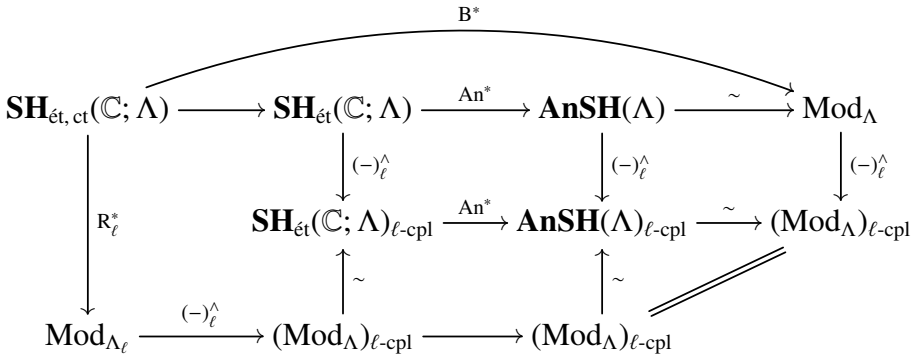
We set $\mathbf{\Gamma}_{\mathrm{B}} = \mathrm{B}_*\Lambda$ and $\Gamma_{\mathrm{B}} = \Omega^\infty_T(\mathbf{\Gamma}_{\mathrm{B}})$.

Definition 6.9. We extend the Betti cohomology theory to any field k endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$ by setting $\Gamma_{\mathrm{B},k} = \sigma_*(\Gamma_{\mathrm{B}})$.

Proposition 6.10. *Let $\sigma : k \hookrightarrow \mathbb{C}$ be a complex embedding. Then, there is a morphism of Weil cohomology theories $\Gamma_{\mathrm{B},k} \to \Gamma_{\ell,k}$ in $\mathrm{WCT}(k;\Lambda)$. (In fact, there is a canonical such a morphism if we take for $\Gamma_{\ell,k}$ the ℓ-adic cohomology theory associated to the algebraic closure k^{alg}/k of k in $\mathbb{C}$.)*

Proof. This is a reformulation of the classical comparison theorem between Betti cohomology and ℓ-adic cohomology. In our framework, it suffices to show that the associated plain realization functors coincide after the appropriate scalar extension, and it is enough to do so for $k = \mathbb{C}$. But

we have a commutative diagram



showing that the symmetric monoidal functors $\mathrm{B}^*(-) \otimes \Lambda_\ell$ and R^*_ℓ are equivalent on $\mathbf{SH}_{\text{ét, ct}}(\mathbb{C}; \Lambda)$, and hence also on $\mathbf{SH}_{\text{ét}}(\mathbb{C}; \Lambda)$ by indization. □

Next, we discuss the de Rham cohomology theory.

Construction 6.11. Assume that k has characteristic zero. We fix a morphism of commutative ring spectra $\Lambda \to k$ and use it to view k-modules as Λ-modules. The Zariski sheafification of the algebraic de Rham complex $\Omega^\bullet_{/k}$, viewed as a presheaf of Λ-modules on Sm_k, defines a Weil cohomology theory Γ_{dR}. This is the algebraic de Rham cohomology theory. We denote by $\mathbf{\Gamma}_{\mathrm{dR}}$ the associated Weil spectrum and $\mathrm{dR}^* : \mathbf{SH}_{\text{ét}}(k; \Lambda) \to \mathrm{Mod}_k$ the associated plain realization. To stress the dependence on k, we would write '$\Gamma_{\mathrm{dR}, k}$', '$\mathbf{\Gamma}_{\mathrm{dR}, k}$' and '$\mathrm{dR}^*_k$'.

The following is a reformulation of the Betti-de Rham comparison theorem of Grothendieck.

Proposition 6.12. *Let $\sigma : k \hookrightarrow \mathbb{C}$ be a complex embedding and fix a morphism $\Lambda \to k$. Then, there is a morphism of Weil cohomology theories $\Gamma_{\mathrm{B}, k} \to \Gamma_{\mathrm{dR}} \otimes_k \mathbb{C}$ in $\mathrm{WCT}(k; \Lambda)$.*

Proof. See for example [Ayo14b, Proposition 2.88 & Corollaire 2.89]. □

The last classical Weil cohomology theory that we mention is Berthelot's rigid cohomology.

Construction 6.13. Assume that K has characteristic zero, but allow k to have any characteristic. We fix a morphism of commutative ring spectra $\Lambda \to \widehat{K}$ and use it to view $\widehat{K}$-modules as Λ-modules. There is a Weil cohomology theory $\Gamma^\dagger_{\mathrm{dR}} \in \mathrm{RigWCT}(K; \Lambda)$ on rigid analytic K-varieties given by overconvergent de Rham cohomology in the sense of [GK04]. A construction of $\Gamma^\dagger_{\mathrm{dR}}$ can be found in [Vez18, Proposition 5.12]. Roughly speaking, one considers the overconvergent de Rham complex $\Omega^{\dagger, \bullet}_{/K}$ which is a presheaf on the category $\mathrm{RigSm}^\dagger_K$ of smooth dagger rigid analytic K-varieties. The étale hypersheafification of $\Omega^{\dagger, \bullet}_{/K}$ being $\mathbb{B}^1$-local, [Vez18, Theorem 4.23] shows that $\mathrm{L}_{\text{ét}}(\Omega^{\dagger, \bullet}_{/K})$ factors uniquely through the forgetful functor $\mathrm{RigSm}^\dagger_K \to \mathrm{RigSm}_K$ yielding the étale hypersheaf $\Gamma^\dagger_{\mathrm{dR}}$. Moreover, by [Vez18, Proposition 5.12], the Weil spectrum $\mathbf{\Gamma}_{\mathrm{rig}} = \xi_* \mathbf{\Gamma}^\dagger_{\mathrm{dR}}$ represents Berthelot's rigid cohomology on algebraic k-varieties. To stress the dependence on K, we would write '$\Gamma^\dagger_{\mathrm{dR}, K}$', '$\Gamma_{\mathrm{rig}, K}$', etc.

The following is a reformulation of a theorem of Grosse-Klönne [GK04, Theorem 2.3].

Proposition 6.14. *Fix a morphism of commutative ring spectra $\Lambda \to K$. The Weil cohomology theories $\Gamma_{\mathrm{dR}, K} \in \mathrm{WCT}(K; \Lambda)$ and $\Gamma^\dagger_{\mathrm{dR}, K} \in \mathrm{RigWCT}(K; \Lambda)$ are related by a morphism of Weil spectra $\mathbf{\Gamma}_{\mathrm{dR}, K} \to \mathrm{Rig}_*(\mathbf{\Gamma}^\dagger_{\mathrm{dR}, K})$ in $\mathrm{WSp}(K; \Lambda)$.*

Proof. The rigid analytification X^{an} of an algebraic K-variety X has a natural dagger structure. Thus, there is a functor $(-)^{\mathrm{an},\dagger} : \mathrm{Sm}_K \to \mathrm{RigSm}_K^{\dagger}$ factoring the usual rigid analytification functor. Moreover, there is an obvious morphism of complexes of presheaves $\Omega^{\bullet}_{/K}(-) \to \Omega^{\dagger,\bullet}_{/K}((-)^{\mathrm{an},\dagger})$. This induces a morphism of Weil cohomology theories $\Gamma_{\mathrm{dR},K} \to \Gamma^{\dagger}_{\mathrm{dR},K} \circ (-)^{\mathrm{an}}$ as needed. □

Using Theorem 3.4, we can construct 'new' Weil cohomology theories using the classical ones. This was undertaken in [Ayo20], and we revisit the construction below.

Construction 6.15. We assume that K has characteristic zero, but we allow its residue field k to have arbitrary characteristic.

(1) Let K^{alg}/K be an algebraic closure of K, and let $\Gamma_{\ell,K}$ be the associated ℓ-adic cohomology theory on algebraic K-varieties. The Weil cohomology theory $\widehat{\Gamma}_{\mathrm{new},\ell}$ on rigid analytic K-varieties corresponds to the Weil spectrum $\mathrm{Rig}^*\mathbf{\Gamma}_{\ell,K}$. Its ring of coefficients is $\mathrm{R}^*_{\ell}(\mathrm{Rig}_*\Lambda)$.
(2) Let $\sigma : K \hookrightarrow \mathbb{C}$ be a complex embedding, and let $\Gamma_{\mathrm{B},K}$ be the associated Betti cohomology theory on algebraic K-varieties. The Weil cohomology theory $\widehat{\Gamma}_{\mathrm{new,B}}$ on rigid analytic K-varieties corresponds to the Weil spectrum $\mathrm{Rig}^*\mathbf{\Gamma}_{\mathrm{B},K}$. Its ring of coefficients is $\mathrm{B}^*(\mathrm{Rig}_*\Lambda)$.
(3) Fix a morphism $\Lambda \to K$. Let $\Gamma_{\mathrm{dR},K}$ be the de Rham cohomology theory on algebraic K-varieties. The Weil cohomology theory $\widehat{\Gamma}_{\mathrm{new,dR}}$ on rigid analytic K-varieties corresponds to the Weil spectrum $\mathrm{Rig}^*\mathbf{\Gamma}_{\mathrm{dR},K}$. Its ring of coefficients is $\mathrm{dR}^*(\mathrm{Rig}_*\Lambda)$.

The Weil spectra $\mathbf{\Gamma}_{\mathrm{new},\ell}$, $\mathbf{\Gamma}_{\mathrm{new,B}}$ and $\mathbf{\Gamma}_{\mathrm{new,dR}}$ on algebraic k-varieties are obtained from $\widehat{\mathbf{\Gamma}}_{\mathrm{new},\ell}$, $\widehat{\mathbf{\Gamma}}_{\mathrm{new,B}}$ and $\widehat{\mathbf{\Gamma}}_{\mathrm{new,dR}}$ respectively by applying ξ_*. To stress the dependence on K, we would write '$\widehat{\Gamma}_{\mathrm{new},\ell,K}$', '$\Gamma_{\mathrm{new},\ell,K}$', etc.

Proposition 6.16. *There is a diagram in* $\mathrm{RigWCT}(K;\Lambda)$ *as follows:*

$$\begin{array}{ccccccc} \widehat{\Gamma}_{\mathrm{new,dR}} & \longrightarrow & \widehat{\Gamma}_{\mathrm{new,dR}} \otimes_K \mathbb{C} & \longleftarrow & \widehat{\Gamma}_{\mathrm{new,B}} & \longrightarrow & \widehat{\Gamma}_{\mathrm{new},\ell} \\ \downarrow & & & & & & \downarrow \\ \Gamma^{\dagger}_{\mathrm{dR}} & & & & & & \Gamma_{\ell}. \end{array} \tag{6.3}$$

Similarly, there is a diagram in $\mathrm{WCT}(k;\Lambda)$ *as follows:*

$$\begin{array}{ccccccc} \Gamma_{\mathrm{new,dR}} & \longrightarrow & \Gamma_{\mathrm{new,dR}} \otimes_K \mathbb{C} & \longleftarrow & \Gamma_{\mathrm{new,B}} & \longrightarrow & \Gamma_{\mathrm{new},\ell} \\ \downarrow & & & & & & \downarrow \\ \Gamma_{\mathrm{rig}} & & & & & & \Gamma_{\ell}. \end{array} \tag{6.4}$$

Proof. The horizontal line in the diagram (6.3) is obtained from the following diagram

$$\Gamma_{\mathrm{dR}} \to \Gamma_{\mathrm{dR}} \otimes_K \mathbb{C} \leftarrow \Gamma_{\mathrm{B}} \to \Gamma_{\ell}$$

by applying the functor $\mathrm{Rig}^* : \mathrm{WCT}(K;\Lambda) \to \mathrm{RigWCT}(K;\Lambda)$. (See Propositions 6.10 and 6.12.) The vertical arrows in the diagram (6.3) are deduced by adjunction from the natural morphisms $\Gamma_{\mathrm{dR}} \to \mathrm{Rig}_*(\Gamma^{\dagger}_{\mathrm{dR}})$ and $\Gamma_{\ell} \to \mathrm{Rig}_*(\Gamma_{\ell})$ provided by Propositions 6.7 and 6.14. Finally, the diagram (6.4) is obtained from the diagram (6.3) by applying ξ_* and using Proposition 6.7. □

7. The connectivity theorem

In this final section, we prove the main new result of this paper, namely the connectivity of the motivic Hopf algebras associated to the classical Weil cohomology theories. More precisely, we will prove the following.

Theorem 7.1. *Let Λ be a connective commutative ring spectrum, k a field whose exponent characteristic is invertible in $\pi_0(\Lambda)$, and K a valued field of characteristic zero whose valuation has height 1 and finite rational rank, and whose residue field is k. Below, ℓ is a prime invertible in k, a complex embedding $\sigma : K \hookrightarrow \mathbb{C}$ is chosen, and a morphism $\Lambda \to K$ is fixed allowing us to view K-modules as Λ-modules.*

(1) The motivic Hopf algebroids of the following Weil cohomology theories

$$\Gamma_\ell, \quad \Gamma_{\mathrm{rig}}, \quad \Gamma_{\mathrm{new},\ell}, \quad \Gamma_{\mathrm{new,B}} \quad \textit{and} \quad \Gamma_{\mathrm{new,dR}} \quad \in \mathrm{WCT}(k;\Lambda)$$

are connective.

(2) The motivic Hopf algebroids of the following Weil cohomology theories

$$\Gamma_\ell, \quad \Gamma^\dagger_{\mathrm{dR}}, \quad \widehat{\Gamma}_{\mathrm{new},\ell}, \quad \widehat{\Gamma}_{\mathrm{new,B}} \quad \textit{and} \quad \widehat{\Gamma}_{\mathrm{new,dR}} \quad \in \mathrm{RigWCT}(K;\Lambda)$$

are connective.

Remark 7.2. Our assumption on the rational rank of K is merely for convenience. It can be removed at the cost of proving Theorems 3.13 and 3.21 without the assumption on the value group made in Situation 3.12. This is possible but would render the proofs even more technical. At least for the classical Weil cohomology theories Γ_ℓ and $\Gamma^\dagger_{\mathrm{dR}}$, one can easily remove the assumption on the rational rank using the continuity property for the motivic Hopf algebras given in Proposition 5.12.

We start by establishing the following reduction.

Lemma 7.3. *It is enough to prove Theorem 7.1 for $\Gamma_{\mathrm{new,B}}$ and $\widehat{\Gamma}_{\mathrm{new,B}}$ under the extra assumption that K is algebraically closed.*

Proof. By Corollary 4.15 and Proposition 6.16, it is enough to treat the Weil cohomology theories $\Gamma_{\mathrm{new,B}}$ and $\widehat{\Gamma}_{\mathrm{new,B}}$. It remains to explain why we can assume that K is algebraically closed. We do this in two steps: first we reduce the general case to the henselian one, and then we reduce the henselian case to the algebraically closed one. For later use, it will be convenient to assume that $\Lambda = \mathbb{S}_{(\ell)}$ is the localization of the sphere spectrum at a prime number ℓ, which we can do without lost of generality.

Step 1: reduction to the case where K is henselian. Let K' be the henselization of K with respect to its valuation. Thus, the ring of integers K'° in K' is the henselization of the ring of integers K° in K, and K and K' have the same completion $\widehat{K}$. We denote by $e : \mathrm{Spec}(K') \to \mathrm{Spec}(K)$ the obvious morphism and, for the purpose of this proof, we denote by $\mathrm{Rig}'^* : \mathbf{SH}_{\text{ét}}(K';\Lambda) \to \mathbf{RigSH}_{\text{ét}}(\widehat{K};\Lambda)$ the rigid analytification functor. Notice that $\mathrm{Rig}_*\Lambda \simeq e_*\mathrm{Rig}'_*\Lambda$ is naturally an $e_*\Lambda$-algebra. It follows that $\mathrm{B}^*(\mathrm{Rig}_*\Lambda)$ is naturally an algebra over $\mathrm{B}^*(e_*\Lambda) = \Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}$. (As usual, we view $\mathrm{Hom}_K(K',\mathbb{C})$ as a profinite set and $\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}$ is the Λ-algebra of locally constant functions on $\mathrm{Hom}_K(K',\mathbb{C})$ with values in Λ.) Since the ring of coefficients of $\Gamma_{\mathrm{new,B}}$ and $\widehat{\Gamma}_{\mathrm{new,B}}$ is $\mathrm{B}^*(\mathrm{Rig}_*\Lambda)$, the Weil spectra $\mathbf{\Gamma}_{\mathrm{new,B}}$ and $\widehat{\mathbf{\Gamma}}_{\mathrm{new,B}}$ have the structure of a $\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}$-algebra. Thus, the motivic Hopf algebroids $\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B}})$ and $\mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B}})$ are algebras over the cosimplicial ring $\Lambda^{\check{\mathrm{C}}\bullet(\mathrm{Hom}_K(K',\mathbb{C}))}$. To prove that these motivic Hopf algebroids are connective, it is enough to show the following.

(1) Given $\sigma' \in \mathrm{Hom}_K(K', \mathbb{C})$, the commutative algebra

$$\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B}})(\Delta^0) \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'\}} = \mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B}})(\Delta^0) \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'\}} \tag{7.1}$$

is connective.

(2) Given $\sigma'_1, \sigma'_2 \in \mathrm{Hom}_K(K', \mathbb{C})$, the commutative algebras

$$\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B}})(\Delta^1) \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})^{\times 2}}} \Lambda^{\{(\sigma'_1,\sigma'_2)\}} \text{ and } \mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B}})(\Delta^1) \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})^{\times 2}}} \Lambda^{\{(\sigma'_1,\sigma'_2)\}} \tag{7.2}$$

are connective.

Now, the algebra (7.1) is obtained by applying $\Gamma(\widehat{K}; \Omega^\infty_T(-))$ to $\widehat{\mathbf{\Gamma}}_{\mathrm{new,B}} \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'\}}$, and the algebras (7.2) are obtained by applying $\Gamma(k; \Omega^\infty_T(-))$ and $\Gamma(\widehat{K}; \Omega^\infty_T(-))$ to

$$(\widehat{\mathbf{\Gamma}}_{\mathrm{new,B}} \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'_1\}}) \otimes (\widehat{\mathbf{\Gamma}}_{\mathrm{new,B}} \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'_2\}}) \quad \text{and}$$

$$(\mathbf{\Gamma}_{\mathrm{new,B}} \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'_1\}}) \otimes (\mathbf{\Gamma}_{\mathrm{new,B}} \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'_2\}}).$$

Thus, we need a good description of the spectra $\widehat{\mathbf{\Gamma}}_{\mathrm{new,B}} \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'\}}$, for $\sigma' \in \mathrm{Hom}_K(K', \mathbb{C})$. We claim that there is an equivalence

$$\widehat{\mathbf{\Gamma}}_{\mathrm{new,B}} \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'\}} \simeq \widehat{\mathbf{\Gamma}}_{\mathrm{new,B'}} \tag{7.3}$$

where $\widehat{\Gamma}_{\mathrm{new,B'}} \in \mathrm{RigWCT}(\widehat{K}; \Lambda)$ is the new Weil cohomology theory obtained from the Betti cohomology theory $\Gamma_{\mathrm{B}'} \in \mathrm{WCT}(K'; \Lambda)$ associated to the complex embedding $\sigma' : K' \to \mathbb{C}$. Indeed, recall from Theorem 3.7 that we have equivalences of ∞-categories

$$\begin{array}{ccc} \mathbf{RigSH}_{\text{ét}}(\widehat{K}; \Lambda) & \xrightarrow[\sim]{\widetilde{\mathrm{Rig}'_*}} & \mathbf{SH}_{\text{ét}}(K'; \mathrm{Rig}'_*\Lambda) \\ & \searrow^{\sim}_{\widetilde{\mathrm{Rig}_*}} & \Big\downarrow{\scriptstyle \sim}\, \widetilde{e}_* \\ & & \mathbf{SH}_{\text{ét}}(K; \mathrm{Rig}_*\Lambda) \end{array}$$

where the vertical arrow is induced by the equivalence

$$\widetilde{e}_* : \mathbf{SH}_{\text{ét}}(K'; \Lambda) \xrightarrow{\sim} \mathbf{SH}_{\text{ét}}(K; e_*\Lambda)$$

from Proposition 5.2. Modulo $\widetilde{\mathrm{Rig}_*}$, the Weil spectrum $\widehat{\mathbf{\Gamma}}_{\mathrm{new,B}}$ is given by $\mathrm{Rig}_*\Lambda \otimes \mathrm{B}_*\Lambda$. Thus, modulo $\widetilde{\mathrm{Rig}'_*}$, the same Weil spectrum $\widehat{\mathbf{\Gamma}}_{\mathrm{new,B}}$ is given by

$$e^*(e_*\mathrm{Rig}'_*\Lambda \otimes \mathrm{B}_*\Lambda) \otimes_{e^*e_*\mathrm{Rig}'_*\Lambda} \mathrm{Rig}'_*\Lambda \simeq \mathrm{Rig}'_*\Lambda \otimes e^*\mathrm{B}_*\Lambda. \tag{7.4}$$

Note that

$$\begin{array}{rcl} \Gamma(K'; \Omega^\infty_T(e^*\mathrm{B}_*\Lambda)) & \simeq & \Gamma(K; \Omega^\infty_T(e_*e^*\mathrm{B}_*\Lambda)) \\ & \simeq & \Gamma(K; \Omega^\infty_T(\mathrm{B}_*e_*e^*\Lambda)) \\ & \simeq & \Gamma(K; \Omega^\infty_T(\mathrm{B}_*\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})})) \\ & \simeq & \Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}. \end{array}$$

In particular, $e^*\mathrm{B}_*\Lambda$ has a natural structure of a $\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}$-algebra, and it is easy to see that this structure induces the one we have on $\widehat{\mathbf{\Gamma}}_{\mathrm{new,B}}$ modulo the equivalence $\widetilde{\mathrm{Rig}'_*}$. For the purpose of this proof, we denote by $\mathrm{B}'^* : \mathbf{SH}_{\text{ét}}(K'; \Lambda) \to \mathrm{Mod}_\Lambda$ the Betti realization associated to the embedding σ', so that $\mathbf{\Gamma}_{\mathrm{B}'} = \mathrm{B}'_*\Lambda$. We have equivalences in $\mathbf{SH}_{\text{ét}}(K; e_*\Lambda)$:

$$\begin{array}{rcl} (e_*e^*\mathrm{B}_*\Lambda) \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'\}} & \simeq & (\mathrm{B}_*e_*e^*\Lambda) \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'\}} \\ & \simeq & (\mathrm{B}_*\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}) \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'\}} \\ & \simeq & \mathrm{B}_*\Lambda, \end{array}$$

where the action of $e_*\Lambda$ on $\mathrm{B}_*\Lambda$ is the one deduced from the equivalence $\mathrm{B}_*\Lambda \simeq e_*\mathrm{B}'_*\Lambda$. Using Proposition 5.2, this yields an equivalence in $\mathbf{SH}_{\text{ét}}(K';\Lambda)$:

$$(e^*\mathrm{B}_*\Lambda) \otimes_{\Lambda^{\mathrm{Hom}_K(K',\mathbb{C})}} \Lambda^{\{\sigma'\}} \simeq \mathrm{B}'_*\Lambda.$$

Combining this with the fact that $\widehat{\mathbf{\Gamma}}_{\mathrm{new},\mathrm{B}}$ is given by (7.4) modulo the equivalence $\widetilde{\mathrm{Rig}'_*}$, we deduce the claimed equivalence (7.3).

It is now easy to conclude. Indeed, recall that we want to check the properties (1) and (2) above. Using the equivalence (7.3), the commutative algebra (7.1) can be identified with the ring of coefficients of $\widehat{\Gamma}_{\mathrm{new},\mathrm{B}'}$ whose connectivity would be granted if Theorem 7.1 was known for K'. Similarly, denoting $\Gamma_{\mathrm{new},\mathrm{B}'_i} \in \mathrm{WCT}(k;\Lambda)$ and $\widehat{\Gamma}_{\mathrm{new},\mathrm{B}'_i} \in \mathrm{RigWCT}(K';\Lambda)$, for $i \in \{1,2\}$, the new Weil cohomology theories obtained from the Betti cohomology theories $\Gamma_{\mathrm{B}'_i}$ associated to the complex embeddings σ'_i, the commutative algebras in (7.2) can be identified with

$$\Gamma(k;\Omega^\infty_T(\mathbf{\Gamma}_{\mathrm{new},\mathrm{B}'_1} \otimes \mathbf{\Gamma}_{\mathrm{new},\mathrm{B}'_2})) \qquad \text{and} \qquad \Gamma(\widehat{K};\Omega^\infty_T(\widehat{\mathbf{\Gamma}}_{\mathrm{new},\mathrm{B}'_1} \otimes \widehat{\mathbf{\Gamma}}_{\mathrm{new},\mathrm{B}'_2})).$$

If $\sigma'_1 = \sigma'_2 = \sigma'$, these algebras coincide with $\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new},\mathrm{B}'})(\Delta^1)$ and $\mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new},\mathrm{B}'})(\Delta^1)$, and we would be done. In general, it would suffices to know that $\widehat{\Gamma}_{\mathrm{new},\mathrm{B}'_1}$ and $\widehat{\Gamma}_{\mathrm{new},\mathrm{B}'_2}$ become equivalent after a common faithfully flat extension of their ring of coefficients. This would follow if the Betti cohomology theories $\Gamma_{\mathrm{B}'_1}$ and $\Gamma_{\mathrm{B}'_2}$ become equivalent after a faithfully flat extension of Λ. To prove this, recall from the beginning of the proof that $\Lambda = \mathbb{S}_{(\ell)}$. Proposition 6.10 implies that $\Gamma_{\mathrm{B}'_1} \otimes \mathbb{S}_\ell$ and $\Gamma_{\mathrm{B}'_2} \otimes \mathbb{S}_\ell$ are equivalent to the ℓ-adic cohomology theories $\Gamma_{\ell,1}$ and $\Gamma_{\ell,2}$ in $\mathrm{WCT}(K';\Lambda)$ associated the algebraic closures $\overline{K}_1/K'$ and $\overline{K}_2/K'$ deduced from the complex embeddings σ'_1 and σ'_2. Lemma 6.4 implies that $\Gamma_{\ell,1}$ and $\Gamma_{\ell,2}$ is equivalent. This finishes the proof since $\mathbb{S}_\ell$ is faithfully flat over $\mathbb{S}_{(\ell)}$ by [Lur18, Corollary 7.3.6.9].

Step 2: reduction to the case where K is algebraically closed. We now assume that K is henselian. Let $K^{\mathrm{alg}} \subset \mathbb{C}$ be the algebraic closure of K in $\mathbb{C}$. The valuation of K extends uniquely to a valuation of K^{alg} and every element of the Galois group G of K^{alg}/K induces an automorphism of the completion $\widehat{K}^{\mathrm{alg}}$ of K^{alg}. Let $e : \mathrm{Spec}(K^{\mathrm{alg}}) \to \mathrm{Spec}(K)$ and $\widehat{e} : \mathrm{Spa}(\widehat{K}^{\mathrm{alg}}) \to \mathrm{Spa}(\widehat{K})$ be the obvious morphisms. For the purpose of this proof, we denote by $\mathrm{Rig}'^* : \mathbf{SH}_{\text{ét}}(K^{\mathrm{alg}};\Lambda) \to \mathbf{RigSH}_{\text{ét}}(K^{\mathrm{alg}};\Lambda)$ the rigid analytification functor. By Lemma 7.4 below, we have an equivalence

$$e^*\mathrm{Rig}_*\Lambda \simeq \mathrm{Rig}'_*\Lambda$$

of commutative algebras in $\mathbf{SH}_{\text{ét}}(K^{\mathrm{alg}};\Lambda)$. Write $\mathrm{B}'^* : \mathbf{SH}_{\text{ét}}(K^{\mathrm{alg}};\Lambda) \to \mathrm{Mod}_\Lambda$ for the plain Betti realization functor associated to the complex embedding $K^{\mathrm{alg}} \subset \mathbb{C}$. The functor

$$e_* : \mathbf{SH}_{\text{ét}}(K^{\mathrm{alg}};\mathrm{Rig}'_*\Lambda) \to \mathbf{SH}_{\text{ét}}(K;\mathrm{Rig}_*\Lambda)$$

takes the algebra $\mathrm{Rig}'_*\Lambda \otimes \mathrm{B}'_*\Lambda$ to

$$\begin{aligned} e_*(e^*\mathrm{Rig}_*\Lambda \otimes \mathrm{B}'_*\Lambda) &\simeq \mathrm{Rig}_*\Lambda \otimes e_*\mathrm{B}'_*\Lambda \\ &\simeq \mathrm{Rig}_*\Lambda \otimes \mathrm{B}_*\Lambda, \end{aligned}$$

where the first equivalence is deduced from the projection formula for the morphism e which is a profinite morphism. This implies that the Weil cohomology theories $\widehat{\Gamma}_{\mathrm{new},\mathrm{B}} \in \mathrm{RigWCT}(K;\Lambda)$ and $\widehat{\Gamma}_{\mathrm{new},\mathrm{B}'} \in \mathrm{RigWCT}(K^{\mathrm{alg}};\Lambda)$ are related by the formula:

$$\widehat{\Gamma}_{\mathrm{new},\mathrm{B}} \simeq \widehat{e}_*(\widehat{\Gamma}_{\mathrm{new},\mathrm{B}'}). \tag{7.5}$$

Letting k^{alg} be the residue field of K^{alg} and $\overline{e} : \mathrm{Spec}(k^{\mathrm{alg}}) \to \mathrm{Spec}(k)$ the obvious morphism, we also have the formula

$$\Gamma_{\mathrm{new,B}} \simeq \overline{e}_*(\Gamma_{\mathrm{new,B'}}) \tag{7.6}$$

relating the Weil cohomology theories $\Gamma_{\mathrm{new,B}} \in \mathrm{WCT}(k;\Lambda)$ and $\Gamma_{\mathrm{new,B'}} \in \mathrm{WCT}(k^{\mathrm{alg}};\Lambda)$. This follows immediately from the commutative square

$$\begin{array}{ccc} \mathbf{SH}_{\text{ét}}(k;\Lambda) & \xrightarrow{\xi^*} & \mathbf{RigSH}_{\text{ét}}(K;\Lambda) \\ \downarrow{\scriptstyle \overline{e}^*} & & \downarrow{\scriptstyle \widehat{e}^*} \\ \mathbf{SH}_{\text{ét}}(k^{\mathrm{alg}};\Lambda) & \xrightarrow{\xi^*} & \mathbf{RigSH}_{\text{ét}}(K^{\mathrm{alg}};\Lambda) \end{array}$$

by considering the associated plain realization functors.

It is now easy to conclude. Indeed, we want to show that $\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B}})$ and $\mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B}})$ are connective knowing that $\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B'}})$ and $\mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B'}})$ are connective. Using the equivalences (7.5) and (7.6), and Proposition 5.13, we are reduced to showing that, given an automorphism g of K^{alg}/K, the Weil cohomology theories $\widehat{\Gamma}_{\mathrm{new,B'}}$ and $g_*\widehat{\Gamma}_{\mathrm{new,B'}}$ become equivalent after a common faithfully flat extension of their rings of coefficients. It is easy to see that $g_*\widehat{\Gamma}_{\mathrm{new,B'}} \in \mathrm{RigWCT}(K^{\mathrm{alg}};\Lambda)$ is the new Weil cohomology theory obtained from the Weil cohomology theory $g_*\Gamma_{\mathrm{B'}} \in \mathrm{WCT}(K^{\mathrm{alg}};\Lambda)$. Thus, it is enough to show that $\Gamma_{\mathrm{B'}}$ and $g_*\Gamma_{\mathrm{B'}}$ become equivalent after a faithfully flat extension of Λ. To prove this, recall from the beginning of the proof that $\Lambda = \mathbb{S}_{(\ell)}$. Proposition 6.10 implies that $\Gamma_{\mathrm{B'}} \otimes \mathbb{S}_\ell$ is equivalent to the ℓ-adic cohomology theory $\Gamma_\ell \in \mathrm{WCT}(K^{\mathrm{alg}};\Lambda)$. Lemma 6.4 implies that Γ_ℓ and $g_*\Gamma_\ell$ are equivalent. This finishes the proof since $\mathbb{S}_\ell$ is faithfully flat over $\mathbb{S}_{(\ell)}$ by [Lur18, Corollary 7.3.6.9]. □

Lemma 7.4. *Assume that the valued field K is henselian. Let K'/K be an algebraic extension of K and endow K' with the unique extension of the valuation of K. Let $e : \mathrm{Spec}(K') \to \mathrm{Spec}(K)$ and $\widehat{e} : \mathrm{Spa}(\widehat{K'}) \to \mathrm{Spa}(\widehat{K})$ be the obvious morphisms. The commutative square of ∞-categories*

$$\begin{array}{ccc} \mathbf{SH}_{\text{ét}}(K;\Lambda) & \xrightarrow{e^*} & \mathbf{SH}_{\text{ét}}(K';\Lambda) \\ \downarrow{\scriptstyle \mathrm{Rig}^*} & & \downarrow{\scriptstyle \mathrm{Rig}^*} \\ \mathbf{RigSH}_{\text{ét}}(K;\Lambda) & \xrightarrow{\widehat{e}^*} & \mathbf{RigSH}_{\text{ét}}(K';\Lambda) \end{array}$$

is right adjointable horizontally and vertically, i.e., the induced natural transformations

$$\mathrm{Rig}^* e_* \to \widehat{e}_* \mathrm{Rig}^* \qquad \textit{and} \qquad e^* \mathrm{Rig}_* \to \mathrm{Rig}_* \widehat{e}^*$$

are equivalences.

Proof. Consider objects $M' \in \mathbf{SH}_{\text{ét}}(K';\Lambda)$ and $N \in \mathbf{RigSH}_{\text{ét}}(K;\Lambda)$. We want to prove that

$$\mathrm{Rig}^* e_*(M') \to \widehat{e}_* \mathrm{Rig}^*(M') \qquad \text{and} \qquad e^* \mathrm{Rig}_*(N) \to \mathrm{Rig}_* \widehat{e}^*(N)$$

are equivalences. It is enough to do this with M' replaced by $M' \otimes \mathbb{Q}$ and $M' \otimes \mathbb{Q}/\mathbb{Z}$, and similarly for N. Said differently, we may assume that M' and N are uniquely divisible or torsion. In the torsion case, the commutative square

$$\begin{array}{ccc} \mathbf{SH}_{\text{ét}}(K;\Lambda)_{\mathrm{tor}} & \xrightarrow{e^*} & \mathbf{SH}_{\text{ét}}(K';\Lambda)_{\mathrm{tor}} \\ \downarrow{\scriptstyle \mathrm{Rig}^*} & & \downarrow{\scriptstyle \mathrm{Rig}^*} \\ \mathbf{RigSH}_{\text{ét}}(K;\Lambda)_{\mathrm{tor}} & \xrightarrow{\widehat{e}^*} & \mathbf{RigSH}_{\text{ét}}(K';\Lambda)_{\mathrm{tor}} \end{array}$$

can be identified, using [AGV22, Theorems 2.10.3 & 2.10.4], with the following commutative square

$$\begin{array}{ccc} \mathrm{Shv}_{\text{ét}}(\acute{\mathrm{E}}\mathrm{t}_K;\Lambda)_{\mathrm{tor}} & \xrightarrow{e^*} & \mathrm{Shv}_{\text{ét}}(\acute{\mathrm{E}}\mathrm{t}_{K'};\Lambda)_{\mathrm{tor}} \\ \downarrow\sim & & \downarrow\sim \\ \mathrm{Shv}_{\text{ét}}(\acute{\mathrm{E}}\mathrm{t}_{\widehat{K}};\Lambda)_{\mathrm{tor}} & \xrightarrow{\hat{e}^*} & \mathrm{Shv}_{\text{ét}}(\acute{\mathrm{E}}\mathrm{t}_{\widehat{K}'};\Lambda)_{\mathrm{tor}} \end{array} \tag{7.7}$$

where the vertical arrows are equivalences since K and K' are henselian. Lemmas 3.8 and 5.4 imply that e_*, $\widehat{e}_*$ and Rig_* preserve torsion objects. Thus, it is enough to prove that the square (7.7) is adjointable horizontally and vertically, which is clear.

We now assume that M' and N are uniquely divisible or, equivalently, that Λ is a $\mathbb{Q}$-algebra. In this case, we can invoke [AGV22, Theorem 2.5.1 & Proposition 2.5.11] to reduce to the case where the extension K'/K is finite. Using [AGV22, Theorems 2.9.6 & 2.9.7], we can even assume that K'/K is étale. Then, the result follows from [AGV22, Propositions 2.2.13 & 2.2.14]. □

We are now ready to finish the proof of Theorem 7.1.

Proof of Theorem 7.1. By Lemma 7.3, we may assume that K is algebraically closed, and we only need to consider $\Gamma_{\mathrm{new,B}}$ and $\widehat{\Gamma}_{\mathrm{new,B}}$. We now claim that it suffices to treat the case of $\widehat{\Gamma}_{\mathrm{new,B}}$. Indeed, assume that $\mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B}})$ is connective. Then, this already implies that $\Gamma_{\mathrm{new,B}}(\mathrm{pt})$ is a connective algebra. By Theorem 5.9, the obvious morphism

$$\check{\mathrm{C}}(\Gamma_{\mathrm{new,B}}(\mathrm{pt})) \to \mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B}})$$

induces an equivalence after ℓ-completion, for every prime ℓ. It follows that the square

$$\begin{array}{ccc} \check{\mathrm{C}}(\Gamma_{\mathrm{new,B}}(\mathrm{pt})) & \longrightarrow & \mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B}}) \\ \downarrow & & \downarrow \\ \check{\mathrm{C}}(\Gamma_{\mathrm{new,B}}(\mathrm{pt}))\otimes\mathbb{Q} & \longrightarrow & \mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B}})\otimes\mathbb{Q} \end{array}$$

is cocartesian in $(\mathrm{Mod}_\Lambda)^{\Delta}$, and hence it is enough to show that $\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B}})\otimes\mathbb{Q}$ is connective. To do so, we may replace Λ by $\Lambda_{\mathbb{Q}}$ and assume that Λ is a $\mathbb{Q}$-algebra. Under this assumption, the connectivity of $\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{new,B}})$ follows from the connectivity of $\mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B}})$ by Theorem 5.15. This proves our claim.

It remains to see that $\mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B}})$ is connective. By Lemma 4.3, it is enough to show that the algebras $\mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B}})(\Delta^0)$ and $\mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B}})(\Delta^1)$ are connective. These are obtained by applying $\Gamma(\mathrm{pt};\Omega^\infty_T(-))$ to the two algebras $\mathrm{Rig}_*\Lambda\otimes\mathbf{\Gamma}_{\mathrm{B}}$ and $\mathrm{Rig}_*\Lambda\otimes\mathbf{\Gamma}_{\mathrm{B}}\otimes\mathbf{\Gamma}_{\mathrm{B}}$ in $\mathbf{SH}_{\text{ét}}(K;\mathrm{Rig}_*\Lambda)$. Unravelling Definition 4.10 and using Proposition 1.11, we deduce an equivalence $\mathbf{\Gamma}_{\mathrm{B}}\otimes\mathbf{\Gamma}_{\mathrm{B}}\simeq\mathbf{\Gamma}_{\mathrm{B}}\otimes(\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{B}})(\Delta^1))$. It follows that

$$\mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B}})(\Delta^1)\simeq\mathrm{R}\Gamma(\mathrm{pt};\Omega^\infty_T(\mathrm{Rig}_*\Lambda\otimes\mathbf{\Gamma}_{\mathrm{B}}))\otimes(\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{B}})(\Delta^1)).$$

Since the motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(\Gamma_{\mathrm{B}})$ is known to be connective by [Ayo14b, Corollaire 2.105], we are left to check that

$$\begin{aligned}\mathcal{H}_{\mathrm{mot}}(\widehat{\Gamma}_{\mathrm{new,B}})(\Delta^0) &= \mathrm{R}\Gamma(\mathrm{pt};\Omega^\infty_T(\mathrm{Rig}_*\Lambda\otimes\mathbf{\Gamma}_{\mathrm{B}})) \\ &\simeq \mathrm{B}^*(\mathrm{Rig}_*\Lambda)\end{aligned}$$

is connective. We claim that the morphism $\Lambda \to \mathrm{Rig}_*\Lambda$ induces an equivalence in $\mathbf{SH}_{\text{ét}}(K;\Lambda)_{\ell\text{-cpl}}$ after ℓ-completion for every prime ℓ. Indeed, since Rig_* is colimit-preserving (see Lemma 3.8), we have $(\mathrm{Rig}_*\Lambda)^\wedge_\ell \simeq (\mathrm{Rig}_*)_{\ell\text{-cpl}}\Lambda_\ell$ where $(\mathrm{Rig}_*)_{\ell\text{-cpl}}$ is the right adjoint to the functor

$$(\mathrm{Rig}^*)_{\ell\text{-cpl}} : \mathbf{SH}_{\text{ét}}(K;\Lambda)_{\ell\text{-cpl}} \to \mathbf{RigSH}_{\text{ét}}(K;\Lambda)_{\ell\text{-cpl}}.$$

It follows from [AGV22, Theorem 2.10.3 & 2.10.4] that the functor $(\mathrm{Rig}^*)_{\ell\text{-cpl}}$ is an equivalence, proving our claim. That said, we have a cocartesian square in $\mathbf{SH}_{\text{ét}}(K;\Lambda)$

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathrm{Rig}_*\Lambda \\ \downarrow & & \downarrow \\ \Lambda_{\mathbb{Q}} & \longrightarrow & \mathrm{Rig}_*\Lambda_{\mathbb{Q}}. \end{array}$$

Thus, we are reduced to showing that $\mathrm{B}^*(\mathrm{Rig}_*\Lambda_{\mathbb{Q}})$ is connective. Said differently, we may assume that Λ is a $\mathbb{Q}$-algebra and even that $\Lambda = \mathbb{Q}$. Using Proposition 6.12, it is enough to show that the algebra

$$\mathrm{R}^*_{\mathrm{dR}}(\mathrm{Rig}_*\mathbb{Q}) = \mathrm{R}\Gamma(\mathrm{pt}; \Omega^\infty_T(\mathrm{Rig}^*\mathbf{\Gamma}_{\mathrm{dR}}))$$

is connective. This algebra is known to be connective by [Ayo20, Corollaire 3.8]. □

Remark 7.5. Part (2) of Theorem 7.1 admits a version for equi-characteristic valued fields K of height 1 which can be deduced from Part (1) of the same theorem. For simplicity, we only discuss the case of the ℓ-adic cohomology theory $\Gamma_\ell \in \mathrm{RigWCT}(K;\Lambda)$, for a prime ℓ invertible in K. One can reduce to the case where K is algebraically closed by arguing as in the proof of Lemma 7.3. We then apply Proposition 6.7(3) and Theorem 5.17 to conclude.

Appendix A. Some computation with Milnor–Witt K-theory

Given a scheme S, we denote by $\mathbf{SH}(S;\Lambda)$ the stable Morel–Voevodsky ∞-category with coefficients in Λ. (This is the Nisnevich local counterpart of the ∞-category $\mathbf{SH}_{\text{ét}}(S;\Lambda)$.) We do not assume in this appendix that the residual characteristics of S are invertible in $\pi_0(\Lambda)$. Our goal is to show the following result which was used in the proof of Theorem 3.13.

Proposition A.1. *Let S be a scheme such that $-1 \in \mathcal{O}(S)$ is a square. Then, for every integer $m \in \mathbb{Z}$, elevation to the m-th power on $\mathbb{G}_{\mathrm{m},S}$ is given by the matrix*

$$\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} : \Lambda \oplus \Lambda(1)[1] \to \Lambda \oplus \Lambda(1)[1]$$

on the associated homological motive in $\mathbf{SH}(S;\Lambda)$.

The decomposition of the homological motive of $\mathbb{G}_{\mathrm{m},S}$ alluded to in the statement of Proposition A.1 is induced by the unit section of the group scheme $\mathbb{G}_{\mathrm{m},S}$. Modulo this decomposition, the multiplication morphism $m : \mathbb{G}_{\mathrm{m},S} \times_S \mathbb{G}_{\mathrm{m},S} \to \mathbb{G}_{\mathrm{m},S}$ induces a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \eta \end{pmatrix} : \Lambda \oplus \Lambda(1)[1] \oplus \Lambda(1)[1] \oplus \Lambda(2)[2] \to \Lambda \oplus \Lambda(1)[1]$$

on the associated homological motives. The morphism $\eta : \Lambda(1)[1] \to \Lambda$ obtained in this way is known as the Hopf map. On the other hand, an element $a \in \mathcal{O}^\times(S)$, viewed as a section of $\mathbb{G}_{\mathrm{m},S}$, gives rise to a matrix of the form

$$\begin{pmatrix} 1 \\ [a] \end{pmatrix} : \Lambda \to \Lambda \oplus \Lambda(1)[1].$$

It follows immediately from the above discussion that the identity

$$[ab] = [a] + [b] + \eta[a][b]$$

holds in the graded ring $\mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, \Lambda(\bullet)[\bullet])$ for all $a, b \in \mathcal{O}^\times(S)$. In fact, by [Dru21], we even have a morphism of graded rings

$$\mathrm{K}^{\text{"MW"}}_\bullet(S) \to \mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, \Lambda(\bullet)[\bullet])$$

where $\mathrm{K}^{\text{"MW"}}_\bullet(S)$ is the naive Milnor–Witt K-theory ring of S, freely generated by symbols $\eta \in \mathrm{K}^{\text{"MW"}}_{-1}(S)$ and $[a] \in \mathrm{K}^{\text{"MW"}}_1(S)$, one for each $a \in \mathcal{O}^\times(S)$, that satisfy Morel's relations in [Mor12, Definition 3.1]:

(1) (Steinberg relation) For $a, b \in \mathcal{O}^\times(S)$ such that $a + b = 1$, we have $[a][b] = 0$.
(2) For $a, b \in \mathcal{O}^\times(S)$, we have $[ab] = [a] + [b] + \eta[a][b]$.
(3) For $a \in \mathcal{O}^\times(S)$, $[u]\eta = \eta[u]$.
(4) $\eta(2 + \eta[-1]) = 0$.

Given $a \in \mathcal{O}^\times(S)$, we set $\langle a\rangle = 1 + \eta[a]$ as in [Mor12].

Remark A.2. Over a base scheme S, extra care is needed when using the Steinberg relation since there could be invertible elements $a \in \mathcal{O}^\times(S) \smallsetminus \{1\}$ such that $1 - a$ is not invertible. However, the proof of [Mor12, Lemma 3.5] does not use the Steinberg relation and hence is valid over a general base S. Thus, the following relations hold in $\mathrm{K}^{\text{"MW"}}_\bullet(S)$ for all $a, b \in \mathcal{O}^\times(S)$:

- $\langle 1\rangle = 1$ and $[1] = 0$;
- $[ab] = [a] + \langle a\rangle[b] = [a]\langle b\rangle + [b]$;
- $\langle ab\rangle = \langle a\rangle\langle b\rangle$;
- $\langle a\rangle$ is central in $\mathrm{K}^{\text{"MW"}}_\bullet(S)$;
- $[ab^{-1}] = [a] - \langle ab^{-1}\rangle[b]$ and, in particular, $[a^{-1}] = -\langle a^{-1}\rangle[a]$.

For $m \in \mathbb{Z}$, we set following [Mor12]:

$$m_\epsilon = \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor \langle -1\rangle$$

where, for a real number x, $\lceil x\rceil$ is the smallest integer $\geq x$ and $\lfloor x\rfloor$ is the largest integer $\leq x$. In particular, for $m \geq 0$, we have

$$m_\epsilon = \overbrace{1 + \langle 1\rangle + 1 + \cdots}^{m \text{ terms}}$$

and $(-m)_\epsilon = -m_\epsilon\langle -1\rangle$. It is also easy to check that $m_\epsilon n_\epsilon = (mn)_\epsilon$ for all $m, n \in \mathbb{Z}$.

If S is the spectrum of a field, Morel's relations imply that $[a][-a] = 0$. Although, this is unreasonable to expect in $\mathrm{K}^{\text{"MW"}}_\bullet(S)$, for a general S, it is nevertheless satisfied in the graded ring $\mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, \Lambda(\bullet)[\bullet])$.

Lemma A.3. *For $a \in \mathcal{O}^\times(S)$, we have $[a][-a] = 0$ in $\mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, \Lambda(2)[2])$.*

Proof. By functoriality, it is enough to prove that $[t][-t] = 0$ in $\mathrm{Hom}_{\mathbf{SH}(S[t,t^{-1}];\Lambda)}(\Lambda, \Lambda(2)[2])$. The cohomological motives of the S-schemes $S[t, t^{-1}]$ and $S[t, t^{-1}, (1-t)^{-1}]$ are equal to $\Lambda \oplus \Lambda(-1)[-1]$ and $\Lambda \oplus \Lambda(-1)[-1] \oplus \Lambda(-1)[-1]$ respectively. By adjunction, the morphism

$$\mathrm{Hom}_{\mathbf{SH}(S[t,t^{-1}];\Lambda)}(\Lambda, \Lambda(2)[2]) \to \mathrm{Hom}_{\mathbf{SH}(S[t,t^{-1},(1-t)^{-1}];\Lambda)}(\Lambda, \Lambda(2)[2]) \tag{A.1}$$

can be identified with the morphism obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} : \Lambda(2)[2] \oplus \Lambda(-1)[-1] \to \Lambda(2)[2] \oplus \Lambda(-1)[-1] \oplus \Lambda(-1)[-1]$$

by applying $\mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, -)$. In particular, the morphism (A.1) is injective, and it is enough to show that $[t][-t] = 0$ in $\mathrm{Hom}_{\mathbf{SH}(S[t,t^{-1},(1-t)^{-1}];\Lambda)}(\Lambda, \Lambda(2)[2])$. We will actually show this relation in $\mathrm{K}_2^{\text{"MW"}}(S[t, t^{-1}, (1-t)^{-1}])$ by repeating the proof in [Mor12, Lemma 3.7]. Indeed, $1 - t$ and $1 - t^{-1}$ are invertible in $\mathcal{O}(S)[t, t^{-1}, (1-t)^{-1}]$. The Steinberg relation gives $[t^{-1}][1 - t^{-1}] = 0$. But we have:

$$[t^{-1}] = -\langle t^{-1}\rangle[t] \qquad \text{and} \qquad [1 - t^{-1}] = \left[\frac{1-t}{-t}\right] = [1-t] - \left\langle \frac{1-t}{-t} \right\rangle [-t].$$

Putting these relations together, and using that $[t][1-t] = 0$, we obtain the identity $[t][-t] = 0$. □

Corollary A.4. *For $a \in \mathcal{O}^\times(S)$, we have*

$$[a][a] = [-1][a] = [a][-1] \qquad \textit{and} \qquad \langle a^2\rangle = 1$$

in $\mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, \Lambda(\bullet)[\bullet])$.

Proof. This is identical to the proof of the second and fourth parts of [Mor12, Lemma 3.7]. We compute using the relation $[a][-a] = 0$:

$$[a][a] = [a][-(-a)] = [a]([-1] + [-a] + \eta[-1][-a]) = [a][-1].$$

The relation $[a][a] = [-1][a]$ is proven similarly. For the relation $\langle a^2\rangle = 1$, we note that

$$1 + \eta[a^2] = 1 + \eta([a] + [a] + \eta[a][a]) = 1 + \eta(2 + \eta[-1])[a]$$

and conclude using the fourth of Morel's relations. □

Corollary A.5. *For $a, b \in \mathcal{O}^\times(S)$, we have $[a][b] = (-1)_\epsilon[b][a]$ in* $\mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, \Lambda(2)[2])$.

Proof. This is identical to the proof of the third part of [Mor12, Lemma 3.7]. Using Lemma A.3 and Corollary A.4, we have of equalities

$$\begin{aligned} 0 &= [ab][-ab] \\ &= ([a] + \langle a\rangle[b])([-a] + \langle -a\rangle[b]) \\ &= \langle a\rangle[b][-a] + \langle -a\rangle[a][b] + \langle -a^2\rangle[b][b] \\ &= \langle a\rangle[b][-a] + \langle -a\rangle[a][b] + \langle -1\rangle[b][-1] \\ &= \langle a\rangle[b]([a] + \langle a\rangle[-1]) + \langle -a\rangle[a][b] + \langle -1\rangle[b][-1] \\ &= \langle a\rangle([b][a] + \langle -1\rangle[a][b]) + [b][-1] + \langle -1\rangle[b][-1]. \end{aligned}$$

So, to conclude, it remains to show that $[b][-1] + \langle -1\rangle[b][-1] = 0$. Using again Lemma A.3 and Corollary A.4, we have

$$\begin{aligned} 0 &= [b][-b] \\ &= [b]([-1] + \langle -1\rangle[b]) \\ &= [b][-1] + \langle -1\rangle[b][b] \\ &= [b][-1] + \langle -1\rangle[b][-1] \end{aligned}$$

as needed. This finishes the proof. □

Corollary A.6. *For $a \in \mathcal{O}^\times(S)$ and $m \in \mathbb{Z}$, we have $[a^m] = m_\epsilon[a]$ in* $\mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, \Lambda(1)[1])$.

Proof. This is identical to the proof of [Mor12, Lemma 3.14]. Arguing by induction, we have for $m \geq 1$:

$$\begin{aligned}
[a^m] &= [a] + [a^{m-1}] + \eta[a][a^{m-1}] \\
&= [a] + (m-1)_\epsilon[a] + (m-1)_\epsilon\eta[a][a] \\
&= [a] + (m-1)_\epsilon[a] + (m-1)_\epsilon\eta[-1][a] \\
&= (1 + (m-1)_\epsilon\langle -1\rangle)[a] \\
&= m_\epsilon[a].
\end{aligned}$$

The case $m < 0$ is obtained by applying the case $m > 0$ to a^{-1}. □

If -1 is a square in S, Corollary A.4 implies that $\langle -1\rangle = 1$ in $\mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, \Lambda)$. It follows that $m_\epsilon = m$ for all $m \in \mathbb{Z}$. This shows that Proposition A.1 follows from the following more general statement.

Proposition A.7. *Let S be any scheme. Then, for every integer $m \in \mathbb{Z}$, elevation to the m-th power on $\mathbb{G}_{\mathrm{m},S}$ is given by the matrix*

$$\begin{pmatrix} 1 & 0 \\ 0 & m_\epsilon \end{pmatrix} : \Lambda \oplus \Lambda(1)[1] \to \Lambda \oplus \Lambda(1)[1]$$

on the associated homological motive in $\mathbf{SH}(S;\Lambda)$.

Proof. It is more convenient to show that elevation to the m-th power induces the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & m_\epsilon \end{pmatrix} : \Lambda \oplus \Lambda(-1)[-1] \to \Lambda \oplus \Lambda(-1)[-1]$$

on cohomological motives (rather than homological motives). The fact that the matrix is diagonal with first entry 1 follows from the fact that elevation to the m-th power preserves the unit section. It remains to determine the last entry of the matrix. Using the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{SH}(S[t,t^{-1}];\Lambda)}(\Lambda, \Lambda(1)[1]) &\simeq \mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, \Lambda(1)[1] \oplus \Lambda) \\
&\simeq \mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, \Lambda(1)[1]) \oplus \mathrm{Hom}_{\mathbf{SH}(S;\Lambda)}(\Lambda, \Lambda) \cdot [t],
\end{aligned}$$

the result follows from the equality $[t^m] = m_\epsilon[t]$ in $\mathrm{Hom}_{\mathbf{SH}(S[t,t^{-1}];\Lambda)}(\Lambda, \Lambda(1)[1])$. □

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