Integral Invariants of 3-Manifolds (Commentary on [111], [114])

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These papers arose from my collaboration with Raoul during my stay at Harvard. I was there as a postdoc in physics, in the group of Arthur Jaffe. It was the critical time when I had to decide what to do with my life, including whether I should complete my move to mathematics. At the time I was working on perturbative topological field theories and one day, while I was alone in my office, I was visited by a tall guy who said he was interested in what I was doing and wished to start a collaboration to apply my results. I listened to him for a while, getting worried when he began explaining to me that his projects included the construction of a time machine and of a teleporter. I tried a diplomatical way out, telling him that, though his projects were certainly interesting, I unfortunately already had work to complete, also involving some collaborators. The guy got furious, accused me of listening to his ideas just to steal them and started moving towards me threateningly. I was fortunately rescued by my office mates who had been alerted by the secretary who had seen this strange looking guy enter the department: she had found them in the library and told them to wait in front of our office door ready to intervene at the first strange noise.

After this experience, I was wary at the arrival of a new very tall, strong looking though elderly visitor who told me he had attended my recent talk and thought I might help him with one of his problems. I was getting rather nervous, when the visitor said, "By the way, I am Raoul Bott." So started our collaboration!

The problem Raoul had was that of understanding the perturbative Chern–Simons 3-manifold invariants extending what he had done in his work with Cliff Taubes on the link invariants [106]. Recall that in his celebrated paper Witten had explained how to get 3-manifold and link invariants out of Chern–Simons theory. It was later realized by many authors that the usual perturbative expansion in quantum field theory would yield invariants at each order in the expansion parameter (h bar). One way to cope with the problem, the Kontsevich integral, corresponds to what physicists call the axial gauge and requires presenting the link as the closure of a braid; the 3-manifold invariants are then recovered as special invariants of the link along which one performs surgery. Another approach uses what physicists call the covariant, or Lorenz, gauge. The corresponding link invariants, as presented in Raoul's paper with Cliff, are obtained as integrals over configuration spaces of a product of 2-forms associated to (certain) pairs of points—the propagators. Geometrically, the propagator is just the pullback of the invariant volume form on the 2-sphere by the map that assigns two distinct points their normalized difference vector.

The case of the 3-manifold invariants requires solving two problems: finding a propagator and understanding the role of the underlying cohomology. An explicit construction for the propagator, depending on the choice of a Riemannian metric, was given by Axelrod and Singer, who also showed how to construct invariants corresponding to the expansion of Chern–Simons theory around an acyclic connection. This construction unfortunately rules out the very important case of the trivial connection, which is certainly not acyclic. Kontsevich proposed a way out in the case of rational homology spheres by removing a point (so killing all cohomology in positive degree) and working with what was termed a singular framing. His construction was at the time - mistakenly! - thought to give trivial invariants.

Our approach generalized the above in two directions. On the one hand, it was Raoul's strong suggestion that the propagator should be thought of as (an appropriate) Poincaré dual of the diagonal inside the Cartesian square of the manifold. This gave a much more general class of propagators than in Axelrod and Singer; for example one can now also have propagators with compact supports (this feature starts being appreciated again in the context of Costello's work). In [114] we applied this to the acyclic case and completed the program started by Axelrod and Singer. For the trivial connection on a rational homology sphere [111] we had to find out a suitable modification of the underlying graphs (Feynman diagrams) to accommodate for the non trivial cohomology: this resulted in allowing special vertices decorated by a (chosen) representative of the top cohomology class. It was a first instance of the computation of an "effective action as a function of zero modes," a theme to which I unexpectedly returned to several years later in collaborations with Felder and with Mnev.

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